# Order and type of indeterminate moment problems and the Valent conjecture

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For an indeterminate moment problem let $p_n$ denote the corresponding orthonormal polynomials. We are interested in determining the growth properties of the function $P(z) = \left(\sum |p_n(z)|^2\right)^{1/2}$ and the summability properties of the sequence $\{p_n(z)\}$. In particular we say that $P(z)$ is of order $\alpha$, $(0 < \alpha \leq 1)$ if $\alpha$ is the infimum of numbers $\alpha'$ for which $P(z) \leq Ce^{K|z|^{\alpha'}}$. We say that $P(z)$ has order $\alpha$ and finite type if $P(z) \leq Ce^{K|z|^{\alpha}}$. The infimum of numbers $K$ for which the latter holds is called the type of function $P(z)$.

In a recent paper (C. Berg, R. Szwarc Adv. Math (2014)) we have related the growth of recurrence coefficients in the recurrence relation satisfied by the polynomials $p_n$ with growth of the function $P(z)$. We were able to study more general types of growth, slower growth in particular. In doing so we have always assumed that the diagonal term in the recurrence relation is "small" with respect to off diagonal coefficients. In a paper by G. Valent (1998) we have encountered a conjecture concerning the order and type of the problem where diagonal term is comparable with off diagonal ones. We were unable to apply our methods directly to that problem. So we had to find new methods.

We have succeded in determining the order and to estimate the type. In the meantime Romanov posted a preprint on Arxiv Math (R. Romanov, Order problem for canonical systems and a conjecture of Valent), where he determined the order in Valent's conjecture by other methods, but he didn't study the type.

Consider a moment problem associated with positive definite sequence $\{s_n\}_{n=0}^{\infty}$. This gives rise to the inner product in the space of all polynomials by the rule

$$\langle x^n, x^m \rangle = s_{n+m}.$$

Denote the corresponding orthonormal polynomials by $p_n(z)$ and those of the second kind by $q_n(z)$. They satisfy the recurrence relation

$$zr_n(z) = b_n r_{n+1}(z) + a_n r_n(z) + b_{n-1} r_{n-1}(z), \quad n \geq 1,$$

with $p_0(z) = 1,\ p_1(z) = (z - a_0)/b_0$, and
$q_0(z) = 0,\ q_1(z) = 1/b_0$. Denote

$$P(z) = \left( \sum |p_n(z)|^2 \right)^{1/2}$$

It is known that the sequence of moments $\{s_n\}_{n=0}^{\infty}$ is indeterminate if and only if $P(z) < \infty$ for any complex (a nonreal) number $z$. Equivalently

$$\sum_{n=0}^{\infty} p_n(0)^2 + q_n(0)^2 < \infty.$$

Moreover the function is of minimal exponential type, i.e. for any $\varepsilon > 0$

$$P(z) \leq C_\varepsilon e^{\varepsilon |z|}$$

We would like to study the order more precisely. For example under what assumptions we may expect

$$P(z) \leq C e^{K|z|^\alpha}$$

with $0 < \alpha < 1$ ?

In C. Berg, R. Szwarc (2014) we have proved the following

**Theorem (1)**

(a) *Assume that* $\sum[p_n^{2\alpha}(0)+q_n^{2\alpha}(0)]<\infty$, *for some number* $0<\alpha<1$. *Then the moment problem is of order at most* $\alpha$, *i.e.*

$$P(z)\le C\exp(K|z|^\alpha).$$

(b) *Let* $\sum \frac{|a_n|}{b_{n-1}}<\infty$ *(for instance* $a_n\equiv 0$*).*
*Assume also that either* $b_{n-1}b_{n+1}\le b_n^2$ *for any* $n$ *large enough or* $b_{n-1}b_{n+1}\ge b_n^2$ *for any such* $n$. *If the moment problem has order* $\alpha$ *with* $0<\alpha<1$ *then*

$$|p_n(z)|^{2\alpha}=O(n^{-1})$$

*for any* $z\in\mathbb{C}$. *In particular for any* $\varepsilon>0$

$$\sum|p_n(z)|^{2\alpha+\varepsilon}<\infty$$

The assumption concerning the coefficients $a_n$ is essential in the proof. In this way we are unable to cover the case of the polynomials corresponding to the Stieltjes moment problems, where one of the orthogonality measures lives on the nonnegative half axis. In that case the coefficients $a_n$ are comparable with $b_n$. The problem considered by Valent, associated with birth and death processes, is of that nature. In order to overcome the obstacle we may use symmetrization.

