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Random Matrices, Free Probability and the Invariant Subspace Problem Relative to a von Neumann Algebra

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1. Introduction

Random matrices have their roots in multivariable analysis in statistics, and since Wigner's pioneering work [Wi] from 1955, they have been a very important tool in mathematical physics. In functional analysis, random matrices and random structures have in the last two decades been used to construct Banach spaces with surprising properties. After Voiculescu in 1990–91 used random matrices to classification problems for von Neumann algebras, they have played a key role in von Neumann algebra theory (cf. [V8]). In this article we will discuss some new applications of random matrices to operator algebra theory, namely applications to classification problems for C^*-algebras and to the invariant subspace problem relative to a von Neumann algebra.

The rest of this lecture is divided into eight sections:

2. Selfadjoint random matrices and Wigner's semicircle law.
3. Free probability and Voiculescu's random matrix model.
4. $\mathrm{Ext}(C^*_r(F_k))$ is not a group for $k \geq 2$.
5. Other applications of random matrices to C^*-algebras.
6. The invariant subspace problem relative to a von Neumann algebra.
7. The Fuglede-Kadison determinant and Brown's spectral distribution measure.
8. Spectral subspaces for operators in II_1-factors.
9. Voiculescu's circular operator Y and the strictly upper triangular operator T.

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2. Selfadjoint random matrices and Wigner's semicircle law

A random matrix X is an $n \times n$ matrix whose entries are real or complex random variables on a probability space $(\Omega, \mathcal{F}, P)$. We denote by $\mathrm{SGRM}(n, \sigma^2)$ the class of selfadjoint random matrices

$$X_n = (X_{ij}^{(n)})_{i,j=1}^n$$

where X_{ij}, $i, j = 1, \dots, n$ are n^2 complex random variables and

$$(X_{ii}^{(n)})_i, \quad (\sqrt{2}\,\mathrm{Re}\,X_{ij}^{(n)})_{i<j}, \quad (\sqrt{2}\,\mathrm{Im}\,X_{ij}^{(n)})_{i<j}$$

are n^2 independent identical distributed real Gaussian random variables with mean value 0 and variance σ^2. In the terminology of Mehta's book [Me], X_n is a Gaussian unitary ensemble (GUE). In the following we put $\sigma^2 = \frac{1}{n}$ which is the normalization used in Voiculescu's random matrix paper [V4]. By results of Gaudin, Mehta and Wigner from 1960–65, the joint distribution of the eigenvalues (in random order) of X has density g given by

$$g_n(\lambda_1, \dots, \lambda_n) = c_n \prod_{i<j} (\lambda_j - \lambda_i)^2 \exp\big(-\frac{n}{2}\sum_{i=1}^n \lambda_i^2\big)$$

where c_n is a normalization constant, and the (average) density for a single eigenvalue is given by

$$h_n(x) = \frac{1}{\sqrt{2n}} \sum_{k=0}^{n-1} \varphi_k\big(\sqrt{\frac{n}{2}}x\big)^2$$

where $\varphi_0, \varphi_1, \dots$ is the sequence of Hermite functions. Moreover,

$$\lim_{n\to\infty} h_n(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\, 1_{[-2,2]}(x), \quad x \in \mathbb{R}$$

(cf. [Me]). This is Wigner's semicircle law for the GUE-case. In the sense of weak convergence of probability measures, the semicircle law can be proved under much more general assumptions on the entries (see Wigner [Wi]). Arnold proved in 1967 that the corresponding strong law also holds, i.e. for almost all ω in the probability space Ω, the empirical eigenvalue distribution of $X_n(\omega)$ converges weakly to the semicircular distribution $\frac{1}{2\pi}\sqrt{4 - x^2}\, 1_{[-2,2]}(x)dx$ as $n \to \infty$. Very interesting research have been carried out on the level spacing of the eigenvalues in the bulk of the spectrum (cf. [Me]) and more recently near the boundary of the spectrum (cf. [TW1], [TW2]) for selfadjoint Gaussian random matrices with real, complex or symplectic entries (the GOE, GUE and GSE cases), but this is outside the scope of the present lecture.

3. Free probability and Voiculescu's random matrix model

Voiculescu proved in 1991 [V4] an extensive generalization of Wigner's semicircle law to families of independent random matrices. In order to state the result, we will need some basic concepts from free probability theory (cf. [V2], [V3] and [VDN]).

Definition 3.1 *[V2]*

1. *A non-commutative probability space is a pair (A, φ) consisting of a unital complex algebra A and a functional $\varphi\colon A \to \mathbb{C}$ such that $\varphi(1_A) = 1$.*
2. *A C^*-probability space is a pair (A, φ) consisting of a unital C^*-algebra A and a state $\varphi\colon A \to \mathbb{C}$ on A.*

The connection to classical probability theory on a probability space $(\Omega, \mathcal{F}, P)$ is obtained by putting

$$A = \bigcap_{p=1}^{\infty} L^p(\Omega)$$

and

$$\varphi(a) = \mathbb{E}(a) = \int_{\Omega} a(\omega) dP(\omega), \quad a \in A$$

or $A' = L^\infty(\Omega, P)$ with the same definition of φ. The latter example is a C^*-probability space. To fit random matrices (of size n) into this framework, one must instead consider the non-commutative algebra

$$A_n = \bigcap_{p=1}^{\infty} L^p(\Omega, M_n(\mathbb{C}))$$

with functional

$$\varphi_n(a) = \mathbb{E}(\mathrm{tr}_n(a)) = \int_{\Omega} \mathrm{tr}_n(a(\omega)) d\omega$$

where $\mathrm{tr}_n = \frac{1}{n}\mathrm{Tr}$ is the normalized trace on $M_n(\mathbb{C})$.

