# The Real World is Complex

---

## An International Symposium in honor of Christian Berg

August 26–28, 2015

Department of Mathematical Sciences
@ University of Copenhagen

# Welcome

We are 33 registered participants from 14 different countries.

# WOP 2009

Group photo WOP, Copenhagen, 2009
Top row:
Van Assche, Rosengren, Kajihara, Sanchez-Moreno, Arvesu, Ranga, Berg, Lazarow
Bottom row:
Szafraniec, Medem, Deano, Christiansen, Koelink, Lopez Lagomasino, Szwarc, Obermaier, Marcellan, Remling
Levin, Yuditskii, Vignat, Ismail, Tookos, Zeng, Anshelevich

# WITPA 2010


WITPA 2010
Standing: Qi, Sasvari, Szafraniec, Christiansen, Rebocho,
zu Castell, Buescu, Koumandos, Perez Sinusia, Pedersen,
Ressel
Sitting: Bondesson, Breuer, Berg, Møller

# OSCA 2012

# WoSFA 2013

SPONSORS

# Special thanks to...

## The Carlsberg Foundation

Their generous support to art and science in Denmark since 1876 is outstanding and unique!


CARLSBERG FOUNDATION

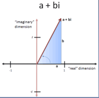
## & The dept of Mathematical Sciences

# Special thanks to...

## The Carlsberg Foundation

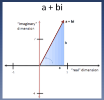
Their generous support to art and science in Denmark since 1876 is outstanding and unique!

## & The dept of Mathematical Sciences




# The Real World is...

$\mathbb{C}$omplex !




# The Real World is...

## $\mathbb{C}$omplex !

This title illustrates an underlying idea in many of Christian's papers.

The application of complex ideas and techniques is one of the central themes throughout his work — also when the problem itself is $\mathbb{R}$eal.

We believe that examples will show up during this conference.

# Practical information

- All lectures take place in Auditorium 10 *

*except perhaps Thursday afternoon

# Practical information

- All lectures take place in Auditorium 10 *
- Lunches (= buffet + drink) are in the Biocenter. Show your badge when passing through the counter.

*except perhaps Thursday afternoon

# Practical information

- All lectures take place in Auditorium 10 *
- Lunches (= buffet + drink) are in the Biocenter. Show your badge when passing through the counter.
- Coffee breaks are just outside this room, at around 10am and 3pm.

*except perhaps Thursday afternoon

# Practical information

- All lectures take place in Auditorium 10 *
- Lunches (= buffet + drink) are in the Biocenter. Show your badge when passing through the counter.
- Coffee breaks are just outside this room, at around 10am and 3pm.
- A group photo will be taken today at noon †

---

*except perhaps Thursday afternoon
†depending upon the weather

# Practical information

- All lectures take place in Auditorium 10 *
- Lunches (= buffet + drink) are in the Biocenter. Show your badge when passing through the counter.
- Coffee breaks are just outside this room, at around 10am and 3pm.
- A group photo will be taken today at noon †
- The conference dinner will take place Thursday evening, starting at 7pm.

* except perhaps Thursday afternoon
† depending upon the weather

# Practical information

- All lectures take place in Auditorium 10 [*]
- Lunches (= buffet + drink) are in the Biocenter. Show your badge when passing through the counter.
- Coffee breaks are just outside this room, at around 10am and 3pm.
- A group photo will be taken today at noon [†]
- The conference dinner will take place Thursday evening, starting at 7pm.
- A certificate of participation is available upon request.

[*] except perhaps Thursday afternoon

[†] depending upon the weather

# Christian Berg

- 1944: Born in Haarslev
- 1963: Graduated high school (Næstved)
- 1968: Cand. Scient (Univ. of Copenhagen)
- 1971: Lic. Scient (Univ. of Copenhagen)
- 1972: Associate Professor (Univ. of Copenhagen)
- 1978: Professor (Univ. of Copenhagen)
- 2014: Professor Emeritus (Univ. of Copenhagen)

# Selected visits

- 1966–67: Nancy

# Selected visits

- 1966–67: Nancy
- 1969–70: Paris

# Selected visits

- 1966–67: Nancy
- 1969–70: Paris
- 1974–75: UCLA

# Selected visits

- 1966–67: Nancy
- 1969–70: Paris
- 1974–75: UCLA
- 1989: Nancy

# Selected visits

- 1966–67: Nancy
- 1969–70: Paris
- 1974–75: UCLA
- 1989: Nancy
- 1998: Paris

# Selected visits

- 1966–67: Nancy
- 1969–70: Paris
- 1974–75: UCLA
- 1989: Nancy
- 1998: Paris
- 2002: Sevilla

