Higher order monotonicity: analytical and probabilistic aspects

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Outline

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4. Classical mean values
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1. Univariate higher order monotonicity

Let $I \subseteq \mathbb{R}$ be an interval, $f : I \longrightarrow \mathbb{R}$ and $h \geq 0$. Put $(E_h f)(s) := f(s+h)$, where defined, and

$$\Delta_h := E_h - E_0 =: -\nabla_h$$

i.e.

$$(\Delta_h f)(s) = f(s+h) - f(s) = -(\nabla_h f)(s) \, .$$

$\{E_h \mid h \geq 0\}$ commutes, hence also $\{\Delta_h\}$ and $\{\nabla_h\}$.

Definition.

(a) f is n-$\uparrow$ (n-*increasing*)
$:\Longleftrightarrow \Delta_h^p f \geq 0 \qquad \forall\, h > 0, \ \forall\, p = 1, \ldots, n$
(not necessarily for $p = 0$, i.e. $f \geq 0$ is not assumed)

(b) f is n-$\downarrow$ (n-*decreasing*)
$:\Longleftrightarrow \nabla_h^p f \geq 0 \qquad \forall\, h > 0, \ \forall\, p = 1, \ldots, n$

Remarks

(a) f is 2-$\uparrow \iff f$ is increasing and convex (and then continuous on $\mathring{I}$)

(b) f is 2-$\downarrow \iff f$ is decreasing and concave

(c) f is n-$\downarrow \iff f(-s)$ is n-$\uparrow$ on $-I$.

(d) For I open and n times differentiable f :

$$f \text{ is } n\text{-}\uparrow \iff f^{(p)} \geq 0 \quad \forall\, p = 1, \ldots, n$$
$$f \text{ is } n\text{-}\downarrow \iff (-1)^p f^{(p)} \geq 0 \quad \forall\, p = 1, \ldots, n$$

Examples.

(i) $I = \mathbb{R}_+$, $f(s) = s^\alpha$, $n < \alpha < n+1, n \in \mathbb{N}_0$
f is $(n+1)$-$\uparrow$, but not $(n+2)$-$\uparrow$

(ii) $I = \mathbb{R}_+$, $f(s) := \frac{s^n}{1+s}$, $n \in \mathbb{N}$
$\Longrightarrow f', \ldots, f^{(n)} \geq 0 > f^{(n+1)}$
i.e. f is n-$\uparrow$, but not $(n+1)$-$\uparrow$

(iii) $I = \mathbb{R}$, $g(s) := e^{-ns}/(1+e^{-s}) = f(e^{-s})$ with f as in (ii)
g is n-$\downarrow$, but certainly not completely monotone, since

$$g(s) = e^{-ns} - e^{-(n+1)s} + e^{-(n+2)s} \mp \ldots \quad \text{for } s > 0$$

(iv) $I = \mathbb{R}_+$, $f(s) = (1-cs)_+^{n-1}, n \in \mathbb{N}, c > 0$
f is n-$\downarrow$, but not $(n+1)$-$\downarrow$

(v) $I =]0, \infty[$, $f(s) = -\log s$ is n-$\downarrow$ $\forall\, n \in \mathbb{N}$ (but not completely monotone!)

Theorem 1.

Let $n \geq 2$, $f : [0,1[\longrightarrow \mathbb{R}_+$. *Then* f *is* n-$\uparrow$ *if and only if there exist uniquely determined* $a_0, \ldots, a_{n-2} \geq 0$ *and a measure* μ *on* $[0,1[$ *such that*

$$f(t) = a_0 + a_1\, t + \ldots + a_{n-2}\, t^{n-2} + \int (t-a)_+^{n-1}\, d\mu(a), \quad 0 \leq t < 1\,.$$

The function f *is continuous and for* $n > 2$ $(n-2)$ *times continuously differentiable on* $[0,1[$, *where* $f^{(m)}$ *is* $(n-m)$-$\uparrow$, $m = 1, \ldots, n-2$. *The right derivative of* $f^{(n-2)}$ *exists and equals* $(n-1)! \cdot \mu([0,t])$, *and is therefore right-continuous and increasing; in particular* $\mu(\{0\}) = f_r^{(n-1)}(0)/(n-1)!$. *The constants* a_j *are given by* $a_j = f^{(j)}(0)/j!$, $j \leq n-2$.

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$f_a(t) := \frac{(t-a)_+}{1-a}$, $0 \leq a < 1$, $0 \leq t \leq 1$ (so $f_0(t) = t$), $f_1 := 1_{\{1\}}$
$K_n := \{f : [0,1] \longrightarrow \mathbb{R}_+ \mid f$ is n-$\uparrow$, $f(1) = 1\}$

Theorem 2.

K_n is a Bauer simplex for $n \geq 2$, and

$$\mathrm{ex}\,(K_n) = \{f_0^j \mid j = 0, 1, \ldots, n-2\} \cup \{f_a^{n-1} \mid a \in [0,1]\}\,.$$

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Definition.

f is *absolutely monotone* ("∞-$\uparrow$") iff f is n-$\uparrow$ $\quad \forall\, n \in \mathbb{N}$.

