On a new concept of conjugate Laplace series

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The Real World is Complex
an international symposium in honor of
Christian Berg
Copenhagen, 2015

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\vartheta + b_k \sin k\vartheta)$$

conjugate series: $-\sum_{k=1}^{\infty} (b_k \cos k\vartheta - a_k \sin k\vartheta)$

$$\sum_{h=0}^{\infty} \sum_{k=0}^{2h} a_{hk} Y_{hk}(\varphi, \vartheta)$$

conjugate series: ?

Riesz, F. e M., 1916: If they are both Fourier-Stjelties series then they are ordinary Fourier series

$$\mu,\ \beta \in M([0,2\pi])$$

$$\begin{cases} \displaystyle\int_0^{2\pi} \cos k\vartheta \ d\mu = \int_0^{2\pi} \sin k\vartheta \ d\beta \\ \displaystyle\int_0^{2\pi} \sin k\vartheta \ d\mu = -\int_0^{2\pi} \cos k\vartheta \ d\beta \quad (k = 1, 2, \ldots) \end{cases}$$

$\Rightarrow \mu$ and β are absolutely continuous

$$\exists\ f,\ g \in L^1(0,2\pi)\ :$$

$$\mu(B) = \int_B f(\vartheta)\, d\vartheta, \quad \beta(B) = \int_B g(\vartheta)\, d\vartheta$$

Riesz, F. e M., 1916: If they are both Fourier-Stjelties series then they are ordinary Fourier series

In its direct applications as well as the generalizations it has inspired, this has proved to be one of the more important theorems of the century. (R.B. Burckel)

Cole-Range, 1972
(see Rudin, W.: Function Theory in the Unit Ball of $\mathbb{C}^n$, 1980)

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Rudin, W.: Fourier analysis on groups, 1962.

Cassisa, C., Fichera, G.: A mechanical interpretation of the Brothers Riesz Theorem, 1990.

Koshi, S.: Recent developments on the F. and M. Riesz theorem, 1995.

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$$\Omega \subset \mathbb{C}, \quad f \in H(\Omega), \quad f|_{\Sigma} = \mu, \ \mu \in M(\Sigma)$$
$$\Rightarrow \mu \text{ is absolutely continuous}$$

(*Fichera*, 1959)

u, v harmonic conjugate in Ω

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

$u|_{\Sigma} = \alpha, \ v|_{\Sigma} = \beta \ \Rightarrow \ \alpha, \beta$ are absolutely continuous

Bochner, 1944.
Multiple Fourier Series.

Stein, Weiss, 1960.

$$R_k \mu = \beta_k \qquad (k = 1, 2, \ldots, n-1)$$

then μ, $\beta_1, \ldots, \beta_{n-1}$ are absolutely continuous.

Muckenhoupt, Stein, 1965

$$\sum_{k=0}^{\infty} a_k P_k^{\lambda}(\cos\vartheta), \quad \frac{1}{(1 - 2t\omega + \omega^2)^{\lambda}} = \sum_{k=0}^{\infty} \omega^k P_k^{\lambda}(t) \quad (\lambda > 0)$$

$$2\lambda \sum_{k=1}^{\infty} \frac{a_k}{k + 2\lambda} \sin\vartheta \, P_{k-1}^{\lambda+1}(\cos\vartheta)$$

$$\Delta_2 u = 0 \ \text{ in } D = \{z \in \mathbb{C} \mid |z| < 1\}$$

$$u(\varrho, \vartheta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \varrho^k (a_k \cos k\vartheta + b_k \sin k\vartheta)$$

