Estimates for Bergman polynomials in domains with corners

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Bergman polynomials

Let Γ be a Jordan curve in $\mathbb{C}$ let $\boxed{G := \operatorname{int}(\Gamma)}$ and consider the Lebesgue space $L^2(G)$ with inner product and norm:

$$\langle f, g\rangle_G := \int_G f(z)\overline{g(z)}dA(z), \quad \|f\|_{L^2(G)} := \langle f, f\rangle_G^{1/2}$$

The Bergman polynomials $\{p_n\}_{n=0}^{\infty}$ of G are the unique orthonormal polynomials w.r.t. the area measure on G:

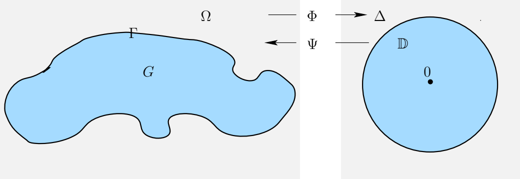
$$\langle p_m, p_n\rangle_G = \int_G p_m(z)\overline{p_n(z)}dA(z) = \delta_{m,n},$$

with

$$p_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots.$$

We will investigate the asymptotic behaviour of $p_n(z)$ on Γ, in cases when Γ has corners.

Associated conformal maps



$$\boxed{\Omega := \overline{\mathbb{C}} \setminus \overline{G}}$$

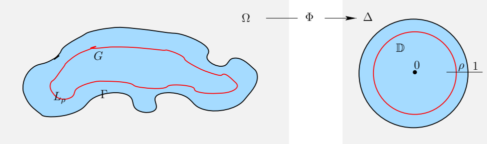
$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots . \qquad \boxed{\operatorname{cap}(\Gamma) = 1/\gamma}$$

$$\Psi(w) = bw + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots . \qquad \boxed{\operatorname{cap}(\Gamma) = b}$$

The Bergman polynomials of G:

$$p_n(z) = \lambda_n z^n + \cdots , \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots .$$

Strong asymptotics when Γ is analytic



Theorem (Carleman, Ark. Mat. Astr. Fys., 1922)

If $\rho < 1$ is the smallest index for which Φ is conformal in $\mathrm{ext}(L_\rho)$, then

$$\boxed{\frac{n+1}{\pi}\,\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n}, \quad \text{where } 0 \leq \alpha_n \leq c_1(\Gamma)\,\rho^{2n},$$

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\,\Phi^n(z)\Phi'(z)\{1 + A_n(z)\}}, \quad n \in \mathbb{N},$$

where

$$|A_n(z)| \leq c_2(\Gamma)\sqrt{n}\,\rho^n, \quad z \in \overline{\Omega}.$$

Strong asymptotics when Γ is smooth

We say that $\Gamma \in C(p,\alpha)$, for some $p \in \mathbb{N}$ and $0 < \alpha < 1$, if Γ is given by $z = g(s)$, where s is the arclength, with $g^{(p)} \in \mathrm{Lip}\alpha$. Then both Φ and $\Psi := \Phi^{-1}$ are p times continuously differentiable in $\overline{\Omega} \setminus \{\infty\}$ and $\overline{\Delta} \setminus \{\infty\}$ respectively, with $\Phi^{(p)}$ and $\Psi^{(p)} \in \mathrm{Lip}\alpha$.

Theorem (Suetin, Proc. Steklov Inst. Math. AMS, 1974)

Assume that $\Gamma \in C(p+1,\alpha)$, *with* $p+\alpha > 1/2$. *Then, for* $n \in \mathbb{N}$,

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n,} \quad \textit{where } 0 \le \alpha_n \le c_1(\Gamma)\,\frac{1}{n^{2(p+\alpha)}},$$

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\{1 + A_n(z)\}},$$

where

$$|A_n(z)| \le c_2(\Gamma)\,\frac{\log n}{n^{p+\alpha}}, \quad z \in \overline{\Omega}.$$

Strong asymptotics for Γ non-smooth

Theorem (St, C. R. Acad. Sci. Paris, 2010 & Constr. Approx., 2013)

Assume that Γ is piecewise analytic without cusps. Then, for $n \in \mathbb{N}$,

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n}, \quad \textit{where} \quad 0 \le \alpha_n \le c(\Gamma)\,\frac{1}{n},$$

and for any $z \in \Omega$,

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\left\{1 + A_n(z)\right\}},$$

where

$$|A_n(z)| \le \frac{c_1(\Gamma)}{\operatorname{dist}(z,\Gamma)\,|\Phi'(z)|}\,\frac{1}{\sqrt{n}} + c_2(\Gamma)\,\frac{1}{n}.$$