Let $\mu$ denote a probability measure (with all moments finite) on nonnegative half axis. Denote the corresponding orthonormal polynomials by $P_n$. Consider the symmetric measure $\nu$ defined by

$$d\nu(x) = \frac{1}{2} d\mu(\sqrt{x}),\ x > 0,\ \nu(\{0\}) = \mu(\{0\}).$$

Let $p_n$ denote the orthonormal polynomials associated with $\nu$. Then by an easy exercise one can check that

$$P_n(x^2) = p_{2n}(x).$$

Moreover if $Q_n$ denote the associated polynomials we have

$$xQ_n(x^2) = p_{2n+1}(x).$$

In this way we get

$$P(z^2) = \left( \sum_{n=0}^{\infty} |P_n(z^2)|^2 \right)^{1/2} = \left( \sum_{n=0}^{\infty} |p_{2n}(z)|^2 \right)^{1/2} =: p(z).$$

Therefore the order and the type of the function $p(z)$ is related to the order and type of $P(z)$. Namely the type doesn't change, but the order of $P(z)$ is one half the order of $p(z)$.

Also the recurrence relations are connected by two equations. Namely let

$$xp_n = c_n p_{n+1} + c_{n-1} p_{n-1}.$$

Then

$$x^2 p_{2n} = c_{2n} c_{2n+1} p_{2n+2} + [c_{2n}^2 + c_{2n-1}^2] p_{2n} + c_{2n-2} c_{2n-2} p_{2n-2}.$$

Therefore since $p_{2n}(x) = P_n(x^2)$ these polynomials should satisfy the same recurrence relation, i.e.

$$b_n = c_{2n} c_{2n+1}, \quad a_n = c_{2n}^2 + c_{2n-1}^2.$$

Once $a_n$ and $b_n$ are given we can solve for $c_n$. The problem is if we can determine the growth of the coefficients $c_n$ by the behaviour of the coefficients $a_n$ and $b_n$. After doing so we could hopefully apply the results of our paper of 2014.

G. Valent studied the polynomials satisfying the recurrence relation

$$xF_n = \mu_{n+1}F_{n+1} + (\mu_n + \lambda_n)F_n + \lambda_{n-1}F_{n-1},$$

where

$$\lambda_n = (pn + e_1)\dots(pn + e_p) \approx p^p n^p$$
$$\mu_n = (pn + d_1)\dots(pn + d_p) \approx p^p n^p.$$

The polynomials are not orthonormal. The orthonormal versions satisfy

$$xP_n = \sqrt{\lambda_n\mu_{n+1}}P_{n+1} + (\mu_n + \lambda_n)P_n + \sqrt{\lambda_{n-1}\mu_n}P_{n-1}.$$

These polynomials correspond to the Stieltjes moment problem. The diagonal terms are comparable with off diagonal ones.

It can be verified that the problem is indeterminate if and only if

$$p < \sum_{i=1}^{p} (e_i - d_i) < p(p-1).$$

When we perform symmetrization we can calculate explicitly the recurrence coefficients $c_n$ corresponding to the symmetric problem. Namely we have

$$c_{2n}^2 = (pn + e_1)\ldots(pn + e_p)$$

$$c_{2n-1}^2 = (pn + d_1)\ldots(pn + d_p).$$

Unfortunately we are unable to apply directly the results of our paper of 2014 because the sequence $c_n$ does not behave regularly enough to meet our assumptions. First of all this sequence is not eventually logarithmically convex or concave. Moreover the summability exponents of the sequences $p_n(0)$ and $q_n(0)$ are different.

