Regularization of some integral equations of the first kind in potential theory

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The real world is complex
an international symposium in honor of Christian Berg
Copenaghen, August 27, 2015

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A. Cialdea, University of Basilicata, Italy

V. Leonessa, University of Basilicata, Italy

E. Dolce, University of Basilicata, Italy

BIEs $\Leftrightarrow$ REDUCTION

$$\begin{cases} \text{Direct method} & \text{Green's representation formula} \\ \text{Indirect method} & \text{layer ansatz} \end{cases}$$

Dirichlet Problem for Laplace Equation in $\mathbb{R}^n$, $n \geq 2$

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \Sigma \end{cases}$$

$\Delta = \sum_{h=1}^{n} \frac{\partial^2}{\partial x_h^2}$

$\Omega \subset \mathbb{R}^n$, $n \geq 2$, bounded domain

$\Sigma = \partial\Omega$ Lyapunov hypersurface: $\Sigma \in C^{1,\lambda}$, $0 < \lambda \leq 1$

f assigned function on Σ

Dirichlet problem for Laplace equation in $\mathbb{R}^2$

Simple layer potential approach

- Muskhelishvili, 1972.

simple layer potential $$u(x) = \frac{1}{2\pi}\int_{\Sigma}\varphi(y)\ln|x-y|d\sigma_y$$

BIE 1^{st} kind: $$\frac{1}{2\pi}\int_{\Sigma}\varphi(y)\ln|x-y|ds_y = f(x), \qquad x\in\Sigma$$

$f \in C^{1,\lambda}(\Sigma)$, $0<\lambda\leq 1$

$$\frac{1}{2\pi}\int_{\Sigma}\varphi(y)\frac{\partial}{\partial s_x}\ln|x-y|\,ds_y = \frac{df}{ds}(x), \quad x\in\Sigma. \tag{1}$$

If B and B' are two Banach spaces and $S : B \to B'$ is a continuous linear operator, we say that S can be reduced on the left if $\exists$ a continuous linear operator $S' : B' \to B$ s. t. $S'S = I + K$, where I stands for the identity operator of B and $K : B \to B$ is compact.

Analogously, one can define an operator S reducible on the right.

Dirichlet problem for Laplace equation in $\mathbb{R}^2$

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If B and B' are two Banach spaces and $S : B \to B'$ is a continuous linear operator, we say that S can be reduced on the left if $\exists$ a continuous linear operator $S' : B' \to B$ s. t. $S'S = I + K$, where I stands for the identity operator of B and $K : B \to B$ is compact.
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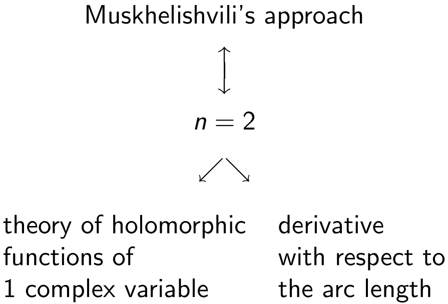
Indirect Method

Dirichlet Problem for Laplace Equation in $\mathbb{R}^2$: Muskhelishvili's approach



Muskhelishvili's approach

$n = 2$

theory of holomorphic functions of 1 complex variable

derivative with respect to the arc length

Dirichlet Problem for Laplace equation in $\mathbb{R}^n$, $n \geq 3$

Simple layer potential approach

- Cialdea, 1988

$\Omega \subset \mathbb{R}^n$ bounded domain; $\mathbb{R}^n - \overline{\Omega}$ is connected, $\Sigma \in C^{1,\lambda}$

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \Sigma, \ f \in W^{1,p}(\Sigma), \ 1 < p < +\infty \end{cases}$$

simple layer potential $$u(x) = \int_\Sigma \varphi(y) s(x,y) d\sigma_y$$ ▶

$\varphi \in L^p(\Sigma)$

$$n = 2 \left\{ \begin{array}{l} \text{theory of holomorphic} \\ \text{functions of} \\ \text{1 complex variable} \end{array} \right. \rightsquigarrow \left. \begin{array}{l} \text{theory of} \\ \text{differential} \\ \text{forms} \end{array} \right\} n > 2$$

$$n = 2 \left\{ \begin{array}{l} \text{derivative with respect to} \\ \text{the arc length} \end{array} \right. \rightsquigarrow \left. \begin{array}{l} \text{exterior} \\ \text{differential} \end{array} \right\} n > 2$$

Neumann Problem for Laplace equation

Ω simply connected domain

- Cialdea, Hsiao, 1995.

