Supplement til SDL, Blok 1 2009

Inst. f. Matematiske Fag, KU
Gerd Grubb
November 2009

SOLUTION OF THE CLASSROOM TEST

Exercise E14.

We are considering the differential equation for $(t, \boldsymbol{y}) \in \mathbb{R}^3$:

$$\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y}), \text{ where } \boldsymbol{f}(\boldsymbol{y}) = \begin{pmatrix} e^{y_1} - 1 - 2y_2 \\ 3y_1 - 4y_2 \end{pmatrix}$$

with the initial condition

$$\boldsymbol{y}(t_0) = \boldsymbol{\eta}.$$

(a). Since all the entering functions are C^∞-functions, the conditions for applying Theorem S1 are satisfied; this assures that for any $\boldsymbol{\eta} \in \mathbb{R}^2$, $t_0 \in \mathbb{R}$, there exists a unique maximal solution $\boldsymbol{\varphi}(t)$ defined on an open interval containing t_0. As in Theorem S4.2 we use the notation $]c^*, d^*[$ for the interval.

The possible ways of behavior of the solution for $t \to d^*$ are given in Corollary S4.3. Since the open set where $(t, \boldsymbol{y})$ runs is $D = \mathbb{R}^3$, the boundary ∂D is the empty set. Then (a) and (c) in Corollary S4.3 cannot happen. Therefore (b) happens, $|t| + |\boldsymbol{\varphi}(t)| \to \infty$ for $t \to d^*$. If d^* is finite, it is $|\boldsymbol{\varphi}(t)|$ that goes to ∞.

(b). That $\mathbf{0}$ is a critical point means that $\boldsymbol{f}(\mathbf{0}) = \mathbf{0}$. We see by insertion of $\boldsymbol{y} = \mathbf{0}$ that

$$\boldsymbol{f}(\mathbf{0}) = \begin{pmatrix} e^0 - 1 - 2 \cdot 0 \\ 3 \cdot 0 - 4 \cdot 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}.$$

(c). Taylor's formula for the exponential function gives that for y_1 in an interval $[-k, k]$ (with $k > 0$),

$$e^{y_1} = 1 + y_1 + \tfrac{1}{2} y_1^2 + o(y_1^2) = 1 + y_1 + h(y_1),$$

where $|h(y_1)| \leq c|y_1|^2$ for some $c > 0$. Then

$$e^{y_1} - 1 = y_1 + h(y_1).$$

Now we can write

$$\boldsymbol{f}(\boldsymbol{y}) = \begin{pmatrix} e^{y_1} - 1 - 2y_2 \\ 3y_1 - 4y_2 \end{pmatrix} = \begin{pmatrix} y_1 + h(y_1) - 2y_2 \\ 3y_1 - 4y_2 \end{pmatrix} = A\boldsymbol{y} + \boldsymbol{g}(\boldsymbol{y}),$$

where

$$A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}, \quad \boldsymbol{g}(\boldsymbol{y}) = \begin{pmatrix} h(y_1) \\ 0 \end{pmatrix}.$$

The eigenvalues of the matrix A are determined as the roots of

$$p_A(\lambda) = (1-\lambda)(-4-\lambda) - (-2)3 = \lambda^2 + 3\lambda + 2,$$

they are found to have the negative values -1 and -2. Moreover, for $|y_1| \le k$,

$$\frac{|\boldsymbol{g}(\boldsymbol{y})|}{|\boldsymbol{y}|} = \frac{|h(y_1)|}{|y_1| + |y_2|} \le \frac{c|y_1|^2}{|y_1|} = c|y_1| \to 0 \text{ for } \boldsymbol{y} \to \boldsymbol{0}.$$

Then the assumptions of Theorem 4.3 in the book are satisfied, so it follows that the null-solution is asymptotically stable.
(*Comment.* It is also OK to indicate $h(y_1)$ by the explicit series $\frac{1}{2!}y_1^2 + \frac{1}{3!}y_1^3 + \dots$, as long as one can show that $h(y_1)/(|y_1| + |y_2|) \to 0$ for $\boldsymbol{y} \to 0$. L'Hospital's rule can be used.

Some people have tried to use Theorem 4.3 with matrix $\begin{pmatrix} 0 & -2 \\ 3 & -4 \end{pmatrix}$ and remainder $\begin{pmatrix} e^{y_1} - 1 \\ 0 \end{pmatrix}$; here the eigenvalues of the matrix do have negative real part, but the remainder does not have the needed limit property, since $(e^{y_1} - 1)/y_1 \to 1$ for $y_1 \to 0$.)

Exercise E15.

