A Random Matrix Approach to the Lack of Projections in $C^*_{\mathrm{red}}(\mathbb{F}_2)$

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Abstract

In 1982 Pimsner and Voiculescu computed the K_0- and K_1-groups of the reduced group C^*-algebra $C^*_{\mathrm{red}}(F_k)$ of the free group F_k on k generators and settled thereby a long standing conjecture: $C^*_{\mathrm{red}}(F_k)$ has no projections except for the trivial projections 0 and 1. Later simpler proofs of this conjecture were found by methods from K-theory or from non-commutative differential geometry. In this paper we provide a new proof of the fact that $C^*_{\mathrm{red}}(F_k)$ is projectionless. The new proof is based on random matrices and is obtained by a refinement of the methods recently used by the first and the third named author to show that the semigroup $\mathrm{Ext}(C^*_{\mathrm{red}}(F_k))$ is not a group for $k \geq 2$. By the same type of methods we also obtain that two phenomena proved by Bai and Silverstein for certain classes of random matrices: "no eigenvalues outside (a small neighbourhood of) the support of the limiting distribution" and "exact separation of eigenvalues by gaps in the limiting distribution" also hold for arbitrary non-commutative selfadjoint polynomials of independent GUE, GOE or GSE random matrices with matrix coefficients.

1 Introduction.

In [HT] the first and the third named author proved the following extension of Voiculescu's random matrix model for a semicircular system:

Let $X_1^{(n)}, \ldots, X_r^{(n)}$ be r independent selfadjoint $n \times n$ random matrices from Gaussian unitary ensembles (GUE) and with the scaling used in Voiculescu's paper [V1]. Moreover, let $x_1, \ldots, x_r$ be a semicircular system in a C^*-probability space (A, τ) with τ faithful. Then, for every polynomial p in r non-commuting variables,

$$\lim_{n\to\infty} \|p(X_1^{(n)}, \ldots, X_r^{(n)})\| = \|p(x_1, \ldots, x_r)\| \tag{1.1}$$

holds almost surely.

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The main steps in the proof of (1.1) were:

STEP 1 (Linearization trick). In order to prove (1.1), it is sufficient to show that for every $m \in \mathbb{N}$, every $\varepsilon > 0$ and every selfadjoint polynomial q in r non-commuting variables with coefficients in $M_m(\mathbb{C})$ and with $\deg(q) = 1$,

$$\sigma(q(X_1^{(n)}, \ldots, X_r^{(n)})) \subseteq \sigma(q(x_1, \ldots, x_r)) + (-\varepsilon, \varepsilon) \tag{1.2}$$

eventually as $n \to \infty$ (almost surely). Here $\sigma(\cdot)$ denotes the spectrum of a matrix or of an element in a C^*-algebra.

STEP 2 (Mean value estimate). If q is a polynomial of first degree with matrix coefficients as in step 1, then for every $\varphi \in C_c^\infty(\mathbb{R}, \mathbb{R})$

$$\mathbb{E}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\varphi(q(X_1^{(n)}, \ldots, X_r^{(n)}))\} = (\mathrm{tr}_m \otimes \tau)\varphi(q(x_1, \ldots, x_r)) + O(\tfrac{1}{n^2}) \tag{1.3}$$

where $\mathrm{tr}_m = \frac{1}{m}\mathrm{Tr}_m$ is the normalized trace on $M_m(\mathbb{C})$.

STEP 3 (Variance estimates). If q is a polynomial of first degree with matrix coefficients as in step 1, then for every $\varphi \in C_c^\infty(\mathbb{R}, \mathbb{R})$,

$$\mathbb{V}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\varphi(q(X_1^{(n)}, \ldots, X_r^{(n)}))\} = O(\tfrac{1}{n^2}). \tag{1.4}$$

Moreover, if $\varphi' = \frac{d\varphi}{dx}$ vanishes in a neighbourhood of $\sigma(q(x_1, \ldots, x_r))$, then

$$\mathbb{V}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\varphi(q(X_1^{(n)}, \ldots, X_r^{(n)}))\} = O(\tfrac{1}{n^4}). \tag{1.5}$$

A standard application of the Borel-Cantelli lemma and the Chebychev inequality to (1.3) and (1.5) gives that if φ' vanishes on a neighbourhood of $\sigma(q(x_1, \ldots, x_r))$, then

$$(\mathrm{tr}_m \otimes \mathrm{tr}_n)\varphi(q(X_1^{(n)}, \ldots, X_r^{(n)})) = (\mathrm{tr}_m \otimes \tau)\varphi(q(x_1, \ldots, x_r)) + O(n^{-\frac{4}{3}}) \tag{1.6}$$

holds almost surely, and from this (1.2) easily follows (cf. [HT, proof of Theorem 6.4]).

In [S], the second named author generalized the above to real and symplectic Gaussian random matrices (the GOE- and GSE-cases). The main new problem in these two cases is that (1.3) no longer holds. However, the following formula holds (cf. [S, Theorem 5.6]):

$$\mathbb{E}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\varphi(q(X_1^{(n)}, \ldots, X_r^{(n)}))\} = (\mathrm{tr}_m \otimes \tau)\varphi(q(x_1, \ldots, x_r)) + \tfrac{1}{n}\Lambda(\varphi) + O(\tfrac{1}{n^2}), \tag{1.7}$$

where $\Lambda \colon C_c^\infty(\mathbb{R}) \to \mathbb{C}$ is a distribution (in the sense of L. Schwartz) depending on the polynomial q and on the scalar field ($\mathbb{R}$ or $\mathbb{H}$). Moreover, by [S, Lemma 5.5],

$$\mathrm{supp}(\Lambda) \subseteq \sigma(q(x_1, \ldots, x_r)). \tag{1.8}$$

Using (1.7) and (1.8) instead of (1.3) the proofs of (1.2) and (1.1) could be completed essentially as in the GUE-case.

The proof of the linearization trick relied on C^*-algebra techniques, namely on Stinespring's theorem and on Arveson's extension theorem for completely positive maps. In the present paper we give a purely algebraic proof of the linearization trick which in turn allows us to work directly with polynomials of degree greater than 1. As a result, we prove in Section 6 and Section 10 that (1.3) for the GUE-case (resp. (1.7) and (1.8) for the GOE- and GSE-cases) holds for selfadjoint polynomials q of *any* degree with coefficients in $M_m(\mathbb{C})$. Also (1.4), (1.5) and (1.6) hold in this generality. Consequently, (1.2) holds for all such polynomials q in all three cases (GUE, GOE and GSE). This is the phenomenon "no eigenvalues outside (a small neighbourhood of) the support of the limiting distribution", which Bai and Silverstein obtained in [BS1] for a different class of selfadjoint random matrices.

Let us next discuss the application to projections in $C^*_{\rm red}(\mathbb{F}_r)$: Recall from [HT, Lemma 8.1] that $C^*_{\rm red}(\mathbb{F}_r)$ has a unital, trace preserving embedding in $C^*(x_1,\dots,x_r,\mathbf{1})$, where $x_1,\dots,x_r$ is a semicircular system. If e is a projection in $M_m(C^*(x_1,\dots,x_r,\mathbf{1}))$, then by standard C^*-algebra techniques (cf. Section 7) there exists a projection f in $M_m(C^*(x_1,\dots,x_r,\mathbf{1}))$ such that $\|e-f\|<1$ and such that f takes the form

$$f=\varphi(q(x_1,\dots,x_r)),$$

where q is a selfadjoint polynomial in r non-commuting variables with coefficients in $M_m(\mathbb{C})$, and φ is a C^∞-function with compact support, such that φ only takes the values 0 and 1 is some neighbourhood of $\sigma(q(x_1,\dots,x_r))$.

Consider now random matrices $X_1^{(n)},\dots,X_r^{(n)}$ as in the GUE-case described above. By (1.6) we have that

$$\begin{aligned}(\mathrm{tr}_m\otimes\mathrm{tr}_n)\varphi(q(X_1^{(n)},\dots,X_r^{(n)})) &= (\mathrm{tr}_m\otimes\tau)\varphi(q(x_1,\dots,x_r))+O(n^{-\frac{4}{3}})\\ &= (\mathrm{tr}_m\otimes\tau)(f)+O(n^{-\frac{4}{3}})\end{aligned}$$

holds almost surely and hence the corresponding unnormalized trace satisfies

$$(\mathrm{Tr}_m\otimes\mathrm{Tr}_n)\varphi(q(X_1^{(n)},\dots,X_r^{(n)}))=n(\mathrm{Tr}_m\otimes\tau)(f)+O(n^{-\frac{1}{3}}). \tag{1.9}$$

Using that the left hand side of (1.9) is an integer for all large $n\in\mathbb{N}$, it is not hard to prove that $(\mathrm{Tr}_m\otimes\tau)(f)$ is an integer (cf. section 7 for details). Moreover, since $\|e-f\|<1$ implies that $e=ufu^*$ for a unitary $u\in M_m(C^*(x_1,\dots,x_r,1))$, we also have

$$(\mathrm{Tr}_m\otimes\tau)(e)\in\mathbb{Z}. \tag{1.10}$$

Hence, using the existence of a unital trace-preserving embedding of $C^*_{\rm red}(\mathbb{F}_r)$ into $C^*(x_1,\dots,x_r,\mathbf{1})$, it follows that:

$$(\mathrm{Tr}_m\otimes\tau)(e)\in\mathbb{Z}\quad\text{for all projections}\quad e\in M_m(C^*_{\rm red}(\mathbb{F}_r)). \tag{1.11}$$

In particular:

$$C^*_{\rm red}(\mathbb{F}_r)\quad\text{has no projections except 0 and 1}. \tag{1.12}$$

The two statements (1.11) and (1.12) were first obtained by Pimsner and Voiculescu in 1982 (cf. [PV]) by proving that $K_0(C^*_{\rm red}(\mathbb{F}_r))=\mathbb{Z}$, where the K_0-class $[\mathbf{1}]$ of the unit in $C^*_{\rm red}(\mathbb{F}_r)$ corresponds to $1\in\mathbb{Z}$. Simpler proofs of $K_0(C^*_{\rm red}(\mathbb{F}_r))=\mathbb{Z}$ were later obtained by Cuntz [Cu1] and Lance [L]. Connes gave in [Co, pp. 269–272] a more direct proof of (1.12) based on Fredholm modules. Connes' argument can be further simplified to a short selfcontained proof without explicit mentioning of Fredholm modules ([Cu2], [CF]).

It is an elementary consequence of (1.11) that for every selfadjoint polynomial q in r non-commuting variables and with coefficients in $M_m(\mathbb{C})$, the spectrum of $q(x_1,\dots,x_r)$ has at most m connected components, that is

$$\sigma(q(x_1,\dots,x_r)) = I_1 \cup \cdots \cup I_j, \qquad \text{(disjoint union)},$$

where each I_i is a compact interval or a one-point set (cf. Proposition 8.1), and $j \le m$. Let

$$\mathbf{1} = e_1 + \cdots + e_j$$

be the corresponding decomposition of the unit in $M_m(C^*(x_1,\dots,x_r,\mathbf{1}))$ into ortogonal projections, and put

$$k_i = (\mathrm{Tr}_m \otimes \tau)(e_i) \in \mathbb{Z}.$$

Let now $0 < \varepsilon < \frac{1}{3}\varepsilon_0$ where ε_0 is the smallest length of the gaps between the sets $I_1,\dots,I_j$. In Section 8 and Section 11 we prove that the number of eigenvalues of $q(X_1^{(n)},\dots,X_r^{(n)})$ in the open intervals $I_i + (-\varepsilon,\varepsilon)$, $i = 1,\dots,j$, are exactly nk_i eventually as $n \to \infty$ (almost surely) in all three cases (GUE, GOE and GSE). This is the phenomenon "Exact separation of eigenvalues" by the gaps in the support of the limiting distribution which Bai and Silverstein obtained in [BS2] for the class of selfadjoint random matrices previously studied in [BS1].

2 Matrix Results.

2.1 Proposition. *Let d, m and m' be positive integers and let p be a polynomial in $M_{m,m'} \otimes \mathbb{C}\langle X_1,\dots,X_r\rangle$ of degree d. Then there exist positive integers $m_1, m_2, \dots, m_{d+1}$ and polynomials*

$$u_j \in M_{m_j,m_{j+1}}(\mathbb{C}) \otimes \mathbb{C}\langle X_1,\dots,X_r\rangle, \qquad (j = 1,2,\dots,d),$$

such that

(i) $m_1 = m$ *and* $m_{d+1} = m'$,

(ii) $\deg(u_j) \le 1$ *for all* j *in* $\{1,2,\dots,r\}$,

(iii) $p = u_1 u_2 \cdots u_d$.

Proof. The proof proceeds by induction on d. Noting that the case $d = 1$ is trivial, we assume that $d \ge 2$ and that the proposition has been verified for all polynomials in $M_{m,m'}(\mathbb{C}) \otimes \mathbb{C}\langle X_1,\dots,X_r\rangle$ og degree at most $d-1$. Given then a polynomial p in

$M_{m,m'}(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \dots, X_r\rangle$ of degree d, we may, setting $X_0 = \mathbf{1}_{\mathbb{C}\langle X_1,\dots,X_r\rangle}$, write p in the form

$$p(X_1, \dots, X_r) = \sum_{0 \le i_1, i_2, \dots, i_d \le r} c(i_1, i_2, \dots, i_d) \otimes X_{i_1} X_{i_2} \cdots X_{i_d},$$

for suitable matrices

$$c(i_1, i_2, \dots, i_d) \in M_{m,m'}(\mathbb{C}), \qquad (i_1, i_2, \dots, i_d \in \{0, 1, \dots, r\}).$$

For any i_1 in $\{0, 1, \dots, r\}$, we put

$$Y(i_1) = \sum_{0 \le i_2, \dots, i_d \le r} c(i_1, i_2, \dots, i_d) \otimes X_{i_2} \cdots X_{i_d}.$$

Note then that

$$\begin{aligned} p(X_1, \dots, X_r) &= \sum_{i_1=0}^{r} (\mathbf{1}_m \otimes X_{i_1}) \Big(\sum_{0 \le i_2, \dots, i_d \le r} c(i_1, i_2, \dots, i_d) \otimes X_{i_2} \cdots X_{i_d} \Big) \\ &= \sum_{i_1=0}^{r} (\mathbf{1}_m \otimes X_{i_1}) Y(i_1) \\ &= \begin{pmatrix} \mathbf{1}_m \otimes X_0 & \mathbf{1}_m \otimes X_1 & \cdots & \mathbf{1}_m \otimes X_r \end{pmatrix} \cdot \begin{pmatrix} Y(0) \\ Y(1) \\ \vdots \\ Y(r) \end{pmatrix} \\ &= u_1(X_1, \dots, X_r) \cdot p'(X_1, \dots, X_r), \end{aligned}$$

where

$$u_1(X_1, \dots, X_r) := \begin{pmatrix} \mathbf{1}_m \otimes X_0 & \mathbf{1}_m \otimes X_1 & \cdots & \mathbf{1}_m \otimes X_r \end{pmatrix} \in M_{m,(r+1)m}(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \dots, X_r\rangle,$$

and

$$p'(X_1, \dots, X_r) := \begin{pmatrix} Y(0) \\ Y(1) \\ \vdots \\ Y(r) \end{pmatrix} \in M_{(r+1)m,m'}(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \dots, X_r\rangle.$$

We note that $\deg(u_1) = 1$ and that $\deg(p') \le d-1$. By the induction hypothesis, there are positive integers $m_2, m_3, \dots, m_{d+1}$ and polynomials u_j in $M_{m_j,m_{j+1}}(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \dots, X_r\rangle$ $(j = 2, 3, \dots, d)$ such that

(i′) $m_2 = (r+1)m$ and $m_{d+1} = m'$,

(ii′) $\deg(u_j) \le 1$, for all j in $\{2, 3, \dots, d\}$,

(iii′) $p' = u_2 u_3 \dots u_d$.

Now, $p = u_1 p' = u_1 u_2 \cdots u_d$ and we have the desired decomposition. ■

2.2 Remark. By inspection of the proof of Proposition 2.1, it is apparent that the decomposition, implicitly given in that proposition, is explicitly given as follows:

$$m_1 = m,\ m_2 = (r+1)m,\ m_3 = (r+1)m_2 = (r+1)^2 m,\ \ldots\ , m_d = (r+1)^{d-1} m,$$

$$m_{d+1} = m'$$

$$u_j = \begin{pmatrix} \mathbf{1}_{m_j} \otimes X_0 & \mathbf{1}_{m_j} \otimes X_1 & \cdots & \mathbf{1}_{m_j} \otimes X_r \end{pmatrix}, \qquad (j = 1, 2, \ldots, d-1),$$

$$u_d = \left(\textstyle\sum_{i_d=0}^{r} c(i_1, \ldots, i_{d-1}, i_d) \otimes X_{i_d} \right)_{0 \le i_1, i_2, \ldots, i_{d-1} \le r},$$

where u_d should be thought of as a block column matrix with (block) rows indexed by the tuples $(i_1, i_2, \ldots, i_{d-1})$ in a certain order.

Note in particular that the polynomials $u_1, u_2, \ldots, u_{d-1}$ are basicly canonical, in the sense that they only depend on p through the degree d and the dimension m. Conversely, the polynomial u_d basicly contains all information about p.

2.3 Proposition. *Let $\mathcal{A}$ be an algebra with unit $\mathbf{1}_{\mathcal{A}}$ and let $d, m, m_1, m_2, \ldots, m_{d+1}$ be positive integers such that $m_1 = m = m_{d+1}$. Put $k = \sum_{j=1}^{d} m_j$.*

Consider further for each j in $\{1, 2, \ldots, d\}$ a matrix u_j from $M_{m_j, m_{j+1}}(\mathcal{A})$, and note that $u_1 u_2 \cdots u_d \in M_m(\mathcal{A})$. For each λ in $M_m(\mathcal{A})$, define the matrix $A(\lambda)$ in $M_k(\mathcal{A})$ by

$$A(\lambda) = \begin{pmatrix} \lambda & -u_1 & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{1}_{m_2} & -u_2 & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{1}_{m_3} & -u_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbf{1}_{m_{d-1}} & -u_{d-1} \\ -u_d & 0 & 0 & \cdots & 0 & \mathbf{1}_{m_d} \end{pmatrix}, \tag{2.1}$$

where $\mathbf{1}_{m_j}$ denotes the unit in $M_{m_j}(\mathcal{A})$. For any λ in $M_m(\mathbb{C})$ we then have

$$\lambda - u_1 u_2 \cdots u_d \quad \text{is invertible in } M_m(\mathcal{A}) \iff A(\lambda) \quad \text{is invertible in } M_k(\mathcal{A}),$$

in which case

$$A(\lambda)^{-1} = B(\lambda) + C,$$

where

$$B(\lambda) = \begin{pmatrix} \mathbf{1}_m \\ u_2 u_3 \cdots u_d \\ u_3 u_4 \cdots u_d \\ u_4 \cdots u_d \\ \vdots \\ u_d \end{pmatrix} (\lambda - u_1 u_2 \cdots u_d)^{-1} \begin{pmatrix} \mathbf{1}_m & u_1 & u_1 u_2 & u_1 u_2 u_3 & \cdots & u_1 u_2 \cdots u_{d-1} \end{pmatrix}$$

and

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{1}_{m_2} & u_2 & u_2u_3 & u_2u_3u_4 & \cdots & u_2u_3\cdots u_{d-1} \\ 0 & 0 & \mathbf{1}_{m_3} & u_3 & u_3u_4 & \cdots & u_3u_4\cdots u_{d-1} \\ 0 & 0 & 0 & \mathbf{1}_{m_4} & u_4 & \cdots & u_4u_5\cdots u_{d-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \mathbf{1}_{m_{d-1}} & u_{d-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mathbf{1}_{m_d} \end{pmatrix}.$$

Note, in particular, that $(\lambda - u_1u_2\cdots u_d)^{-1}$ *is the (block-) entry at position* $(1,1)$ *of* $A(\lambda)^{-1}$.

Proof. At first assume that $A(\lambda)$ is invertible with inverse $F(\lambda)$. We write $F(\lambda)$ in block matrix form as

$$F(\lambda) = \big(f_{i,j}(\lambda)\big)_{1\le i,j\le d},$$

corresponding to the block matrix form of $A(\lambda)$:

$$A(\lambda) = \big(a_{i,j}(\lambda)\big)_{1\le i,j\le d},$$

specified above.

$>$From the equality $\mathbf{1}_k = A(\lambda)F(\lambda)$, we get, in particular, the identities

$$\sum_{i=1}^{d} a_{i,j}(\lambda) f_{i,1}(\lambda) = \delta_{j,1}, \qquad (j = 1, 2, \dots, n), \tag{2.2}$$

where

$$\delta_{i,j} = \begin{cases} \mathbf{1}_{m_i}, & \text{if } i = j, \\ \mathbf{0}_{m_i\times m_j} & \text{if } i \ne j. \end{cases}$$

For $j = 1$, (2.2) becomes

$$\lambda f_{1,1}(\lambda) - u_1 f_{2,1}(\lambda) = \mathbf{1}_{m_1}, \tag{2.3}$$

and for j in $\{2, 3, \dots, d-1\}$, we get

$$f_{j,1}(\lambda) - u_j f_{j+1,1}(\lambda) = \mathbf{0}_{m_j\times m_1}, \quad \text{i.e.,} \quad f_{j,1}(\lambda) = u_j f_{j+1,1}(\lambda). \tag{2.4}$$

Finally, for $j = d$, (2.2) yields

$$-u_d f_{1,1}(\lambda) + f_{d,1}(\lambda) = \mathbf{0}_{m_d\times m_1} \quad \text{i.e.,} \quad f_{d,1}(\lambda) = u_d f_{1,1}(\lambda). \tag{2.5}$$

Then, by successive applications of the formulae (2.4) and (2.5), we find that

$$f_{2,1}(\lambda) = u_2 f_{3,1}(\lambda) = u_2u_3 f_{4,1}(\lambda) = \cdots = u_2u_3\cdots u_{d-1} f_{d,1}(\lambda) = u_2u_3\cdots u_d f_{1,1}(\lambda).$$

Inserting this in (2.4), we obtain

$$\mathbf{1}_{m_1} = \lambda f_{1,1}(\lambda) - u_1\big(u_2u_3\cdots u_d f_{1,1}(\lambda)\big) = \big(\lambda - u_1u_2\cdots u_d\big) f_{1,1}(\lambda).$$

To verify that also $f_{1,1}(\lambda)(\lambda - u_1 u_2 \cdots u_d) = \mathbf{1}_{m_1}$, we consider the equality $F(\lambda)A(\lambda) = \mathbf{1}_k$, from which

$$\sum_{i=1}^{d} f_{1,i}(\lambda) a_{i,j}(\lambda) = \delta_{1,j}, \qquad (j = 1, 2, \dots d).$$

For $j = 1$ we obtain

$$f_{1,1}(\lambda)\lambda - f_{1,d}(\lambda)u_d = \mathbf{1}_{m_1}, \tag{2.6}$$

and for j in $\{2, 3, \dots, d\}$,

$$f_{1,j-1}(\lambda)u_{j-1} + f_{1,j}(\lambda) = \mathbf{0}_{m_1 \times m_j} \quad \text{i.e.,} \quad f_{1,j}(\lambda) = f_{1,j-1}(\lambda)u_{j-1}. \tag{2.7}$$

By successive applications of (2.7),

$$f_{1,d}(\lambda) = f_{1,d-1}(\lambda)u_{d-1} = f_{1,d-2}(\lambda)u_{d-2}u_{d-1} = \cdots = f_{1,1}(\lambda)u_1 u_2 \cdots u_{d-1},$$

and inserting this in (2.6), we obtain

$$\mathbf{1}_{m_1} = f_{1,1}(\lambda)\lambda - \big(f_{1,1}(\lambda)u_1 u_2 \cdots u_{d-1}\big)u_d = f_{1,1}(\lambda)\big(\lambda - u_1 u_2 \cdots u_d\big),$$

as desired.

Assume next that $(\lambda - u_1 u_2 \cdots u_d)$ is invertible in $M_m(\mathcal{A})$ and consider the matrices $B(\lambda)$ and C introduced in Proposition 2.3. At first we show that

$$A(\lambda)\big(B(\lambda) + C\big) = \mathbf{1}_k.$$

It is easily seen that

$$A(\lambda) \begin{pmatrix} \mathbf{1}_m \\ u_2 u_3 \cdots u_d \\ u_3 u_4 \cdots u_d \\ u_4 \cdots u_d \\ \vdots \\ u_d \end{pmatrix} = \begin{pmatrix} \lambda - u_1 u_2 \cdots u_d \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

so that

$$A(\lambda)B(\lambda) = \begin{pmatrix} \mathbf{1}_m \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} 1 & u_1 & u_1 u_2 & u_1 u_2 u_3 & \cdots & u_1 u_2 \cdots u_{d-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & u_1 & u_1 u_2 & u_1 u_2 u_3 & \cdots & u_1 u_2 \cdots u_{d-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

It thus remains to verify that

$$A(\lambda)C = \begin{pmatrix} 0 & -u_1 & -u_1u_2 & -u_1u_2u_3 & \cdots & -u_1u_2\cdots u_{d-1} \\ 0 & \mathbf{1}_{m_2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{1}_{m_3} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \mathbf{1}_{m_4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \mathbf{1}_{m_d} \end{pmatrix}. \tag{2.8}$$

To this end, note that the first column in $A(\lambda)C$ consists entirely of zeroes and that the second column in $A(\lambda)C$ equals that of $A(\lambda)$.

Note next that for j in $\{3, 4, \dots, d\}$, the entry at position $(1, j)$ is

$$\big[A(\lambda)C\big]_{1,j} = \begin{pmatrix} \lambda & -u_1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 \\ u_2u_3\cdots u_{j-1} \\ u_3u_4\cdots u_{j-1} \\ \vdots \\ u_{j-1} \\ \mathbf{1}_{m_j} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = -u_1u_2\cdots u_{j-1}.$$

Next, if $i \in \{2, 3, \dots, d-1\}$ and $j \in \{3, 4, \dots, d\}$, then the entry of $A(\lambda)C$ at position (i, j) is

$$\begin{aligned} \big[A(\lambda)C\big]_{i,j} &= \begin{pmatrix} 0 & \cdots & 0 & \mathbf{1}_{m_i} & -u_i & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 \\ u_2u_3\cdots u_{j-1} \\ u_3u_4\cdots u_{j-1} \\ \vdots \\ u_{j-1} \\ \mathbf{1}_{m_j} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ &= \begin{cases} 0, & \text{if } i > j, \\ \mathbf{1}_{m_i}, & \text{if } i = j, \\ u_iu_{i+1}\cdots u_{j-1} - u_iu_{i+1}\cdots u_{j-1} = 0, & \text{if } i \le j-1. \end{cases} \end{aligned}$$

Finally, for j in $\{3,4,\dots,k\}$, the entry at position (d,j) is

$$\big[A(\lambda)C\big]_{d,j} = \begin{pmatrix} -u_d & 0 & \cdots & 0 & \mathbf{1}_{m_d} \end{pmatrix} \begin{pmatrix} 0 \\ u_2u_3\cdots u_{j-1} \\ u_3u_4\cdots u_{j-1} \\ \vdots \\ u_{j-1} \\ \mathbf{1}_{m_j} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{cases} 0, & \text{if } j<d, \\ \mathbf{1}_{m_d} & \text{if } j=d. \end{cases}$$

Hence (2.8) holds, and therefore $A(\lambda)(B(\lambda)+C)=\mathbf{1}_k$.

