Radii of convexity of some Lommel and Struve functions

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Abstract

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The talk is based on a joint work [`arXiv.1410.5217`] with Nihat Yağmur:

1. Á. Baricz, N. Yağmur: Geometric properties of some Lommel and Struve functions. *Ramanujan J.* (in press).

• Let $\mathbb{D}_r$ be the open disk $\{z \in \mathbb{C} : |z| < r\}$, where $r > 0$. As usual, with $\mathcal{A}$ we denote the class of analytic functions $f : \mathbb{D}_r \to \mathbb{C}$ which satisfy the usual normalization conditions $f(0) = f'(0) - 1 = 0$. Let us denote by $\mathcal{S}$ the class of functions belonging to $\mathcal{A}$ which are univalent in $\mathbb{D}_r$ and let $\mathcal{K}(\alpha)$ be the subclass of $\mathcal{S}$ consisting of functions which are convex of order α in $\mathbb{D}_r$, where $0 \leq \alpha < 1$.

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• The analytic characterization of this class of functions is

$$\mathcal{K}(\alpha) = \left\{ f \in \mathcal{S} \ : \ \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \ \text{ for all } \ z \in \mathbb{D}_r \right\},$$

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• The real number

$$r_\alpha^c(f) = \sup\left\{ r > 0 \ : \ \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \ \text{ for all } \ z \in \mathbb{D}_r \right\},$$

is called the radius of convexity of order α of the function f. It is worth mentioning that $r^c(f) = r_0^c(f)$ is the largest radius such that the image region $f(\mathbb{D}_{r^c(f)})$ is a convex domain in $\mathbb{C}$.

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$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\, {}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; -\frac{z^2}{4}\right),$$

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Observe that

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A common feature of these functions is that they are solutions of inhomogeneous Bessel differential equations. Indeed, the Lommel function of the first kind $s_{\mu,\nu}$ is a solution of

$$z^2w''(z) + zw'(z) + (z^2-\nu^2)w(z) = z^{\mu+1}$$

while the Struve function $\mathbf{H}_\nu$ obeys

$$z^2w''(z) + zw'(z) + (z^2-\nu^2)w(z) = \frac{4\left(\frac{z}{2}\right)^{\nu+1}}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}.$$

• In 1970 Steinig investigated the real zeros of the Struve function $\mathbf{H}_{\nu}$, while in 1972 he examined the sign of $s_{\mu,\nu}(z)$ for real μ, ν and positive z. He showed, among other things, that for $\mu < \frac{1}{2}$ the function $s_{\mu,\nu}$ has infinitely many changes of sign on $(0, \infty)$.

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• Motivated by those results, in this talk we are interested on the radii of convexity of certain analytic functions related to the classical special functions under discussion. Since neither $s_{\mu-\frac{1}{2},\frac{1}{2}}$, nor $\mathbf{H}_\nu$ belongs to $\mathcal{A}$, first we perform some natural normalizations.

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Clearly the functions f_μ, g_μ, h_μ, u_ν, v_ν and w_ν belong to the class $\mathcal{A}$.

Lemma (Á. Baricz, S. Koumandos)

Let

$$\varphi_k(z) = {}_1F_2\left(1; \frac{\mu-k+2}{2}, \frac{\mu-k+3}{2}; -\frac{z^2}{4}\right)$$

where $z \in \mathbb{C}$, $\mu \in \mathbb{R}$ *and* $k \in \{0, 1, \dots\}$ *such that* $\mu - k$ *is not in* $\{0, -1, \dots\}$. *Then,* φ_k *is an entire function of order* $\rho = 1$.