It is easy to compute from the recurrence relation

$$xp_n = c_n p_{n+1} + c_{n-1} p_{n-1},$$

that

$$\begin{aligned} p_{2n}(0) &= (-1)^n \frac{c_0 c_2 \dots c_{2n-2}}{c_1 c_3 \dots c_{2n-1}}, \\ q_{2n+1}(0) &= (-1)^n \frac{c_1 c_3 \dots c_{2n-1}}{c_0 c_2 \dots c_{2n}}. \end{aligned}$$

We can easily deduce that

$$p_{2n}^2(0) \sim n^{-[p-(E-D)/p]}, \qquad q_{2n-1}^2(0) \sim n^{-(E-D)/p}.$$

Let $v_n = p_{2n}^2(0)$ and $u_n = q_{2n+1}^2(0)$. Denote

$$\alpha_0 = \frac{p}{E-D}, \qquad \beta_0 = \frac{1}{p-(E-D)/p}.$$

Then

$$u_n^{\alpha_0} \sim v_n^{\beta_0} \sim \frac{1}{n}.$$

# Theorem (1)

(a) *Assume that* $\sum[p_n^{2\alpha}(0) + q_n^{2\alpha}(0)] < \infty$, *for some number* $0 < \alpha < 1$. *Then the moment problem is of order at most* $\alpha$, *i.e.*

$$P(z) \leq C\exp(K|z|^{\alpha}).$$

(b) *Let* $\sum \frac{|a_n|}{b_{n-1}} < \infty$ *(for instance* $a_n \equiv 0$*).*
*Assume also that either* $b_{n-1}b_{n+1} \leq b_n^2$ *for any* $n$ *large enough or* $b_{n-1}b_{n+1} \geq b_n^2$ *for any such* $n$. *If the moment problem has order* $\alpha$ *with* $0 < \alpha < 1$ *then*

$$|p_n(z)|^{2\alpha} = O(n^{-1})$$

*for any* $z \in \mathbb{C}$. *In particular for any* $\varepsilon > 0$

$$\sum |p_n(z)|^{2\alpha+\varepsilon} < \infty$$

Let

$$\begin{aligned} A_n(z) &= \phantom{-1 +{}} z \sum_{i=0}^{n-1} q_i(0) q_i(z), \\ B_n(z) &= -1 + z \sum_{i=0}^{n-1} q_i(0) p_i(z), \\ C_n(z) &= \phantom{-}1 + z \sum_{i=0}^{n-1} p_i(0) q_i(z), \\ D_n(z) &= \phantom{-1 +{}} z \sum_{i=0}^{n-1} p_i(0) p_i(z), \end{aligned}$$

denote partial sums of Nevanlinna functions $A(z)$, $B(z)$, $C(z)$ and $D(z)$.

By [B. Simon, Adv. Math. 1998] we have

$$\begin{pmatrix} A_{n+1}(z) & B_{n+1}(z) \\ C_{n+1}(z) & D_{n+1}(z) \end{pmatrix} = \Bigg[ I + z \\ \times \begin{pmatrix} -p_n(0)q_n(0) & q_n^2(0) \\ -p_n^2(0) & p_n(0)q_n(0) \end{pmatrix} \Bigg] \begin{pmatrix} A_n(z) & B_n(z) \\ C_n(z) & D_n(z) \end{pmatrix}.$$

This follows from recurrence relation through Christoffel-Darboux formula. Therefore

$$\begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} = \prod \Bigg[ I + z \begin{pmatrix} -p_n(0)q_n(0) & q_n^2(0) \\ -p_n^2(0) & p_n(0)q_n(0) \end{pmatrix} \Bigg]$$

The product is not commutative. The multiplication continues leftwards. We have $p_n(0)q_n(0) = 0$ and $p_{2n+1}(0) = q_{2n}(0) = 0$. Hence the following Proposition will be useful.