We are considering the differential equation for $(t, \boldsymbol{y}) \in \mathbb{R}^4$:

$$\boldsymbol{y}' = A\boldsymbol{y}, \text{ where } A = \begin{pmatrix} 5 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 7 \end{pmatrix}.$$

(a). The eigenvalues of A are determined as the roots of the polynomial

$$\begin{aligned} p_A(\lambda) &= \det(A - \lambda E) = (5-\lambda)(2-\lambda)(7-\lambda) - 1\cdot(2-\lambda)\cdot(-1) \\ &= (2-\lambda)((5-\lambda)(7-\lambda)+1) = (2-\lambda)(\lambda^2 - 12\lambda + 36) = (2-\lambda)(\lambda-6)^2. \end{aligned}$$

Here 2 is a simple root, 6 a double root.

For $\lambda = 2$, the eigenvectors are found as the nontrivial solutions of

$$\begin{pmatrix} 3 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 5 \end{pmatrix} \boldsymbol{y} = \boldsymbol{0},$$

and it is seen that they are multiples of the vector $(0, 1, 0)$. So the eigenspace is spanned by this vector, and it is the same as the generalized eigenspace X_1 for $\lambda = 2$, since the eigenvalue is simple.

For $\lambda = 6$, the eigenvectors are found as the nontrivial hsolutions of

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -4 & 0 \\ -1 & 0 & 1 \end{pmatrix} \boldsymbol{y} = \boldsymbol{0},$$

and it is seen that they are multiples of the vector $(1, 0, 1)$. So the eigenspace is spanned by this vector. Since $\lambda = 6$ has multiplicity 2, the generalized eigenspace has dimension 2, and another vector in it is found by calculating

$$(A - 6E)^2 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -4 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 0 & -4 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and finding a solution of $(A - 6E)^2 \boldsymbol{y} = 0$ that is linearly independent of $(1, 0, 1)$; here we can for example take $(1, 0, 0)$. The generalized eigenspace X_2 is then the span of $(1, 0, 1)$ and $(1, 0, 0)$, and, even simpler, it is the span of $(1, 0, 0)$ and $(0, 0, 1)$.

(b). To find the fundamental matrix e^{tA} we use the formulas on page 66 of the book. For $\boldsymbol{v}_1 \in X_1$, $\boldsymbol{v}_1 = (0, x_2, 0)$

$$e^{tA} \boldsymbol{v}_1 = e^{2t} \boldsymbol{v}_1 = \begin{pmatrix} 0 \\ e^{2t} x_2 \\ 0 \end{pmatrix}.$$

For $\boldsymbol{v}_2 \in X_2$, $\boldsymbol{v}_2 = (x_1, 0, x_3)$,

$$e^{tA} \boldsymbol{v}_2 = e^{6t}(E + t(A - 6E)) \boldsymbol{v}_2 = e^{6t} \begin{pmatrix} 1-t & 0 & t \\ 0 & 1-4t & 0 \\ -t & 0 & 1+t \end{pmatrix} \begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix}$$

$$= \begin{pmatrix} (1-t)e^{6t} & 0 & te^{6t} \\ 0 & 0 & 0 \\ -te^{6t} & 0 & (1+t)e^{6t} \end{pmatrix} \begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix}.$$

Adding the formulas, we find

$$e^{tA} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} (1-t)e^{6t} & 0 & te^{6t} \\ 0 & e^{2t} & 0 \\ -te^{6t} & 0 & (1+t)e^{6t} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

so

$$e^{tA} = \begin{pmatrix} (1-t)e^{6t} & 0 & te^{6t} \\ 0 & e^{2t} & 0 \\ -te^{6t} & 0 & (1+t)e^{6t} \end{pmatrix}.$$

(c). To find a solution of the nonhomogeneous problem

$$\boldsymbol{y}' = A\boldsymbol{y} + \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix}, \quad \boldsymbol{y}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

we use that this is the sum of the solution $\boldsymbol{\varphi}$ of the homogeneous problem with the given initial value and the solution $\boldsymbol{\psi}$ of the nonhomogeneous problem which is $\mathbf{0}$ at $t = 0$. The first function is

$$\boldsymbol{\varphi}(t) = e^{tA} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{6t} \\ e^{2t} \\ e^{6t} \end{pmatrix}.$$

The second function is

$$\boldsymbol{\psi}(t) = \int_0^t e^{(t-s)A} \begin{pmatrix} 0 \\ e^s \\ 0 \end{pmatrix} ds = \begin{pmatrix} 0 \\ \int_0^t e^{2(t-s)} e^s \, ds \\ 0 \end{pmatrix};$$

here

$$\int_0^t e^{2(t-s)} e^s \, ds = e^{2t} \Big[-e^{-s} \Big]_0^t = e^{2t} - e^t.$$

Then we find the solution to be:

$$\boldsymbol{\varphi} + \boldsymbol{\psi} = \begin{pmatrix} e^{6t} \\ 2e^{2t} - e^t \\ e^{6t} \end{pmatrix}.$$

(*Comment.* If we exchange the second and third coordinate, the matrix gets the form

$$\begin{pmatrix} 5 & 1 & 0 \\ -1 & 7 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

and it is seen clearly how the problem breaks up into two easy problems, for (x_1, x_2) resp. x_3. One could answer the problem using this transformation.)