To verify that also

$$\big(B(\lambda)+C\big)A(\lambda)=\mathbf{1}_k,$$

at first note that

$$\begin{pmatrix} \mathbf{1}_m & u_1 & u_1u_2 & u_1u_2u_3 & \cdots & u_1u_2\cdots u_{d-1} \end{pmatrix} A(\lambda) = \begin{pmatrix} \lambda - u_1u_2\dots u_d & 0 & \cdots & 0 \end{pmatrix}$$

so that

$$B(\lambda)A(\lambda) = \begin{pmatrix} \mathbf{1}_m \\ u_2u_3\cdots u_d \\ u_3u_4\cdots u_d \\ \vdots \\ u_d \end{pmatrix} \begin{pmatrix} \mathbf{1}_m & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{1}_m & 0 & \cdots & 0 \\ u_2u_3\cdots u_d & 0 & \cdots & 0 \\ u_3u_4\cdots u_d & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_d & 0 & \cdots & 0 \end{pmatrix}.$$

It thus remains to show that

$$CA(\lambda) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ -u_2u_3\cdots u_d & \mathbf{1}_{m_2} & 0 & \cdots & 0 \\ -u_3u_4\cdots u_d & 0 & \mathbf{1}_{m_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -u_d & 0 & 0 & \cdots & \mathbf{1}_{m_d} \end{pmatrix}.$$

This follows easily by considerations similar to those described above. ■

2.4 Corollary. *Let p be a polynomial in $M_m(\mathbb{C})\otimes\mathbb{C}\langle X_1,\dots,X_r\rangle$ of degree d, and let $x_1,\dots,x_r$ be elements in a unital algebra $\mathcal{A}$. As in Proposition 2.1, choose a factorization of p into polynomials of first degree,*

$$p = u_1u_2\cdots u_d.$$

Put

$$v_j = u_j(x_1,\dots,x_r), \qquad (j=1,2,\dots,d),$$

and let $\lambda \in M_m(\mathbb{C})$*. Then* $\lambda \otimes \mathbf{1}_{\mathcal{A}} - p(x_1,\dots,x_r)$ *is invertible in* $M_m(\mathcal{A})$ *iff the matrix*

$$A(\lambda, v_1,\dots,v_r) = \begin{pmatrix} \lambda\otimes \mathbf{1}_n & -v_1 & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{1}_{m_2}\otimes \mathbf{1}_n & -v_2 & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{1}_{m_3}\otimes \mathbf{1}_n & -v_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbf{1}_{m_{d-1}}\otimes \mathbf{1}_n & -v_{d-1} \\ -v_d & 0 & 0 & \cdots & 0 & \mathbf{1}_{m_d}\otimes \mathbf{1}_n \end{pmatrix},$$

is invertible in $M_k(\mathcal{A})$*.*

By application of Corollary 2.4, one can give a purely algebraic proof of the following "linearization trick" which was obtained in [HT] by use of Stinespring's Theorem and Arveson's Extension Theorem for completely positive maps:

2.5 Corollary. *[HT, Theorem 2.2] Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be unital* C^**-algebras, and let* $x_1,\dots,x_r \in \mathcal{A}_{sa}$*,* $y_1,\dots,y_r \in \mathcal{B}_{sa}$*. If for all* $m \in \mathbb{N}$ *and all* $a_0, a_1,\dots,a_r \in M_m(\mathbb{C})_{sa}$ *we have that*

$$\sigma\Big(a_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_r a_i \otimes x_i\Big) \supseteq \sigma\Big(a_0 \otimes \mathbf{1}_{\mathcal{B}} + \sum_r a_i \otimes y_i\Big), \tag{2.9}$$

then there exists a unital $*$*-homomorphism*

$$\phi : C^*(\mathbf{1}_{\mathcal{A}}, x_1,\dots,x_r) \to C^*(\mathbf{1}_{\mathcal{B}}, y_1,\dots,y_r),$$

such that $\phi(x_i) = y_i$ *for* $i = 1,\dots,r$*.*

Proof. As in step I of the proof of [HT, Theorem 2.2], a simple 2×2-matrix argument shows that if (2.9) holds, then it also holds for arbitrary elements $a_0, a_1,\dots,a_r \in M_m(\mathbb{C})$. That is, for every polynomial q of degree at most 1 in $M_m(\mathbb{C}) \otimes \mathbb{C}\langle X_1,\dots,X_r\rangle$ one has that

$$\sigma(q(x_1,\dots,x_r)) \supseteq \sigma(q(y_1,\dots,y_r)). \tag{2.10}$$

Now, let $p \in \mathbb{C}\langle X_1,\dots,X_r\rangle$ be a polynomial of degree $d \geq 1$, and as in Proposition 2.1 (with $m = 1$), choose a factorization

$$p = u_1 u_2 \cdots u_d.$$

For $j = 1,\dots,d$, put

$$v_j = u_j(x_1,\dots,x_r)$$

and

$$w_j = u_j(y_1,\dots,y_r).$$

Then, with the notation of Corollary 2.4, for $\lambda \in \mathbb{C}$ we have that

$$\lambda \mathbf{1}_{\mathcal{A}} - p(x_1,\dots,x_r) \in \mathcal{A}_{inv} \Leftrightarrow A(\lambda, v_1,\dots,v_d) \in M_k(\mathcal{A})_{inv}$$

and

$$\lambda \mathbf{1}_{\mathcal{B}} - p(y_1,\dots,y_r) \in \mathcal{B}_{inv} \Leftrightarrow A(\lambda, w_1,\dots,w_d) \in M_k(\mathcal{B})_{inv}.$$

Since the u_i's have degree 1,

$$q := A(\lambda, u_1, \ldots, u_d)$$

is a polynomial of degree 1 in $M_k(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \ldots, X_r\rangle$. Note that $A(\lambda, v_1, \ldots, v_d) = q(x_1, \ldots, x_r)$ and $A(\lambda, w_1, \ldots, w_d) = q(y_1, \ldots, y_r)$. Hence, by (2.10),

$$\sigma(A(\lambda, v_1, \ldots, v_d)) \supseteq \sigma(A(\lambda, w_1, \ldots, w_d)).$$

In particular,

$$A(\lambda, v_1, \ldots, v_d) \in M_k(\mathcal{A})_{inv} \Rightarrow A(\lambda, w_1, \ldots, w_d) \in M_k(\mathcal{B})_{inv}.$$

Altogether, we have shown that

$$\lambda \mathbf{1}_{\mathcal{A}} - p(x_1, \ldots, x_r) \in \mathcal{A}_{inv} \Rightarrow \lambda \mathbf{1}_{\mathcal{B}} - p(y_1, \ldots, y_r) \in \mathcal{B}_{inv},$$

i.e.

$$\sigma(p(x_1, \ldots, x_r)) \supseteq \sigma(p(y_1, \ldots, y_r)) \tag{2.11}$$

holds for all polynomials $p \in \mathbb{C}\langle X_1, \ldots, X_r\rangle$. In particular, the spectral radii, $r(p(x_1, \ldots, x_r))$ and $r(p(y_1, \ldots, y_r))$ satisfy

$$r(p(x_1, \ldots, x_r)) \geq r(p(y_1, \ldots, y_r)). \tag{2.12}$$

Applying (2.12) to the self-adjoint polynomial p^*p, we get that

$$\|p(x_1, \ldots, x_r)\|^2 \geq \|p(y_1, \ldots, y_r)\|^2.$$

Hence, the map

$$\phi_0 : p(x_1, \ldots, x_r) \mapsto p(y_1, \ldots, y_r), \qquad (p \in \mathbb{C}\langle X_1, \ldots, X_r\rangle),$$

is well-defined and extends by continuity to a unital $*$-homomorphism ϕ from $C^*(\mathbf{1}_{\mathcal{A}}, x_1, \ldots, x_r)$ into $C^*(\mathbf{1}_{\mathcal{B}}, y_1, \ldots, y_r)$ with $\phi(x_i) = y_i$, $i = 1, \ldots, r$. ∎

3 Norm estimates.

In this section we consider a fixed self-adjoint polynomial p in r non-commuting variables with coefficients in $M_m(\mathbb{C})$, i.e. $p \in (M_m(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \ldots, X_r\rangle)_{sa}$, and for each $n \in \mathbb{N}$, we let $X_1^{(n)}, \ldots, X_r^{(n)}$ be stochastically independent random matrices from SGRM $(n, \frac{1}{n})$. Define self-adjoint random matrices $(Q_n)_{n=1}^{\infty}$ by

$$Q_n(\omega) = p(X_1^{(n)}(\omega), \ldots, X_r^{(n)}(\omega)), \qquad (\omega \in \Omega), \tag{3.1}$$

where $(\Omega, \mathcal{F}, P)$ denotes the underlying probability space.

With $d = \deg(p)$ we may, according to Proposition 2.1, choose $m_1, \ldots, m_{d+1} \in \mathbb{N}$ with $m = m_1 = m_{d+1}$, and polynomials $u_j \in M_{m_j, m_{j+1}}(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \ldots, X_r\rangle$ of first degree,

$j = 1, \dots, d$, such that $p = u_1 u_2 \cdots u_d$. For each $n \in \mathbb{N}$ define random matrices $u_j^{(n)}$, $j = 1, \dots, d$, by

$$u_j^{(n)}(\omega) = u_j(X_1^{(n)}(\omega), \dots, X_r^{(n)}(\omega)), \qquad (\omega \in \Omega).$$

For $\lambda \in M_m(\mathbb{C})$ we put $\mathrm{Im}\lambda = \frac{1}{2\mathrm{i}}(\lambda - \lambda^*)$ as in [HT, Section 3].

Since $Q_n(\omega)$ is self-adjoint, $\lambda \otimes \mathbf{1}_n - Q_n(\omega)$ is invertible for every $\lambda \in M_m(\mathbb{C})$ with $\mathrm{Im}\lambda$ positive definite (cf. [HT, Lemma 3.1]). Then, according to Corollary 2.4, the random matrix

$$A_n(\lambda) = \begin{pmatrix} \lambda \otimes \mathbf{1}_n & -u_1^{(n)} & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{1}_{m_2} \otimes \mathbf{1}_n & -u_2^{(n)} & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{1}_{m_3} \otimes \mathbf{1}_n & -u_3^{(n)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbf{1}_{m_{d-1}} \otimes \mathbf{1}_n & -u_{d-1}^{(n)} \\ -u_d^{(n)} & 0 & 0 & \cdots & 0 & \mathbf{1}_{m_d} \otimes \mathbf{1}_n \end{pmatrix} \tag{3.2}$$

is (point-wise) invertible in $M_k(\mathbb{C})$, where $k = \sum_{i=1}^d m_i$.

3.1 Lemma. *For every $p \in \mathbb{N}$ there exist constants $C_{1,p}, C_{2,p} \geq 0$, such that for all $m \in \mathbb{N}$ and for all $\lambda \in M_m(\mathbb{C})$ with $\mathrm{Im}\lambda$ positive definite,*

$$\sup_{n \in \mathbb{N}} \mathbb{E}\{\|A_n(\lambda)^{-1}\|^p\} \leq C_{1,p} + C_{2,p}\|(\mathrm{Im}\lambda)^{-1}\|^p.$$

Proof. Let $p \in \mathbb{N}$. According to Proposition 2.3 we may write

$$A_n(\lambda)^{-1} = C_n + B_n^{(1)}(\lambda \otimes \mathbf{1}_n - Q_n)^{-1} B_n^{(2)},$$

where

$$C_n = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{1}_{m_2} \otimes \mathbf{1}_n & u_2^{(n)} & u_2^{(n)} u_3^{(n)} & u_2^{(n)} u_3^{(n)} u_4^{(n)} & \cdots & u_2^{(n)} u_3^{(n)} \cdots u_{d-1}^{(n)} \\ 0 & 0 & \mathbf{1}_{m_3} \otimes \mathbf{1}_n & u_3^{(n)} & u_3^{(n)} u_4^{(n)} & \cdots & u_3^{(n)} u_4^{(n)} \cdots u_{d-1}^{(n)} \\ 0 & 0 & 0 & \mathbf{1}_{m_4} \otimes \mathbf{1}_n & u_4^{(n)} & \cdots & u_4^{(n)} u_5^{(n)} \cdots u_{d-1}^{(n)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \mathbf{1}_{m_{d-1}} \otimes \mathbf{1}_n & u_{d-1}^{(n)} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mathbf{1}_{m_d} \otimes \mathbf{1}_n \end{pmatrix},$$

$$B_n^{(1)} = \begin{pmatrix} \mathbf{1}_m \otimes \mathbf{1}_n \\ u_2^{(n)} u_3^{(n)} \cdots u_d^{(n)} \\ u_3^{(n)} u_4^{(n)} \cdots u_d^{(n)} \\ \vdots \\ u_d^{(n)} \end{pmatrix},$$

and

$$B_n^{(2)} = \begin{pmatrix} \mathbf{1}_m \otimes \mathbf{1}_n & u_1^{(n)} & u_1^{(n)}u_2^{(n)} & \dots & u_1^{(n)}u_2^{(n)}\cdots u_{d-1}^{(n)} \end{pmatrix}.$$

By [HT, Lemma 3.1], $\|(\lambda \otimes \mathbf{1}_n - Q_n)^{-1}\| \le \|(\mathrm{Im}\lambda)^{-1}\|$, and therefore

$$\begin{aligned} \mathbb{E}\{\|A_n(\lambda)^{-1}\|^p\} &\le \mathbb{E}\{(\|C_n\| + \|B_n^{(1)}\|\|B_n^{(2)}\|\|(\mathrm{Im}\lambda)^{-1}\|)^p\} \\ &\le \mathbb{E}\{(2\,\max\{\|C_n\|, \|B_n^{(1)}\|\|B_n^{(2)}\|\|(\mathrm{Im}\lambda)^{-1}\|\})^p\} \\ &\le 2^p\,\mathbb{E}\{\|C_n\|^p + \|B_n^{(1)}\|^p\|B_n^{(2)}\|^p\|(\mathrm{Im}\lambda)^{-1}\|^p\}. \end{aligned} \tag{3.3}$$

With

$$K_n = \max\{1, \|u_1^{(n)}\|, \|u_2^{(n)}\|, \dots, \|u_d^{(n)}\|\} \tag{3.4}$$

one easily proves that

$$\|C_n\| \le d - 1 + d^2K_n^{d-2} \le d^2(1 + K_n^d),$$

implying that

$$\|C_n\|^p \le 2^pd^{2p}(1 + K_n^{dp}) \le 2^pd^{2p}(1 + K_n^{2pd}). \tag{3.5}$$

Moreover,

$$\|B_n^{(1)}\|^p\|B_n^{(2)}\|^p \le d^{2p}K_n^{2p(d-1)} \le d^{2p}K_n^{2pd}. \tag{3.6}$$

Now, according to (3.4),

$$K_n^{2pd} \le 1 + \sum_{j=1}^{d} \|u_j^{(n)}\|^{2pd}. \tag{3.7}$$

Since $u_j^{(n)}$ is of first degree, we may choose $a_0^{(j)}, \dots, a_r^{(j)} \in M_{m_j,m_{j+1}}(\mathbb{C})$ such that

$$u_j^{(n)} = a_0^{(j)} \otimes \mathbf{1}_n + \sum_{i=1}^{r} a_i^{(j)} \otimes X_i^{(n)}.$$

Hence

$$\|u_j^{(n)}\|^{2pd} \le (1 + r)^{2pd}\,\max\{\|a_0^{(j)}\|, \|a_1^{(j)}\|\|X_1^{(n)}\|, \dots, \|a_r^{(j)}\|\|X_r^{(n)}\|\}^{2pd}. \tag{3.8}$$

According to [S, Lemma 6.4],

$$\sup_n(\mathbb{E}\{\|X_i^{(n)}\|^{2pd}\}) < \infty,$$

and combining this fact with (3.3), (3.5), (3.6), (3.7) and (3.8) we obtain the desired estimate. ■

3.2 Lemma. *Let $x_1, \dots, x_r$ be a semicircular system in a C^*-probability space $(\mathcal{A}, \tau)$, and let*

$$q = p(x_1, \dots, x_r). \tag{3.9}$$

Then again, with $u_j = u_j(x_1, \dots, x_r)$, $j = 1, \dots, d$, the matrix

$$A(\lambda) = \begin{pmatrix} \lambda \otimes \mathbf{1}_{\mathcal{A}} & -u_1 & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{1}_{m_2} \otimes \mathbf{1}_{\mathcal{A}} & -u_2 & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{1}_{m_3} \otimes \mathbf{1}_{\mathcal{A}} & -u_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbf{1}_{m_{d-1}} \otimes \mathbf{1}_{\mathcal{A}} & -u_{d-1} \\ -u_d & 0 & 0 & \cdots & 0 & \mathbf{1}_{m_d} \otimes \mathbf{1}_{\mathcal{A}} \end{pmatrix}$$

is invertible for every $\lambda \in M_m(\mathbb{C})$ with $\mathrm{Im}\lambda > 0$. Moreover, for every $p \in \mathbb{N}$ there exist constants $C'_{1,p}$ and $C'_{2,p}$ such that

$$\|A(\lambda)^{-1}\|^p \leq C'_{1,p} + C'_{2,p} \|(\mathrm{Im}\lambda)^{-1}\|^p \tag{3.10}$$

holds for every $\lambda \in M_m(\mathbb{C})$ with $\mathrm{Im}\lambda$ positive definite.

Proof. It follows again form [HT, Lemma 3.1] that $\lambda \otimes \mathbf{1}_{\mathcal{A}} - q$ is invertible. Hence, by Proposition 2.3, $A(\lambda)$ is invertible with

$$A(\lambda)^{-1} = C + B^{(1)} (\lambda \otimes \mathbf{1}_{\mathcal{A}} - q)^{-1} B^{(2)},$$

where

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \mathbf{1}_{m_2} \otimes \mathbf{1}_{\mathcal{A}} & u_2 & u_2 u_3 & u_2 u_3 u_4 & \dots & u_2 u_3 \cdots u_{d-1} \\ 0 & 0 & \mathbf{1}_{m_3} \otimes \mathbf{1}_{\mathcal{A}} & u_3 & u_3 u_4 & \dots & u_3 u_4 \cdots u_{d-1} \\ 0 & 0 & 0 & \mathbf{1}_{m_4} \otimes \mathbf{1}_{\mathcal{A}} & u_4 & \dots & u_4 u_5 \cdots u_{d-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \mathbf{1}_{m_{d-1}} \otimes \mathbf{1}_{\mathcal{A}} & u_{d-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & \mathbf{1}_{m_d} \otimes \mathbf{1}_{\mathcal{A}} \end{pmatrix},$$

$$B^{(1)} = \begin{pmatrix} \mathbf{1}_m \otimes \mathbf{1}_{\mathcal{A}} \\ u_2 u_3 \cdots u_d \\ u_3 u_4 \cdots u_d \\ \vdots \\ u_d \end{pmatrix},$$

and

$$B^{(2)} = \begin{pmatrix} \mathbf{1}_m \otimes \mathbf{1}_{\mathcal{A}} & u_1 & u_1 u_2 & \dots & u_1 u_2 \cdots u_{d-1} \end{pmatrix}.$$

Then, since $\|(\lambda \otimes \mathbf{1}_{\mathcal{A}} - q)^{-1}\| \leq \|(\mathrm{Im}\lambda)^{-1}\|$, we have as in the proof of Lemma 3.1 that

$$\|A(\lambda)^{-1}\|^p \leq 2^p (\|C\|^p + \|B^{(1)}\|^p \|B^{(2)}\|^p \|(\mathrm{Im}\lambda)^{-1}\|^p),$$

and the claim follows. ∎

4 The master equation and master inequality.

In this section we prove generalizations of the master equation and master inequality from [HT]. These generalizations will allow us to handle self-adjoint polynomials $p(X_1^{(n)}, \dots, X_r^{(n)})$ of arbitrary degree in r independent random matrices from SGRM $(n, \frac{1}{n})$.

Let r and n be positive integers. As in [HT, Section 3], we shall consider the *real* vector space $(M_n(\mathbb{C})_{sa})^r$, which we denote by $\mathcal{E}_{r,n}$. We equip $\mathcal{E}_{r,n}$ with the inner product $\langle \cdot, \cdot \rangle_e$ given by

$$\langle (A_1, \dots, A_r), (B_1, \dots, B_r) \rangle_e = \mathrm{Tr}_n\Big(\sum_{j=1}^{r} A_j B_j \Big), \qquad ((A_1, \dots, A_r), (B_1, \dots, B_r) \in \mathcal{E}_{r,n}),$$

and we denote the corresponding norm by $\| \cdot \|_e$. Still following [HT], we consider the linear isomorphism Ψ_0 between $M_n(\mathbb{C})_{sa}$ and $\mathbb{R}^{n^2}$ given by

$$\Psi_0((a_{uv})_{1 \le u,v \le n}) = \big((a_{uu})_{1 \le u \le n}, (\sqrt{2}\mathrm{Re}(a_{uv}))_{1 \le u < v \le n}, (\sqrt{2}\mathrm{Im}(a_{ub}))_{1 \le u < v \le n} \big), \tag{4.1}$$

for $(a_{uv})_{1 \le u,v \le n}$ in $M_n(\mathbb{C})_{sa}$. We consider further the natural extension $\Psi \colon \mathcal{E}_{r,n} \to \mathbb{R}^{rn^2}$ of Ψ_0 given by

$$\Psi(A_1, \dots, A_r) = (\Psi_0(A_1), \dots, \Psi_0(A_r)), \qquad (A_1, \dots, A_r \in M_n(\mathbb{C})_{sa}).$$

We note that Ψ is an isometry between $(\mathcal{E}_{r,n}, \| \cdot \|_e)$ and $\mathbb{R}^{rn^2}$ equipped with its usual Hilbert space norm. Accordingly, we shall identify $\mathcal{E}_{r,n}$ with $\mathbb{R}^{rn^2}$ via Ψ.

In the following we consider a fixed self-adjoint polynomial p from $M_m(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \dots, X_r \rangle$ of degree d and the corresponding polynomials

$$u_j = u_j(X_1, \dots, X_r) \in M_{m_j, m_{j+1}}(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \dots, X_r \rangle, \qquad (j = 1, 2, \dots, d),$$

introduced in Proposition 2.1. We put $k = m_1 + m_2 + \cdots m_d$.

We consider further independent random matrices $X_1^{(n)}, \dots, X_r^{(n)}$ from SGRM$(n, \frac{1}{n})$ and a fixed matrix λ from $M_m(\mathbb{C})$, such that $\mathrm{Im}(\lambda)$ is positive definite. We may then consider the (random) matrix $A(\lambda, X_1^{(n)}, \dots, X_r^{(n)})$ defined in Corollary 2.4. Since the polynomials $u_1, \dots, u_d$ are of degree 1, we may write

$$A(\lambda, X_1^{(n)}, \dots, X_r^{(n)}) = \begin{pmatrix} \lambda & 0 \\ 0 & \mathbf{1}_{k-m} \end{pmatrix} \otimes \mathbf{1}_n - a_0 \otimes \mathbf{1}_n - \sum_{j=1}^{r} a_i \otimes X_i^{(n)},$$

for suitable matrices $a_0, a_1, \dots, a_r$ in $M_k(\mathbb{C})$. We put

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \mathbf{1}_{k-m} \end{pmatrix} \quad \text{and} \quad S_n = a_0 \otimes \mathbf{1}_n + \sum_{j=1}^{r} a_j \otimes X_j^{(n)}, \tag{4.2}$$

so that

$$A(\lambda, X_1^{(n)}, \dots, X_r^{(n)}) = \Lambda \otimes \mathbf{1}_n - S_n.$$

According to Corollary 2.4, $\Lambda\otimes\mathbf{1}_n - S_n$ is invertible, and hence we may consider the $k\times k$ matrix

$$H_n(\lambda) = (\mathrm{id}_k\otimes\mathrm{tr}_n)\big[(\Lambda\otimes\mathbf{1}_n - S_n)^{-1}\big]. \tag{4.3}$$

The following lemma generalizes [HT, Lemma 3.5]:

4.1 Lemma. *With $H_n(\lambda)$ as defined in* (4.3), *we have for any j in $\{1,2,\dots,r\}$ the formula*

$$\mathbb{E}\big\{H_n(\lambda)a_jH_n(\lambda)\big\} = \mathbb{E}\big\{(\mathrm{id}_k\otimes\mathrm{tr}_n)\big[(\mathbf{1}_k\otimes X_j^{(n)})\cdot(\Lambda\otimes\mathbf{1}_n - S_n)^{-1}\big]\big\}.$$

Proof. For any $v_1,\dots,v_r\in M_n(\mathbb{C})_{sa}$ we may consider the matrix $A(\lambda,v_1,\dots,v_r)$ described in Corollary 2.4, and we clearly have that

$$A(\lambda,v_1,\dots,v_r) = \Lambda\otimes\mathbf{1}_n - -a_0\otimes\mathbf{1}_n - \sum_{j=1}^{r} a_j\otimes v_j,$$

with $\Lambda, a_1,\dots,a_r$ as above. According to Corollary 2.4, we may then consider the mapping $\tilde{F}\colon \mathcal{E}_{r,n}\to M_k(\mathbb{C})\otimes M_n(\mathbb{C})$ given by

$$\tilde{F}(v_1,\dots,v_r) = \big((\Lambda-a_0)\otimes\mathbf{1}_n - \textstyle\sum_{j=1}^{r} a_j\otimes v_j\big)^{-1}, \qquad ((v_1,\dots,v_r)\in\mathcal{E}_{r,n}).$$

We consider furthermore the mapping $F\colon\mathbb{R}^{rn^2}\to M_k(\mathbb{C})\otimes M_n(\mathbb{C})$, given by

$$F = \tilde{F}\circ\Psi^{-1}.$$

Note then that

$$(\Lambda\otimes\mathbf{1}_n - S_n)^{-1} = \tilde{F}(X_1^{(n)},\dots,X_r^{(n)}) = F(\Psi(X_1^{(n)},\dots,X_r^{(n)})), \tag{4.4}$$

where $\Psi(X_1^{(n)},\dots,X_r^{(n)}) = (\gamma_1,\gamma_2,\dots,\gamma_{rn^2})$ with $\gamma_1,\gamma_2,\dots,\gamma_{rn^2}\sim$ i.i.d. $N(0,\frac{1}{n})$. According to the proof of [HT, Lemma 3.1], we have that

$$\big\|\tilde{F}(v_1,\dots,v_r)\big\| \le h(\|v_1\|,\dots,\|v_r\|)(1+\|(\mathrm{Im}\lambda)^{-1}\|), \tag{4.5}$$

for some polynomial h in $\mathbb{C}[X_1,\dots,X_r]$. From (4.4) and (4.5) it follows firstly that that the expectations in Lemma 4.1 are well-defined. In addition, (4.5) shows that the function F is a polynomially bounded function of rn^2 real variables. In order to apply [HT, Lemma 3.3], we need to check that the partial derivatives of F are polynomially bounded as well. To this end, consider the standard orthonormal basis for $M_n(\mathbb{C})_{sa}$:

$$e_{u,u}^{(n)}, \qquad (1\le u\le n)$$

$$f_{u,v}^{(n)} = \tfrac{1}{\sqrt{2}}\big(e_{u,v}^{(n)} + e_{v,u}^{(n)}\big) \qquad (1\le u<v\le n),$$

$$g_{u,v}^{(n)} = \tfrac{\mathrm{i}}{\sqrt{2}}\big(e_{u,v}^{(n)} - e_{v,u}^{(n)}\big) \qquad (1\le u<v\le n),$$

where $\{e^{(n)}_{u,v} \mid 1 \le u,v \le n\}$ are the standard $n \times n$ matrix units. The correponding orthonormal basis for $\mathcal{E}_{r,n}$ is

$$e^{(n)}_{j,u,u} = (0,\dots,0,e^{(n)}_{u,u},0,\dots,0), \qquad (1 \le j \le r,\ 1 \le u \le n)$$

$$f^{(n)}_{j,u,v} = (0,\dots,0,f^{(n)}_{u,v},0,\dots,0), \qquad (1 \le j \le r,\ 1 \le u < v \le n),$$

$$g^{(n)}_{j,u,v} = (0,\dots,0,g^{(n)}_{u,v},0,\dots,0), \qquad (1 \le j \le r,\ 1 \le u < v \le n),$$

with the non-zero entry in the j'th slot. Note that the images by Ψ of these basis vectors is exactly the standard orthonormal basis for $\mathbb{R}^{rn^2}$. Hence, the partial derivatives of F at a point ξ in $\mathbb{R}^{rn^2}$ are, setting $(v_1,\dots,v_r) = \Psi^{-1}(\xi)$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F\big(\xi + t\Psi(e^{(n)}_{j,u,u})\big) \\
&\quad= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \tilde{F}\big((v_1,\dots,v_r) + te^{(n)}_{j,u,u}\big) \\
&\quad= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \tilde{F}\big(v_1,\dots,v_{j-1},v_j + te^{(n)}_{u,u},v_{j+1},\dots,v_r\big) \\
&\quad= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \big((\Lambda - a_0)\otimes \mathbf{1}_n - \textstyle\sum_{i=1}^r a_i \otimes v_i - t(a_j \otimes e^{(n)}_{u,u})\big)^{-1} \\
&\quad= \big((\Lambda - a_0)\otimes \mathbf{1}_n - \textstyle\sum_{i=1}^r a_i \otimes v_i\big)^{-1}\big(a_j \otimes e^{(n)}_{u,u}\big)\big((\Lambda - a_0)\otimes \mathbf{1}_n - \textstyle\sum_{i=1}^r a_i \otimes v_i\big)^{-1},
\end{aligned} \tag{4.6}$$

where the last equality uses [HT, Lemma 3.2]. We find similarly that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F\big(\xi + t\Psi(f^{(n)}_{j,u,v})\big) = \big((\Lambda - a_0)\otimes \mathbf{1}_n - \textstyle\sum_{i=1}^r a_i \otimes v_i\big)^{-1}\big(a_j \otimes f^{(n)}_{u,v}\big)\big((\Lambda - a_0)\otimes \mathbf{1}_n - \textstyle\sum_{i=1}^r a_i \otimes v_i\big)^{-1},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F\big(\xi + t\Psi(g^{(n)}_{j,u,v})\big) = \big((\Lambda - a_0)\otimes \mathbf{1}_n - \textstyle\sum_{i=1}^r a_i \otimes v_i\big)^{-1}\big(a_j \otimes g^{(n)}_{u,v}\big)\big((\Lambda - a_0)\otimes \mathbf{1}_n - \textstyle\sum_{i=1}^r a_i \otimes v_i\big)^{-1}.$$

Appealing once more to (4.5), it follows that the partial derivatives of F are polynomially bounded as well. Hence, we may apply [HT, Lemma 3.3] to F and the i.i.d. Gaussian variables $\Psi(X^{(n)}_1,\dots,X^{(n)}_r) = (\gamma_1,\gamma_2,\dots,\gamma_{rn^2})$. For any j in $\{1,2,\dots,r\}$ put

$$X^{(n)}_{j,u,u} = (X^{(n)}_j)_{kk}, \qquad (1 \le u \le n),$$

$$Y^{(n)}_{j,u,v} = \sqrt{2}\mathrm{Re}(X^{(n)}_j)_{u,v}, \qquad (1 \le u < v \le n),$$

$$Z^{(n)}_{j,kl} = \sqrt{2}\mathrm{Im}(X^{(n)}_j)_{u,v}, \qquad (1 \le u < v \le n),$$

and note that these random variables are the coefficients of $\Psi(X^{(n)}_1,\dots,X^{(n)}_r)$ w.r.t. the standard orthonormal basis for $\mathbb{R}^{rn^2}$, in the sense that

$$\Psi(X^{(n)}_1,\dots,X^{(n)}_r) = \sum_{j=1}^r \Big(\sum_{u=1}^n X^{(n)}_{j,u,u}\Psi(e^{(n)}_{j,u,u}) + \sum_{1\le u<v\le n} Y^{(n)}_{j,u,v}\Psi(f^{(n)}_{j,u,v}) + \sum_{1\le u<v\le n} Z^{(n)}_{j,u,v}\Psi(g^{(n)}_{j,u,v})\Big).$$

It follows thus from the Gaussian Poincaré Inequality and (4.6) that

$$\begin{aligned}\mathbb{E}\big\{X_{j,u,u}^{(n)}\big(\Lambda\otimes\mathbf{1}_n-S_n\big)^{-1}\big\} &= \mathbb{E}\big\{X_{j,u,u}^{(n)}F(\Psi(X_1^{(n)},\dots,X_r^{(n)}))\big\} \\ &= \frac{1}{n}\mathbb{E}\Big\{\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}F(\Psi(X_1^{(n)},\dots,X_r^{(n)}))-t\Psi(e_{j,u,u})\Big\} \\ &= \frac{1}{n}\mathbb{E}\big\{\big(\Lambda\otimes\mathbf{1}_n-S_n\big)^{-1}\big(a_j\otimes e_{u,u}^{(n)}\big)\big(\Lambda\otimes\mathbf{1}_n-S_n\big)^{-1}\big\},\end{aligned}$$

and similarly we get that

$$\mathbb{E}\big\{Y_{j,u,v}^{(n)}\cdot\big(\Lambda\otimes\mathbf{1}_n-S_n\big)^{-1}\big\} = \frac{1}{n}\mathbb{E}\big\{(\Lambda\otimes\mathbf{1}_n-S_n)^{-1}(a_j\otimes f_{u,v}^{(n)})(\Lambda\otimes\mathbf{1}_n-S_n)^{-1}\big\}$$

$$\mathbb{E}\big\{Z_{j,u,v}^{(n)}\cdot\big(\Lambda\otimes\mathbf{1}_n-S_n\big)^{-1}\big\} = \frac{1}{n}\mathbb{E}\big\{(\Lambda\otimes\mathbf{1}_n-S_n)^{-1}(a_j\otimes g_{u,v}^{(n)})(\Lambda\otimes\mathbf{1}_n-S_n)^{-1}\big\}.$$

>From this point, the proof is completed exactly as in the proof of [HT, Lemma 3.5]. ■

Lemma 4.1 implies the following analogue of [HT, Theorem 3.6]. The proof is the same as in [HT] and will therefore be omitted.

