Indeterminate Hamburger moment problems

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Overview

1. Basic facts about Hamburger moment problems

2. Orthonormal polynomials. Classical examples

3. Three term recurrence relation and Favard's Theorem

4. Indeterminate moment problems and the Nevanlinna parametrization

5. Order and type of indeterminate moment problems

6. Determination of the order as exponent of convergence

7. Symmetric indeterminate moment problems

8. Multi-zeta values

9. Inverse of indeterminate Hankel matrices

10. Results in case of eventually q-increasing sequences

What is the Hamburger moment problem?

A Hamburger moment sequence is a sequence $(s_n)_{n\geq 0}$ of real numbers of the form

$$s_n = \int_{-\infty}^{\infty} x^n \, d\mu(x), \quad , n = 0, 1, \ldots, \quad (*)$$

where μ is a positive measure on $\mathbb{R}$. It is called normalized if $s_0 = \mu(\mathbb{R}) = 1$.
The Hamburger moment problem has two parts:

1) Characterize the set $H(\mathbb{R})$ of Hamburger moment sequences
2) For $(s_n) \in H(\mathbb{R})$ describe how to find a measure μ satisfying $(*)$.

In 2) a difficulty occurs: There can be exactly one solution—the determinate case
or there can be several solutions—the indeterminate case.
In the latter case, the set of solutions is an infinite compact convex set V of measures in the weak topology.

A family of examples

For $c > 0$ consider the symmetric probability density

$$f_c(x) = ((2/c)\Gamma(1/c))^{-1} \exp(-|x|^c), \quad x \in \mathbb{R}$$

with moments

$$s_{2n} = \int_{-\infty}^{\infty} f_c(x) x^{2n}\, dx = \frac{\Gamma((2n+1)/c)}{\Gamma(1/c)}, \quad s_{2n+1} = 0$$

(s_n) is determinate for $c \geq 1$, indeterminate for $0 < c < 1$.

$$f_c(x)(1 + \lambda \cos(|x|^c \tan((c\pi)/2))$$

has the same moments when $0 < c < 1$, $\lambda \in [-1, 1]$.
For $c = 2$ it is the Gaussian (or normal) distribution with moments

$$s_{2n} = 1 \cdot 3 \cdot 5 \ldots \cdot (2n-1).$$

Infinite Hankel matrices

Given a sequence $(s_n)_{n\geq 0}$ of real numbers, we consider the infinite Hankel matrix $\mathcal{H} = \{s_{j+k}\}_{j,k=0}^{\infty}$

$$\mathcal{H} = \begin{pmatrix} s_0 & s_1 & s_2 & \cdots \\ s_1 & s_2 & s_3 & \cdots \\ s_2 & s_3 & s_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Hamburger proved around 1920 that $(s_n) \in H(\mathbb{R})$ if and only if $\mathcal{H}$ is positive semi-definite in the sense that

$$\langle \mathcal{H}v, v\rangle = \sum_{j,k=0}^{\infty} s_{j+k} v_j v_k \geq 0.$$

for all real sequences $v = (v_n)$ with only finitely many non-zero entries.

Photo of Hans Ludwig Hamburger (1889-1956)

Interesting biography of Hamburger in MacTutor History of Mathematics archive at St. Andrews

The finite truncations

An equivalent formulation is that all the finite sections of $\mathcal{H}$:

$$\mathcal{H}_N = \{s_{j+k}\}, 0 \le j, k \le N, \quad N = 0, 1, 2, \ldots$$

are positive semi-definite matrices.
For $(s_n) \in H(\mathbb{R})$ we then have

$$D_N = \det \mathcal{H}_N \ge 0, \quad N = 0, 1, \ldots,$$

and two possibilities occur: Either $D_N > 0$ for all N and then all solutions to the moment problem have infinite support, or there exists $n_0 \in \mathbb{N}_0$ such that

$$D_N > 0, 0 \le N < n_0, \quad D_N = 0, N \ge n_0,$$

and in this case the moment problem is determinate and the unique solution is discrete with n_0 mass points. (degenerate case)

Orthonormal polynomials for non-degenerate Hamburger moment sequences

We will only consider non-degenerate moment sequences, i.e., the case where $D_N > 0$ for all N.
In this case there is a unique sequence $(P_n)_{n\geq 0}$ of polynomials such that

1. P_n is a polynomial of degree n with positive leading coefficient
2. $$\int_{-\infty}^{\infty} P_n(x)P_m(x)\,d\mu(x) = \delta_{nm}.$$

This follows from the Gram-Schmidt procedure applied to the monomials $1, x, x^2, \ldots$.

