Zero distribution of incomplete Padé and Hermite-Padé approximations

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1. Definitions and auxiliary results.
    1. Padé approximants.
    2. Hermite-Padé approximants.
    3. Incomplete Padé approximants.
2. Zero distribution results.
3. Zeros of Padé approximants.
4. Zeros of incomplete Padé approximants.
5. Zeros of Hermite-Padé approximants.

Padé approximants

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, \quad |z| < R_0(f).$$

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Definition

Let $n, m \in \mathbb{Z}_+$, $n \geq m$, be given. There exist polynomials q, p such that

i) $\deg p \leq n - m$, $\deg q \leq m$, $q \not\equiv 0$,

ii) $[qf - p](z) = Az^{n+1} + \cdots$.

$$f(z)=\sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, \quad |z|<R_0(f).$$

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i) $\deg p \leq n-m$, $\deg q \leq m$, $q \not\equiv 0$,

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$$\pi_{n,m}=\frac{p}{q}, \text{ defines a unique rational function.}$$

Padé approximant of type (n,m) of f.

We denote $\pi_{n,m} = \dfrac{p_{n,m}}{q_{n,m}}$ after canceling out common factors and $q_{n,m}$ is normalized in the form:

$$q_{n,m}(z) = \prod_{|\xi_{n,k}|\leq 1} (z - \xi_{n,k}) \prod_{|\xi_{n,k}|>1} \left(1 - \frac{z}{\xi_{n,k}}\right).$$

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It is known

$$\pi_{n,m}(z) - \pi_{n-1,m}(z) = \frac{A_{n,m} z^n}{q_{n,m}(z) q_{n-1,m}(z)}$$

and

$$R_m(f) = \frac{1}{\limsup_{n\to\infty} |A_{n,m}|^{1/n}}, \quad \text{Hadamard formula.}$$

Let $\mathcal{Q}_m(f)$ be the polynomial whose zeros are the poles of f in $D_m(f)$.

Theorem (Montessus de Ballore)

Let $\mathcal{P}_m(f)$ be the set of poles of f. Assume that $R_0(f) > 0$ and that f has exactly m poles in $D_m(f)$ (counting multiplicities), then

$$\limsup_{n\to\infty} \|f - \pi_{n,m}\|_K^{1/n} = \frac{\|z\|_K}{R_m(f)},$$

where $K \subset D_m(f) \setminus \mathcal{P}_m(f)$ is a compact subset. Additionally

$$\limsup_{n\to\infty} \|\mathcal{Q}_m(f) - q_{n,m}\|^{1/n} = \frac{\max\{|\eta| : \eta \in \mathcal{P}_m(f)\}}{R_m(f)},$$

where $\|\cdot\|$ denotes the coefficient norm in the space of polynomials.

Convergence in σ-content

$\mathcal{U}(B)$ denotes the class of all coverings of A by at most a numerable set of disks.

$$\sigma(B) = \inf\left\{\sum_{i=1}^{\infty}|U_i| : \{U_i\} \in \mathcal{U}(B)\right\}, \ |U_i| \text{ radius of the disk } U_i.$$

$$\sigma - \lim_{n\to\infty}\varphi_n = \varphi, \text{ in } D \Leftrightarrow \lim_{n\to\infty}\sigma\{z \in K : |\varphi_n(z) - \varphi(z)| > \varepsilon\} = 0, \ K \subset D.$$

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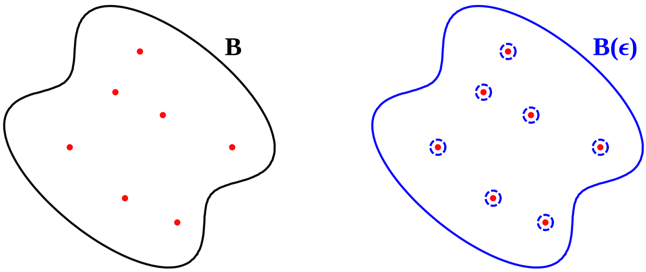
Take an arbitrary $\varepsilon > 0$ and define $J_\varepsilon = \cup_{n\geq m-1} J_{n,\varepsilon}$ where

- $J_{n,\varepsilon}$ is the $\varepsilon/6mn^2$-neighborhood of the set of zeros of $q_{n,m}$.
- $J_{m-1,\varepsilon}$ is the set of poles of $\varepsilon/6m$.