Then $K_\infty := \bigcap_{n\geq 1} K_n$ is also a Bauer simplex, with

$\mathrm{ex}(K_\infty) = \{f_0^j \mid j \in \overline{\mathbb{N}}_0\}$, $f_0^\infty := \lim_n f_0^n = f_1 = 1_{\{1\}}$.

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There are counterparts for n-$\downarrow$ functions:

$$L_n := \{g : [0,1] \longrightarrow \mathbb{R}_+ \mid g \text{ is } n\text{-}\downarrow,\ g(0) = 1\}$$

$$H_n := \{g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ \mid g \text{ is } n\text{-}\downarrow,\ g(0) = 1\}$$

Both L_n and H_n are Bauer simplices, and

$$\mathrm{ex}(L_n) = \{g_1^j \mid j = 0, \ldots, n-2\} \cup \{g_c^{\,n-1} \mid 1 \leq c \leq \infty\}$$

$$\text{where } g_c(s) := (1 - cs)_+, \quad g_\infty := 1_{\{0\}}$$

$$\mathrm{ex}(H_n) = \{g_c^{n-1} \mid c \in [0, \infty]\} .$$

The resulting integral representation for functions in H_n goes back to Schoenberg and Williamson. Functions in $H_\infty := \bigcap_{n \geq 1} H_n$ are called *completely monotone*. Both H_∞ and $L_\infty := \bigcap_{n \geq 1} L_n$ are Bauer simplices.

Remark. $f(s) = e^{-s}$ is extreme in H_∞ but not in L_∞ :

$$f(s) = \sum_{j=0}^{\infty} \frac{1}{e \cdot j!} (1 - s)^j$$

2. Multivariate distribution and survival functions

Let $\mu \in M_+^1(\mathbb{R}^d)$, i.e. a probability measure.

$$\hat{\mu}(a) := \mu(]-\infty, a]) \quad , \quad a \in \mathbb{R}^d$$

$$\check{\mu}(a) := \mu([a, \infty[) \quad , \quad a \in \mathbb{R}^d$$

$\hat{\mu}, \check{\mu}$ are called *distribution* resp. *survival function* of μ ("d.f." resp. "s.f."), with special monotonicity properties as we'll see now.

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Let $A_1, \ldots A_n \neq \emptyset$, $A := A_1 \times \ldots \times A_n$, and $\varphi : A \longrightarrow \mathbb{R}$. For $a, b \in A$ put

$$\begin{aligned} D_a^b \varphi \quad := \quad & \varphi(b) - \varphi(a_1, b_2, \ldots, b_n) - \ldots - \varphi(b_1, \ldots, b_{n-1}, a_n) \\ & + \varphi(a_1, a_2, b_3, \ldots, b_n) + \varphi(a_1, b_2, a_3, b_4, \ldots, b_n) \\ & + \ldots + \varphi(b_1, \ldots, b_{n-2}, a_{n-1}, a_n) \\ & - \varphi(a_1, a_2, a_3, b_4, \ldots, b_n) - \ldots \pm \ldots + (-1)^n \varphi(a) \end{aligned}$$

[Note: If $a_i = b_i$ for some i then $D_a^b\,\varphi = 0$.]

In the following: $A_i \subseteq \overline{\mathbb{R}}\ \forall\, i$; and for $a, b \in A$

$$a < b :\Longleftrightarrow a_i < b_i \quad \forall\, i\,.$$

Definition.

φ is *fully n-increasing* (abbrev. " $\underset{n}{\Uparrow}$ ") iff $D_a^b\varphi \geq 0$ for all $a < b$ in A , and if this also applies if some of the variables in φ are fixed.

For $A = \mathbb{R}^n$ and $\varphi = \hat{\mu}$ we have

$$D_a^b\,\hat{\mu} = \mu(]a, b]) \geq 0 \quad \forall\, a < b$$

so that $\hat{\mu}$ is $\underset{n}{\Uparrow}$, and this is in fact the crucial property of d.f.'s:

Correspondence Theorem

Let $A_i \subseteq \overline{\mathbb{R}}$ be non-empty, $1 \leq i \leq n$, with $\sup A_i \in A_i \; \forall\, i,\; A := A_1 \times \ldots \times A_n$. For $\varphi : A \longrightarrow \mathbb{R}_+$ we have

$$\exists \;\; \mu \in M_+(\overline{A}) \;\text{ with }\; \varphi(a) = \mu([-\infty, a] \cap \overline{A}) \quad \forall\, a \in A$$

$$\iff \varphi \;\text{ is }\; \underset{n}{\Uparrow} \;\text{ and right-continuous}$$

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$$\Longleftrightarrow \varphi \;\text{ is }\; \underset{n}{\Uparrow} \;\text{ and right-continuous}$$

If in the last definition we require $D_a^b\,\varphi \ge 0$ for all $a > b$ in A, then φ is called *fully n-max increasing*, the decisive property of survival functions:

