PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 7, Pages 1913–1923
S 0002-9939(04)07317-4
Article electronically published on February 13, 2004

GORENSTEIN DERIVED FUNCTORS

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(Communicated by Bernd Ulrich)

Abstract. Over any associative ring R it is standard to derive $\mathrm{Hom}_R(-,-)$ using projective resolutions in the first variable, or injective resolutions in the second variable, and doing this, one obtains $\mathrm{Ext}^n_R(-,-)$ in both cases. We examine the situation where projective and injective modules are replaced by Gorenstein projective and Gorenstein injective ones, respectively. Furthermore, we derive the tensor product $-\otimes_R-$ using Gorenstein flat modules.

1. Introduction

When R is a two-sided Noetherian ring, Auslander and Bridger [2] introduced in 1969 the G-dimension, $\text{G-dim}_R M$, for every *finite* (that is, finitely generated) R-module M. They proved the inequality $\text{G-dim}_R M \leqslant \mathrm{pd}_R M$, with equality $\text{G-dim}_R M = \mathrm{pd}_R M$ when $\mathrm{pd}_R M < \infty$, along with a generalized Auslander-Buchsbaum formula (sometimes known as the Auslander-Bridger formula) for the G-dimension.

The (finite) modules with G-dimension zero are called *Gorenstein projectives.* Over a general ring R, Enochs and Jenda in [6] defined Gorenstein projective modules. Avramov, Buchweitz, Martsinkovsky and Reiten proved that if R is two-sided Noetherian, and G is a finite Gorenstein projective module, then the new definition agrees with that of Auslander and Bridger; see the remark following [4, Theorem (4.2.6)]. Using Gorenstein projective modules, one can introduce the Gorenstein projective dimension for arbitrary R-modules. At this point we need to introduce:

1.1 (Notation). Throughout this paper, we use the following notation:

- R is an associative ring. All modules are—if not specified otherwise—*left* R-modules, and the category of all R-modules is denoted $\mathcal{M}$. We use $\mathcal{A}$ for the category of abelian groups (that is, $\mathbb{Z}$-modules).
- We use $\mathcal{GP}$, $\mathcal{GI}$ and $\mathcal{GF}$ for the categories of *Gorenstein projective*, *Gorenstein injective* and *Gorenstein flat* R-modules; please see [6] and [8], or Definition 2.7 below.
- Furthermore, for each R-module M we write $\mathrm{Gpd}_R M$, $\mathrm{Gid}_R M$ and $\mathrm{Gfd}_R M$ for the Gorenstein projective, Gorenstein injective, and Gorenstein flat dimension of M, respectively.

Received by the editors May 14, 2002 and, in revised form, April 16, 2003.
2000 *Mathematics Subject Classification.* Primary 13D02, 13D05, 13D07, 13H10, 16E05, 16E10, 16E30.
Key words and phrases. Gorenstein dimensions, homological dimensions, derived functors, Tor-modules, Ext-modules.

Now, given our base ring R, the usual right derived functors $\mathrm{Ext}_R^n(-,-)$ of $\mathrm{Hom}_R(-,-)$ are important in homological studies of R. The material presented here deals with the Gorenstein right derived functors $\mathrm{Ext}_{\mathcal{GP}}^n(-,-)$ and $\mathrm{Ext}_{\mathcal{GI}}^n(-,-)$ of $\mathrm{Hom}_R(-,-)$.

More precisely, let N be a fixed R-module. For an R-module M that has a *proper left $\mathcal{GP}$-resolution* $\boldsymbol{G} = \cdots \to G_1 \to G_0 \to 0$ (please see 2.1 below for the definition of proper resolutions), we define

$$\mathrm{Ext}_{\mathcal{GP}}^n(M,N) \ := \ \mathrm{H}^n(\mathrm{Hom}_R(\boldsymbol{G},N)).$$

From 2.4 it will follow that $\mathrm{Ext}_{\mathcal{GP}}^n(-,N)$ is a well-defined contravariant functor, defined on the full subcategory, $\mathsf{LeftRes}_{\mathcal{M}}(\mathcal{GP})$, of $\mathcal{M}$, consisting of all R-modules that have a proper left $\mathcal{GP}$-resolution.