## Proposition

*Assume $u_n \in \ell^\alpha$ and $v_n \in \ell^\beta$ for $0 < \alpha, \beta < 1$. Let*

$$M(z) = \prod_{n=1}^{\infty} \left[ I + z \begin{pmatrix} 0 & u_n \\ 0 & 0 \end{pmatrix} \right] \left[ I - z \begin{pmatrix} 0 & 0 \\ v_n & 0 \end{pmatrix} \right],$$

*where the product continues leftwards. Assume additionally that $u_n \leq C v_n^{\beta/\alpha}$ and that the sequence $v_n$ is decreasing. Then*

$$\|M(z)\| \leq C \exp(K|z|^\gamma), \qquad \gamma = \frac{2}{\alpha^{-1} + \beta^{-1}} = \frac{2\alpha\beta}{\alpha + \beta}.$$

Let

$$U_n = \begin{pmatrix} 0 & u_n \\ 0 & 0 \end{pmatrix}, \quad V_n = \begin{pmatrix} 0 & 0 \\ v_n & 0 \end{pmatrix}$$

Then $U_m U_n = V_m V_n = 0$ and

$$U_n V_m = \begin{pmatrix} u_n v_m & 0 \\ 0 & 0 \end{pmatrix}, \quad V_n U_m = \begin{pmatrix} 0 & 0 \\ 0 & u_n v_m \end{pmatrix}.$$

We want to analyze the infinite product

$$M(z) = \prod_{n=1}^{\infty} (I + zU_n)(I - zV_n).$$

Let's expand the product to get

$$M(z) = 1 + \sum_{n=1}^{\infty} M_n z^n.$$

We will analyze the coefficient $M_{2n}$.

It must be of the form $(-1)^n$ times

$$\sum_{k_1 \leq k_2 < \ldots < k_{2n-1} \leq k_{2n}} U_{k_{2n}} V_{k_{2n-1}} \ldots U_{k_2} V_{k_1} \\ + \sum_{k_1 < k_2 \leq \ldots \leq k_{2n-1} < k_{2n}} V_{k_{2n}} U_{k_{2n-1}} \ldots V_{k_2} U_{k_1}$$

Let's focus on the upper left corner of $M_{2n}$. This matrix entry is equal to $(-1)^n$ times

$$a_{2n} := \sum_{k_1 \leq k_2 < \ldots < k_{2n-1} \leq k_{2n}} v_{k_1} u_{k_2} \ldots v_{k_{2n-1}} u_{k_{2n}}. \tag{1}$$

Consider the case $\alpha < \beta$. By assumptions we have $u_n \leq C v_n^{\beta/\alpha}$ and the sequence $v_n$ is decreasing. Then for $i \leq j$ we get

$$v_i u_j \leq C v_i v_j^{\beta/\alpha} \leq C v_i^{\beta/\gamma} v_j^{\beta/\gamma},$$

where $\gamma$ is the harmonic mean of $\alpha$ and $\beta$.
Applying this to (1) we can estimate the coefficient by

$$\widetilde{a}_{2n} := C^n \sum_{k_1 \leq k_2 < \ldots < k_{2n-1} \leq k_{2n}} v_{k_1}^{\beta/\gamma} v_{k_2}^{\beta/\gamma} \ldots v_{k_{2n-1}}^{\beta/\gamma} v_{k_{2n}}^{\beta/\gamma}. \qquad (2)$$

The number $\widetilde{a}_{2n}/C^n$ is not greater than the McLaurin coefficient of order $2n$ of the product

$$F(z) = \prod_{n=1}^{\infty} \left(1 + v_n^{\beta/\gamma} z\right)^2 . \qquad (3)$$

Since $v_n^{\beta/\gamma} \in \ell^\gamma$ the order of this product is less than or equal to $\gamma$. Indeed, assume $t_n \in \ell^\gamma$ for $0 < \gamma < 1$. Then

$$1 + t_n|z| \leq (1 + t_n^\gamma |z|^\gamma)^{1/\gamma} \leq e^{\frac{1}{\gamma} t_n^\gamma |z|^\gamma}.$$

Hence

$$\prod_{n=1}^{\infty} (1 + t_n|z|) \leq e^{\frac{1}{\gamma} \sum t_n^\gamma \, |z|^\gamma}.$$

The upper left corner of $M(z)$ involves only even powers of $z$. Thus we proved that the upper left corner has order less than or equal to $\gamma$.