4.2 Theorem. (Master equation) *Let λ be a matrix in $M_m(\mathbb{C})$ such that $\mathrm{Im}(\lambda)$ is positive definite, and let Λ and S_n be the matrices introduced in (4.2). Then with*

$$H_n(\lambda) = (\mathrm{id}_k\otimes\mathrm{tr}_n)\big[(\Lambda\otimes\mathbf{1}_n-S_n)^{-1}\big]$$

we have the formula

$$\mathbb{E}\Big\{\sum_{i=1}^r a_iH_n(\lambda)a_iH_n(\lambda)+(a_0-\Lambda)H_n(\lambda)+\mathbf{1}_m\Big\}=0. \qquad (4.7)$$

We next prove the following analogue of [HT, Theorem 4.5]:

4.3 Theorem. (Master inequality) *Let λ be a matrix in $M_m(\mathbb{C})$ such that $\mathrm{Im}(\lambda)$ is positive definite, and let Λ and S_n be the matrices introduced in (4.2). Then with*

$$H_n(\lambda) = (\mathrm{id}_k\otimes\mathrm{tr}_n)\big[(\Lambda\otimes\mathbf{1}_n-S_n)^{-1}\big]$$

and

$$G_n(\lambda)=\mathbb{E}\{H_n(\lambda)\},$$

we have the estimate

$$\Big\|\sum_{i=1}^r a_iG_n(\lambda)a_iG_n(\lambda)+(a_0-\Lambda)G_n(\lambda)+\mathbf{1}_k\Big\| \le \frac{C}{n^2}\Big(C_{1,4}+C_{2,4}\big\|(\mathrm{Im}\lambda)^{-1}\big\|^4\Big),$$

where $C=k^3(\sum_{i=1}^r\|a_i\|^2)^2$ and $C_{1,4},C_{2,4}$ are the constants introduced in Lemma 3.1.

Proof. Setting $K_n(\lambda) = H_n(\lambda) - G_n(\lambda) = H_n(\lambda) - \mathbb{E}\{H_n(\lambda)\}$, we find exactly as in [HT, proof of Theorem 4.5] that

$$\sum_{i=1}^{r} a_i G_n(\lambda) a_i G_n(\lambda) + (a_0 - \Lambda) G_n(\lambda) + \mathbf{1}_k = -\mathbb{E}\Big\{ \sum_{i=1}^{r} a_i K_n(\lambda) a_i K_n(\lambda) \Big\},$$

and from this

$$\begin{aligned} \Big\| \sum_{i=1}^{r} a_i G_n(\lambda) a_i G_n(\lambda) + (a_0 - \Lambda) G_n(\lambda) + \mathbf{1}_k \Big\| &\le \sum_{i=1}^{r} \|a_i\|^2 \mathbb{E}\big\{ \|K_n(\lambda)\|^2 \big\} \\ &\le \Big(\sum_{i=1}^{r} \|a_i\|^2 \Big) \sum_{u,v=1}^{k} \mathbb{E}\big\{ |K_{n,u,v}(\lambda)|^2 \big\} \quad (4.8) \\ &= \Big(\sum_{i=1}^{r} \|a_i\|^2 \Big) \sum_{u,v=1}^{k} \mathbb{V}\big\{ H_{n,u,v}(\lambda) \big\}, \end{aligned}$$

where $K_{n,u,v}(\lambda)$ (resp. $H_{n,u,v}(\lambda)$), $1 \le u, v \le k$, are the entries of $K_n(\lambda)$ (resp. $H_n(\lambda)$). As in [HT, proof of Theorem 4.5] we note that

$$H_{n,u,v}(\lambda) = f_{n,u,v}\big(X_1^{(n)}, \dots, X_r^{(n)}\big),$$

where $f_{n,u,v} \colon \mathcal{E}_{r,n} \to \mathbb{C}$ is the function given by

$$f_{n,u,v}(v_1, \dots, v_r) = k(\mathrm{tr}_k \otimes \mathrm{tr}_n)\big[(e_{u,v}^{(k)} \otimes \mathbf{1}_n)\big((\Lambda - a_0) \otimes \mathbf{1}_n - \textstyle\sum_{i=1}^{r} a_i \otimes v_i\big)^{-1}\big],$$

for $v = (v_1, \dots, v_r) \in \mathcal{E}_{r,n}$. For any unit vector $w = (w_1, \dots, w_r)$ from $\mathcal{E}_{r,n}$, we find as in [HT] that

$$\Big| \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f_{n,u,v}(v + tw) \Big| \le \frac{1}{n} \big\| \textstyle\sum_{i=1}^{r} a_i \otimes w_i \big\|_{2,\mathrm{Tr}_k \otimes \mathrm{Tr}_n}^2 \big\| ((\Lambda - a_0) \otimes \mathbf{1}_n - \textstyle\sum_{i=1}^{r} a_i \otimes v_i)^{-1} \big\|^4,$$

and here, by arguing as in the proof of [HT, Lemma 4.4],

$$\big\| \textstyle\sum_{i=1}^{r} a_i \otimes w_i \big\|_{2,\mathrm{Tr}_k \otimes \mathrm{Tr}_n}^2 \le k \big\| \textstyle\sum_{i=1}^{r} a_i^* a_i \big\| \le k \textstyle\sum_{i=1}^{r} \|a_i\|^2,$$

so that

$$\Big| \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f_{n,u,v}(v + tw) \Big| \le \frac{k}{n} \big(\textstyle\sum_{i=1}^{r} \|a_i\|^2 \big) \big\| ((\Lambda - a_0) \otimes \mathbf{1}_n - \textstyle\sum_{i=1}^{r} a_i \otimes v_i)^{-1} \big\|^4.$$

Consequently,

$$\begin{aligned} \big\| \mathrm{grad} f_{n,u,v}(v) \big\|^2 &= \max \Big\{ \Big| \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f_{n,u,v}(v + tw) \Big|^2 \;\Big|\; w \in \mathcal{E}_{r,n},\ \|w\|_e = 1 \Big\} \\ &\le \frac{k}{n} \big(\textstyle\sum_{i=1}^{r} \|a_i\|^2 \big) \big\| ((\Lambda - a_0) \otimes \mathbf{1}_n - \textstyle\sum_{i=1}^{r} a_i \otimes v_i)^{-1} \big\|^4. \end{aligned}$$

Combining this with (4.5), it is clear that grad $f_{n,u,v}$ (as well as $f_{n,u,v}$ itself) is polynomially bounded as a function of rn^2 real variables. Hence we may apply the Gaussian Poincaré Inequality in the form of [HT, Corollary 4.2] as follows:

$$\begin{aligned}\mathbb{V}\big\{H_{n,u,v}(\lambda)\big\} &= \mathbb{V}\big\{f_{n,u,v}(X_1^{(n)},\dots,X_r^{(n)})\big\} \le \frac{1}{n}\mathbb{E}\Big\{\big\|\mathrm{grad} f_{n,u,v}(X_1^{(n)},\dots,X_r^{(n)})\big\|^2\Big\} \\ &\le \frac{k}{n^2}\big(\textstyle\sum_{i=1}^r \|a_i\|^2\big)\mathbb{E}\Big\{\big\|\big((\Lambda-a_0)\otimes \mathbf{1}_n - \sum_{i=1}^r a_i\otimes X_i^{(n)}\big)^{-1}\big\|^4\Big\} \qquad (4.9)\\ &\le \frac{k}{n^2}\big(\textstyle\sum_{i=1}^r \|a_i\|^2\big)\cdot\Big(C_{1,4}+C_{2,4}\big\|(\mathrm{Im}\lambda)^{-1}\big\|^4\Big),\end{aligned}$$

where $C_{1,4}$ and $C_{2,4}$ are the constants given in Lemma 3.1. Since (4.9) holds for all u,v in $\{1,2,\dots,k\}$, we find in combination with (4.8) that

$$\Big\|\sum_{i=1}^r a_iG_n(\lambda)a_iG_n(\lambda)+(a_0-\Lambda)G_n(\lambda)+\mathbf{1}_k\Big\| \le \frac{k^3}{n^2}\Big(\sum_{i=1}^r\|a_i\|^2\Big)^2\cdot\Big(C_{1,4}+C_{2,4}\big\|(\mathrm{Im}\lambda)^{-1}\big\|^4\Big),$$

and this is the desired estimate. ■

5 Estimation of $\|G_n(\lambda)-G(\lambda)\|$.

As in the two previous sections, for each $n\in\mathbb{N}$ we consider stochastically independent random matrices $X_1^{(n)},\dots,X_r^{(n)}$ from GUE$(n,\frac{1}{n})$, and we let

$$Q_n = p(X_1^{(n)},\dots,X_r^{(n)}), \tag{5.1}$$

where p is a fixed self-adjoint polynomial from $M_m(\mathbb{C})\otimes\mathbb{C}\langle X_1,\dots,X_r\rangle$. We let $A_n(\lambda)$ be given by (3.2), where $\lambda\in M_m(\mathbb{C})$, and Im$\lambda$ is positive definite. Then we may write

$$A_n(\lambda)=\Lambda\otimes\mathbf{1}_n - S_n, \tag{5.2}$$

where

$$\Lambda=\begin{pmatrix}\lambda & 0\\ 0 & \mathbf{1}_{k-m}\end{pmatrix}, \tag{5.3}$$

and

$$S_n = a_0\otimes\mathbf{1}_n+\sum_{i=1}^r a_i\otimes X_i^{(n)} \tag{5.4}$$

for suitable matrices $a_0,a_1,\dots,a_r\in M_k(\mathbb{C})$. Note that, according to (3.2), the a_i's are block matrices of the form

$$a_i=\begin{pmatrix}0 & a_i^{(1)} & 0 & 0 & \cdots & 0\\ 0 & 0 & a_i^{(2)} & 0 & \cdots & 0\\ 0 & 0 & 0 & a_i^{(3)} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & a_i^{(d-1)}\\ a_i^{(d)} & 0 & 0 & \cdots & 0 & 0\end{pmatrix}, \tag{5.5}$$

where $a_i^{(j)} \in M_{m_j,m_{j+1}}(\mathbb{C})$, $j = 1,\dots,d-1$, and $a_i^{(d)} \in M_{m_d,m_1}(\mathbb{C})$. As in the previous section, let

$$H_n(\lambda) = (\mathrm{id}_k \otimes \mathrm{tr}_n)[(\Lambda \otimes \mathbf{1}_n - S_n)^{-1}], \tag{5.6}$$

and let

$$G_n(\lambda) = \mathbb{E}\{H_n(\lambda)\}. \tag{5.7}$$

Next, let $x_1,\dots,x_r$ be a semicircular system in a C^*-probability space $(\mathcal{A},\tau)$ with τ faithful, and put

$$q = p(x_1,\dots,x_r), \tag{5.8}$$

$$s = a_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_{i=1}^{r} a_i \otimes x_i, \tag{5.9}$$

and

$$G(\lambda) = (\mathrm{id}_k \otimes \tau)[(\Lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}]. \tag{5.10}$$

Note that, according to Lemma 2.3, $\Lambda \otimes \mathbf{1}_{\mathcal{A}} - s$ is invertible. Finally, for every $\mu \in M_k(\mathbb{C})$, such that $\mu \otimes \mathbf{1}_{\mathcal{A}} - s$ is invertible, put

$$\tilde{G}(\mu) = (\mathrm{id}_k \otimes \tau)[(\mu \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}]. \tag{5.11}$$

5.1 Lemma. *(i) The $\mathcal{R}$-transform of s w.r.t. amalgamation over $M_k(\mathbb{C}) \otimes \mathbf{1}_{\mathcal{A}}$ is given by*

$$\mathcal{R}(z) = a_0 + \sum_{i=1}^{r} a_i z a_i, \qquad (z \in M_k(\mathbb{C})).$$

(ii) If $\mu \in M_k(\mathbb{C})$ is invertible, and $\|\mu^{-1}\| < \frac{1}{\|s\|}$, then $\tilde{G}(\mu)$ is well-defined and invertible, and

$$a_0 + \sum_{i=1}^{r} a_i \tilde{G}(\mu) a_i + \tilde{G}(\mu)^{-1} = \mu.$$

(iii) Let $\mu \in M_k(\mathbb{C})$ be invertible, and let $R, T \in M_k(\mathbb{C})$ be block diagonal matrices of the form

$$R = \mathrm{diag}(r_1 \mathbf{1}_{m_1}, r_2 \mathbf{1}_{m_2}, \dots, r_d \mathbf{1}_{m_d}),$$

$$T = \mathrm{diag}(t_1 \mathbf{1}_{m_1}, t_2 \mathbf{1}_{m_2}, \dots, t_d \mathbf{1}_{m_d}),$$

where $r_1,\dots,r_d,t_1,\dots,t_d \in \mathbb{C} \setminus \{0\}$ satisfy

$$r_1 t_2 = r_2 t_3 = \cdots = r_{d-1} t_d = r_d t_1 = 1.$$

If $\|(R\mu T)^{-1}\| < \frac{1}{\|s\|}$, then $\tilde{G}(\mu)$ is well-defined and invertible, and

$$a_0 + \sum_{i=1}^{r} a_i \tilde{G}(\mu) a_i + \tilde{G}(\mu)^{-1} = \mu.$$

Proof. (i) is essentially due to Lehner [Le]. One just have to exchange a_i^* with a_i in the proof of [Le, Prop. 4.1].

In order to prove (ii), note that if $\|\mu^{-1}\| < \frac{1}{\|s\|}$, then

$$\mu \otimes \mathbf{1}_{\mathcal{A}} - s = (\mu \otimes \mathbf{1}_{\mathcal{A}})(\mathbf{1}_k \otimes \mathbf{1}_{\mathcal{A}} - (\mu^{-1} \otimes \mathbf{1}_{\mathcal{A}})s),$$

where $\|(\mu^{-1} \otimes \mathbf{1}_{\mathcal{A}})s\| < 1$. Hence, $\mu \otimes \mathbf{1}_{\mathcal{A}} - s$ is invertible, and $\tilde{G}(\mu)$ is well-defined. If in addition $\|\mu^{-1}\| < \frac{1}{2\|s\|}$, then we get from Neumann's series that

$$\begin{aligned} \|\tilde{G}(\mu)\| &\leq \|(\mu \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}\| \\ &\leq \|\mu^{-1}\| \|(\mathbf{1}_k \otimes \mathbf{1}_{\mathcal{A}} - (\mu^{-1} \otimes \mathbf{1}_{\mathcal{A}})s)^{-1}\| \\ &\leq 2\|\mu^{-1}\|. \end{aligned} \tag{5.12}$$

Now, $\tilde{G}(\mu)$ is the Cauchy transform of s w.r.t. amalgamation over $M_k(\mathbb{C}) \otimes \mathbf{1}_{\mathcal{A}}$ (cf. [V4], [Le]). Hence, the maps

$$z \mapsto \mathcal{R}(z) + z^{-1}$$

and

$$\mu \mapsto \tilde{G}(\mu)$$

are inverses of each other, when z and μ are invertible, and $\|z\|$ and $\|\mu^{-1}\|$ are sufficiently small. Thus, according to (i) and (5.12), there is a $\delta \in \left(0, \frac{1}{2\|s\|}\right)$, sucht that when $\mu \in M_k(\mathbb{C})$ is invertible with $\|\mu^{-1}\| < \delta$, then

$$\begin{cases} \tilde{G}(\mu) \text{ is invertible, and} \\ a_0 + \sum_{i=1}^r a_i \tilde{G}(\mu) a_i + \tilde{G}(\mu)^{-1} = \mu. \end{cases}$$

This statement is equivalent to the identity

$$\tilde{G}(\mu)\Big(\mu - a_0 - \sum_{i=1}^r a_i \tilde{G}(\mu) a_i\Big) = \mathbf{1}_k. \tag{5.13}$$

It is easily seen that

$$\mathcal{U} = \Big\{\mu \in GL_k(\mathbb{C}) \mid \|\mu^{-1}\| < \frac{1}{\|s\|}\Big\}$$

is an open, connected set in $M_k(\mathbb{C})$. Hence, by uniqueness of analytic continuation, (5.13) holds for all $\mu \in \mathcal{U}$. Therefore, for every $\mu \in \mathcal{U}$, $\tilde{G}(\mu)$ is invertible with inverse

$$\tilde{G}(\mu)^{-1} = \mu - a_0 - \sum_{i=1}^r a_i \tilde{G}(\mu) a_i.$$

This proves (ii).

Finally, to prove (iii), observe that by (5.5) and (5.12),

$$R a_i T = a_i, \qquad (i = 0, 1, \ldots, r). \tag{5.14}$$

If $\|(R\mu T)^{-1}\| < \frac{1}{\|s\|}$, then we get from (ii) that $\tilde{G}(R\mu T)$ is well-defined and invertible, and

$$a_0 + \sum_{i=1}^{r} a_i \tilde{G}(R\mu T) a_i + \tilde{G}(R\mu T)^{-1} = R\mu T. \tag{5.15}$$

According to (5.14), $(R \otimes \mathbf{1}_{\mathcal{A}}) s (T \otimes \mathbf{1}_{\mathcal{A}}) = s$. Hence,

$$R\mu T \otimes \mathbf{1}_{\mathcal{A}} - s = (R \otimes \mathbf{1}_{\mathcal{A}})(\mu \otimes \mathbf{1}_{\mathcal{A}} - s)(T \otimes \mathbf{1}_{\mathcal{A}}).$$

Then, since $R\mu T \otimes \mathbf{1}_{\mathcal{A}} - s$ is invertible, so is $\mu \otimes \mathbf{1}_{\mathcal{A}} - s$, and

$$(\mu \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1} = (T \otimes \mathbf{1}_{\mathcal{A}})(R\mu T \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}(R \otimes \mathbf{1}_{\mathcal{A}}).$$

It follows that $\tilde{G}(\mu)$ is well-defined, and

$$\tilde{G}(\mu) = T\tilde{G}(R\mu T)R$$

is invertible with inverse

$$\tilde{G}(\mu)^{-1} = R^{-1}\tilde{G}(R\mu T)^{-1}T^{-1}.$$

Taking (5.14) and (5.15) into account, we find that

$$\begin{aligned}
a_0 &+ \sum_{i=1}^{r} a_i \tilde{G}(\mu) a_i + \tilde{G}(\mu)^{-1} \\
&= R^{-1}\Big(Ra_0T + \sum_{i=1}^{r}(Ra_iT)\tilde{G}(R\mu T)(Ra_iT) + \tilde{G}(R\mu T)^{-1}\Big)T^{-1} \\
&= R^{-1}\Big(a_0 \sum_{i=1}^{r} a_i \tilde{G}(R\mu T) a_i + \tilde{G}(R\mu T)^{-1}\Big)T^{-1} \\
&= R^{-1}R\mu T T^{-1} \\
&= \mu.
\end{aligned}$$

This proves (iii). ■

In the following we let

$$\mathcal{O} = \{\lambda \in M_m(\mathbb{C}) \,|\, \mathrm{Im}\lambda \text{ is positive definite}\},$$

and as before, for $\lambda \in \mathcal{O}$ we put

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \mathbf{1}_{k-m} \end{pmatrix}.$$

5.2 Lemma. *There is a constant C', depending only on $s = a_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_{i=1}^{r} a_i \otimes x_i$, such that:*

(i) For all $\lambda \in \mathcal{O}$,
$$\|(\Lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}\| \le C'(1 + \|(\mathrm{Im}\lambda)^{-1}\|).$$
Moreover, $G(\lambda)$ *is invertible, and*
$$a_0 + \sum_{i=1}^{r} a_i G(\lambda) a_i + G(\lambda)^{-1} = \Lambda.$$

(ii) Let $\lambda \in \mathcal{O}$, *and suppose that* $\mu \in M_k(\mathbb{C})$ *satisfies*
$$\|\mu - \Lambda\| < \frac{1}{2C'(1 + \|(\mathrm{Im}\lambda)^{-1}\|)}.$$
Then $\mu \otimes \mathbf{1}_{\mathcal{A}} - s$ *is invertible, and*
$$\|(\mu \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}\| < 2C'(1 + \|(\mathrm{Im}\lambda)^{-1}\|).$$
Moreover, $\tilde{G}(\mu)$ *is invertible, and*
$$a_0 + \sum_{i=1}^{r} a_i \tilde{G}(\mu) a_i + \tilde{G}(\mu)^{-1} = \mu.$$

Proof. (i) With $C'_{1,1}$ and $C'_{2,1}$ as in Lemma 3.2, put $C' = \max\{C'_{1,1}, C'_{2,1}\}$. Then by Lemma 3.2,
$$\|(\Lambda \otimes \mathbf{1}_n - s)^{-1}\| \le C'(1 + \|(\mathrm{Im}\lambda)^{-1}\|).$$
Put
$$\mathcal{O}' = \{\lambda \in \mathcal{O} \mid \|\lambda^{-1}\| < \min\{1, \|s\|^{-d}\}\}.$$
Then $\mathcal{O}'$ is a non-empty, open subset of $\mathcal{O}$. At first we will show that the remaining part of (i) holds for all $\lambda \in \mathcal{O}'$. Let $\lambda \in \mathcal{O}'$, and put
$$\alpha = \|\lambda^{-1}\|^{\frac{1}{d}} < \min\Big\{1, \frac{1}{\|s\|}\Big\}.$$
Then with
$$(r_1, \dots, r_d) = (\alpha^{d-1}, \alpha^{d-2}, \dots, \alpha, 1)$$
and
$$(t_1, \dots, t_d) = (1, \alpha^{1-d}, \alpha^{2-d}, \dots, \alpha^{-1}),$$
put
$$R = \mathrm{diag}(r_1 \mathbf{1}_{m_1}, \dots, r_d \mathbf{1}_{m_d})$$
and
$$T = \mathrm{diag}(t_1 \mathbf{1}_{m_1}, \dots, t_d \mathbf{1}_{m_d}).$$
Then $r_1 t_2 = r_2 t_3 = \cdots = r_{d-1} t_d = r_d t_1 = 1$, and
$$R \Lambda T = \begin{pmatrix} \alpha^{d-1}\lambda & 0 \\ 0 & \alpha^{-1}\mathbf{1}_{k-m} \end{pmatrix},$$

whence

$$\|(R\Lambda T)^{-1}\| = \max\{\alpha^{d-1}\|\lambda\|, \alpha\} = \alpha < \frac{1}{\|s\|}.$$

Therefore, by Lemma 5.1, $G(\lambda) = \tilde{G}(\Lambda)$ is invertible and satisfies

$$a_0 + \sum_{i=1}^{r} a_i G(\lambda) a_i + G(\lambda)^{-1} = \Lambda.$$

It follows that

$$G(\lambda)\Big(\Lambda - a_0 - \sum_{i=1}^{r} a_i G(\lambda) a_i\Big) = \mathbf{1}_k \tag{5.16}$$

holds for all $\lambda \in \mathcal{O}'$, but then, since $\mathcal{O}$ is open and connected, it follows from uniqueness of analytic continuation that (5.16) holds for all $\lambda \in \mathcal{O}$. That is, for every such λ, $G(\lambda)$ is invertible and satisfies

$$a_0 + \sum_{i=1}^{r} a_i G(\lambda) a_i + G(\lambda)^{-1} = \Lambda.$$

(ii) Suppose that $\mu \in M_k(\mathbb{C})$ and

$$\|(\mu \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}\| < 2C'(1 + \|(\text{Im}\lambda)^{-1}\|).$$

According to (i), $\|(\Lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}\| \leq C'(1 + \|\text{Im}\lambda\|^{-1})$. Put $x = \Lambda \otimes \mathbf{1}_{\mathcal{A}} - s$ and $y = \mu \otimes \mathbf{1}_{\mathcal{A}} - s$. Then

$$\|x^{-1}(x - y)\| < \frac{1}{2}.$$

Therefore, $y = x(\mathbf{1} - x^{-1}(x - y))$ is invertible, and by Neumann's series,

$$\|y^{-1}\| \leq \Big\| \sum_{n=0}^{\infty} (x^{-1}(x - y))^n \Big\| \|x^{-1}\| \leq 2\|x^{-1}\|.$$

Hence,

$$\|(\mu \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}\| \leq 2C'(1 + \|(\text{Im}\lambda)^{-1}\|).$$

Put

$$\mathcal{O}'' = \bigcup_{\lambda \in \mathcal{O}} \Big\{\mu \in M_k(\mathbb{C}) \,|\, \|\mu - \Lambda\| < \frac{1}{2C'(1 + \|(\text{Im}\lambda)^{-1}\|)}\Big\}.$$

