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On Hyperbolically Monotone Functions

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I have always admired Christian as a very elegant writer in mathematics. We have had communications since the late 1970's when I was in Lund. We also met in Umeå in the beginning of the 1980's. Christian was interested in potential theory and moment problems whereas I was interested in infinite divisibility but there were connections. Later on Christian had a student Grete Ebbesen who wrote a thesis on my stuff and I visited Copenhagen in the mid 1980's in connection with that. I also attended a 2010 workshop that Henrik Pedersen and Christian arranged here. I was asked to talk about hyperbolic complete monotonicity. We also met in Hammamet in Tunisia in 2013.

My talk here is a vague continuation of my 2010 talk. It contains new recent results but no proofs.

Def 1. *Let $k \geq 1$ be an integer. A function $f(x) \geq 0$ on $(0,\infty)$ is called hyperbolically monotone of order k (HM$_k$) if for each $u > 0$ the function*

$$h(w) = f(uv)f(u/v), \text{ where } w = v + v^{-1},$$

is such that $(-1)^j h^{(j)}(w) \geq 0$ for $j = 1, 2, \ldots, k-1$ and moreover $(-1)^{k-1} h^{(k-1)}(w)$ is non-increasing.

This definition was first introduced for $k = \infty$ by myself in [2] in 1992 (and called hyperbolic complete monotonicity, HCM) but it had its inspiration from papers by O. Thorin from the 1970's. It was useful in connection with generalized gamma convolutions.

Apparently we have, for the corresponding classes of functions,

$$\mathsf{HCM} = \mathsf{HM}_\infty \subset \ldots \subset \mathsf{HM}_3 \subset \mathsf{HM}_2 \subset \mathsf{HM}_1 = \mathsf{HM}$$

The HM_k class is closed wrt multiplication, i.e.

$$f, g \in \mathsf{HM}_k \Rightarrow f \cdot g \in \mathsf{HM}_k.$$

For $k = \infty$, it was found early that f is HCM if and only if f can be represented as a limit of functions of the type

$$Cx^{\beta-1} \prod_{i=1}^{N} (1 + c_i x)^{-\gamma_i}, \qquad c_i, \gamma_i > 0, \beta \in \mathbb{R}.$$

For $k = 1$ we have that f is HM_1 if, for each $u > 0$,

$$h(w) = f(uv)f(u/v) \downarrow \quad \text{as} \quad w \uparrow .$$

It is not difficult to see that this only means that $\log f(e^y)$ is concave and hence there is a simple canonical representation for HM_1 functions. We have that f is HM_1 if and only if

$$f(x) = C \exp(-\int_{x_0}^{x} \frac{g(y)}{y} dy),$$

with g non-decreasing, C a constant, and x_0 suitable chosen.

An important property for HM$_k$:

Prop 1. *Let X and Y be independent random variables with HM$_k$ probability densities. Then $X \cdot Y$ and X/Y also have densities that are HM$_k$. Moreover, X^q has an HM$_k$ density for $|q| \geq 1$.*

Equivalently: the HM$_k$ class is closed wrt multiplicative convolution. This result is from [3].

Considering $\log X$ and $\log Y$, we see that Prop 1 for $k = 1$ is equivalent to the well-known fact that logconcavity is preserved under convolution.

Two examples of HM_k functions:

Example 1. Let $f(x) = C(x-a)^{\alpha-1}(b-x)^{\beta-1}\mathbf{1}_{(a,b)}(x)$ This is a beta density. It can be shown to be HM_k provided that $\min(\alpha, \beta) \geq k$. In particular the uniform density on (a, b) is HM_1.

Example 2. Let $f(x) = (1 + Ax + Bx^2)^{-\gamma}$ with $A, B \geq 0$. It can be shown to be HM_k for all $\gamma > 0$ provided that $B \leq A^2/K$, where $K = 4/(1 + \binom{k}{2}^{-1})$. In particular $f(x)$ is HCM if $B \leq A^2/4$ in which case $1 + Ax + Bx^2 = (1 + c_1 x)(1 + c_2 x)$ with $c_1, c_2 \geq 0$.

In Example 2 the functions f are such that f^p is HM_k for all $p > 0$ but that is not true for the functions in Example 1 for $k > 1$.

A canonical representation for the HM_k functions for $2 \leq k < \infty$ has been open since long. No full solution is given here.

However, I have been successful in finding representations of all HM_k functions f such that also f^p is HM_k for all $p > 0$. For $k = 1$ and $k = \infty$ this property holds automatically which one can see from the earlier representations.

Let $\psi(x) = -xf'(x)/f(x)$. This is the derivative of $\log f(e^{-y})$ after the substitution $x = e^{-y}$.

