Asymptotics of Chebyshev polynomials

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Chebyshev polynomials

Let $\boldsymbol{E} \subset \mathbb{C}$ be an infinite, compact set of points.

For any function f, define

$$\|f\|_{\boldsymbol{E}} := \sup\{|f(z)| : z \in \boldsymbol{E}\}.$$

The *Chebyshev polynomial of degree n* is the monic polynomial T_n with

$$\|T_n\|_{\boldsymbol{E}} = \inf\{\|P\|_{\boldsymbol{E}} : \deg(P) = n \text{ and } P \text{ is monic}\}.$$

One can show that this minimizer is indeed unique.

When $\boldsymbol{E} \subset \mathbb{R}$, the polynomial T_n is real.

Two examples

1) When $\boldsymbol{E}$ is the unit circle, $\partial\mathbb{D}$, the Chebyshev polynomials are given by

$$T_n(z) = z^n \quad \text{for all } n.$$

So in that case, $\|T_n\| = 1$ — and we have $\mathrm{Cap}(\boldsymbol{E}) = 1$.

2) When $\boldsymbol{E} = [-1, 1]$, the case considered by Chebyshev, the polynomials in question are

$$T_0(x) = 1, \quad T_n(x) = 2^{-n+1}\cos(n\theta) \quad \text{for } n \geq 1,$$

with $x = \cos(\theta)$.

These are of course the familiar Chebyshev of the first kind.

Note that $\|T_n\| = 2^{-n+1}$ for $n \geq 1$ — and $\mathrm{Cap}(\boldsymbol{E}) = 1/2$.

The alternation theorem

We say that P_n, a real degree n polynomial, has an *alternating set* in $\boldsymbol{E} \subset \mathbb{R}$ if there exists $\{x_j\}_{j=0}^n \subset \boldsymbol{E}$ with

$$x_0 < x_1 < \ldots < x_n$$

so that

$$P_n(x_j) = (-1)^{n-j} \|P_n\|_{\boldsymbol{E}}, \quad j = 0, \ldots, n.$$

Theorem

The Chebyshev polynomial of degree n for $\boldsymbol{E} \subset \mathbb{R}$ has an alternating set in $\boldsymbol{E}$. Conversely, any monic polynomial with an alternating set in $\boldsymbol{E}$ is the Chebyshev polynomial for $\boldsymbol{E}$.

This theorem implies several simple facts about T_n's zeros.

They are 1) all simple, 2) lie in $\mathrm{cvh}(\boldsymbol{E})$, and 3) each gap of $\boldsymbol{E}$ contains at most one zero.

Bounds on the norm

Szegő realized that Chebyshev polynomials are intimately connected with potential theory.

He proved that

$$\|T_n\|_{\boldsymbol{E}} \geq \mathrm{Cap}(\boldsymbol{E})^n$$

which is optimal for subsets of $\mathbb{C}$ (as Example 1 shows).

For real subsets, Schiefermayr established the bound

$$\|T_n\|_{\boldsymbol{E}} \geq 2\,\mathrm{Cap}(\boldsymbol{E})^n$$

which is also optimal (cf. Example 2).

Upper bounds of the form $\|T_n\|_{\boldsymbol{E}} \leq K\,\mathrm{Cap}(\boldsymbol{E})^n$ for some constant $K \geq 2$ are clearly interesting.

Totik (and Widom) proved that this holds true whenever $\boldsymbol{E} \subset \mathbb{R}$ is a *finite gap set*.

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Recently, C–Simon–Zinchenko established such bounds for all infinite gap sets of *Parreau–Widom type*.

Norm asymptotics

As is clear from the previous slides, we have

$$\lim_{n\to\infty} \|T_n\|_{\boldsymbol{E}}^{1/n} = \mathrm{Cap}(\boldsymbol{E}),$$

at least for all finite gap sets.

But in fact this holds for *all* compact, non-polars sets in $\mathbb{C}$ (proven by Faber, Fekete, and Szegő).

When $\boldsymbol{E}$ is an analytic Jordan curve (e.g., the unit circle), we can do even better.

In this case, Faber proved the refined asymptotic formula

$$\lim_{n\to\infty} \frac{\|T_n\|_{\boldsymbol{E}}}{\mathrm{Cap}(\boldsymbol{E})^n} = 1.$$

This result is much stronger, but restricted to a special case.

Asymptotics of T_n

Now, Faber proved more. He also proved that

$$\frac{T_n(z)B(z)^n}{\mathrm{Cap}(\boldsymbol{E})^n} \to 1$$

uniformly on $\boldsymbol{\Omega}$, the unbounded component of $\overline{\mathbb{C}} \setminus \boldsymbol{E}$.