For any set $B \subset \mathbb{C}$, we write $B(\varepsilon) = B \setminus J_\varepsilon$.



$$\varphi_n \rightrightarrows \varphi \text{ in } K(\varepsilon), \text{ for every compact } K \subset D$$

$$\Updownarrow$$

$$\sigma - \lim_{n\to\infty} \varphi_n = \varphi, \text{ in } D.$$

Theorem (Gonchar)

Assume that f has at most m poles in $D_m(f)$, then

$$\sigma - \lim_{n\to\infty} \pi_{n,m} = f, \ \textit{in } D_m(f).$$

Or equivalently,

$$\limsup_{n\to\infty} \|f - \pi_{n,m}\|_{K(\varepsilon)}^{1/n} = \frac{\|z\|_{K(\varepsilon)}}{R_m(f)}, \quad K \subset D_m(f) \ \textit{compact subset.}$$

$$\limsup_{n\to\infty} \|q_{n,m} f - p_{n,m}\|_{K}^{1/n} = \frac{\|z\|_{K}}{R_m(f)}, \quad K \subset D_m(f) \setminus \mathcal{P}_m(f) \ \textit{compact subset.}$$

Definition

$\boldsymbol{f} = (f_1, \ldots, f_d)$, $\boldsymbol{m} = (m_1, \ldots, m_d) \in \mathbb{Z}_+^d \setminus \{\boldsymbol{0}\}$, $|\boldsymbol{m}| = m_1 + \cdots + m_d$. For each $n \geq \max\{m_1, \ldots, m_d\}$, there exist polynomials Q, P_j, $j = 1, \ldots, d$ such that

i) $\deg P_j \leq n - m_j$, $j = 1, \ldots, d$, $\deg Q \leq |\boldsymbol{m}|$, $Q \not\equiv 0$,

ii) $[Qf_j - P_j](z) = A_j z^{n+1} + \cdots$.

$R_{n,\boldsymbol{m}} = (P_1/Q, \ldots, P_d/Q)$ Hermite-Padé approximant of $\boldsymbol{f}$ and is not uniquely determined.

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We cancel common factors of Q simultaneously with all the P_j

$$\mathbf{R}_{n,\mathbf{m}} = (R_{n,\mathbf{m},1}, \ldots, R_{n,\mathbf{m},d}) = (P_{n,\mathbf{m},1}, \ldots, P_{n,\mathbf{m},d})/Q_{n,\mathbf{m}},$$

where $Q_{n,m}$ is normalized in the same way above.

$$Q_{n,\mathbf{m}}(z) = \prod_{|\xi_{n,k}| \leq 1} (z - \xi_{n,k}) \prod_{|\xi_{n,k}| > 1} \left(1 - \frac{z}{\xi_{n,k}}\right).$$

Definition (J. Cacoq, B. de la Calle Ysern, G. López Lagomasino)

Fix $m^* \leq m$. *Let* $n \geq m$. $R_{n,m}$ *is an incomplete Padé approximant of type* (n, m, m^*) *if* $R_{n,m}$ *is the quotient of any two polynomials* P *and* Q *that verify*

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- (n, m, m^*), $n \geq m \geq m^*$, $\pi_{n,m^*}, \ldots, \pi_{n,m}$ are incomplete Padé approximants of type (n, m, m^*) of f.

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- Padé-type approximants where $m - m^*$ zeros of Q are fixed and m^* are left free are also incomplete Padé approximants.
- $R_{n,\mathbf{m},k}$, $k = 1, \ldots, d$, is an incomplete Padé approximant of type $(n, |\mathbf{m}|, m_k)$ with respect to f_k.