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = f & \text{on } \Sigma,\ f \in L^p(\Sigma),\ \displaystyle\int_\Sigma f\, d\sigma = 0 \end{cases}$$

$$\text{double layer pot.: } u(x) = \int_\Sigma \psi(y)\, \frac{\partial}{\partial \nu_y} s(x,y)\, d\sigma_y,\ \ \psi \in W^{1,p}(\Sigma)$$

- Cialdea, 1988.

$$\frac{\partial}{\partial \nu_x}\left(\int_\Sigma \psi(y) \frac{\partial}{\partial \nu_y} s(x,y)\, d\sigma_y \right) d\sigma_x = d_x \int_\Sigma d\psi(y) \wedge s_{n-2}(x,y)$$

$$s_{n-2}(x,y) = \sum_{j_1 < \ldots < j_{n-2}} s(x,y) dx^{j_1} \ldots dx^{j_{n-2}} dy^{j_1} \ldots dy^{j_{n-2}}$$

$$d_x \int_\Sigma d\psi(y) \wedge s_{n-2}(x,y) = f\, d\sigma \qquad (2)$$

We note that the method of Cialdea does not require the use of pseudo-differential operators nor the use of hypersingular integrals, differently from other methods, see e.g. Chapter 4 in

- Lifanov, Poltavskii, Vainikko: Hypersingular integral equations and their applications. In Differential and Integral Equations and Their Applications, 4. Boca Raton: Chapman & Hall/CRC; 2004.

Generalization to other PDEs

in a simply connected domain

- Cialdea: *The simple layer potential for the biharmonic equation in n variables*, Rend. Mat. Acc. Lincei, IX, vol. II, fasc. 2, 115–127, 1991.
- Cialdea, Hsiao: *Regularization for some boundary integral equations of the first kind in Mechanics*, Rend. Acc. Naz. XL, Mem. Mat., XIX, fasc. 1, 25–42, 1995.
- M.: *Regularization for integral equations of the first kind in the theory of thermoelastic pseudo-oscillations*, AMIM, 9, 2, 29–51, 2004.
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Recent generalizations to other PDEs

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- Cialdea, Dolce, Nanni, M.: *On an integral equation of the first kind arising in the theory of Cosserat.* , Intern. J. Math. 24(5), 21 pages, 2013.
- Cialdea, Dolce, M.: *A complement to potential theory in the Cosserat theory.*, Math. Methods Appl. Sci., 38(3), 537–547, 2015.
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- Cialdea, Leonessa, M.: *The Dirichlet problem for second order divergence form elliptic operators with variable coefficients: the simple layer potential ansatz*, submitted.

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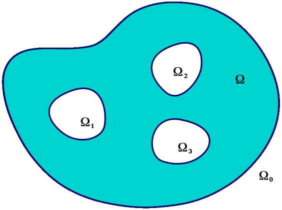
Generalization in multiply connected domains

Ω multiply connected domain of $\mathbb{R}^n$, $n \geq 2$

$$\Omega = \Omega_0 \setminus \bigcup_{j=1}^{m} \overline{\Omega}_j$$

$$\overline{\Omega}_j \subset \Omega_0, \quad \overline{\Omega}_j \cap \overline{\Omega}_k = \emptyset,\ j \neq k.$$



- Cialdea, Leonessa, M.: *On the Dirichlet and the Neumann problems for Laplace equation in multiply connected domains.* Complex Var. Elliptic Equ. 57(10), 1035–1054, 2012. doi:10.1080/17476933.2010.534156
- Cialdea, Leonessa, M.: *Integral representations for solutions of some BVPs for the Lamé system in multiply connected domains.* Bound. Value Probl., 2011:53, 25 pages, 2011. doi:10.1186/1687-2770-2011-53
- Cialdea, Leonessa, M.: *On the Dirichlet problem for the Stokes system in multiply connected domains.* Abstr. Appl. Anal., Art. ID 765020, 12 pages, 2013. doi:10.1155/2013/765020 2013.