A determinant formula for P_n

P_n can be calculated from the moments s_n via the formula

$$P_n(x) = \frac{1}{\sqrt{D_{n-1}D_n}} \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}, \quad D_n = \det \mathcal{H}_n.$$

The polynomials Q_n of the second kind are defined by

$$Q_n(x) = \int_{-\infty}^{\infty} \frac{P_n(x) - P_n(y)}{x - y}\, d\mu(y).$$

The classical examples of orthogonal polynomials

The classical orthogonal polynomials are

Probability measure	Moment sequence	Orthogonal pol.
$(1/\sqrt{\pi})e^{-x^2}\,dx$	$\begin{cases} s_{2n} = 1\cdot 3\cdots(2n-1) \\ s_{2n+1} = 0 \end{cases}$	Hermite
$e^{-x}1_{]0,\infty[}(x)\,dx$	$s_n = n!$	Laguerre
$1_{]0,1[}(x)\,dx$	$s_n = 1/(n+1)$	Legendre
$e^{-a}\sum_{k=0}^{\infty} \frac{a^k}{k!}\delta_k$	$s_n = e^{-a}\sum_{k=0}^{\infty} \frac{k^n a^k}{k!}$	Charlier

On the interval $[-1,1]$ we consider the density

$$(1-x)^{\alpha}(1+x)^{\beta}, \quad \alpha, \beta > -1.$$

The corresponding orthogonal polynomials are called Jacobi polynomials. All these moment problems are determinate.

Lognormal distribution—Stieltjes-Wigert polynomials

If a random variable X has a normal distribution, then e^X follows a so-called log-normal distribution. Stieltjes discovered that it is indeterminate. In modern notation fix $0 < q < 1$, then

$$s_n = q^{-n^2/2} = \int_0^\infty x^n w_q(x)/x\,dx, \quad n \geq 0,$$

where

$$w_q(x) = \frac{1}{\sqrt{2\pi \log(1/q)}} \exp\left(-\frac{(\log x)^2}{2\log(1/q)}\right).$$

The orthonormal polynomials are

$$P_n(x) = (-1)^n \frac{q^{n/2}}{\sqrt{(q;q)_n}} \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{k^2-k/2} x^k,$$

with

$$\binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \quad (z;q)_n = \prod_{j=0}^{n-1}(1 - zq^j).$$

Three-term recurrence relation

The orthonormal polynomials (P_n) satisfy the three-term recurrence relation

$$xP_n(x) = b_nP_{n+1}(x) + a_nP_n(x) + b_{n-1}P_{n-1}(x), n \geq 0, \quad \text{(3trl)}$$

with the initial conditions $P_{-1}(x) = 0, P_0(x) = 1$.
Here

$$a_n = \int_{-\infty}^{\infty} xP_n^2(x)\, d\mu(x) \in \mathbb{R}, \quad b_n = \int_{-\infty}^{\infty} xP_n(x)P_{n+1}(x)\, d\mu(x) > 0.$$

Conversely (Favard's Theorem 1935): Given two sequences $(a_n), (b_n)$ of real numbers with $b_n > 0$, then (3trl) and the initial conditions determine a sequence of polynomials (P_n), which are the orthonomal polynomials associated with a normalized non-degenerate Hamburger moment sequence (s_n).

How to get s_n from $(a_n),(b_n)$?

We form the symmetric Jacobi matrix

$$J = \begin{pmatrix} a_0 & b_0 & 0 & 0 & \cdots \\ b_0 & a_1 & b_1 & 0 & \cdots \\ 0 & b_1 & a_2 & b_2 & \cdots \\ 0 & 0 & b_2 & a_3 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Then, using the standard inner product $\langle \cdot, \cdot \rangle$ in ℓ^2

$$s_n = \langle J^n \delta_0, \delta_0 \rangle, \quad \delta_0 = (1, 0, 0, \ldots).$$

The polynomials of the second kind $(Q_n(x))$ satisfy (3trl) with $Q_{-1}(x) = -1, Q_0(x) = 0, b_{-1} = 1$.