Given $n \geq m \geq m^*$, $R_{n,m}$ is not unique so we choose one candidate and cancel out common factors

$$R_{n,m} = \frac{P_{n,m}}{Q_{n,m}}.$$

Normalizing $Q_{n,m}$

$$Q_{n,m}(z) = \prod_{|\xi_{n,k}|\leq 1} (z - \xi_{n,k}) \prod_{|\xi_{n,k}|>1} \left(1 - \frac{z}{\xi_{n,k}}\right).$$

Incomplete Padé approximants

For $m \geq m^*$,

$$\sigma - \lim_{n\to\infty} \pi_{n,m} = f, \text{ in compact subsets of } D_{m^*}(f).$$

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$$R_{n,m}(z) - R_{n-1,m}(z) = \frac{A_{n,m} z^n q_{n,m-m^*}(z)}{Q_{n,m}(z) Q_{n-1,m}(z)},$$

where $A_{n,m}$ are constant and $q_{n,m-m^*}$, $\deg q_{n,m-m^*} \leq m - m^*$.

$$R_m^*(f) = \frac{1}{\limsup_{n\to\infty} |A_{n,m}|^{1/n}}, \quad D_m^*(f) = \{z : |z| < R_m^*(f)\}.$$

Theorem (J. Cacoq, B. de la Calle Ysern, G. López Lagomasino)

$$R_m^*(f) > 0 \Longrightarrow R_0(f) > 0.$$

$$D_{m^*}(f) \subset D_m^*(f) \subset D_m(f)$$

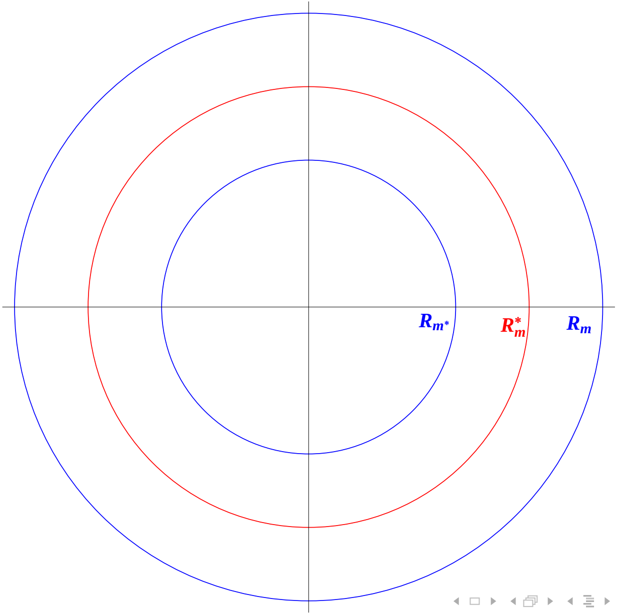
and $D_m^*(f)$ is the largest disk such that

$$\sigma - \lim_{n\to\infty} R_{n,m} = f.$$

Moreover, $\{R_{n,m}\}$ is pointwise divergent in $\{z : |z| > R_m^*(f)\}$ except on a set of σ-content zero.

- J. Cacoq, B. de la Calle Ysern, G. López Lagomasino, *Incomplete Padé approximation and convergence of row sequences of Hermite-Padé approximants*, J. Approx. Theory, **170** (2013), 59–77.
- J. Cacoq, B. de la Calle Ysern, G. López Lagomasino, *Direct and inverse results on row sequences of Hermite-Padé approximants*, Constr. Approx., **38** (2013), 133–160.

Incomplete Padé approximants



- α_n and α Borel measures on $\overline{\mathbb{C}}$

$$\alpha_n \xrightarrow{*} \alpha, \quad n \to \infty,$$

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- $\Omega = \overline{\mathbb{C}} \setminus E$ connected. If $\operatorname{cap} E > 0$, $g_\Omega(z)$ is the **generalized Green function** of Ω with pole at $z = \infty$.