Suppose $\inf A_i \in A_i \;\forall\, i$. Then

$$\exists \;\; \mu \in M_+(\overline{A}) \;\text{ with }\; \varphi(a) = \mu([a, \infty] \cap \overline{A}) \quad \forall\, a \in A$$

$$\Longleftrightarrow \varphi \;\text{ is fully } n\text{-max increasing and left-continuous}$$

Remark. On an abelian semigroup $(S,+)$ a function $f : S \longrightarrow \mathbb{R}$ is *positive definite* iff

$$\sum_{i,j=1}^{k} c_i \, c_j f(s_i + s_j) \geq 0$$

$\forall\, k \in \mathbb{N},\ \forall\, c_i \in \mathbb{R}, \forall\, s_i \in S$. Now $\varphi : A \longrightarrow \mathbb{R}_+$ is fully n-(resp. n-max) increasing iff φ is positive definite on $(A, \wedge)$, resp. on $(A, \vee)$.

3. Connections between 1. and 2.

The equivalence (i) $\iff$ (ii) in the following result is due to P. Morillas.

Theorem 3.

Let $n \geq 2, I \subseteq \mathbb{R}$ *an interval,* $f : I \longrightarrow \mathbb{R}$. *Then there are equivalent:*

(i) f *is* n-$\uparrow$

(ii) *If* $\varphi : \{0,1\}^n \longrightarrow I$ *is* $\underset{n}{\Uparrow}$ *then so is* $f \circ \varphi$.

(iii) $\forall\, k \in \{1, \ldots, n\}, t \in I, h > 0$ *with* $t + kh \in I$

$$D_{\mathbf{0}_k}^{\mathbf{1}_k} f\left(t + h \sum_{i=1}^{k} x_i\right) \geq 0$$

Corollary.

(a) f, g n-$\uparrow$, $g : I \longrightarrow J$, $f : J \longrightarrow \mathbb{R}$
$\Longrightarrow$ *also* $f \circ g$ *is* n-$\uparrow$.

(b) f, g n-$\uparrow$ *on* I , *both non-negative*
$\Longrightarrow$ *also* $f \cdot g$ *is* n-$\uparrow$.

Proof.

(a) follows immediately from (i) $\Longleftrightarrow$ (ii) in the preceding theorem

(b) If $\varphi : \{0, 1\}^n \longrightarrow I$ is $\underset{n}{\Uparrow}$, then so are $f \circ \varphi$ and $g \circ \varphi$; both are positive definite, hence also $(f \circ \varphi) \cdot (g \circ \varphi) = (f \cdot g) \circ \varphi$.$\square$

As a consequence, we can characterize the (univariate) functions operating on multivariate d.f.'s:

Theorem 4.

Let $f : [0,1] \longrightarrow [0,1]$ *be continuous at* 1, $f(0) = 0$, $f(1) = 1$; $n \geq 2$. *Then there are equivalent:*

(i) f *is* n-$\uparrow$

(ii) $f \circ F$ *is a d.f.* $\forall$ *n-dimensional d.f.* F

(iii) $f\left(\sum_{i=1}^{n} x_i/n\right)$ *is a d.f. on* $[0,1]^n$.

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Another very direct connection is the following

Theorem 5.

Let $u = (u_1, \ldots, u_n) \in \,]0, \infty[^n$, $u_0 := u_1 + \ldots u_n$, *and let* $f : [0, u_0] \longrightarrow \mathbb{R}$ be given. Then $f\left(\sum_{i=1}^{n} x_i\right)$ *is* $\underset{n}{\Uparrow}$ *on* $[0, u]$ *iff* f *is* n-$\uparrow$.

Corollary.

For $n \geq 2$ there is no d.f. on $\mathbb{R}_+^n$ of the form $f\left(\sum_{i=1}^n x_i\right)$, for some $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$.

For, f would then be increasing, convex, and bounded, hence constant.

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For, f would then be increasing, convex, and bounded, hence constant.

n-$\uparrow$ functions operate also on survival functions:

If $f : I \longrightarrow \mathbb{R}$ is n-$\uparrow$, and $\psi : A \longrightarrow I$ is fully n-max increasing then so is also $f \circ \psi$.

And for n-$\downarrow$ functions we have:

Theorem 6.

Let $g : [0,1] \longrightarrow [0,1]$ *be continuous at* $0, g(0) = 1, g(1) = 0,$ $n \geq 2$. *Then there are equivalent:*

(i) g *is* n-$\downarrow$

(ii) $g \circ (1 - G)$ *is a s.f.* $\forall$ *n-dimensional s.f.* G

(iii) $g\left(\sum_{i=1}^{n} x_i/n\right)$ *is a s.f. on* $[0,1]^n$.

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(iii) $g\left(\sum_{i=1}^{n} x_i/n\right)$ *is a s.f. on* $[0,1]^n$.

