Approximation of solutions to multidimensional parabolic equations by Approximate Approximations

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$$\frac{\partial u}{\partial t} - \Delta_{\mathbf{x}} u + 2\mathbf{b} \cdot \nabla_{\mathbf{x}} u + c\, u = f(\mathbf{x}, t), \qquad u(\mathbf{x}, 0) = g(\mathbf{x}) \tag{1}$$

$(\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R}_+$ with $\mathbb{R}_+ = [0, \infty)$, $\mathbf{b} \in \mathbb{C}^n$, $c \in \mathbb{C}$.

$$[\mathbf{P}, \mathbf{Q}] = \{\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n : P_j \le x_j \le Q_j,\ j = 1, ..., n\} = \prod_{j=1}^{n} [P_j, Q_j]$$

$\operatorname{supp} f \subseteq [\mathbf{P}, \mathbf{Q}] \times \mathbb{R}_+$, $\operatorname{supp} g \subseteq [\mathbf{P}, \mathbf{Q}]$.

F.L. - V. Maz'ya - G. Schmidt, *Approximation of solutions to multidimensional parabolic equations by Approximate Approximations*, to appear on Appl. Comput. Harmon. Anal., available on line at http://dx.doi.org/10.1016/J.acha.2015.06.001

The solution of (1) can be written as

$$u(\mathbf{x},t) = \mathcal{P}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]}\, g(\mathbf{x},t) + \mathcal{H}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]}\, f(\mathbf{x},t)$$

where

$$\mathcal{P}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]} g(\mathbf{x},t) = \frac{\mathrm{e}^{-ct}}{(4\pi\, t)^{n/2}} \int\limits_{[\mathbf{P},\mathbf{Q}]} \mathrm{e}^{-|\mathbf{x}-\mathbf{y}-2\mathbf{b}t|^2/(4t)}\, g(\mathbf{y})d\mathbf{y}\,,$$

$$\mathcal{H}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]} f(\mathbf{x},t) = \int\limits_0^t \frac{\mathrm{e}^{-cs}ds}{(4\pi\, s)^{n/2}} \int\limits_{[\mathbf{P},\mathbf{Q}]} \mathrm{e}^{-|\mathbf{x}-\mathbf{y}-2\mathbf{b}s|^2/(4s)}\, f(\mathbf{y},t-s)d\mathbf{y}$$

$$= \int\limits_0^t (\mathcal{P}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]}\, f(\cdot,s))(\mathbf{x},t-s)ds.$$

How to compute $\mathcal{P}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]}\, g$ and $\mathcal{H}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]}\, f$?

V. Maz'ya, G. Schmidt, *Approximate Approximations*, Math. Surveys and Monographs, 141, AMS 2007.

$\mathcal{D}$ is a positive fixed parameter, h is the step, $\eta \in \mathcal{S}(\mathbb{R}^n)$
Take a smooth extension $\widetilde{g}$ of g in a neighborhood Ω_{rh} of $[\mathbf{P}, \mathbf{Q}] \subset \mathbb{R}^n$

$$\mathcal{N}_h^{(r)} g(\mathbf{x}) := \frac{1}{\mathcal{D}^{n/2}} \sum_{h\mathbf{m} \in \Omega_{rh}} \widetilde{g}(h\mathbf{m})\, \eta\left(\frac{\mathbf{x} - h\mathbf{m}}{h\sqrt{\mathcal{D}}}\right)$$

$\Omega_{rh} = \prod_{j=1}^{n} I_j$ with $I_j = (P_j - rh\sqrt{\mathcal{D}}, Q_j + rh\sqrt{\mathcal{D}})$

Moment conditions

$$\int_{\mathbb{R}^n} \eta(\mathbf{x})\, \mathbf{x}^{\alpha} d\mathbf{x} = \delta_{0,\alpha},\ 0 \le |\alpha| < N$$

Error estimate

$\forall\, \epsilon > 0\, \exists\, \mathcal{D} > 0$:

$$|g(\mathbf{x}) - \mathcal{N}_h^{(r)} g(\mathbf{x})| = \mathcal{O}((h\sqrt{\mathcal{D}})^N) + \epsilon$$

for all $\mathbf{x} \in [\mathbf{P}, \mathbf{Q}]$

$\mathcal{D}$, $\mathcal{D}_0$ are positive fixed parameters, h, τ are the steps, $\widetilde{\eta} \in \mathcal{S}(\mathbb{R})$, $\eta \in \mathcal{S}(\mathbb{R}^n)$
Take a smooth extension $\widetilde{f}$ of f in a neighborhood of $[\mathbf{P}, \mathbf{Q}] \times [0, T]$, $T > 0$

$$\mathcal{M}^{(r)}_{h,\tau} f(\mathbf{x}, t) := \frac{1}{\mathcal{D}_0^{1/2} \mathcal{D}^{n/2}} \sum_{\substack{\tau i \in \widetilde{\Omega}_{r_0 \tau} \\ h\mathbf{m} \in \Omega_{rh}}} \widetilde{f}(h\mathbf{m}, \tau i)\, \widetilde{\eta}\left(\frac{t - \tau i}{\tau\sqrt{\mathcal{D}_0}}\right) \eta\left(\frac{\mathbf{x} - h\mathbf{m}}{h\sqrt{\mathcal{D}}}\right)$$