Characterizations of indeterminacy

The following is a classical result

Theorem

Given the data $(s_n), (a_n), (b_n)$, the following conditions are equivalent:

(i) $\sum_{n=0}^{\infty} \left(P_n^2(0) + Q_n^2(0)\right) < \infty$,

(ii) $P(z) = \left(\sum_{n=0}^{\infty} |P_n(z)|^2\right)^{1/2} < \infty, \quad z \in \mathbb{C}$.

(iii) *J has deficiency indices $(1,1)$ in ℓ^2.*

If (i)–(iii) hold (the indeterminate case), then $Q(z) = \left(\sum_{n=0}^{\infty} |Q_n(z)|^2\right)^{1/2} < \infty$ for $z \in \mathbb{C}$, and the series for P, Q are uniformly convergent on compact subsets of $\mathbb{C}$.

In the following we will only discuss the indeterminate case, hence by Carleman's Theorem $\sum(1/b_n) < \infty$.
Warning: $\sum(1/b_n) < \infty$ is not sufficient for indeterminacy.

A theorem of B, Chen and Ismail characterizing indeterminacy

Let λ_N be the smallest eigenvalue of the positive definite matrix $\mathcal{H}_N$,

$$\lambda_N = \min\{\langle \mathcal{H}_N v, v\rangle \mid v \in \mathbb{R}^{N+1}, ||v|| = 1\} > 0.$$

Therefore $\lambda_N \geq \lambda_{N+1}$, hence

$$\lambda_\infty := \lim_{N\to\infty} \lambda_N \text{ exists, and } \quad \lambda_\infty \geq 0.$$

The number λ_∞ characterizes determinacy:

Theorem (B-C-I, Math. Scand. 2002)

$\lambda_\infty = 0$ *(resp.* $\lambda_\infty > 0$*) if and only if* (s_n) *is determinate (resp. indeterminate).*

How to find the measure μ in the determinate case?

By a theorem of Markov (1895) we have

$$\lim_{n\to\infty} \frac{Q_n(z)}{P_n(z)} = \int \frac{d\mu(x)}{z-x}, \quad z \in \mathbb{C} \setminus \mathbb{R}$$

and then one shall invert the Stieltjes transform

$$\int \frac{d\mu(x)}{z-x}, \quad z \in \mathbb{C} \setminus \mathbb{R}$$

in order to find the measure μ.
Of course all this can be difficult to carry out.

In the indeterminate case this is of course even more difficult since there are many μ.

How to find μ in the indeterminate case

The following polynomials

$$\begin{aligned} A_n(z) &= z\sum_{k=0}^{n-1} Q_k(0)Q_k(z), \\ B_n(z) &= -1 + z\sum_{k=0}^{n-1} Q_k(0)P_k(z), \\ C_n(z) &= 1 + z\sum_{k=0}^{n-1} P_k(0)Q_k(z), \\ D_n(z) &= z\sum_{k=0}^{n-1} P_k(0)P_k(z). \end{aligned}$$

tend to entire functions denoted A, B, C, D, when n tends to infinity. They are needed in the Nevanlinna parametrization of the set V of all solutions to an indeterminate moment problem.

Nevanlinna parametrization of solutions $\nu \in V$ to $s_n = \int x^n \, d\nu(x), n = 0, 1, \ldots$

The formula

$$\int \frac{d\nu_\varphi(x)}{x - z} = -\frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)}\,, \quad z \in \mathbb{C} \setminus \mathbb{R}\,,$$

expresses the Stieltjes transform of any solution $\nu = \nu_\varphi$ in terms of a parameter φ running through $\mathcal{P} \cup \{\infty\}$, where $\mathcal{P}$ denotes the set of Pick functions, i.e., the holomorphic functions in the upper half-plane $\mathbb{H}$ with values in $\overline{\mathbb{H}}$.

$$\begin{pmatrix} A(z) & C(z) \\ B(z) & D(z) \end{pmatrix}$$ Nevanlinna matrix of the moment problem

Its determinant is identically 1.