For a s.f. $G(a) = \mu([a, \infty])$ of some $\mu \in M_+^1(\overline{\mathbb{R}}^n)$ we call

$$\psi(a) := 1 - G(a) = \mu([a, \infty]^c) =: \mu^*(a)$$

the *co-survival function* of μ . Their characteristic property is to be

$$\textit{fully n-max decreasing} \quad (\text{“} \underset{n}{\Downarrow} \text{”})$$

$$\Longleftrightarrow D_a^b \, \psi \leq 0 \qquad \forall \, a > b$$

(plus left continuity).

4. Classical mean values

For $x \in \mathbb{R}_+^n$ and $t \in \mathbb{R}$ let

$$M_t(x) := \left(\frac{1}{n}\sum_{i=1}^n x_i^t\right)^{1/t} \quad \text{for } t \neq 0\,,\; M_0(x) := \left(\prod_{i=1}^n x_i\right)^{1/n} ,$$

completed by $M_{-\infty}(x) := \min_{1 \leq i \leq n} x_i$, $M_\infty(x) := \max_{1 \leq i \leq n} x_i$.

Then $\overline{\mathbb{R}} \ni t \longmapsto M_t(x)$ is continuous, and strictly increasing for non-constant x_i . Of course $M_t(\mathbf{0}) = 0$, $M_t(\mathbf{1}) = 1$, and M_t is (in some sense) increasing. Is it a d.f. on $[0,1]^n$?

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Here's a complete answer:

Theorem 7.

Let $n \ge 2$. *Then*

(i) $M_t \mid [0,1]^n$ *is a* d.f. $\iff t \in \left[-\infty, \frac{1}{n-1}\right] \cup \left\{\frac{1}{n-2}, \ldots, \frac{1}{2}, 1\right\}$

(ii) M_t *is "max-infinitely divisible"* (i.e. $\sqrt[k]{M_t}$ is a d.f. $\forall\, k \in \mathbb{N}$) $\iff t \in [-\infty, 0]$.

Consider again the general situation where $\emptyset \neq A_i \subseteq \overline{\mathbb{R}}$ and $A := A_1 \times \ldots \times A_n,\ \varphi : A \longrightarrow \mathbb{R}_+$. If φ is $\underset{n}{\Uparrow}$ and infinitely divisible, i.e. $\sqrt[k]{\varphi}$ is $\underset{n}{\Uparrow}$ $\forall\, k \in \mathbb{N}$, then $\{\varphi > 0\} \subseteq A$ is a subsemigroup w.r.t. "$\wedge$" and

$$\log \varphi = \lim_{k \to \infty} k \left(\sqrt[k]{\varphi} - 1\right)$$

is still $\underset{n}{\Uparrow}$ on $\{\varphi > 0\}$.

$$\psi := -\log \varphi$$

is then by definition *fully n-decreasing*, and $\psi \geq 0$ for $0 < \varphi \leq 1$. Furthermore ψ is negative definite and completely alternating on the semigroup $(\{\varphi > 0\}, \wedge)$. Similarly, for a fully n-max increasing $\varphi : A \longrightarrow \mathbb{R}_+$ (typically a survival function) which is inf. div. (called "*min-infinitely divisible*"), its negative logarithm $\psi = -\log \varphi$ is fully n-max decreasing, i.e. $\underset{n}{\Downarrow}$.

Look at M_t for $-\infty < t < 0$ (for $t = -1$ this is the harmonic mean): with $s := -t$ we have

$$-\log\ M_t(x) = \frac{1}{s}\log\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^s\right]$$

where $[...]$ is clearly fully n-decreasing, and $\log$ is a Bernstein function, preserving negative definiteness. Therefore M_t is an infinitely divisible d.f.

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where $[\ldots]$ is clearly fully n-decreasing, and $\log$ is a Bernstein function, preserving negative definiteness. Therefore M_t is an infinitely divisible d.f.

For $t \geq 1 \quad M_t(x) = \left(\frac{1}{n} \sum\limits_{1}^{n} x_i^t \right)^{1/t}$ is $\underset{n}{\Downarrow}$, hence a co-survival function. Of course, also M_∞ is $\underset{n}{\Downarrow}$.

All M_t's are (positively) homogeneous:

$$M_t(sx) = s \cdot M_t(x)\ , \quad s \geq 0\ , \quad x \in \mathbb{R}_+^n\ .$$

Let

$$\hat{H} \ := \ \{\varphi : \mathbb{R}^n_+ \longrightarrow \mathbb{R} \mid \varphi \text{ is } \underset{n}{\Uparrow}\ , \ \text{right continuous}$$
$$\text{and homogeneous}, \ \varphi(\mathbf{1}_n) = 1\}$$
$$H \ := \ \left\{\mu \in M_+(\mathbb{R}^n_+) \mid \mu \ \text{ is homogeneous}, \ \mu([0,1]^n) = 1\right\}$$

then $H \sim \hat{H}$ via $\mu \longmapsto \hat{\mu}$, both are Bauer simplices and

$$\mu \in \ \text{ex}\,(H) \Longleftrightarrow \mu = \lambda_c \ \text{ for some } \ c \in C$$

with $C := \left\{c \in \mathbb{R}^n_+ \mid \max c_i = 1\right\}$, where λ_c is the image of $\lambda\!\!\lambda \mid \mathbb{R}_+$ under $s \longmapsto s \cdot c$, and $\hat{\lambda}_c(x) = \lambda_c([0,x]) = \min(x_i/c_i)$.