# Selected visits

- 1966–67: Nancy
- 1969–70: Paris
- 1974–75: UCLA
- 1989: Nancy
- 1998: Paris
- 2002: Sevilla
- 2007–08: Wrocław

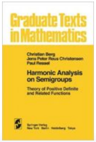
# Research

Christian has been active in research for almost 50 years.

# Research

Christian has been active in research for almost 50 years.

- 115 research papers (with 30 co-authors)
- 2 monographs (with Forst and Christensen–Ressel)

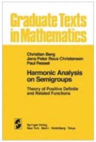




- numerous conferences, workshops, etc.

# Research

Christian has been active in research for almost 50 years.

- 115 research papers (with 30 co-authors)
- 2 monographs (with Forst and Christensen–Ressel)


- numerous conferences, workshops, etc.

He has supervised 24 master's students and 3 PhD students.

# Early publications

2: Shephards approximation theorem for convex bodies and the Milman theorem. *Math. Scand.* **25** (1969), 19–24.

5: Quelques propriétés de la topologie fine dans la théorie du potentiel et des processus standard. *Bull. Sci. Math.* **95** (1971), 27–31.

16: On the support of the measures in a symmetric convolution semigroup. *Math. Zeitschrift* **148** (1976), 141–146.

18: Potential theory on the infinite dimensional torus. *Inventiones Math.* **32** (1976), 49–100.

# Selected publications

44:

*The Annals of Probability*
1988, Vol. 16, No. 2, 910–913

## THE CUBE OF A NORMAL DISTRIBUTION IS INDETERMINATE

By Christian Berg

*University of Copenhagen*

It is established that if $X$ is a stochastic variable with a normal distribution, then $X^{2n+1}$ has an indeterminate distribution for $n \geq 1$. Furthermore, the distribution of $|X|^{\alpha}$ is determinate for $0 < \alpha \leq 4$ while indeterminate for $\alpha > 4$.

**1. Introduction.** Let $\mathscr{M}^*$ denote the set of probability measures on the real axis having moments of all orders. The $k$th moment of $\mu \in \mathscr{M}^*$ is the number

$$s_k(\mu) = \int x^k \, d\mu(x), \qquad k = 0, 1, \dots .$$

Two distributions $\mu, \nu \in \mathscr{M}^*$ are called equivalent if $s_k(\mu) = s_k(\nu)$ for $k = 0, 1, 2, \dots$, and $\mu$ is called *determinate* (in the Hamburger sense), if the equivalence class $[\mu]$ containing $\mu$ is equal to $\{\mu\}$, and *indeterminate* otherwise.

It is well known that there exist indeterminate distributions. This observation goes back to Stieltjes (1894/1895), who proved that the distributions on $(0, \infty)$

# Selected publications

52:

Acta Math., 167 (1991), 207–227

## Rotation invariant moment problems

by

CHRISTIAN BERG and MARCO THILL

*University of Copenhagen*
*Copenhagen, Denmark*

*University of Copenhagen*
*Copenhagen, Denmark*

### 0. Introduction

An important theorem of Marcel Riesz, cf. [14], states that the polynomials are dense in $L^2(\mu)$, when $\mu$ is a determinate measure on the real line. In the indeterminate case Riesz also characterized the measures $\mu$ for which the polynomials are dense in $L^2(\mu)$. They are the so-called Nevanlinna extremal measures, introduced in Nevanlinna [11].

It does not seem to be known whether the polynomials are dense in $L^2(\mu)$, when $\mu$ is a determinate measure on $\mathbf{R}^d$, $d>1$, cf. the expository paper by Fuglede [7], as well as the research problems book [8, p. 529], where Devinatz poses the problem as question 1 and ascribes it to the physicist John Challifour (1978).

