Asymptotic values and analytic sets

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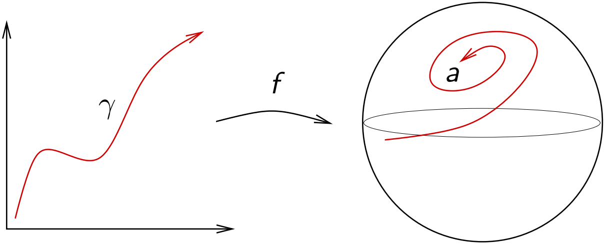
Asymptotic values

$f : \mathbb{R}^d \to \mathbb{R}^d$ continuous, $a \in \mathbb{R}^d \cup \{\infty\}$ is an asymptotic value if there exists a continuous path γ, $\gamma \to \infty$, along which

$$\lim_{\substack{x \to \infty \\ x \in \gamma}} f(x) = a.$$

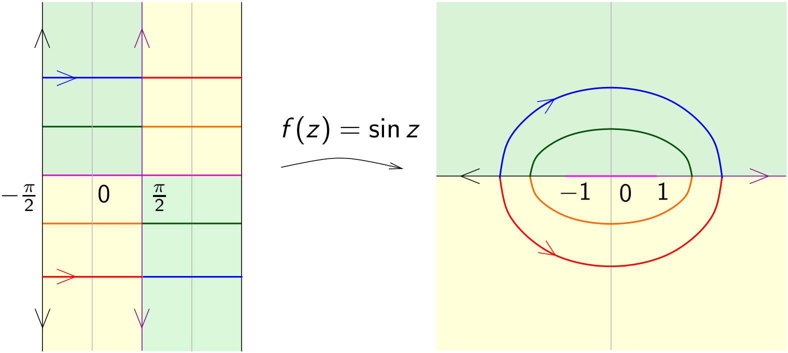


- γ is an asymptotic path,
- $\mathrm{As}(f)$ denotes the set of asymptotic values.

A remark and an example

- When $d = 1$, only two asymptotic directions $\Longrightarrow$ not interesting!
- Consider $d = 2$, and $f : \mathbb{C} \to \mathbb{C}$ holomorphic, by Liouville's theorem $\infty \in \mathrm{As}(f)$.

The sine function, $\mathrm{As}(\sin z) = \{\infty\}$.



More examples

Modifications of the sine function give extra asymptotic values

- $g(z) = \dfrac{\sin z}{z}$ holomorphic in $\mathbb{C}$,

$$\mathrm{As}(g) = \{0, \infty\},$$

with $\mathbb{R}^+$ and $\mathbb{R}^-$ as asymptotic paths for 0.

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- The entire function

$$f(z) = \int_0^z \frac{\sin \zeta}{\zeta} d\zeta,$$

by integrating on any continuous path that joins 0 and z. Then,

$$\mathrm{As}(f) = \left\{-\frac{\pi}{2}, \frac{\pi}{2}, \infty\right\},$$

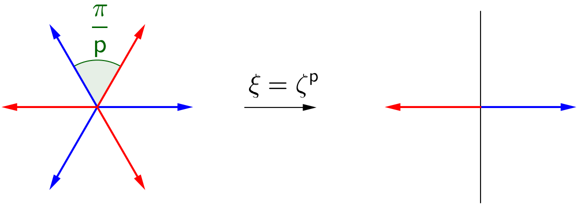
with $\mathbb{R}^+$ asymptotic path for $\frac{\pi}{2}$ and $\mathbb{R}^-$ for $-\frac{\pi}{2}$.

Even more examples

For $p \in \mathbb{N}$, define $f(z) = \int_0^z \frac{\sin \zeta^p}{\zeta^p} d\zeta$, holomorphic in $\mathbb{C}$.

The half-lines mapped by $\zeta \to \zeta^p$ onto the real half-axes are **asymptotic paths** of f with finite asymptotic values.