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- If $C_R = \{z \in \mathbb{C} : |z| = R\}$ and $G = \overline{\mathbb{C}} \setminus \{|z| < R\}$,

$$\operatorname{cap} C_R = R, \quad g_G(z) = \log|z| - \log R.$$

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- p polynomial

$$\Theta_p = \frac{1}{\deg p} \sum_{\zeta:\ p(\zeta)=0} \delta_\zeta.$$

Grothmann Theorem

$\{P_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ are sequences such that $\deg P_n \leq d_n$. Suppose

$$\limsup_{n\to\infty} \left(\frac{1}{d_n} \log \|P_n\|_E \right) \leq 0, \tag{1}$$

$$\lim_{n\to\infty} \Theta_{P_n}(K) = 0, \quad K \subset \mathring{E}, \text{ compact subset} \tag{2}$$

and there exists a compact subset $M \subset \overline{\mathbb{C}} \setminus E$ with

$$\liminf_{n\to\infty} \left\{ \max_{z\in M} \left(\frac{1}{d_n} \log |P_n(z)| - g_\Omega(z) \right) \right\} \geq 0. \tag{3}$$

$\Omega = \overline{\mathbb{C}} \setminus E$. Then

$$\Theta_{P_n} \xrightarrow{*} \mu_E,\; n \in \mathbb{N}$$

where μ_E is the equilibrium measure of the set E.

Corollary

$\Lambda \subset \mathbb{N}$, $\{P_n\}_{n\in\Lambda}$ *sequence of monic polynomials with* $\deg P_n = n$, $n \in \Lambda$.

$$\boxed{\limsup_{n\in\Lambda} \|P_n\|_E^{1/n} \leq \operatorname{cap} E,} \tag{4}$$

and

$$\lim_{n\in\Lambda} \Theta_{P_n}(K) = 0, \quad K \subset \mathring{E} \textit{ compact subset.} \tag{5}$$

Then

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$$\pi_{n,m} = \frac{p_{n,m}}{q_{n,m}} \begin{cases} \deg p_{n,m} \leq n - m, \ \deg q_{n,m} \leq m, \ q_{n,m} \not\equiv 0, \\ [q_{n,m} f - p_{n,m}](z) = A z^{n+1} + \cdots . \end{cases}$$

Jentzsch-Szegő-type Theorem

$$\Theta_{p_{n,m}} \xrightarrow{*} \mu_{C_{R_m(f)}}, \quad n \in \Lambda,$$

$\mu_{C_{R_m(f)}}$ is the equilibrium measure of the circle $C_{R_m(f)}$.

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We know

$$\pi_{n,m}(z) - \pi_{n-1,m}(z) = \frac{A_{n,m} z^n}{q_{n,m}(z) q_{n-1,m}(z)}$$

and

$$\limsup_{n\to\infty} |A_{n,m}|^{1/n} = \frac{1}{R_m(f)}.$$

Let $\Lambda \subset \mathbb{N}$ be such that $\lim_{n\in\Lambda} |A_{n,m}|^{1/n} = \dfrac{1}{R_m(f)} \Rightarrow |A_{n,m}| > 0,\ n \geq N.$

Zero distribution of Padé approximants

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- Convergence in σ-content, normalization of $q_{n,m}$ ($|b_{n,m}| \leq 1$) and Bernstein-Walsh lemma

$$\limsup_{n\in\Lambda} \left\| \frac{p_{n,m}}{c_{n,m}} \right\|_{C_{R_m(f)}}^{1/n} = \limsup_{n\in\Lambda} \left\| \frac{b_{n-1,m}}{A_{n,m}} p_{n,m} \right\|_{C_{R_m(f)}}^{1/n} \leq \mathrm{cap}\, C_{R_m(f)} = R_m(f).$$