More on the geometry of Γ

By Γ being piecewise analytic without cusps we mean that Γ consists of N analytic arcs that meet at points z_j, where they form exterior angles $\omega_j\pi$, with $0 < \omega_j < 2$, $j = 1, \dots, N$.
Our results below will be given in terms of

$$\boxed{\widehat{\omega} := \max\{1, \omega_1, \dots, \omega_N\}}.$$

A standard argument

Consider $L_{1/n} := \{z \in \Omega : |\Phi(z)| = 1 + 1/n\}$, $n \in \mathbb{N}$, and use:

(i) The well-known estimate for the distance of $L_{1/n}$ from Γ,

$$\operatorname{dist}(z, \Gamma) \geq c(\Gamma)\frac{1}{n^{\widehat{\omega}}}, \quad z \in L_{1/n}.$$

(ii) The double inequality, which is a simple consequence of Koebe's 1/4-theorem,

$$\frac{1}{4}\frac{|\Phi(z)| - 1}{\operatorname{dist}(z, \Gamma)} \leq |\Phi'(z)| \leq 4\frac{|\Phi(z)| - 1}{\operatorname{dist}(z, \Gamma)}, \quad z \in \Omega \setminus \{\infty\}.$$

The above, in conjunction with the estimate for $A_n(z)$, lead easily to

$$\boxed{\|p_n\|_{L^\infty(G)} \leq \|p_n\|_{L_{1/n}} \leq c(\Gamma)n^{\widehat{\omega}}}.$$

Other known estimates

The previous estimate agrees with the one obtained by Abdullaev, under somewhat weaker assumptions:

Theorem (Abdullaev, J. Ukrain. Mat., 2000)

If Γ is a quasiconformal curve such that $\Phi \in \mathrm{Lip}\alpha$, $1/2 < \alpha \leq 1$. Then,

$$\|p_n\|_{L^\infty(G)} \leq c(\Gamma) n^{1/\alpha}.$$

To see the relevance, note that our assumptions above, regarding the geometry of Γ, imply the relation $\widehat{\omega} = 1/\alpha$.

Other known estimates

The "standard argument" result is also in agreement with the following norm-comparison estimate, which is valid for any polynomial P_n of degree n.

Theorem (Pritsker, J. Math. Anal. Appl., 1997)

Assume that Γ *consists of* N *Dini-smooth arcs that meet at points* z_j*, where they form exterior angles* $\omega_j\pi$*, with* $0 < \omega_j \leq 2$*,* $j = 1, \ldots, N$*. Then,*

$$\|P_n\|_{L^\infty(G)} \leq c(\Gamma) n^{\widehat{\omega}} \|P_n\|_{L_2(G)},$$

where $\widehat{\omega}$ *has the same meaning as above.*

This estimate was shown to be sharp by Totik and Varga in Proc. London Math. Soc. (2015), under the slightly stronger assumption that the arcs constituting Γ are C^{1+} smooth.

Uniform estimate on Γ

However, as our first result shows, the standard argument has led to a non-optimal exponent of n in the upper bound for $\|p_n\|_{L^\infty(G)}$.

Theorem (St, Contemp. Math., 2015)

Assume that Γ is piecewise analytic without cusps and recall the notation $\widehat{\omega} := \max\{1, \omega_1, \ldots, \omega_N\}$. Then,

$$\|p_n\|_{L^\infty(G)} \leq c(\Gamma) n^{\widehat{\omega}-1/2}.$$

Pointwise estimate on Γ

The next theorem gives a pointwise estimate for $|p_n(z)|$, $z \in \Gamma$.

Theorem (St, Contemp. Math., 2015)

Assume that Γ is piecewise analytic without cusps. Then, for any $z \in \Gamma$ away from corners,

$$|p_n(z)| \leq c(\Gamma, z) n^{1/2}.$$

If z_j is a corner of Γ with exterior angle $\omega_j \pi$, $0 < \omega_j < 2$, then

$$|p_n(z_j)| \leq c(\Gamma, z) n^{\omega_j - 1/2} \sqrt{\log n}.$$

It is interesting to note that the above yields the following limit

$$\lim_{n \to \infty} p_n(z_j) = 0,$$

provided $0 < \omega_j < 1/2$.