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$$\varphi_k(z) = \prod_{n\geq 1}\left(1 - \frac{z^2}{z_{\mu,k,n}^2}\right),$$

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where $z_{\mu,k,n}$ *is the nth positive zero of the function* φ_k *and the infinite product is absolutely convergent. Moreover, for* z, μ *and* k *as above, we have*

$$(\mu - k + 1)\varphi_{k+1}(z) = (\mu - k + 1)\varphi_k(z) + z\varphi_k'(z),$$

$$\sqrt{z}s_{\mu-k-\frac{1}{2},\frac{1}{2}}(z) = \frac{z^{\mu-k+1}}{(\mu - k)(\mu - k + 1)}\varphi_k(z).$$

Lemma (Á. Baricz, S. Ponnusamy and S. Singh)

If $|\nu| \leq \frac{1}{2}$, *then the Hadamard factorization of the real entire function* $\mathcal{H}_\nu : \mathbb{R} \to (-\infty, 1]$, *defined by* $\mathcal{H}_\nu(x) = \sqrt{\pi} 2^\nu x^{-\nu-1} \Gamma\left(\nu + \frac{3}{2}\right) \mathbf{H}_\nu(x)$,

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where $h_{\nu,n}$ stands for the nth positive zero of the Struve function $\mathbf{H}_\nu$. The above infinite product is absolutely convergent and if $|\nu| \leq \frac{1}{2}$ and $x \neq h_{\nu,n}$, $n \in \{1, 2, \dots\}$, then the Mittag-Leffler expansion of the Struve function $\mathbf{H}_\nu$ is as follows

$$\frac{\mathbf{H}_{\nu-1}(x)}{\mathbf{H}_\nu(x)} = \frac{2\nu + 1}{x} + \sum_{n \geq 1} \frac{2x}{x^2 - h_{\nu,n}^2}. \tag{2}$$

• We also recall that by definition the real entire function ϕ, defined by

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$$\phi(z) = cz^d e^{-\alpha z^2+\beta z}\prod_{n=1}^{\omega}\left(1-\frac{z}{z_n}\right)e^{\frac{z}{z_n}}, \quad 0\leq\omega\leq\infty,$$

where c and β are real, z_n's are real and nonzero for all $n\in\{1,2,\ldots,\omega\}$, $\alpha\geq 0$, d is a nonnegative integer and $\sum_{n=1}^{\omega} z_i^{-2} < \infty$. If $\omega = 0$, then, by convention, the product is defined to be 1.

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Lemma (Á. Baricz, N. Yağmur)

The zeros of the Lommel function $s_{\mu-\frac{1}{2},\frac{1}{2}}$ and its derivative interlace when $\mu\in(-1,1)$, $\mu\neq 0$.

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The zeros of the function $\mathbf{H}_\nu$ and its derivative interlace when $|\nu|\leq\frac{1}{2}$.

Proof.

It is known that the zeros of $\mathbf{H}_\nu$ are all real and simple when $|\nu| \leq \frac{1}{2}$, see Steinig.

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and thus the Laguerre inequality (3) for $n = 1$ is equivalent to

$$\pi 2^{2\nu} x^{-2\nu-4}\left[x^2\left(\mathbf{H}_\nu'(x)\right)^2 - x^2\mathbf{H}_\nu(x)\mathbf{H}_\nu''(x) - (\nu+1)\left(\mathbf{H}_\nu(x)\right)^2\right] > 0.$$

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This implies that

$$\left(\mathbf{H}_\nu'(x)\right)^2 - \mathbf{H}_\nu(x)\mathbf{H}_\nu''(x) > \frac{(\nu+1)\left(\mathbf{H}_\nu(x)\right)^2}{x^2} > 0 \tag{4}$$

for $|\nu| \leq \frac{1}{2}$ and $x \in \mathbb{R}$, that is, the function $x \longmapsto \mathbf{H}_\nu'(x)/\mathbf{H}_\nu(x)$ is decreasing on $(0,\infty)\setminus\{h_{\nu,n} : n \in \mathbb{N}\}$.

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Let $\mu \in (-1, 1) \setminus \{0\}$, $\xi_{\mu,1}$ *and* $\xi'_{\mu,1}$ *be the first positive zeros of* $s_{\mu-\frac{1}{2},\frac{1}{2}}$ *and* $s'_{\mu-\frac{1}{2},\frac{1}{2}}$.