Let

$$\hat{H} := \{\varphi : \mathbb{R}^n_+ \longrightarrow \mathbb{R} \mid \varphi \text{ is } \underset{n}{\Uparrow}, \text{ right continuous and homogeneous}, \varphi(\mathbf{1}_n) = 1\}$$

$$H := \left\{\mu \in M_+(\mathbb{R}^n_+) \mid \mu \text{ is homogeneous}, \mu([0,1]^n) = 1\right\}$$

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Similarly, with $Z := \overline{\mathbb{R}}^n_+ \setminus \{\infty_n\}$, put

$$K := \{\mu \in M_+(Z) \mid \mu \text{ is homogeneous}, \mu([\mathbf{1}_n, \infty_n]^c) = 1\}$$

$$K^* := \left\{\varphi : \mathbb{R}^n_+ \longrightarrow \mathbb{R} \mid \varphi \text{ is homogeneous and } \underset{n}{\Downarrow}, \varphi(\mathbf{1}_n) = 1\right\},$$

then likewise $K \sim K^*$ via $\mu \longmapsto \mu^*$, and

$$\operatorname{ex}(K) = \left\{ \lambda_{1/c} \mid c \in C \right\} \ , \quad \frac{1}{c} := \left(\frac{1}{c_1}, \ldots, \frac{1}{c_n} \right) \in Z$$

with $\lambda^*_{1/c}(x) = \lambda_{1/c}([x, \infty_n]^c) = \max(c_i \, x_i)$.

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with $\lambda^*_{1/c}(x) = \lambda_{1/c}([x, \infty_n]^c) = \max(c_i\, x_i)$.

Remark. Functions in K^* are automatically continuous. This does not hold for homogeneous $\underset{n}{\Uparrow}$ functions:

$\varphi(x,y) := (x+y) \cdot 1_{]0,\infty[^2}(x,y)$ is an example.

then likewise $K \sim K^*$ via $\mu \longmapsto \mu^*$, and

$$\operatorname{ex}(K) = \{\lambda_{1/c} \mid c \in C\} \ , \quad \frac{1}{c} := \left(\frac{1}{c_1}, \ldots, \frac{1}{c_n}\right) \in Z$$

with $\lambda^*_{1/c}(x) = \lambda_{1/c}([x, \infty_n]^c) = \max(c_i\, x_i)$.

Remark. Functions in K^* are automatically continuous. This does not hold for homogeneous $\underset{n}{\Uparrow}$ functions:

$\varphi(x,y) := (x+y) \cdot 1_{]0,\infty[^2}(x,y)$ is an example.

So in particular each mean value M_t which is a d. f. allows a spectral representation in terms of the "lower end of the scale", i.e. it is a unique mixture of $x \longmapsto \min(x_i/c_i),\ c \in C$.

For $t \geq 1$ each M_t is $\underset{n}{\Downarrow}$, the co-s.f. of some (homogeneous) measure on Z . It is a unique mixture of $x \longmapsto \max(c_i\, x_i),\ c \in C$, i.e. from the "upper end of the scale".

Examples.

$n = 2$

(a) $t = -1$

$$M_{-1}(x,y) = \left[\frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right)\right]^{-1} = \frac{2xy}{x+y} = \int_C \left(\frac{x}{c_1} \wedge \frac{y}{c_2}\right) d\gamma_{-1}(c)$$

where $\gamma_{-1} \in M_+^1(C)$ is given by the density $4u/(1+u)^3$ on both parts of $([0,1] \times \{1\}) \cup (\{1\}) \times [0,1]) = C$.

(b) $t = 0$

$$M_0(x,y) = \sqrt{xy} = \int_C \left(\frac{x}{c_1} \wedge \frac{y}{c_2}\right) d\gamma_0(c)$$

where γ_0 has the density $1/(4\sqrt{u})$ on C .

(c) $t = \frac{1}{2}$

$$M_{1/2}(x,y) = \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2 = \int_C \left(\frac{x}{c_1} \wedge \frac{y}{c_2}\right) d\gamma_{1/2}(c)$$

where $\gamma_{1/2}$ has density $1/(8\sqrt{u})$, plus $\frac{1}{4}\left[\varepsilon_{(1,0)} + \varepsilon_{(0,1)}\right]$.