On sharpness

The following result settles, in a certain sense, the question of sharpness of the pointwise estimate, and hence that of the uniform estimate.

Theorem (Totik & Varga, Proc. London Math. Soc., 2015)

Assume that Γ has a C^{1+} smooth corner of exterior angle $\omega\pi$, with $1 \leq \omega < 2$ at the point z. Then, for infinitely many n,

$$|p_n(z)| \geq c(\Gamma, z) n^{\omega - 1/2}.$$

An estimate inside Γ

Theorem (Maymeskul, Saff & St, Numer. Math., 2002)

Assume, as above, that Γ *is piecewise analytic without cusps and let* $s := \min_{j=1,\dots,N}\{\omega_j/(2-\omega_j)\}$. *Then, for any* $z \in B$, *where* B *is a compact set in* G, *it holds that*

$$|p_n(z)| \le C(\Gamma, B)\frac{1}{n^s}, \quad n \in \mathbb{N}.$$

Furthermore, if $1/(2-\omega_j) \notin \mathbb{N}$, *then for each* $\varepsilon > 0$, *there exists a subsequence* Λ_ε *such that*

$$|p_n(z)| \ge C(\Gamma, B)\frac{1}{n^{s+\frac{1}{2}+\varepsilon}}, \quad n \in \Lambda_\varepsilon.$$

We note that a well-known result of Fejer implies that all the zeros of $p_n(z)$, $n \in \mathbb{N}$, are contained in the convex hull of G. This result was refined by Saff to the interior of the convex hull.

A result of Lehman

For the statement of a conjecture regarding the behaviour of $p_n(z)$ on Γ, we need a result of Lehman, for the asymptotics of both Φ and Φ'. Assume that $\omega\pi$, $0 < \omega < 2$, is the opening of the exterior angle at a point $z \in \Gamma$. Then, for any ζ near z:

$$\Phi(\zeta) = \Phi(z) + a_1(\zeta - z)^{1/\omega} + o(|\zeta - z|^{1/\omega}),$$

and

$$\Phi'(\zeta) = \frac{1}{\omega} a_1(\zeta - z)^{1/\omega - 1} + o(|\zeta - z|^{1/\omega - 1}),$$

with $a_1 \neq 0$.

A Conjecture for the asymptotics of p_n on Γ

Conjecture

Assume that Γ is piecewise analytic without cusps. Then, at any point z of Γ with exterior angle $\omega\pi$, $0 < \omega < 2$, it holds that

$$p_n(z) = \frac{\omega(n+1)^{\omega-1/2} a_1^{\omega} \Phi^{n+1-\omega}(z)}{\sqrt{\pi}\,\Gamma(\omega+1)} \{1 + \beta_n(z)\},$$

with $\lim_{n\to\infty} \beta_n(z) = 0$.

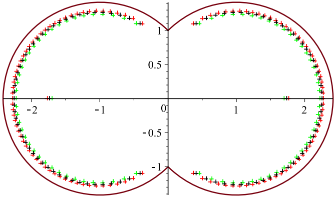
The two intersecting circles

Consider the case where G is defined by the two intersecting circles $|z-1| = \sqrt{2}$ and $|z+1| = \sqrt{2}$. Then,

$$\Phi(z) = \frac{1}{2}\left(z - \frac{1}{z}\right).$$

We test the conjecture numerically for $z = i$ (corner) and $z = 1 + \sqrt{2}$ (non-corner).

The two intersecting circles



Zeros of the Bergman polynomials $p_n(z)$, with $n = 80, 100, 120$.

Theorem (Saff & St, JAT 2015)

Let ν_n denote the normalised counting measure of zeros of p_n. Then

$$\nu_n \xrightarrow{*} \mu_\Gamma, \quad n \to \infty, \quad n \in \mathbb{N},$$

where μ_Γ denotes the equilibrium measure on Γ.

The reluctance of the zeros to approach the points $\pm i$, is due to the fact that $\boxed{d\mu_\Gamma(z) = |\Phi'(z)|ds}$, where s denotes the arclength on Γ.