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$$1 - \nu + \sqrt{r}\frac{(2-\nu)\mathbf{H}'_\nu(\sqrt{r}) + \sqrt{r}\mathbf{H}''_\nu(\sqrt{r})}{(1-\nu)\mathbf{H}_\nu(\sqrt{r}) + \sqrt{r}\mathbf{H}'_\nu(\sqrt{r})} = 2\alpha.$$

Moreover, we have the inequalities $r^c_\alpha(u_\nu) < h'_{\nu,1} < h_{\nu,1}$,

Theorem (Á. Baricz, N. Yağmur)

Let $|\nu| \leq \frac{1}{2}$. Moreover, let $h_{\nu,1}$ and $h'_{\nu,1}$ denote the first positive zeros of $\mathbf{H}_\nu$ and $\mathbf{H}'_\nu$, respectively. Then, the following assertions are true:

1. *The radius of convexity of order α of the function u_ν is the smallest positive root of the equation*
$$1 + \frac{r\,\mathbf{H}''_\nu(r)}{\mathbf{H}'_\nu(r)} + \left(\frac{1}{\nu+1} - 1\right)\frac{r\,\mathbf{H}'_\nu(r)}{\mathbf{H}_\nu(r)} = \alpha.$$
2. *The radius of convexity of order α of the function v_ν is the smallest positive root of the equation*
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$$1 - \nu + \sqrt{r}\frac{(2-\nu)\mathbf{H}'_\nu(\sqrt{r}) + \sqrt{r}\mathbf{H}''_\nu(\sqrt{r})}{(1-\nu)\mathbf{H}_\nu(\sqrt{r}) + \sqrt{r}\mathbf{H}'_\nu(\sqrt{r})} = 2\alpha.$$

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Proof.

If $|\nu| \leq \frac{1}{2}$, then the Hadamard factorization of the transcendental entire function $\mathcal{H}_\nu$, defined by

$$\mathcal{H}_\nu(z) = \sqrt{\pi} 2^\nu z^{-\nu-1} \Gamma\left(\nu + \frac{3}{2}\right) \mathbf{H}_\nu(z),$$

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where $h_{\nu,n}$ stands for the nth positive zero of the Struve function $\mathbf{H}_\nu$.

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where $h_{\nu,n}$ stands for the nth positive zero of the Struve function $\mathbf{H}_\nu$. The infinite sum representation of the function $\mathbf{H}_\nu'$ is

$$\mathbf{H}_\nu'(z) = \frac{(\nu+1) z^\nu}{\sqrt{\pi} 2^\nu \Gamma\left(\nu + \frac{3}{2}\right)} \sum_{n \geq 0} \frac{(-1)^n (2n + \nu + 1)}{(\nu+1) 2^{2n} \left(\frac{3}{2}\right)_n \left(\nu + \frac{3}{2}\right)_n} z^{2n}, \tag{6}$$

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$$\mathbf{H}'_\nu(z) = \frac{(\nu+1)z^\nu}{\sqrt{\pi}2^\nu\Gamma\left(\nu+\frac{3}{2}\right)}\sum_{n\geq 0}\frac{(-1)^n(2n+\nu+1)}{(\nu+1)2^{2n}\left(\frac{3}{2}\right)_n\left(\nu+\frac{3}{2}\right)_n}z^{2n}, \tag{6}$$

which can be rewritten as

$$\frac{\sqrt{\pi}2^\nu\Gamma\left(\nu+\frac{3}{2}\right)}{(\nu+1)}z^{-\nu}\mathbf{H}'_\nu(z) = 1 + \sum_{n\geq 1}\frac{(-1)^n(2n+\nu+1)}{(\nu+1)2^{2n}\left(\frac{3}{2}\right)_n\left(\nu+\frac{3}{2}\right)_n}z^{2n}.$$

Proof.

The above entire function is of growth order $\rho = \frac{1}{2}$ since as $n \longrightarrow \infty$

$$\frac{n \log n}{\log(\nu+1) + 2n\log 2 + \log\left(\frac{3}{2}\right)_n + \log\left(\nu + \frac{3}{2}\right)_n - \log(2n+\nu+1)} \longrightarrow \frac{1}{2}.$$

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Applying Hadamard's Theorem we can write the infinite product representation of $\mathbf{H}'_\nu$ as follows