Since $\mathcal{O}$ is connected, $\mathcal{O}''$ is a connected, open subset of $M_k(\mathbb{C})$. In order to prove (ii), we must show that

$$\tilde{G}(\mu)\Big(\mu - a_0 - \sum_{i=1}^{r} a_i \tilde{G}(\mu) a_i\Big) = \mathbf{1}_k \tag{5.17}$$

holds for all $\mu \in \mathcal{O}''$. Again, by uniqness of analytic continuation, it suffices to show that (5.17) holds for all μ in a non-empty open subset of $\mathcal{O}''$. Choose $\lambda \in \mathcal{O}$ such that $\|\lambda^{-1}\| < \min\{1, \|s\|^{-d}\}$, and define block matrices $R, T \in M_k(\mathbb{C})$ as in the proof of (i). Then $\|(R\Lambda T)^{-1}\| < \frac{1}{\|s\|}$. Since $x \mapsto x^{-1}$ is continuous on $GL_k(\mathbb{C})$, we may choose $\delta \in \left(0, \frac{1}{2C'(1+\|(\mathrm{Im}\lambda)^{-1}\|)}\right)$, such that if $\|\mu - \Lambda\| < \delta$, then μ is invertible, and

$$\|(R\mu T)^{-1}\| < \frac{1}{\|s\|}.$$

Put

$$\mathcal{O}''' = \{\mu \in M_k(\mathbb{C}) \,|\, \|\mu - \Lambda\| < \delta\}.$$

Since $\delta < \frac{1}{2C'(1+\|(\mathrm{Im}\lambda)^{-1}\|)}$, $\mathcal{O}''' \subseteq \mathcal{O}''$. Moreover, we get from Lemma 5.1 that when $\mu \in \mathcal{O}'''$, then $\tilde{G}(\mu)$ is invertible, and

$$\tilde{G}(\mu)^{-1} = \mu - a_0 - \sum_{i=1}^{r} a_i \tilde{G}(\mu) a_i.$$

That is, (5.17) holds for all $\mu \in \mathcal{O}'''$ and therefore for all $\mu \in \mathcal{O}''$. ■

Let $\lambda \in \mathcal{O}$, and put $\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \mathbf{1}_{k-m} \end{pmatrix}$. According to Theorem 4.3, there is a constant $C_1 \geq 0$ such that

$$\left\| \Big(\Lambda - a_0 - \sum_{i=1}^{r} a_i G_n(\lambda) a_i\Big) G_n(\lambda) - \mathbf{1}_k \right\| \leq \frac{C_1}{n^2}(1 + \|(\mathrm{Im}\lambda)^{-1}\|^4). \tag{5.18}$$

Put

$$B_n(\lambda) = \Lambda - a_0 - \sum_{i=1}^{r} a_i G_n(\lambda) a_i, \qquad (\lambda \in \mathcal{O}).$$

Then, by Neumann's Lemma and (5.18), if

$$\frac{C_1}{n^2}(1 + \|(\mathrm{Im}\lambda)^{-1}\|^4) < \frac{1}{2}, \tag{5.19}$$

then $B_n(\lambda)G_n(\lambda)$ is invertible with $\|(B_n(\lambda)G_n(\lambda))^{-1}\| \leq 2$. Hence $G_n(\lambda)$ is invertible too with

$$\begin{aligned} \|G_n(\lambda)^{-1}\| &\leq \|(B_n(\lambda)G_n(\lambda))^{-1}\| \|B_n(\lambda)\| \\ &\leq 2\|B_n(\lambda)\| \\ &\leq 2\Big(\|\lambda\| + 1 + \|a_0\| + \sum_{i=1}^{r} \|a_i\|^2 \|G_n(\lambda)\|\Big). \end{aligned} \tag{5.20}$$

Taking Lemma 3.1 into account we find that for some constant $C_2 \geq 0$,

$$\|G_n(\lambda)^{-1}\| \leq C_2(\|\lambda\| + 1)(1 + \|(\mathrm{Im}\lambda)^{-1}\|). \tag{5.21}$$

By (5.18), if $\lambda \in \mathcal{O}$ satisfies (5.19), then

$$\begin{aligned}\|\Lambda - a_0 - \sum_{i=1}^{r} a_i G_n(\lambda) a_i - G_n(\lambda)^{-1}\| &\leq \frac{C_1}{n^2}(1 + \|(\mathrm{Im}\lambda)^{-1}\|^4)\|G_n(\lambda)^{-1}\| \\ &\leq \frac{C_1}{n^2}(1 + \|(\mathrm{Im}\lambda)^{-1}\|^4)C_2(\|\lambda\| + 1)(1 + \|(\mathrm{Im}\lambda)^{-1}\|) \\ &\leq \frac{C_3}{n^2}(1 + \|\lambda\|)(1 + \|(\mathrm{Im}\lambda)^{-1}\|^5)\end{aligned} \tag{5.22}$$

for some constant $C_3 \geq 0$. For $\lambda \in \mathcal{O}$ fulfilling (5.19) define $\Lambda_n(\lambda) \in M_k(\mathbb{C})$ by

$$\Lambda_n(\lambda) = a_0 + \sum_{i=1}^{r} a_i G_n(\lambda) a_i + G_n(\lambda)^{-1}. \tag{5.23}$$

Note that

$$\Lambda - \Lambda_n(\lambda) = B_n(\lambda) - G_n(\lambda)^{-1}, \tag{5.24}$$

and therefore, by (5.22),

$$\|\Lambda - \Lambda_n(\lambda)\| \leq \frac{C_3}{n^2}(1 + \|\lambda\|)(1 + \|(\mathrm{Im}\lambda)^{-1}\|^5). \tag{5.25}$$

Let C' be as in Lemma 5.2. Then, if also

$$\frac{2C'C_3}{n^2}(1 + \|\lambda\|)(1 + \|(\mathrm{Im}\lambda)^{-1}\|^5)(1 + \|(\mathrm{Im}\lambda)^{-1}\|) < 1, \tag{5.26}$$

then

$$\|\Lambda_n(\lambda) - \Lambda\| < \frac{1}{2C'(1 + \|(\mathrm{Im}\lambda)^{-1}\|)}. \tag{5.27}$$

Hence, by Lemma 5.2, $\tilde{G}(\Lambda_n(\lambda))$ is well-defined and invertible and satisfies

$$a_0 + \sum_{i=1}^{r} a_i \tilde{G}(\Lambda_n(\lambda)) a_i + \tilde{G}(\Lambda_n(\lambda)^{-1} = \Lambda_n(\lambda). \tag{5.28}$$

Put

$$C_4 = \max\{2C_1, 2C'C_3\}$$

and

$$V_n = \Big\{\lambda \in \mathcal{O} \,\Big|\, \frac{C_4}{n^2}(1 + \|\lambda\|)(1 + \|(\mathrm{Im}\lambda)^{-1}\|^5)(1 + \|(\mathrm{Im}\lambda)^{-1}\|) < 1\Big\}. \tag{5.29}$$

Then for all $\lambda \in V_n$, (5.21) and (5.26) hold, and hence also (5.27) and (5.28) hold. Observe that the set

$$U_n = \Big\{\mathrm{i}t\mathbf{1}_m \,\Big|\, t > 0,\ \frac{C_4}{n^2}(1 + t)(1 + t^{-5})(1 + t^{-1}) < 1\Big\}$$

is contained in V_n, and that the function

$$f : t \mapsto (1 + t)(1 + t^{-5})(1 + t^{-1}) \tag{5.30}$$

is strictly convex on $]0,\infty[$ and satisfies

$$\begin{aligned} f(t) &\to \infty, \quad t \to 0^+, \\ f(t) &\to \infty, \quad t \to \infty. \end{aligned} \tag{5.31}$$

Therefore the set

$$\mathcal{I}_n = \Big\{ t > 0 \,\Big|\, \frac{C_4}{n^2}(1+t)(1+t^{-5})(1+t^{-1}) < 1 \Big\} \tag{5.32}$$

is either empty or an open, bounded interval. In particular, U_n is arc-wise connected.

For $\lambda \in \mathcal{O}$ put $\varepsilon(\lambda) = \|(\mathrm{Im}\lambda)^{-1}\|^{-1}$. Then, as in [HT, Proof of Proposition 5.6], we find that $\mathrm{i}\varepsilon(\lambda)\mathbf{1}_m \in U_n$ for all $\lambda \in V_n$, and that the line segment connecting λ and $\mathrm{i}\varepsilon(\lambda)\mathbf{1}_m$ is contained in V_n. Hence, either $V_n = \emptyset$ or V_n is connected.

For $\lambda \in V_n$ we get from (5.23) and (5.28) that

$$\sum_{i=1}^{r} a_i G_n(\lambda) a_i + G_n(\lambda)^{-1} = \sum_{i=1}^{r} a_i \tilde{G}(\Lambda_n(\lambda)) a_i + \tilde{G}(\Lambda_n(\lambda))^{-1}. \tag{5.33}$$

In the following we will show that (5.33) implies that $G_n(\lambda) = \tilde{G}(\Lambda_n(\lambda))$ for all $\lambda \in V_n$.

5.3 Lemma. *Let $z, w \in GL_k(\mathbb{C})$, and suppose that*

$$\sum_{i=1}^{r} a_i z a_i + z^{-1} = \sum_{i=1}^{r} a_i w a_i + w^{-1}. \tag{5.34}$$

If there exists $T \in GL_k(\mathbb{C})$, such that

$$\sum_{i=1}^{r} \|w a_i\| \|T a_i z T^{-1}\| < 1, \tag{5.35}$$

then $z = w$.

Proof. By (5.34),

$$w\Big(\sum_{i=1}^{r} a_i z a_i + z^{-1}\Big) z = w\Big(\sum_{i=1}^{r} a_i w a_i + w^{-1}\Big) z,$$

i.e.

$$\sum_{i=1}^{r} w a_i (z - w) a_i z = z - w.$$

Therefore,

$$\sum_{i=1}^{r} w a_i (z-w) T^{-1} (T a_i z T^{-1}) = (z-w) T^{-1},$$

which implies that

$$\Big(\sum_{i=1}^{r} \|w a_i\| \|T a_i z T^{-1}\|\Big) \|(z-w)T^{-1}\| \geq \|(z-w)T^{-1}\|.$$

Hence, if (5.35) holds, then $)\|(z-w)T^{-1}\| = 0$, and thus $z = w$. ∎

5.4 Lemma. *For all $n \in \mathbb{N}$ and all $\lambda \in \mathcal{O}$ with $\|(\mathrm{Im}\lambda)^{-1}\| \leq 1$ there exists $T \in GL_k(\mathbb{C})$, depending only on $\|(\mathrm{Im}\lambda)^{-1}\|$, and a constant $C_5 > 0$, depending only on $a_0, \ldots, a_r$, such that*

$$\|Ta_i G_n(\lambda) T^{-1}\| \leq C_5 \|(\mathrm{Im}\lambda)^{-1}\|^{\frac{1}{d}}, \qquad (i = 0, 1, \ldots, r).$$

Proof. With the same notation as in the proof of Lemma 3.1,

$$\begin{aligned} \|G_n(\lambda)\| &\leq \mathbb{E}\{\|A_n(\lambda)^{-1}\|\} \\ &\leq \mathbb{E}\{\|C_n\|\} + \mathbb{E}\{\|B_n^{(1)}\|\|B_n^{(2)}\|\}\|(\mathrm{Im}\lambda)^{-1}\|, \end{aligned}$$

where

$$C_{1,1} := \sup_{n \in \mathbb{N}} \mathbb{E}\{\|C_n\|\} < \infty,$$

and

$$C_{2,1} := \sup_{n \in \mathbb{N}} \mathbb{E}\{\|B_n^{(1)}\|\|B_n^{(2)}\|\} < \infty.$$

In the same way we get for $T \in GL_k(\mathbb{C})$ that

$$\begin{aligned} \|Ta_i G_n(\lambda) T^{-1}\| &\leq \mathbb{E}\{\|Ta_i C_n T^{-1}\|\} + \mathbb{E}\{\|Ta_i B_n^{(1)}\|\|B_n^{(2)} T^{-1}\|\}\|(\mathrm{Im}\lambda)^{-1}\| \\ &\leq \mathbb{E}\{\|Ta_i C_n T^{-1}\|\} + C_{2,1}\|a_i\|\|T\|\|T^{-1}\|\|(\mathrm{Im}\lambda)^{-1}\|. \end{aligned} \tag{5.36}$$

By Lemma 2.3, C_n has at most $\frac{1}{2}d(d-1)$ non-zero block entries and takes the form

$$C_n = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & * & * & * & * & \cdots & * \\ 0 & 0 & * & * & * & \cdots & * \\ 0 & 0 & 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & * & * \\ 0 & 0 & 0 & 0 & \cdots & 0 & * \end{pmatrix}.$$

Combining this with (5.5), we find that $a_i C_n$ is a strictly upper triangular $d \times d$ block matrix. Let $\beta \geq 1$, and put

$$T = \mathrm{diag}(\beta \mathbf{1}_{m_1}, \beta^2 \mathbf{1}_{m_2}, \ldots, \beta^d \mathbf{1}_{m_d}). \tag{5.37}$$

Then $Ta_i C_n T^{-1}$ is obtained from $a_i C_n$ by multiplying the (μ, ν)'th block entry by $\beta^{\mu-\nu}$. Since $a_i C_n$ is strictly upper triangular,

$$\begin{aligned} \|Ta_i C_n T^{-1}\| &\leq \sum_{\mu<\nu} \|[Ta_i C_n T^{-1}]_{\mu,\nu}\| \\ &= \sum_{\mu<\nu} \beta^{\mu-\nu} \|[a_i C_n]_{\mu,\nu}\| \\ &\leq \beta^{-1} \sum_{\mu<\nu} \|[a_i C_n]_{\mu,\nu}\|, \end{aligned}$$

where $[x]_{\mu,\nu}$ denotes the (μ,ν)'th block entry of a matrix $x \in M_k(\mathbb{C})$. Hence,

$$\|Ta_iC_nT^{-1}\| \le \frac{d^2}{\beta}\|a_iC_n\|$$

Since $\mathbb{E}\{\|C_n\|\} \le C_{1,1}$, we get from (5.36) that

$$\|Ta_iG_n(\lambda)T^{-1}\| \le \Big(\frac{d^2}{\beta}C_{1,1} + C_{2,1}\|T\|\|T^{-1}\|\|(\mathrm{Im}\lambda)^{-1}\|\Big)\|a_i\| \qquad (5.38)$$

Moreover, since $\beta \ge 1$, we have that $\|T\| = \beta^d$ and $\|T^{-1}\| = \beta^{-1}$. Now, if $\|(\mathrm{Im}\lambda)^{-1}\| \le 1$, put $\beta = \|(\mathrm{Im}\lambda)^{-1}\|^{-\frac{1}{d}} \ge 1$. Then by (5.38),

$$\|Ta_iG_n(\lambda)T^{-1}\| \le \frac{\|a_i\|}{\beta}(C_{1,1}d^2 + C_{2,1}).$$

Put $C_5 = \Big(\sum_{i=1}^r \|a_i\|\Big)(C_{1,1}d^2 + C_{2,1})$. Then

$$\|Ta_iG_n(\lambda)T^{-1}\| \le \frac{C_5}{\beta} = C_5\|(\mathrm{Im}\lambda)^{-1}\|^{\frac{1}{d}}. \qquad \blacksquare$$

5.5 Lemma. *There is a positive integer N, such that for all $n \ge N$,*

$$G_n(\lambda) = \tilde{G}(\Lambda_n(\lambda)), \qquad (\lambda \in V_n).$$

Proof. Let $\lambda \in V_n$, and put $z = G_n(\lambda)$ and $w = \tilde{G}(\Lambda_n(\lambda))$. According to (5.33),

$$\sum_{i=1}^r a_iza_i + z^{-1} = \sum_{i=1}^r a_iwa_i + w^{-1}.$$

Moreover, by Lemma 5.2 (ii) and (5.27) we have that

$$\begin{aligned}\|w\| &\le \|(\Lambda_n(\lambda)\otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}\| \\ &\le 2C'(1 + \|(\mathrm{Im}\lambda)^{-1}\|).\end{aligned}$$

Thus, if $\|(\mathrm{Im}\lambda)^{-1}\| < 1$, then $\|w\| < 4C'$. Moreover, by Lemma 5.4 there exist a constant C_5 and $T \in GL_k(\mathbb{C})$, such that

$$\|Ta_izT^{-1}\| \le C_5\|(\mathrm{Im}\lambda)^{-1}\|^{\frac{1}{d}}.$$

Hence

$$\sum_{i=1}^r \|wa_i\|\|Ta_izT^{-1}\| < 4C'C_5\Big(\sum_{i=1}^r \|a_i\|\Big)\|(\mathrm{Im}\lambda)^{-1}\|^{\frac{1}{d}}.$$

Put

$$\varepsilon = \min\Big\{\Big(4C'C_5\sum_{i=1}^r \|a_i\|\Big)^{-d}, 1\Big\}$$

and

$$V_n' = \{\lambda \in V_n \mid \|(\mathrm{Im}\lambda)^{-1}\| < \varepsilon\}. \tag{5.39}$$

Then for all $\lambda \in V_n'$,

$$\sum_{i=1}^{r} \|wa_i\| \|Ta_i z T^{-1}\| < 1,$$

and therefore, by Lemma 5.4, $z = w$. That is, for all $\lambda \in V_n'$,

$$G_n(\lambda) = \tilde{G}(\Lambda_n(\lambda)). \tag{5.40}$$

Recall from the proof of Lemma 5.2 that

$$U_n = \left\{ \mathrm{i}t\mathbf{1}_m \,\middle|\, t > 0,\ \frac{C_4}{n^2}(1+t)(1+t^{-5})(1+t^{-1}) < 1 \right\}$$

is a subset of V_n. Hence, if

$$\frac{C_4}{n^2}\Big(1+\frac{\varepsilon}{2}\Big)\Big(1+\Big(\frac{\varepsilon}{2}\Big)^{-5}\Big)\Big(1+\Big(\frac{\varepsilon}{2}\Big)^{-1}\Big) < 1, \tag{5.41}$$

then $\mathrm{i}\frac{\varepsilon}{2}\mathbf{1}_m \in V_n'$. Choose $N \in \mathbb{N}$, such that

$$N^2 > C_4\Big(1+\frac{\varepsilon}{2}\Big)\Big(1+\Big(\frac{\varepsilon}{2}\Big)^{-5}\Big)\Big(1+\Big(\frac{\varepsilon}{2}\Big)^{-1}\Big).$$

Then for all $n \geq N$, (5.41) holds, and hence V_n' is a non-empty open subset of V_n. Since V_n is open, and $\lambda \mapsto G_n(\lambda) - \tilde{G}(\Lambda_n(\lambda))$ is analytic, (5.40) holds for all $\lambda \in V_n$ when $n \geq N$. ■

5.6 Theorem. *There exist $N \in \mathbb{N}$ and a constant $C_6 > 0$, both depending only on $a_0, \dots, a_r$, such that for all $\lambda \in \mathcal{O}$ and all $n \geq N$,*

$$\|G_n(\lambda) - G(\lambda)\| \leq \frac{C_6}{n^2}(1+\|\lambda\|)(1+\|(\mathrm{Im}\lambda)^{-1}\|^7). \tag{5.42}$$

Proof. Let N be as in Lemma 5.5. Then for $n \geq N$ and $\lambda \in V_n$,

$$\begin{aligned}
\|G_n(\lambda) - G(\lambda)\| &= \|\tilde{G}(\Lambda_n(\lambda)) - G(\lambda)\| \\
&\leq \|(\Lambda_n(\lambda)\otimes \mathbf{1}_{\mathcal{A}} - s)^{-1} - (\Lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}\| \\
&= \|(\Lambda_n(\lambda)\otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}(\Lambda - \Lambda_n(\lambda))(\Lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}\| \\
&\leq \|(\Lambda_n(\lambda)\otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}\| \|\Lambda - \Lambda_n(\lambda)\| \|(\Lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}\|.
\end{aligned}$$

Since (5.27) holds for all $\lambda \in V_n$, we get from Lemma 5.2 that

$$\|G_n(\lambda) - G(\lambda)\| \leq 2(C')^2(1+\|(\mathrm{Im}\lambda)^{-1}\|)^2 \|\Lambda - \Lambda_n(\lambda)\|.$$

Hence, by (5.25), for all $\lambda \in V_n$,

$$\begin{aligned}
\|G_n(\lambda) - G(\lambda)\| &\leq \frac{2(C')^2 C_3}{n^2}(1+\|\lambda\|)(1+\|(\mathrm{Im}\lambda)^{-1}\|)^2(1+\|(\mathrm{Im}\lambda)^{-1}\|^5) \\
&\leq \frac{C_6^{(1)}}{n^2}(1+\|\lambda\|)(1+\|(\mathrm{Im}\lambda)^{-1}\|^7)
\end{aligned}$$

for some constant $C_6^{(1)} > 0$. Next, if $\lambda \in \mathcal{O} \setminus V_n$, i.e.

$$\frac{C_4}{n^2}(1 + \|\lambda\|)(1 + \|(\text{Im}\lambda)^{-1}\|^5)(1 + \|(\text{Im}\lambda)^{-1}\|) \geq 1,$$

then

$$\|G_n(\lambda) - G(\lambda)\| \leq \frac{C_4}{n^2}(1 + \|\lambda\|)(1 + \|(\text{Im}\lambda)^{-1}\|^5)(1 + \|(\text{Im}\lambda)^{-1}\|)(\|G_n(\lambda)\| + \|G(\lambda)\|). \tag{5.43}$$

Put $C'' = \max\{C_{1,1}, C_{2,1}, C'_{1,1}, C'_{2,1}\}$, where $C_{1,1}, C_{2,1}, C'_{1,1}, C'_{2,1}$ refer to the constants from Lemma 3.1 and Lemma 3.2. Then

$$\|G_n(\lambda)\| + \|G(\lambda)\| \leq 2C''(1 + \|(\text{Im}\lambda)^{-1}\|),$$

and hence, by (5.43), there is a constant $C_6^{(2)} > 0$, such that for all $\lambda \in \mathcal{O} \setminus V_n$,

$$\|G_n(\lambda) - G(\lambda)\| \leq \frac{C_6^{(2)}}{n^2}(1 + \|\lambda\|)(1 + \|(\text{Im}\lambda)^{-1}\|^7).$$

Thus, with $C_6 = \max\{C_6^{(1)}, C_6^{(2)}\}$, (5.42) holds. ■

6 The spectrum of Q_n.

As in the previous we consider a fixed polynomial $p \in (M_m(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \ldots, X_r\rangle)_{sa}$ and define Q_n and q by (3.1) and (3.9), respectively. For $\lambda \in \mathbb{C}$ with $\text{Im}\lambda > 0$ put

$$g(\lambda) = (\text{tr}_m \otimes \tau)[(\lambda \mathbf{1}_m \otimes \mathbf{1}_{\mathcal{A}} - q)^{-1}], \tag{6.1}$$

$$g_n(\lambda) = \mathbb{E}\{(\text{tr}_m \otimes \text{tr}_n)[(\lambda \mathbf{1}_m \otimes \mathbf{1}_n - Q_n)^{-1}]\}. \tag{6.2}$$

By application of Proposition 2.3, we find that with $E = \mathbf{1}_m \oplus 0_{k-m} \in M_k(\mathbb{C})$,

$$g(\lambda) = \frac{k}{m}\,\text{tr}_k(EG(\lambda\mathbf{1}_m)E),$$

and

$$g_n(\lambda) = \frac{k}{m}\,\text{tr}_k(EG_n(\lambda\mathbf{1}_m)E).$$

Hence for every $n \geq N'$,

$$|g_n(\lambda) - g(\lambda)| \leq \frac{k}{m}\frac{C_7}{n^2}(1 + \|\lambda\|)(1 + \|(\text{Im}\lambda)^{-1}\|^7). \tag{6.3}$$

6.1 Theorem. *For every $\phi \in C_c^\infty(\mathbb{R}, \mathbb{R})$,*

$$\mathbb{E}\{(\text{tr}_m \otimes \text{tr}_n)\phi(Q_n)\} = (\text{tr}_m \otimes \tau)\phi(q) + O\Big(\frac{1}{n^2}\Big).$$

Proof. This follows from (6.3) by minor modifications of [HT, Proof of Theorem 6.2]. ■

We are now going to prove

6.2 Theorem. *Let $\phi \in C^\infty(\mathbb{R},\mathbb{R})$ such that ϕ is constant outside a compact subset of $\mathbb{R}$, and suppose*

$$\mathrm{supp}(\phi) \cap \sigma(q) = \emptyset.$$

Then

$$\begin{aligned} \mathbb{E}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\phi(Q_n)\} &= O(n^{-2}), &(6.4)\\ \mathbb{V}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\phi(Q_n)\} &= O(n^{-4}), &(6.5)\end{aligned}$$

and

$$P(|(\mathrm{tr}_m \otimes \mathrm{tr}_n)\phi(Q_n)| < n^{-\frac{4}{3}}, \text{ eventually as } n \to \infty) = 1. \tag{6.6}$$

In the proof of this theorem we shall need:

6.3 Proposition. *Let $m, r \in \mathbb{N}$, let $p \in M_m(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \dots, X_r\rangle$ with $p = p^*$, and for each $n \in \mathbb{N}$, let $X_1^{(n)}, \dots, X_r^{(n)}$ be stocastically independent random matrices from* $\mathrm{SGRM}(n, \frac{1}{n})$. *Then for every compactly supported C^1-function $\phi : \mathbb{R} \to \mathbb{C}$ there is a constant $C \geq 0$ such that*

$$\mathbb{V}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\phi(Q_n)\} \leq \frac{C}{n^2}\mathbb{E}\Big\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)[|\phi'|^2(Q_n)] \cdot \Big(1 + \sum_{j=1}^r \|X_j^{(n)}\|^{d-1}\Big)^2\Big\}, \tag{6.7}$$

where

$$Q_n = p(X_1^{(n)}, \dots, X_r^{(n)}),$$

and $d = \deg(p)$.