- Let η s.t. $\eta^p = \pm 1$. Then, $\lim_{\substack{z \to \infty \\ z \in \eta \mathbb{R}^+}} f(z) = \eta \int_0^\infty \frac{\sin t^p}{t^p} dt = \eta I$.



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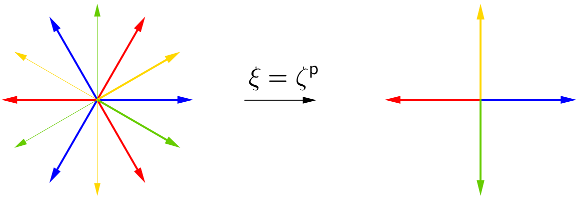

In fact, $\#\mathrm{As}(f) = 2p + 1$.

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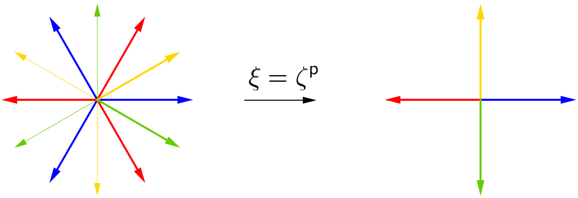

In fact, $\#\mathrm{As}(f) = 2p + 1$.

Order of growth

Question: Is there a relation between the rate of growth of f and the set $\mathrm{As}(f)$?

The order of growth of an entire function f is defined as,

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r},$$

where $M_f(r) = \max_{|z|=r} |f(z)|$ is the maximum modulus.

Examples

1. $f(z) = p(z)$, polynomial $\implies \rho_p = 0$, $\mathrm{As}(p) = \{\infty\}$.
2. $f(z) = \sin z \implies \rho_{\sin} = 1$, $\mathrm{As}(\sin z) = \{\infty\}$.
3. $f(z) = \int_0^z \frac{\sin \zeta^p}{\zeta^p} d\zeta \implies \rho_f = p$, $\#\mathrm{As}(f) = 2p + 1$.

Ahlfors Theorem

Denjoy-Carleman-Ahlfors

If $f : \mathbb{C} \to \mathbb{C}$ holomorphic then

$$\# \operatorname{As}(f) \leq 2\rho_f + 1.$$

The “1” corresponds to ∞.

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Consequences

- If $\rho_f < 1/2$ then $\operatorname{As}(f) = \{\infty\}$, i.e. f has no finite asymptotic value.
- If $\rho_f < \infty$ then $\operatorname{As}(f)$ is finite.

Only finite sets are sets of a.v. of entire functions with finite order of growth.

Infinite order of growth

For any order of growth:

Mazurkiewicz 30's

$f : \mathbb{C} \to \mathbb{C}$ holomorphic then $\mathrm{As}(f)$ is an analytic set in the sense of Suslin.

Heins 50's

Given $A \subset \mathbb{C}$ (Suslin) analytic set there exists $f : \mathbb{C} \to \mathbb{C}$ holomorphic such that $\mathrm{As}(f) = A \cup \{\infty\}$.

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Analytic sets characterize the set of a.v. of entire functions.

Analytic sets in $\mathbb{R}^d$

- The collection of Borel sets (in $\mathbb{R}^d$) is the smallest σ-algebra that contains all open and closed sets.
- The inverse image under a continuous function of a Borel set is Borel.
- ⚠ The continuous image of a Borel set is not necessarily Borel!!

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Analytic sets in $\mathbb{R}^d$ are the images of Borel sets under continuous functions.