Illustration about Pick functions, Georg Pick, 1859-1942

The Pick functions are holomorphic functions $f : \mathbb{H} \to \mathbb{C}$ satisfying $\Im f(z) \geq 0$ for $z \in \mathbb{H}$. They can be described as the functions of the form

$$f(z) = a + bz + \int_{-\infty}^{\infty} \frac{1 + tz}{t - z} d\tau(t), \quad z \in \mathbb{H},$$

where $a \in \mathbb{R}, b \geq 0$ and τ is a positive measure on $\mathbb{R}$ with finite total mass.

Note that the formula also defines f in the lower half-plane and $f(\overline{z}) = \overline{f(z)}$. It is only defined in points z of the real axis if z is outside the support of τ.

Examples: $f(z) = a \in \mathbb{R}$, $f(z) = z$, $f(z) = -1/z$, $f(z) = \log(z), f(z) = z^{\alpha}, 0 < \alpha < 1$ both defined in $\mathbb{C} \setminus (-\infty, 0]$.

The set $V = \{\nu_\varphi \mid \varphi \in \mathcal{P} \cup \{\infty\}\}$ of solutions is big

There are many Pick functions: By looking at classes of measures τ defining Pick functions one can get:

To any indeterminate moment problem, there are always "many" solution in V of the following types (B.-Christensen, 1981):

- measures with a C^∞-density
- discrete
- continuous singular

Many in the sense of category or in the sense of dense subsets in the weak topology.

More about Nevanlinna matrices

Marcel Riesz (1923): $A, \ldots, D$ have minimal exponential type, i.e. each of these functions satisfy

$$\forall \epsilon > 0 \; \exists C_\epsilon > 0 \quad : \; |f(z)| \le C_\epsilon e^{\epsilon |z|}, \quad z \in \mathbb{C}.$$

B. and Pedersen (1994): $A, \ldots, D, P, Q$ have the same order ρ and type τ called order and type of the moment problem.

By M. Riesz: $\rho \le 1$ and if $\rho = 1$, then $\tau = 0$.

B. and Pedersen (2005): If $\rho = 0$ then $A, \ldots, D, P, Q$ have the same logarithmic order $\rho^{[1]}$ and logarithmic type $\tau^{[1]}$ called logarithmic order and logarithmic type of the moment problem.

Definition of order, type, log-order,...

The order and type of a function f are defined in terms of its maximum modulus

$$M_f(r) := \max_{|z|\le r} |f(z)|, \quad r \ge 0$$

as

$$\rho_f = \inf\{\alpha > 0 \mid M_f(r) \le_{as} e^{r^\alpha}\}$$

$$\tau_f = \inf\{c > 0 \mid M_f(r) \le_{as} e^{cr^{\rho_f}}\}.$$

and logarithmic order and type as (if $\rho_f = 0$)

$$\rho_f^{[1]} = \inf\{\alpha > 0 \mid M_f(r) \le_{as} r^{(\log r)^\alpha}\}$$

$$\tau_f^{[1]} = \inf\{c > 0 \mid M_f(r) \le_{as} r^{c(\log r)^{\rho_f^{[1]}}}\}.$$

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Calculation of Nevanlinna matrices

Nevanlinna matrices and their use to describe all the solutions to indeterminate moment problems were discovered by Nevanlinna in 1922.

The first explicit examples of Nevanlinna matrices were calculated around 1994:

Ismail-Masson: Order 0, logarithmic order 1. The functions are related to theta-functions.

B.-Valent: Order 1/4. The functions are related to trigonometric functions of order 4.

A Question: Is it possible to calculate the order (and type) of the moment problem directly from the recurrence coefficients or the moments?

Some preliminary results about order and type

Proposition

Consider an indeterminate Hamburger moment problem corresponding to sequences $(a_n), (b_n)$ from the three-term recurrence relation. If another moment problem is given in terms of $(\tilde{a}_n), (\tilde{b}_n)$, and if $\tilde{a}_n = a_n, \tilde{b}_n = b_n$ for $n \geq n_0$, then the second problem is also indeterminate and the two problems have the same order and type.