Theorem 1. *Let* $2 \le k < \infty$. *Then* $f^p(x)$ *is HM*$_k$ *for all* $p > 0$ *if and only if, with* $\psi(x) = -xf'(x)/f(x)$,

$$(-1)^{j-1}(x^{2j-1}\psi^{(j)}(x))^{(j-1)} \ge 0, \quad j = 1, 2, \dots, k. \tag{1}$$

Remark 1. All ψ satisfying (1) (not necessarily nonnegative) constitute a *convex cone*.

Theorem 2. *Moreover the condition (1) holds if and only if*

$$\psi'(x) = a + \frac{b}{x^2} + \int_{(0,\infty)} \phi_k'(x;\lambda) E_k(d\lambda), \tag{2}$$

where $a, b \geq 0$ *and* E_k *is a nonnegative measure with density* $e_k(\lambda) = (-1)^{k-1}(\lambda^{2k-1}\psi^{(k)}(\lambda))^{(k-1)}$ *(possibly degenerate), and*

$$\phi_k(x;\lambda) = \frac{(-1)^{k-1}}{(2k-2)!}\left(\frac{(x-\lambda)^{2k-2}}{\lambda^k x^{k-1}} H(x-\lambda) - P_k(x;\lambda)\right) \tag{3}$$

with H = *Heaviside and* $P_k(x;\lambda)$ = *Polynomial part of* $\frac{(x-\lambda)^{2k-2}}{\lambda^k x^{k-1}}$. *Here* $P_k(x;\lambda)$ *includes all terms* $a_j(\lambda)x^j$, $j = 1, 2, \ldots, k-1$ *but no term of the form* $a_j(\lambda)x^j$, $j \leq 0$.

Remark 2. The fact that

$$(-1)^{k-1}(x^{2k-1}\phi_k^{(k)}(x;\lambda))^{(k-1)} = Dirac(x-\lambda)$$

is a main reason why (2) holds. Moreover, it can be checked that

$$(-1)^{j-1}(x^{2j-1}\phi_k^{(j)}(x;\lambda))^{(j-1)} \geq 0, \;\; j = 1, 2, \ldots, k-1.$$

Thus these ϕ_k are some kind of *fundamental solutions* to (1).

Remark 3. By integration of (2) we get a formula for $\psi(x)$. However, the integral $\int_0^\infty \phi_k(x;\lambda)e_k(\lambda)d\lambda$ may be divergent and one must consider instead $\int_0^\infty(\phi_k(x;\lambda) - d)e_k(\lambda)d\lambda$, where d may depend on λ. By further integration we get a formula for $f(x)$.

Remark 4. For $\psi'(x) = a + bx^{-2}$, we get by integration

$$\psi(x) = ax - bx^{-1} + c,$$

where $c \in \mathbb{R}$, and then

$$f'(x)/f(x) = -a + bx^{-2} - cx^{-1}$$

and hence

$$f(x) = C\,\mathsf{exp}(-ax - bx^{-1} - c\log x) = Cx^{-c}\,\mathsf{exp}(-ax - bx^{-1})$$

which is HCM.

Remark 5. To prove Thms 1 and 2 it is first shown that the assumption that $f^p(x)$ is HM$_k$ for all $p > 0$ implies Condition (1). We look at $f^p(uv)f^p(u/v) = \exp(-p(-\log(f(uv)f(u/v))))$. Let

$$\Delta = D_w(-\log(f(uv)f(u/v))) = (\psi(uv) - \psi(u/v))/(v - v^{-1}).$$

Here $\Delta = u\psi'(u) + \frac{1}{3!}(u^3\psi^{(2)}(u))'(w-2) + \frac{1}{5!}(u^5\psi^{(3)}(u))^{(2)}(w-2)^2 + \ldots.$

Except for the factorials $(2j-1)!$ and u instead of x, the coefficients are those in (1) after sign changes. Thus (1) follows.

Condition (1) then leads to (2). It is then checked that, for any λ, $\phi_k(x;\lambda)$ is such that $\exp(-p\int_{x_0}^{x} y^{-1}\phi_k(y;\lambda)dy)$ is HM$_k$ for all $p > 0$. It suffices to check that $\Delta_k = \frac{\phi_k(uv;\lambda)-\phi_k(u/v;\lambda)}{v-v^{-1}}$ is such that $(-1)^j D_w^{(j)}\Delta_k \geq 0, j = 0, 1, \ldots, k-1$. Thus (2) $\Rightarrow$ (1) as well.