- Borel sets are those analytic sets whose complement is also analytic.
- Analytic sets are Lebesgue measurable

Quasiregular maps in $\mathbb{R}^d$

$K \geq 1$, $\Omega \subset \mathbb{R}^d$ open, $f : \Omega \to \mathbb{R}^d$ continuous. f is K-quasiregular in Ω iff

1) $f \in W^{1,d}_{\text{loc}}(\Omega)$ ($\Longrightarrow f'(x)$ exists ae x),
2) $|f'(x)|^d \leq K J_f(x)$ ae $x \in \Omega$,

(where $f'(x)$ is a linear operator, $|\cdot|$ its norm and $J_f(x)$ its determinant)

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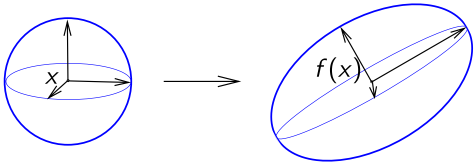
Geometrically: If $f'(x) \neq 0$, in the tangent space



Remarks

- A K-quasiregular homeomorphism is a K-quasiconformal map.
- For $d = 2$, f is K-quasiregular iff $f = h \circ g$ with h holomorphic and g K-quasiconformal.

  Since 1-qc functions are conformal, f is 1-quasiregular iff it is holomorphic.
- If $K = 1$, for any $d \geq 3$, a non-constant 1-quasiregular map is a sense-preserving Möbius transformation.

The interesting cases are $K > 1$ and $d \geq 3$

Properties of quasiregular maps

shared with holomorphic functions

$f : \Omega \to \mathbb{R}^d$ quasiregular then,

- f is open ($\Longrightarrow$ the maximum principle holds) and discrete.
- $\dim(B_f) \leq d - 2$ where B_f is the branching set.
- If $f : \mathbb{R}^d \to \mathbb{R}^d$, f is unbounded (Liouville's Theorem).
- Picard's Theorem for quasiregular maps:

Theorem (Rickman)

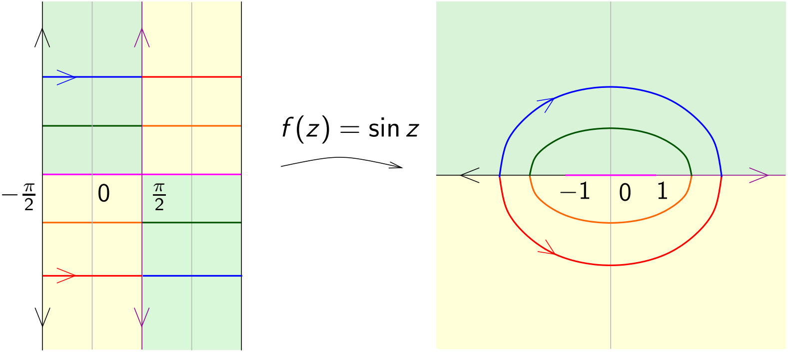
$f : \mathbb{R}^d \to \mathbb{R}^d$, a non-constant K-quasiregular map. Then f can omit at most $q = q(d, K)$ points.

Moreover, for $d \geq 3$, $q(d, K) \nearrow \infty$ as $K \nearrow \infty$.

Example of a qr map

Drasin's sine function

It is a higher dimensional analog of the holormophic sine.



Example of a qr map

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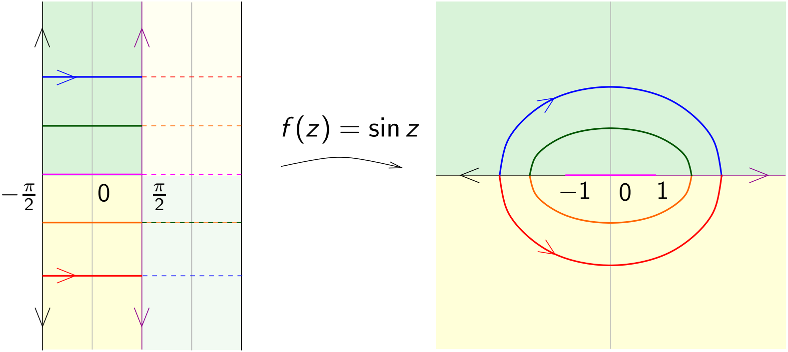
Consider a fundamental domain,