(d) $t = 2$

$M_2(x,y) = \sqrt{\frac{x^2+y^2}{2}} = \frac{1}{\sqrt{2}} \cdot \| (x,y) \|_2$ is $\underset{2}{\Downarrow}$. We have

$$\begin{aligned} \sqrt{x^2+y^2} &= \sqrt{2}\int_C [(c_1 x) \vee (c_2 y)] d\gamma_2(c) \\ &= \int_0^1 [(ux) \vee y + x \vee (uy)] \frac{du}{(1+u^2)^{3/2}} \, . \end{aligned}$$

(e) $t = 1$ is a special case:

$$M_1(x,y) = \frac{x+y}{2} \text{ is both } \underset{2}{\Uparrow} \text{ and } \underset{2}{\Downarrow}, \ \gamma_1 = \frac{1}{2}\left(\varepsilon_{(1,0)} + \varepsilon_{(0,1)}\right)$$

$$M_1 = \hat{\mu},\ \mu = \frac{1}{2}\left(\lambda\!\!\lambda \text{ on } (\{0\} \times \mathbb{R}_+) + \lambda\!\!\lambda \text{ on } (\mathbb{R}_+ \times \{0\})\right)$$

$$M_1 = \nu^*,\ \nu = \frac{1}{2}\left(\lambda\!\!\lambda \text{ on } (\{\infty\} \times \mathbb{R}_+) + \lambda\!\!\lambda \text{ on } (\mathbb{R}_+ \times \{\infty\})\right).$$

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$$M_1 = \nu^*,\; \nu = \frac{1}{2}(\lambda\!\!\lambda \text{ on } (\{\infty\} \times \mathbb{R}_+) + \lambda\!\!\lambda \text{ on } (\mathbb{R}_+ \times \{\infty\})) .$$

As a consequence of our general discussion above we can state the following

Corollary.

Each homogeneous $\underset{n}{\Downarrow}$ *function on* $\mathbb{R}_+^n$ *is the restriction of a seminorm, i.p. it is continuous.*

(... because $x \longmapsto \max(c_i|x_i|)$ is a seminorm for $c_i \geq 0$).

5. Stable tail dependence functions

Definition.

A d.f. F on $\mathbb{R}^n_+$ is called a *simple multivariate extreme value distribution* (function) iff

$$(F(tx))^t = F(x) \quad \forall\, x \in \mathbb{R}^n_+, \quad \forall\, t > 0$$

and if F has standard *Fréchet margins*, defined by the (1-dimensional) d.f. $\exp(-1/u)$ for $u > 0$.

Then

$$F(x) = 0 \iff x_i = 0 \ \text{ for some } \ i$$

$$F(x) < 1 \quad \forall\, x \in \mathbb{R}^n_+$$

$$\sqrt[n]{F(x)} = F(nx) \text{ is a d.f. } \forall\, n \in \mathbb{N}$$

and therefore

$$\log F = \lim_n n\left(\sqrt[n]{F} - 1\right)$$

is also $\underset{n}{\Uparrow}$ on $]0,\infty[^n, \log F \leq 0$, and finally

$$f(x) := -\log F\left(\frac{1}{x}\right) \text{ is } \underset{n}{\Downarrow} \text{ on } \mathbb{R}^n_+ \text{ , homogeneous and } f \geq 0 \,.$$

f is called a *stable tail dependence function.*

Theorem 8.

$f : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ is a stable tail dependence function iff it is homogeneous, $\underset{n}{\Downarrow}$ and $f(e_1) = \ldots = f(e_n) = 1$, (e_i denoting the unit vectors in $\mathbb{R}^n$).

Examples.

1. The t-norms $\left(\sum x_i^t\right)^{1/t} = n^{1/t} \cdot M_t$, for $t \geq 1$.
2. $f(x,y) := x + y - \frac{xy}{x+y}$ on $\mathbb{R}^2_+$.
3. $f(x,y) := \frac{1}{2}\left(x + y + \sqrt{x^2 + y^2 + 2\vartheta xy}\right)$, with $|\vartheta| \leq 1$.

All these functions extend to norms on $\mathbb{R}^n$.

6. Linearly order symmetric d.f.'s and exchangeable Marshall-Olkin distributions

We saw above that homogeneous d.f.'s on $[0,1]^n$ have the form

$$F_\gamma(x) = \int_C \hat{\lambda}_c(x)\, d\gamma(c) \quad , \quad \gamma \in M_+^1(C)$$

Then F is exchangeable if γ is.
Let now γ be exchangeable and concentrated on
$C \cap \{0,1\}^n = \{0,1\}^n \setminus \{0\}$, equivalently

$$\gamma = \sum_{k=1}^{n} p_k \gamma_k \quad , \quad \gamma_k \text{ uniform on } \{e_\alpha \mid |\alpha| = k\},$$

with $e_\alpha := \sum_{i\in\alpha} e_i$, $\emptyset \neq \alpha \subseteq \{1,\dots,n\}$, and $p_k \geq 0$, $\sum p_k = 1$.
Put $\varphi_k := F_{\gamma_k}$, then

$$\varphi_k(x) = \frac{1}{\binom{n}{k}} \sum_{|\alpha|=k} \min_{i\in\alpha} x_i \;,\; k = 1,\dots,n$$

are "normalized min sums", i.p.$\varphi_1(x) = \frac{1}{n}\sum_{i=1}^{n} x_i, \varphi_n(x) = \min_{1\le i\le n} x_i$.
With $x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$ as ordered components of x it turns out that

$$F = F_\gamma \text{ for some exchangeable } \gamma \in M_+^1\left(\{0,1\}^n \setminus \{0\}\right)$$

iff $F(x) = \sum_{i=1}^{n} a_i\, x_{(i)}$ for some $a_i \ge 0$, a socalled *linearly order symmetric d.f.* where the sequence $f(i) := a_i$ is characterized by being $(n-1)$-$\downarrow, \ge 0$ with $\sum_{i=1}^{n} f(i) = 1$.