$\widetilde{\Omega}_{r_0\tau} = (-r_0\tau\sqrt{\mathcal{D}_0}, T + r_0\tau\sqrt{\mathcal{D}_0},)$; $\Omega_{rh} = \prod_{j=1}^n I_j, I_j = (P_j - rh\sqrt{\mathcal{D}}, Q_j + rh\sqrt{\mathcal{D}})$

Moment conditions

$$\int_{\mathbb{R}} \widetilde{\eta}(t)\, t^s dt = \delta_{0,s}, \; 0 \le s < N$$

$$\int_{\mathbb{R}^n} \eta(\mathbf{x})\, \mathbf{x}^\alpha d\mathbf{x} = \delta_{0,\alpha}, \; 0 \le |\alpha| < N$$

Error estimate

$\forall\, \epsilon > 0\; \exists\, \mathcal{D}_0,\, \mathcal{D} > 0$:

$$|f(\mathbf{x}, t) - \mathcal{M}^{(r)}_{h,\tau} f(\mathbf{x}, t)| = \mathcal{O}((h\sqrt{\mathcal{D}})^N + (\tau\sqrt{\mathcal{D}_0})^N) + \epsilon$$

for all $\mathbf{x} \in [\mathbf{P}, \mathbf{Q}]$ and $t \in [0, T]$, $T > 0$.

Examples of generating functions:

- $\eta_2(\mathbf{x}) = \pi^{-n/2} \mathrm{e}^{-|\mathbf{x}|^2}, \qquad N = 2, \quad \epsilon = \mathcal{O}(\mathrm{e}^{-\pi^2 \mathcal{D}})$

$$\mathcal{D} = 2: \quad \mathrm{e}^{-\pi^2 \mathcal{D}} = 2.68 \times 10^{-9}; \qquad \mathcal{D} = 4: \quad \mathrm{e}^{-\pi^2 \mathcal{D}} = 7.15 \times 10^{-18}$$

- $\eta_{2M}(\mathbf{x}) = \pi^{-n/2} L_{M-1}^{(n/2)}(|\mathbf{x}|^2) \mathrm{e}^{-|\mathbf{x}|^2}$, $N = 2M$
with the generalized Laguerre polynomials

$$L_k^{(\gamma)}(y) = \frac{\mathrm{e}^y y^{-\gamma}}{k!} \left(\frac{d}{dy} \right)^k (\mathrm{e}^{-y} y^{k+\gamma}), \quad \gamma > -1$$

- $\eta_{2M}(\mathbf{x}) = \prod_{j=1}^{n} \widetilde{\eta}_{2M}(x_j); \quad \widetilde{\eta}_{2M}(x) = \pi^{-1/2} L_{M-1}^{(1/2)}(x^2) \mathrm{e}^{-x^2}, \qquad N = 2M$

$$u(\mathbf{x},t) = \mathcal{P}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]}\, g(\mathbf{x},t) + \mathcal{H}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]}\, f(\mathbf{x},t)\,,$$

$$g \longrightarrow \mathcal{N}^{(r)}_h g\,; \qquad f \longrightarrow \mathcal{M}^{(r)}_{h,\tau} f$$

$$u_{h,\tau}(\mathbf{x},t) = \mathcal{P}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]}(\mathcal{N}^{(r)}_h g)(\mathbf{x},t) + \mathcal{H}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]}(\mathcal{M}^{(r)}_{h,\tau} f)(\mathbf{x},t)$$

provides an approximation of $u(\mathbf{x},t)$ such that

$$|u(\mathbf{x},t) - u_{h,\tau}(\mathbf{x},t)| = \mathcal{O}((h\sqrt{\mathcal{D}})^N + (\tau\sqrt{\mathcal{D}_0})^N) + \epsilon$$

for all $\mathbf{x} \in [\mathbf{P},\mathbf{Q}]$ and $t \in [0,T],\ T > 0$.