In this paper we shall settle the question in the negative. There exist rotation

# Selected publications

73:

Arab J. Math. Sc.
Volume 4, Number 2, December 1998
pp. 1-18

**FROM DISCRETE TO ABSOLUTELY CONTINUOUS SOLUTIONS OF INDETERMINATE MOMENT PROBLEMS**

CHRISTIAN BERG

ABSTRACT. We consider well-known families of discrete solutions to indeterminate moment problems and show how they can be used in a simple way to generate absolutely continuous solutions to the same moment problems. The following cases are considered: log-normal, generalized Stieltjes -Wigert, q-Laguerre and discrete q-Hermite II.

95:

Arab Journal of Mathematical Sciences (2011) **17**, 75–88

King Saud University
**Arab Journal of Mathematical Sciences**
www.ksu.edu.sa
www.sciencedirect.com

ORIGINAL ARTICLE

**Fibonacci numbers and orthogonal polynomials**

**Christian Berg**

*Department of Mathematics, University of Copenhagen, Universitetsparken 5, DK-2100 København Ø, Denmark*

# Selected publications

89:

Mediterr. j. math. 1 (2004), 433–439
1660-5446/040433-7, DOI 10.1007/s00009-004-0022-6


Mediterranean Journal of Mathematics

**Integral Representation of Some Functions Related to the Gamma Function**

Christian Berg

**Abstract.** We prove that the functions $\Phi(x) = [\Gamma(x+1)]^{1/x}(1+1/x)^x/x$ and $\log\Phi(x)$ are Stieltjes transforms.

**Mathematics Subject Classification (2000).** Primary 33B15; Secondary 26A48.
**Keywords.** Complete monotonicity, Gamma function.

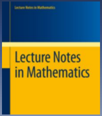
112:

Mediterr. J. Math. 10 (2013), 1685–1696
DOI 10.1007/s00009-013-0272-2
1660-5446/13/041685-12, *published online* February 27, 2013


Mediterranean Journal of Mathematics

**Complete Monotonicity of a Difference Between the Exponential and Trigamma Functions and Properties Related to a Modified Bessel Function**

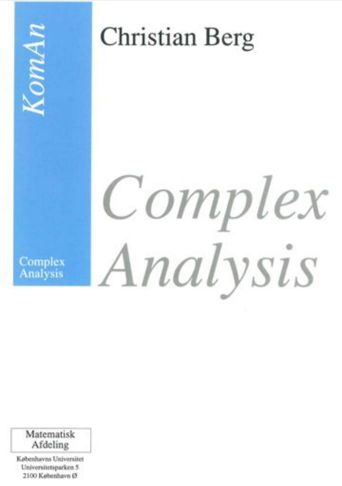
Feng Qi* and Christian Berg

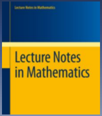



# Lecture notes

Among the math students in Copenhagen, Christian is world famous for his lecture notes.

Complex Analysis, Measure Theory, General Topology, etc.






# With pictures...

Augustin Louis Cauchy (1789–1857), French

6

1.1

## §1. Introduction

### 1.1. Preliminaries.

In these notes the reader is assumed to have a basic knowledge of the complex numbers, here denoted $\mathbb{C}$, including the basic algebraic operations with complex numbers as well as the geometric representation of complex numbers in the euclidean plane.

We will therefore without further explanation view a complex number $x + iy \in \mathbb{C}$ as representing a point or a vector $(x, y)$ in $\mathbb{R}^2$, and according to our need we shall speak about a complex number or a point in the complex plane. A set of complex numbers can be conceived as a set of points in $\mathbb{R}^2$.

Let us recall some basic notions:

A complex number $z = x + iy \in \mathbb{C}$ has a *real part* $x = \mathrm{Re}(z)$ and an *imaginary part* $y = \mathrm{Im}(z)$, and it has an *absolute value* (also called its *modulus*) $r = |z| = \sqrt{x^2 + y^2}$. We recall the important *triangle inequality* for $z, w \in \mathbb{C}$

$$||z| - |w|| \le |z - w| \le |z| + |w|.$$

For a non-zero complex number $z$ we denote by $\arg(z)$ the set of its arguments, i.e. the set of real numbers $\theta$ such that

$$z = r(\cos\theta + i\sin\theta).$$

The pair of numbers $(r, \theta)$ for $\theta \in \arg(z)$ are also called *polar coordinates* for the complex number $z$. More about this will be discussed in Section 5.

Every complex number $z = x + iy$ with $x, y \in \mathbb{R}$ has a complex conjugate number $\bar{z} = x - iy$, and we recall that $|z|^2 = z\bar{z} = x^2 + y^2$.