**Remark.** It is sufficient to assume $v_j \leq D v_i$ for $i < j$ and a constant $D$.

In a similar way we can deal with the case $\alpha > \beta$. To this end we rewrite (1) as

$$a_{2n} = \sum_{k_1 \le k_2 < \ldots < k_{2n-1} \le k_{2n}} v_{k_1} [u_{k_2} v_{k_3}] \ldots [u_{k_{2n-2}} v_{k_{2n-1}}] u_{k_{2n}}$$
$$\le D_1 D_2 \sum_{k_2 < k_3 \le \ldots \le k_{2n-2} < k_{2n-1}} u_{k_2} v_{k_3} \ldots u_{k_{2n-2}} v_{k_{2n-1}}$$

where $D_1 = \sum u_k$, $D_2 = \sum v_k$, and apply the estimate $u_n \le C v_n^{\beta/\alpha}$ to get

$$u_i v_j \le C v_i^{\beta/\alpha} v_j \le C v_i^{\beta/\gamma} v_j^{\beta/\gamma}.$$

Thus $a_{2n}$ can be estimated by

$$\widehat{a}_{2n} = D_1 D_2 C^{n-1} \sum_{k_2 < k_3 \le \ldots \le k_{2n-2} < k_{2n-1}} v_{k_2}^{\beta/\gamma} v_{k_3}^{\beta/\gamma} \ldots v_{k_{2n-2}}^{\beta/\gamma} v_{k_{2n-1}}^{\beta/\gamma}.$$

Then number $\widehat{a}_{2n}/(D_1 D_2 C^{n-1})$ is not greater than the McLaurin coefficient of order $2n-2$ of the function $F(z)$ given by (3).

In the same way we can deal with the lower right entry of $M_{2n}$. This quantity is equal to $(-1)^n$ times

$$\sum_{k_1 \leq k_2 < \ldots < k_{2n-1} \leq k_{2n}} u_{k_1} v_{k_2} \ldots u_{k_{2n-1}} v_{k_{2n}}. \tag{4}$$

We can follow the same lines in order to conclude that the the lower right entry of $M(z)$ is of order less than or equal to $\gamma$. Much in the same way we can treat other two entries of $M(z)$, which involve only odd power of $z$.

The lower left entry of $M_{2n+1}$ is equal to $(-1)^{n+1}$ times

$$\sum_{k_1\le k_2<\ldots<k_{2n-1}\le k_{2n}<k_{2n+1}} v_{k_1}u_{k_2}\ldots v_{k_{2n-1}}u_{k_{2n}}v_{k_{2n+1}},$$

while the upper right entry is equal to $(-1)^n$ times

$$\sum_{k_1<k_2\le\ldots\le k_{2n-1}<k_{2n}\le k_{2n+1}} u_{k_1}v_{k_2}\ldots u_{k_{2n-1}}v_{k_{2n}}u_{k_{2n+1}}.$$

We can obtain estimates for these entries relying on the estimates obtained for diagonal entries. For example

$$\begin{aligned}&\sum_{k_1\le k_2<\ldots<k_{2n-1}\le k_{2n}<k_{2n+1}} v_{k_1}u_{k_2}\ldots v_{k_{2n-1}}u_{k_{2n}}v_{k_{2n+1}}\\ &\qquad\le\ D_2\sum_{k_1\le k_2<\ldots<k_{2n-1}\le k_{2n}} v_{k_1}u_{k_2}\ldots v_{k_{2n-1}}u_{k_{2n}}=D_2a_{2n},\end{aligned}$$

where $D_2=\sum v_k$. But instead of doing this we can as well rely on Berg and Pedersen result stating that all four functions in Nevanlinna matrix have the same growth and type.