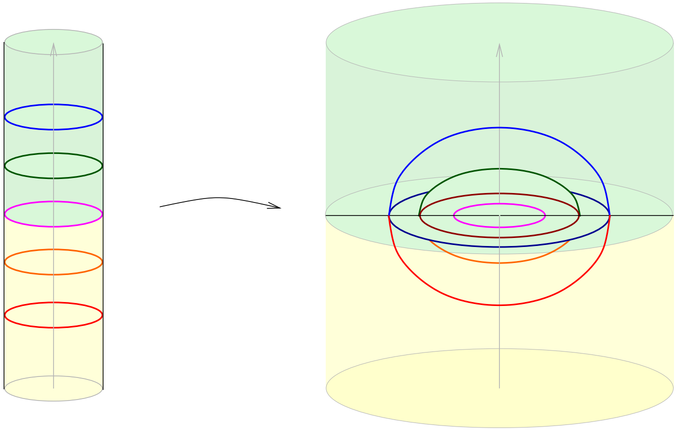


and rotate it around the vertical axis.

Example of a qr map

Drasin's sine function

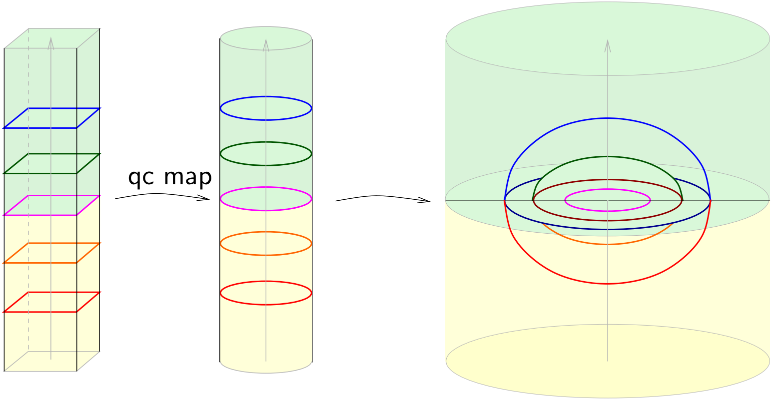
The rotation gives,



Example of a qr map

Drasin's sine function

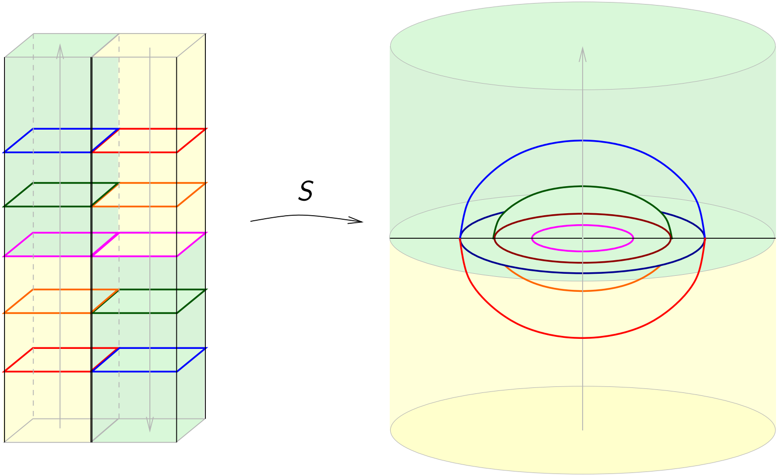
To extend the function to $\mathbb{R}^d$, map the cylinder onto a square based prism in a bilipschitz way.



Example of a qr map

Drasin's sine function

Extend the function by symmetry



Order of growth

The order of growth of a qr map is

$$\rho_f = \limsup_{r\to\infty} \frac{(d-1)\log\log M_f(r)}{\log r}$$

where $M_f(r) = \sup_{|x|=r} |f(x)|$.