For a fixed R-module M' there is a similar definition of the functor $\mathrm{Ext}_{\mathcal{GI}}^n(M',-)$, which is defined on the full subcategory, $\mathsf{RightRes}_{\mathcal{M}}(\mathcal{GI})$, of $\mathcal{M}$, consisting of all R-modules that which have a proper right $\mathcal{GI}$-resolution. Now, the best one could *hope* for is the existence of isomorphisms,

$$\mathrm{Ext}_{\mathcal{GP}}^n(M,N) \ \cong \ \mathrm{Ext}_{\mathcal{GI}}^n(M,N),$$

which are functorial in each variable $M \in \mathsf{LeftRes}_{\mathcal{M}}(\mathcal{GP})$ and $N \in \mathsf{RightRes}_{\mathcal{M}}(\mathcal{GI})$. The aim of this paper is to show a slightly weaker result.

When R is n-Gorenstein (meaning that R is both left and right Noetherian, with self-injective dimension $\leqslant n$ from both sides), Enochs and Jenda [9, Theorem 12.1.4] have proved the existence of such functorial isomorphisms $\mathrm{Ext}_{\mathcal{GP}}^n(M,N) \cong \mathrm{Ext}_{\mathcal{GI}}^n(M,N)$ for all R-modules M and N.

It is important to note that for an n-Gorenstein ring R, we have $\mathrm{Gpd}_R M < \infty$, $\mathrm{Gid}_R M < \infty$, and also $\mathrm{Gfd}_R M < \infty$ for all R-modules M; please see [9, Theorems 11.2.1, 11.5.1, 11.7.6]. For any ring R, [12, Proposition 2.18] (which is restated in this paper as Proposition 3.1) implies that the category $\mathsf{LeftRes}_{\mathcal{M}}(\mathcal{GP})$ contains all R-modules M with $\mathrm{Gpd}_R M < \infty$; that is, every R-module with finite G-projective dimension has a proper left $\mathcal{GP}$-resolution. Also, every R-module with finite G-injective dimension has a proper right $\mathcal{GI}$-resolution. So $\mathsf{RightRes}_{\mathcal{M}}(\mathcal{GI})$ contains all R-modules N with $\mathrm{Gid}_R N < \infty$.

Theorem 3.6 in this text proves that the functorial isomorphisms $\mathrm{Ext}_{\mathcal{GP}}^n(M,N) \cong \mathrm{Ext}_{\mathcal{GI}}^n(M,N)$ hold over *arbitrary* rings R, provided that $\mathrm{Gpd}_R M < \infty$ and $\mathrm{Gid}_R N < \infty$. By the remarks above, this result generalizes that of Enochs and Jenda.

Furthermore, Theorems 4.8 and 4.10 give similar results about the Gorenstein left derived of the tensor product $- \otimes_R -$, using proper left $\mathcal{GP}$-resolutions and proper left $\mathcal{GF}$-resolutions. This has also been proved by Enochs and Jenda [9, Theorem 12.2.2] in the case when R is n-Gorenstein.

2. Preliminaries

Let $T\colon \mathcal{C} \to \mathcal{E}$ be any additive functor between abelian categories. One usually derives T using resolutions consisting of projective or injective objects (if the category $\mathcal{C}$ has enough projectives or injectives). This section is a very brief note on how to derive functors T with resolutions consisting of objects in some subcategory $\mathcal{X} \subseteq \mathcal{C}$. The general discussion presented here will enable us to give very short proofs of the main theorems in the next section.