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where $h'_{\nu,n}$ denotes the nth positive zero of the function $\mathbf{H}'_\nu$. Using (5) and (7) we have

$$\frac{z\mathbf{H}'_\nu(z)}{\mathbf{H}_\nu(z)} = \nu + 1 - \sum_{n\geq 1}\frac{2z^2}{h^2_{\nu,n} - z^2}, \quad 1 + \frac{z\mathbf{H}''_\nu(z)}{\mathbf{H}'_\nu(z)} = \nu + 1 - \sum_{n\geq 1}\frac{2z^2}{h'^2_{\nu,n} - z^2}.$$

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where $h'_{\nu,n}$ denotes the nth positive zero of the function $\mathbf{H}'_\nu$. Using (5) and (7) we have

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and consequently

$$\begin{aligned} 1 + \frac{zu''_\nu(z)}{u'_\nu(z)} &= 1 + \frac{z\,\mathbf{H}''_\nu(z)}{\mathbf{H}'_\nu(z)} + \left(\frac{1}{\nu+1} - 1\right)\frac{z\mathbf{H}'_\nu(z)}{\mathbf{H}_\nu(z)} \\ &= 1 - \left(\frac{1}{\nu+1} - 1\right)\sum_{n\geq 1}\frac{2z^2}{h^2_{\nu,n} - z^2} - \sum_{n\geq 1}\frac{2z^2}{h'^2_{\nu,n} - z^2}. \end{aligned}$$

Proof.

Now, suppose that $\nu \in [-\frac{1}{2}, 0]$. By using an elementary inequality for all $z \in \mathbb{D}_{h'_{\nu,1}}$ we obtain the inequality

$$\mathrm{Re}\left(1 + \frac{zu''_\nu(z)}{u'_\nu(z)}\right) \geqslant 1 - \left(\frac{1}{\nu+1} - 1\right) \sum_{n\geq 1} \frac{2r^2}{h^2_{\nu,n} - r^2} - \sum_{n\geq 1} \frac{2r^2}{h'^2_{\nu,n} - r^2},$$

where $|z| = r$.

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where $|z| = r$. Moreover, observe that if we use another elementary inequality then we get that the above inequality is also valid when $\nu \in \left[0, \frac{1}{2}\right]$. Here we used that the zeros $h_{\nu,n}$ and $h'_{\nu,n}$ interlace.

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$$\inf_{z\in\mathbb{D}_r}\left\{\mathrm{Re}\left(1+\frac{zu''_{\nu}(z)}{u'_{\nu}(z)}\right)\right\} = 1+\frac{ru''_{\nu}(r)}{u'_{\nu}(r)}.$$

Proof.

Now, suppose that $\nu \in [-\frac{1}{2}, 0]$. By using an elementary inequality for all $z \in \mathbb{D}_{h'_{\nu,1}}$ we obtain the inequality

$$\mathrm{Re}\left(1 + \frac{zu''_\nu(z)}{u'_\nu(z)}\right) \geqslant 1 - \left(\frac{1}{\nu+1} - 1\right) \sum_{n\geq 1} \frac{2r^2}{h_{\nu,n}^2 - r^2} - \sum_{n\geq 1} \frac{2r^2}{h'^2_{\nu,n} - r^2},$$

where $|z| = r$. Moreover, observe that if we use another elementary inequality then we get that the above inequality is also valid when $\nu \in \left[0, \frac{1}{2}\right]$. Here we used that the zeros $h_{\nu,n}$ and $h'_{\nu,n}$ interlace. The above inequality implies for $r \in \left(0, h'_{\nu,1}\right)$

$$\inf_{z\in\mathbb{D}_r} \left\{ \mathrm{Re}\left(1 + \frac{zu''_\nu(z)}{u'_\nu(z)}\right) \right\} = 1 + \frac{ru''_\nu(r)}{u'_\nu(r)}.$$

On the other hand, the function $U_\nu : \left(0, h'_{\nu,1}\right) \longrightarrow \mathbb{R}$, defined by

$$U_\nu(r) = 1 + \frac{ru''_\nu(r)}{u'_\nu(r)},$$

is strictly decreasing since □

Proof.