Second Order Formula: $\eta_2(\mathbf{x}) = \mathrm{e}^{-|\mathbf{x}|^2}\pi^{-n/2}$; $\widetilde{\eta}_2(t) = \mathrm{e}^{-t^2}\pi^{-1/2}$

$$\mathcal{P}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]}(\mathcal{N}^{(r)}_h g)(\mathbf{x},t) = \frac{1}{(\pi\mathcal{D})^{n/2}} \sum_{h\mathbf{m}\in\Omega_{rh}} \widetilde{g}(h\mathbf{m})\, \mathcal{P}^{(h^2\mathcal{D}c,\, h\sqrt{\mathcal{D}}\mathbf{b})}_{[\mathbf{P_m},\mathbf{Q_m}]}(\mathrm{e}^{-|\cdot|^2})\Big(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}, \frac{t}{h^2\mathcal{D}}\Big)$$

approximates $\mathcal{P}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]} g$ in $[\mathbf{P},\mathbf{Q}]\times[0,T]$;

$$\mathcal{H}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]}(\mathcal{M}^{(r)}_{h,\tau} f)(\mathbf{x},t) = \frac{1}{(\pi\mathcal{D}_0)^{1/2}(\pi\mathcal{D})^{n/2}} \sum_{\substack{\tau i\in\widetilde{\Omega}_{r_0\tau} \\ h\mathbf{m}\in\Omega_{rh}}} \widetilde{f}(h\mathbf{m},\tau i)$$
$$\times \int_0^t \mathrm{e}^{-\frac{(\sigma-(t-\tau i))^2}{\tau^2\mathcal{D}_0}} \mathcal{P}^{(h^2\mathcal{D}c,\, h\sqrt{\mathcal{D}}\mathbf{b})}_{[\mathbf{P_m},\mathbf{Q_m}]}(\mathrm{e}^{-|\cdot|^2})\Big(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}, \frac{\sigma}{h^2\mathcal{D}}\Big)\, d\sigma$$

approximates $\mathcal{H}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]} f$ in $[\mathbf{P},\mathbf{Q}]\times[0,T]$.

$$\mathbf{P_m} = \frac{\mathbf{P}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}; \qquad \mathbf{Q_m} = \frac{\mathbf{Q}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}$$

$$\mathcal{P}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]}(\mathrm{e}^{-|\cdot|^2})(\mathbf{x},t) = \mathrm{e}^{-c\,t}\prod_{j=1}^{n}\left(\phi^{(b_j)}_{P_j}(x_j,t) - \phi^{(b_j)}_{Q_j}(x_j,t)\right)$$

with the analytic expression

$$\phi^{(b)}_P(x,t) = \frac{\mathrm{e}^{-(x-2bt)^2/(1+4t)}}{2\sqrt{1+4t}}\,\mathrm{erfc}\left(\sqrt{\frac{1+4t}{4t}}\Big(P - \frac{x-2bt}{1+4t}\Big)\right).$$

Here erfc denotes the complementary error function

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty}\mathrm{e}^{-t^2}\,dt.$$

$$\mathrm{erfc}(-\infty) = 2, \qquad \mathrm{erfc}(\infty) = 0$$

$\Longrightarrow$ all type of unbounded rectangular domains are covered

At the point of the uniform grid $\{(h\mathbf{k}, \tau\ell)\}$:

$$\mathcal{P}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]}(\mathcal{N}^{(r)}_h g)(h\mathbf{k}, \tau\ell) = \sum_{h\mathbf{m}\in\Omega_{rh}} \widetilde{g}(h\mathbf{m})\, a_{\mathbf{m},\mathbf{k},\ell}$$

$$a_{\mathbf{m},\mathbf{k},\ell} = \frac{1}{(\pi\mathcal{D})^{n/2}} \mathcal{P}^{(h^2\mathcal{D}c,\, h\sqrt{\mathcal{D}}\mathbf{b})}_{[\mathbf{P_m},\mathbf{Q_m}]}(\mathrm{e}^{-|\cdot|^2})\left(\frac{\mathbf{k}-\mathbf{m}}{\sqrt{\mathcal{D}}}, \frac{\tau\ell}{h^2\mathcal{D}}\right)$$

How to compute the sum?

Suppose that $\widetilde{g}(\mathbf{x})$ allows a *separated representation*:

$$\widetilde{g}(\mathbf{x}) = \sum_{p=1}^{P} \alpha_p \prod_{j=1}^{n} g_j^{(p)}(x_j) + \mathcal{O}(\epsilon)$$

Instead of computing $\{a_{\mathbf{m},\mathbf{k},\ell}\}$ and the $n-$dimensional sum

$$\sum_{h\mathbf{m}\in\Omega_{rh}} \widetilde{g}(h\mathbf{m})\, a_{\mathbf{m},\mathbf{k},\ell}$$

it suffices to compute the one-dimensional sums

$$S_j^{(p)}(k,t) = \sum_{hm_j\in I_j} g_j^{(p)}(hm_j)\Big(\phi_{P_{\mathbf{m}_j}}^{(h\sqrt{\mathcal{D}}b_j)}\big(\frac{k-m_j}{\sqrt{\mathcal{D}}},\frac{t}{h^2\mathcal{D}}\big) - \phi_{Q_{\mathbf{m}_j}}^{(h\sqrt{\mathcal{D}}b_j)}\big(\frac{k-m_j}{\sqrt{\mathcal{D}}},\frac{t}{h^2\mathcal{D}}\big)\Big)$$

and to sum up

$$\mathcal{P}_{[\mathbf{P},\mathbf{Q}]}^{(c,\mathbf{b})}(\mathcal{N}_h^{(r)} g)(h\mathbf{k},\tau\ell) \approx \frac{\mathrm{e}^{-c\tau\ell}}{\mathcal{D}^{n/2}} \sum_{p=1}^{P} \alpha_p \prod_{j=1}^{n} S_j^{(p)}(k_j,\tau\ell)$$