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- Rouché Theorem

$$\lim_{n\in\Lambda} \Theta_{p_{n,m}}(K) = 0, \quad K \subset D_{R_m}(f) \text{ compact subset.}$$

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Corollary

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Theorem (B. de la Calle Ysern, J. Mínguez Ceniceros)

$\{R_{n,m}\}_{n \geq m}$ is a sequence of incomplete Padé approximants of type (n, m, m^*) for f. Then, there exists a subsequence $\Lambda \subset \mathbb{N}$ such that

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$$\limsup_{n\to\infty} |A_{n,m}|^{1/n} = \frac{1}{R_m^*(f)}, \quad D_m^*(f) = \{z : |z| < R_m^*(f)\}.$$

We take Λ such that $\lim_{n\in\Lambda} |A_{n,m}|^{1/n} = \dfrac{1}{R_m^*(f)}$

$$P_{n,m}(z)Q_{n-1,m}(z) - P_{n-1,m}(z)Q_{n,m}(z) = A_{n,m}z^n q_{n,m-m^*}(z)$$

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Proposition (Exact rate of convergence)

For each nondegenerate continuum $Q \subset D_m^*(f) \setminus (\{0\} \cup \mathcal{P}_f)$, it holds that

$$\limsup_{k\to\infty} \|Q_{n_k,m}(f - R_{n_k,m})\|_Q^{1/n_k} = \frac{\|z\|_Q}{R_m^*(f)} < 1.$$

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$$\Downarrow$$

Grothmann Theorem

Theorem (B. de la Calle Ysern, J. Mínguez Ceniceros)

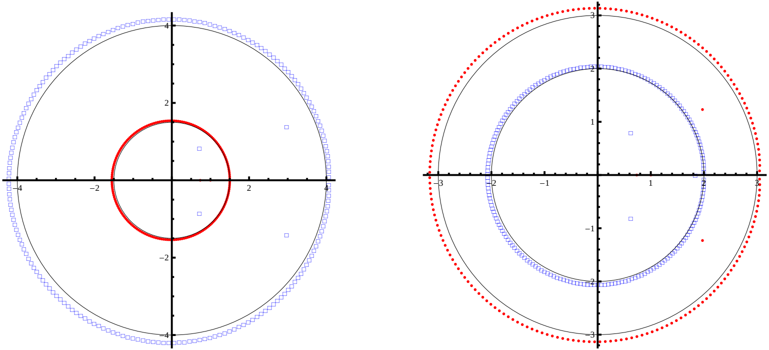
$\boldsymbol{f} = (f_1, \ldots, f_d)$, $\boldsymbol{m} = (m_1, \ldots, m_d) \in \mathbb{Z}_+^d \setminus \{\boldsymbol{0}\}$. $\{R_{n,\boldsymbol{m}}\}$ *a sequence of Hermite-Padé approximants of type* $(n, \boldsymbol{m})$ *for* $\boldsymbol{f}$. *Then, for each* $k = 1, \ldots, d$, *there exists a subsequence* $\Lambda_k \subset \mathbb{N}$ *such that*

$$\Theta_{P_{n,\boldsymbol{m},k}} \xrightarrow{*} \mu_{R^*_{|\boldsymbol{m}|}(f_k)}, \quad n \in \Lambda_k,$$

where $\mu_{R^*_{|\boldsymbol{m}|}(f_k)}$ *is the equilibrium measure of the circle* $C_{R^*_{|\boldsymbol{m}|}(f_k)}$.

$$f_1(z) = \frac{1}{1-2z} + \frac{1}{3/2-z} + \log(3-z), \quad f_2(z) = \frac{1}{1-z} + \frac{1}{2-z} + \log(4-z)$$

$\mathbf{m} = (1, 2)$ $\qquad\qquad$ $\mathbf{m} = (2, 1)$



$R_3^*(f_1) = 3/2,\ R_3^*(f_2) = 4$ $\qquad\qquad$ $R_3^*(f_1) = 3,\ R_3^*(f_2) = 2$

Thank you very much for your attention