The estimate $\gamma$ is optimal in the Valent case. Indeed, assume that $\alpha$ is the order for the function $\prod(1+zu_n)$ and the sequence $u_n$ is decreasing. Let's analyze the upper left entry given by (1), i.e.

$$a_{2n} := \sum_{k_1 \le k_2 < \ldots < k_{2n-1} \le k_{2n}} v_{k_1} u_{k_2} \ldots v_{k_{2n-1}} u_{k_{2n}}.$$

. Assume $\alpha < \beta$. For $i < j$ we have

$$v_i u_j \ge \widetilde{C} u_i^{\alpha/\beta} u_j \ge \widetilde{C} u_i^{\alpha/\gamma} u_j^{\alpha/\gamma},$$

where $\widetilde{C} = C^{-\alpha/\beta}$. Thus

$$a_{2n} \ge \widetilde{C}^n \sum_{k_1 \le k_2 < \ldots < k_{2n-1} \le k_{2n}} u_{k_1}^{\alpha/\gamma} u_{k_2}^{\alpha/\gamma} \ldots u_{k_{2n-1}}^{\alpha/\gamma} u_{k_{2n}}^{\alpha/\gamma}.$$

The multisum represents the MacLaurin coefficient of order $2n$ of the function

$$\prod_{k=1}^{\infty}\left(1+ru_k^{\alpha/\gamma}\right)^2, \quad r>0.$$

But since $\alpha<\gamma<\beta$ we have

$$1+ru_k^{\alpha/\gamma}\geq(1+r^{\gamma/\alpha}u_k)^{\alpha/\gamma}.$$

Therefore the order of the upper left entry is greater than or equal to the order of the function

$$\prod_{k=1}^{\infty}(1+r^{\gamma/\alpha}u_k)^2.$$

By our assumption made at the begining of this part the order of the latter is equal $\gamma$.

These computations are sufficient to determine the order of the problem of Valent. Indeed, for

$$\alpha > \alpha_0 = \frac{p}{E-D}, \quad \beta > \beta_0 = \frac{1}{p - \frac{E-D}{p}}$$

we have

$$\gamma = \frac{2}{\alpha^{-1} + \beta^{-1}} > \frac{2}{\alpha_0^{-1} + \beta_0^{-1}} = \frac{2}{p} =: \gamma_0.$$

The optimality of $2/p$ has been explained on the previous slide.

The computations are not precise enough to determine the type. To this end we will have to rely on straightforward evaluations of the McLaurin coefficients.
It can be shown that

$$\begin{aligned} u_n = p_{2n}(0)^2 &\approx c_1 n^{-[p-(E-D)/p]}, \\ v_n = q_{2n+1}(0)^2 &\approx c_2 n^{-(E-D)/p}, \end{aligned}$$

where $c_1 c_2 = p^{-p}$. The approximation is precise enough to preserve the order and type of the studied functions. Therefore in our computations we will use the values on the right hand side.

We were after the determining the behaviour of the McLaurin coefficient

$$a_{2n} = \sum_{k_1 \le k_2 < \ldots < k_{2n-1} \le k_{2n}} v_{k_1} u_{k_2} \ldots v_{k_{2n-1}} u_{k_{2n}}.$$

Instead we will study the coefficient

$$p^{-pn} \sum_{k_1 \le k_2 < \ldots < k_{2n-1} \le k_{2n}} {k_1}^{-1/\beta_0} {k_2}^{-1/\alpha_0} \ldots k_{2n-1}^{-1/\beta_0} k_{2n}^{-1/\alpha_0}.$$

We know already that this quantity behaves like

$$\widetilde{a}_{2n} = p^{-pn} \sum_{k_1 \le k_2 < \ldots < k_{2n-1} \le k_{2n}} {k_1}^{-1/\gamma_0} {k_2}^{-1/\gamma_0} \ldots k_{2n-1}^{-1/\gamma_0} k_{2n}^{-1/\gamma_0},$$

where

$$\gamma_0 = \frac{2}{\alpha_0^{-1} + \beta_0^{-1}} = \frac{2}{p}.$$

Just to recall (for those who didn't pay attention) it suffices to use the inequality

$$i^{-1/\beta} j^{-1/\alpha_0} \leq i^{-1/\gamma_0} j^{-1/\gamma_0}, \quad i < j,$$

in case $\alpha_0 \leq \beta_0$, or

$$i^{-1/\alpha_0} j^{-1/\beta_0} \leq i^{-1/\gamma_0} j^{-1/\gamma_0}, \quad i < j,$$

in case $\alpha_0 \geq \beta_0$,

Taking $k_{2i-1} = k_{2i}$ gives

$$\begin{aligned}\sum_{k_1<k_2<\ldots<k_n} & {k_1}^{-2/\gamma_0} {k_2}^{-2/\gamma_0} \ldots k_n^{-2/\gamma_0} \\ & \leq \sum_{k_1\leq k_2<\ldots<k_{2n-1}\leq k_{2n}} {k_1}^{-1/\gamma_0} {k_2}^{-1/\gamma_0} \ldots k_{2n-1}^{-1/\gamma_0} k_{2n}^{-1/\gamma_0}.\end{aligned}$$