$\Rightarrow$ Computing time and memory requirement proportional to dimension n

At the point of the uniform grid $\{(h\mathbf{k}, \tau\ell)\}$:

$$\mathcal{H}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]}(\mathcal{M}^{(r)}_{h,\tau} f)(h\mathbf{k}, \tau\ell) = \sum_{\substack{\tau i \in \widetilde{\Omega}_{r_0 \tau} \\ h\mathbf{m} \in \Omega_{rh}}} \widetilde{f}(h\mathbf{m}, \tau i) b_{\mathbf{m},i,\mathbf{k},\ell}$$

$$b_{\mathbf{m},i,\mathbf{k},\ell} = \frac{1}{(\pi\mathcal{D}_0)^{1/2}(\pi\mathcal{D})^{n/2}} \int_0^{\tau\ell} \mathrm{e}^{-\frac{(\sigma-\tau(\ell-i))^2}{\tau^2\mathcal{D}_0}} \mathcal{P}^{(h^2\mathcal{D}c, h\sqrt{\mathcal{D}}\mathbf{b})}_{[\mathbf{P_m},\mathbf{Q_m}]}(\mathrm{e}^{-|\cdot|^2})\Big(\frac{\mathbf{k}-\mathbf{m}}{\sqrt{\mathcal{D}}}, \frac{\sigma}{h^2\mathcal{D}}\Big)\, d\sigma$$

How to compute $b_{\mathbf{m},i,\mathbf{k},\ell}$?
How to compute the sum?

The second term involves additionally an integration. We use a quadrature based on the classical trapezoidal rule. It is known that it is exponentially converging for rapidly decaying smooth functions on the real line.
First we make the substitution (Takahasi - Mori 1974)

$$\sigma = \frac{\tau\ell}{1+\mathrm{e}^{-\pi \sinh \xi}}\,,$$

which transforms to an integral over $\mathbb{R}$ with doubly exponentially decaying integrand. Then we apply the classical trapezoidal rule with step κ and sufficiently large S:

$$b_{\mathbf{m},i,\mathbf{k},\ell} \approx \frac{\pi\tau\ell\kappa}{2\pi^{n/2}}\sum_{s=-S}^{S}\omega_s\,\mathrm{e}^{-(\ell/(1+\mathrm{e}^{\pi\,\sinh(s\kappa)})-i)^2/\mathcal{D}_0}\,\big(\mathcal{P}_{[\mathbf{P_m},\mathbf{Q_m}]}^{(h^2\mathcal{D}c,h\sqrt{\mathcal{D}}\mathbf{b})}\mathrm{e}^{-|\cdot|^2}\big)\Big(\frac{\mathbf{k}-\mathbf{m}}{\sqrt{\mathcal{D}}},\frac{\tau\ell}{a_s h^2\mathcal{D}}\Big)$$

where

$$\omega_s = \frac{\cosh(s\kappa)}{1+\cosh(\pi\sinh(s\kappa))}\,,\quad a_s = 1+\mathrm{e}^{-\pi\,\sinh(s\kappa)}\,.$$

Then, for the second term one gets

$$\mathcal{H}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]}(\mathcal{M}^{(r)}_{h,\tau} f)(h\mathbf{k},\tau\ell) \approx$$

$$\frac{\tau\ell\kappa\sqrt{\pi}}{2\sqrt{\mathcal{D}_0\mathcal{D}^n\pi^n}} \sum_{s=-S}^{S} \omega_s \sum_{\substack{\tau i\in\widetilde{\Omega}_{r_0\tau} \\ h\mathbf{m}\in\Omega_{rh}}} \mathrm{e}^{-(\ell/(1+\mathrm{e}^{\pi\,\sinh(s\kappa)})-i)^2/\mathcal{D}_0}\, \widetilde{f}(h\mathbf{m},\tau i)$$

$$\times\, \mathcal{P}^{(h^2\mathcal{D}c,h\sqrt{\mathcal{D}}\mathbf{b})}_{[\mathbf{P_m},\mathbf{Q_m}]}(\mathrm{e}^{-|\cdot|^2})\left(\frac{\mathbf{k}-\mathbf{m}}{\sqrt{\mathcal{D}}},\frac{\tau\ell}{a_s h^2\mathcal{D}}\right)$$