Proposition

Consider an indeterminate Hamburger moment problem of order ρ and type τ corresponding to the sequences $(a_n), (b_n)$ from the three-term recurrence relation. For $c > 0$ the moment problem corresponding to the sequences $(ca_n), (cb_n)$ is also indeterminate with order $\rho(c) = \rho$ and type $\tau(c) = \tau/c^\rho$.

Theorem (B. - Szwarc, 2014)

For a moment problem and $0 < \alpha \leq 1$ the following conditions are equivalent:

(i) $(P_n^2(0)), (Q_n^2(0)) \in \ell^\alpha$,

(ii) $(P_n^2(z)), (Q_n^2(z)) \in \ell^\alpha$ *for all* $z \in \mathbb{C}$.

If the conditions are satisfied, the moment problem is indeterminate and the two series indicated in (ii) converge locally uniformly. Furthermore, $(1/b_n) \in \ell^\alpha$ *and*

$$P(z) := \left(\sum_{n=0}^{\infty} |P_n(z)|^2\right)^{1/2} \leq C \exp(K|z|^\alpha),$$

$$C = \left(\sum_{n=0}^{\infty}(P_n^2(0) + Q_n^2(0))\right)^{1/2}, \; K = \frac{1}{\alpha}\sum_{n=0}^{\infty}(|P_n(0)|^{2\alpha}+|Q_n(0)|^{2\alpha}).$$

In particular the moment problem has order $\rho \leq \alpha$*, and if the order is* α*, then the type* $\tau \leq K$.

The Livšic function I

In 1939 Livšic considered the function

$$F(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{s_{2n}}$$

which is entire of minimal exponential type for any indeterminate problem because $\lim n/\sqrt[2n]{s_{2n}} = 0$. This holds by Carleman's criterion giving that

$$\sum_{n=0}^{\infty} 1/\sqrt[2n]{s_{2n}} < \infty.$$

Moreover, $\sqrt[2n]{s_{2n}}$ is increasing for $n \geq 1$.
Livšic proved that $\rho_F \leq \rho$, where ρ is the order of the moment problem.
We shall discuss the **Question:** Is $\rho_F = \rho$?

The Livšic function II

B.-Szwarc, Adv. Math. 2014: The answer is yes under certain "regularity" conditions.

Pruckner-Romanov-Woracek, Constr. Approx. 2016 Examples where $\rho_F < \rho$.

The results of B.-Szwarc build on refinements/extensions of work by Berezanskiĭ (1956), where (b_n) is assumed log-concave, i.e., $b_n^2 \geq b_{n-1}b_{n+1}$.

Definition: We say that (b_n) is regular if (b_n) is either eventually log-convex, i.e., $b_n^2 \leq b_{n-1}b_{n+1}$ for $n \geq n_0$, or eventually log-concave, i.e. $b_n^2 \geq b_{n-1}b_{n+1}$ for $n \geq n_0$.
Examples of regular (b_n) with $\sum(1/b_n) < \infty$:

$$b_n = n^\alpha, \quad b_n = n\log^\alpha n,\ \alpha > 1, \quad b_n = a^{n^\alpha}, a > 1, \alpha > 0.$$

Results from B.-Szwarc: Adv. Math. 2014

Definition: For a sequence (z_n) of complex numbers for which $|z_n| \to \infty$, we introduce the exponent of convergence

$$\mathcal{E}(z_n) = \inf\left\{\alpha > 0 \mid \sum_{n=n^*}^{\infty} \frac{1}{|z_n|^\alpha} < \infty\right\},$$

where $n^* \in \mathbb{N}$ is such that $|z_n| > 0$ for $n \geq n^*$.

Theorem

Assume that (b_n) is regular and that

$$\sum_{n=1}^{\infty} \frac{1+|a_n|}{\sqrt{b_n b_{n-1}}} < \infty.$$

Then the moment problem is indeterminate of order $\rho = \mathcal{E}(b_n)$. If the order is $\rho = 0$, then the logarithmic order of the moment problem is $\rho^{[1]} = \mathcal{E}(\log b_n)$.