On the other hand

$$\begin{aligned}\sum_{k_1\leq k_2<\ldots<k_{2n-1}\leq k_{2n}} & {k_1}^{-1/\gamma_0} {k_2}^{-1/\gamma_0} \ldots k_{2n-1}^{-1/\gamma} k_{2n}^{-1/\gamma_0} \\ & \leq \left[\sum_{k_1<k_2<\ldots<k_n} {k_1}^{-1/\gamma_0} {k_2}^{-1/\gamma_0} \ldots k_n^{-1/\gamma_0}\right]^2.\end{aligned}$$

In this way we can estimate the function $F(r) := \sum \widetilde{a}_{2n} r^{2n}$ by

$$\prod_{n=1}^{\infty}\left(1+n^{-2/\gamma_0}p^{-p}r^2\right) \le F(r) \le \left[\prod_{n=1}^{\infty}\left(1+n^{-1/\gamma_0}p^{-p/2}r\right)\right]^2.$$

It is known (see R. P. Boas, Entire functions) that the product

$$\prod_{n=1}^{\infty}\left(1+n^{-1/\varrho}r\right)$$

has order $\varrho$ and type $\pi/\sin(\pi\varrho)$. Hence the order of $F(r)$ is equal $\gamma_0 = \frac{2}{p}$ and the type $T$ satisfies

$$p^{-p\gamma_0/2}\frac{\pi}{\sin(\pi\gamma_0/2)} \le T \le p^{-p\gamma_0/2}\frac{2\pi}{p\sin\pi\gamma_0},$$

i.e.

$$\frac{\pi}{p\sin(\pi/p)} \le T \le \frac{2\pi}{p\sin(2\pi/p)},$$

The order and the type do not depend on $E$ and $D$ because the coefficients $\tilde{a}_{2n}$ depend only on $p$.
According to Valent's conjecture we should have

$$T = \int_0^1 \frac{du}{(1-u^p)^{2/p}}.$$

This number satisfies the inequality on previous slide.
Moreover Valent proved validity of his conjecture for $p = 3$ and $p = 4$.

As we see in order to determine the type precisely we have to study the growth of the quantity

$$s_n := \sum_{k_1 \le k_2 < \ldots < k_{2n-1} \le k_{2n}} {k_1}^{-1/\gamma} {k_2}^{-1/\gamma} \ldots k_{2n-1}^{-1/\gamma} k_{2n}^{-1/\gamma}.$$

It can be shown that this quantity behaves similarly to

$$\widetilde{s}_n := \sum_{k_1 < k_2 < \ldots < k_n} {k_1}^{-2/\gamma} {k_2}^{-2/\gamma} \ldots k_n^{-2/\gamma} (k_2 - k_1) \ldots (k_n - k_{n-1}).$$

Indeed, observe that

$$s_n \le \sum_{k_1 < k_3 < \ldots < k_{2n-3}} {k_1}^{-2/\gamma} {k_3}^{-2/\gamma} \ldots k_{2n-3}^{-2/\gamma} \\ (k_3 - k_1) \ldots (k_{2n-3} - k_{2n-5}) \sum_{k_{2n-3} < k_{2n-1} \le k_{2n}} k_{2n-1}^{-1/\gamma} k_{2n}^{-1/\gamma}$$

But

$$\sum_{k_{2n-3}<k_{2n-1}\leq k_{2n}} k_{2n-1}^{-1/\gamma} k_{2n}^{-1/\gamma} \quad \leq \quad c_\gamma \sum_{k_{2n-3}<k_{2n-1}} k_{2n-1}^{-1/\gamma} k_{2n-1}^{1-1/\gamma}$$