Suppose that $\widetilde{f}(\mathbf{x},t)$ allows a *separated representation*:

$$\widetilde{f}(\mathbf{x},t) = \sum_{p=1}^{P} \beta_p \prod_{j=1}^{n} f_j^{(p)}(x_j,t) + \mathcal{O}(\epsilon)$$

Then, for the second term one gets

$$\mathcal{H}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]}(\mathcal{M}^{(r)}_{h,\tau} f)(h\mathbf{k},\tau\ell) \approx$$

$$\frac{\tau\ell\kappa\sqrt{\pi}}{2\sqrt{\mathcal{D}_0\mathcal{D}^n}} \sum_{s=-S}^{S} \omega_s \mathrm{e}^{-c\,\tau\ell/a_s} \sum_{\tau i\in\widetilde{\Omega}_{r_0\tau}} \mathrm{e}^{-(\ell/(1+\mathrm{e}^{\pi\,\sinh(s\kappa)})-i)^2/\mathcal{D}_0} \sum_{p=1}^{P} \beta_p \prod_{j=1}^{n} T_j^{(p)}(k_j,\tau\ell,\tau i,a_s)$$

where

$$T_j^{(p)}(k_j,\tau\ell,\tau i,a_s) = \sum_{hm_j\in I_j} f_j^{(p)}(hm_j,\tau i)$$
$$\times\Big(\phi^{(h\sqrt{\mathcal{D}}b_j)}_{P_{\mathbf{m}_j}}\Big(\frac{k_j-m_j}{\sqrt{\mathcal{D}}},\frac{\tau\ell}{a_s h^2\mathcal{D}}\Big) - \phi^{(h\sqrt{\mathcal{D}}b_j)}_{Q_{\mathbf{m}_j}}\Big(\frac{k_j-m_j}{\sqrt{\mathcal{D}}},\frac{\tau\ell}{a_s h^2\mathcal{D}}\Big)\Big)$$

$\Rightarrow$ Computing time and memory requirement proportional to dimension n

Our method combines the basis functions introduced by Approximate Approximations with an approach from Khoromskij for computing volume potentials (Preprint MPI MiS 2008,J. Comp. Appl. Math. 2010):
piecewise constant and piecewise linear approximations of the density
numerical test for $n = 3$, approximation order $\mathcal{O}(h^3)$

$\Longrightarrow$ We join the tensor product approach with high order cubature formulas of approximate approximations

Results on fast computation of volume potentials:

- F.L., V. Maz'ya, G. Schmidt, *On the fast computation of high dimensional volume potentials*, Math. Comp. (2011)
- F.L., V. Maz'ya, G. Schmidt, *Accurate cubature of volume potentials over high-dimensional half-spaces*, J. Math. Sci. (2011)
- F.L., V. Maz'ya, G. Schmidt, *Fast cubature of volume potentials over rectangular domains by Approximate Approximations*, Appl. Comput. Harmon. Anal. (2014)
- F.L., V. Maz'ya, G. Schmidt, *On the computation of high-dimensional potentials of advection-diffusion operators*, Mathematika (2015)

We get similar formulas for high order approximation if we assume that $\eta(\mathbf{x})$ is the product of univariate basis functions of the form Gaussians times special polynomials

$$\eta(\mathbf{x}) = \prod_{j=1}^{n} \eta_{2M}(x_j); \quad \eta_{2M}(x_j) = \frac{(-1)^{M-1}}{2^{2M-1}\sqrt{\pi}(M-1)!} \frac{H_{2M-1}(x_j)\mathrm{e}^{-x_j^2}}{x_j}$$

where H_k are the Hermite polynomials

$$H_k(x) = (-1)^k \mathrm{e}^{x^2} \left(\frac{d}{dx}\right)^k \mathrm{e}^{-x^2}$$

and $\widetilde{\eta}(t) = \eta_{2M}(t)$.

$$\mathcal{P}^{(c,\mathbf{b})}_{[\mathbf{P},\mathbf{Q}]}\left(\prod_{j=1}^{n} \eta_{2M}(x_j)\right) = \text{"}\textit{separated} \quad \textit{representation}\text{"}$$

Theorem

Let $M \geq 1$. We obtain

$$(\mathcal{P}_{[P,Q]}^{(c,\mathbf{b})}(\prod_{j=1}^{n} \eta_{2M}(\cdot)))(\mathbf{x}, t) = \mathrm{e}^{-c\,t} \prod_{j=1}^{n}(\Phi_M(4t, x_j - 2b_j t, P_j) - \Phi_M(4t, x_j - 2b_j t, Q_j))$$

where

$$\Phi_M(t, x, p) = \frac{\mathrm{e}^{-x^2/(1+t)}}{2\sqrt{\pi}} \Big(\mathrm{erfc}\,(F(t,x,p)) \mathcal{R}_M(t,x) - \frac{\mathrm{e}^{-F^2(t,x,p)}}{\sqrt{\pi}} \mathcal{Q}_M(t,x,p) \Big),$$

with

$$F(t,x,p) = \sqrt{\frac{1+t}{t}} \Big(p - \frac{x}{1+t} \Big)$$

$\mathcal{R}_M$ and $\mathcal{Q}_M$ are polynomials in x of degree $2M-2$ and $2M-3$ (provided that $M > 1$), respectively.

