Indeterminate moment problems and Christian Berg

The real world is complex

Erik Koelink

Radboud Universiteit

Christian Berg

in Munich 2005

in Luminy 2007

in Sousse 2013

Tak til arrangørerne

Moment problems: determinate and indeterminate

Suppose that we have a moment sequence $(m_k)_{k\in\mathbb{N}}$,
$m_k = \int_{\mathbb{R}} x^k \, d\mu(x)$, $m_0 = 1$. Does the sequence determine the measure μ?

If yes, then the moment problem is determinate.

If no, then the moment is indeterminate.

Hamburger moment problem (Hans Ludwig Hamburger 1889–1956).

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Stieltjes moment problem

Suppose that we have a moment sequence $(m_k)_{k\in\mathbb{N}}$, $m_k = \int_0^\infty x^k \, d\mu(x)$. Does the sequence determine the measure μ with $\mathrm{supp}(\mu) \subset [0,\infty)$?
If yes, then the Stieltjes moment problem is determinate.
If no, then the Stieltjes moment is indeterminate.
(Thomas Joannes Stieltjes jr 1856–1894).

Solving problems and/or building up theory

Sir Timothy Gowers
The Two Cultures of Mathematics

(i) The point of solving problems is to understand mathematics better.
(ii) The point of understanding mathematics is to become better able to solve problems.

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Two approaches

- Understand the general theory of (in-)determinate moment problems
- Understand as well as possible explicit (in-)determinate moment problems

Christian's work on smallest eigenvalues of Hankel matrices

paper with Yang Chen and Mourad Ismail (Math. Scand. 2002)
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$$H_{k,l} = m_{k+l}, \quad H = \begin{pmatrix} m_0 & m_1 & m_2 & \cdots \\ m_1 & m_2 & m_3 & \cdots \\ m_2 & m_3 & m_4 & \\ \vdots & \vdots & & \ddots \end{pmatrix}, \quad H_N = (H_{k,l})_{0 \le k,l \le N}$$

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H_N positive definite matrix with smallest positive eigenvalue λ_N.
The sequence $(\lambda_N)_N$ is decreasing.

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H_N positive definite matrix with smallest positive eigenvalue λ_N. The sequence $(\lambda_N)_N$ is decreasing.

Theorem (BCI)

$\lim_{N\to\infty} \lambda_N = 0 \iff$ the moment problem is determinate

Theorem (BS)

For any decreasing sequence (τ_N) of positive numbers with $\lim_{N\to\infty} \tau_N = 0$ and $\tau_0 < 1$, there exists a determinate moment problem with $\lambda_N \ge \frac{1}{2}\tau_N$.

Christian's work on smallest eigenvalues of Hankel matrices

Corresponding orthonormal polynomials $(P_n)_{n=0}^{\infty}$

$$\int_{\mathbb{R}} P_n(x) P_m(x) \, d\mu(x) = \delta_{m,n}$$

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$$\int_{\mathbb{R}} P_n(x)P_m(x)\,d\mu(x) = \delta_{m,n} \qquad \Longrightarrow$$

$$xP_n(x) = b_n P_{n+1}(x) + a_n P_n(x) + b_{n-1}P_{n-1}(x),$$
$$xP_0(x) = b_0 P_1(x) + a_0 P_0(x),$$
$$P_0(x) = 1, \quad b_n > 0, \quad a_n \in \mathbb{R}$$

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Theorem

Let $\{e_n\}_{n\in\mathbb{N}}$ standard ONB of $\ell^2(\mathbb{N})$, $D \subset \ell^2(\mathbb{N})$ subspace of finite linear combinations of standard ONB. Define Jacobi operator $J\colon D \to \ell^2(\mathbb{N})$ by

$$Je_n = b_ne_{n+1} + a_ne_n + b_{n-1}e_{n-1}, \qquad Je_0 = b_0e_1 + a_0e_0,$$

then (J, D) is a symmetric operator. (J, D) is essentially self-adjoint if and only if the moment problem is determinate.

Part of proof of Theorem (BCI)

$$\lambda_N = \min\left\{\frac{\langle H_N v, v\rangle}{\|v\|^2} : v \in \mathbb{C}^{N+1} \setminus \{0\}\right\}$$

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$$\lambda_N = \min\left\{\frac{\langle H_N v, v\rangle}{\|v\|^2} : v \in \mathbb{C}^{N+1} \setminus \{0\}\right\} \quad \Longrightarrow \quad \lambda_{N+1} \leq \lambda_N$$

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$$\implies \quad \frac{1}{\lambda_N} = \max\left\{\frac{\langle (H_N)^{-1} v, v\rangle}{\|v\|^2} : v \in \mathbb{C}^{N+1} \setminus \{0\}\right\}$$

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Lemma

$(H_N)^{-1} = A_N$ where

$$K_N(z,w) = \sum_{n=0}^{N} P_n(z)P_n(w) = \sum_{k,l=0}^{N} (A_N)_{k,l} z^k w^l$$

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Proof.