2.1 (Proper Resolutions). Let $\mathcal{X} \subseteq \mathcal{C}$ be a full subcategory. A *proper left $\mathcal{X}$-resolution* of $M \in \mathcal{C}$ is a complex $\boldsymbol{X} = \cdots \to X_1 \to X_0 \to 0$ where $X_i \in \mathcal{X}$, together with a morphism $X_0 \to M$, such that $\boldsymbol{X}^+ := \cdots \to X_1 \to X_0 \to M \to 0$ is also a complex, and such that the sequence

$$\mathrm{Hom}_{\mathcal{C}}(X, \boldsymbol{X}^+) \;=\; \cdots \to \mathrm{Hom}_{\mathcal{C}}(X, X_1) \to \mathrm{Hom}_{\mathcal{C}}(X, X_0) \to \mathrm{Hom}_{\mathcal{C}}(X, M) \to 0$$

is exact for every $X \in \mathcal{X}$. We sometimes refer to $\boldsymbol{X}^+ = \cdots \to X_1 \to X_0 \to M \to 0$ as an *augmented* proper left $\mathcal{X}$-resolution. We do not require that $\boldsymbol{X}^+$ itself is exact. Furthermore, we use $\mathsf{LeftRes}_{\mathcal{C}}(\mathcal{X})$ to denote the full subcategory of $\mathcal{C}$ consisting of those objects that have a proper left $\mathcal{X}$-resolution. Note that $\mathcal{X}$ is a subcategory of $\mathsf{LeftRes}_{\mathcal{C}}(\mathcal{X})$.

Proper right $\mathcal{X}$-resolutions are defined dually, and the full subcategory of $\mathcal{C}$ consisting of those objects that have a proper right $\mathcal{X}$-resolution is $\mathsf{RightRes}_{\mathcal{C}}(\mathcal{X})$.

The importance of working with *proper* resolutions comes from the following:

Proposition 2.2. *Let $f\colon M \to M'$ be a morphism in $\mathcal{C}$, and consider the diagram*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & M & \longrightarrow 0 \\
& \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & \\
\cdots \longrightarrow & X'_2 & \longrightarrow & X'_1 & \longrightarrow & X'_0 & \longrightarrow & M' & \longrightarrow 0
\end{array}$$

where the upper row is a complex with $X_n \in \mathcal{X}$ for all $n \geqslant 0$, and the lower row is an augmented proper left $\mathcal{X}$-resolution of M'. Then the following conclusions hold:

(i) *There exist morphisms $f_n\colon X_n \to X'_n$ for all $n \geqslant 0$, making the diagram above commutative. The chain map $\{f_n\}_{n\geqslant 0}$ is called a lift of f.*

(ii) *If $\{f'_n\}_{n\geqslant 0}$ is another lift of f, then the chain maps $\{f_n\}_{n\geqslant 0}$ and $\{f'_n\}_{n\geqslant 0}$ are homotopic.*

Proof. The proof is an exercise; please see [9, Exercise 8.1.2]. □

Remark 2.3. A few comments are in order:

- In our applications, the class $\mathcal{X}$ contains all projectives. Consequently, all the augmented proper left $\mathcal{X}$-resolutions occurring in this paper will be exact. Also, all augmented proper right $\mathcal{Y}$-resolutions will be exact, when $\mathcal{Y}$ is a class of R-modules containing all injectives.
- Recall (see [15, Definition 1.2.2]) that an *$\mathcal{X}$-precover* of $M \in \mathcal{C}$ is a morphism $\varphi\colon X \to M$, where $X \in \mathcal{X}$, such that the sequence

$$\mathrm{Hom}_{\mathcal{C}}(X', X) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(X',\varphi)} \mathrm{Hom}_{\mathcal{C}}(X', M) \longrightarrow 0$$

  is exact for every $X' \in \mathcal{X}$. Hence, in an augmented proper left $\mathcal{X}$-resolution $\boldsymbol{X}^+$ of M, the morphisms $X_{i+1} \to \mathrm{Ker}(X_i \to X_{i-1})$, $i > 0$, and $X_0 \to M$ are $\mathcal{X}$-precovers.
- What we have called *proper* $\mathcal{X}$-resolutions, Enochs and Jenda [9, Definition 8.1.2] simply call *$\mathcal{X}$-resolutions*. We have adopted the terminology *proper* from [3, Section 4].