Definition 3.2 *[V2], [V3]*

1. *A family $(a_i)_{i\in 1}$ of elements in a non-commutative probability space is a* free family *if for all $n \in \mathbb{N}$ and all polynomials $p_1, \ldots, p_n \in \mathbb{C}[X]$, one has*

$$\varphi(p_1(a_{i_1}) \cdot \ldots \cdot p_n(a_{i_n})) = 0$$

whenever $i_1 \neq i_2 \neq \cdots \neq i_n$ (neighbouring indices are different) and $\varphi(p_k(a_{i_k})) = 0$ for $k = 1, \ldots, n$.

2. *A family* $(X_i)_{i\in j}$ *of elements in a* C^**-probability space* (A,φ) *is called a semicircular family if* $(x_i)_{i\in I}$ *is a free family,* $x_i = x_i^*$ *and*

$$\varphi(x_i^k) = \frac{1}{2\pi}\int_{-2}^{2} t^k\sqrt{4-t^2}dt = \frac{1}{k+1}\binom{2k}{k}$$

for all $k\in\mathbb{N}$ *and all* $i\in I$.

We can now formulate Voiculescu's generalization of Wigner's semicircle law:

Theorem 3.3 *[V4] Let* I *be an index set and let for each* $n\in\mathbb{N}$, $(X_i^{(n)})_{i\in I}$ *be a family of independent SGRM*$(n,\frac{1}{n})$*-distributed selfadjoint random matrices. Then asymptotically as* $n\to\infty$ $(X_i^{(n)})_{i\in I}$ *is a semicircular family, i.e. if* $(X_i)_{i\in I}$ *is a semicircular family index by* I *in a* C^**-probability space* (A,φ) *then*

$$\lim_{n\to\infty}\mathbb{E}\,\mathrm{tr}_n(X_{i_1}^{(n)}\cdot\ldots\cdot X_{i_p}^{(n)}) = \varphi(x_{i_1}\cdot\ldots\cdot x_{i_p}) \tag{3.1}$$

for all $p\in\mathbb{N}$ *and all* $i_1,\ldots,i_p\in I$.

The corresponding strong law: For almost all $\omega\in\Omega$, one has

$$\lim_{n\to\infty}\mathrm{tr}_n(X_{i_1}^{(n)}(\omega)\cdot\ldots\cdot X_{i_p}^{(n)}(\omega)) = \varphi(x_{i_1}\cdot\ldots\cdot x_{i_p}) \tag{3.2}$$

was proved independently by Hiai and Petz [HP2] and Thorbjørnsen [T].

4. $\mathrm{Ext}(C_r(F_k))$ is not a group for $k\geq 2$

Very recently Thorbjørnsen and the lecturer proved that the strong version (3.2) of Voiculescu's random matrix model also holds for the operator norm:

Theorem 4.1 *[HT4] Let* $r\in\mathbb{N}$ *and let for each* $n\in\mathbb{N}$ $(X_1^{(n)},\ldots,X_r^{(n)})$ *be a set of* r *independent SGRM*$(n,\frac{1}{n})$*-distributed selfadjoint random matrices. Let further* $(x,\ldots,x_r)$ *be a semicircular system in a* C^**-probability space* (A,φ)*, where* φ *is a faithful state on* A*. Then there is a null set* $N\subseteq\Omega$ *such that for all* $\omega\in\Omega\backslash N$ *and all non-commutative polynomials* P *in* r *variables*

$$\lim_{n\to\infty}\|P(X_1^{(n)}(\omega),\ldots,X_r^{(n)}(\omega))\| = \|P(x_1,\ldots,x_r)\|.$$

Let Γ be a countable (discrete) group. The reduced group C^*-algebra $C_r^*(\Gamma)$ is the C^*-subalgebra of $B(\ell^2(\Gamma))$ generated by the set of unitaries $\{\lambda(\gamma)\mid\gamma\in\Gamma\}$, where $\lambda\colon\Gamma\to B(\ell^2(\Gamma))$ is the left regular representation. By the methods of [V3] it follows that for the free group F_k on k generators, $C_r^*(F_k)$ can be embedded in $C^*(x_1,\ldots,x_k,1)$, where $x_1,\ldots,x_k$ is a free semicircular family in a C^*-probability space (A,φ) with φ faithful. Hence as a corollary of Theorem 4.1 we have

Corollary 4.2 *[HT4] Let $k \in \mathbb{N}$, $k \geq 2$. Then $C_r^*(F_k)$ can be embedded in the quotient C^*-algebra $\prod M_n(\mathbb{C})/\sum M_n(\mathbb{C})$ where*

$$\begin{aligned} \prod M_n(\mathbb{C}) &= \left\{ (x_n)_{n=1}^{\infty} \mid x_n \in M_n(\mathbb{C}),\ \sup_n \|x_n\| < \infty \right\} \\ \sum M_n(\mathbb{C}) &= \left\{ (x_n)_{n=1}^{\infty} \mid x_n \in M_n(\mathbb{C}),\ \lim_{n\to\infty} \|x_n\| = 0 \right\}. \end{aligned}$$

In particular $C_r^*(F_k)$ is a MF-algebra in the sense of Blackadar and Kirchberg [BK].