Proof. Choose monomials $m_j \in \mathbb{C}\langle X_1, \dots, X_r\rangle$, $j = 1, \dots, N$, and choose $\alpha_j \in M_m(\mathbb{C})$, $j = 0, \dots, N$, such that

$$p = \alpha_0 \otimes 1 + \sum_{j=1}^N \alpha_j \otimes m_j. \tag{6.8}$$

Consider a fixed $n \in \mathbb{N}$, and let $\mathcal{E}_{r,n}$ be the real vector space $(M_n(\mathbb{C})_{sa})^r$ equipped with the Euclidean norm $\|\cdot\|_e$ (cf. [HT, Section 3]). Then define $f : \mathcal{E}_{r,n} \to \mathbb{C}$ by

$$f(v_1, \dots, v_r) = (\mathrm{tr}_m \otimes \mathrm{tr}_n)[\phi(p(v_1, \dots, v_r))], \qquad (v_1, \dots, v_r \in M_n(\mathbb{C})_{sa}).$$

According to [HT, (4.4)],

$$\mathbb{V}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)[\phi(Q_n)]\} \leq \frac{1}{n}\,\mathbb{E}\{\|(\mathrm{grad} f)(X_1^{(n)}, \dots, X_r^{(n)})\|_e^2\}. \tag{6.9}$$

Now, let $v = (v_1, \dots, v_r) \in \mathcal{E}_{r,n}$, and let $w = (w_1, \dots, w_r) \in \mathcal{E}_{r,n}$ with $\|w\|_e = 1$. As in [HT, Proof of Proposition 4.7] we find that

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(v+tw)\right|^2 \leq \frac{1}{m^2n^2}\|\phi'(p(v))\|^2_{2,\mathrm{Tr}_m\otimes\mathrm{Tr}_n}\left\|\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p(v+tw)\right\|^2_{2,\mathrm{Tr}_m\otimes\mathrm{Tr}_n}, \tag{6.10}$$

where

$$\|\phi'(p(v))\|^2_{2,\mathrm{Tr}_m\otimes\mathrm{Tr}_n} = mn\cdot(\mathrm{tr}_m\otimes\mathrm{tr}_n)[|\phi'|^2(p(v))].$$

According to (6.8),

$$\begin{aligned}\left\|\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p(v+tw)\right\|_{2,\mathrm{Tr}_m\otimes\mathrm{Tr}_n} &\leq \sum_{j=1}^N \left\|\alpha_j\otimes\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} m_j(v+tw)\right\|_{2,\mathrm{Tr}_m\otimes\mathrm{Tr}_n}\\ &= \sum_{j=1}^N \|\alpha_j\|_{2,\mathrm{Tr}_m}\left\|\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} m_j(v+tw)\right\|_{2,\mathrm{Tr}_n}.\end{aligned}$$

Making use of the fact that $\|w_j\|^2_{2,\mathrm{Tr}_n} \leq 1$, $j = 1, \dots, r$, we find that with $d_j = \deg(m_j)$,

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} m_j(v+tw)\right\|_{2,\mathrm{Tr}_n} \leq d_j\cdot\max_{1\leq i\leq r}\|v_i\|^{d_j-1} \leq d\cdot\Big(1+\sum_{i=1}^r\|v_i\|^{d-1}\Big),$$

and it follows that

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p(v+tw)\right\|_{2,\mathrm{Tr}_m\otimes\mathrm{Tr}_n} \leq d\cdot\Big(1+\sum_{i=1}^r\|v_i\|^{d-1}\Big)\Big(\sum_{j=1}^N\|\alpha_j\|_{2,\mathrm{Tr}_m}\Big).$$

Then by insertion into (6.10),

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(v+tw)\right|^2 \leq \frac{1}{mn}&(\mathrm{tr}_m\otimes\mathrm{tr}_n)[|\phi'|^2(p(v))]\cdot\\ &d^2\cdot\Big(1+\sum_{i=1}^r\|v_i\|^{d-1}\Big)^2\Big(\sum_{j=1}^N\|\alpha_j\|_{2,\mathrm{Tr}_m}\Big)^2.\end{aligned}$$

Since w was arbitrary this implies that with

$$C = \frac{1}{m}\cdot d^2\cdot\Big(\sum_{j=1}^N\|\alpha_j\|_{2,\mathrm{Tr}_m}\Big)^2,$$

$$\|(\mathrm{grad}\,f)(v)\|_e^2 \leq \frac{C}{n}(\mathrm{tr}_m\otimes\mathrm{tr}_n)[|\phi'|^2(p(v))]\cdot\Big(1+\sum_{i=1}^r\|v_i\|^{d-1}\Big)^2.$$

Then by (6.9),

$$\mathbb{V}\{(\mathrm{tr}_m\otimes\mathrm{tr}_n)[\phi(Q_n)]\}\le\frac{C}{n^2}\mathbb{E}\Big\{(\mathrm{tr}_m\otimes\mathrm{tr}_n)[|\phi'|^2(Q_n)]\cdot\Big(1+\sum_{i=1}^r\|X_i^{(n)}\|^{d-1}\Big)^2\Big\},$$

as desired. ■

6.4 Proposition. *There exist universal constants $\gamma(k)\ge 0$, $k\in\mathbb{N}$, such that for every $n\in\mathbb{N}$ and for every $X_n\in\mathrm{SGRM}(n,\frac{1}{n})$,*

$$\mathbb{E}\{1_{(\|X_n\|>3)}\cdot\|X_n\|^k\}\le\gamma(k)n\mathrm{e}^{-\frac{n}{2}}. \tag{6.11}$$

Proof. Let k, $n\in\mathbb{N}$ and $X_n\in\mathrm{SGRM}(n,\frac{1}{n})$. Define $F:[0,\infty[\to[0,1]$ by

$$F(t)=P(\|X_n\|\le t),\qquad (t\ge 0).$$

Recall from [S, Proof of Lemma 6.4] that for all $\varepsilon>0$ one has that

$$1-F(2+\varepsilon)\le 2n\exp\Big(-\frac{n\varepsilon^2}{2}\Big). \tag{6.12}$$

Integrating by parts as in [Fe, Lemma V.6.1] we get that

$$\begin{aligned}\mathbb{E}\{1_{(\|X_n\|>3)}\cdot\|X_n\|^k\}&=\int_3^\infty t^k\mathrm{d}F(t)\\&=3^k(1-F(3))+k\int_3^\infty t^{k-1}(1-F(t))\mathrm{d}t.\end{aligned}$$

According to (6.12), $1-F(3)\le 2n\mathrm{e}^{-\frac{n}{2}}$ and

$$\begin{aligned}\int_3^\infty t^{k-1}(1-F(t))\mathrm{d}t&=\int_0^\infty(3+t)^{k-1}(1-F(3+t))\mathrm{d}t\\&\le 2n\int_0^\infty(3+t)^{k-1}\exp\Big(-\frac{n}{2}(1+t)^2\Big)\mathrm{d}t\\&=2n\mathrm{e}^{-\frac{n}{2}}\int_0^\infty(3+t)^{k-1}\exp\Big(-\frac{n}{2}(2t+t^2)\Big)\mathrm{d}t.\end{aligned}$$

Hence (6.11) holds with

$$\gamma(k)=2\cdot 3^k+2\int_0^\infty(3+t)^{k-1}\exp\Big(-\frac{n}{2}(2t+t^2)\Big)\mathrm{d}t<\infty. \qquad ■$$

Proof of Theorem 6.2. (6.4) follows from Theorem 6.1 as in [HT, Proof of Lemma 6.3]. To prove (6.5), note that by Proposition 6.3,

$$\mathbb{V}\{(\mathrm{tr}_m\otimes\mathrm{tr}_n)\phi(Q_n)\}\le\frac{C}{n^2}\mathbb{E}\Big\{(\mathrm{tr}_m\otimes\mathrm{tr}_n)[|\phi'|^2(Q_n)]\cdot(r+1)\Big(1+\sum_{i=1}^r\|X_i^{(n)}\|^{2d-2}\Big)\Big\}. \tag{6.13}$$

Let $\Omega_i = \{\omega \in \Omega \,|\, \|X_i^{(n)}(\omega)\| \le 3\}$, $i = 1, \dots, r$. Then by Proposition 6.4,

$$\begin{aligned}
&\mathbb{E}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)[|\phi'|^2(Q_n)]\|X_i^{(n)}\|^{2d-2}\} \\
&\qquad \le \int_{\Omega_i} 3^{2d-2}(\mathrm{tr}_m \otimes \mathrm{tr}_n)[|\phi'|^2(Q_n)]\mathrm{d}P + \int_{\Omega\setminus\Omega_i} \|\phi'\|_\infty^2 \|X_i^{(n)}\|^{2d-2}\mathrm{d}P \\
&\qquad \le 3^{2d-2}\mathbb{E}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)[|\phi'|^2(Q_n)]\} + \|\phi'\|_\infty^2\|\gamma(2d-2)n\mathrm{e}^{-\frac{n}{2}}.
\end{aligned}$$

Applying (6.4) to $|\phi'|^2$ we get that

$$\mathbb{E}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)[|\phi'|^2(Q_n)]\} = O(n^{-2}),$$

and hence by (6.13),

$$\mathbb{V}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\phi(Q_n)\} = O\Big(n^{-4} + \frac{1}{n}\mathrm{e}^{-\frac{n}{2}}\Big) = O(n^{-4}),$$

which proves (6.5).

Finally, (6.6) follows from (6.4) and (6.5) as in [HT, Proof of Lemma 6.3]. ■

As in [HT, proof of Theorem 6.4], (6.6) implies the following:

6.5 Theorem. *For any $\varepsilon > 0$ and for almost all $\omega \in \Omega$,*

$$\sigma(Q_n(\omega)) \subseteq \sigma(q) +]-\varepsilon, \varepsilon[,$$

eventually as $n \to \infty$.

7 No projections in $C^*_{red}(\mathbb{F}_r)$ – a new proof.

7.1 Theorem. *([V3], [PV]). Let $m, r \in \mathbb{N}$, let $x_1, \dots, x_r$ be a semicircular system in $(\mathcal{A}, \tau)$, and let e be a projection in $M_m(C^*(\mathbf{1}_\mathcal{A}, x_1, \dots, x_r))$. Then $(\mathrm{Tr}_m \otimes \tau)e \in \mathbb{N}_0$. In particular, $C^*_{red}(\mathbb{F}_r)$ contains no projections but the trivial ones, i.e. $\mathcal{P}(C^*_{red}(\mathbb{F}_r)) = \{0, \mathbf{1}\}$.*

Proof. Choose $p \in M_m(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \dots, X_r\rangle$, such that $p = p^*$ and

$$\|e - p(x_1, \dots, x_r)\| < \frac{1}{8}.$$

Put $q = p(x_1, \dots, x_r)$. By [Da, Proposition 2.1] the Hausdorff distance between the spectra $\sigma(e)$ and $\sigma(q)$ is at most $\|e - q\|$. Hence $\sigma(q) \subseteq]-\frac{1}{8}, \frac{1}{8}[\cup]\frac{7}{8}, \frac{9}{8}[$.

Choose $\phi \in C_c^\infty(\mathbb{R})$ such that $0 \le \phi \le 1$, $\phi|_{]-\frac{1}{4},\frac{1}{4}[} = 0$ and $\phi|_{]\frac{3}{4},\frac{5}{4}[} = 1$. $\phi(q)$ is a projection, and

$$\|\phi(q) - q\| < \frac{1}{8}.$$

Consequently,

$$\|\phi(q) - e\| < \frac{1}{4} < 1,$$

implying that $\phi(q)$ is equivalent to e. In particular,

$$(\mathrm{Tr}_m \otimes \tau)e = (\mathrm{Tr}_m \otimes \tau)\phi(q).$$

For each $n \in \mathbb{N}$, let $X_1^{(n)}, \dots, X_r^{(n)}$ be stochastically independent random matrices from $\mathrm{SGRM}(n, \frac{1}{n})$, and put

$$Q_n = p(X_1^{(n)}, \dots, X_r^{(n)}).$$

We know from Theorem 6.5 that there is a P-null set $N \subseteq \Omega$ such that for all $\omega \in \Omega \setminus N$,

$$\sigma(Q_n(\omega)) \subseteq \sigma(q) + \,]-\tfrac{1}{8}, \tfrac{1}{8}[\, \subseteq \,]-\tfrac{1}{4}, \tfrac{1}{4}[\, \cup \,]\tfrac{3}{4}, \tfrac{5}{4}[$$

holds eventually as $n \to \infty$.

In particular, when $\omega \in \Omega \setminus N$, there is an $N(\omega) \in \mathbb{N}$ such that $\phi(Q_n(\omega))$ is a projection for all $n \geq N(\omega)$, and therefore

$$(\mathrm{Tr}_m \otimes \mathrm{Tr}_n)\phi(Q_n(\omega)) \in \mathbb{Z}. \tag{7.1}$$

Put

$$Z_n(\omega) = (\mathrm{tr}_m \otimes \mathrm{tr}_n)\phi(Q_n(\omega)) - (\mathrm{tr}_m \otimes \tau)\phi(q), \qquad (\omega \in \Omega).$$

According to Theorem 6.1, $\mathbb{E}\{Z_n\} = O\Big(\frac{1}{n^2}\Big)$. Moreover, since ϕ' vanishes in a neighbourhood of $\sigma(q)$, we get, as in the proof of Theorem 6.2, that

$$\mathbb{V}\{Z_m\} = \mathbb{V}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\phi(Q_n)\} = O\Big(\frac{1}{n^4}\Big).$$

As previously noted this implies that

$$P(|Z_n| < n^{-\frac{4}{3}}, \text{ eventually as } n \to \infty) = 1.$$

Hence, we may assume that

$$(\mathrm{Tr}_m \otimes \mathrm{Tr}_n)\phi(Q_n(\omega)) = n(\mathrm{Tr}_m \otimes \tau)\phi(q) + O(n^{-\frac{1}{3}}), \tag{7.2}$$

holds for almost all $\omega \in \Omega \setminus N$ as well.

Now choose $\omega \in \Omega \setminus N$ and $n_0 \in \mathbb{N}$ such that (7.1) and (7.2) hold when $n \geq n_0$. Take $C \geq 0$ such that for all $n \in \mathbb{N}$,

$$|(\mathrm{Tr}_m \otimes \mathrm{Tr}_n)\phi(Q_n(\omega)) - n(\mathrm{Tr}_m \otimes \tau)\phi(q)| \leq C \cdot n^{-\frac{1}{3}}.$$

Then

$$\mathrm{dist}(n(\mathrm{Tr}_m \otimes \tau)\phi(q), \mathbb{Z}) \leq C \cdot n^{-\frac{1}{3}},$$
$$\mathrm{dist}((n+1)(\mathrm{Tr}_m \otimes \tau)\phi(q), \mathbb{Z}) \leq C \cdot (n+1)^{-\frac{1}{3}},$$

and hence by subtraction,

$$\mathrm{dist}((\mathrm{Tr}_m \otimes \tau)\phi(q), \mathbb{Z}) \leq C(n^{-\frac{1}{3}} + (n+1)^{-\frac{1}{3}})$$

for all $n \geq n_0$. This implies that $(\mathrm{Tr}_m \otimes \tau)\phi(q) \in \mathbb{Z}$.

The last statement of Theorem 7.1 follows from this and the fact that $C^*_{red}(\mathbb{F}_r)$ has a unital trace-preserving embedding into $\mathcal{A}_0 = C^*(\mathbf{1}_{\mathcal{A}}, x_1 \dots, x_r)$ (cf. [HT, Lemma 8.1]). ■

7.2 Remark. The last statement of Theorem 7.1 was originally proved in [PV] by application of methods from K-theory, and also the first statement of Theorem 7.1 may be obtained using K-theory. Indeed, in [V3] it was shown that $K_0(\mathcal{A}_0) = \mathbb{Z}\,[\mathbf{1}_{\mathcal{A}}]_0$, where $\mathcal{A}_0 = C^*(\mathbf{1}_{\mathcal{A}}, x_1, \dots, x_r)$.

8 Gaps in the spectrum of q.

As in the previous sections, consider a semicircular system $x_1, \dots, x_r$ in $(\mathcal{A}, \tau)$. Take $p \in M_m(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \dots, X_r\rangle$, such that $p = p^*$ and put

$$q = p(x_1, \dots, x_r).$$

The following is an easy consequence of Theorem 7.1:

8.1 Proposition. *$\sigma(q)$ is a union of at most m disjoint connected sets, each of which is a compact interval or a one-point set.*

Proof. $\mathbb{R} \setminus \sigma(q)$ is a union of disjoint open intervals. If $\mathbb{R} \setminus \sigma(q)$ had more than $m+1$ connected components, one could choose $m+1$ non-zero orthogonal projections $e_1, \dots, e_{m+1} \in M_m(C^*(\mathbf{1}_{\mathcal{A}}, x_1, \dots, x_r))$. Since

$$(\mathrm{Tr}_m \otimes \tau)e_j \in \{1, \dots, m\}, \qquad (1 \leq j \leq m+1),$$

we would get that

$$m = (\mathrm{Tr}_m \otimes \tau)(\mathbf{1}_m \otimes \mathbf{1}) \geq \sum_{j=1}^{m+1} (\mathrm{Tr}_m \otimes \tau)e_j \geq m+1$$

– a contradiction. Consequently, $\mathbb{R} \setminus \sigma(q)$ has at most $m+1$ connected components, and $\sigma(q)$ is a union of at most m disjoint non-empty compact intervals. ■

Now, for each $n \in \mathbb{N}$, let $X_1^{(n)}, \dots, X_r^{(n)}$ be stochastically independent random matrices from $\mathrm{SGRM}(n, \frac{1}{n})$, and put

$$Q_n = p(X_1^{(n)}, \dots, X_r^{(n)}).$$

8.2 Theorem. *Let ε_0 denote the smallest distance between disjoint connected components of $\sigma(q)$, let $\mathfrak{J}$ be a connected component of $\sigma(q)$, let $0 < \varepsilon < \frac{1}{3}\varepsilon_0$, and let $\mu_q \in \mathrm{Prob}(\mathbb{R})$ denote the distribution of q w.r.t. $\mathrm{tr}_m \otimes \tau$. Then $\mu_q(\mathfrak{J}) = \frac{k}{m}$ for some $k \in \{1, \dots, m\}$, and for almost all $\omega \in \Omega$, the number of eigenvalues of $Q_n(\omega)$ in $\mathfrak{J}+]-\varepsilon, \varepsilon[$ is $k \cdot n$, eventually as $n \to \infty$.*

Proof. Take $\phi \in C_c^\infty(\mathbb{R})$ such that $0 \le \phi \le 1$, $\phi|_{\mathfrak{J}+]-\varepsilon,\varepsilon[} = 1$, and $\phi|_{\mathbb{R}\setminus(\mathfrak{J}+]-2\varepsilon,2\varepsilon[)} = 0$. Then, $\phi(q) \in M_m(C^*(\mathbf{1}_{\mathcal{A}}, x_1, \dots, x_r))$, and hence, by Theorem 7.1,

$$\mu_q(\mathfrak{J}) = (\mathrm{tr}_m \otimes \tau)1_{\mathfrak{J}}(q) = (\mathrm{tr}_m \otimes \tau)\phi(q) = \frac{k}{m}$$

for some $k \in \{1, \dots, m\}$.

As in the proof of Theorem 7.1 there is a P-null set $N \subset \Omega$ such that for all $\omega \in \Omega \setminus N$,

$$\sigma(Q_n(\omega)) \subseteq \sigma(q) +] - \varepsilon, \varepsilon[,$$

eventually as $n \to \infty$, and

$$(\mathrm{tr}_m \otimes \mathrm{tr}_n)\phi(Q_n(\omega)) = \frac{k}{m} + O(n^{-\frac{4}{3}}). \tag{8.1}$$

In particular, for all $\omega \in \Omega \setminus N$ there exists $N(\omega) \in \mathbb{N}$ such that $\phi(Q_n(\omega))$ is a projection for all $n \ge N(\omega)$.

For $\omega \in \Omega \setminus N$ and $n \ge N(\omega)$ take $k_n(\omega) \in \{0, \dots, m \cdot n\}$ such that

$$(\mathrm{tr}_m \otimes \mathrm{tr}_n)\phi(Q_n(\omega)) = \frac{k_n(\omega)}{m \cdot n}.$$

Note that $k_n(\omega)$ is the number of eigenvalues of $Q_n(\omega)$ in $\mathfrak{J}+] - \varepsilon, \varepsilon[$. (8.1) implies that

$$k_n(\omega) = k \cdot n + O(n^{-\frac{1}{3}}),$$

and hence $k_n(\omega) = k \cdot n$ for n sufficiently big. ■

9 The real and symplectic cases.

In the sections 9 and 10 we will generalize the results of section 4-6 to Gaussian random matrices with real or sympletic entries. The case of polynomials of degree 1 was treated by the second named author in [S].

The sympletic numbers $\mathbb{H}$ can be expressed as

$$\mathbb{H} = \mathbb{R} + j\mathbb{R} + k\mathbb{R} + l\mathbb{R},$$

where $j^2 = k^2 = l^2 = 1$ and

$$jk = -kj = l, \quad kl = -lk = j, \quad lj = -jl = k.$$

$\mathbb{H}$ can be realized as a subring of $M_2(\mathbb{C})$ with unit $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ by putting

$$j = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \quad k = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad l = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

By this realization of $\mathbb{H}$, the complectification of $\mathbb{H}$ becomes $\mathbb{H}^{\mathbb{C}} = \mathbb{H} + i\mathbb{H} = M_2(\mathbb{C})$.

Following the notation of [S] we will consider the following random matrix ensembles:

(i) $\mathrm{GRM}^{\mathbb{R}}(n, \sigma^2)$ is the set of random matrices $Y\colon\ \Omega \to M_n(\mathbb{R})$ fulfilling that the entries of Y, Y_{uv}, $1 \leq u, v \leq n$, constitute a set of n^2 i.i.d. random variables with distribution $N(0, \sigma^2)$.

(ii) $\mathrm{GRM}^{\mathbb{H}}(n, \sigma^2)$ is the set of random matrices $Y \to M_n(\mathbb{H})$ of the form

$$Y = 1 \otimes Y^{(1)} + j \otimes Y^{(2)} + k \otimes Y^{(3)} + l \otimes Y^{(4)}$$

where $Y^{(1)}, Y^{(2)}, Y^{(3)}, Y^{(4)}$ are stocastically independent random matrices from $\mathrm{GRM}^{\mathbb{R}}(n, \frac{\sigma^2}{4})$.

(iii) The ensemble $\mathrm{GOE}(n, \sigma^2)$ (resp. $\mathrm{GOE}^*(n, \sigma^2)$) from [S] can be described as the set of selfadjoint random matrices, which have the same distribution as

$$\frac{1}{\sqrt{2}}(Y + Y^*), \qquad (\text{resp. } \frac{1}{i\sqrt{2}}(Y - Y^*))$$

where $Y \in \mathrm{GRM}^{\mathbb{R}}(n, \sigma^2)$.

(iv) The ensemble $\mathrm{GSE}(n, \sigma^2)$ (resp. $\mathrm{GSE}^*(n, \sigma^2)$) from [S] can be described as the set of selfadjoint random matrices having the same distribution as

$$\frac{1}{\sqrt{2}}(Y + Y^*), \ (\text{resp. } \frac{1}{i\sqrt{2}}(Y - Y^*))$$

where $Y \in \mathrm{GRM}^{\mathbb{H}}(n, \sigma^2)$.

We shall prove the formulas (1.2), (1.4), (1.5), (1.6) for selfadjoint polynomials of arbitrary degree in $r + s$ stocastically independent selfadjoint random matrices

$$X_1^{(n)}, \ldots, X_{r+s}^{(n)}, \quad (r, s \geq 0, \quad r + s \geq 1),$$

where in the real case

$$X_1^{(n)}, \dots, X_r^{(n)} \in \mathrm{GOE}(n, \tfrac{1}{n}), \quad X_{r+1}^{(n)}, \dots, X_{r+s}^{(n)} \in \mathrm{GOE}^*(n, \tfrac{1}{n}), \tag{9.1}$$

and in the symplectic case

$$X_1^{(n)}, \dots, X_r^{(n)} \in \mathrm{GSE}(n, , \tfrac{1}{n}), \quad X_{r+1}^{(n)}, \dots, X_{r+s}^{(n)} \in \mathrm{GSE}^*(n, , \tfrac{1}{n}). \tag{9.2}$$

The symplectic case of (1.2), (1.4), (1.5) and (1.6) can easily be reduced to the real case by use of the methods from [S, Section 7]. Therefore, in the following we will only consider the real case.

Let $r, s \in \mathbb{N}_0$ with $r+s \geq 1$, and for each $n \in \mathbb{N}$, let $X_1^{(n)}, \dots, X_{r+s}^{(n)}$ be independent random matrices such that $X_1^{(n)}, \dots, X_r^{(n)} \in \mathrm{GOE}(n, \frac{1}{n})$ and $X_{r+1}^{(n)}, \dots, X_{r+s}^{(n)} \in GOE^*(n, \frac{1}{n})$. As in the previous sections, we let $p \in (M_m(\mathbb{C}) \otimes C\langle X_1, \dots, X_{r+s}\rangle)_{sa}$ and define random matrices $(Q_n)_{n=1}^{\infty}$ by

$$Q_n(\omega) = p(X_1^{(n)}(\omega), \dots, X_{r+s}^{(n)}(\omega)), \qquad (\omega \in \Omega). \tag{9.3}$$

With $d = \deg(p)$ we may choose $m_1, \dots, m_{d+1} \in \mathbb{N}$ with $m = m_1 = m_{d+1}$ and polynomials $u_j \in M_{m_j, m_{j+1}}(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \dots, X_{r+s}\rangle$ of first degree, $j = 1, \dots, d$, such that $p = u_1 u_2 \cdots u_d$. For each $n \in \mathbb{N}$ define random matrices $u_j^{(n)}, j = 1, \dots, d$, by

$$u_j^{(n)}(\omega) = u_j(X_1^{(n)}(\omega), \dots, X_{r+s}^{(n)}(\omega)), \qquad (\omega \in \Omega).$$

Since $Q_n(\omega)$ is self-adjoint, $\lambda \otimes \mathbf{1}_n - Q_n(\omega)$ is invertible for every $\lambda \in M_m(\mathbb{C})$ with $\mathrm{Im}\lambda > 0$. Then, according to Proposition 2.3, the random matrix

$$A_n(\lambda) = \begin{pmatrix} \lambda \otimes \mathbf{1}_n & -u_1^{(n)} & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{1}_{m_2} \otimes \mathbf{1}_n & -u_2^{(n)} & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{1}_{m_3} \otimes \mathbf{1}_n & -u_3^{(n)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbf{1}_{m_{d-1}} \otimes \mathbf{1}_n & -u_{d-1}^{(n)} \\ -u_d^{(n)} & 0 & 0 & \cdots & 0 & \mathbf{1}_{m_d} \otimes \mathbf{1}_n \end{pmatrix} \tag{9.4}$$

is (point-wise) invertible in $M_k(\mathbb{C})$, where $k = \sum_{i=1}^{d} m_i$.

Choose $a_0, \dots, a_{r+s} \in M_k(\mathbb{C})$ taking the form

$$a_i = \begin{pmatrix} 0 & a_{i_1} & 0 & \cdots & 0 \\ 0 & 0 & a_{i2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_{i_{d-1}} \\ a_{i_d} & 0 & \cdots & 0 & 0 \end{pmatrix},$$

such that with

$$S_n = a_0 \otimes \mathbf{1}_n + \sum_{i=1}^{r+s} a_i \otimes X_i^{(n)},$$

and $\Lambda = \lambda \oplus 1_{k-m}$ we have:

$$A_n(\lambda) = \Lambda \otimes \mathbf{1}_n - S_n.$$

As in section 6 put

$$\mathcal{O} = \{\lambda \in M_m(\mathbb{C}) \mid \mathrm{Im}\lambda \text{ is positive definite}\}$$

For $\lambda \in \mathcal{O}$ we put

$$H_n(\lambda) = (\mathrm{id}_k \otimes \mathrm{tr}_n)[(\Lambda \otimes \mathbf{1}_n - S_n)^{-1}] \tag{9.5}$$

and

$$G_n(\lambda) = \mathbb{E}\{H_n(\lambda)\}. \tag{9.6}$$

By [S, Lemma 6.4], Lemma 3.1 also holds in the real case, possibly with new constants $C_{1,p}$ and $C_{2,p}$. Hence, $G_n(\lambda)$ is well-defined and it is easy to check, that $\lambda \mapsto G_n(\lambda)$ is an analytic map from $\mathcal{O}$ to $M_k(\mathbb{C})$.

As in [S], we will let A^{-t} denote the transpose of the inverse of an invertible matrix A.

9.1 Theorem. *There is a constant $\widetilde{C}_1 \geq 0$, such that for every $n \in \mathbb{N}$ and for all $\lambda \in \mathcal{O}$,*

$$\left\| \sum_{j=1}^{r+s} a_j G_n(\lambda) a_i G_n(\lambda) + (a_0 - \Lambda) G_n(\lambda) + \mathbf{1}_k + \frac{1}{n} R_n(\lambda) \right\| \leq \frac{\widetilde{C}_1}{n^2}(1 + \|(\mathrm{Im}\lambda)^{-1}\|^4), \tag{9.7}$$

where

$$R_n(\lambda) = \sum_{j=1}^{r+s} \sum_{u,v=1}^{k} \varepsilon_j a_j e_{uv}^{(k)} \mathbb{E}\{(\mathrm{id}_k \otimes \mathrm{tr}_n)[(\Lambda \otimes \mathbf{1}_n - S_n)^{-t}(e_{uv}^{(k)} a_j \otimes \mathbf{1}_n)(\Lambda \otimes \mathbf{1}_n - S_n)^{-1}]\}, \tag{9.8}$$

$\varepsilon_j = 1$, $1 \leq j \leq r$, and $\varepsilon_j = -1$, $r+1 \leq j \leq r+s$.

