Approximation of semigroups generated by differential operators associated with Markov operators

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joint work with
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References

F. Altomare, M. Cappelletti Montano, V. L., Ioan Rașa
On differential operators associated with Markov operators,
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10.1016/j.jfa.2014.01.001

Differential operators associated with Markov operators

Given

- a convex compact subset K of $\mathbb{R}^d$ ($d \geq 1$) with non-empty interior
- a Markov operator T on $\mathscr{C}(K)$ (i.e., a positive linear operator T on $\mathscr{C}(K)$ such that $T(\mathbf{1}) = \mathbf{1}$, $\mathbf{1}$ being the constant function of value 1)

it is possible to associate with T an elliptic second-order differential operator W_T, defined by setting, for every $u \in \mathscr{C}^2(K)$,

$$W_T(u) := \frac{1}{2} \sum_{i,j=1}^{d} \alpha_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \tag{1}$$

where, for each $i, j = 1, \ldots, d$ and $x \in K$,

$$\alpha_{ij}(x) := T(pr_i pr_j)(x) - (pr_i pr_j)(x), \tag{2}$$

pr_i being the i-th coordinate function (i.e., $pr_i(x) = x_i$ for every $x \in K$).

Differential operators associated with Markov operators

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Difficulties

- *The boundary ∂K of K is generally non-smooth, due to the presence of possible sides and corners.*
- *W_T degenerates on the set*

$$\partial_T K := \{x \in K \mid T(f)(x) = f(x) \text{ for every } f \in \mathscr{C}(K)\} \tag{4}$$

of all interpolation points for T which contains the set $\partial_e K$ of the extreme points of K if, in addition,

$$T(h) = h \quad \text{for every } h \in \{pr_1, \ldots, pr_d\}. \tag{5}$$

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Interests

Operators of the form (3) are of concern in the study of several diffusion problems arising in biology, financial mathematics and other fields.

Our main aim

Proving that, under suitable hypotheses on T, the operator $(W_T, \mathscr{C}^2(K))$ is closable and its closure generates a Markov semigroup $(T(t))_{t\geq 0}$ on $\mathscr{C}(K)$.

$$W_T \quad \xrightarrow[\text{via asymptotic formula}]{} \quad (B_n)_{n\geq 1}$$

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Our approach

Trotter-Schnabl-type theorem

Let $(L_n)_{n\geq 1}$ be a sequence of linear contractions [a] *on a Banach space $(E, \|\cdot\|)$ over $\mathbb{R}$ or $\mathbb{C}$ and let $(\rho(n))_{n\geq 1}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} \rho(n) = 0$. Let (A_0, D_0) be a linear operator defined on a subspace D_0 of E and assume that*

(i) *there exists a family $(E_i)_{i\in I}$ of finite dimensional subspaces of D_0 which are invariant under L_n and whose union $\bigcup_{i\in I} E_i$ is dense in E.*

(ii) $\lim_{n\to\infty} \dfrac{L_n(u) - u}{\rho(n)} = A_0(u)$ *for every $u \in D_0$.*

Then (A_0, D_0) is closable and its closure $(A, D(A))$ is the generator of a contractive C_0-semigroup $(T(t))_{t\geq 0}$ on E.

[a] $\|L_n\| \leq 1$ for every $n \geq 1$.

Moreover, if $t \geq 0$ and if $(k(n))_{n\geq 1}$ is a sequence of positive integers satisfying $\lim_{n\to\infty} k(n)\rho(n) = t$, then, for every $f \in E$,

$$T(t)(f) = \lim_{n\to\infty} L_n^{k(n)}(f). \tag{6}$$

Furthermore, $\bigcup_{i\in I} E_i$ is a core [a] for $(A, D(A))$.

[a] A core for a linear operator $A : D(A) \longrightarrow \mathscr{C}(K)$ is a linear subspace of $D(A)$ which is dense in $D(A)$ with respect to the graph norm $\|u\|_A := \|A(u)\|_\infty + \|u\|_\infty$ $(u \in D(A))$.