The invariant $\mathrm{Ext}(A)$ for a C^*-algebra A was introduced by Brown, Douglas and Fillmore in [BDF]. $\mathrm{Ext}(A)$ is the set of all essential extensions B of A by the compact operators K on the Hilbert space $\ell^2(\mathbb{N})$, and it has a natural semigroup structure. Voiculescu proved in [V1] that $\mathrm{Ext}(A)$ is always a unital semigroup, and by Choi and Effros [CE] $\mathrm{Ext}(A)$ is a group, when A is a nuclear C^*-algebra. Andersen [An] provided in 1978 the first example of a C^*-algebra A for which $\mathrm{Ext}(A)$ is not a group. The C^*-algebra in [An] is generated by $C_r^*(F_2)$ and a projection $p \in B(\ell^2(F_2))$. Since then it has been an open problem whether $\mathrm{Ext}(C_r^*(F_2))$ is a group (see [V6, Sect.5] for a more detailed discussion about this problem). It is well known that a proof of Corollary 4.2 would provide a negative solution to this problem (see [V6, 5.12], [V5] and [Ro]). The argument works for all $k \geq 2$. Hence we have

Corollary 4.3 *[HT4] For all $k \in \mathbb{N}$, $k \geq 2$, $\mathrm{Ext}(C_r^*(F_k))$ is not a group.*

Remarks 4.4
a) Corollaries 4.2 and 4.3 also hold for $k = \infty$.
b) $C_r^*(F_k)$ is not quasidiagonal (cf [Ro]) but the non-invertible extension B of $C_r^*(F_k)$ obtained from Corollary 4.2 is quasidiagonal.
c) $C_r^*(F_k)$ is an exact C^*-algebra, but for any non-invertible extension B of $C_r^*(F_k)$ by the compact operators, B cannot be exact. This follows from the Lifting theorem in [EH]. Other examples of non-exact extensions of exact C^*-algebras by K are given in [Ki2].

In the rest of this section, I will briefly outline the main steps in the proof of Theorem 4.1. From (3.2) it follows that for all non-commutative polynomials P in r variables

$$\liminf_{n\to\infty} \|P(X_1^{(n)}(\omega), \ldots, X_r^{(n)}(\omega))\| \geq \|P(x_1, \ldots, x_r)\| \tag{4.1}$$

for almost all $\omega \in \Omega$ (see [T]), so we "only" have to prove that

$$\limsup_{n\to\infty} \|P(X_1^{(n)}(\omega), \ldots, X_r^{(n)}(\omega))\| \leq \|P(x_1, \ldots, x_r)\| \tag{4.2}$$

for almost all $\omega \in \Omega$. Even the case $r = 1$ and $P(x) = x$ is a difficult task. It corresponds to proving that if X_n is $\mathrm{SGRM}(n, \frac{1}{n})$-distributed, $n = 1, 2, \ldots$ then for almost all $\omega \in \Omega$,

$$\begin{aligned} \limsup_{n\to\infty} \lambda_{\max}(X_n(\omega)) &\leq 2 \\ \liminf_{n\to\infty} \lambda_{\min}(X_n(\omega)) &\geq -2 \end{aligned}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the smallest and largest eigenvalue of $X_n(\omega)$. This problem was settled by Bai and Yin [BY] in 1988 using Geman's combinatorial method [Ge]. (See also [Ba, Thm. 2.12] and [HT1, Thm. 3.1]).

Lemma 4.5 *(The linearization trick) [HT4] In order to prove* (4.2) *it is sufficient to show that for all $m \in \mathbb{N}$ and all selfadjoint $m \times m$-matrices $a_0, \ldots, a_r$ and all $\varepsilon > 0$,*

$$\sigma\big(a_0 \otimes 1 + \sum_{i=1}^{r} a_i \otimes X_i^{(n)}(\omega)\big) \subseteq \sigma\big(a_0 \otimes 1 + \sum_{i=1}^{r} a_i \otimes x_i\big) +]-\varepsilon, \varepsilon[\tag{4.3}$$

holds eventually as $n \to \infty$ for almost all $\omega \in \Omega$. Here $\sigma(T)$ denotes the spectrum of a matrix or an operator T.

Lemma 4.6 *[HT4] Let $a_0, \ldots, a_r$ be as above, and put*

$$\begin{aligned} S_n &= a_0 \otimes 1 + \sum_{i=1}^{r} a_i \otimes X_i^{(n)} \\ s &= a_0 \otimes 1 + \sum_{i=1}^{r} a_i \otimes x_i. \end{aligned}$$

Moreover, let G_n, G be the matrix valued Stieltjes transforms of S_n and S, i.e. for $\lambda \in M_n(\mathbb{C})$, and $\operatorname{Im}\lambda = \frac{1}{2i}(\lambda - \lambda^*)$ positive definite

$$\begin{aligned} G_n(\lambda) &= \mathbb{E}((id_m \otimes \mathrm{tr}_n)((\lambda \otimes 1 - S_n)^{-1})) \\ G(\lambda) &= (id_m \otimes \varphi)((\lambda \otimes 1 - s)^{-1}). \end{aligned}$$

Then $G_n(\lambda)$ and $G(\lambda)$ are invertible and

$$a_0 + \sum_{i=1}^{r} a_i G(\lambda) a_i + G(\lambda)^{-1} = \lambda \tag{4.4}$$

$$\big\| a_0 + \sum a_i G_n(\lambda) a_i + G_n(\lambda)^{-1} - \lambda \big\| \leq \frac{C}{n^2}(K + \|\lambda\|)^2 \|(Im\lambda)^{-1}\|^5 \tag{4.5}$$

where $C = \frac{\pi^2 m^3}{8}\big(\sum_{i=1}^{r} \|a_i\|^2\big)^2$ and $K = \|a_0\| + 4\sum_{i=1}^{r} \|a_i\|$.