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \qquad v(0) = 0$$

$$v(\varrho, \vartheta) = -\sum_{k=1}^{\infty} \varrho^k (b_k \cos k\vartheta - a_k \sin k\vartheta)$$

$$\varrho = 1$$

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\vartheta + b_k \sin k\vartheta), \quad -\sum_{k=1}^{\infty} (b_k \cos k\vartheta - a_k \sin k\vartheta)$$

$$\Delta_2 u = 0 \quad \text{in } B = \{x \in \mathbb{R}^n \mid |x| < 1\}$$

$$u(x) = \sum_{h=0}^{\infty} |x|^h \sum_{k=1}^{p_{nh}} a_{hk} Y_{hk}\left(\frac{x}{|x|}\right)$$

$$v(x) = ?$$

k-form

$$u = \frac{1}{k!}\, u_{s_1 \dots s_k} dx^{s_1} \dots dx^{s_k}$$

differential $d : C^1_k \to C^0_{k+1}$

$$du = \frac{1}{k!} \frac{\partial}{\partial x_j} u_{s_1 \dots s_k} dx^j dx^{s_1} \dots dx^{s_k}$$

adjoint $* : C^h_k \to C^h_{n-k}$

$$*u = \frac{1}{(n-k)!}\, u^*_{i_1 \dots i_{n-k}} dx^{i_1} \dots dx^{i_{n-k}}$$

$$u^*_{i_1 \dots i_{n-k}} = \frac{1}{k!}\, \delta^{1 \dots\dots\dots n}_{s_1 .. s_k i_1 .. i_{n-k}} u_{s_1 \dots s_k}$$

co-differential $\delta : C^1_k \to C^0_{k-1}$

$$\delta u = (-1)^{n(k+1)+1} * d * u$$

$$-(d\delta + \delta d) u = \Delta u = \frac{1}{k!}\, \Delta u_{s_1 \dots s_k} dx^{s_1} \dots dx^{s_k}$$

$\Omega \subset \mathbb{R}^n$

$u \in C^1_k(\Omega),\ v \in C^1_{k+2}(\Omega)$ are conjugate if

$$\begin{cases} du = \delta v \\ \delta u = 0, dv = 0 \end{cases}$$

$$n = 2 : \quad u + i(*v) \text{ is holomorphic}$$

C. (1997)

If $u \in C^1_k(\Omega),\ v \in C^1_{k+2}(\Omega)$ are conjugate and $u|_\Sigma = \alpha$, $v|_\Sigma = \beta$, then α and β are absolutely continuous

$$U \in C^1_0(\Omega) \oplus \ldots \oplus C^1_n(\Omega) \qquad U = \sum_{k=0}^{n} u_k \qquad u_k \in C^1_k(\Omega)$$

$$dU = \sum_{k=0}^{n-1} du_k, \qquad \delta U = \sum_{k=1}^{n} \delta u_k$$

U is self-conjugate if $dU = \delta U$

$$\delta u_1 = 0, \quad du_k = \delta u_{k+2} \ (k = 0, \ldots, n-2), \quad du_{n-1} = 0$$

$$U = u + v, \quad u \in C^1_k(\Omega), \quad v \in C^1_{k+2}(\Omega)$$

$$dU = \delta U \Leftrightarrow \begin{cases} du = \delta v \\ \delta u = 0, \ dv = 0 \end{cases}$$

Particular cases:

$$n = 2, \quad U = u_0 + u_2,$$
$$u_0 \equiv u, \quad u_2 = v\, dxdy$$
$$dU = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy; \; \delta U = \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy$$

$$dU = \delta U \iff u + iv \text{ is holomorphic}$$

$$n = 3, \quad U = u_0 + u_2$$
$$u_0 \equiv u, \quad u_2 = v_1 dx^2 dx^3 + v_2 dx^3 dx^1 + v_3 dx^1 dx^2$$

$$dU = \delta U \iff \operatorname{div}(v_1, v_2, v_3) = 0, \; \operatorname{grad} u = \operatorname{curl}(v_1, v_2, v_3)$$