The two intersecting circles: $z = i$

Then, $\Phi(i) = i$, $\omega = 1/2$, $\Gamma(3/2) = \sqrt{\pi}/2$ and $a_1 = 1/(2i)$. Thus, in this case the conjecture takes the form $\boxed{p_n(i) = (i^n/\sqrt{2}\pi)\{1 + \beta_n\}}$.

n	$\lvert\beta_n\rvert$	n	$\lvert\beta_n\rvert$
100	0.057 121	101	0.037 299
102	0.056 990	103	0.037 428
104	0.056 864	105	0.037 554
106	0.056 741	107	0.037 675
108	0.056 623	109	0.037 793
110	0.056 508	111	0.037 907
112	0.056 396	113	0.038 017
114	0.056 288	115	0.038 125
116	0.056 183	117	0.038 229
118	0.056 081	119	0.038 312
120	0.055 981		

The two intersecting circles: $z = 1 + \sqrt{2}$

Now $\omega = 1$, $\Phi(z) = 1$, $\beta_n \in \mathbb{R}$, and the conjecture takes the form

$$p_n(1+\sqrt{2}) = \sqrt{\frac{n+1}{\pi}\,\frac{2+\sqrt{2}}{(1+\sqrt{2})^2}}\,\{1+\beta_n\}.$$

n	β_n	n	β_n
100	0.000 596	111	0.000 986
101	0.001 095	112	0.000 784
102	0.000 930	113	0.000 557
103	0.001 410	114	0.000 466
104	0.001 163	115	0.000 184
105	0.001 557	116	0.000 261
106	0.001 246	117	0.000 429
107	0.001 525	118	0.000 447
108	0.001 224	119	0.000 822
109	0.001 325	120	0.000 722
110	0.001 054		

Faber polynomials of the second kind

We consider the polynomial part of $\Phi^n(z)\Phi'(z)$ and denote the resulting series by $\{G_n\}_{n=0}^{\infty}$. Thus,

$$\Phi^n(z)\Phi'(z) = G_n(z) - H_n(z), \quad z \in \Omega,$$

with

$$G_n(z) = \gamma^{n+1} z^n + \cdots \quad \text{and} \quad H_n(z) = O\left(1/|z|^2\right), \quad z \to \infty.$$

$G_n(z)$ is the so-called Faber polynomial of the 2nd kind (of degree n). We also consider the auxiliary polynomial

$$q_{n-1}(z) := G_n(z) - \frac{\gamma^{n+1}}{\lambda_n} p_n(z).$$

Observe that $q_{n-1}(z)$ has degree at most $n-1$, but it can be identical to zero, as the special case $G = \{z : |z| < 1\}$ shows.

Related asymptotics

Theorem (Pritsker, JAT 2002)

Assume that Γ is rectifiable and let $z \in \Gamma$ be formed by two analytic arcs meeting with exterior angle $\omega\pi$, $0 < \omega \leq 2$. Then,

$$G_n(z) = \frac{\omega(n+1)^{\omega-1} a_1^{\omega} \Phi^{n+1-\omega}(z)}{\Gamma(\omega+1)} \{1 + o(1)\},$$

For polygonal Γ the above result was established by Szegő in Math. Z., 1925.

Theorem (St, Constr. Approx., 2013)

If Γ is piecewise analytic without cusps, then it holds

$$\frac{\lambda_n}{\gamma^{n+1}} = \sqrt{\frac{n+1}{\pi}} \left\{ 1 + O\left(\frac{1}{n}\right) \right\} \quad \text{and} \quad \|q_{n-1}\|_{L^2(G)} = O\left(\frac{1}{n}\right).$$

Theoretical support of conjecture

The above two theorems suggest the conjecture, because

$$p_n(z) = \frac{\lambda_n}{\gamma^{n+1}}\{G_n(z) - q_{n-1}(z)\}.$$

A useful Lemma

Under the assumption Γ is piecewise analytic without cusps, the following three inequalities hold for any polynomial P_n of degree n:

(i) The uniform estimate

$$\|P_n\|_{L^\infty(G)} \le c(\Gamma) n^{\widehat{\omega}} \|P_n\|_{L^2(G)}.$$

(ii) The pointwise estimate

$$|P_n(z)| \le c(\Gamma, z) n^{\omega} \sqrt{\log n}\, \|P_n\|_{L^2(G)},$$

valid for any $z \in \Gamma$, formed from arcs meeting with exterior angle $\omega\pi$.

(iii) The pointwise estimate

$$|P_n(z)| \le c(\Gamma, z) n \|P_n\|_{L^2(G)},$$

valid for $z \in \Gamma$, not a corner point (hence $\omega = 1$).