$$\begin{aligned} U_\nu'(r) &= -\left(\frac{1}{\nu+1}-1\right)\sum_{n\geq1}\frac{4rh_{\nu,n}^2}{\left(h_{\nu,n}^2-r^2\right)^2}-\sum_{n\geq1}\frac{4rh_{\nu,n}'^2}{\left(h_{\nu,n}'^2-r^2\right)^2} \\ &< \sum_{n\geq1}\frac{4rh_{\nu,n}^2}{\left(h_{\nu,n}^2-r^2\right)^2}-\sum_{n\geq1}\frac{4rh_{\nu,n}'^2}{\left(h_{\nu,n}'^2-r^2\right)^2}<0 \end{aligned}$$

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&< \sum_{n\geq1}\frac{4rh_{\nu,n}^2}{\left(h_{\nu,n}^2-r^2\right)^2}-\sum_{n\geq1}\frac{4rh_{\nu,n}'^2}{\left(h_{\nu,n}'^2-r^2\right)^2}<0
\end{aligned}
$$

for $\nu \in \left[0, \frac{1}{2}\right]$ and $r \in \left(0, h_{\nu,1}'\right)$,

Proof.

$$
\begin{aligned}
U'_\nu(r) &= -\left(\frac{1}{\nu+1}-1\right)\sum_{n\geq1}\frac{4rh^2_{\nu,n}}{\left(h^2_{\nu,n}-r^2\right)^2}-\sum_{n\geq1}\frac{4rh'^2_{\nu,n}}{\left(h'^2_{\nu,n}-r^2\right)^2}\\
&< \sum_{n\geq1}\frac{4rh^2_{\nu,n}}{\left(h^2_{\nu,n}-r^2\right)^2}-\sum_{n\geq1}\frac{4rh'^2_{\nu,n}}{\left(h'^2_{\nu,n}-r^2\right)^2}<0
\end{aligned}
$$

for $\nu\in\left[0,\frac{1}{2}\right]$ and $r\in\left(0,h'_{\nu,1}\right)$, and also we have $U'_\nu(r)<0$ for $\nu\in\left[-\frac{1}{2},0\right]$ and $r>0$.

Proof.

$$\begin{aligned} U'_\nu(r) &= -\left(\frac{1}{\nu+1}-1\right)\sum_{n\geq1}\frac{4rh_{\nu,n}^2}{\left(h_{\nu,n}^2-r^2\right)^2}-\sum_{n\geq1}\frac{4rh_{\nu,n}'^2}{\left(h_{\nu,n}'^2-r^2\right)^2} \\ &< \sum_{n\geq1}\frac{4rh_{\nu,n}^2}{\left(h_{\nu,n}^2-r^2\right)^2}-\sum_{n\geq1}\frac{4rh_{\nu,n}'^2}{\left(h_{\nu,n}'^2-r^2\right)^2}<0 \end{aligned}$$

for $\nu\in\left[0,\frac{1}{2}\right]$ and $r\in\left(0,h'_{\nu,1}\right)$, and also we have $U'_\nu(r)<0$ for $\nu\in\left[-\frac{1}{2},0\right]$ and $r>0$. Here we used that the zeros $h_{\nu,n}$ and $h'_{\nu,n}$ interlace for all $n\in\mathbb{N}$, $|\nu|\leq\frac{1}{2}$ and $r<\sqrt{h_{\nu,n}h'_{\nu,n}}$ we have that $h_{\nu,n}^2\left(h_{\nu,n}'^2-r^2\right)^2<h_{\nu,n}'^2\left(h_{\nu,n}^2-r^2\right)^2$.

Proof.

$$\begin{aligned} U_\nu'(r) &= -\left(\frac{1}{\nu+1}-1\right)\sum_{n\geq 1}\frac{4rh_{\nu,n}^2}{\left(h_{\nu,n}^2-r^2\right)^2}-\sum_{n\geq 1}\frac{4rh_{\nu,n}'^2}{\left(h_{\nu,n}'^2-r^2\right)^2} \\ &< \sum_{n\geq 1}\frac{4rh_{\nu,n}^2}{\left(h_{\nu,n}^2-r^2\right)^2}-\sum_{n\geq 1}\frac{4rh_{\nu,n}'^2}{\left(h_{\nu,n}'^2-r^2\right)^2}<0 \end{aligned}$$