For us

- $(E, \|\cdot\|) = (\mathscr{C}(K), \|\cdot\|_\infty)$
- $(A_0, D_0) = (W_T, \mathscr{C}^2(K))$
- $L_n = B_n$

Generation and approximation result

Moreover, if $t \geq 0$ and if $(k(n))_{n\geq 1}$ is a sequence of positive integers satisfying $\lim_{n\to\infty} k(n)\rho(n) = t$, then, for every $f \in E$,

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- $L_n = B_n$

It is well-known that for every $x \in K$ there exists a (unique) probability Borel measure $\tilde{\mu}_x^T$ on K such that, for every $f \in \mathscr{C}(K)$,

$$T(f)(x) = \int_K f \, d\tilde{\mu}_x^T \,. \tag{7}$$

Then, for every $n \geq 1$, we define the n-th Bernstein-Schnabl operator B_n associated with T by setting, for every $f \in \mathscr{C}(K)$ and $x \in K$,

$$B_n(f)(x) := \int_K \cdots \int_K f\left(\frac{x_1 + \ldots + x_n}{n}\right) d\tilde{\mu}_x^T(x_1) \cdots d\tilde{\mu}_x^T(x_n). \tag{8}$$

Bernstein-Schnabl operators associated with T

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- B_n is a positive linear operator from $\mathscr{C}(K)$ into $\mathscr{C}(K)$.
- $B_n(\mathbf{1}) = \mathbf{1} \Longrightarrow \|B_n\| = 1$ for every $n \geq 1$.
- $B_1 = T$.
- If $K = [0,1]$ and $T = T_1$ is the canonical projection, i.e.

$$T_1(f)(x) = xf(1) + (1-x)f(0) \quad (f \in \mathscr{C}([0,1]), x \in [0,1]), \tag{9}$$

then B_n's turn into the classical Bernstein operators on $[0,1]$.

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Properties of B_n's

Assume that

$$T(h) = h \quad \text{for every } h \in \{pr_1, \dots, pr_d\}. \tag{10}$$

Theorem 1

For every $i, j = 1, \dots, d$ and $n \geq 1$,

$$B_n(pr_i) = pr_i \quad \text{and} \quad B_n(pr_i pr_j) = \frac{1}{n} T(pr_i pr_j) + \frac{n-1}{n} pr_i pr_j. \tag{11}$$

Moreover, for every $f \in \mathscr{C}(K)$,

$$\lim_{n \to \infty} B_n(f) = f \quad \text{uniformly on } K. \tag{12}$$

Finally, for every $n \geq 1$ and $f \in \mathscr{C}(K)$,

$$B_n(f) = f \quad \text{on } \partial_T K. \tag{13}$$

Properties of B_n's

For every $m \geq 1$ denote by $P_m(K)$ the (restriction to K of all) polynomials of degree at most m. Moreover, we set

$$P_\infty(K) := \bigcup_{m \geq 1} P_m(K). \tag{14}$$

Observe that P_∞ is a subalgebra of $\mathscr{C}(K)$ and it is dense in $\mathscr{C}(K)$.

Theorem 2

If

$$T(P_m(K)) \subset P_m(K) \quad \text{for every } m \geq 1, \tag{15}$$

then

$$B_n(P_m(K)) \subset P_m(K) \tag{16}$$

for every $n, m \geq 1$.

In the Trotter-Schnabl-type theorem:

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Asymptotic formula for B_n's

Theorem 3

For every $u \in \mathscr{C}^2(K)$,

$$\lim_{n\to\infty} n(B_n(u) - u) = W_T(u) \quad \textit{uniformly on } K. \tag{17}$$

In the Trotter-Schnabl-type theorem:

(ii) $L_n = B_n$ and $\rho(n) = 1/n$

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Main result

Theorem 4

Let K be a convex compact subset of $\mathbb{R}^d$, $d \geq 1$, having non-empty interior and consider a Markov operator T on $\mathscr{C}(K)$ preserving coordinate functions. Furthermore, assume that

$$T(P_m(K)) \subset P_m(K) \qquad \text{for every } m \geq 2. \tag{18}$$

Then the operator $(W_T, \mathscr{C}^2(K))$ is closable and its closure $(A_T, D(A_T))$ generates a Markov semigroup $(T(t))_{t\geq 0}$ on $\mathscr{C}(K)$ such that, if $t \geq 0$ and $(k(n))_{n\geq 1}$ is a sequence of positive integers satisfying $\lim_{n\to\infty} k(n)/n = t$,

$$T(t)(f) = \lim_{n\to\infty} B_n^{k(n)}(f) \quad \text{uniformly on } K \tag{19}$$

for every $f \in \mathscr{C}(K)$.
Moreover, $P_\infty(K)$, and hence $\mathscr{C}^2(K)$ as well, is a core for $(A_T, D(A_T))$.