Consider a symmetric indeterminate Hamburger moment problem, where $a_n = 0$. Then there are symmetric solutions $\mu \in V$, but in V there are also non-symmetric solutions.
Then

$$P_{2n+1}(0) = Q_{2n}(0) = 0$$

so the condition for indeterminacy

$$\sum P_n^2(0) + Q_n^2(0) < \infty$$

reduces to the behaviour of

$$v_n := P_{2n}^2(0), \quad u_n := Q_{2n-1}^2(0), \quad n \geq 1.$$

One possibility is

$$v_n \approx n^{-1/\beta}, \quad u_n \approx n^{-1/\alpha}, \quad 0 < \alpha, \beta < 1.$$

A result of B.-Szwarc from Sbornik Math. 2017

Consider a symmetric indeterminate Hamburger moment problem with the rather precise behaviour

$$v_n = P_{2n}^2(0) = c_1 n^{-1/\beta}(1+\delta_n), \quad u_n = Q_{2n-1}^2(0) = c_2 n^{-1/\alpha}(1+\varepsilon_n),$$

where $c_1, c_2 > 0, 0 < \alpha, \beta < 1, |\delta_n|, |\varepsilon_n| \leq K/n$ for some constant $K > 0$ independent of n. Let γ be the harmonic mean of α, β, i.e.,

$$1/\gamma = \frac{1}{2}(1/\alpha + 1/\beta).$$

Then the order and type of the moment problem is determined by the order and type of a new kind of special function $G_s(z)$, which we introduce next:

Multi-zeta values

For real $s > 1$ and $n \geq 1$ define the special multi-zeta value

$$\zeta_n(s) := \sum_{1 \leq k_1 \leq k_2 < \ldots < k_{2n-1} \leq k_{2n}} (k_1 k_2 \ldots k_{2n-1} k_{2n})^{-s}.$$

Observe that the inequalities between the indices k_j are alternating between $\leq$ and $<$: $k_{2j-1} \leq k_{2j} < k_{2j+1} \leq k_{2j+2}$. For $n = 1$:

$$\zeta_1(s) := \sum_{1 \leq k_1 \leq k_2} (k_1 k_2)^{-s} < \left(\sum_{n=1}^{\infty} n^{-s} \right)^2.$$

We shall consider the entire function

$$G_s(z) = 1 + \sum_{n=1}^{\infty} \zeta_n(s) z^n.$$

and we call its order $\rho_s := \rho_{G_s}$ and its type $\tau_s = \tau_{G_s}$.

Continuation of the result from Sbornik 2017

Consider the symmetric indeterminate Hamburger moment problem with the rather precise behaviour

$$v_n = P_{2n}^2(0) = c_1 n^{-1/\beta}(1+\delta_n), \quad u_n = Q_{2n-1}^2(0) = c_2 n^{-1/\alpha}(1+\varepsilon_n),$$

where $c_1, c_2 > 0, 0 < \alpha, \beta < 1, |\delta_n|, |\varepsilon_n| \le K/n$ for some constant $K > 0$ independent of n.
Then the order and type of the moment problem is the order and type of the function

$$G_{1/\gamma}(c_1 c_2 z^2), \quad 1/\gamma = \frac{1}{2}(1/\alpha + 1/\beta)$$

hence

$$\rho = \gamma, \quad \tau = (c_1 c_2)^{\gamma/2} \tau_{1/\gamma}.$$

Order and type of $G_s(z)$

It is relatively easy to prove that the order of G_s is $\rho_s = 1/(2s)$. The type τ_s was unknown until recently when Ivan Bochkov (St. Petersburg, ArXiv 2018) proved

$$\tau_s = B(1/(2s), 1 - 1/s), \quad (I)$$

in accordance with a 20 years old conjecture of G. Valent about birth and death processes with polynomial rates.

For an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ of order $0 < \rho < \infty$ the type is given by (a classical formula)

$$\tau_f = \frac{1}{e\rho} \limsup_{n\to\infty} \left(n|a_n|^{\rho/n} \right).$$

In particular

$$\tau_s = \frac{2s}{e} \limsup_{n\to\infty} \left(n\ \zeta_n(s)^{1/2ns} \right), \quad (II).$$

Bochkov proved that $I = II$.