Lamè system

$\Omega \subset \mathbb{R}^n$, $n \geq 2$, bounded domain (open connected set)

$u(x) = (u_1(x), \ldots, u_n(x))$

$$\Delta u + k \nabla \mathrm{div}\ u = 0$$

$k > 1 - 2/n$

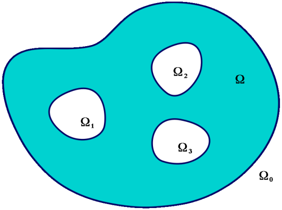
$$\Delta u = \sum_{h=1}^{n} \frac{\partial^2 u_i}{\partial x_h^2}, \quad \nabla \cdot = (\frac{\partial \cdot}{\partial x_1}, \ldots, \frac{\partial \cdot}{\partial x_n}), \quad \mathrm{div}\ u = \sum_{h=1}^{n} \frac{\partial u_h}{\partial x_h}.$$

Dirichlet Problem for the Lamè system

Ω multiply connected domain of $\mathbb{R}^n$, $n \geq 2$
$\Sigma = \partial\Omega \in C^{1,\lambda}$, $0 < \lambda \leq 1$.

$$u(x) = (u_1(x), \ldots, u_n(x))$$

$$\begin{cases} \Delta u + k\nabla \mathrm{div}\, u = 0 & \text{in } \Omega, \\ u = f & \text{on } \Sigma, \end{cases}$$



$k > 1 - 2/n$

$f \in [W^{1,p}(\Sigma)]^n$, $1 < p < +\infty$

Dirichlet BVP for Lamè system

Simple layer potential approach

simple layer potential $u(x) = \int_\Sigma \Gamma(x,y)\,\varphi(y)\,d\sigma_y, \quad \varphi \in [L^p(\Sigma)]^n$

$$\Gamma = \Big(\Gamma_{ij}(x,y)\Big)_{n\times n}$$

$$\Gamma_{ij}(x,y)=\begin{cases} \dfrac{1}{2\pi}\left(-\dfrac{k+2}{2(k+1)}\delta_{ij}\ln|x-y|+\dfrac{k}{2(k+1)}\dfrac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\right) & n=2\\ \dfrac{1}{\omega_n}\left(-\dfrac{k+2}{2(k+1)}\delta_{ij}\dfrac{|x-y|^{2-n}}{2-n}+\dfrac{k}{2(k+1)}\dfrac{(x_i-y_i)(x_j-y_j)}{|x-y|^n}\right) & n\geq 3 \end{cases}$$

ω_n being the hypersurface measure of the unit sphere in $\mathbb{R}^n$.

$$\text{BIS } 1^{st} \text{ kind:} \quad \int_\Sigma \Gamma(x,y)\varphi(y)d\sigma_y = f(x)$$

$$\text{singular integral system:} \int_\Sigma d_x[\Gamma(x,y)]\varphi(y)\,d\sigma_y = df(x) \qquad (3)$$

$$R : [L^p(\Sigma)]^n \to [L^p_1(\Sigma)]^n$$

$$R\varphi = \int_\Sigma d_x[\Gamma(x,y)]\varphi(y)\,d\sigma_y$$

$$R\varphi = df$$

$$\varphi = (\varphi_1, \dots, \varphi_n) \in [L^p(\Sigma)]^n, \qquad df = (df_1, \dots, df_n) \in [L^p_1(\Sigma)]^n$$