2.4 (Derived Functors). Consider an additive functor $T\colon \mathcal{C} \to \mathcal{E}$ between abelian categories. Let us assume that T is covariant, say. Then (as usual) we can define the n^{th} left derived functor

$$\mathrm{L}_n^{\mathcal{X}} T\colon \mathsf{LeftRes}_{\mathcal{C}}(\mathcal{X}) \to \mathcal{E}$$

of T, with respect to the class $\mathcal{X}$, by setting $\mathrm{L}_n^{\mathcal{X}}T(M) = \mathrm{H}_n(T(\boldsymbol{X}))$, where $\boldsymbol{X}$ is any proper left $\mathcal{X}$-resolution of $M \in \mathsf{LeftRes}_{\mathcal{C}}(\mathcal{X})$. Similarly, the n^{th} right derived functor

$$\mathrm{R}^n_{\mathcal{X}}T \colon \mathsf{RightRes}_{\mathcal{C}}(\mathcal{X}) \to \mathcal{E}$$

of T with respect to $\mathcal{X}$ is defined by $\mathrm{R}^n_{\mathcal{X}}T(N) = \mathrm{H}_n(T(\boldsymbol{Y}))$, where $\boldsymbol{Y}$ is any proper right $\mathcal{X}$-resolution of $N \in \mathsf{RightRes}_{\mathcal{C}}(\mathcal{X})$. These constructions are well-defined and functorial in the arguments M and N by Proposition 2.2.

The situation where T is contravariant is handled similarly. We refer to [9, Section 8.2] for a more detailed discussion on this matter.

2.5 (Balanced Functors). Next we consider yet another abelian category $\mathcal{D}$, together with a full subcategory $\mathcal{Y} \subseteq \mathcal{D}$ and an additive functor $F \colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ in *two* variables. We will assume that F is contravariant in the first variable, and covariant in the second variable.

Actually, the variance of the variables of F is not important, and the definitions and results below can easily be modified to fit the situation where F is covariant in both variables, say.

For fixed $M \in \mathcal{C}$ and $N \in \mathcal{D}$ we can then consider the two right derived functors as in 2.4:

$$\mathrm{R}^n_{\mathcal{X}}F(-,N) \colon \mathsf{LeftRes}_{\mathcal{C}}(\mathcal{X}) \to \mathcal{E} \qquad \text{and} \qquad \mathrm{R}^n_{\mathcal{Y}}F(M,-) \colon \mathsf{RightRes}_{\mathcal{D}}(\mathcal{Y}) \to \mathcal{E}.$$

If furthermore $M \in \mathsf{LeftRes}_{\mathcal{C}}(\mathcal{X})$ and $N \in \mathsf{RightRes}_{\mathcal{D}}(\mathcal{Y})$, we can ask for a sufficient condition to ensure that

$$\mathrm{R}^n_{\mathcal{X}}F(M,N) \;\cong\; \mathrm{R}^n_{\mathcal{Y}}F(M,N),$$

functorial in M and N. Here we wrote $\mathrm{R}^n_{\mathcal{X}}F(M,N)$ for the functor $\mathrm{R}^n_{\mathcal{X}}F(-,N)$ applied to M. Another, and perhaps better, notation could be

$$\mathrm{R}^n_{\mathcal{X}}F(-,N)[M].$$

Enochs and Jenda have in [5] developed a machinery for answering such questions. They operate with the term *left/right balanced functor* (hence the headline), which we will not define here (but the reader might consult [5, Definition 2.1]). Instead we shall focus on the following result:

Theorem 2.6. *Consider the functor $F \colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which is contravariant in the first variable and covariant in the second variable, together with the full subcategories $\mathcal{X} \subseteq \mathcal{C}$ and $\mathcal{Y} \subseteq \mathcal{D}$. Assume that we have full subcategories $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{Y}}$ of $\mathsf{LeftRes}_{\mathcal{C}}(\mathcal{X})$ and $\mathsf{RightRes}_{\mathcal{D}}(\mathcal{Y})$, respectively, satisfying:*