Remark. The order of growth of Drasin's sine function, S, is $\rho_S = d-1$.

Order of growth and asymptotic values

As in Ahlfors Theorem in the holomorphic case:

Theorem (Rickman-Vuorinen)

$f : \mathbb{R}^d \to \mathbb{R}^d$, K-qr map. If $\rho_f < c(d, K)$ then $As(f) = \{\infty\}$.

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⚠ Although,

Theorem (Drasin)

For every $d \geq 3$, there exists a qr map $f : \mathbb{R}^d \to \mathbb{R}^d$ with $\rho_f = d - 1$ and $As(f) = \mathbb{R}^d \cup \{\infty\}$.

Asymptotic values and analytic sets

Theorem (Mazurkiewicz, C-Qu)

If $f : \mathbb{R}^d \to \mathbb{R}^d$ *is qr, then* $As(f)$ *is an analytic set that contains* ∞.

Theorem (C-Qu)

For $d \geq 3$ *and any analytic set* $A \subset \mathbb{R}^d$ *there exists a qr map* $f : \mathbb{R}^d \to \mathbb{R}^d$ *with* $\rho_f = d - 1$ *and* $As(f) = A \cup \{\infty\}$.

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In $d \geq 3$, analytic sets characterize the set of a.v. of quasiregular functions even for those with finite order of growth.

Comments on the proofs

- The first statement is already contained in Mazurkiewicz's work although not explicitly stated. We give an alternative proof, which is "dual" to that of Mazurkiewicz's.
- The proof of the second theorem is based on a modification of Drasin's arguments.
- Main difference with Drasin's work: more control on the asymptotic paths to show that no other asymptotic values are attained.

Another characterization of analytic sets

Suslin $\mathcal{A}$-operation

Theorem (Suslin)

$A \subset \mathbb{R}^d$ analytic iff there exists a collection of closed sets $\{E_{n_1,\dots,n_k}\}$, labelled with $\{n_1,\dots,n_k\}$, finite sequences of natural numbers, such that

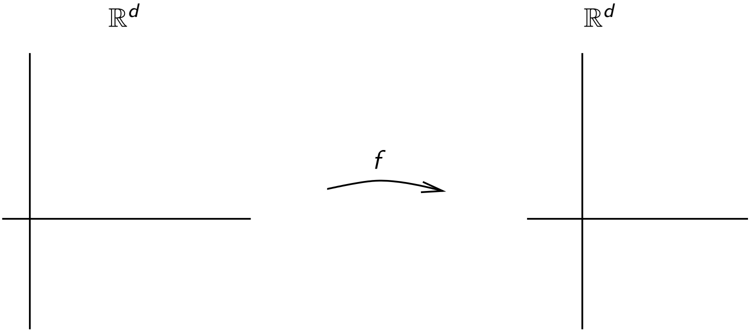
$$A = \bigcup_{\{n_1,n_2,\dots\}\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k\geq 1} E_{n_1,\dots,n_k},$$

(the Suslin $\mathcal{A}$-operation applied to $\{E_{n_1,\dots,n_k} : n_1,\dots,n_k \in \mathbb{N}, k \in \mathbb{N}\}$.)

Proof of the first result

As(f) is an analytic set

f qr, hence discrete and open.