Proof. We may assume that

$$X_j^{(n)} = \frac{1}{\sqrt{2}}(Y_j^{(n)} + Y_j^{(n)^*}), \quad (1 \leq j \leq r),$$
$$X_j^{(n)} = \frac{1}{i\sqrt{2}}(Y_j^{(n)} - Y_j^{(n)^*}), \quad (r+1 \leq j \leq r+s),$$

where $Y_1^{(n)}, \ldots, Y_{r+s}^{(n)}$ are $r+s$ stocastically independent random matrices from $\mathrm{GRM}^{\mathbb{R}}(n, \frac{1}{n})$. Then

$$S_n = a_0 \otimes \mathbf{1}_n + \sum_{j=1}^{r+s}(b_j \otimes Y_j^{(n)} + c_j \otimes Y_j^{(n)^*}),$$

where

$$\begin{aligned} b_j = c_j &= \tfrac{1}{\sqrt{2}} a_j, \quad (1 \leq j \leq r), \\ b_j = -c_j &= \tfrac{1}{i\sqrt{2}} a_j, \quad (r+1 \leq j \leq r+s). \end{aligned} \tag{9.9}$$

Following now the proof of [S, Theorem 2.1] we get that

$$\mathbb{E}\big\{(a_0-\lambda)H_n(\lambda)+\sum_{i=1}^{r+s}(b_jH_n(\lambda)c_jH_n(\lambda)+c_jH_n(\lambda)b_jH_n(\lambda))+\mathbf{1}_k\big\}=-\frac{1}{n}R_n(\lambda), \quad (9.10)$$

where

$$\begin{aligned}R_n(\lambda) &= \sum_{j=1}^{r+s}\sum_{u,v=1}^{k} b_je_{uv}^{(k)}\mathbb{E}\{(\mathrm{id}_m\otimes\mathrm{tr}_n)(\lambda\otimes\mathbf{1}_n-S_n)^{-t}(e_{uv}^{(k)}b_j\otimes\mathbf{1}_n)(\lambda\otimes\mathbf{1}_n-S_n)^{-1}\}\\ &+ \sum_{j=1}^{r+s}\sum_{u,v=1}^{k} c_je_{uv}^{(k)}\mathbb{E}\{(\mathrm{id}_m\otimes\mathrm{tr}_n)(\lambda\otimes\mathbf{1}_n-S_n)^{-t}(e_{uv}^{(k)}c_j\otimes\mathbf{1}_n)(\lambda\otimes\mathbf{1}_n-S_n)^{-1}\}\end{aligned}$$

Hence by (9.10),

$$\mathbb{E}\{((a_0-\lambda)H_n(\lambda)+\sum_{j=1}^{r+s}a_jH_n(\lambda)a_jH_n(\lambda)+\mathbf{1}_k\}=-\frac{1}{n}R_n(\lambda) \quad (9.11)$$

where

$$R_n(\lambda)=\sum_{j=1}^{r+s}\sum_{u,v=1}^{k}\varepsilon_ja_je_{uv}^{(k)}\mathbb{E}\{(\mathrm{id}_m\otimes\mathrm{tr}_n)(\lambda\otimes\mathbf{1}_n-S_n)^{-t}(e_{uv}^{(k)}a_j\otimes\mathbf{1}_n)(\lambda\otimes\mathbf{1}_n-S_n)^{-1}\} \quad (9.12)$$

and where $\varepsilon_j=1$, $1\le j\le r$ and $\varepsilon_j=-1$, $r+1\le j\le r+s$. Now, combining the method of proof from [S, proof of Theorem 2.4] with the proof of Theorem 4.3 of this paper, one finds that

$$\left\|\sum_{j=1}^{r+s}a_jG_n(\lambda)a_jG_n(\lambda)+(a_0-\lambda)G_n(\lambda)+\mathbf{1}_k+\frac{1}{n}R_n(\lambda)\right\|\le\frac{\widetilde{C}}{n^2}(C_{1,4}+C_{2,4}\|(\mathrm{Im}\lambda)^{-1}\|^4)$$

for some constant $\widetilde{C}>0$ depending only on $a_0,\dots,a_r$. Hence (9.7) holds with $\widetilde{C}_1=\widetilde{C}\cdot(C_{1,4}+C_{2,4})$. ■

9.2 Corollary. *There is a constant $\widetilde{C}_2\ge 0$, such that for every $\lambda\in\mathcal{O}$,*

$$\left\|\sum_{j=1}^{r+s}a_jG_n(\lambda)a_jG_n(\lambda)+(a_0-\Lambda)G_n(\lambda)+\mathbf{1}_k\right\|\le\frac{\widetilde{C}_2}{n}(1+\|(\mathrm{Im}\lambda)^{-1}\|^4).$$

Proof. By (9.12) and Lemma 3.1 (for the real case)

$$\begin{aligned}\|R_n(\lambda)\| &\le k^2\Big(\sum_{j=1}^{r+s}\|a_j\|^2\Big)\mathbb{E}\{\|(\Lambda\otimes\mathbf{1}_n-S_n)^{-1}\|^2\}\\ &\le k^2\Big(\sum_{j=1}^{r+s}\|a_j\|^2\Big)(C_{2,1}+C_{2,2}\|(\mathrm{Im}\lambda)^{-1}\|^2)\\ &\le C''(\mathbf{1}+\|(\mathrm{Im}\lambda)^{-1}\|^2)\end{aligned} \quad (9.13)$$

for a constant C'' depending only on $a_0, \dots, a_r$. Hence by Theorem 9.1,

$$\Big\| \sum_{i=1}^{r+s} a_i G_n(\lambda) a_i G_n(\lambda) + (a_0 - \Lambda) G_n(\lambda) + \mathbf{1}_k \| \leq \frac{C''}{n} (\mathbf{1} + \|(\mathrm{Im}\lambda)^{-1}\|^2) + \frac{\widetilde{C}_1}{n^2} (1 + \|(\mathrm{Im}\lambda)^{-1}\|^4)$$
$$\leq \frac{\widetilde{C}_2}{n} (\mathbf{1} + \|(\mathrm{Im}\lambda)^{-1}\|^4)$$

for a constant $\widetilde{C}_2 \geq 0$. ■

Let $(x_1, \dots, x_{r+s})$ be a semicircular system in a C^*-probability space $(\mathcal{A}, \tau)$, where τ is a faithfull state on $\mathcal{A}$. Put

$$s = a_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_{j=1}^{r+s} a_j \otimes x_j, \tag{9.14}$$

$$G(\lambda) = (\mathrm{id}_k \otimes \tau)[(\Lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}], \qquad (\lambda \in \mathcal{O}), \tag{9.15}$$

and put

$$\widetilde{G}(\mu) = (\mathrm{id}_k \otimes \tau)[(\mu \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}] \tag{9.16}$$

for all $\mu \in M_k(\mathbb{C})$ with $\mu \otimes \mathbf{1}_{\mathcal{A}} - s$ is invertible.

9.3 Theorem. *There is an $N \in \mathbb{N}$ and a constant $\widetilde{C}_3$ both depending only on $a_0, \dots, a_r$ such that for all $\lambda \in \mathcal{O}$*

$$\|G_n(\lambda) - G(\lambda)\| \leq \frac{\widetilde{C}_3}{n} (1 + \|\lambda\|)(1 + \|(\mathrm{Im}\lambda)^{-1}\|^7),$$

where $G(\lambda)$ is definded by (9.15).

Proof. This follows from Corollary 9.2 exactly as Theorem 5.6 followed from Theorem 4.3. One just has to replace n^2 by n in the proofs in Section 5. ■

9.4 Remark. From the proof of Theorem 9.3, i.e. from section 5 with n^2 replaced by n (cf. the formulas (5.18) through (5.33) and Lemma 5.5), it follows that there exist positive constants C_2, C_3 and C_4 such that when V_n denotes the set

$$V_n = \Big\{ \lambda \in \mathcal{O} \,\Big|\, \frac{C_4}{n^2} (1 + \|\lambda\|)(1 + \|(\mathrm{Im}\lambda)^{-1}\|^5)(1 + \|(\mathrm{Im}\lambda)^{-1}\|) < 1 \Big\}, \tag{9.17}$$

then for all $\lambda \in V_n$, $G_n(\lambda)$ is invertible and the following estimate holds:

$$\|G_n(\lambda)^{-1}\| \leq C_2 (1 + \|\lambda\|)(1 + \|(\mathrm{Im}\lambda)^{-1}\|). \tag{9.18}$$

Moreover, if one defines $\Lambda_n(\lambda)$ by

$$\Lambda_n(\lambda) = a_0 + \sum_{j=1}^{r+s} a_j G_n(\lambda) a_j + G_n(\lambda)^{-1}, \tag{9.19}$$

then for $\lambda \in V_n$,

$$\|\Lambda_n(\lambda) - \Lambda\| \le \frac{C_3}{n^2}(1 + \|\lambda\|)(1 + \|(\mathrm{Im}\lambda)^{-1}\|^5), \tag{9.20}$$

and

$$\|\Lambda_n(\lambda) - \Lambda\| \le \frac{1}{2C'(1 + \|(\mathrm{Im}\lambda)^{-1}\|)}, \tag{9.21}$$

where C' is the constant from Lemma 5.2. Finally, $\widetilde{G}(\Lambda_n(\lambda))$ is well-defined, and

$$\widetilde{G}(\Lambda_n(\lambda)) = G_n(\lambda), \qquad (\lambda \in V_n). \tag{9.22}$$

As above, consider a semicircular system $x_1, \dots, x_{r+s}$ in a C^*-probability space $(\mathcal{A}, \tau)$, where τ is a faithful state. It is no loss of generality to assume that

$$\mathcal{A} = C^*(x_1, \dots, x_{r+s}).$$

Note that the random matrices $X_1^{(n)}, \dots, X_r^{(n)} \in \mathrm{GOE}(n, \frac{1}{n})$ are symmetric, whereas $X_{r+1}^{(n)}, \dots, X_{r+s}^{(n)} \in \mathrm{GOE}^*(n, \frac{1}{n})$ are skew-symmetric. This is the reason for the following choice of "transposition" in $\mathcal{A}$ and $M_k(\mathcal{A})$.

9.5 Lemma. *(1) There is a unique bounded linear map $a \mapsto a^t$ of $\mathcal{A}$ onto itself such that*

(a) $x_j^t = x_j, \qquad (1 \le j \le r)$,

(b) $x_j^t = -x_j, \qquad (r+1 \le j \le r+s)$,

(c) $(ab)^t = b^t a^t, \qquad (a, b \in \mathcal{A})$.

Moreover, $(a^t)^t = a$ and $\|a^t\| = \|a\|$ for all $a \in \mathcal{A}$.

(2) Define a map $a \to a^t$ of $M_k(\mathcal{A})$ onto itself by

$$((a_{uv})_{u,v=1}^k)^t = (a_{vu}^t)_{u,v=1}^k$$

Then $(ab)^t = b^t a^t$, $(a, b \in M_k(\mathcal{A}))$. Moreover, $(a^t)^t = a$ and $\|a^t\| = \|a\|$ for all $a \in M_k(\mathcal{A})$.

Proof. (1) By the proof of [S, lemma 5.2(ii)] the is τ-prepreserving *-automorphism ψ of $\mathcal{A} = C^*(x_1, \dots, x_r, \mathbf{1})$, such that

$$\psi(x_j) = x_j, \quad 1 \le j \le r$$
$$\psi(x_j) = -x_j, \quad r+1 \le j \le r+s$$

Moreover, by [S, lemma 5.2(i)], there is a canjugate linear *-isomorphism φ of $\mathcal{A}$ such that $\tau \circ \varphi = \bar{\tau}$ and $\varphi(x_j) = x_j$, $1 \le j \le r+s$. Put now

$$a^t = \psi \circ \varphi(a^*), \qquad a \in \mathcal{A}$$

Then it is clear, that $a \to a^t$ satisfies all the conditions of (1). Also $a \to a^t$ is unique by boundedness and (a), (b), (c).
(2) It is elementary to check that the map on $M_k(\mathcal{A})$ defined in (2) is involutive and reverses the product. Hence it is a *-isomorphism of $M_k(\mathcal{A})$ on the opposite algebra $M_k(\mathcal{A})^{op}$, in particular it is an isometri. ■

For an invertible element $a \in M_k(\mathcal{A})$ we let a^{-t} denote the operator $(a^{-1})^t = (a^t)^{-1}$. In analogy with (9.8) we can now put

$$R(\lambda) = \sum_{j=1}^{r+s} \varepsilon_j a_j e_{uv}^{(k)} (\mathrm{id}_k \otimes \tau)[(\Lambda \otimes 1_{\mathcal{A}} - s)^{-t}(e_{uv}^{(k)} a_j \otimes \mathbf{1}_{\mathcal{A}})(\Lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}] \tag{9.23}$$

Note that by lemma 3.2

$$\|R(\lambda)\| \le \widetilde{C}''(1 + \|(\mathrm{Im}\lambda)^{-1}\|^2), \qquad \lambda \in \mathcal{O} \tag{9.24}$$

for a constant $\widetilde{C}'' \ge 0$.

9.6 Theorem. *There is a constant $\widetilde{C}_4 \ge 0$ such that for all $\lambda \in \mathcal{O}$ and all $n \in \mathbb{N}$,*

$$\|R_n(\lambda) - R(\lambda)\| \le \frac{\widetilde{C}_4}{n}(1 + \|(\mathrm{Im}\lambda)^{-1}\|^{14}).$$

Before proving Theorem 9.6 we will show how the main result of this section (Theorem 9.7 below) can be derived from Theorem 9.6 and the previous results of this section.
As in [S, section 4] we put

$$L(\lambda) = (\mathrm{id}_m \otimes \tau)[(\lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-t}(R(\lambda)G(\lambda)^{-1} \otimes \mathbf{1}_{\mathcal{A}})(\lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}].$$

9.7 Theorem. *There is a constant $\widetilde{C}_5 \ge 0$ such that for all $\lambda \in \mathcal{O}$ and all $n \in \mathbb{N}$,*

$$\left\| G_n(\lambda) - G(\lambda) + \frac{1}{n}L(\lambda) \right\| \le \frac{\widetilde{C}_5}{n^2}(1 + \|\lambda\|)(1 + \|(\mathrm{Im}\lambda)^{-1}\|^{17}).$$

for all $\lambda \in \mathcal{O}$ and all $n \in \mathbb{N}$.

Proof. The proof follows the proof of [S, Theorem 4.4]. As in Remark 9.4 we put

$$V_n = \left\{ \lambda \in \mathcal{O} \,\middle|\, \frac{C_4}{n^2}(1 + \|\lambda\|)(1 + \|(\mathrm{Im}\lambda)^{-1}\|^5)(1 + \|(\mathrm{Im}\lambda)^{-1}\|) < 1 \right\},$$

and we let $\Lambda_n(\lambda)$ be given by (9.19). Then by Theorem 9.1 and (9.18), for all $\lambda \in \Lambda_n$,

$$\begin{aligned} & \left\| \Lambda_n(\lambda) - \Lambda + \frac{1}{n}R_n(\lambda)G_n(\lambda)^{-1} \right\| \\ = & \left\| \Big(\sum_{j=1}^{r+s} a_j G_n(\lambda) a_j G_n(\lambda) + (a_0 - \Lambda)G_n(\lambda) + \mathbf{1}_k + \frac{1}{n}R_n(\lambda) \Big) G_n(\lambda)^{-1} \right\| \\ \le & \frac{\widetilde{C}_5}{n^2}(1 + \|\lambda\|)(1 + \|(\mathrm{Im}\lambda)^{-1}\|^6), \end{aligned} \tag{9.25}$$

for some constant $\widetilde{C}_5 \geq 0$. By Lemma 5.2,

$$\|G(\lambda)\| \leq C'(\mathbf{1} + \|(\mathrm{Im}\lambda)^{-1}\|, \qquad (\lambda \in \mathcal{O}) \tag{9.26}$$

Moreover, by Lemma 5.2, $G(\lambda)$ is invertible and

$$G(\lambda)^{-1} = \Lambda - a_0 - \sum_{i=1}^{r+s} a_i G(\lambda) a_i. \tag{9.27}$$

Hence, by (9.26),

$$\|G(\lambda)^{-1}\| \leq \widetilde{C}'(1 + \|\lambda\|)(\mathbf{1} + \|(\mathrm{Im}\lambda)^{-1}\|, \qquad (\lambda \in \mathcal{O}) \tag{9.28}$$

for some constant $\widetilde{C}' \geq 0$. Applying Theorem 9.3, (9.18) and (9.28), we have for $\lambda \in V_n$,

$$\begin{aligned} \|G_n(\lambda)^{-1} - G(\lambda)^{-1}\| &= \|G_n(\lambda)^{-1}(G(\lambda) - G_n(\lambda))G(\lambda)^{-1}\| \\ &\leq \frac{\widetilde{C}_6}{n}(1 + \|\lambda\|)^3(\mathbf{1} + \|(\mathrm{Im}\lambda)^{-1}\|^9) \end{aligned}$$

for a constant $\widetilde{C}_6 \geq 0$. Combining this with Theorem 9.6 and (9.13), we now have

$$\begin{aligned} \|R_n(\lambda)G_n(\lambda)^{-1} - R(\lambda)G(\lambda)^{-1}\| &\leq \|R_n(\lambda)\|\|G_n(\lambda)^{-1} - G(\lambda)^{-1}\| + \|R_n(\lambda) - R(\lambda)\|\|G(\lambda)^{-1}\| \\ &\leq \frac{\widetilde{C}_7}{n}(1 + \|\lambda\|)^3(1 + \|(\mathrm{Im}\lambda)^{-1}\|^{15}) \end{aligned}$$

for a constant $\widetilde{C}_7 \geq 0$. Hence by(9.25),

$$\left\|\Lambda_n(\lambda) - \Lambda + \frac{1}{n}R(\lambda)G(\lambda)^{-1}\right\| \leq \frac{\widetilde{C}_8}{n^2}(1 + \|\lambda\|)^3(\mathbf{1} + \|(\mathrm{Im}\lambda)^{-1}\|^{15}) \tag{9.29}$$

for a constant $\widetilde{C}_8 \geq 0$. By lemma 5.2 and (9.21), we have for $\lambda \in V_n$,

$$\|(\Lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}\| \leq C'(\mathbf{1} + \|(\mathrm{Im}\lambda)^{-1}\|), \tag{9.30}$$

$$\|(\Lambda_n(\lambda) \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}\| \leq 2C'(1 + \|(\mathrm{Im}\lambda)^{-1}\|). \tag{9.31}$$

Proceeding as in [S, (4.24) and (4.25)], one now gets, using (9.20), (9.22), (9.24) and the above estimates, that

$$\|G_n(\lambda) - G(\lambda) - \frac{1}{n}L(\lambda)\| \leq \frac{1}{n^2}(1 + \|\lambda\|)^3 P_{\mathbf{1}}(\|\mathrm{Im}\lambda\|^{-1})$$

for a polynomical $P_{\mathbf{1}}$ of degree 17. Finally, if $\lambda \in \mathcal{O}\backslash V_n$, then one obtains exactly as in [S, proof of Theorem 4.4] that

$$\left\|G_n(\lambda) - G(\lambda) - \frac{1}{n}L(\lambda)\right\| \leq (1 + \|\lambda\|)^2 P_2(\|(\mathrm{Im}\lambda)^{-1}\|)$$

for a polynomial P_2 of degree 13. Put $P = P_1 + P_2$. Then P is of degree 17, and

$$\left\|G_n(\lambda) - G(\lambda) - \frac{1}{n}L(\lambda)\right\| \leq (1 + \|\lambda\|)^3 P(\|(\mathrm{Im}\lambda)^{-1}\|)$$

for all $\lambda \in \mathcal{O}$. This proves Theorem 9.7. ■

We now return to the proof of Theorem 9.6. The proof will be devided into a series of lemmas. The first lemma is a simpel but very useful observation:

9.8 Lemma. *Let A be a unital algebra, and let $x, z \in \mathrm{GL}(A)$ and $y \in A$. Then*

$$\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$$

is invertible in $M_2(A)$ with inverse

$$\begin{pmatrix} x^{-1} & -x^{-1}yz^{-1} \\ 0 & z^{-1} \end{pmatrix}.$$

Let $\lambda \in \mathcal{O}$ and $x \in M_k(\mathbb{C})$ and as usual put

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \mathbf{1}_{k-m} \end{pmatrix}.$$

Moreover, we put

$$\pi_n(\lambda, x) = \begin{pmatrix} \Lambda^t \otimes \mathbf{1}_n - S_n^t & x \otimes \mathbf{1}_n \\ 0 & \Lambda \otimes \mathbf{1}_n - S_n \end{pmatrix}, \tag{9.32}$$

$$H_n(\lambda, x) = (\mathrm{id}_{2k} \otimes \mathrm{tr}_n)[\pi_n(\lambda, x)^{-1}], \tag{9.33}$$

$$G_n(\lambda, x) = \mathbb{E}\{H_n(\lambda, x)\}, \tag{9.34}$$

and

$$\pi(\lambda, x) = \begin{pmatrix} \Lambda^t \otimes \mathbf{1}_{\mathcal{A}} - s^t & x \otimes \mathbf{1}_{\mathcal{A}} \\ 0 & \Lambda \otimes \mathbf{1}_{\mathcal{A}} - s \end{pmatrix}, \tag{9.35}$$

$$G(\lambda, x) = (\mathrm{id}_{2k} \otimes \tau)[\pi(\lambda, x)^{-1}] \tag{9.36}$$

Finally, we put

$$\widehat{s} = \begin{pmatrix} s^t & 0 \\ 0 & s \end{pmatrix}$$

and

$$\widetilde{G}(\mu) = (\mathrm{id}_{2k} \otimes \tau)((\mu \otimes \mathbf{1}_{\mathcal{A}} - \hat{s})^{-1}) \tag{9.37}$$

whenever $\mu \otimes \mathbf{1}_{\mathcal{A}} - \widehat{s}$ is invertible in $M_{2k}(\mathcal{A})$. Note that

$$G(\lambda, x) = \widetilde{G}\begin{pmatrix} \Lambda^t & x \\ 0 & \Lambda \end{pmatrix}. \tag{9.38}$$

The idea is now to estimate $\|G_n(\lambda, x) - G(\lambda, x)\|$ by the methods of section 5 (with n^2 replaced by n). The estimate we obtain in Lemma 9.15 below combined with Lemma 9.8 will then complete the proof of Theorem 9.6.

9.9 Lemma. *(i) The R-transform of $\widehat{s}$ with respect to amalgamation over $M_{2k}(\mathbb{C})$ is*

$$\widehat{R}(z) = \hat{a}_0 + \sum_{i=1}^{r+s} \hat{a}_i z \hat{a}_i, \quad z \in M_{2k}(\mathbb{C})$$

where

$$\widehat{a}_i = \begin{pmatrix} a_i^t & 0 \\ 0 & a_i \end{pmatrix}, \qquad (i = 0, \ldots, r+s).$$

(ii) Let $\mu \in M_{2k}(\mathbb{C})$. If μ is invertible and $\|\mu^{-1}\| < \frac{1}{\|S\|}$, then $\widetilde{G}(\mu)$ is well-defined and invertible. Moreover,

$$\widehat{a}_0 + \sum_{i=1}^{r+s} \widehat{a}_0 \widetilde{G}(\mu) \widehat{a}_i + \widetilde{G}(\mu)^{-1} = \mu.$$

(iii) Let $\mu \in M_{2k}(\mathbb{C})$. If μ is invertible and if

$$\left\| \left(\begin{pmatrix} T^t & 0 \\ 0 & R \end{pmatrix} \mu \begin{pmatrix} R^t & 0 \\ 0 & T \end{pmatrix} \right)^{-1} \right\| < \frac{1}{\|s\|},$$

for some choice of block diagonal matrices R and T of the form

$$\begin{aligned} R &= \operatorname{diag}(r_1 \mathbf{1}_{m_1}, r_2 \mathbf{1}_{m_2}, \ldots, r_d \mathbf{1}_{m_k}), \\ T &= \operatorname{diag}(t_1 \mathbf{1}_{m_1}, t_2 \mathbf{1}_{m_2}, \ldots, t_d \mathbf{1}_{m_k}), \end{aligned}$$

where $r_1, \ldots, r_d, t_1, \ldots, t_d \in \mathbb{C} \backslash \{0\}$ satisfy

$$r_1 t_2 = r_2 t_3 = \ldots = r_{d-1} t_d = r_d t_1 = 1,$$

then $\widetilde{G}(\mu)$ is well-defined and invertible and satisfies

$$\widehat{a}_0 + \sum_{i=1}^{r+s} \widehat{a}_i \widetilde{G}(\mu) \widehat{a}_i + \widetilde{G}(\mu)^{-1} = \mu.$$

Proof. Observe, that with R and T as in (iii),

$$\begin{pmatrix} T^t & 0 \\ 0 & R \end{pmatrix} \widehat{a}_i \begin{pmatrix} R^t & 0 \\ 0 & T \end{pmatrix} = \begin{pmatrix} (Ra_iT)^t & 0 \\ 0 & Ra_iT \end{pmatrix} = \widehat{a}_i$$

for $i = 0, \ldots, r+s$ because $Ra_iT = a_i$ by (5.14). The rest of the proof of Lemma 9.10 is a straightforward generalization of the proof of Lemma 5.1. ■

Let B denote the open unitball in $M_k(\mathbb{C})$ i.e.

$$B = \{x \in M_k(\mathbb{C}) \mid \|x\| < 1\}.$$

9.10 Lemma. *There is a constant $\widetilde{C}$ depending only on $a_0, \ldots, a_{r+s}$, such that:*

(i) For all $\lambda \in \mathcal{O}$ and $x \in B$

$$\|\pi(\lambda, x)^{-1}\| \leq \widetilde{C}(\mathbf{1} + \|(\operatorname{Im}\lambda)^{-2}\|).$$

Moreover, for all such λ and x, $G(\lambda, x)$ is invertible, and

$$\widehat{a}_0 + \sum_{i=1}^{r} \widehat{a}_i G(\lambda, x) \widehat{a}_i + G(\lambda, x)^{-1} = \begin{pmatrix} \Lambda^t & x \\ 0 & \Lambda \end{pmatrix}.$$

(ii) Let $(\lambda, x) \in \mathcal{O} \times B$ and assume that $\mu \in M_{2k}(\mathbb{C})$ satisfies

$$\left\| \mu - \begin{pmatrix} \Lambda^t & x \\ 0 & \Lambda \end{pmatrix} \right\| < \frac{1}{2\widehat{C}(1 + \|(\mathrm{Im}\lambda)^{-2}\|)}.$$

Then $\mu \otimes \mathbf{1}_{\mathcal{A}} - \hat{s}$ is invertible, and

$$\|(\mu \otimes \mathbf{1}_{\mathcal{A}} - \hat{s})^{-1}\| < 2\widehat{C}(1 + \|(\mathrm{Im}\lambda)^{-2}\|).$$

Moreover, $\widetilde{G}(\mu)$ is invertible and

$$\hat{a}_0 + \sum_{i=1}^{r+s} \hat{a}_i \widetilde{G}(\mu) \hat{a}_i + \widetilde{G}(\mu)^{-1} = \mu.$$

Proof. Since $a \mapsto a^t$ is an isometry of $M_k(\mathcal{A})$ and since $(a^{-1})^t = (a^t)^{-1}$, when a is invertible, we have that

$$\|(\Lambda^t \otimes \mathbf{1}_{\mathcal{A}} - s^t)^{-1}\| = \|(\Lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}\|.$$

Hence for $\lambda \in \mathcal{O}$ and $x \in B$, we get by Lemma 9.8 and Lemma 3.2 that

$$\begin{aligned} \|\pi(\lambda, x)^{-1}\| &= \left\| \begin{pmatrix} \Lambda^t \otimes \mathbf{1}_{\mathcal{A}} - s^t & x \otimes \mathbf{1}_{\mathcal{A}} \\ 0 & \Lambda \otimes \mathbf{1}_{\mathcal{A}} - s \end{pmatrix}^{-1} \right\| \\ &\leq \|(\Lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}\| + \|(\Lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}\|^2 \|x\| \\ &\leq C'_{1,1} + C'_{2,1}\|(\mathrm{Im}\lambda)^{-1}\| + C'_{2,1} + C'_{2,2}\|(\mathrm{Im}\lambda)^{-1}\|^2 \\ &\leq \widehat{C}(\mathbf{1} + \|(\mathrm{Im}\lambda^{-1}\|^2) \end{aligned}$$

for a constant $\widetilde{C}$ depending only on $C'_{i,j}$, $i, j = 1, 2$. Put

$$\mathcal{O}' = \{\lambda \in \mathcal{O} \mid \|\lambda^{-1}\| < \min\{1, (2\|s\|)^{-d}\}\},$$

and for a fixed $\lambda \in \mathcal{O}'$ put

$$\alpha = \|\lambda^{-1}\|^{\frac{1}{d}} < \min\left\{1, \frac{1}{2\|s\|}\right\}.$$

Next, let

$$\begin{aligned} (r_1, \dots, r_d) &= (\alpha^{d-1}, \alpha^{d-2}, \dots, \alpha, \mathbf{1}), \\ (t_1, \dots, t_d) &= (\mathbf{1}, \alpha^{1-d}, \alpha^{2-d}, \dots, \alpha^{-1}), \end{aligned}$$

and

$$\begin{aligned} R &= \mathrm{diag}(r_1 \mathbf{1}_{m_1}, \dots, r_d \mathbf{1}_{m_d}), \\ T &= \mathrm{diag}(t_1 \mathbf{1}_{m_1}, \dots, t_d \mathbf{1}_{m_d}). \end{aligned}$$

Then, as in the proof of Lemma 5.2, we get that

$$\|(R\Lambda T)^{-1}\| = \alpha < \frac{1}{2\|s\|}.$$

Hence by Lemma 9.8, we have for $x \in B$ that

$$\begin{aligned}\left\| \left(\begin{pmatrix} T^t & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} \Lambda^t & x \\ 0 & \Lambda \end{pmatrix} \begin{pmatrix} R^t & 0 \\ 0 & T \end{pmatrix} \right)^{-1} \right\| &= \left\| \begin{pmatrix} (R\Lambda T)^t & T^t \times T \\ 0 & R\Lambda T \end{pmatrix}^{-1} \right\| \\ &\leq \|(R\Lambda T)^{-1}\| + \|(R\Lambda T)^{-1}\|^2 \|T\|^2 \|x\| \\ &\leq \frac{1}{2\|s\|} + \frac{\|T\|^2\|x\|}{4\|s\|^2}.\end{aligned}$$