- *(i)* $\mathcal{X} \subseteq \widetilde{\mathcal{X}}$ *and* $\mathcal{Y} \subseteq \widetilde{\mathcal{Y}}$.
- *(ii) Every $M \in \widetilde{\mathcal{X}}$ has an augmented proper left $\mathcal{X}$-resolution $\cdots \to X_1 \to X_0 \to M \to 0$, such that $0 \to F(M,Y) \to F(X_0,Y) \to F(X_1,Y) \to \cdots$ is exact for all $Y \in \mathcal{Y}$.*
- *(iii) Every $N \in \widetilde{\mathcal{Y}}$ has an augmented proper right $\mathcal{Y}$-resolution $0 \to N \to Y^0 \to Y^1 \to \cdots$, such that $0 \to F(X,N) \to F(X,Y^0) \to F(X,Y^1) \to \cdots$ is exact for all $X \in \mathcal{X}$.*

Then we have functorial isomorphisms

$$\mathrm{R}^n_{\mathcal{X}}F(M,N) \;\cong\; \mathrm{R}^n_{\mathcal{Y}}F(M,N),$$

for all $M \in \widetilde{\mathcal{X}}$ and $N \in \widetilde{\mathcal{Y}}$.

Proof. Please see [5, Proposition 2.3]. That the isomorphisms are functorial follows from the construction. The functoriality becomes more clear if one consults the proof of [9, Proposition 8.2.14], or the proofs of [14, Theorems 2.7.2 and 2.7.6]. $\square$

In the next paragraphs we apply the results above to special categories $\mathcal{X}$, $\widetilde{\mathcal{X}}$, $\mathcal{C}$ and $\mathcal{Y}$, $\widetilde{\mathcal{Y}}$, $\mathcal{D}$, including the categories mentioned in 1.1. For completeness we include a definition of Gorenstein projective, Gorenstein injective and Gorenstein flat modules:

Definition 2.7. A *complete projective resolution* is an exact sequence of projective modules,

$$\boldsymbol{P} \;=\; \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots\,,$$

such that $\mathrm{Hom}_R(\boldsymbol{P}, Q)$ is exact for every projective R-module Q. An R-module M is called *Gorenstein projective* (*G-projective* for short), if there exists a complete projective resolution $\boldsymbol{P}$ with $M \cong \mathrm{Im}(P_0 \to P_{-1})$. *Gorenstein injective* (*G-injective* for short) modules are defined dually.

A *complete flat resolution* is an exact sequence of flat (left) R-modules,

$$\boldsymbol{F} \;=\; \cdots \to F_1 \to F_0 \to F_{-1} \to \cdots\,,$$

such that $I \otimes_R \boldsymbol{F}$ is exact for every injective right R-module I. An R-module M is called *Gorenstein flat* (*G-flat* for short), if there exists a complete flat resolution $\boldsymbol{F}$ with $M \cong \mathrm{Im}(F_0 \to F_{-1})$.

3. Gorenstein deriving $\mathrm{Hom}_R(-,-)$

We now return to categories of *modules*. We use $\widetilde{\mathcal{GP}}$, $\widetilde{\mathcal{GI}}$ and $\widetilde{\mathcal{GF}}$ to denote the class of R-modules with finite Gorenstein projective dimension, finite Gorenstein injective dimension, and finite Gorenstein flat dimension, respectively.

Recall that every projective module is Gorenstein projective. Consequently, $\mathcal{GP}$-precovers are always surjective, and $\widetilde{\mathcal{GP}}$ contains all modules with finite projective dimension.

We now consider the functor $\mathrm{Hom}_R(-,-)\colon \mathcal{M} \times \mathcal{M} \to \mathcal{A}$, together with the categories

$$\mathcal{X} \;=\; \mathcal{GP}\;,\; \widetilde{\mathcal{X}} \;=\; \widetilde{\mathcal{GP}} \qquad \text{and} \qquad \mathcal{Y} \;=\; \mathcal{GI}\;,\; \widetilde{\mathcal{Y}} \;=\; \widetilde{\mathcal{GI}}.$$

In this case we define, in the sense of section 2.4,

$$\mathrm{Ext}^n_{\mathcal{GP}}(-,N) \;=\; \mathrm{R}^n_{\mathcal{GP}}\mathrm{Hom}_R(-,N) \qquad \text{and} \qquad \mathrm{Ext}^n_{\mathcal{GI}}(M,-) \;=\; \mathrm{R}^n_{\mathcal{GI}}\mathrm{Hom}_R(M,-),$$

for fixed R-modules M and N. We wish, of course, to apply Theorem 2.6 to this situation. Note that by [12, Proposition 2.18], we have:

Proposition 3.1. *If M is an R-module with $\mathrm{Gpd}_R M < \infty$, then there exists a short exact sequence $0 \to K \to G \to M \to 0$, where $G \to M$ is a $\mathcal{GP}$-precover of M (please see Remark 2.3), and $\mathrm{pd}_R K = \mathrm{Gpd}_R M - 1$ (in the case where M is Gorenstein projective, this should be interpreted as $K = 0$).*

Consequently, every R-module with finite Gorenstein projective dimension has a proper left $\mathcal{GP}$-resolution (that is, there is an inclusion $\widetilde{\mathcal{GP}} \subseteq \mathsf{LeftRes}_{\mathcal{M}}(\mathcal{GP})$).

Furthermore, we will need the following from [12, Theorem 2.13]:

Theorem 3.2. *Let M be any R-module with $\mathrm{Gpd}_R M < \infty$. Then*

$$\mathrm{Gpd}_R M \;=\; \sup\{n \geqslant 0 \mid \mathrm{Ext}^n_R(M,L) \neq 0 \text{ for some } R\text{-module } L \text{ with } \mathrm{pd}_R L < \infty\}.$$

Remark 3.3. It may be useful to compare Theorem 3.2 to the classical projective dimension, which for an R-module M is given by

$$\mathrm{pd}_R M \;=\; \{n \geqslant 0 \mid \mathrm{Ext}_R^n(M,L) \neq 0 \text{ for some } R\text{-module } L\}.$$

It also follows that if $\mathrm{pd}_R M < \infty$, then every projective resolution of M is actually a proper left $\mathcal{GP}$-resolution of M.

Lemma 3.4. *Assume that M is an R-module with finite Gorenstein projective dimension, and let $\boldsymbol{G}^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $\mathcal{GP}$-resolution of M (which exists by Proposition 3.1). Then* $\mathrm{Hom}_R(\boldsymbol{G}^+, H)$ *is exact for all Gorenstein injective modules H.*

Proof. We split the proper resolution $\boldsymbol{G}^+$ into short exact sequences. Hence it suffices to show exactness of $\mathrm{Hom}_R(\boldsymbol{S}, H)$ for all Gorenstein injective modules H and all short exact sequences

$$\boldsymbol{S} = 0 \to K \to G \to M \to 0\ ,$$

where $G \to M$ is a $\mathcal{GP}$-precover of some module M with $\mathrm{Gpd}_R M < \infty$ (recall that $\mathcal{GP}$-precovers are always surjective). By Proposition 3.1, there is a special short exact sequence,

$$\boldsymbol{S}' = \ 0 \longrightarrow K' \xrightarrow{\ \iota\ } G' \xrightarrow{\ \pi\ } M \longrightarrow 0\ ,$$

where $\pi\colon G' \to M$ is a $\mathcal{GP}$-precover and $\mathrm{pd}_R K' < \infty$.

It is easy to see (as in Proposition 2.2) that the complexes $\boldsymbol{S}$ and $\boldsymbol{S}'$ are homotopy equivalent, and thus so are the complexes $\mathrm{Hom}_R(\boldsymbol{S}, H)$ and $\mathrm{Hom}_R(\boldsymbol{S}', H)$ for every (Gorenstein injective) module H. Hence it suffices to show the exactness of $\mathrm{Hom}_R(\boldsymbol{S}', H)$ whenever H is Gorenstein injective.