Main result

Finally, for every $t \geq 0$ *and* $m \geq 1$,

$$T(t)(P_m(K)) \subset P_m(K) \tag{20}$$

and, if $t \geq 0$ *and* $f \in \mathscr{C}(K)$,

$$T(t)(f) = f \qquad \text{on } \partial_T K. \tag{21}$$

Application

Consider the abstract Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = A_T(u(\cdot,t))(x) & x \in K, \quad t \geq 0, \\ \\ u(x,0) = u_0(x) & u_0 \in D(A_T), \quad x \in K. \end{cases} \tag{22}$$

Since $(A_T, D(A_T))$ generates the Markov semigroup $(T(t))_{t\geq 0}$, (22) admits a unique solution $u : K \times [0,+\infty[\longrightarrow \mathbb{R}$ given by

$$u(x,t) = T(t)(u_0)(x) \quad (x \in K, t \geq 0). \tag{23}$$

Hence, by Theorem 4,

$$u(x,t) = T(t)(u_0)(x) = \lim_{n\to\infty} B_n^{k(n)}(u_0)(x), \tag{24}$$

where $(k(n))_{n\geq 1}$ is a sequence of positive integers satisfying $\lim_{n\to\infty} k(n)/n = t$, and the limit is uniform with respect to $x \in K$.

Application

Recall that

$$A_T = W_T \quad \text{on } \mathscr{C}^2(K). \tag{25}$$

Therefore, if $u_0 \in P_m(K)$ ($m \geq 1$) then $u(x,t)$ is the unique solution to the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{d} \alpha_{ij}(x)\dfrac{\partial^2 u}{\partial x_i \partial x_j}(x,t) & x \in K, \quad t \geq 0, \\ \\ u(x,0) = u_0(x) & x \in K \end{cases} \tag{26}$$

and

$$u(\cdot,t) \in P_m(K) \qquad \text{for every } t \geq 0. \tag{27}$$

Application

In

F. Altomare, M. Cappelletti Montano, V. L., Ioan Rașa
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- $(B_n)_{n\geq 1}$ preserves Hölder continuous functions $\Longrightarrow$ the same holds for $(T(t))_{t\geq 0}$

Further properties both for $(B_n)_{n\geq 1}$ and $(T(t))_{t\geq 0}$ have been investigated in

F. Altomare, M. Cappelletti Montano, V. L., Ioan Rașa
Markov Operators, Positive Semigroups and Approximation Processes
De Gruyter Studies in Mathematics, v. **61**, 2014.

Examples of W_T satisfying Theorem 4

$$T(h) = h \quad \text{for every} \quad h \in \{pr_1, \ldots, pr_d\} \tag{Hp1}$$

$$T(P_m(K)) \subset P_m(K) \quad \text{for every} \quad m \geq 2 \tag{Hp2}$$

Examples of W_T satisfying Theorem 4

1. Consider the d-dimensional simplex

$$K_d := \left\{ (x_1, \dots, x_d) \in \mathbb{R}^d \;\middle|\; x_i \geq 0 \text{ for every } i = 1, \dots, d \text{ and } \sum_{i=1}^{d} x_i \leq 1 \right\} \tag{28}$$

and the projection T_d on K_d defined by

$$T_d(f)(x) := \left(1 - \sum_{i=1}^{d} x_i\right) f(v_0) + \sum_{i=1}^{d} x_i f(v_i) \tag{29}$$

$(f \in \mathscr{C}(K_d),\ x = (x_1, \dots, x_d) \in K_d)$, where

$$v_0 := (0, \dots, 0), \quad v_1 := (1, 0, \dots, 0), \dots, v_d := (0, \dots, 0, 1) \tag{30}$$

are the vertices of the simplex.
T_d is a Markov operator that satisfies (Hp1) and (Hp2) (note that $T_d(\mathscr{C}(K_d)) \subset P_1(K_d)$).

Examples of W_T satisfying Theorem 4

Then Theorem 4 applies to the differential operator associated with T_d given by

$$\begin{aligned} W_{T_d}(u)(x) &= \frac{1}{2}\sum_{i,j=1}^{d} x_i(\delta_{ij} - x_j)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) \\ &= \frac{1}{2}\sum_{i=1}^{d} x_i(1 - x_i)\frac{\partial^2 u}{\partial x_i^2}(x) - \sum_{1\le i<j\le d} x_i x_j \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \end{aligned} \tag{31}$$

$(u \in \mathscr{C}^2(K_d),\ x = (x_1, \ldots, x_d) \in K_d)$; δ_{ij} stands for the Kronecker symbol.

The operator W_{T_d} falls into the class of the so called Fleming-Viot operators.

The coefficients of W_{T_d} vanish on the vertices of the simplex.