Example 3. Let $f(x) = \exp(-x^\alpha)$, where $0 < |\alpha| < 1$. It is known to be HCM so it is also HM_k for any integer $k \geq 1$. We have

$$\psi(x) = -xf'(x)/f(x) = \alpha x^\alpha, \ \psi'(x) = \alpha^2 x^{\alpha-1}.$$

Consider $k = 2$. We have

$$\phi_2(x;\lambda) = -\frac{1}{2}\frac{(x-\lambda)^2}{\lambda^2 x}H(x-\lambda) + \frac{1}{2}\frac{x}{\lambda^2}, \ \phi_2'(x;\lambda) = -\frac{(x-\lambda)(x+\lambda)}{2\lambda^2 x^2}H(x-\lambda) + \frac{1}{2}\frac{1}{\lambda^2}$$

Furthermore,

$$e_k(\lambda) = (-1)^{k-1}(\lambda^{2k-1}\psi^{(k)}(\lambda))^{(k-1)} = \alpha^2\lambda^\alpha \prod_{i=1}^{k-1}(i^2 - \alpha^2)$$

and thus $e_2(\lambda) = \alpha^2(1-\alpha^2)\lambda^\alpha$.

It is not difficult to check by Maple that

$$\psi'(x) = \int_0^\infty \phi_2'(x;\lambda)e_2(\lambda)d\lambda.$$

For $0 < \alpha < 1$, we also have that $\psi(x) = \int_0^\infty \phi_2(x;\lambda)e_2(\lambda)d\lambda$.

However for $-1 < \alpha < 0$ that integral is not convergent but

$$\psi(x) = \int_0^\infty (\phi_2(x;\lambda) - d)e_2(\lambda)d\lambda, \quad \text{where } d = 1/\lambda.$$

Example 4 (Ex 2, ctd). Let $f(x) = (1 + Ax + Bx^2)^{-\gamma}$. It is HM_k for all $\gamma > 0$ provided that $B \le A^2/K$, where $K = 4/(1 + \binom{k}{2}^{-1})$. We take $k = 3$ and then $K = 4/(1 + 3^{-1}) = 3$. We put $A = 1$ and $B = A^2/K = 1/3$. Then

$$\psi(x) = -xf'(x)/f(x) = \gamma \frac{x(3 + 2x)}{3 + 3x + x^2}.$$

We also have

$$e_3(x) = (-1)^{3-1}(x^{2\cdot 3-1}\psi^{(3)}(x))^{(3-1)} = \gamma \cdot 3240 \frac{x^4(3 + 2x)(x + 3)(x + 2)(x + 1)}{(3 + 3x + x^2)^6} \ge 0.$$

One can check by Maple that with $\phi_k(x;\lambda)$ as in (3) in Thm 2,

$$\psi'(x) = \int_0^\infty \phi_3'(x;\lambda)e_3(\lambda)d\lambda \quad \text{and} \quad \psi(x) = \int_0^\infty \phi_3(x;\lambda)e_3(\lambda)d\lambda.$$

Thus no convergence producing term d is needed here.

Final Comment 1. There are relations between Theorem 1 for $k = \infty$ and a classical characterization by Widder.

Prop 2 (Widder, 1938). *A function $g(x)$ is the Stieltjes transform of a nonnegative measure if and only if*

$$(-1)^{j-1}(x^j g(x))^{(2j-1)} \geq 0, \quad j = 1, 2, \ldots.$$

Widder sometimes also considers the NASC

$$(-1)^{j-1}(x^{2j-1} g^{(j-1)}(x))^{(j)} \geq 0, \quad j = 1, 2, \ldots.$$

Via Theorem 1 we are led to the different alternative NASC

$$(-1)^{j}(x^{2j-1} g^{(j)}(x))^{(j-1)} \geq 0, \quad j = 1, 2, \ldots.$$

For $g(x) = (x{+}a)^{-1}$ the expressions above are in order as follows:

$$(2j-1)!\frac{a^j}{(x+a)^{2j}}, \quad (2j-1)!\frac{a^j x^{j-1}}{(x+a)^{2j}}, \quad (2j-1)!\frac{a^{j-1} x^j}{(x+a)^{2j}}.$$

The last expression has after division by $j!(j-2)!$ the integral 1 over $(0,\infty)$ and tends, for any $a > 0$, to Dirac$(x-a)$ as $j \to \infty$. Hence for

$$g(x) = \int_{(0,\infty)} \frac{f(a)}{x+a} da,$$

we have that, as $j \to \infty$,

$$\frac{(-1)^j}{j!(j-2)!}(x^{2j-1} g^{(j)}(x))^{(j-1)} \to \int_0^\infty f(a)\text{Dirac}(x-a)da = f(x)$$

Final Comment 2. When talking about GGCs and hyperbolic monotonicity, I have often ended by mentioning an open problem. Prove that an α-stable probability density on $(0, \infty)$, with Laplace transform $\exp(-s^{\alpha})$, is HCM when $0 < \alpha \leq 1/2$.

It has been open since the end of the 1970's. Christian also had G Ebbesen to look at this HCM-problem in the 1980's. Myself I have failed several times to solve it. However, it has recently been proved to be true in a nice way by Bosch & Simon (2015). Simon (now in Lille) is a PhD-student of Hirsch who was Christian's study comrade in Paris in the early 1970's.

References

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