Theorem

Let us define $R'^{\xi}: [L^p_1(\Sigma)]^n \to [L^p(\Sigma)]^n$ *to be the singular integral operator*

$$R'^{\xi}_i(\psi)(x) = (k-\xi)\mathcal{K}^{\xi}_{jj}(\psi)(x)\nu_i(x) + \nu_j(x)\mathcal{K}^{\xi}_{ij}(\psi)(x) + \xi\nu_j(x)\mathcal{K}^{\xi}_{ji}(\psi)(x), \tag{4}$$

where $\psi = (\psi_1,\ldots,\psi_n) \in [L^p_1(\Sigma)]^n$, $\xi \in \mathbb{R}$

$$\mathcal{K}^{\xi}_{js}(\psi)(x) = \Theta_s(\psi_j)(x) - \frac{1}{(n-2)!}\delta^{123\ldots n}_{hij_3\ldots j_n}\int_\Sigma \partial_{x_s} K^{\xi}_{hj}(x,y)\wedge\psi_i(y)\wedge dy^{j_3}\ldots dy^{j_n}$$

$$K^{\xi}_{hj}(x,y) = \left[\frac{k(\xi+1)(2-n)}{2(k+1)}\frac{(y_h-x_h)(y_j-x_j)}{|y-x|^2} + \frac{k-(2+k)\xi}{2(k+1)}\delta_{hj}\right]s(x,y)$$

$$\Theta_h(\psi_j)(x) = *\left(\int_\Sigma d_x[s_{n-2}(x,y)]\wedge\psi_j(y)\wedge dx^h\right)$$

$\nu(x) = (\nu_1(x),\ldots,\nu_n(x))$ *denotes the outwards unit normal vector at a point* x *on* Σ.

Then

$$R'^{\xi} R\varphi = \frac{1}{4}\varphi + (T^{\xi})^2\varphi \tag{5}$$

where

$$T^{\xi}\varphi(x) = \int_{\Sigma} L^{\xi}_{x}[\Gamma(x,y)]\,\varphi(y)\,d\sigma_y \tag{6}$$

and

$$L^{\xi}_{i,x}[\Gamma^{j}(x,y)] =$$

$$-\frac{1}{\omega_n}\left\{\left[\frac{2+(1-\xi)k}{2(1+k)}\delta_{ij} + \frac{nk(\xi+1)}{2(k+1)}\,\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\right]\frac{(x_p-y_p)\nu_p(x)}{|x-y|^n}\right.$$
$$\left. + \frac{k-(2+k)\xi}{2(k+1)}\left[\frac{(x_j-y_j)\nu_i(x)-(x_i-y_i)\nu_j(x)}{|x-y|^n}\right]\right\}.$$

ω_n hypersurface measure of the unit sphere in $\mathbb{R}^n$, $\Gamma^j(\cdot,\cdot)$ j-th column of Γ, $\forall j$.

▶

1. *if* $\xi = k/(2+k)$ $L_{i,y}^{k/(2+k)}[\Gamma^j(x,y)] = \mathcal{O}(|x-y|^{1-n+\lambda})$,
2. $\xi \neq k/(2+k)$ *the kernels* $L_{i,y}^{\xi}[\Gamma^j(x,y)]$ *have a strong singularity on* Σ.

Reduction Theorem

The operator

$$R : [L^p(\Sigma)]^n \to [L_1^p(\Sigma)]^n$$

$$R\varphi = \int_\Sigma d_x[\Gamma(x,y)]\varphi(y)\, d\sigma_y$$

can be reduced on the left. A reducing operator is given by $R' := R'^{\xi}$, *with* $\xi = k/(2+k)$.

Existence Theorem

Given $\omega \in [L_1^p(\Sigma)]^n$, *there exists a solution of the singular integral system*

$$\int_\Sigma d_x[\Gamma(x,y)]\,\varphi(y)\,d\sigma_y = \omega(x), \quad \varphi \in [L^p(\Sigma)]^n,\ x \in \Sigma$$

if, and only if,

$$\int_\Sigma \gamma \wedge \omega_i = 0, \quad i = 1, \dots, n \tag{7}$$

for every $\gamma \in L_{n-2}^q(\Sigma)$ $(q = p/(p-1))$ *s.t.* γ *is a weakly closed* $(n-2)$*-form.*

If $\omega_i = df_i \Rightarrow$ conditions (7) are satisfied and then $\exists$ a solution of $R\varphi = df$.