Moisil-Theodorescu system

$$n = 4, \quad U = u_0 + u_2 + u_4$$
$$u_0 \equiv f_0,$$
$$u_2 = f_1(dx^0dx^1 - dx^2dx^3) +$$
$$f_2(dx^0dx^2 - dx^3dx^1) + f_3(dx^0dx^3 - dx^1dx^2),$$
$$u_4 = f_0dx^0dx^1dx^2dx^3$$

$$dU = \delta U \iff \begin{cases} \frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} = 0 \\ \frac{\partial f_0}{\partial x_1} + \frac{\partial f_1}{\partial x_0} - \frac{\partial f_2}{\partial x_3} + \frac{\partial f_3}{\partial x_2} = 0 \\ \frac{\partial f_0}{\partial x_2} + \frac{\partial f_1}{\partial x_3} + \frac{\partial f_2}{\partial x_0} - \frac{\partial f_3}{\partial x_1} = 0 \\ \frac{\partial f_0}{\partial x_3} - \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} + \frac{\partial f_3}{\partial x_0} = 0 \end{cases} \quad \text{Fueter system}$$

(Cimmino system ...)

$$U = w_h dx^h$$

$$dU = \delta U \iff \mathrm{div}(w_1, \ldots, w_n) = 0, \ \mathrm{curl}(w_1, \ldots, w_n) = 0$$

harmonic vectors

$$U = u_k$$

$$dU = \delta U \iff du_k = 0, \ \delta u_k = 0$$

harmonic forms

$\Omega \subset \mathbb{R}^n \quad \Sigma = \partial\Omega$

C.

$U \in C^1_0(\Omega) \oplus \ldots \oplus C^1_n(\Omega), \qquad dU = \delta U$

$$\begin{cases} U\big|_\Sigma = \alpha \in M_0(\Sigma) \oplus \ldots \oplus M_{n-1}(\Sigma) \\ *U\big|_\Sigma = \widetilde{\alpha} \in M_0(\Sigma) \oplus \ldots \oplus M_{n-1}(\Sigma) \end{cases}$$

$$\Rightarrow \alpha,\ \widetilde{\alpha} \ \text{ absolutely continuous}$$

$$\Delta_2 u = 0 \ \text{ in } B = \{x \in \mathbb{R}^n \mid |x| < 1\}$$

$$u(x) = \sum_{h=0}^{\infty} |x|^h \sum_{k=1}^{p_{nh}} a_{hk} Y_{hk}\left(\frac{x}{|x|}\right)$$

$$v = \sum_{h=1}^{\infty} \sum_{k=1}^{p_{nh}} \frac{a_{hk}}{(h+2)(n+h-2)} dY_{hk}\left(\frac{x}{|x|}\right) \wedge d(|x|^{h+2})$$

$$\begin{cases} du = \delta v \\ dv = 0, \delta u = 0 \end{cases} \quad \Rightarrow \quad U = u + v \text{ is self-conjugate}$$

$$*v = \sum_{h=1}^{\infty} \sum_{k=1}^{p_{nh}} \frac{a_{hk}}{(h+2)(n+h-2)} * \left(dY_{hk}\left(\frac{x}{|x|}\right) \wedge d(|x|^{h+2}) \right)$$

$$\sum_{h=0}^{\infty} \sum_{k=1}^{p_{nh}} a_{hk} Y_{hk}(x) \qquad (|x| = 1)$$

$$u(x) = \sum_{h=0}^{\infty} |x|^h \sum_{k=1}^{p_{nh}} a_{hk} Y_{hk}\left(\frac{x}{|x|}\right)$$

$$*v = \sum_{h=1}^{\infty} \sum_{k=1}^{p_{nh}} \frac{a_{hk}}{(h+2)(n+h-2)} * \left(dY_{hk}\left(\frac{x}{|x|}\right) \wedge d(|x|^{h+2})\right)$$

$$\sum_{h=1}^{\infty} \sum_{k=1}^{p_{nh}} \frac{a_{hk}}{(h+2)(n+h-2)} * \left(dY_{hk}\left(\frac{x}{|x|}\right) \wedge d(|x|^{h+2})\right) \Big|_{\Sigma}$$

$$x_h = x_h(\varrho, \varphi_1, \dots, \varphi_{n-1}) \qquad h = 1, \dots, n$$

$$g_{ij} = \frac{\partial x}{\partial \varphi_i} \times \frac{\partial x}{\partial \varphi_j}, \quad i, j = 1, \dots, n-1;$$