Moreover, $\|T\| = \alpha^{1-d} \leq \alpha^{-d} = \|\lambda^{-1}\|^{-1}$. Thus, if $\|x\| < 2\|s\|\ \|\lambda^{-1}\|^2$, then

$$\left\| \begin{pmatrix} T^t & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} \Lambda^t & x \\ 0 & \Lambda \end{pmatrix} \begin{pmatrix} R^t & 0 \\ 0 & T \end{pmatrix}^{-1} \right\| < \frac{1}{\|s\|},$$

so by (9.38) and Lemma 9.10 (iii),

$$\hat{a}_0 + \sum_{i=1}^{r} \hat{a}_i G(\lambda, x) \hat{a}_i + G(\lambda, x)^{-1} = \begin{pmatrix} \Lambda^t & x \\ 0 & \Lambda \end{pmatrix}. \qquad (9.39)$$

Since

$$\{(\lambda, x) \in \mathcal{O}' \times B \mid \|x\| < 2\|s\|\ \|\lambda^{-1}\|^2\}$$

is a non-empty open subset of $\mathcal{O} \times B$, we can use uniqueness of analytic continuation as in the proof of Lemma 5.2 and obtain that $G(\lambda, x)$ is invertible for all $(x, \lambda) \in \mathcal{O} \times B$ and that these all satisfy (9.39). This proves (i). The proof of (ii) is a straightforward generalization of the proof of Lemma 5.2 (ii). ■

For $\lambda \in \mathcal{O}$ and $x \in M_k(\mathbb{C})$ we let $\pi_n(\lambda, x)$, $H_n(\lambda, x)$ and $G_n(\lambda, x)$ be given by (9.32), (9.33) and (9.34), respectively. Moreover, we put

$$R_n(\lambda, x) = \sum_{i=1}^{r+s} \sum_{u,u=1}^{2k} \hat{a}_i e_{uv}^{(2k)} \mathbb{E}\{(\mathrm{id}_{2k} \otimes \mathrm{tr}_n)[\pi_n(x, \lambda)^{-t}(e_{uv}^{(2k)} \hat{a}_i \otimes \mathbf{1}_n) \pi_n(x, \lambda)^{-1}]\}. \qquad (9.40)$$

9.11 Lemma. *There is a constant $\widehat{C}_1 \geq 0$ only depending on $a_0, \ldots, a_{r+s}$, such that for all $\lambda \in \mathcal{O}$ and all $x \in B$,*

$$\begin{aligned}\| \sum_{i=1}^{r+s} \hat{a}_i G_n(\lambda, x) \hat{a}_i G_n(\lambda, x) + \left(\hat{a}_0 - \begin{pmatrix} \Lambda^t & x \\ 0 & \Lambda \end{pmatrix} \right) G_n(\lambda, x) + \mathbf{1}_n \Big) + \frac{1}{n} R_n(\lambda, x) \| \\ \leq \frac{\widehat{C}_1}{n^2} (\mathbf{1} + \|(\mathrm{Im}\lambda)^{-1}\|^8)\end{aligned} \qquad (9.41)$$

Proof. This is a fairly straightforward generalization of the proof of Theorem 9.1. The master equation (9.11) now becomes

$$\mathbb{E}\left\{ \sum_{i=1}^{r+s} \hat{a}_i H_n(\lambda, x) \hat{a}_i H_n(\lambda, x) + \left(\hat{a}_0 - \begin{pmatrix} \Lambda^t & x \\ 0 & \Lambda \end{pmatrix} \right) H_n(\lambda, x) + \mathbf{1}_n \right\} = -\frac{1}{n} R_n(\lambda, x). \quad (9.42)$$

Since $\|x\| < 1$, we get from Lemma 9.8 and Lemma 3.1 that

$$\begin{aligned}
\mathbb{E}\{\|\pi_n(\lambda, x)^{-1}\|^p\} &= \mathbb{E}\left\{\left\|\begin{pmatrix} \Lambda^t \otimes \mathbf{1}_n - s_n^t & x \otimes \mathbf{1}_n \\ 0 & \Lambda \otimes \mathbf{1}_n - s_n \end{pmatrix}^{-1}\right\|^p\right\} \\
&\leq \mathbb{E}\{(\|\Lambda \otimes \mathbf{1}_n - s_n\| + \|\Lambda \otimes \mathbf{1}_n - s_n\|^2)^p\} \\
&\leq 2^p \mathbb{E}\{\|\Lambda \otimes \mathbf{1}_n - s_n\|^p + \|\Lambda \otimes \mathbf{1}_n - s_n\|^{2p}\} \\
&\leq 2^p (C_{1,p} + C_{2,p}\|(\mathrm{Im}\lambda)^{-1}\|^p + C_{1,2p} + C_{2,2p}\|(\mathrm{Im}\lambda)^{-1}\|^{2p}) \\
&\leq C_p'(1 + \|(\mathrm{Im}\lambda)^{-1}\|^{2p}),
\end{aligned} \tag{9.43}$$

for a constant C_p' which only depends on the constants $C_{i,j}$, $i = 1, 2$, $j = p, 2p$. Applying now (9.43) for $p = 4$, Lemma 9.11 follows from (9.42) exactly as in the proof of Theorem 9.1. ■

9.12 Corollary. *There is a constant $\widehat{C}_2 \geq 0$, such that for all $\lambda \in \mathcal{O}$ and all $x \in B$,*

$$\left\|\sum_{i=1}^{r+s} \hat{a}_i G_n(\lambda, x) \hat{a}_i G_n(\lambda, x) + \left(\hat{a}_0 - \begin{pmatrix} \Lambda^t & x \\ 0 & \Lambda \end{pmatrix}\right) G_n(\lambda, x) + \mathbf{1}_n\right\| \leq \frac{\widehat{C}_2}{n}(1 + \|(\mathrm{Im}\lambda)^{-1}\|^8).$$

Proof. By (9.40) and (9.43) (for $p = 2$), we have that

$$\|R_n(\lambda, x)\| \leq \widehat{C}''(1 + \|(\mathrm{Im}\lambda)^{-1}\|^4),$$

for some constant $\widehat{C}'' \geq 0$. The corollary now follows immediately from Lemma 9.12 (cf. the proof of Corollary 9.2). ■

Note that by (9.33), (9.34) and (9.43),

$$\|G_n(\lambda, x)\| \leq C_p'(1 + \|(\mathrm{Im}\lambda)^{-1}\|^2), \qquad ((\lambda, x) \in \mathcal{O} \times B), \tag{9.44}$$

and by (9.36) and Lemma 9.11 (i),

$$\|G(\lambda, x)\| \leq \widehat{C}(1 + \|(\mathrm{Im}\lambda)^{-1}\|^2), \qquad ((\lambda, x) \in \mathcal{O} \times B). \tag{9.45}$$

Proceeding now as in (5.18)-(5.23) with n^2 replaced by n (see also Remark 9.4), one finds that after suitable changes of the exponents, that there exists positive constants C_2, C_3, C_4 and C_6, such that when $\widehat{V}_n$ denotes the set

$$\widehat{V}_n = \left\{\lambda \in \mathcal{O} \mid \frac{C_4}{n}(1 + \|\lambda\|)(1 + \|(\mathrm{Im}\lambda)^{-1}\|^{10})(1 + \|(\mathrm{Im}\lambda)^{-1}\|^2) < 1\right\}, \tag{9.46}$$

then for $(\lambda, x) \in V_n \times B$, $G_n(\lambda, x)$ is invertible and

$$\|G_n(\lambda)^{-1}\| \leq C_2(1 + \|\lambda\|)(1 + \|(\mathrm{Im}\lambda)^{-1}\|^2). \tag{9.47}$$

Moreover, if one defines $\Lambda_n(\lambda, x)$ by

$$\Lambda_n(\lambda, x) = \hat{a}_0 + \sum_{i=1}^{r+s} \hat{a}_i G_n(\lambda, x) \hat{a}_i + G_n(\lambda, x)^{-1}, \tag{9.48}$$

then for $(\lambda, x) \in \widehat{V}_n \times \mathcal{O}$,

$$\left\| \Lambda_n(\lambda, x) - \begin{pmatrix} \Lambda^t & x \\ 0 & \Lambda \end{pmatrix} \right\| \le \frac{C_3}{n^2}(1 + \|\lambda\|)(1 + \|(\mathrm{Im}\lambda)^{-1}\|^{10}), \tag{9.49}$$

and

$$\left\| \Lambda_n(\lambda, x) - \begin{pmatrix} \Lambda^t & x \\ 0 & \Lambda \end{pmatrix} \right\| \le \frac{1}{2\widehat{C}(1 + \|(\mathrm{Im}\lambda)^{-1}\|^2)}, \tag{9.50}$$

where $\widehat{C}$ is the constant from Lemma 9.10 (ii). Hence by Lemma 9.10 (ii), $\Lambda_n(\lambda, x) \otimes \mathbf{1}_{\mathcal{A}} - \hat{s}$ is invertible, and

$$\|(\Lambda_n(\lambda, x) \otimes \mathbf{1}_{\mathcal{A}} - \hat{s})^{-1}\| \le 2\widehat{C}(1 + \|(\mathrm{Im}\lambda)^{-1}\|^2). \tag{9.51}$$

Moreover, $\widetilde{G}(\Lambda_n(\lambda, x))$ is well-defined, invertible and

$$\hat{a}_0 + \sum_{i=1}^{r+s} \hat{a}_i \widetilde{G}(\Lambda_n(\lambda, x))\hat{a}_i + \widetilde{G}(\Lambda_n(\lambda, x))^{-1} = \Lambda_n(\lambda, x). \tag{9.52}$$

Recall that for $\lambda \in \mathcal{O}$,

$$G_n(\lambda) = \mathbb{E}\{(\mathrm{id}_k \otimes \mathrm{tr}_n)((\Lambda \otimes \mathbf{1}_n - s_n)^{-1})\}.$$

9.13 Lemma. *There is a constant $C_5 \ge 0$, independent of λ and n, such that when $\lambda \in \mathcal{O}$ and $\|(\mathrm{Im}\lambda)^{-1}\| \le 1$, there exist $R, S \in GL(k, \mathbb{C})$, such that*

$$\|RG_n(\lambda)a_iR^{-1}\| \le C_5\|(\mathrm{Im}\lambda)^{-1}\|^{\frac{1}{d}}, \tag{9.53}$$

and

$$\|Sa_iG_n(\lambda)S^{-1}\| \le C_5\|(\mathrm{Im}\lambda)^{-1}\|^{\frac{1}{d}}, \tag{9.54}$$

for $n \in \mathbb{N}$ and $1 \le i \le r + s$.

Proof. Lemma 3.1 holds in the real case too (possibly with change of constants). Therefore Lemma 5.4 also holds in the real case, which proves (9.54). Moreover, by the proof of Lemma 5.4, the matrix S given by

$$S = \mathrm{diag}(\beta\mathbf{1}_{m_1}, \beta^2\mathbf{1}_{m_2}, \dots, \beta^d\mathbf{1}_{m_d}), \tag{9.55}$$

where $\beta = \|(\mathrm{Im}\lambda)^{-1}\|^{\frac{1}{d}} \ge 1$, satisfies (9.54). As in the proof of Lemma 3.1, write

$$(\lambda \otimes \mathbf{1}_n - S_n)^{-1} = C_n + B_n^{(1)}(\lambda \otimes \mathbf{1}_n - Q_n)^{-1}B_n^{(2)}.$$

Then by (5.5) and the positions of the non-zero entries of C_n, we get, that $G_n(\lambda)a_i$ is a $d \times d$ block matrix of the form

$$G_n(\lambda)a_i = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ * & 0 & * & \cdots & * \\ * & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & * \\ * & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Let

$$R = \mathrm{diag}(\beta^d \mathbf{1}_{m_1}, \beta \mathbf{1}_{m_2}, \beta^2 \mathbf{1}_{m_5}, \ldots, \beta^{d-1}\mathbf{1}_{m_d}), \tag{9.56}$$

where as before $\beta = \|(\mathrm{Im}\lambda)^{-1}\|^{-\frac{1}{d}}$. Then the map $G_n(\lambda)a_i \to RG_n(\lambda)a_iR^{-1}$ multiplies the upper diagonal entries of $[G_n(\lambda)a_i]_{uv}$, $2 \le u < v \le d$, by β^{u-v} and it multiplies the entries $[G_n(\lambda)a_i]_{u1}$, $2 \le u \le d$ by β^{u-1-d}. Thus,

$$\|[RG_n(\lambda)a_iR^{-1}]_{uv}\| \le \beta^{-1}\|[G_n(\lambda)a_i]_{uv}\|$$

for all $u, v \in \{1, \ldots, d\}$. The rest of the proof of (9.53) is now a simple modification of the proof of Lemma 5.4. ■

9.14 Lemma. *There is a positive integer N, such that for all $n \ge N$,*

$$\widetilde{G}(\Lambda_n(\lambda, x)) = G_n(\lambda, x), \qquad (\lambda \in \widehat{V}_n,\ x \in B).$$

Proof. Let $(\lambda, x) \in \widehat{V}_n \times B$, and at first assume that $\|(\mathrm{Im}\lambda)^{-1}\| \le 1$. Put

$$z = G_n(\lambda, x) \qquad \text{and} \qquad w = \widetilde{G}(\Lambda_n(\lambda, x)). \tag{9.57}$$

By (9.48) and (9.52), z and w are invertible, and

$$\sum_{i=1}^{r+s} \hat{a}_i z \hat{a}_i + z^{-1} = \sum_{i=1}^{r+s} \hat{a}_i w \hat{a}_i + w^{-1}. \tag{9.58}$$

Put

$$T = \begin{pmatrix} R^{-t} & 0 \\ 0 & S \end{pmatrix}, \tag{9.59}$$

where $R, S \in GL(k, \mathbb{C})$ are the matrices from Lemma 9.13 given by (9.55) and (9.56). We will show that if $\|(\mathrm{Im}\lambda)^{-1}\|$ and $\|x\|$ are sufficiently small, then

$$\sum_{i=1}^{r+s} \|w\hat{a}_i\| \|T\hat{a}_i z T^{-1}\| < 1, \tag{9.60}$$

and thus, by the proof of Lemma 5.3, it follows, that $z = w$.

By Lemma 9.8 we have for $\lambda \in \mathcal{O}$ and $x \in B$ that

$$\begin{aligned} G_n(\lambda, x) &= \mathbb{E}\Big\{(\mathrm{id}_{2k} \otimes \mathrm{tr}_n)\Big[\begin{pmatrix} \lambda^t \otimes \mathbf{1}_n - S_n^t & x \otimes \mathbf{1}_n \\ 0 & \lambda \otimes \mathbf{1}_n - S_n \end{pmatrix}^{-1}\Big]\Big\} \\ &= \begin{pmatrix} G_n(\lambda)^t & K_n(\lambda, x) \\ 0 & G_n(\lambda) \end{pmatrix} \end{aligned}, \tag{9.61}$$

where

$$K_n(\lambda, x) = -\mathbb{E}\{(\mathrm{id}_k \otimes \mathrm{tr}_n)[(\lambda^t \otimes \mathbf{1}_n - s_n^t)^{-1}(x \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1}_n - s_n)^{-1}]\}. \tag{9.62}$$

With R, S and T as above we have that

$$T\hat{a}_i G_n(\lambda, x)T^{-1} = \begin{pmatrix} (RG_n(\lambda)a_i R^{-1})^t & R^{-t}a_i^t K_n(\lambda, x)S^{-1} \\ 0 & Sa_i G_n(\lambda)S^{-1} \end{pmatrix}.$$

Hence, by Lemma 9.13,

$$\|T\hat{a}_i G_n(\lambda, x)T^{-1}\| \leq C_5\|(\mathrm{Im}\lambda)^{-1}\|^{\frac{1}{d}} + \|R^{-1}\|\|a_i^t K_n(\lambda, x)\|\|S^{-1}\|.$$

By (9.55) and (9.56), $\|R^{-1}\| = \|S^{-1}\| = \frac{1}{\beta} \leq 1$. (9.62) and Lemma 3.1 imply that

$$\|K_n(\lambda, x)\| \leq (C_{1,2} + C_{2,2}\|(\mathrm{Im}\lambda)^{-1}\|^2)\|x\|$$

Hence

$$\|T\hat{a}_i G_n(\lambda, x)T^{-1}\| \leq C_5\|(\mathrm{Im}\lambda)^{-1}\|^{\frac{1}{d}} + \|x\|\|a_i\|(C_{1,1} + C_{2,2}\|(\mathrm{Im}\lambda)^{-1}\|^2). \tag{9.63}$$

Moreover, by (9.37) and (9.51),

$$\|\widetilde{G}(\Lambda_n(\lambda, x))\| \leq 2\widehat{C}(1 + \|(\mathrm{Im}\lambda)^{-1}\|^2. \tag{9.64}$$

By (9.62) and (9.63), there is a $\delta \in (0, 1)$, such that when $\|(\mathrm{Im}\lambda)^{-1}\| < \delta$ and $\|x\| < \delta$, then for all $n \in \mathbb{N}$,

$$\sum_{i=1}^{r+s} \|\widetilde{G}(\Lambda_n(\lambda, x))\hat{a}_i\|\|T\hat{a}_i G_n(\lambda, x)T^{-1}\| < 1.$$

That is, (9.60) holds, and therefore $z = w$, which shows that $G_n(\lambda, x) = \widetilde{G}(\Lambda_n(\lambda, x))$ when (λ, x) belongs to the set

$$\mathcal{U}_n = \{\lambda \in \widehat{V}_n \mid \|(\mathrm{Im}\lambda)^{-1}\| < \delta\} \times \{x \in M_k(\mathbb{C}) \mid \|x\| < \delta\}.$$

Exactly as for the sets V_n in section 5, we can prove that $\widehat{V}_n$ is connected and that there exists $N \in \mathbb{N}$, such that $\{\lambda \in \widehat{V}_n \mid \|(\mathrm{Im}\lambda)^{-1}\| < \delta\}$ is non-empty for all $n \geq \mathbb{N}$. Lemma 9.14 now follows by uniqueness of analytic continuation. ■

9.15 Lemma. *There is a constant $C_6 \geq 0$ such that for $\lambda \in \mathcal{O}$, $x \in B$ and $n \in \mathbb{N}$*

$$\|G_n(\lambda, x) - G(\lambda, x)\| \leq \frac{C_6}{n}(1 + \|\lambda\|)(1 + \|(\mathrm{Im}\lambda)^{-1}\|^{14}).$$

Proof. At first assume that $\lambda \in \widehat{V}_n$. Put

$$\mu_n = \Lambda_n(\lambda, x) \qquad \text{and} \qquad \mu = \begin{pmatrix} \Lambda^t & x \\ 0 & \Lambda \end{pmatrix}.$$

According to Lemma 9.14 and (9.38) we then have that

$$\widetilde{G}(\mu_u) = G_n(\lambda, x) \qquad \text{and} \qquad \widetilde{G}(\mu) = G(\lambda, x).$$

Hence,

$$\begin{array}{rcl} \|G_n(\lambda,x)-G(\lambda,x)\| & = & \|\widetilde{G}(\mu_n)-\widetilde{G}(\mu)\| \\ & \leq & \|(\mu_n\otimes\mathbf{1}_{\mathcal{A}}-\hat{s})^{-1}-(\mu\otimes\mathbf{1}_{\mathcal{A}}-\hat{s})^{-1}\| \\ & \leq & \|(\mu_n\otimes\mathbf{1}_{\mathcal{A}}-s)^{-1}\|\|\mu_n-\mu\|\|(\mu\otimes\mathbf{1}_{\mathcal{A}}-\hat{s})^{-1}\| \end{array} . \qquad (9.65)$$

By Lemma 9.10 (i),

$$\|(\mu\otimes\mathbf{1}_{\mathcal{A}}-\hat{s})^{-1}\|=\|\pi(\lambda,x)^{-1}\|\leq\widehat{C}(1+\|(\mathrm{Im}\lambda)^{-1}\|^2),$$

and by (9.51) and (9.49),

$$\|(\mu_n\otimes\mathbf{1}_{\mathcal{A}}-\hat{s})^{-1}\|\leq 2\widehat{C}(1+\|(\mathrm{Im}\lambda)^{-1}\|^2),$$

and

$$\|\mu_n-\mu\|\leq\frac{C_3}{n}(1+\|\lambda\|)(1+\|(\mathrm{Im}\lambda^{-1}\|^{10}).$$

Inserting these estimates in (9.65) we get that

$$\|G_n(\lambda,x)-G(\lambda,x)\|\leq\frac{C_6^{(1)}}{n}(1+\|\lambda\|)(1+\|(\mathrm{Im}\lambda)^{-1}\|^{14})$$

for some constant $C_6^{(1)}\geq 0$. If $\lambda\notin\widehat{V}_n$, then by (9.46),

$$1\leq\frac{C_4}{n}(1+\|\lambda\|)(1+\|(\mathrm{Im}\lambda)^{-1}\|^{10})(1+\|(\mathrm{Im}\lambda)^{-1}\|^2).$$

Therefore

$$\begin{array}{rl} & \|G_n(\lambda,x)-G(\lambda,x)\| \\ \leq & \frac{C_4}{n}(1+\|\lambda\|)(1+\|(\mathrm{Im}\lambda)^{-1}\|^{10})(1+\|(\mathrm{Im}\lambda)^{-1}\|^2)(\|G_n(\lambda,x)\|+\|G(\lambda,x)\|). \end{array}$$

Taking (9.44) and (9.45) into account we then get the estimate

$$\|G_n(\lambda,x)-G(\lambda,x)\|\leq\frac{C_6^{(2)}}{n}(1+\|\lambda\|)(1+\|(\mathrm{Im}\lambda)^{-1}\|^{14})$$

for a constant $C_6^{(2)}\geq 0$. This proves Lemma 9.15 with $C_6=\max\{C_6^{(1)},C_6^{(2)}\}$. ■

Proof of Theorem 9.6 Let $\lambda\in\mathcal{O}$ and $x\in B$. By (9.61) and (9.62),

$$G_n(\lambda,x)=\begin{pmatrix} G_n(\lambda)^t & K_n(\lambda,x) \\ 0 & G_n(\lambda)\end{pmatrix},$$

where $K_n(\lambda,x)$ is given by (9.62). Similarly,

$$G(\lambda,x)=\begin{pmatrix} G(\lambda)^t & K(\lambda,x) \\ 0 & G(\lambda)\end{pmatrix},$$

where

$$K(\lambda, x) = -(\mathrm{id}_k \otimes \tau)[(\Lambda^t \otimes \mathbf{1}_{\mathcal{A}} - s^t)^{-1}(x \otimes \mathbf{1}_{\mathcal{A}})(\Lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1}] \tag{9.66}$$

Hence, by Lemma 9.15,

$$\|K_n(\lambda, x) - K(\lambda, x)\| \leq \frac{C_6}{n}(1 + \|\lambda\|)(1 + \|(\mathrm{Im}\lambda)^{-1}\|^{14}).$$

But $K_n(\lambda, x)$ and $K(\lambda, x)$ are well-defined for all $x \in M_k(\mathbb{C})$. Moreover, since they are linear in x, it follows that

$$\|K_n(\lambda, x) - K(\lambda, x)\| \leq \frac{C_6}{n}(1 + \|\lambda\|)(1 + \|(\mathrm{Im}\lambda)^{-1}\|^{14})\|x\| \tag{9.67}$$

for all $\lambda \in \mathcal{O}$ and all $x \in M_k(\mathbb{C})$. By (9.12) and (9.23),

$$R_n(\lambda) = \sum_{j=1}^{r+s} \sum_{u,v=1}^{k} \varepsilon_j a_j e_{uv}^{(k)} K_n(\lambda, e_{uv}^{(k)} a_j),$$

and

$$R(\lambda) = \sum_{j=1}^{r+s} \sum_{u,v=1}^{k} \varepsilon_j a_j e_{uv}^{(k)} K(\lambda, e_{uv}^{(k)} a_j).$$

Hence by (9.67)

$$\|R_n(\lambda) - R(\lambda)\| \leq \frac{\widetilde{C}_4}{n}(1 + \|\lambda\|)(1 + \|(\mathrm{Im}\lambda)^{-1}\|^{14})$$

for same constant $\widetilde{C}_4 \geq 0$. ■

10 The spectrum of Q_n – the real case.

With the same notation as in the previous section, $x_1, \ldots, x_{r+s}$ is a semicircular system in a C^*-probability space $(\mathcal{A}, \tau)$ with τ faithful, $X_1^{(n)}, \ldots, X_{r+s}^{(n)}$ are stocasticly independent random matrices, for which,

$$X_1^{(n)}, \ldots, X_r^{(n)} \in \mathrm{GOE}\Big(n, \frac{1}{n}\Big) \qquad \text{and} \qquad X_{r+1}^{(n)}, \ldots, X_{r+s}^{(n)} \in \mathrm{GOE}^*\Big(n, \frac{1}{n}\Big).$$

Let $p \in M_m(\mathbb{C}) \otimes \mathbb{C} < X_1, \ldots, X_{r+s} >$ and put

$$q = p(x_1, \ldots, x_{r+s}), \quad Q_n = p(X_1^{(n)}, \ldots, X_{r+s}^{(n)}).$$

Moreover define $g, g_n : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ by

$$g(\lambda) = (\mathrm{tr}_m \otimes \tau)[(\lambda \mathbf{1}_m \otimes \mathbf{1}_{\mathcal{A}} - q)^{-1}], \qquad (\lambda \in \mathbb{C} \setminus \mathbb{R}),$$

$$g_n(\lambda) = \mathbb{E}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)[(\lambda \mathbf{1}_m \otimes \mathbf{1}_n - Q_n)^{-1}]\}, \qquad (\lambda \in \mathbb{C} \setminus \mathbb{R}).$$

Let $E = \mathbf{1}_m \oplus O_{k-m} \in M_k(\mathbb{C})$. Then for $\lambda \in \mathbb{C} \setminus \mathbb{R}$, we have

$$g(\lambda) = \frac{k}{m} \operatorname{tr}_k(EG(\lambda \mathbf{1}_m)E),$$

and

$$g_n(\lambda) = \frac{k}{m} \operatorname{tr}_k(EG_n(\lambda \mathbf{1}_m)E).$$

Now, define $l : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ by

$$l(\lambda) = \frac{k}{m} \operatorname{tr}_k(EL(\lambda \mathbf{1}_m)E), \qquad (\lambda \in \mathbb{C} \setminus \mathbb{R}).$$

Then l is analytic, and applying Theorem 9.7 we find that there is a constant $C \geq 0$ such that

$$\Big| g(\lambda) - g_n(\lambda) + \frac{1}{n} l(\lambda) \Big| \leq \frac{C}{n^2}(1 + |\lambda|)^2 (1 + |\mathrm{Im}\lambda|^{-17}) \tag{10.1}$$

when $\mathrm{Im}\lambda > 0$. Moreover, by arguing as in the proof of [S, Theorem 4.5] the inequality (10.1) also holds when $\mathrm{Im}\lambda < 0$.

10.1 Lemma. *There is a distribution* $\Delta \in \mathcal{D}'_c(\mathbb{R})$ *with* $\operatorname{supp}(\Lambda) \subseteq \sigma(q)$, *such that for any* $\phi \in C_c^\infty(\mathbb{R})$,

$$\Delta(\phi) = \lim_{y \to 0^+} \frac{\mathrm{i}}{2\pi} \int_{\mathbb{R}} \phi(x)[l(x + \mathrm{i}y) - l(x - \mathrm{i}y)] \mathrm{d}x. \tag{10.2}$$

Proof. At first we prove that l has an analytic continuation to $\mathbb{C} \setminus \sigma(q)$. We know that for any $\lambda \in \mathbb{C} \setminus \sigma(q)$, $\lambda \mathbf{1}_m \otimes \mathbf{1}_{\mathcal{A}} - q$ is invertible. Thus with $\Lambda = (\lambda \mathbf{1}_m) \oplus \mathbf{1}_{k-m} \in M_k(\mathbb{C})$ we know from lemma 3.2 that $\Lambda \otimes \mathbf{1}_{\mathcal{A}} - s$ is invertible. But then $\Lambda^t \otimes \mathbf{1}_{\mathcal{A}} - s^t$ is also invertible. It follows that $\lambda \to R(\lambda \mathbf{1}_m)$ and $\lambda \to G(\lambda \mathbf{1}_m)$ have analytic continuations to $\mathbb{C} \setminus \sigma(q)$. Moreover, $G(\lambda \mathbf{1}_m)$ is invertible for all $\lambda \in \mathbb{C} \setminus \sigma(q)$. Indeed, $\mathbb{C} \setminus \sigma(q)$ is connected, and we have seen that for all λ belonging to some open non-empty subset of $\mathbb{C} \setminus \sigma(q)$, the identity

$$\Big(a_0 + \sum_{i=1}^{r+s} a_i G(\lambda \mathbf{1}_m) a_i - \Lambda \Big) G(\lambda \mathbf{1}_m) + \mathbf{1}_k = 0. \tag{10.3}$$

holds. Then, by uniqueness of analytic continuation, (10.3) must hold for all $\lambda \in \mathbb{C} \setminus \sigma(q)$. In particular, $G(\lambda \mathbf{1}_m)$ is invertible for such λ. We conclude that l is well-defined and analytic in all of $\mathbb{C} \setminus \sigma(q)$.