The equality (4.4) was proved by Lehner (cf. [Le, Prop.4.1] using Voiculescu's R-transform with amalgamation [V7]. The inequality (4.5) is more difficult. It relies on the concentration phenomena used in Banach space theory, in form of [P1, Theorem 4.7]. (See [Mi] for a general discussion of the concentration phenomena.) Next we derive from (4.4) and (4.5) that

$$\|G_n(\lambda) - G(\lambda)\| \leq \frac{4C}{n^2}(K + \|\lambda\|)^2 \|(\operatorname{Im}\lambda)^{-1}\|^7 \tag{4.6}$$

when $\lambda \in M_m(\mathbb{C})$ and $\operatorname{Im}\lambda$ is positive definite. The estimate (4.6) implies that for every $f \in C_c^\infty(\mathbb{R})$

$$\mathbb{E}((\mathrm{tr}_m \otimes \mathrm{tr}_n)(f(S_n))) = (\mathrm{tr}_m \otimes \varphi)(f(s)) + O\big(\frac{1}{n^2}\big) \tag{4.7}$$

for $n \to \infty$. Moreover a second application of the concentration phenomena gives

$$\mathrm{Var}((\mathrm{tr}_m \otimes \mathrm{tr}_n)(f(S_n))) \leq \frac{\pi^2}{8n^2}\mathbb{E}((\mathrm{tr}_m \otimes \mathrm{tr}_n)(f'(S_n)^2)) \tag{4.8}$$

where Var denotes the variance. Now let g be a $C^\infty(\mathbb{R})$-function with values in $[0,1]$ such that g vanishes on $\sigma(S)$ and g is 1 on the complement of $\sigma(S)+]-\varepsilon,\varepsilon[$. By applying (4.7) and (4.8) to $f = g - 1$, one gets

$$\mathbb{E}((\mathrm{tr}_m \otimes \mathrm{tr}_n)(g(S_n)) \;=\; O\big(\frac{1}{n^2}\big) \tag{4.9}$$

$$\mathrm{Var}((\mathrm{tr}_m \otimes \mathrm{tr}_n)g(S_n)) \;=\; O\big(\frac{1}{n^4}\big). \tag{4.10}$$

By a standard application of the Borel-Cantelli lemma (4.9) and (4.10) imply

$$(\mathrm{tr}_m \otimes \mathrm{tr}_n)(g(S_n(\omega))) = O(n^{-4/3})$$

almost surely. Hence the number of eigenvalues for $S_n(\omega)$ outside $\sigma(s)+]-\varepsilon,\varepsilon[$ is $O(n^{-1/3})$[1] almost surely, but being an integer, the number has to vanish eventually as $n \to \infty$ for almost all $\omega \in \Omega$. Hence (4.3) holds.

5. Other applications of random matrices to C^*-algebras

A C^*-algebra A is called exact if for every short exact sequence of C^*-algebras

$$0 \to J \to B \to B/J \to 0$$

the sequence

$$0 \to A \otimes_{\min} J \to A \otimes_{\min} B \to A \otimes_{\min} (B/J) \to 0$$

is exact (cf. [Ki1], [Wa]). The class of exact C^*-algebras is very large: All nuclear C^*-algebras are exact and the reduced group C^*-algebra $C_r^*(\Gamma)$ is exact for any discrete subgroup Γ of a connected locally compact group (cf. [Ki2]). In 1991 the

[1] tr_m and tr_n are the normalized traces on $M_n(\mathbb{C})$ and $M_n(\mathbb{C})$.

lecturer proved that 2-quasitraces on unital exact C^*-algebras are traces (cf. [Haa1]). Combined with results of Handelman [Han] and Blackadar and Rørdam [BR], this implies that

$$\text{Every stably finite exact unital } C^*\text{-algebra has a tracial state.} \tag{5.1}$$

$$\text{Every state on the } K_0\text{-group, } K_0(A) \text{ of an exact unital } C^*\text{-algebra } A \text{ is induced by a tracial state on } A. \tag{5.2}$$

Later, Thorbjørnsen and the lecturer found new proofs based on random matrices for (5.1) and (5.2). The key step in the proof was to show:

Theorem 5.1 *[HT2] Let A be an exact unital C^*-algebra, and let $a_1, \dots, a_r \in A$ be elements in A for which*

$$\sum_{i=1}^{r} a_i^* a_i = c\mathbf{1_A} \qquad \textit{where } c > 1 \tag{5.3}$$

$$\sum_{i=1}^{r} a_i a_i^* \leq \mathbf{1_A} \tag{5.4}$$

and let $Y_1^{(n)}, \dots, Y_r^{(n)}$ be random $n \times n$-matrices whose entries are rn^2 independent identically distributed complex Gaussian random variables with density $\frac{n}{\pi}\exp(-n|z|^2)$, $z \in \mathbb{C}$. Put

$$S_n = \sum_{i=1}^{r} a_i \otimes Y_i^{(n)} \tag{5.5}$$

and let $\sigma(S_n^* S_n)$ be the spectrum of $S_n^* S_n$ as a function of $\omega \in \Omega$ (the underlying probability space). Then for almost all $\omega \in \Omega$

$$\limsup_{n\to\infty} \max(\sigma(S_n^* S_n)) \leq (\sqrt{c}+1)^2 \tag{5.6}$$

$$\liminf_{n\to\infty} \min(\sigma(S_n^* S_n)) \geq (\sqrt{c}-1)^2 \tag{5.7}$$

The result is a kind of generalization of the results of Geman 1980 [Ge] and Silverstein 1985 [Si] on the asymptotic behaviour of the largest and smallest eigenvalue of a random matrix of Wishart type. The estimates (5.6) and (5.7) were proved by careful moment estimates and lengthy combinatorial arguments. With Theorem 4.1 at hand, a much simpler proof of (5.6) and (5.7) can now be obtained (cf. [HT4]).

Theorem 5.1 is not true in the general non-exact case (cf. [HT3]). It is unknown whether (5.1) or (5.2) hold for general C^*-algebras. Both problems are equivalent to Kaplansky's problem from the 50'es: Is every AW*-factor of type II_1 a von Neumann factor of type II_1?

Let me end this section by discussing another application of Theorem 4.1: Junge and Pisier proved in [JP] that

$$B(H) \otimes_{\max} B(H) \neq B(H) \otimes_{\min} B(H). \tag{5.8}$$

In the proof they consider a sequence of constants $C(k)$, $k \in \mathbb{N}$: For fixed $k \in \mathbb{N}$ $C(k)$ is the infimum of all $C > 0$ for which there exists a sequence of k-tuples of unitary matrices $(u_1^{(m)}, \dots, u_k^{(m)})_{m\in\mathbb{N}}$ of size $n(m) \in \mathbb{N}$, such that for all $m \neq m'$:

$$\|\sum_{i=1}^{k} u_i^{(m)} \otimes u_i^{(m')}\| \leq C.$$

To obtain (5.8), Junge and Pisier proved that $\lim_{k\to\infty} \frac{C(k)}{k} = 0$. Subsequently, Pisier [P3] proved that $C(k) \geq 2\sqrt{k-1}$ for all $k \in \mathbb{N}$ and Valette [V] proved using Ramanujan graphs, that $C(k) \leq 2\sqrt{k-1}$ when k is of the form $k = p+1$ for an odd prime number p. It is an easy consequence of Corollary 4.2 that $C(k) \leq 2\sqrt{k-1}$ for all $k \geq 2$ and hence $C(k) = 2\sqrt{k-1}$ for all $k \geq 2$ (see [HT4]).

6. The invariant subspace problem relative to a von Neumann algebra

The invariant subspace problem for operators on general Banach spaces were settled by Enflo [E] and Read [Re] in the 80'es, but for Hilbert spaces the problem is still open:

Problem 6.1 *[Hal] Let H be a separable infinite dimensional Hilbert space, and let $T \in B(H)$. Does there exist a non-trivial closed T-invariant subspace of H?*

More generally, one has the invariant subspace problem relative to a von Neumann algebra:

Problem 6.2 *Let $M \subseteq B(H)$ be a von Neumann algebra on a separable Hilbert space H, and let $T \in M$. Does there exist a non-trivial closed T-invariant subspace K for T, such that K is affiliated with M (i.e. K is of the form $K = P(H)$ for a projection $P \in M$)?*

The problem is only interesting when $\dim(M) = +\infty$ and when M is a factor, i.e. when the center of M is just $\mathbb{C}1_M$.

The infinite dimensional factors were divided into 4 types by Murray and von Neumann in the late 30'es (cf. [KR, Vol.2]).

Type I_∞: These are isomorphic to $B(K)$ for some infinite dimensional Hilbert space.
Type II_1: M has a tracial state, i.,e. there exists a functional $\mathrm{tr}\colon M \to \mathbb{C}$, such that $\mathrm{tr}(1_M) = 1$, $\mathrm{tr}(S^*S) \geq 0$ and $\mathrm{tr}(ST) = \mathrm{tr}(TS)$ for all $S, T \in M$.
Tupe II_∞: $M \simeq N\widehat{\otimes}B(K)$ where N is type II_1 and $\dim K = +\infty$.
Type III: All other infinite dimensional factors.

In all 4 cases, problem 2 remains open (the Type I_∞ case is of course equivalent to Problem 7.1). We will in the following address the invariant subspace problem relative to a factor of type II_1.

7. The Fuglede-Kadison determinant and Brown's spectral distribution measure

Let M be a II_1-factor. Then M has a unique tracial state tr, and tr is normal and faithful (see eg. [KR, Vol.2, Sect.8]. The *Fuglede-Kadison determinant* $\Delta\colon M \to [0,\infty)$ can be defined (cf. [FK]) by:

$$\Delta(T) = \lim_{\varepsilon \downarrow 0} \exp(\mathrm{tr}(\log(T^*T + \varepsilon 1)^{\frac{1}{2}})), \quad t \in M. \tag{7.1}$$

If T is invertible, one has

$$\Delta(T) = \exp(\mathrm{tr}(\log |T|))$$

where $|T| = (T^*T)^{\frac{1}{2}}$. Moreover Δ has the following properties:

$$\begin{aligned} \Delta(ST) &= \Delta(S)\Delta(T), \quad S,T \in M \\ \Delta(T) &= \Delta(T^*) = \Delta(|T|), \quad T \in M \\ \Delta(U) &= 1, \quad \text{when } U \in M \text{ is unitary.} \end{aligned}$$

Δ is an upper semi-continuous function on M but it is not continuous in the norm-topology on M.