$$\mathcal{R}_M(t,x) = \sum_{k=0}^{M-1} \frac{1}{(1+t)^{k+1/2}} \frac{(-1)^k}{4^k k!} H_{2k}\left(\frac{x}{\sqrt{1+t}}\right),$$

$$\mathcal{Q}_1(t,x,p) = 0,$$

$$\mathcal{Q}_M(t,x,p) = 2\sum_{k=1}^{M-1} \frac{(-1)^k}{k!\,4^k} \sum_{\ell=1}^{2k} \frac{(-1)^\ell}{t^{\ell/2}} \Bigg(H_{2k-\ell}(p) H_{\ell-1}\Big(\frac{p-x}{\sqrt{t}}\Big) - \binom{2k}{\ell} H_{2k-\ell}\Big(\frac{x}{\sqrt{1+t}}\Big) \frac{H_{\ell-1}\big(F(t,x,p)\big)}{(1+t)^{k+1/2}} \Bigg),\ M > 1$$

H_k Hermite polynomials

The values of $\mathcal{R}_M(t,x)$ and $\mathcal{Q}_M(t,x,p)$ for $M = 1,2,3$ are given by

$$\mathcal{R}_1(t,x) = \frac{1}{\sqrt{1+t}}; \qquad \mathcal{Q}_1(t,x,p) = 0\,,$$

$$\mathcal{R}_2(t,x) = \frac{1}{\sqrt{1+t}} + \frac{1}{2(1+t)^{3/2}} - \frac{x^2}{(1+t)^{5/2}}\,,$$

$$\mathcal{Q}_2(t,x,p) = \frac{\sqrt{t}}{(1+t)}\Big(\frac{x}{1+t} + p\Big),$$

$$\mathcal{R}_3(t,x) = \mathcal{R}_2(t,x) + \frac{3}{8(1+t)^{5/2}} - \frac{3\,x^2}{2(1+t)^{7/2}} + \frac{x^4}{2(1+t)^{9/2}}\,,$$

$$\mathcal{Q}_3(t,x,p) = -\frac{\sqrt{t}}{4(1+t)}\Big(\frac{2x^3}{(1+t)^3} + \frac{2px^2 - 5x}{(1+t)^2} + \frac{(2p^2-5)x - 3p}{1+t} + p(2p^2-7)\Big)$$

Thus we get a computable approximation of the initial value problem (1)

$$u_{h,\tau}(h\mathbf{k},\tau\ell) \approx \frac{\mathrm{e}^{-c\tau\ell}}{\mathcal{D}^{n/2}} \sum_{p=1}^{P} \alpha_p \prod_{j=1}^{n} S_j^{(p)}(k_j,\tau\ell) +$$

$$\frac{\tau\ell\kappa\pi}{2\sqrt{\mathcal{D}_0\mathcal{D}^n}} \sum_{s=-S}^{S} \omega_s \mathrm{e}^{-c\,\tau\ell/a_s} \sum_{\tau i\in\widetilde{\Omega}_{r_0\tau}} \eta_{2M}\Big(\frac{\ell/(1+\mathrm{e}^{\pi\,\sinh(\kappa s)})-i}{\sqrt{\mathcal{D}_0}}\Big)$$

$$\sum_{p=1}^{P} \beta_p \prod_{j=1}^{n} T_j^{(p)}(k_j,\tau\ell,\tau i,a_s)\,,$$

which has the order $\mathcal{O}((h\sqrt{\mathcal{D}})^{2M} + (\tau\sqrt{\mathcal{D}_0})^{2M})$ for $(\mathbf{x},t) \in [\mathbf{P},\mathbf{Q}] \times [0,T]$, $T > 0$

where now the one-dimensional sums are given by

$$S_j^{(p)}(k,t) =$$
$$\sum_{hm_j \in I_j} g_j^{(p)}(hm_j)(\Phi_M(\frac{4t}{h^2\mathcal{D}}, \frac{k-m_j}{\sqrt{\mathcal{D}}} - \frac{2b_j t}{h\sqrt{\mathcal{D}}}, P_{\mathbf{m}_j}) - \Phi_M(\frac{4t}{h^2\mathcal{D}}, \frac{k-m_j}{\sqrt{\mathcal{D}}} - \frac{2b_j t}{h\sqrt{\mathcal{D}}}, Q_{\mathbf{m}_j}))$$