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Examples of W_T satisfying Theorem 4

2. Let $S : \mathscr{C}(K_d) \longrightarrow \mathscr{C}(K_d)$ be the Markov operator defined by

$$S(f)(x) := \begin{cases} \left(1 - \dfrac{x_1}{1 - \sum\limits_{i=2}^{d} x_i}\right) f(0, x_2, \ldots, x_d) \\ + \dfrac{x_1}{1 - \sum\limits_{i=2}^{d} x_i} f\left(1 - \sum\limits_{i=2}^{d} x_i, x_2, \ldots, x_d\right) & \text{if } \sum\limits_{i=2}^{d} x_i \neq 1, \\ f(0, x_2, \ldots, x_d) & \text{if } \sum\limits_{i=2}^{d} x_i = 1 \end{cases} \quad (32)$$

$(f \in \mathscr{C}(K_d),\ x = (x_1, \ldots, x_d) \in K_d)$.

One has

$$S(pr_1 pr_j) = \begin{cases} \left(1 - \sum_{i=2}^{d} pr_i\right) pr_1 & \text{if } j = 1, \\ pr_1 pr_j & \text{if } 1 < j \leq d \end{cases} \tag{33}$$

and $S(pr_i pr_j) = pr_i pr_j$ for every $1 < i \leq j \leq d$.
Then S verifies (Hp1) and (Hp2) since, if $m_1, \ldots, m_d$ are positive integers,

$$S\left(pr_1^{m_1} \cdots pr_d^{m_d}\right) = \begin{cases} pr_2^{m_2} \cdots pr_d^{m_d} & \text{if } m_1 = 0, \\ \left(\mathbf{1} - \sum_{i=2}^{d} pr_i\right)^{m_1 - 1} pr_1 \; pr_2^{m_2} \cdots pr_d^{m_d} & \text{if } m_1 \geq 1. \end{cases} \tag{34}$$

Examples of W_T satisfying Theorem 4

Therefore Theorem 4 applies to the differential operator associated with S given by

$$W_S(u)(x) = \frac{1}{2} x_1 \left(1 - \sum_{i=1}^{d} x_i\right) \frac{\partial^2 u}{\partial x_1^2}(x) \tag{35}$$

$(u \in \mathscr{C}^2(K_d),\ x = (x_1, \dots, x_d) \in K_d)$.

Note that W_S degenerates on the faces $\{x = (x_1, \dots, x_d) \in K_d \ \mid \ x_1 = 0\}$ and $\left\{x = (x_1, \dots, x_d) \in K_d \ \mid \ \sum_{i=1}^{d} x_i = 1\right\}$.

Examples of W_T satisfying Theorem 4

3. Consider the convex combination of the above operators, that is the Markov operator $V := \dfrac{T_d + S}{2}$. Then V satisfies (Hp1) and (Hp2) and Theorem 4 applies to the differential operator

$$W_V(u)(x) = \frac{1}{4}\left(\left(2x_1(1-x_1) - x_1\sum_{i=2}^{d} x_i\right)\frac{\partial^2 u}{\partial x_1^2}(x) + \sum_{i=2}^{d} x_i(1-x_i)\frac{\partial^2 u}{\partial x_i^2}(x) - \sum_{1\le i<j\le d} x_i x_j \frac{\partial^2 u}{\partial x_i \partial x_j}(x)\right) \tag{36}$$

$(u \in \mathscr{C}^2(K_d),\ x = (x_1, \ldots, x_d) \in K_d)$.

Observe that W_V degenerates on the vertices of K_d as well.

4. Let $Q_d := [0,1]^d$, $d \geq 1$, and for every $i = 1, \dots, d$ consider a Markov operator U_i on $\mathscr{C}([0,1])$ satisfying (Hp1) and (Hp2).

If $U := \bigotimes_{i=1}^{d} U_i$ is the tensor product of the family $(U_i)_{1 \leq i \leq d}$, then U is a Markov operator on $\mathscr{C}(Q_d)$ satisfying (Hp1) and (Hp2) because

$$U\left(pr_1^{m_1} \cdots pr_d^{m_d}\right) = \left(U_1\left(e_1^{m_1}\right) \circ pr_1\right) \cdots \left(U_d\left(e_1^{m_d}\right) \circ pr_d\right)$$

for every positive integers $m_1, \dots, m_d$.

Therefore, Theorem 4 applies to the differential operator

$$W_U(u)(x) = \frac{1}{2} \sum_{i=1}^{d} \alpha_i(x) \frac{\partial^2 u}{\partial x_i^2}(x), \tag{37}$$

$(u \in \mathscr{C}^2(Q_d),\ x = (x_1, \dots, x_d) \in Q_d)$, where

$$\alpha_i(x) := U_i(e_2)(x_i) - x_i^2 \qquad (1 \leq i \leq d). \tag{38}$$

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Thank you for your attention