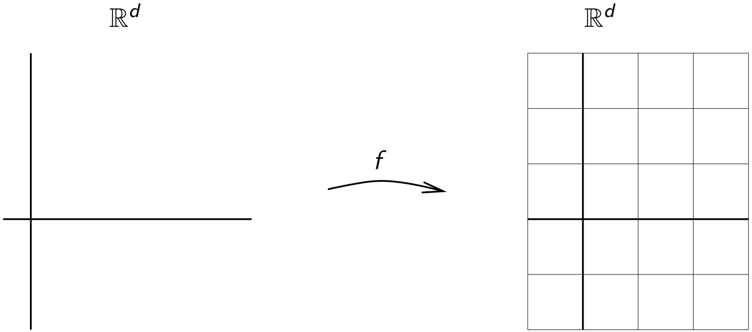


Proof of the first result

As(f) is an analytic set

f qr, hence discrete and open. Partition of $\mathbb{R}^d$ by dyadic cubes:

$$X_{n_1}, \qquad n_1 \in \mathbb{N},$$



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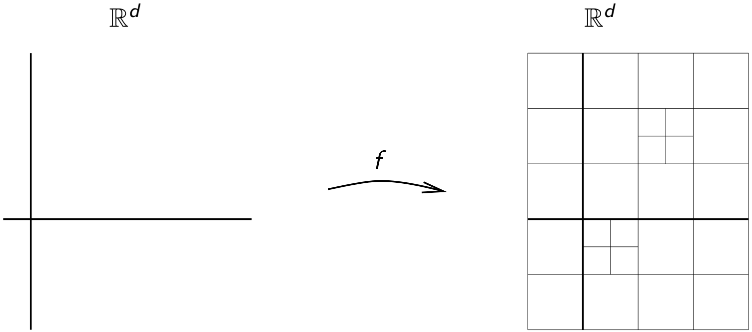
$$X_{n_1,n_2,n_3,\ldots},\ n_1 \in \mathbb{N},\ n_j \in \{1,\ldots,2^d\},\ j > 1.$$



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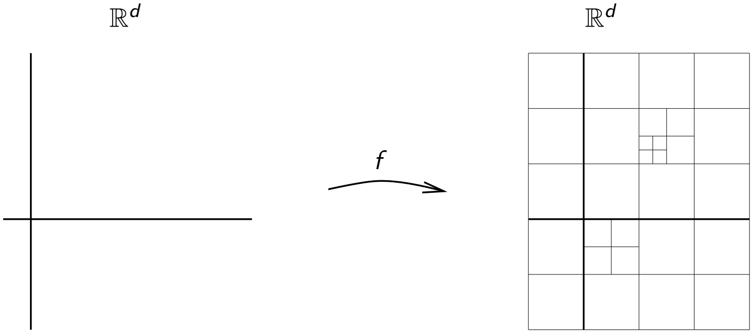

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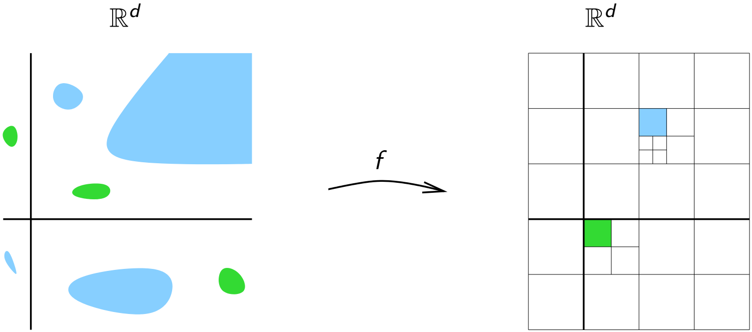
$X_{n_1,\ldots,n_k}$ admissible iff $f^{-1}(X_{n_1,\ldots,n_k})$ has an unbounded connected component.