The next step is to prove that l satisfies (a) and (b) of [S, Theorem 5.4]. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and put $\Lambda = (\lambda \mathbf{1}_m) \oplus \mathbf{1}_{k-m} \in M_k(\mathbb{C})$. According to the proof of lemma 3.2,

$$(\Lambda \otimes \mathbf{1}_{\mathcal{A}} - s)^{-1} = C + B(\lambda),$$

where $(Q \otimes \mathbf{1}_{\mathcal{A}})C = C(Q \otimes \mathbf{1}_{\mathcal{A}}) = 0$, and

$$\|B(\lambda)\| \leq C'_{2,1} \|(\lambda \mathbf{1}_m \otimes \mathbf{1}_{\mathcal{A}} - q)^{-1}\| \leq C'_{2,1} |\mathrm{Im}\lambda|^{-1}.$$

Moreover, if $|\lambda| > \|q\|$, then

$$\|B(\lambda)\| \leq \frac{C'_{2,1}}{|\lambda| - \|q\|}.$$

Now,

$$EL(\lambda\mathbf{1}_m)E = (\mathrm{id}_k\otimes\tau)[(E\otimes\mathbf{1}_\mathcal{A})(\Lambda\otimes\mathbf{1}_\mathcal{A}-s)^{-1}(R(\lambda\mathbf{1}_m)G(\lambda\mathbf{1}_m)^{-1}\otimes\mathbf{1}_\mathcal{A})(\Lambda\otimes\mathbf{1}_\mathcal{A}-s)^{-1}(E\otimes\mathbf{1}_\mathcal{A})],$$

implying that

$$\|EL(\lambda\mathbf{1}_m)E\| \leq C_{2,1}^2|\mathrm{Im}\lambda|^{-2}\|R(\lambda\mathbf{1}_m)G(\lambda\mathbf{1}_m)^{-1}\|, \tag{10.4}$$

and if $|\lambda| > \|q\|$, then

$$\|EL(\lambda\mathbf{1}_m)E\| \leq \left(\frac{C_{2,1}}{|\lambda| - \|q\|}\right)^2 \|R(\lambda\mathbf{1}_m)G(\lambda\mathbf{1}_m)^{-1}\|. \tag{10.5}$$

We have seen that

$$\|R(\lambda\mathbf{1}_m)\| \leq k^2 \sum_{i=1}^{r+s} \|a_i\|^2\|(\Lambda\otimes\mathbf{1}_\mathcal{A} - s)^{-1}\|^2, \tag{10.6}$$

where

$$\|(\Lambda\otimes\mathbf{1}_\mathcal{A} - s)^{-1}\| \leq C'_{1,1} + C'_{2,1}|\mathrm{Im}\lambda|^{-1}, \tag{10.7}$$

and if $|\lambda| > \|q\|$, then

$$\|(\Lambda\otimes\mathbf{1}_\mathcal{A} - s)^{-1}\| \leq C'_{1,1} + \frac{C'_{2,1}}{|\lambda| - \|q\|}. \tag{10.8}$$

Also, there is a constant $C_1 \geq 0$ such that

$$\begin{aligned}
\|G(\lambda\mathbf{1}_m)^{-1}\| &\leq \|\Lambda\| + \|a_0\| + \sum_{i=1}^{r+s}\|a_i\|\|G(\lambda\mathbf{1}_m)\|\|a_i\| & (10.9)\\
&\leq \|\Lambda\| + \|a_0\| + \sum_{i=1}^{r+s}\|a_i\|\|(\Lambda\otimes\mathbf{1}_\mathcal{A} - s)^{-1}\|\|a_i\| & (10.10)\\
&\leq C_1(1+|\lambda|)(1+|\mathrm{Im}\lambda|^{-1}), & (10.11)
\end{aligned}$$

and if $|\lambda| > \|q\|$, then

$$\|G(\lambda\mathbf{1}_m)^{-1}\| \leq C_1(1+|\lambda|)\left(1 + \frac{1}{|\lambda| - \|q\|}\right). \tag{10.12}$$

(10.5), (10.6), (10.8) and (10.12) imply that

$$|l(\lambda)| \leq O\Big(\frac{1}{|\lambda|}\Big) \quad \text{as} \quad |\lambda| \to \infty. \tag{10.13}$$

Combining (10.4), (10.6), (10.7) and (10.11) we find that for some constant $C_2 \geq 0$,

$$|l(\lambda)| \leq \|EL(\lambda \mathbf{1}_m)E\| \leq C_2(1+|\lambda|)(|\mathrm{Im}\lambda|^{-2} + |\mathrm{Im}\lambda|^{-5}). \tag{10.14}$$

Choose $a, b \in \mathbb{R}$, $a < b$, such that $\sigma(q) \subseteq [a, b]$. Put $K = [a-1, b+1]$ and

$$D = \{\lambda \in \mathbb{C} | 0 < \mathrm{dist}(\lambda, K) \leq 1\}.$$

By (10.14), there is a constant $C_3 \geq 0$ such that for any $\lambda \in D$,

$$|l(\lambda)| \leq C_3 \cdot \max\{1, (\mathrm{dist}(\lambda, K))^{-5}\} = C_3 \cdot (\mathrm{dist}(\lambda, K))^{-5}.$$

(10.13) implies that l is bounded on $\mathbb{C} \setminus D$. Therefore C_3 may be chosen such that for all $\lambda \in \mathbb{C} \setminus \sigma(q)$

$$|l(\lambda)| \leq C_3 \cdot \max\{1, (\mathrm{dist}(\lambda, K))^{-5}\}. \tag{10.15}$$

By (10.13) and (10.15), l satisfies (a) and (b) of [S, Theorem 5.4], and the lemma follows. ■

Knowing that (10.1) holds we are now able to prove:

10.2 Theorem. *Let $\phi \in C_c^\infty(\mathbb{R})$. Then*

$$\mathbb{E}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\phi(Q_n)\} = (\mathrm{tr}_m \otimes \tau)\phi(q) + \frac{1}{n}\Delta(\phi) + O\Big(\frac{1}{n^2}\Big).$$

Proof. The result follows from a simple modification of the proof of [S, Theorem 5.6]. ■

10.3 Lemma. *Let $n \in \mathbb{N}$, and let $X_n \in \mathrm{GOE}(n, \frac{1}{n}) \cup \mathrm{GOE}^*(n, \frac{1}{n})$. Then*

(i) for all $\varepsilon > 0$, $P(\|X_n\| > \sqrt{2}(2+\varepsilon)) < 2n \exp\Big(-\frac{n\varepsilon^2}{2}\Big)$,

(ii) there is a sequence of constants not depending on n, $(\gamma'(k))_{k=1}^\infty$, such that for all $k \in \mathbb{N}$,

$$\mathbb{E}\{1_{(\|X_n\| > 3\sqrt{2})}\|X_n\|^k\} \leq \gamma'(k) n \mathrm{e}^{-\frac{n}{2}}.$$

Proof. We may assume that $X_n = \frac{1}{\sqrt{2}}(Y_n + \overline{Y})$ or $X_n = \frac{1}{\sqrt{2}\mathrm{i}}(Y_n - \overline{Y})$ for some $Y_n \in \mathrm{SGRM}(n, \frac{1}{n})$. Hence, by [S, Proof of Lemma 6.4],

$$P(\|X_n\| > \sqrt{2}(2+\varepsilon)) \leq P(\|Y_n\| > 2+\varepsilon) \leq 2n \exp\Big(-\frac{n\varepsilon^2}{2}\Big)$$

holds for all $\varepsilon > 0$. Then by application of the proof of Proposition 6.4 to the random variable $\frac{1}{\sqrt{2}}\|X_n\|$ we get that

$$\mathbb{E}\Big\{1_{\big(\frac{1}{\sqrt{2}}\|X_n\| > 3\big)}\Big(\frac{1}{\sqrt{2}}\|X_n\|\Big)^k\Big\} \leq \gamma(k) n \mathrm{e}^{-\frac{n}{2}}.$$

Hence (ii) holds with $\gamma'(k) = 2^{k/2}\gamma(k)$. ■

10.4 Lemma. *Let $\Delta \in \mathcal{D}'_c(\mathbb{R})$ be as in Lemma 10.1. Then $\Delta(1) = 0$.*

Proof. With $d = \deg(p)$, $x_0 = \mathbf{1}_{\mathcal{A}}$, and $X_0^{(n)} = \mathbf{1}_n$ we may choose $c_{i_1,\dots,i_d} \in M_m(\mathbb{C})$, $0 \le i_1, \dots, i_d \le r+s$, such that

$$q = p(x_1, \dots, x_{r+s}) = \sum_{0 \le i_1,\dots,i_d \le r+s} c_{i_1,\dots,i_d} \otimes x_{i_1} \cdots x_{i_d}$$

and

$$Q_n = p(X_1^{(n)}, \dots, X_{r+s}^{(n)}) = \sum_{0 \le i_1,\dots,i_d \le r+s} c_{i_1,\dots,i_d} \otimes X_{i_1}^{(n)} \cdots X_{i_d}^{(n)}.$$

Put

$$R = (3\sqrt{2})^d \sum_{0 \le i_1,\dots,i_d \le r+s} \|c_{i_1,\dots,i_d}\|.$$

Then

$$(\|Q_n\| > R) \subseteq \Big(\sum_{0 \le i_1,\dots,i_d \le r+s} \|c_{i_1,\dots,i_d}\| \|X_{i_1}^{(n)}\| \cdots \|X_{i_d}^{(n)}\| > R \Big) \subseteq \bigcup_{i=1}^{r+s} (\|X_i^{(n)}\| > 3\sqrt{2}),$$

implying that

$$P(\|Q_n\| > R) \le r \cdot P(\|X_1^{(n)}\| > 3\sqrt{2}) + s \cdot P(\|X_{r+1}^{(n)}\| > 3\sqrt{2}).$$

Now, by Lemma 10.3,

$$P(\|X_i^{(n)}\| > 3\sqrt{2}) \le 2n \cdot \exp\Big(-\frac{n}{2}\Big), \qquad (i = 1, \dots, r+s),$$

and thus

$$P(\|Q_n\| > R) \le 2(r+s)n \cdot \exp\Big(-\frac{n}{2}\Big).$$

Consequently,

$$\mathbb{E}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n) 1_{]-\infty,R[\cup]R,\infty[}(Q_n)\} \le P(\|Q_n\| > R) \le 2(r+s)n \cdot \exp\Big(-\frac{n}{2}\Big). \qquad (10.16)$$

Now, let $\phi \in C_c^\infty(\mathbb{R})$ such that $0 \le \phi \le 1$ and $\phi|_{[-R,R]} = 1$. Then $\phi(x) = 1$ for all x in a neighbourhood of $\sigma(q) \supseteq \mathrm{supp}(\Lambda)$. Hence $\Delta(\phi) = \Delta(1)$, and we have that

$$\mathbb{E}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\phi(Q_n)\} = 1 + \frac{1}{n}\Delta(1) + O\Big(\frac{1}{n^2}\Big),$$

where

$$\mathbb{E}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\phi(Q_n)\} = 1 + \mathbb{E}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)(\phi - 1)(Q_n)\}$$

and

$$|\mathbb{E}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)(\phi - 1)(Q_n)\}| \leq \mathbb{E}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)1_{]-\infty,R[\cup]R,\infty[}(Q_n)\} \leq 2(r+s)n \cdot \exp\Big(-\frac{n}{2}\Big).$$

Altogether we have that

$$\mathbb{E}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)(\phi - 1)(Q_n)\} = \frac{1}{n}\Delta(1) + O\Big(\frac{1}{n^2}\Big), \tag{10.17}$$

where the left hand side is of the order $n \cdot \exp(-\frac{n}{2}) = O\Big(\frac{1}{n^2}\Big)$. Hence, the $\frac{1}{n}$-term appearing on the right hand side of (10.17) must be zero. ■

10.5 Proposition. *Let $\phi \in C^\infty(\mathbb{R},\mathbb{R})$ such that ϕ is constant outside a compact subset of $\mathbb{R}$. Suppose that*

$$\mathrm{supp}(\phi) \cap \sigma(q) = \emptyset.$$

Then

$$\mathbb{V}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\phi(Q_n)\} = O(n^{-4}),$$

and

$$P(|(\mathrm{tr}_m \otimes \mathrm{tr}_n)\phi(Q_n)| \leq n^{-\frac{4}{3}}, \text{ eventually as } n \to \infty) = 1.$$

Proof. Taking Lemma 10.3 (ii) into account this result follows as in the complex case (cf. proof of Theorem 6.2). ■

Taking Theorem 10.2, Lemma 10.4, Proposition 10.5 and Proposition 6.3 into account, we find, as in Section 6:

10.6 Theorem. *Let $\varepsilon > 0$. Then for almost every $\omega \in \Omega$,*

$$\sigma(Q_n(\omega)) \subseteq \sigma(q) +]-\varepsilon, \varepsilon[,$$

eventually as $n \to \infty$.

10.7 Remark. By [S, Section 7] and the remarks in the beginning of section 9, Theorem 10.2, Proposition 10.5 and Theorem 10.6 can easily be generalized to the symplectic case i.e. to the case where $X_1^{(n)}, \dots, X_{r+s}^{(n)}$ are stochasticly independent random matrices for which

$$X_1^{(n)}, \dots, X_r^{(n)} \in \mathrm{GSE}\Big(n, \frac{1}{n}\Big) \qquad \text{and} \qquad X_{r+1}^{(n)}, \dots, X_{r+s}^{(n)} \in \mathrm{GSE}^*\Big(n, \frac{1}{n}\Big).$$

11 Gaps in the spectrum of q — the real case

In this section we shall prove that Theorem 8.2 holds in the GOE $\cup$ GOE*-case as well. That is, if $p \in (M_m(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \dots, X_{r+s}\rangle)_{\text{sa}}$, if $x_1, \dots, x_{r+s}$ is a semicircular system in $(\mathcal{A}, \tau)$, and if for each $n \in \mathbb{N}$, $X_1^{(n)}, \dots, X_{r+s}^{(n)}$ are stochastically independent random matrices from $\text{GOE}(n, \frac{1}{n}) \cup \text{GOE}^*(n, \frac{1}{n})$ as in Section 9, then with $q = p(x_1, \dots, x_{r+s})$ and $Q_n = p(X_1^{(n)}, \dots, X_{r+s}^{(n)})$ we have:

11.1 Theorem. *Let ε_0 denote the smallest distance between disjoint connected components of $\sigma(q)$, let $\mathfrak{J}$ be a connected component of $\sigma(q)$, let $0 < \varepsilon < \frac{1}{3}\varepsilon_0$, and let $\mu_q \in \text{Prob}(\mathbb{R})$ denote the distribution of q w.r.t. $\text{tr}_m \otimes \tau$. Then $\mu_q(\mathfrak{J}) = \frac{k}{m}$ for some $k \in \{1, \dots, m\}$, and for almost all $w \in \Omega$, the number of eigenvalues of $Q_n(w)$ in $\mathfrak{J}+]-\varepsilon, \varepsilon[$ is $k \cdot n$, eventually as $n \to \infty$.*

Proof. Take $\phi \in C_c^\infty(\mathbb{R})$, such that $\phi_{|\mathfrak{J}+]-\varepsilon,\varepsilon[} = 1$ and $\phi_{|\mathbb{R}\setminus(\mathfrak{J}+]-2\varepsilon,2\varepsilon[)} = 0$. Then $\phi(q)$ is a non-zero projection in $M_m(C^*(\mathbf{1}_\mathcal{A}, x_1, \dots, x_n))$ and hence, by Theorem 7.1,

$$\mu_q(\mathfrak{J}) = \int_\mathbb{R} \varphi \, d\mu_q = (\text{tr}_m \otimes \tau)\phi(q) = \frac{k}{m} \tag{11.1}$$

for some $k \in \{1, \dots, m\}$. By Theorem 10.6 there is a P-null set $N \subset \Omega$, such that for all $\omega \in \Omega \backslash N$

$$\sigma(Q_n(\omega)) \subseteq \sigma(q)+] - \varepsilon, \varepsilon[,$$

eventually as $n \to \infty$. In particular, for all $\omega \in \Omega\backslash N$ there exists $N(\omega) \in \mathbb{N}$ such that $\phi(\mathbb{Q}_n(\omega))$ is a projection for all $n \geq N(w)$. For $\omega \in \Omega\backslash N$ and $n \geq N(\omega)$ take $K_n(\omega) \in \{0, \dots, m \cdot n\}$, such that

$$(\text{tr}_m \otimes \text{tr}_n)\varphi(Q_n(\omega)) = \frac{K_n(\omega)}{m \cdot n}. \tag{11.2}$$

Let $\Delta \colon C_c^\infty(\mathbb{R}) \to \mathbb{C}$ be the distribution from Lemma 10.1, and put

$$Z_n = (\text{tr}_m \otimes \text{tr}_n)\phi(Q_n) - (\text{tr}_m \otimes \tau)\phi(q) - \frac{1}{n}\Delta(\phi).$$

Then by Theorem 10.2, $\mathbb{E}(Z_n) = O(\frac{1}{n^2})$. Moreover, since φ' vanishes in a neighbourhood of $\sigma(q)$, we get as in the proof of Theorem 6.2 that

$$\mathbb{V}(Z_n) = O(\tfrac{1}{n^4})$$

and

$$Z_n = O(n^{-\frac{4}{3}}) \quad \text{almost surely.}$$

Hence there exists a P-null set $N' \subseteq N$, such that

$$(\text{tr}_m \otimes \text{tr}_n)\phi(Q_n(\omega)) = (\text{tr}_m \otimes \tau)\phi(q) + \frac{1}{n}\Delta(\phi) + O(n^{-\frac{4}{3}})$$

holds for all $\omega \in \Omega \setminus N'$.

Taking (11.1) and (11.2) into account, we get after multiplication by mn that for $\omega \in \Omega \backslash N$ and $n \geq N(\omega)$,

$$K_n(\omega) = nk + m\Delta(\varphi) + O(n^{-\frac{1}{3}}). \tag{11.3}$$

Therefore there exists $C > 0$ such that $\mathrm{dist}(m\Delta(\varphi), \mathbb{Z}) \leq C\, n^{-\frac{1}{3}}$ for all $n \geq N(\omega)$, which implies that $m\Delta(\varphi) \in \mathbb{Z}$.

We next use an argument based on homotopy to show that $\Delta(\varphi) = 0$. By definition

$$Q_n = p(X_1^{(n)}, \dots, X_{r+s}^{(n)})$$

where $X_1^{(n)}, \dots, X_r^{(n)} \in \mathrm{GOE}(n, \frac{1}{n})$, $X_{r+1}^{(n)}, \dots, X_{r+s}^{(n)} \in \mathrm{GOE}^*(n, \frac{1}{n})$ form a set of $r+s$ independent random matrices. We may without loss of generality assume that there exist $Y_1^{(n)}, \dots, Y_r^{(n)} \in \mathrm{GOE}^*(n, \frac{1}{n})$ and $Y_{r+1}^{(n)}, \dots, Y_{r+s}^{(n)} \in \mathrm{GOE}(n, \frac{1}{n})$ such that $X_1^{(n)}, \dots, X_{r+s}^{(n)}$, $Y_1^{(n)}, \dots, Y_{r+s}^{(n)}$ form a set of $2(r+s)$ independent random matrices. For $j = 1, \dots, r+s$ put

$$X_j^{(n)}(t) = \cos t\, X_j^{(n)} + \sin t\, Y_j^{(n)}, \quad (0 \leq t \leq \tfrac{\pi}{4}).$$

It is a simple observation that if $Z \in \mathrm{GOE}(n, \frac{1}{n})$ and $W \in \mathrm{GOE}^*(n, \frac{1}{n})$, then $\frac{1}{\sqrt{2}}(Z+W) \in \mathrm{SGRM}(n, \frac{1}{n})$. Hence

$$(X_1^{(n)}(t), \dots, X_{r+s}^{(n)}(t)), \quad (0 \leq t \leq \tfrac{\pi}{4})$$

defines a path which connects the given set of random matrices $X_1^{(n)}, \dots, X_{r+s}^{(n)}$ (at $t = 0$) with a set of $r+s$ independent $\mathrm{SGRM}(n, \frac{1}{n})$ random matrices (at $t = \frac{\pi}{4}$). Put

$$Q_n(t) = q(X_1^{(n)}(t), \dots, X_{r+s}^{(n)}(t)), \quad (0 \leq t \leq \tfrac{\pi}{4}).$$

Let $x_1, \dots, x_{r+s}, y_1, \dots, y_{r+s}$ be a semicircular system in a C^*-probability space $(\mathcal{A}, \tau)$ with τ faithful. Put

$$x_j(t) = \cos t\, x_j + \sin t\, y_j, \quad (0 \leq t \leq \tfrac{\pi}{4}).$$

Since an orthogonal transformation of a semicircular system is again a semicircular system (cf. [VDN, Proposition 5.12]), $x_1(t), \dots, x_{r+s}(t)$ is a semicircular system for each $t \in [0, \frac{\pi}{4}]$. Hence the operators

$$q(t) = q(z_1(t), \dots, x_{r+s}(t)), \quad (0 \leq t \leq \tfrac{\pi}{4})$$

form a norm continuous path in $\mathcal{A}$ for which $\sigma(q(t)) = \sigma(q)$. Moreover

$$(\lambda, t) \to (\lambda \mathbf{1}_m \otimes \mathbf{1}_{\mathcal{A}} - q(t))^{-1} \tag{11.4}$$

is norm continuous on $(\mathbb{C} \backslash \sigma(q)) \times [0, \frac{\pi}{4}]$. For $t \in [0, \frac{\pi}{4}]$, $Q_n(t)$ can be expressed as a polynomial in $X_1^{(n)}, \dots, X_{r+s}^{(n)}, Y_1^{(n)}, \dots, Y_{r+s}^{(n)}$, and $q(t)$ can be expressed as the same polynomial in $x_1, \dots, x_{r+s}, y_1, \dots, y_{r+s}$. Hence, by Lemma 10.1 and Theorem 10.2, there exists for each $t \in [0, \frac{\pi}{4}]$ a distribution $\Lambda_t \colon C_r^\infty(\mathbb{C}) \to \mathbb{C}$, such that for all $\psi \in C_c^\infty(\mathbb{R})$:

$$\mathbb{E}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\psi(Q_n(t))\} = (\mathrm{tr}_m \otimes \tau)\psi(q(t)) + \frac{1}{n}\Lambda_t(\psi) + O(\tfrac{1}{n^2}).$$

Since $\sigma(q(t)) = \sigma(q)$, $0 \le t \le \frac{\pi}{4}$, we get by the first part of this proof that

$$m\Delta_t(\varphi) \in \mathbb{Z}, \quad (0 \le t \le \tfrac{\pi}{4}), \tag{11.5}$$

where φ is the function chosen in the beginning of the proof. Moreover, by Theorem 6.1,

$$\mathbb{E}\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\varphi(Q_n(\tfrac{\pi}{4})\} = (\mathrm{tr}_m \otimes \tau)\varphi(q(\tfrac{\pi}{4})) + O(\tfrac{1}{n^2}),$$

which implies that

$$\Delta_{\pi/4}(\varphi) = 0. \tag{11.6}$$

We next prove that $t \to \Delta_t(\varphi)$ is a continuous function:
Let

$$\ell_t(\lambda) = \Delta_t\Big(\frac{1}{\lambda - x}\Big), \quad (\lambda \in \mathbb{C}\backslash\sigma(q))$$

be the Stieltjes transformation of Δ_t, $0 \le t \le \frac{\pi}{4}$ (cf. [S, Lemma 5.4]). By a simple modification of the proof of [S, Lemma 5.6], we get that

$$\Delta_t(\varphi) = \frac{1}{2\pi i}\int_{\partial R} \ell_t(\lambda) d\lambda \tag{11.7}$$

where ∂R is the boundary of the rectangle

$$R = (\mathcal{J} + [-\varepsilon, \varepsilon]) \times [-1, 1]$$

with counter clockwise orientation. Since $(\lambda, t) \to (\lambda\mathbf{1}_m \otimes \mathbf{1}_{\mathcal{A}} - q(t))^{-1}$ is norm continuous on $(\mathbb{C}\backslash\sigma(q)) \times [0, \frac{\pi}{4}]$, it is obvious from the explicit formula for the Stieltjes transform $\ell_t(\lambda)$ (cf. (10.2)) that $(\lambda, t) \to \ell_t(\lambda)$ is a continuous function on $(\mathbb{C}\backslash\sigma(q)) \times [0, \frac{\pi}{4}]$. Hence by (11.7), $\Delta_t(\varphi)$ is a continous funtion of $t \in [0, \frac{\pi}{4}]$. Together with (11.5) and (11.6) this shows that $\Delta(\varphi) = \Delta_0(\varphi) = 0$. Hence by (11.3) we have for all $\omega \in \Omega \setminus N'$ and all $n \ge N(\omega)$ that

$$K_n(\omega) = nk + O(n^{-\frac{1}{3}}),$$

and since $K_n(\omega) \in \mathbb{N}$, it follows that $k_n(\omega) = nk$ eventually as $n \to \infty$. □

11.2 Remark. Using again [S, Section 7], Theorem 11.1 can also be generalized to the symplectic case (cf. remark 10.7).

References

[BS1] Z.D. Bai and J.W. Silverstein, No eigenvalues outside the support of the limiting spectral distribution of large dimensional sample matrices. Annals Prob. 26 (1998), 316–345.

[BS2] Z.D. Bai and J.W. Silverstein, Exact separation of eigenvalues of large dimensional sample covariance matrices. Annals Prob. 27 (1999), 1536–1555.

[CF] J.M. Cohen and A. Figa-Talamanea, Idempotents in the reduced C^*-algebra of a free group. Proc. Amer. Math. Soc. 103 (1988), 779–782.

[Co] A. Connes, Non commutative differential geometry. Publ. Math. Inst. Hautes Etudes Sci. 62 (1985), 257–360.

[Cu1] J. Cuntz, K-theoretical amenability for discrete groups, J. Reine Angew. Math. 344 (1983), 181–195.

[Cu2] J. Cuntz, Personal communication (1986).

[Da] K.H. Davidson, The distance between unitary orbits of normal operators, Acta Sci. Math. 50 (1986), 213–223.

[Fe] W. Feller, An Introduction to Probability Theory and its Applications, Volume II, Second Edition. John Wiley & Sons (1971).

[HT] U. Haagerup and S. Thorbjørnsen, A new application of random matrices: $\mathrm{Ext}(C^*_{\mathrm{red}}(F_2))$ is not a group. Preprint SDU (2002). To appear in Ann. Math.

[L] E.C. Lance, K-theory for certain group C^*-algebras, Acta Math. 151 (1983), 209–230.

[Le] F. Lehner, Computing norms of free operators with matrix coefficients, Amer. J. Math. 121 (1999), 453–486.

[PV] M. Pimsner and D. Voiculescu, K-groups of reduced crossed products by free groups, J. Operator Theory 8 (1982), 131–156.

[S] H. Schultz, Non-commutative polynomials of independent Gaussian random matrices. The real and symplectic cases. To appear in Probability Theory and Related Fields. (Available at `http://www.imada.sdu.dk/~schultz/master.pdf`)

[T] S. Thorbjørnsen, Mixed moments of Voiculescu's Gaussian random matrices, Journ. Funct. Anal. 176 (2000), 213–246.

[V1] D. Voiculescu, Circular and semicircular systems and free product factors, "Operator Algebras, Unitary representations, Algebras, and Invariant Theory", Progress in Math. Vol. 92, Birkhäuser, 1990, 45–60.

[V2] D. Voiculescu, Limit laws for random matrices and free products, Inventiones Math. 104 (1991), 201–220.

[V3] D. Voiculescu, The K-groups of the C^*-algebra of a semicircular family. K-theory 7 (1993), 5–7.

[V4] D. Voiculescu, Operations on certain non-commutative random variables, "Recent advandces in operator algebras, Orleans 1992", Asterique 232 (1995), 243–275.

[VDN] D. Voiculescu, K. Dykema and A. Nica, Free Random Variables, CMR Monograph Series 1, American Mathematical Society, 1992.

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