Dirichlet BVP for the Lamè system

Ω multiply connected domain

Representation Theorem

For any $f \in [W^{1,p}(\Sigma)]^n$ *there exists a unique solution of the following BVP*

$$\begin{cases} \Delta v + k\nabla \mathrm{div}\ v = 0 & \textit{in } \Omega, \\ v = f & \textit{on } \Sigma. \end{cases}$$

It is given by

$$v(x) = \int_\Sigma \Gamma(x,y)\,\varphi(y)\,d\sigma_y$$

where the density $\varphi \in [L^p(\Sigma)]^n$ *solves the singular integral system*

$$R\varphi \equiv \int_\Sigma d_x[\Gamma(x,y)]\varphi(y)\,d\sigma_y = df\,.$$

Remark

$$R\varphi = df$$

$$R'R\varphi = \frac{1}{4}\varphi + K^2\varphi \qquad \text{is not an equivalent reduction}$$

equivalent reduction: $\mathcal{N}(R') = \{0\}$

$R\varphi = \psi \quad \Leftrightarrow \quad R'R\varphi = R'\psi$

R' still provides a kind of equivalence.

$$\mathcal{N}(R'R) = \mathcal{N}(R)$$

If ψ is s. t. $\exists$ a solution of $R\varphi = \psi$, then

$$R\varphi = \psi \iff R'R\varphi = R'\psi.$$

Traction problem for Lamè system

Ω multiply connected domain of $\mathbb{R}^n$

$$\begin{cases} \Delta w + k\nabla \mathrm{div}\ w = 0 & \text{in } \Omega, \\ Lw = f & \text{on } \Sigma, \end{cases} \tag{8}$$

$f \in [L^p(\Sigma)]^n$ s.t.

$$\int_\Sigma f \cdot \alpha \, d\sigma = 0, \qquad \forall\ \alpha \in \mathcal{R}$$

stress operator: $L_i w = (k-1)(\mathrm{div}\ w)\,\nu_i + \nu_j \partial_j w_i + \nu_j \partial_i w_j \quad i = 1, \dots, n,$

double layer pot. $w(x) = \int_\Sigma L_y[\Gamma(x,y)]\,\psi(y) d\sigma_y,\ \psi \in [W^{1,p}(\Sigma)]^n$

Traction problem for Lamè system

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double layer pot. $w(x) = \int_\Sigma L_y[\Gamma(x,y)]\,\psi(y) d\sigma_y,\ \psi \in [W^{1,p}(\Sigma)]^n$

Traction problem for Lamè system

multiply connected domains

Theorem

Given $f \in [L^p(\Sigma)]^n$, $1 < p < +\infty$, *the traction problem admits a solution (determined up to an additive rigid displacement)* $\Leftrightarrow$

$$\int_{\Sigma_j} f \cdot \alpha \, d\sigma_x = 0, \quad \forall \alpha \in \mathcal{R}, \ j = 1, \dots, m.$$

Moreover, double layer potential is a solution of (8) if, and only if, its density ψ *is given by*

$$\psi(x) = \int_\Sigma \Gamma(x, y) \phi(y) \, d\sigma_y, \quad x \in \Sigma,$$

where ϕ *is a solution of the singular integral system*

$$\frac{1}{4}\phi + T^2 \phi = f,$$

and where T *is defined by operator* (6) T^ξ *with* $\xi = 1$.

Traction problem for Lamè system

multiply connected domains

Theorem

Given $f \in [L^p(\Sigma)]^n$ *satisfying* $\int_\Sigma f \cdot \alpha \, d\sigma = 0, \ \forall \ \alpha \in \mathcal{R}$, *the traction problem*

$$\begin{cases} \Delta w + k\nabla \mathrm{div}\ w = 0 & in\ \Omega, \\ Lw = f & on\ \Sigma, \end{cases}$$

admits a solution uniquely determined up to an additive rigid displacement

$$w(x) = \int_\Sigma L_y[\Gamma(x,y)]\psi(y)\, d\sigma_y + \sum_{j=1}^{m} \sum_{h=1}^{n(n+1)/2} \frac{1}{|\Sigma_j|} \int_{\Sigma_j} f(t) v_h(t)\, d\sigma_t \int_{\Sigma_j} \Gamma(x,y) v_h(y)\, d\sigma_y,$$

where $\{v_h(x) \, : \, h = 1, \dots, n(n+1)/2\}$ *is an orthonormal basis of the space of all rigid displacements* $\mathcal{R}$.