Theorem 7.1 *(L.G. Brown 1983 [Br]). Let M be a II_1-factor and let $T \in M$. Then the function*

$$\varphi\colon \lambda \to \frac{1}{2\pi} \log \Delta(T - \lambda 1), \quad \lambda \in \mathbb{C}$$

is subharmonic and its Laplacian taken in distribution sense

$$\mu_T = \left(\frac{\partial^2}{\partial \lambda_1^2} + \frac{\partial^2}{\partial \lambda_2^2} \right) \varphi \tag{7.2}$$

($\lambda_1 = Re\,\lambda$, $\lambda_2 = Im\,\lambda$) is a probability measure in $\mathbb{C}$ concentrated on the spectrum $\sigma(T)$ og T.

Definition 7.2 *The above measure μ_T is called Brown's spectral distribution measure for T or just the Brown measure for T.*

Example 7.3
a) The Fuglede-Kadison determinant and the Brown measure also make sense for $M = M_n(\mathbb{C})$, and $\mathrm{tr} = \frac{1}{n}\mathrm{Tr}$ the normalized trace on $M_n(\mathbb{C})$. In this case one gets

$$\begin{aligned} \Delta(T) &= \sqrt[n]{|\det T|} \\ \mu_T &= \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i} \end{aligned}$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of T repeated according to root multiplicity, and δ_{λ_i} is the Dirac measure at λ_i.
b) If T is a normal operator (i.e. $T^*T = TT^*$) in a factor of type II_1, T has a spectral resolution

$$T = \int_{\sigma(T)} \lambda dE(\lambda).$$

In this case μ_T is equal to $\mathrm{tr} \circ E$.

Methods for computing Brown measures have been developed by Larsen and the lecturer [HL] and by Biane and Lehner [BL].

8. Spectral subspaces for operators in II_1-factors

In 1968, Apostol [Ap] and Foias [Fo1], [Fo2] introduced the notion of spectral subspaces for certain well behaved operators on Banach spaces, the decomposable operators (see [LN] for a modern treatment of this theory):

Definition 8.1 *[LN, Definition 1.1.1] An operator T on a Banach space X is called decomposable if for any open covering $\mathbb{C} = V \cup W$ of the complex plane, there exist closed T-invariant subspaces Y, Z of X such that*

$$\begin{aligned} X &= Y + Z && (8.1)\\ \sigma(T|_Y) &\subseteq V \quad \textit{and } \sigma(T|_Z) \subseteq W. && (8.2) \end{aligned}$$

If $T \in B(X)$ is decomposable, it has a *spectral capacity*, i.e. there exists a map E from the closed subsets of $\mathbb{C}$ into the closed T-invariant subspaces of X, such that

$$\begin{aligned} E(\emptyset) &= 0 \quad \text{and } E(\mathbb{C}) = X && (8.3)\\ X &= E(\bar{V}_1) + \cdots + E(\bar{V}_N) \quad \text{for every finite} && (8.4)\\ & \quad\ \text{open covering } \mathbb{C} = V_1 \cup V_2 \cup \cdots \cup V_n && \\ E(\cap_{n=1}^{\infty} F_n) &= \cap_{n=1}^{\infty} E(F_n), \quad F_n \subseteq \mathbb{C} \text{ closed} && (8.5)\\ \sigma(T_{|E(F)}) &\subseteq F, \quad F \subseteq \mathbb{C} \text{ closed.} && (8.6) \end{aligned}$$

Moreover, a spectral capacity is unique (cf. [LN, Sect.1]).

In this section we will discuss a new method for constructing spectral subspaces of operators which works for all operators in "almost all" II_1-factors, regardless of whether the operator is decomposable in the above sense.

Definition 8.2 *A II_1-factor M on a separable Hilbert space has the* embedding property *if it can be embedded in the ultrapower R^ω of the hyperfinite II_1-factor R for some free ultrafilter ω on the natural numbers.*

All II_1-factors of current interest have this embedding property, and in fact no counterexamples are known. The question whether every II_1-factor on a separable Hilbert space can be embedded in R^ω was first raised by Connes in 1976 [Co] (see also [Ki2] and [HW] for further discussions about this problem).

Let M be a II_1-factor, $M \subseteq B(H)$, and let $T \in M$. If $K \subseteq H$ is a non-trivial closed T-invariant subspace affiliated with M, and $P = P_K$ is the orthogonal projection on M, then according to the decomposition, $H = K \oplus K^\perp$ we can write

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}, \tag{8.7}$$

where $T_{11} = PTP$ and $T_{22} = (1-P)T(1-P)$ are elements of the II_1-factors $M_1 = PMP$ and $M_2 = (1-P)M(1-P)$. Let $\mu_{T_{11}}$ and $\mu_{T_{22}}$ be the Brown measures of T_{11} and T_{22} computed relative to M_1 and M_2 (respectively) then by [Br]:

$$\mu_T = a\mu_{T_{11}} + (1-a)\mu_{T_{22}} \tag{8.8}$$

where $a = \mathrm{tr}_M(P)$.

