Polynomial inequalities on general sets

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Bernstein (1912) had $2n$, the sharp n is due to M. Riesz (1914)

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What if we have two or more intervals?

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For any θ lying in $Int(E)$ there are $T_n \not\equiv 0$ such that

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Jordan arc: homeomorphic image of $[0, 1]$

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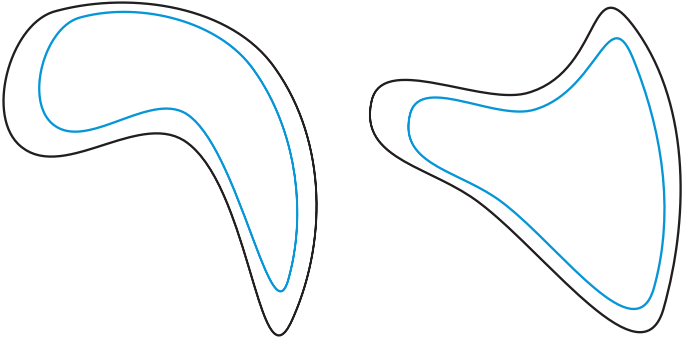

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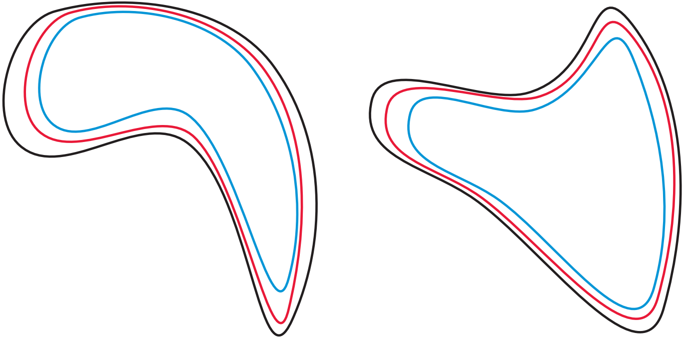

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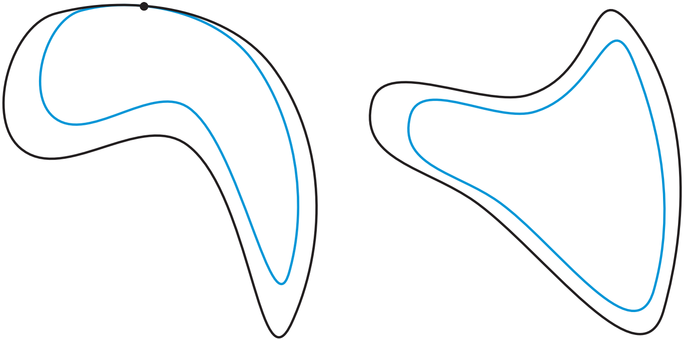

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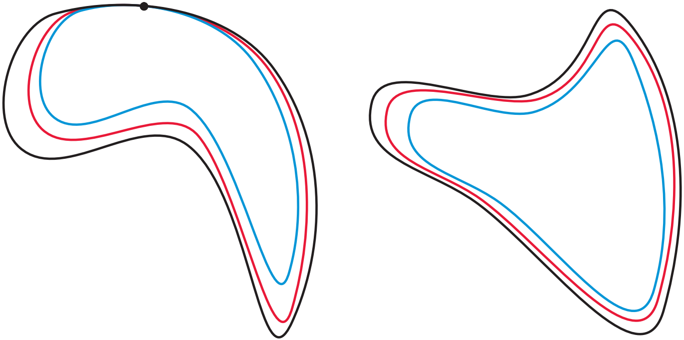

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Similar result holds for lemniscates enclosing E

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- get rid of the special properties of P_n
- apply the sharp form of Hilbert's lemniscate theorem to approximate E by such a σ

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What if E is more complicated than the union of two intervals of equal length?

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$$M_k = 2\pi^2 \left(\lim_{x \to a_k,\ x \in E} \sqrt{|x - a_k|}\,\omega_E(x) \right)^2$$

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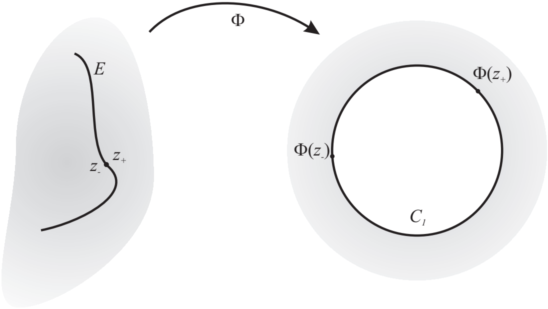

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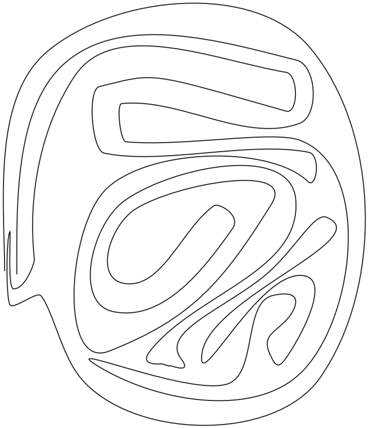

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First proved for analytic arcs by Nagy and Kalmykov

Sharp: if z not an endpoint of E, then for some polynomials

$$|P_n'(z)| > (1 - o(1))n \max\big(|\Phi'(z_+)|, |\Phi'(z_+)|\big) \|P_n\|_E$$

Jordan arcs, Markov

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Theorem

$$\|P_n'\|_{\tilde{E}} \le (1+o(1))n^2 2\Omega_w^2 \|P_n\|_E$$

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Sharp

Higher derivatives:

$$\|P_n^{(k)}\|_{\tilde{E}} \le (1+o(1))n^{2k} \frac{(2\Omega_w^2)^k}{(2k-1)!!} \|P_n\|_E$$