To see the result about order consider first the canonical product

$$P_s(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n^s}\right), \quad 1 < s, z \in \mathbb{C}.$$

It is a classical fact that $\rho_{P_s} = 1/s$ and type $\tau_{P_s} = \pi/\sin(\pi/s)$.

Determination of the order ρ_s

Choosing $k_1 = k_2 = l_1, k_3 = k_4 = l_2, \ldots k_{2n-1} = k_{2n} = l_n$ we get

$$\zeta_n(s) > \sum_{1 \le l_1 < l_2 < \ldots < l_n} (l_1 l_2 \ldots l_n)^{-2s},$$

which is an ordinary multi-zeta value, showing that for $r > 0$

$$\prod_{n=1}^{\infty} \left(1 + \frac{r^2}{n^{2s}}\right) < 1 + \sum_{n=1}^{\infty} \gamma_n(s) r^{2n} < \prod_{n=1}^{\infty} \left(1 + \frac{r}{n^s}\right)^2$$

i.e.,

$$P_{2s}(r^2) < G_s(r^2) < P_s(r)^2.$$

The first and third function have order $1/s$ and so does $G_s(z^2)$. Therefore $\rho_{G_s} = 1/(2s)$. Since the type of $G_s(z)$ and $G_s(z^2)$ are the same called τ_s, we get

$$\frac{\pi}{\sin(\pi/(2s))} \le \tau_s \le \frac{2\pi}{\sin(\pi/s)} = \frac{\pi}{\sin(\pi/(2s) \cos(\pi/(2s))}.$$

Matrix inverse of the infinite Hankel matrix $\mathcal{H}$

Let

$$\mathcal{H} = \left\{s_{j+k}\right\}_{j,k=0}^{\infty} \quad \mathcal{H}_N = \left\{s_{j+k}\right\}_{j,k=0}^{N}$$

be the infinite Hankel matrix and its sections only assuming that $\mathcal{H}_N$ is positive definite for all N. (determinate or not.)
It has been noticed by e.g. Aitken (paper by Collar(1939)) that the inverse of $\mathcal{H}_N$ is

$$\mathcal{A}_N = \left\{a_{j,k}^{(N)}\right\}_{j,k=0}^{N}, \quad K_N(x,y) = \sum_{n=0}^{N} P_n(x)P_n(y) = \sum_{j,k=0}^{N} a_{j,k}^{(N)} x^j y^k.$$

In the indeterminate case we can consider

$$K(x,y) = \sum_{n=0}^{\infty} P_n(x)P_n(y) = \sum_{j,k=0}^{\infty} a_{j,k} x^j y^k,$$

and it is true that $a_{j,k}^{(N)} \to a_{j,k}$ for $N \to \infty$.

Is $\mathcal{HA} = \mathcal{AH} = \mathcal{I}$ in the indeterminate case ?

The answer is yes for Stieltjes-Wigert polynomials, i.e. $s_n = q^{-(1/2)n^2}, 0 < q < 1$ (B-Szwarc 2011) in the sense of absolute convergence:

$$\sum_{k=0}^{\infty} |s_{j+k}||a_{k,l}| < \infty, \quad j, l \geq 0.$$

We say that the moment problem has property (ac) if this holds, and we say that property (aci) holds if in addition $\mathcal{HA} = \mathcal{I}$. (Since both matrices are symmetric, the equation $\mathcal{AH} = \mathcal{I}$ is then automatic).

We checked that the same result holds for a number of concrete cases.

In submitted work with R. Szwarc, we have searched for some general results, which I will describe now.