The main result of [Haa2] is

Theorem 8.3 *[Haa2] Let M be II_1-factor with the embedding property, and let $T \in M$. Then for every Borel set $B \subseteq \mathbb{C}$ there is a unique T-invariant subspace K affiliated with M, such that $\mu_{T_{11}}$ is concentrated on B and $\mu_{T_{22}}$ is concentrated on $\mathbb{C}\backslash B$, where T_{11} and T_{22} are defined as in* (8.7)*. Moreover, $\mathrm{Tr}_M(P_K) = \mu_T(B)$, where $P_K \in M$ is the projection onto K.*

Remark 8.4 *If T is decomposable and B is closed, then the subspace K coincide with the spectral subspace $E(B)$ characterized by* (8.3)–(8.6)*. However, already in the hyperfinite II_1-factor R, there are operators T which are not decomposable.*

Corollary 8.5 *[Haa2] Let $T \in M$, where M is a II_1-factor with the embedding property. If the Brown measure μ_T of T is not concentrated in a single point, then T has a non-trivial closed invariant subspace affiliated with M.*

Remark 8.6 *Corollary 8.5 reduced the invariant subspace problem for II_1-factors M with the embedding problem to operators $T \in M$ for which $\mu_T = \delta_0$ (the Dirac-measure at 0). It can be shown that $\mu_T = \delta_0$ if and only if*

$$\lim_{n\to\infty} ((T^*)^n T^n)^{\frac{1}{n}} = 0$$

in the strong operator topology on M (cf. [Haa2]).

In the rest of this section, I will briefly outline the proof of Theorem 8.3.

Let M be a II_1-factor and let $T \in M$. Define the modified spectrum $\sigma'(T)$ and modified spectral radius $r'(T)$ by

$$\begin{aligned} \sigma'(T) &= \mathrm{supp}(\mu_T) \\ r'(T) &= \max\{|\lambda| \mid \lambda \in \sigma'(T)\}. \end{aligned}$$

Then $\sigma'(T) \subseteq \sigma(T)$ and $r'(T) \leq r(T)$.

The classical spectral radius formula

$$r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$$

has a modified version (cf. [Haa2]):

$$r'(T) = \lim_{p\to\infty} (\lim_{n\to\infty} \|T^n\|_{\frac{p}{n}}^{\frac{1}{n}})$$

where $\|S\|_p = \mathrm{tr}_M(|S|^p)^{\frac{1}{p}}$, $p > 0$.

Spectral subspace lemma 8.7 *[Haa2] Let M be a II_1-factor. (Here we do not need the embedding property.) Let $T \in M$ and let $F \subseteq \mathbb{C}$ be a closed set. Then*

(a) There exists a maximal closed T-invariant subspace K affiliated with M such that $\sigma'(T_{|K}) \subseteq F$, where $\sigma'(T_{|K})$ is the modified spectrum of the operator $T_{|K}$ considered as an element of the II_1-factor $P_K M P_K$ (P_K is the projection of H onto K).

(b) Let $K(F)$ be the subspace K defined by (a). Then

$$\mathrm{tr}_M(P_{K(F)}) \leq \mu(F)$$

for all closed subsets F of $\mathbb{C}$.

Random distortion lemma 8.8 *[Haa2] Let M be a II_1-factor with the embedding property and let $T \in M$. Then*

(a) There exists natural numbers $k(1) < k(2) < \dots$ and $T_n \in M_{k(n)}(\mathbb{C})$ wuch that

$$\sup_{n\in\mathbb{N}} \|T_n\| < \infty. \tag{8.9}$$

(b) For every non-commutative polynomial p in two variables

$$\lim_{n\to\infty} \mathrm{tr}_{k(n)}(p(T_n, T_n^*)) = \mathrm{tr}(p(T, T^*)) \tag{8.10}$$

where $\mathrm{tr}_{k(n)}$ is the normalized trace on $M_n(\mathbb{C})$.

(c) Furthermore, there exists a sequence $T'_n \in M_{k(n)}(\mathbb{C})$ such that

$$\lim_{n\to\infty} \|T'_n - T_n\|_p = 0 \quad \textit{for some } p > 0 \tag{8.11}$$

$$\lim_{n\to\infty} \Delta(T'_n - \lambda 1) = \Delta(T - \lambda 1) \quad \textit{for almost all } \lambda \in \mathbb{C} \tag{8.12}$$

$$\lim_{n\to\infty} \mu_{T'_n} = \mu_T \quad \textit{weakly in } Prob(\mathbb{C}). \tag{8.13}$$

The embedding property is needed in (b). To pass from (b) to (c) we use a random distortion argument where we put

$$T'_n = T_n + \varepsilon_n X_n Y_n^{-1}$$

where X_n, Y_n are random Gaussian matrices with independent entries and $\varepsilon_n \to 0$. Subsequently Sniady proved [Sn1] that by using a different random distortion, one can obtain a stronger result, namely in (c), (8.11) can be replaced by

$$\lim_{n\to\infty} \|T_n' - T_n\|_\infty = 0$$

where $\|\cdot\|_\infty$ is the operator norm.