and

$$T_j^{(p)}(k,t,\tau i, a_s) = \sum_{hm_j \in I_j} f_j^{(p)}(hm_j, \tau i)$$
$$\times \Big(\Phi_M(\frac{4t}{a_s h^2\mathcal{D}}, \frac{k-m_j}{\sqrt{\mathcal{D}}} - \frac{2b_j t}{a_s h\sqrt{\mathcal{D}}}, P_{\mathbf{m}_j}) - \Phi_M(\frac{4t}{a_s h^2\mathcal{D}}, \frac{k-m_j}{\sqrt{\mathcal{D}}} - \frac{2b_j t}{a_s h\sqrt{\mathcal{D}}}, Q_{\mathbf{m}_j})\Big)$$

$$\frac{\partial u}{\partial t} - \Delta_{\mathbf{x}} u = 0, \quad (\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R}_+; \quad u(\mathbf{x}, 0) = g(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n \tag{2}$$

supp $g \subseteq [-1, 1]^n$

$$g(\mathbf{x}) = \prod_{j=1}^{n} w(x_j), \qquad w \in C^N([-1, 1])$$

Hestenes reflection principle

$$\widetilde{w}(x) = \begin{cases} \sum_{s=1}^{N+1} c_s w(-\alpha_s(x-1)+1), & 1 < x \leq 1 + \frac{2}{A} \\ w(x), & -1 \leq x \leq 1 \\ \sum_{s=1}^{N+1} c_s w(-\alpha_s(x+1)-1), & -1 - \frac{2}{A} \leq x < -1 \end{cases}$$

where $\{\alpha_1, ..., \alpha_{N+1}\}$ are different positive constants, $A = \max_{1 \leq s \leq N+1} \alpha_s$ and $\{c_1, ..., c_{N+1}\}$ satisfy the system

$$\sum_{s=1}^{N+1} c_s (-\alpha_s)^k = 1, \quad k = 0, ..., N.$$

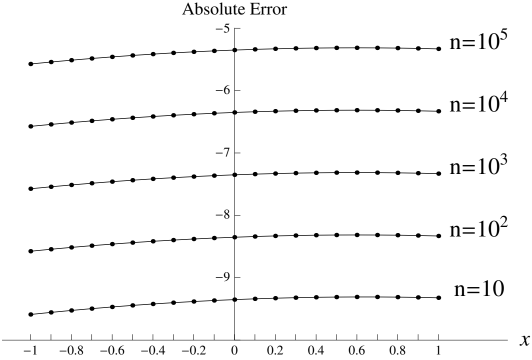

Figure: Absolute errors, using $\log_{10}$ scale on the vertical axes, for the solution of (2) with $w(x) = \mathrm{e}^{-x^2+ax}$, $a = 2.97109077126449$, and the Hestenes extension corresponding to $\alpha_s = 1/2^s$ using $N = 6$, $h = 0.0125$, $(x, 0, ..., 0) \in \mathbb{R}^n$, $t = 1$, $x \in [-1, 1]$.

In our case: $\epsilon_n = \mathcal{O}(n\,\epsilon_1)$, $\quad t_n = \mathcal{O}(n\,t_1)$
In the general case: $\epsilon_n = \mathcal{O}(n\,P\,\epsilon_1)$, $\quad t_n = \mathcal{O}(n\,P\,t_1)$

	h^{-1}	$M=1$ error	rate	$M=2$ error	rate	$M=3$ error	rate
$n=3$	80	3.468E-03		4.231E-06		4.904E-09	
	160	8.655E-04	2.00	2.640E-07	4.00	7.633E-11	6.00
	320	2.162E-04	2.00	1.649E-08	4.00	1.141E-12	6.06
$n=10$	80	1.154E-02		1.403E-05		1.624E-08	
	160	2.875E-03	2.00	8.753E-07	4.00	2.529E-10	6.00
	320	7.182E-04	2.00	5.468E-08	4.00	3.782E-12	6.06
$n=100$	80	0.121E+00		1.400E-04		1.625E-07	
	160	2.908E-02	2.06	8.735E-06	4.00	1.670E-09	6.60
	320	7.194E-03	2.01	5.457E-07	4.00	2.006E-11	6.37

Table: Absolute errors and approximation rates for the solution of (2) with $w(x) = e^{(x+a)^2}$, $a = 0.575770212624068$, at the point $\mathbf{x} = (0.3, 0, \ldots 0)$, $t = 2$ using the extension $\widetilde{w}(x) = w(x)$.

	h^{-1}	$M=2$ error	$M=2$ rate	$M=3$ error	$M=3$ rate
	80	1.395E-03		1.625E-06	
$n=1000$	160	8.727E-05	3.99	1.669E-08	6.60
	320	5.455E-06	3.99	2.007E-10	6.37
	80	1.386E-02		1.617E-05	
$n=10000$	160	8.724E-04	3.99	1.669E-07	6.60
	320	5.455E-05	3.99	2.007E-09	6.37
	80	0.130E+00		1.625E-04	
$n=100000$	160	8.690E-03	3.90	1.669E-06	6.60
	320	5.453E-04	3.99	2.007E-08	6.37