$$g_{ni} = g_{in} = \frac{\partial x}{\partial \varphi_i} \times \frac{\partial x}{\partial \varrho} \quad i = 1, \dots, n-1; \quad g_{nn} = \frac{\partial x}{\partial \varrho} \times \frac{\partial x}{\partial \varrho}.$$

$$g^{ij} g_{js} = \delta^i_s \qquad g = \det(g_{ij})_{i,j=1,\dots,n}$$

$$\sum_{h=1}^{\infty} \sum_{k=0}^{p_{nh}} \sqrt{\frac{h}{n+h-2}}\, a_{hk} \psi_{hk}$$

$$\psi_{hk} =$$

$$\frac{1}{\sqrt{h(n+h-2)}} \sum_{j=1}^{n-1} (-1)^{n-1-j} \sqrt{g}\, g^{jj} \frac{\partial Y_{hk}}{\partial \varphi_j}\, d\varphi^1 \dots \widehat{j} \dots d\varphi^{n-1}$$

$n = 3:$

$$\sum_{h=1}^{\infty} \sum_{k=0}^{2n} a_{hk} Y_{hk}$$

$$\sum_{h=1}^{\infty} \sum_{k=0}^{2n} \frac{a_{hk}}{(n+1)} \left[\frac{1}{\sin\varphi} \frac{\partial Y_{hk}}{\partial \vartheta} d\varphi - \sin\varphi \frac{\partial Y_{hk}}{\partial \varphi} d\vartheta \right]$$

$$\psi_{hk} =$$

$$\frac{1}{\sqrt{h(n+h-2)}} \sum_{j=1}^{n-1} (-1)^{n-1-j} \sqrt{g}\, g^{jj} \frac{\partial Y_{hk}}{\partial \varphi_j}\, d\varphi^1 \ldots \widehat{j} \ldots d\varphi^{n-1}$$

$$L^2_{n-2}(\Sigma) \qquad\qquad (\gamma, \psi) = \int_{+\Sigma} \gamma \wedge \underset{\Sigma}{*} \psi$$

$$(\psi_{hk}, \psi_{rs}) = \begin{cases} 1 & \text{if } h = r \text{ and } k = s \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{h=1}^{\infty}\sum_{k=0}^{p_{nh}} a_{hk} Y_{hk}, \quad a_{hk} = \int_{\Sigma} f\, Y_{hk} d\sigma, \quad f \in L^2(\Sigma)$$

$$\Rightarrow \ \exists\ g \in L^2_{n-2}(\Sigma)\ : a_{hk} = \sqrt{\frac{n+h-2}{h}} \int_{+\Sigma} g \wedge \underset{\Sigma}{*} \psi_{hk}$$

$$\sum_{h=1}^{\infty}\sum_{k=0}^{p_{nh}} \sqrt{\frac{h}{n+h-2}}\, a_{hk}\psi_{hk} = \sum_{h=1}^{\infty}\sum_{k=0}^{p_{nh}} (g, \psi_{hk})\, \psi_{hk}$$

$$(g, \gamma) = 0, \qquad \forall\ \gamma \in C^{\infty}_{n-2}(\mathbb{R}^n)\ :\ d\gamma = 0 \text{ on } \Sigma$$

Brothers Riesz Theorem for Laplace series (C.)

$$\sum_{h=0}^{\infty}\sum_{k=1}^{p_{nh}} a_{hk} Y_{hk}, \qquad a_{hk} = \int_{\Sigma} Y_{hk}\, d\mu \qquad \mu \in M(\Sigma)$$

$$\exists\, \beta \in M_{n-2}(\Sigma)\ :\ \sum_{h=1}^{\infty}\sum_{k=0}^{p_{nh}} \sqrt{\frac{h}{n+h-2}}\, a_{hk}\psi_{hk} = \sum_{h=1}^{\infty}\sum_{k=0}^{p_{nh}} \left(\int_{+\Sigma} \beta \wedge \underset{\Sigma}{*}\, \psi_{hk}\right)\psi_{hk}$$

$$\int_{+\Sigma} \beta \wedge \underset{\Sigma}{*}\, \gamma = 0 \qquad \forall\, \gamma \in C^{\infty}_{n-2}(\mathbb{R}^n)\ :\ d\gamma = 0 \text{ on } \Sigma\ ,$$