The random distortion lemma is used to reduce the proof of Theorem 8.3 to the case of $M = M_n(\mathbb{C})$ by an ultraproduct argument. For $M = M_n(\mathbb{C})$, Theorem 8.3 is a corollary of Jordan's normal form.

9. Voiculescu's circular operator Y and the strictly upper triangular operator T

Prior to the proof of theorem 8.3, Dykema and the lecturer had constructed invariant subspaces for special operators in factors of type II_1. An example of particular interest is Voiculescu's circular operator Y, which can be written as

$$Y = \frac{1}{\sqrt{2}}(X_1 + iX_2)$$

where (X_1, X_2) is a semicircular system (cf. Section 3.). The von Neumann algebra $M = VN(Y)$ generated by Y is isomorphic to $L(F_2)$ (the von Neumann associated to a free group on two generators) which is a factor of type II_1. The operator Y is far from being normal and for some time it was considered a possible counterexample for the invariant subspace problem relative to the II_1-factor it generates. In [HL] Larsen and the lecturer proved that

$$\sigma(Y) = \overline{D} \quad \text{(the closed unit disc in } \mathbb{C}) \tag{9.1}$$

The Brown measure μ_Y of Y is the uniform distribution on $\overline{D}$, i.e. it has constant density $\frac{1}{\pi}$. (9.2)

Theorem 9.1 *[DH1] For each $r \in (0,1)$ there is a unique projection $p \in M = VN(Y)$ such that*

$$pYp = Yp \quad \textit{(i.e. the range of } p \textit{ is } Y\textit{-invariant)} \tag{9.3}$$

$$\sigma(pYp) \subseteq \{z \in \mathbb{C} \mid |z| \le r\} \tag{9.4}$$

$$\sigma((1-p)Y(1-p)) \subseteq \{z \in \mathbb{C} \mid r \le |z| \le 1\} \tag{9.5}$$

where the spectra in (9.4) *and* (9.5) *are computed relative to pMp and $(1-p)M(1-p)$. Moreover*

$$\mathrm{tr}_M(p) = r^2. \tag{9.6}$$

This result was generalized to arbitrary R-diagonal elements by Sniady and Speicher [SS]. Later Dykema and the lecturer proved

Theorem 9.2 *[DH2] Voiculescu's circular operator is decomposable in the sense of Apostol and Foias (see Definition 8.1).*

In [DH2] we also considered the "strictly upper triangular operator" T. It is defined in terms of its random matrix model:

Theorem/Definition 9.3 *[DH2] Let for each $n \in \mathbb{N}$ T_n denote the strictly upper triangular random matrix*

$$T_n = \begin{pmatrix} 0 & t_{11}^{(n)} & \cdots & t_{1n}^{(n)} \\ \ddots & & \ddots & t_{n-1,n}^{(n)} \\ 0 & & & 0 \end{pmatrix} \tag{9.7}$$

for which the entries $(t_{ij}^{(n)})_{i<j}$ are $\frac{n(n-1)}{2}$ independent identically distributed complex Gaussian random variables with densities $\frac{n}{\pi}\exp(-n|z|^2)$, $z \in \mathbb{C}$. Then there is an operator T in a II_1-factor M such that T_n converges in *-moments to T, i.e.

$$\mathrm{tr}_M(P(T,T^*)) = \lim_{n\to\infty} \mathbb{E}\,\mathrm{tr}_n(P(T_n,T_n^*)) \tag{9.8}$$

for every non-commutative polynomial P. T is called the strictly upper triangular operator.

The strictly upper triangular operator is quasi nilpotent, i.e. $\sigma(T) = \{0\}$, and therefore its Brown measure μ_T is equal to δ_0. In view of remark 8.6 it could be a candidate for a counterexample to the invariant subspace problem relative to a II_1-factor. However, this is not the case:

Dykema and the lecturer proved in [DH2] that

$$\mathrm{tr}((T^*T)^n) = \frac{n^n}{(n+1)!}, \qquad n \in \mathbb{N} \tag{9.9}$$

and in [Sn2], Sniady proved

$$\mathrm{tr}(((T^k)^*T^k)^n) = \frac{n^{nk}}{(nk+1)!}, \qquad n,k \in \mathbb{N}, \tag{9.10}$$

a formula which was conjectured in [DH2].

Based on (9.10) and its proof, we recently proved

Theorem 9.4 *[DH3] Let T be as above. Put $S_k = k((T^k)^kT^k)^{\frac{1}{k}}$ and let $F\colon [0,\pi] \to [0,1]$ be the strictly increasing function given by $F(0) = 0$, $F(\pi) = 1$ and*

$$F\left(\frac{\sin v}{v}\exp(v\cot v)\right) = 1 - \frac{v}{\pi} + \frac{1}{\pi}\frac{\sin^2 v}{v}, \quad 0 < v < \pi. \tag{9.11}$$

Then $F(S_k)$ converges in strong operator topology to the "diagonal operator" D_0 with matrix model

$$D_{0,n} = \begin{pmatrix} \frac{1}{n} & & & 0 \\ & \frac{2}{n} & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}. \tag{9.12}$$

In particular $D_0 \in VN(T)$. Moreover $VN(T)$ is isomorphic to $L(F_2)$ and the ranges of the projections $1_{[0,t]}(D_0)$, $0 < t < 1$, form an uncountable family of non-trivial invariant subspaces for T affiliated with $VN(T)$.

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