Here, $B : \boldsymbol{\Omega} \to \mathbb{D}$ is the Riemann map with $B(\infty) = 0$ and positive "derivative" at ∞.

For more complicated sets, $\boldsymbol{\Omega}$ is no longer simply connected and one needs to modify the function B.

The 'new' B is *multivalued* and *character automorphic*.

Hence $T_n(z)B(z)^n \, \mathrm{Cap}(\boldsymbol{E})^{-n}$ cannot have a limit — for its character is n dependent.

Widom's minimizers

Widom [Adv. Math. '69] had the idea that there should be functions F_χ defined for each χ in the character group and continuous in χ so the 'limit' of

$$\frac{T_n(z)B(z)^n}{\mathrm{Cap}(\boldsymbol{E})^n}$$

is the F_χ, call it F_n, associated with the character of B^n.

As a function of n, this limit is *almost periodic*.

He even found a candidate for these functions!

Let F_χ be the function which among all character automorphic functions G on $\boldsymbol{\Omega}$ with character χ and with $G(\infty) = 1$ minimizes

$$\sup_{z \in \boldsymbol{\Omega}} |G(z)|.$$

Widom's theorems

Let $\boldsymbol{E} \subset \mathbb{C}$ be a finite union of disjoint analytic Jordan curves.

Theorem

If F_n is the unique minimizer with same character as B^n, then

$$\lim_{n\to\infty}\left[\frac{T_n(z)B(z)^n}{\mathrm{Cap}(\boldsymbol{E})^n} - F_n(z)\right] = 0$$

uniformly on compact subsets of the universal cover of $\boldsymbol{\Omega}$.

Theorem

In the setting as above,

$$\lim_{n\to\infty}\frac{\|T_n\|_E}{\mathrm{Cap}(\boldsymbol{E})^n\|F_n\|_{\boldsymbol{\Omega}}} = 1.$$

Since $|B(z)| \to 1$ and $\|F_n\|_{\boldsymbol{\Omega}}$ is taken as $z \to \boldsymbol{E}$, these two asymptotics results fit together.

Widom's conjecture

Let $\boldsymbol{E} \subset \mathbb{R}$ be a finite union of disjoint, compact intervals.

Theorem

If F_n is the unique minimizer with same character as B^n, then

$$\lim_{n\to\infty} \frac{\|T_n\|_E}{\mathrm{Cap}(\boldsymbol{E})^n \|F_n\|_{\boldsymbol{\Omega}}} = 2.$$

Widom conjectured that in this case one also has

$$\lim_{n\to\infty} \left[\frac{T_n(z)B(z)^n}{\mathrm{Cap}(\boldsymbol{E})^n} - F_n(z) \right] = 0$$

uniformly on compact subsets of the universal cover of $\boldsymbol{\Omega}$.

But he had no proof.....

After 45 years, this is now a theorem of C–Simon–Zinchenko.

Motivation

Widom was clearly motivated by the example of $\boldsymbol{E} = [-1, 1]$.

Indeed, in that case

$$B(z) = z - \sqrt{z^2 - 1}, \quad B(z)^{-1} = z + \sqrt{z^2 - 1}$$

and a familiar formula for the Chebyshev polynomials is

$$T_n(z) = 2^{-n}\Big(B(z)^n + B(z)^{-n}\Big).$$

Recall also that $\|T_n\| = 2^{-n+1} = 2\,\mathrm{Cap}(\boldsymbol{E})^n$.

We see that $|B|$ is 1 (resp. < 1) on $\boldsymbol{E}$ (resp. off $\boldsymbol{E}$).

Thus off $\boldsymbol{E}$, only B^{-n} contributes to the asymptotics while on $\boldsymbol{E}$, there are points with $B = 1$ so both terms contribute and the norm is twice as large as one might have expected.

This explains where Widom's conjecture came from.

The proof

All details can be found in the paper

– *Asymptotics of Chebyshev polynomials, I. Subsets of* $\mathbb{R}$,

by Christiansen, Simon, and Zinchenko.

@ `arXiv` : 1505.02604 [math.CA]

We consider the sets

$$\boldsymbol{E}_n := T_n^{-1}\Big(-\|T_n\|_{\boldsymbol{E}}, \|T_n\|_{\boldsymbol{E}}\Big)$$

and a crucial formula is

$$\frac{2T_n(z)}{\|T_n\|_{\boldsymbol{E}}} = B_n(z)^n + B_n(z)^{-n},$$

where B_n is short for the B-function for $\boldsymbol{E}_n$.

Thanks for your attention!