As distance between two complex numbers $z, w$ we use $d(z, w) = |z - w|$, which equals the euclidean distance in $\mathbb{R}^2$, when $\mathbb{C}$ is interpreted as $\mathbb{R}^2$. With this distance $\mathbb{C}$ is organized as a metric space, but as already remarked, this is the same as the euclidean plane. The concepts of open, closed and bounded subsets of $\mathbb{C}$ are therefore exactly the same as for the corresponding subsets of $\mathbb{R}^2$. In this exposition—with a minor exception in Section 9—formal knowledge of the theory of metric spaces is not needed, but we need basic topological notions from euclidean spaces.

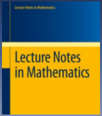
To $a \in \mathbb{C}$ and $r > 0$ is attached the open (circular) disc with centre $a$ and radius $r > 0$, defined as

$$K(a, r) = \{z \in \mathbb{C} \mid |a - z| < r\}.$$

As a practical device we introduce $K'(a, r)$ as the punctured disc

$$K'(a, r) = K(a, r) \setminus \{a\} = \{z \in \mathbb{C} \mid 0 < |a - z| < r\}.$$

7




# ...and names...



Bernhard Riemann (1826–1866), German





## §4. Applications of Cauchy's integral formula

In section 4.2 we shall apply the Cauchy integral formula to prove that a holomorphic function is differentiable infinitely often and is equal to the sum of its Taylor series. In the proof we need the concept of uniform convergence, which is treated in section 4.1. The set of holomorphic functions is stable with respect to local uniform convergence. This clearly makes it an important concept, which will be developed in section 4.4.

Liouville's theorem and the fundamental theorem of algebra appear unexpectedly.

### 4.1. Sequences of functions.

Let $M$ be an arbitrary non-empty set. In the applications $M$ will typically be a subset of $\mathbb{C}$.

A sequence of functions $f_n : M \to \mathbb{C}$ is said to *converge pointwise* to the function $f : M \to \mathbb{C}$ if for all $x \in M$

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Using quantifiers this can be expressed:

$$\forall x \in M\ \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \in \mathbb{N}\ (n \geq N \Rightarrow |f(x) - f_n(x)| \leq \varepsilon). \quad (1)$$

**Examples:** 1) Let $M = \mathbb{R}$ and $f_n(x) = |\sin x|^n$, $n = 1, 2, \ldots$. Then

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{for } x = \frac{\pi}{2} + p\pi,\ p \in \mathbb{Z} \\ 0 & \text{for } x \in \mathbb{R} \setminus \{\frac{\pi}{2} + \mathbb{Z}\pi\}, \end{cases}$$

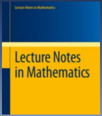
i.e. $f_n : \mathbb{R} \to \mathbb{C}$ converges pointwise to the indicator function $f = 1_{\frac{\pi}{2}+\mathbb{Z}\pi}$ for the set $\frac{\pi}{2} + \mathbb{Z}\pi$, meaning the function which is 1 on the set and 0 on the complement of the set.

2) Let $M = \mathbb{C}$ and $f_n(z) = z^n/n!$, $n = 1, 2, \ldots$. Then

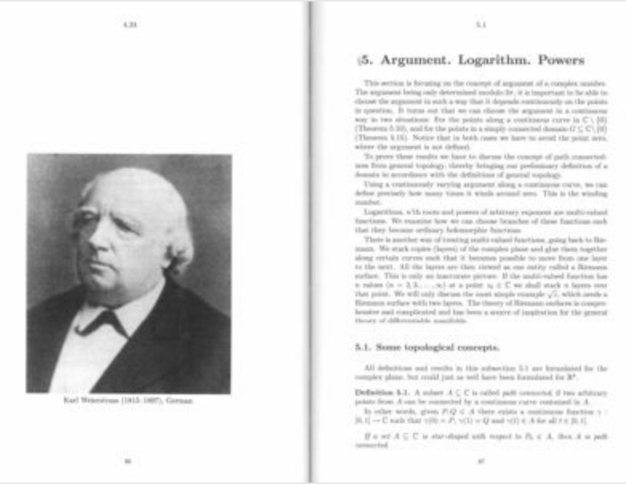
$$\lim_{n\to\infty} f_n(z) = 0 \quad \text{for all} \quad z \in \mathbb{C},$$

hence $f_n$ converges pointwise to the zero function. In fact, the power series for exp being convergent for all $z \in \mathbb{C}$, the $n$'th term will converge to 0.