After changing symbols $k_i := k_{2i-1}$ we obtain

$$s_n \leq c_\gamma \sum_{k_1<k_2<\ldots<k_n} {k_1}^{-2/\gamma} {k_2}^{-2/\gamma} \ldots k_n^{-2/\gamma} (k_2-k_1) \ldots (k_{n-1}-k_{n-2}) k_n .$$

On the other hand we have

$$\begin{aligned} s_n \geq \sum_{k_2<k_4<\ldots<k_{2n}} & {k_2}^{-2/\gamma} {k_4}^{-2/\gamma} \ldots k_{2n}^{-2/\gamma} \\ & \qquad\qquad k_2 (k_4 - k_2) \ldots (k_{2n} - k_{2n-2}) \end{aligned}$$

After changing symbols $k_i := k_{2i}$ we obtain

$$s_n \geq \sum_{k_1<k_2<\ldots<k_n} k_1{}^{-2/\gamma} k_2{}^{-2/\gamma} \ldots k_n^{-2/\gamma} k_1(k_2-k_1)\ldots(k_n-k_{n-1}).$$

The upper estimate is slightly different:

$$s_n \leq c_\gamma \sum_{k_1<k_2<\ldots<k_n} k_1{}^{-2/\gamma} k_2{}^{-2/\gamma} \ldots k_n^{-2/\gamma} (k_2-k_1)\ldots(k_{n-1}-k_{n-2})k_n.$$

By $\frac{2}{\gamma} = p > 2$ we obtain

$$\sum_{k_n>k_{n-1}} k_n^{-2/\gamma} k_n \leq e_\gamma k_{n-1}^{2-2/\gamma} \leq e_\gamma (n-1)^{2-2/\gamma} \ \leq d_\gamma n^{2-2/\gamma}.$$

Summarizing, we get

$$s_n \leq c_\gamma d_\gamma n^{2-2/\gamma} \sum_{k_1<\ldots<k_{n-1}} k_1^{-2/\gamma} \ldots k_{n-1}^{-2/\gamma}(k_2-k_1)\ldots(k_{n-1}-k_{n-2}).$$

On the other hand we have

$$s_n \geq \sum_{k_1<k_2<\ldots<k_n} {k_1}^{-2/\gamma} {k_2}^{-2/\gamma} \ldots k_n^{-2/\gamma}(k_2-k_1)\ldots(k_n-k_{n-1}).$$

$$\widetilde{s}_n \leq s_n \leq c_\gamma d_\gamma n^{2-2/\gamma} \widetilde{s}_{n-1}.$$

The estimate is sharp enough to determine the type, as the functions

$$g(r) = \sum_{n=0} \widetilde{s}_n r^n, \qquad h(r) = \sum_{n=1} \widetilde{s}_{n-1} r^n$$

have the same behaviour since

$$rg(r) = h(r).$$

In the Valent setting (after symmetrization) we have

$$\begin{aligned} c_{2n}^2 &= (pn+e_1)(pn+e_2)\dots(pn+e_p), \\ c_{2n-1}^2 &= (pn+d_1)(pn+d_2)\dots(pn+d_p). \end{aligned}$$

Denote

$$E = e_1 + \ldots + e_p, \quad D = d_1 + \ldots + d_p.$$

We know that the problem is indeterminate only if

$$1 < \frac{E-D}{p} < p-1.$$

We already know that the order and type do not depend on the parameters $e_k$ and $d_k$. Set

$$d_1 = \ldots = d_p = 0, \quad e_1 = \ldots = e_p = \frac{p}{2}.$$

Then

$$\frac{E - D}{p} = \frac{p}{2}.$$

In that case

$$p_{2n}^2(0) \approx n^{-p/2}, \quad q_{2n+1}^2(0) \approx n^{-p/2}.$$

Also

$$c_n = \left(\frac{p}{2}\right)^{p/2} (n+1)^{p/2}.$$

Therefore it suffices to determine the type of the moment problem associated with the recurrence relation

$$x r_n = (n+1)^{p/2} r_{n+1} + n^{p/2} r_{n-1} \ n \geq 1.$$