Let $0 \le k \le N$, then

$$y^k = \int_{\mathbb{R}} K_N(x,y)x^k \, d\mu(x)$$

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$$= \sum_{j=0}^{N} \left(\sum_{i=0}^{N} m_{i+k} (A_N)_{i,j} \right) y^j \quad \implies \quad \sum_{i=0}^{N} H_{k,i}(A_N)_{i,j} = \delta_{k,j}$$

□

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$$A_N \text{ positive definite} \implies \frac{1}{\lambda_N} \leq \mathrm{tr}(A_N)$$

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Assume moment problem is *indeterminate*. Then

$$\forall\, z \in \mathbb{C} : \qquad \sum_{n=0}^{\infty} |P_n(z)|^2 < \infty$$

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$$\lambda_N \geq \frac{1}{\rho_0} > 0$$

Some other results

- (BCI) Converse can also be proved in this fashion (also proved by Hamburger)
- (BS) Better estimate

$$\frac{1}{\lambda_N} \geq (1 - |z|^2) \sum_{n=0}^{N} |P_n(z)|^2, \qquad |z| < 1.$$

- (BS) If $b = \limsup_{n\to\infty} b_n < \infty$, then

$$\limsup_{N\to\infty} \sqrt[N]{\lambda_N} \leq \frac{b^2}{1+b^2}$$

- (BS) For any decreasing sequence (τ_N) of positive numbers with $\lim_{N\to\infty} \tau_N = 0$ and $\tau_0 = 1$, there exists a determinate moment problem with $\lambda_N \leq \tau_N$.

Christian and the Nevanlinna parametrisation

Nevanlinna parametrisation

Consider an indeterminate moment problem. Then there exist four entire functions A, B, C and D so that the solutions to the moment problem are in $1-1$ correspondence with Pick functions ϕ (i.e. holomorphic on the upper half plane and preserving the upper half plane) extended with $\{\infty\}$ by the Stieltjes transform

$$\int_{\mathbb{R}} \frac{1}{z-x}\, d\mu_\phi(x) = \frac{A(z)\phi(z) - C(z)}{B(z)\phi(z) - D(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}$$

Rolf Nevanlinna
1895–1980

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Theorem (Berg-Pedersen)

A, B, C and D have the same order and type.

For entire function f the order ρ and type σ

$$M(r)=\sup_{|z|\le r}|f(z)|, \quad \rho=\limsup_{r\to\infty}\frac{\log\log M(r)}{\log r}, \quad \sigma=\limsup_{r\to\infty}\frac{\log M(r)}{r^\rho}$$

N-extremal measures

N-extremal points

The N-extremal points are the solutions $\mu = \mu_t$ to the moment problem for which $\phi(z) = t \in \mathbb{R} \cup \{\infty\}$. Characterised by

- the polynomials are dense in $L^2(\mu)$ or
- μ corresponds to the spectral measure of self-adjoint extension of (J, D)

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N-extremal measures

μ_t discrete probability measure supported on zeros of entire function $z \mapsto B(z)t - C(z)$ (or $B(z)$ for $t = \infty$).
Problem: describe the support and the weights explicitly!

N-extremal measures for continuous q^{-1}-Hermite pol's

Continuous q^{-1}-Hermite polynomials

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Alternative approach for this N-extremal measures based on study of symmetric q^{-1} Al-Salam–Chihara polynomials by Christiansen-K in Constr. Approx. 2008.

Symmetric q^{-1} Al-Salam–Chihara polynomials

$$P_n(u; a, b \mid q) = a^{-n}(ab; q)_n \, {}_3\varphi_2\left(\begin{matrix} q^{-n}, au, a/u \\ ab, 0 \end{matrix}; q, q\right)$$

$$Q_n(y; \beta \mid q) = i^n P_n(ie^{-y}; \beta^{\frac{1}{2}}, -\beta^{\frac{1}{2}} \mid q^{-1})$$

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$$2x h_n^{(\beta)}(x \mid q) = h_{n+1}(x \mid q) + q^{-n}(1 - q^n)(1 + \beta q^{1-n}) h_{n-1}^{(\beta)}(x \mid q)$$

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continuous q^{-1}-Hermite polynomials: $h_n(x\mid q)=h_n^{(0)}(x\mid q)$

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Askey scheme: hence solution to 2nd order operator

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$L \colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ with ONB $\{e_l\}_{l \in \mathbb{Z}}$, $\alpha \in \mathbb{R} \setminus \{0\}$,

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$$a_l = \frac{\alpha q^{l+\frac12}}{1+\alpha^2 q^{2l+1}}\sqrt{\frac{(\beta+\alpha^2q^{2l})(1+\alpha^2q^{2l})}{(1+\alpha^2q^{2l})(1+\beta\alpha^2q^{2l})}}$$

$$b_l = \frac{\alpha^2(1+q)(1-\beta q)q^{2l-1}}{(1+\alpha^2q^{2l+1})(1+\alpha^2q^{2l-1})}$$

N-extremal measures for continuous q^{-1}-Hermite pol's

Theorem (Christiansen-K)

L is a compact operator with $\sigma(L) = \overline{q^{\mathbb{N}} \cup -\beta q^{1+\mathbb{N}}}$ (indep of α).
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Corollary

$\lambda_\alpha^{(0)}$ for $\alpha \in (q, 1]$ are the N-extremal measures for the continuous q^{-1}-Hermite pol's

Other, but related, issues

- ad Christiansen-K
  - other proof of summation formula for Bailey's ${}_6\psi_6$-formula
  - extends previous results of Christiansen-Ismail (TAMS 2006)
- Christian's historical interest in moment problem
- How to generate explicit measures in the indeterminate case?
- Work on *m*-canonical measures (partly joint Durán)
- Multidimensional moment problems (joint with Thill)
- $\cdots$ on indeterminate case
- Matrix-valued moment problem
- Bootstrap method with Mourad Ismail (CJM 1996)
- Etc, etc, etc $\cdots$

The end

Tillykke og i mange år