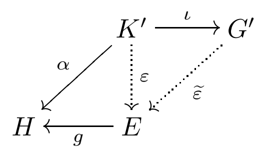
Now let H be any Gorenstein injective module. We need to prove the exactness of

$$\mathrm{Hom}_R(G',H) \xrightarrow{\ \mathrm{Hom}_R(\iota,H)\ } \mathrm{Hom}_R(K',H) \longrightarrow 0\ .$$

To see this, let $\alpha\colon K' \to H$ be any homomorphism. We wish to find $\varrho\colon G' \to H$ such that $\varrho\iota = \alpha$. Now pick an exact sequence

$$0 \longrightarrow \widetilde{H} \longrightarrow E \xrightarrow{\ g\ } H \longrightarrow 0\ ,$$

where E is injective, and $\widetilde{H}$ is Gorenstein injective (the sequence in question is just a part of the complete injective resolution that defines H). Since $\widetilde{H}$ is Gorenstein injective and $\mathrm{pd}_R K' < \infty$, we get $\mathrm{Ext}_R^1(K', \widetilde{H}) = 0$ by [7, Lemma 1.3], and thus a lifting $\varepsilon\colon K' \to E$ with $g\varepsilon = \alpha$:



Next, injectivity of E gives $\widetilde{\varepsilon}\colon G' \to E$ with $\widetilde{\varepsilon}\iota = \varepsilon$. Now $\varrho = g\widetilde{\varepsilon}\colon G' \to H$ is the desired map. $\square$

With a similar proof we get:

Lemma 3.5. *Assume that N is an R-module with finite Gorenstein injective dimension, and let $\boldsymbol{H}^+ = 0 \to N \to H^0 \to H^1 \to \cdots$ be an augmented proper right $\mathcal{GI}$-resolution of N (which exists by the dual of Proposition 3.1). Then $\mathrm{Hom}_R(G, \boldsymbol{H}^+)$ is exact for all Gorenstein projective modules G.* □

Comparing Lemmas 3.4 and 3.5 with Theorem 2.6, we obtain:

Theorem 3.6. *For all R-modules M and N with $\mathrm{Gpd}_R M < \infty$ and $\mathrm{Gid}_R N < \infty$, we have isomorphisms*

$$\mathrm{Ext}^n_{\mathcal{GP}}(M,N) \;\cong\; \mathrm{Ext}^n_{\mathcal{GI}}(M,N),$$

which are functorial in M and N. □

3.7 (Definition of GExt). Let M and N be R-modules with $\mathrm{Gpd}_R M < \infty$ and $\mathrm{Gid}_R N < \infty$. Then we write

$$\mathrm{GExt}^n_R(M,N) \;:=\; \mathrm{Ext}^n_{\mathcal{GP}}(M,N) \;\cong\; \mathrm{Ext}^n_{\mathcal{GI}}(M,N)$$

for the isomorphic abelian groups in Theorem 3.6 above.

Naturally we want to compare GExt with the classical Ext. This is done in:

Theorem 3.8. *Let M and N be any R-modules. Then the following conclusions hold:*

(i) *There are natural isomorphisms $\mathrm{Ext}^n_{\mathcal{GP}}(M,N) \cong \mathrm{Ext}^n_R(M,N)$ under each of the conditions*

$$(\dagger)\ \ \mathrm{pd}_R M < \infty \qquad \textit{or} \qquad (\ddagger)\ \ M \in \mathsf{LeftRes}_{\mathcal{M}}(\mathcal{GP}) \ \ \textit{and}\ \ \mathrm{id}_R N < \infty.$$

(ii) *There are natural isomorphisms $\mathrm{Ext}^n_{\mathcal{GI}}(M,N) \cong \mathrm{Ext}^n_R(M,N)$ under each of the conditions*

$$(\dagger)\ \ \mathrm{id}_R N < \infty \qquad \textit{or} \qquad (\ddagger)\ \ N \in \mathsf{RightRes}_{\mathcal{M}}(\mathcal{GI}) \ \ \textit{and}\ \ \mathrm{pd}_R M < \infty.$$

(iii) *Assume that $\mathrm{Gpd}_R M < \infty$ and $\mathrm{Gid}_R N < \infty$. If either $\mathrm{pd}_R M < \infty$ or $\mathrm{id}_R N < \infty$, then*

$$\mathrm{GExt}^n_R(M,N) \cong \mathrm{Ext}^n_R(M,N)$$

is functorial in M and N.