Table: Absolute errors and approximation rates for the solution of (2) with $w(x)=\mathrm{e}^{(x+a)^2}$, $a=0.575770212624068$, at the point $\mathbf{x}=(0.3,0,\ldots 0)$, $t=2$ using the Hestenes extension $\alpha_s=1/2^s$.

$$\frac{\partial u}{\partial t} - \Delta_{\mathbf{x}} u = f(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R}_+, \; u(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \mathbb{R}^n \tag{3}$$

with $\operatorname{supp} f \subseteq [-1, 1]^n \times \mathbb{R}_+$.

$$f(\mathbf{x}, t) = (\frac{\partial}{\partial t} - \Delta_{\mathbf{x}}) \prod_{j=1}^{n} w(x_j) v(t) = \sum_{p=1}^{n} \prod_{j=1}^{n} f_j^{(p)}(x_j, t), \mathbf{x} = (x_1, \ldots, x_n) \in [-1, 1]^n;$$

$$f_j^{(p)}(x, t) = w(x) \quad \text{if} \quad j \neq p, \quad f_j^{(j)}(x, t) = \frac{1}{n} v'(t) w(x) - v(t) w''(x)$$

where $\operatorname{supp} w \subseteq [-1, 1]$ and $\operatorname{supp} v \subseteq \mathbb{R}_+$. If $w(\pm 1) = w'(\pm 1) = 0$ and $v(0) = 0$, then the solution of (3) is

$$u(\mathbf{x}, t) = \prod_{j=1}^{n} w(x_j) v(t).$$

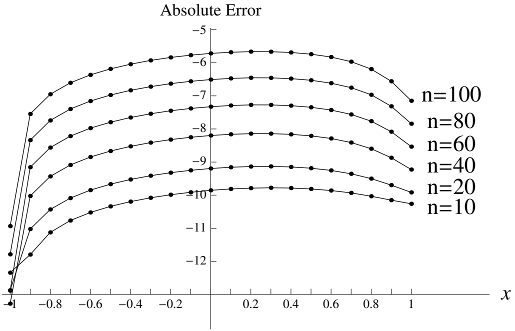

Figure: Absolute errors, using $\log_{10}$ scale on the vertical axes, for the solution of (3) with $w(x) = \mathrm{e}^x(x^2-1)^2$ and $v(t) = t$, at the point $(x, 0.1, ..., 0.1, 2) \in \mathbb{R}^n \times \mathbb{R}_+$ using the approximation formula of order $N = 6$ with $h = \tau = 1/160$ and the Hestenes extension corresponding to $\alpha_s = 1/2^s$.
The other parameters were $\mathcal{D} = \mathcal{D}_0 = 4$ and $\kappa = 0.02$, $S = 305$ in the trapezoidal rule.

In our case: $\epsilon_n = \mathcal{O}(n^2\,\epsilon_1)$
In the general case: $\epsilon_n = \mathcal{O}(n\,P\,\epsilon_1)$

	h^{-1}	τ^{-1}	$M=1$ error	$M=1$ rate	$M=2$ error	$M=2$ rate	$M=3$ error	$M=3$ rate
$n=3$	80	40	0.799E-03		0.383E-05		0.175E-08	
	160	80	0.214E-03	1.90	0.244E-06	3.97	0.277E-10	5.98
	320	160	0.553E-04	1.95	0.154E-07	3.98	0.485E-12	5.83
	640	320	0.137E-04	2.01	0.955E-09	4.00	0.933E-14	5.70
$n=10$	80	40	0.831E-02		0.148E-05		0.335E-08	
	160	80	0.208E-02	1.99	0.917E-07	4.00	0.523E-10	6.00
	320	160	0.521E-03	1.99	0.572E-08	4.00	0.809E-12	6.01
	640	320	0.131E-03	1.99	0.337E-09	4.08	0.133E-14	9.24
$n=100$	80	40			0.239E-01		0.455E-04	
	160	80			0.149E-02	4.00	0.710E-06	6.00
	320	160			0.931E-04	4.00	0.110E-07	6.01
	640	320			0.579E-05	4.00	0.196E-10	9.13
$n=200$	80	40					0.269E+00	
	160	80			0.891E+01		0.420E-02	6.00
	320	160			0.556E+00	4.00	0.651E-04	6.01
	640	320			0.347E-01	4.00	0.122E-06	9.06

Table: Absolute errors and approximation rates for the solution of (3) with $w(x) = \mathrm{e}^{x}(x^2-1)^2$ and $v(t) = 1 - \mathrm{e}^{-t}$, at the point $\mathbf{x} = (-0.2, 0.1, ..., 0.1)$; $t = 4$ using the Hestenes extension corresponding to $\alpha_s = 1/s$.

Thank you for your attention!