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Consider such a d.f.:

$$F(x) = \sum_{i=1}^{n} a_i\, x_{(i)} \ ,$$

then $H(x) := -F(-x) = \sum_{i=1}^{n} a_i\, x_{[i]}$, $x_{[1]} \ge x_{[2]} \ge \ldots$, is $\underset{n}{\Downarrow}$,

showing H to be a co-s.f., therefore negative definite on $(\mathbb{R}^n_+, \vee)$, so that $\exp(-H)$ is an infinitely divisible s.f..
For $F = \varphi_k$ and $\psi_k(x) := -\varphi_k(-x)$ we get "normalized max-sums"

$$\psi_k(x) = \frac{1}{\binom{n}{k}} \sum_{|\alpha|=k} \max_{i\in\alpha} x_i \ ,$$

and any co-s.f. of the form

$$\psi(x) = \sum_{i=1}^{n} a_i \, x_{[i]} \qquad \text{(with } (a_i)_{i=1}^n \ (n-1)\text{-}\!\downarrow)$$

can be written uniquely as

$$\psi = \sum_{k=1}^{n} p_k \, \psi_k \ , \quad p_k \geq 0 \ .$$

The corresponding s.f.'s $\exp(-\psi)$ are precisely those of exchangeable Marshall-Olkin distributions.

7. Extended multivariate higher order monotonicity

Motivation: if F is an m-dimensional d.f., G an n-dimensional d.f., $f : [0,1]^2 \longrightarrow [0,1]$, when is $f \circ (F \times G)$ a d.f. on $\mathbb{R}^{m+n}$?
Let $I_1, \ldots, I_d$ be intervals in $\mathbb{R}$, $I := I_1 \times \ldots \times I_d$, $f : I \longrightarrow \mathbb{R}$.
Again, for $s \in I$, $h \in \mathbb{R}_+^d$ such that also $s + h \in I$ put
$(E_h f)(s) := f(s+h)$ and $\Delta_h := E_h - E_0 =: -\nabla_h$.

Definition.

Let $\boldsymbol{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$. $f : I \longrightarrow \mathbb{R}$ is called

$$\boldsymbol{n}\text{-}\!\uparrow \quad (\boldsymbol{n}\textit{-increasing})$$

iff

$$\Delta_h^{\boldsymbol{p}} f(s) := \left(\Delta_{h_1 e_1}^{p_1} \ldots \Delta_{h_d e_d}^{p_d} f \right)(s) \geq 0$$

$\forall\, s \in I, h \in\,]0,\infty[^d$, $\forall\, \boldsymbol{p} = (p_1, \ldots, p_d) \in \mathbb{N}_0^d$, $0 \lneqq \boldsymbol{p} \leq \boldsymbol{n}$ such that $s_j + p_j h_j \in I_j \ \ \forall j \leq d$.

If instead

$$\left(\nabla_h^{\boldsymbol{p}} f\right)(s) := \left(\nabla_{h_1 e_1}^{p_1} \ldots \nabla_{h_d e_d}^{p_d} f\right)(s) \geq 0$$

f is called $\boldsymbol{n}$-$\downarrow$ ($\boldsymbol{n}$-*decreasing*).
We'll say f is $\boldsymbol{n}$ times (continuously) differentiable if

$$f_{\boldsymbol{n}} := \frac{\partial^{|\boldsymbol{n}|} f}{\partial s_1^{n_1} \ldots \partial s_d^{n_d}} \qquad |\boldsymbol{n}| := n_1 + \ldots + n_d$$

exists (and is continuous). If this holds

$$f \text{ is } \boldsymbol{n}\text{-}\uparrow \iff f_{\boldsymbol{p}} \geq 0 \qquad \forall\, 0 \lneqq \boldsymbol{p} \leq \boldsymbol{n}$$

and

$$f \text{ is } \boldsymbol{n}\text{-}\downarrow \iff (-1)^{|\boldsymbol{p}|} f_{\boldsymbol{p}} \geq 0 \qquad \forall\, 0 \lneqq \boldsymbol{p} \leq \boldsymbol{n}\ .$$

Since

$$\left(\Delta_h^{\mathbf{1}_d} f\right)(s) = D_s^{s+h} f$$

we see that

$$f \text{ is } \underset{d}{\Uparrow} \iff f \text{ is } \mathbf{1}_d\text{-}\uparrow$$

$$f \text{ is } \underset{d}{\Downarrow} \iff -f \text{ is } \mathbf{1}_d\text{-}\downarrow$$