General results about $\mathcal{A}$ and $\mathcal{H}$

B.-Szwarc , Constr. Approx. 2011: $\mathcal{A}$ is a positive definite trace class operator on ℓ^2. Therefore

$$|a_{k,l}| \le c_k c_l, \quad c_k := \sqrt{a_{k,k}}.$$

Furthermore, $c_k \to 0$ very rapidly, in the sense

$$\lim_{k\to\infty} k \sqrt[k]{c_k} = 0$$

$\mathcal{H}$ does not define an operator on ℓ^2 in the classical sense that the columns are the images of the standard bases vectors $\delta_k = (\delta_{k,n})_{n\ge 0}$. This is because

$$\sum_{j=0}^{\infty} s_{j+k}^2 = \infty, \quad k = 0, 1, \dots.$$

In fact, $s_{2n} \ge 1$ for n sufficiently large because by Carleman

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt[2n]{s_{2n}}} < \infty.$$

We say the moment problem has property

(cs) if $\sum_{n=0}^{\infty} c_{2n}s_{2n} < \infty$.

(cs*) if $\sum_{l=0}^{\infty} c_l|s_{l+m}| < \infty$ for all $m \geq 0$.

Theorem

For an indeterminate moment problem the following holds:
(i) (cs*) $\implies$ (aci) $\implies$ (ac),
(ii) (cs*) $\implies$ (cs).
(iii) (ac) *and* (aci) *are equivalent for Stieltjes problems and for symmetric problems.*

We do not know if some of the other implications can be reversed.

A symmetric moment problem is given by $a_n = 0$ and $b_n > 0$.
We say that (b_n) is eventually q-increasing if there exist a number $0 < q < 1$ and $n_0 \in \mathbb{N}$ such that

$$\frac{b_{n-1}}{b_n} \leq q, \quad n \geq n_0.$$

Theorem

Consider a symmetric moment problem given in terms of an eventually q-increasing sequence (b_n). We then have
(i) The moment problem is indeterminate of order 0.
(ii) $(c_n\sqrt{s_{2n}})$ is a bounded sequence.
(iii) Property (cs^*) holds (and hence also (aci)).

Results for symmetric moment problems II

The property of (b_n) being eventually q-increasing can be weakened to include indeterminate problems of order $\rho > 0$.

Theorem

Consider a symmetric moment problem given in terms of (b_n) *satisfying*

$$\frac{b_{n-1}}{b_n} \le e^{-f(n)}, \quad n \ge n_0,$$

where $f(n) > 0$ *for* $n \ge n_0$ *and* $\alpha := \liminf nf(n) > 1$. *We then have*
(i) The moment problem is indeterminate of order $\le 1/\alpha$.
(ii) Property (cs^*) *holds (and hence also* (aci)*) if* $k(\alpha) < 1$, *where* k *is certain decreasing function. In particular* $k(\alpha) < 1$ *if* $\alpha > 1.68746$.

Examples of b_n for which the previous result holds

The assumptions of the previous theorem are satisfied for the following sequences, where it is assumed that n is so large that the expressions are defined and positive:

$$\begin{array}{llll}
b_n = e^{n^\gamma}, & 0 < \gamma < 1, & f(n) = \gamma n^{\gamma-1}, & \alpha = \infty \\
b_n = e^{(\log n)^\gamma}, & \gamma > 1, & f(n) = \dfrac{\gamma}{n} \log^{\gamma-1} n, & \alpha = \infty \\
b_n = n^\gamma, & \gamma > 1.68746, & f(n) = \dfrac{\gamma}{n}, & \alpha = \gamma.
\end{array}$$

Further examples

There exists a symmetric indeterminate moment problem for which (ac) does not hold.
Such a case is the symmetrized version of a birth-death process with cubic rates studied by Gilewicz, Leopold and Valent in 2005.

For the symmetric moment problem with $b_n = (n+1)^c$, which is indeterminate iff $c > 1$, we have
(i) (cs*) holds if c>3/2 (improving the previous theorem from 1.68... to 1.5)
(ii) (cs) does not hold for $1 < c < 3/2$.

- C. Berg and R. Szwarc, *The smallest eigenvalue of Hankel matrices*, Constr. Approx. **34** (2011), 107–133.
- C. Berg and R. Szwarc, *On the order of indeterminate moment problems*, Advances in Mathematics **250** (2014), 105–143.
- C. Berg and R. Szwarc, *Symmetric moment problems and a conjecture of Valent*. Sbornik: Mathematics **208:3** (2017), 335–359.
- C. Berg and R. Szwarc, *Inverse of infinite Hankel moment matrices*. Submitted.

Thank you for your attention