Proof of the first result

As(f) is an analytic set

Let

$$S_{n_1,\dots,n_k} = \begin{cases} X_{n_1,\dots,n_k}, & X_{n_1,\dots,n_k} \text{ admissible,} \\ \varnothing, & \text{otherwise,} \end{cases}$$

and define

$$B = \bigcup_{\{n_1,n_2,\dots\}\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k\geq 1} S_{n_1,\dots,n_k},$$

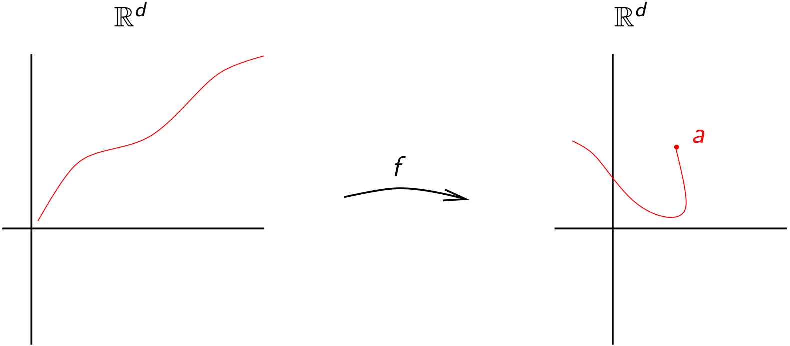
an analytic obtained by the Suslin $\mathcal{A}$-operation.

Claim

$$B = \mathrm{As}(f) \setminus \{\infty\}.$$

Reason

$a \in \mathrm{As}(f) \setminus \{\infty\}$,



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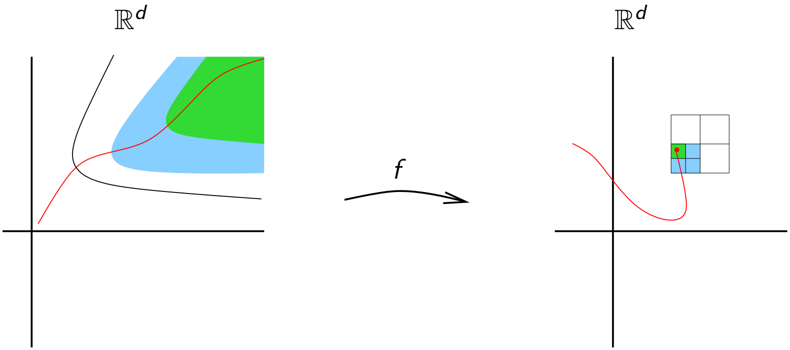

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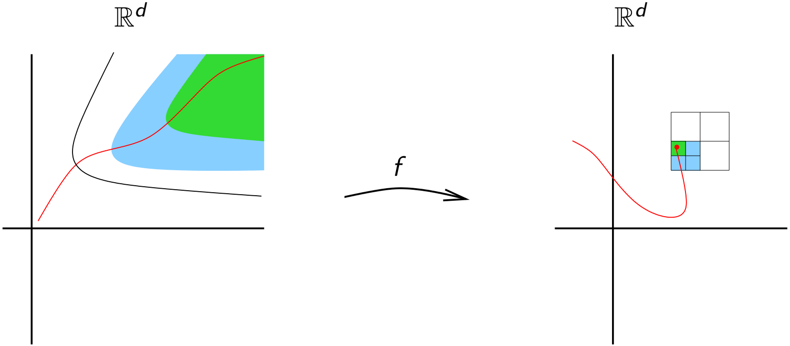
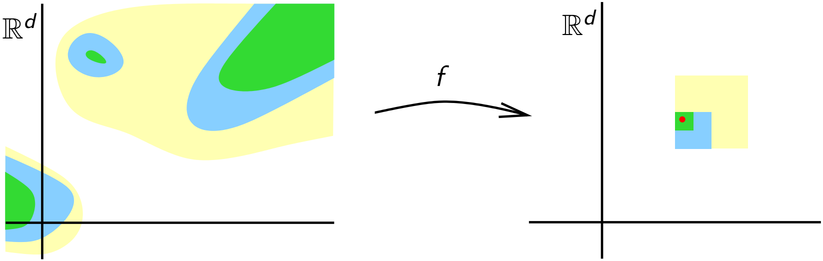


$$a = \bigcap_{k \geq 1} X_{n_1,\ldots,n_k} = \bigcap_{k \geq 1} S_{n_1,\ldots,n_k} \in B.$$

Converse

$b \in B \in \mathbb{R}^d \Longrightarrow b = \bigcap_{k\geq 1} X_{n_1,\ldots,n_k}$, with $X_{n_1,\ldots,n_k}$ admissible.