$\Rightarrow \mu$ and β are absolutely continuous

$$a_{hk} = \sqrt{\frac{n+h-2}{h}} \int_{+\Sigma} \beta \wedge \underset{\Sigma}{*}\, \psi_{hk} \iff \int_{\Sigma} Y_{hk}\, d\mu = \frac{1}{h} \int_{+\Sigma} \beta \wedge dY_{hk}$$

$$(h = 1, 2, \ldots ; k = 1, \ldots, p_{nh})$$

$$n = 2 : \qquad \begin{cases} \displaystyle\int_0^{2\pi} \cos h\vartheta \; d\mu = -\int_0^{2\pi} \sin h\vartheta \; d\beta \\ \displaystyle\int_0^{2\pi} \sin h\vartheta \; d\mu = \int_0^{2\pi} \cos h\vartheta \; d\beta \qquad (h = 1, 2, \ldots) \end{cases}$$

$$\int_{+\Sigma} \beta \wedge \underset{\Sigma}{*}\, \gamma = 0 \qquad \forall\, \gamma \in C^{\infty}(\mathbb{R}^2) \;:\; d\gamma = 0 \text{ on } \Sigma$$

$$\int_0^{2\pi} d\beta = 0$$

Caramuta, Silverio, C. (submitted)
$n = 3$:

- construction of a "conjugate Poisson kernel";
- Abel summability of conjugate Laplace series of measures

$$\frac{1}{\pi}\left(\int_{\Sigma}\frac{2-|x-y|}{|x+y|^2}\,M_{iy}\left(\frac{1}{|x-y|}\right)\,d\mu_y\right)dx^i, \quad a.e.\ x\in\Sigma\,;$$

- "Riesz inequalities";
- Convergence in L^p norm of conjugate Laplace series of functions belonging to $L^p(\Sigma)$ $(1 < p < \infty)$;

$$f \in L^p(\Sigma)\ (1 < p < \infty), \qquad a_{hk} = \int_\Sigma f\, Y_{hk}\, d\sigma$$

$$\sum_{h=1}^{\infty} \sum_{k=0}^{2h} \frac{a_{hk}}{(h+1)} \left[\frac{1}{\sin\varphi} \frac{\partial Y_{hk}}{\partial \vartheta} d\varphi - \sin\varphi \frac{\partial Y_{hk}}{\partial \varphi} d\vartheta \right]$$

converges in $L^p_1(\Sigma)$ iff

$$\sum_{h=0}^{\infty} \sum_{k=0}^{2h} a_{hk} Y_{hk}(\varphi, \vartheta) \text{ converges in } L^p(\Sigma)$$

$$f \in L^p(\Sigma), \alpha > 1/2 \Rightarrow (C, \alpha)\text{-summability in } L^p_1(\Sigma)$$

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- "Riesz inequalities";
- Convergence in L^p norm of conjugate Laplace series of functions belonging to $L^p(\Sigma)$ $(1 < p < \infty)$;
- pointwise convergence results.

$$\int_{\Sigma} \frac{|f(y) - f(x)|}{|y - x|^2 |y + x|} d\sigma_y < \infty$$

$$\int_{\Sigma_+} \frac{|f(y) - f(x)|}{|y - x|^2} d\sigma_y < \infty, \quad \int_{\Sigma_-} \frac{|f(y)|}{|y + x|} d\sigma_y < \infty$$

$\Rightarrow$ (C,1)-summability at x.

(e.g. $f \in L^\infty(\Sigma)$; $|f(y) - f(x)| \leqslant H|y - x|^h$)

$$f \in BV(\Sigma), \quad \mu = df, \quad \mu^i(D) = \int_D \mu \wedge dy^i \qquad D \subset \Sigma$$

$$x \in \Sigma \ : \ \int_{\Sigma} \frac{d|\mu^i|_y}{|x - y|\sqrt{|x + y|}} < \infty \qquad (i = 1, 2, 3)$$

$\Rightarrow$ convergence at x.

Thank you for your attention !