for $\nu\in\left[0,\frac{1}{2}\right]$ and $r\in\left(0,h_{\nu,1}'\right)$, and also we have $U_\nu'(r)<0$ for $\nu\in\left[-\frac{1}{2},0\right]$ and $r>0$. Here we used that the zeros $h_{\nu,n}$ and $h_{\nu,n}'$ interlace for all $n\in\mathbb{N}$, $|\nu|\leq\frac{1}{2}$ and $r<\sqrt{h_{\nu,n}h_{\nu,n}'}$ we have that $h_{\nu,n}^2\left(h_{\nu,n}'^2-r^2\right)^2<h_{\nu,n}'^2\left(h_{\nu,n}^2-r^2\right)^2$. Since $\lim_{r\searrow 0}U_\nu(r)=1>\alpha$ and $\lim_{r\nearrow h_{\nu,1}'}U_\nu(r)=-\infty$, in view of the minimum principle for harmonic functions it follows that for $z\in\mathbb{D}_{r_4}$ we have

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for $\nu \in \left[0, \frac{1}{2}\right]$ and $r \in \left(0, h'_{\nu,1}\right)$, and also we have $U'_\nu(r) < 0$ for $\nu \in \left[-\frac{1}{2}, 0\right]$ and $r > 0$. Here we used that the zeros $h_{\nu,n}$ and $h'_{\nu,n}$ interlace for all $n \in \mathbb{N}$, $|\nu| \leq \frac{1}{2}$ and $r < \sqrt{h_{\nu,n}h'_{\nu,n}}$ we have that $h^2_{\nu,n}\left(h'^2_{\nu,n}-r^2\right)^2 < h'^2_{\nu,n}\left(h^2_{\nu,n}-r^2\right)^2$. Since $\lim_{r\searrow 0} U_\nu(r) = 1 > \alpha$ and $\lim_{r\nearrow h'_{\nu,1}} U_\nu(r) = -\infty$, in view of the minimum principle for harmonic functions it follows that for $z \in \mathbb{D}_{r_4}$ we have

$$\mathrm{Re}\left(1+\frac{zu''_\nu(z)}{u'_\nu(z)}\right) > \alpha$$

Proof.

$$\begin{aligned} U_\nu'(r) &= -\left(\frac{1}{\nu+1}-1\right)\sum_{n\geq1}\frac{4rh_{\nu,n}^2}{\left(h_{\nu,n}^2-r^2\right)^2}-\sum_{n\geq1}\frac{4rh_{\nu,n}'^2}{\left(h_{\nu,n}'^2-r^2\right)^2} \\ &< \sum_{n\geq1}\frac{4rh_{\nu,n}^2}{\left(h_{\nu,n}^2-r^2\right)^2}-\sum_{n\geq1}\frac{4rh_{\nu,n}'^2}{\left(h_{\nu,n}'^2-r^2\right)^2}<0 \end{aligned}$$

for $\nu \in \left[0, \frac{1}{2}\right]$ and $r \in \left(0, h_{\nu,1}'\right)$, and also we have $U_\nu'(r) < 0$ for $\nu \in \left[-\frac{1}{2}, 0\right]$ and $r > 0$. Here we used that the zeros $h_{\nu,n}$ and $h_{\nu,n}'$ interlace for all $n \in \mathbb{N}$, $|\nu| \leq \frac{1}{2}$ and $r < \sqrt{h_{\nu,n}h_{\nu,n}'}$ we have that $h_{\nu,n}^2\left(h_{\nu,n}'^2-r^2\right)^2 < h_{\nu,n}'^2\left(h_{\nu,n}^2-r^2\right)^2$. Since $\lim_{r\searrow0} U_\nu(r) = 1 > \alpha$ and $\lim_{r\nearrow h_{\nu,1}'} U_\nu(r) = -\infty$, in view of the minimum principle for harmonic functions it follows that for $z \in \mathbb{D}_{r_4}$ we have

$$\operatorname{Re}\left(1+\frac{zu_\nu''(z)}{u_\nu'(z)}\right) > \alpha$$

if and only if r_4 is the unique root of

$$1+\frac{ru_\nu''(r)}{u_\nu'(r)} = \alpha$$

situated in $\left(0, h_{\nu,1}'\right)$. □

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