Since

$$\left(\Delta_h^{\mathbf{1}_d} f\right)(s) = D_s^{s+h} f$$

we see that

$$\begin{aligned} f \text{ is } \underset{d}{\Uparrow} &\iff f \text{ is } \mathbf{1}_d\text{-}\uparrow \\ f \text{ is } \underset{d}{\Downarrow} &\iff -f \text{ is } \mathbf{1}_d\text{-}\downarrow \end{aligned}$$

Remark. f $\boldsymbol{n}$-$\uparrow \implies f(\cdot, s_2, s_3, \ldots)$ is n_1-$\uparrow$ in s_1, etc. The converse doesn't hold: let $a > 0$, then

$$f(s_1, s_2) := (s_1 s_2 - a)_+$$

is 2-$\uparrow$ in s_1 for fixed s_2, and 2-$\uparrow$ in s_2 for fixed s_1. However, f is <u>not</u> $(2,2)$-$\uparrow$.

We noticed above that for $n \geq 2$

$$K_n := \{f : [0,1] \longrightarrow \mathbb{R}_+ \mid f \text{ is } n\text{-}\uparrow, f(1) = 1\}$$

is a Bauer simplex with extreme points

$$E_n := \left\{f_0^j \mid j = 0, \ldots, n-2\right\} \cup \left\{f_a^{n-1} \mid 0 \leq a \leq 1\right\}$$

where $f_a(s) = (s-a)_+/(1-a)$ for $0 \leq a < 1$, $f_1 = 1_{\{1\}}$.
In analogy to dimension 1 we introduce

$$K_{\boldsymbol{n}} := \left\{f : [0,1]^d \longrightarrow \mathbb{R}_+ \mid f \text{ is } \boldsymbol{n}\text{-}\uparrow, f(\mathbf{1}_d) = 1\right\}$$

Then we have a far reaching extension of the above univariate result:

Theorem 9.

For $\boldsymbol{n} \geq \mathbf{2}_d$ $K_{\boldsymbol{n}}$ *is a Bauer simplex and* $\mathrm{ex}(K_{\boldsymbol{n}}) = E_{n_1} \otimes \ldots \otimes E_{n_d}$.

Theorem 9.

For $\boldsymbol{n} \geq \mathbf{2}_d$ $K_{\boldsymbol{n}}$ *is a Bauer simplex and* $\mathrm{ex}(K_{\boldsymbol{n}}) = E_{n_1} \otimes \ldots \otimes E_{n_d}$.

Functions in $K_{\underline{\infty}_d} := \bigcap_{\boldsymbol{n} \in \mathbb{N}^d} K_{\boldsymbol{n}}$ are by definition *absolutely monotone*.

Theorem 10.

(i) $f : [0,1[^d \longrightarrow \mathbb{R}_+$ is absolutely monotone iff f is analytic with non-negative coefficients.

(ii) $K_{\underline{\infty}_d}$ is a Bauer simplex, and $\mathrm{ex}(K_{\underline{\infty}_d}) = \left\{ f_0^{j_1} \otimes \ldots \otimes f_0^{j_d} \mid j_i \in \overline{\mathbb{N}}_0 \ \forall_i \right\}$.

We saw already that $f\left(\sum_{i=1}^{d} x_i\right)$ is $\underset{d}{\Uparrow}$ iff f is d-$\uparrow$; more generally

$$f\left(\sum_{i=1}^{d} x_i\right) \text{ is } \boldsymbol{n}\text{-}\uparrow \iff f \text{ is } |\boldsymbol{n}|\text{-}\uparrow \ .$$

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$$f\left(\sum_{i=1}^{d} x_i\right) \text{ is } \boldsymbol{n}\text{-}\uparrow \Longleftrightarrow f \text{ is } |\boldsymbol{n}|\text{-}\uparrow \ .$$

One very important consequence of the "simplicial structure" of $\boldsymbol{n}$-$\uparrow$ functions is a clear picture about when higher order monotonic functions operate on each other. Here a special (but typical) basic case:

Theorem 11.

$\varphi : [0,1]^{d_1} \longrightarrow [0,1]$ $\boldsymbol{m}$-$\uparrow$, $\psi : [0,1]^{d_2} \longrightarrow [0,1]$ $\boldsymbol{n}$-$\uparrow$, and

$$f : [0,1]^2 \longrightarrow \mathbb{R} \quad (|\boldsymbol{m}|, |\boldsymbol{n}|)\text{-}\uparrow \Longrightarrow f \circ (\varphi \times \psi) \text{ is } (\boldsymbol{m}, \boldsymbol{n})\text{-}\uparrow$$

This is an answer to our above question: Given a 2-dimensional d.f. F and a 3-dimensional d.f. G, $f \circ (F \times G)$ is a 5-dimensional d.f. provided f is $(2,3)$-$\uparrow$, $f(1,1) = 1$, and f is continuous in $(1,1)$.

This is an answer to our above question: Given a 2-dimensional d.f. F and a 3-dimensional d.f. G, $f \circ (F \times G)$ is a 5-dimensional d.f. provided f is $(2,3)$-$\uparrow$, $f(1,1) = 1$, and f is continuous in $(1,1)$.

Many thanks

for your attention!