In both examples the functions $f_n$ are continuous, but the limit function is not continuous in the first example.




# ...years and nationality!

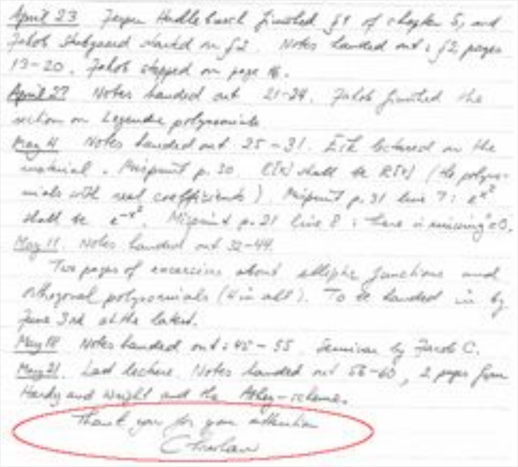


Karl Weierstrass (1815–1897), German

## §5. Argument. Logarithm. Powers

### 5.1. Some topological concepts.

# The ‘Berg’ font


April 23 Jesper Hedlebark finished §1 of chapter 5, and Jakob Stegaard started on §2. Notes handed out: §2 pages 13–20. Jakob stopped on page 16.
April 27 Notes handed out 21–24. Jakob finished the section on Legendre polynomials.
May 4 Notes handed out 25–31. Erik lectured on the material. Misprint p. 30 C[x] shall be R[x] (the polynomials with real coefficients). Misprint p. 31 line 7: $e^{x^2}$ shall be $e^{-x^2}$. Misprint p. 21 line 8: there is missing $\neq 0$.
May 11 Notes handed out 32–44.
Two pages of exercises about elliptic functions and orthogonal polynomials (4 in all). To be handed in by June 3rd at the latest.
May 18 Notes handed out: 45–55. Seminar by Jacob C.
May 21 Last lecture. Notes handed out 56–60, 2 pages from Hardy and Wright and the Atkey-scheme.
Thank you for your attention
Christian

Lecture Notes in Mathematics

# Student notes

holds because the term corresponding to $k=p$ is zero.

Hence we get that if $n$ is odd, ie $n=2p+1$

$$P_{2p+1}'(0) = \frac{(-1)^p \left(\frac{1}{2}\right)_{2p+1-p}}{p!\,0!} \cdot 2 = \frac{(-1)^p}{p!} \cdot 2 \cdot \left(\tfrac{1}{2}\right)_{p+1},$$

and if $n$ is even, ie $n=2p$, then

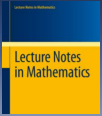
$$P_{2p}'(0) = 0.$$

The last equation could also have been seen by noticing that $P_{2p}'(x)$ must be odd since $P_{2p}(x)$ is even.

We now wish to find some other formulas for $P_n(x)$. The first one is called <u>Rodrigues'</u> formula

<u>Theorem 2.2</u> For all $n \in \mathbb{N}$, we have the following formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n, \quad x \in \mathbb{C}$$




# Student notes

holds because the term corresponding to $k=p$ is zero.

Hence we get that if $n$ is odd, ie $n=2p+1$

$$P'_{2p+1}(0) = \frac{(-1)^p \left(\frac{1}{2}\right)_{2p+1-p}}{p!\;0!}\cdot 2 = \frac{(-1)^p}{p!}\cdot 2\cdot\left(\frac{1}{2}\right)_{p+1}, \qquad (2.4)$$

and if $n$ is even, ie $n=2p$, then

$$P'_{2p}(0) = 0. \qquad (2.5)$$

The last equation could also have been seen by noticing that $P'_{2p}(x)$ must be odd since $P_{2p}(x)$ is even.

We now wish to find some other formulas for $P_n(x)$. The first one is called Rodrigues' formula (O. Rodrigues 1794–1851).

Theorem 2.2 For all $n \in \mathbb{N}$ we have the following formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \quad x \in \mathbb{C}$$

# The Berg family

# The Berg family

We wish you all a profiterable conference with many good discussions and talks!