Remarks on the plane

for Laplace equation

If $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) is a bounded simply connected domain

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \Sigma,\ f \in W^{1,p}(\Sigma) \end{cases}$$

is solvable by means of

$$u(x) = \int_\Sigma \varphi(y) s(x,y)(x,y)\, d\sigma_y, \quad \varphi \in L^p(\Sigma).$$

This is not always true for $n = 2$. In fact, there are some boundaries for which it is not possible to represent the solution of the Dirichlet problem by means of a simple layer potential. In particular it is not possible to represent the constant function.

If this occurs in the simply connected domain $\Omega \subset \mathbb{R}^2$ we say that its boundary Σ is exceptional.

The simplest example of such a domain is given by the unit disc D, where

$$\int_{\partial D} \ln |x - y|\, ds_y = 0, \quad |x| \leq 1.$$

Remarks on the plane

for the Lamè system

Proposition

Let C_r be the circle of radius r centered at the origin.
If $r = \exp(k/(2(k+2)))$, C_r is exceptional for the operator $\Delta + k\nabla\mathrm{div}$, i.e. in B_r (the ball of radius r centered at the origin), for such a value of r, we cannot represent any smooth solution of the system $\Delta u + k\nabla\mathrm{div}\ u = 0$ by means of a simple layer potential.

We have showed that also in some multiply connected domains one cannot represent any constant vectors by a simple layer potential and that this happens if, and only if, the exterior boundary Σ_0 (considered as the boundary of the simply connected domain Ω_0) is exceptional.

Theorem

Let $\Omega \subset \mathbb{R}^2$ be an $(m+1)$-connected domain. The following conditions are equivalent:

(i) *there exists an Hölder continuous vector function $\varphi \not\equiv 0$ such that*

$$\int_\Sigma \Gamma(x,y)\,\varphi(y)\; ds_y = 0, \quad \forall\; x \in \Sigma;$$

(ii) *there exists a constant vector which cannot be represented in Ω by a simple layer potential (i.e., there exists $c \in \mathbb{R}^2$ such that $c \notin \mathcal{S}^p$);*

(iii) *Σ_0 is exceptional;*

(iv) *Let* $\varphi_1, \ldots, \varphi_{2m+2}$ *be linearly independent functions of*

$$\mathcal{P}=\left\{\varphi \in [L^p(\Sigma)]^2 : \int_\Sigma \varphi_j(y) \frac{\partial}{\partial s_x} \Gamma_{ij}(x,y) ds_y = 0, a.e.\ x \in \Sigma, i = 1,2\right\}$$

and let $c_{jk} = (\alpha_{jk}, \beta_{jk}) \in \mathbb{R}^2$ *be given by*

$$\int_\Sigma \Gamma(x,y)\, \varphi_j(y)\, ds_y = c_{jk},$$

$x \in \Sigma_k,\ j = 1, \ldots, 2m+2,\ k = 0, 1, \ldots, m.$ *Then*

$$\det \mathcal{C} = 0\,,$$

where

$$\mathcal{C} = \begin{pmatrix} \alpha_{1,0} & \cdots & \alpha_{2m+2,0} \\ \cdots & \cdots & \cdots \\ \alpha_{1,m} & \cdots & \alpha_{2m+2,m} \\ \beta_{1,0} & \cdots & \beta_{2m+2,0} \\ \cdots & \cdots & \cdots \\ \beta_{1,m} & \cdots & \beta_{2m+2,m} \end{pmatrix} .$$

THANK YOU FOR YOUR ATTENTION!

$$L_i^{\xi} u = (k - \xi)(\mathrm{div}\ u)\,\nu_i + \nu_j \partial_j u_i + \xi \nu_j \partial_i u_j\,,$$

$\xi \in \mathbb{R}$

If $\xi = 1$, L^1 is the classical stress operator

If $\xi = k/(2+k)$, $L^{k/(2+k)}$ is pseudostress operator

◂ back

$$s(x,y)=\begin{cases} \dfrac{1}{2\pi}\ln|x-y| & n=2, \\ \dfrac{1}{(2-n)\omega_n}|x-y|^{2-n} & n\geq 3 \end{cases}$$

ω_n hypersurface measure of the unit sphere in $\mathbb{R}^n$

◂ back