Since f is discrete, $\bigcap_{k\geq 1} f^{-1}\left(X_{n_1,\ldots,n_k}\right)$ discrete set that contains ∞.



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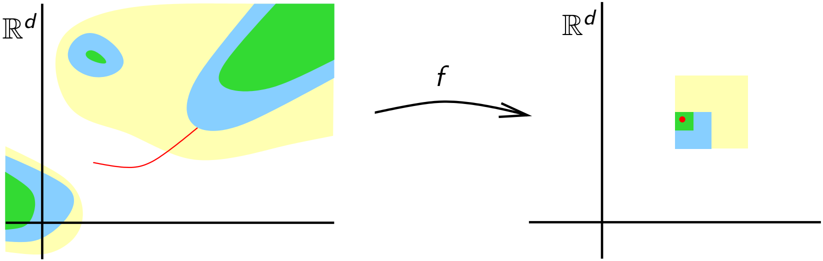

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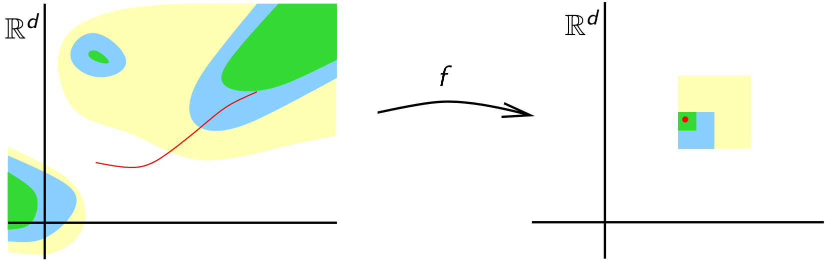

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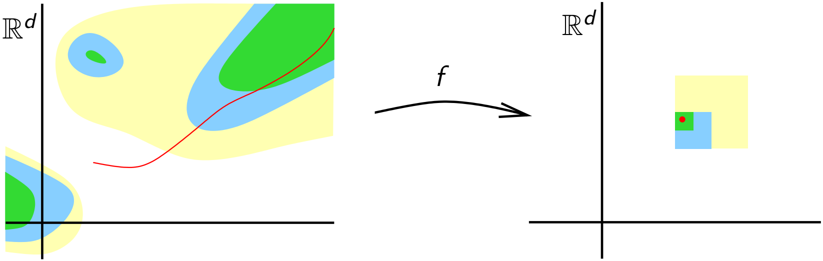

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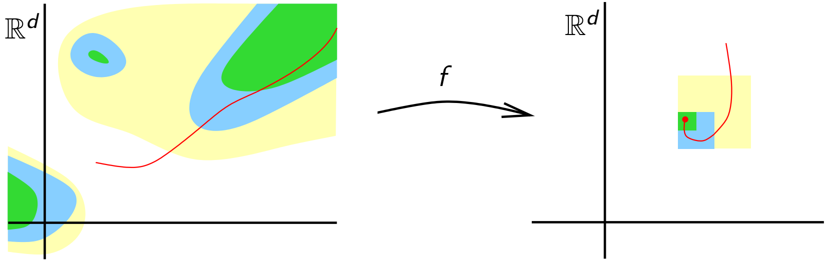

$\gamma \in \mathbb{R}^d$, $\gamma \to \infty$ and

$$\lim_{\substack{x \to \infty \\ x \in \gamma}} f(x) = b \Longrightarrow b \in \mathrm{As}(f) \setminus \{\infty\}.$$

The shortest path between two truths in the real domain passes through the complex domain.

J. Hadamard