Proof. (*i*) Assume that $\mathrm{pd}_R M < \infty$, and pick any projective resolution $\boldsymbol{P}$ of M. By Remark 3.3, $\boldsymbol{P}$ is also a proper left $\mathcal{GP}$-resolution of M, and thus

$$\mathrm{Ext}^n_{\mathcal{GP}}(M,N) \;=\; \mathrm{H}^n(\mathrm{Hom}_R(\boldsymbol{P},N)) \;=\; \mathrm{Ext}^n_R(M,N).$$

In the case where $M \in \mathsf{LeftRes}_{\mathcal{M}}(\mathcal{GP})$ and $\mathrm{id}_R N = m < \infty$, we see that Gorenstein projective modules are acyclic for the functor $\mathrm{Hom}_R(-,N)$, that is, $\mathrm{Ext}^i_R(G,N) = 0$ (the usual Ext) for every Gorenstein projective module G, and every integer $i > 0$.

This is because, if G is a Gorenstein projective module, and $i > 0$ is an integer, then there exists an exact sequence $0 \to G \to Q^0 \to \cdots \to Q^{m-1} \to C \to 0$, where $Q^0, \ldots, Q^{m-1}$ are projective modules. Breaking this exact sequence into short exact ones, and applying $\mathrm{Hom}_R(-,N)$, we get $\mathrm{Ext}^i_R(G,N) \cong \mathrm{Ext}^{m+i}_R(C,N) = 0$, as claimed.

Therefore [11, Chapter III, Proposition 1.2A] implies that $\mathrm{Ext}^n_R(-,N)$ can be computed using (proper) left Gorenstein projective resolutions of the argument in the first variable, as desired.

The proof of (*ii*) is similar. The claim (*iii*) is a direct consequence of (*i*) and (*ii*), together with the Definition 3.7 of $\mathrm{GExt}^n_R(-,-)$. □

4. Gorenstein deriving $-\otimes_R -$

In dealing with the tensor product we need, of course, both left and right R-modules. Thus the following addition to Notation 1.1 is needed:

If $\mathcal{C}$ is any of the categories in Notation 1.1 ($\mathcal{M}$, $\mathcal{GP}$, etc.), we write ${}_R\mathcal{C}$, respectively, $\mathcal{C}_R$, for the category of left, respectively, right, R-modules with the property describing the modules in $\mathcal{C}$.

Now we consider the functor $-\otimes_R -\colon \mathcal{M}_R \times {}_R\mathcal{M} \to \mathcal{A}$. For fixed $M \in \mathcal{M}_R$ and $N \in {}_R\mathcal{M}$ we define, in the sense of section 2.4:

$$\mathrm{Tor}_n^{\mathcal{GP}_R}(-,N) := \mathrm{L}_n^{\mathcal{GP}_R}(-\otimes_R N) \quad \text{and} \quad \mathrm{Tor}_n^{{}_R\mathcal{GP}}(M,-) := \mathrm{L}_n^{{}_R\mathcal{GP}}(M\otimes_R -),$$

together with

$$\mathrm{Tor}_n^{\mathcal{GF}_R}(-,N) := \mathrm{L}_n^{\mathcal{GF}_R}(-\otimes_R N) \quad \text{and} \quad \mathrm{Tor}_n^{{}_R\mathcal{GF}}(M,-) := \mathrm{L}_n^{{}_R\mathcal{GF}}(M\otimes_R -).$$

The first two Tor*s* use proper left Gorestein *projective* resolutions, and the last two Tor*s* use proper left Gorenstein *flat* resolutions. In order to compare these different Tor*s*, we wish, of course, to apply (a version of) Theorem 2.6 to different combinations of

$$(\mathcal{X}\ ,\ \widetilde{\mathcal{X}}) \ = \ (\mathcal{GP}_R\ ,\ \widetilde{\mathcal{GP}}_R) \quad \text{or} \quad (\mathcal{GF}_R\ ,\ \widetilde{\mathcal{GF}}_R),$$

and

$$(\mathcal{Y}\ ,\ \widetilde{\mathcal{Y}}) \ = \ ({}_R\mathcal{GP}\ ,\ {}_R\widetilde{\mathcal{GP}}) \quad \text{or} \quad ({}_R\mathcal{GF}\ ,\ {}_R\widetilde{\mathcal{GF}}),$$

namely, the covariant-covariant version of Theorem 2.6, instead of the stated contravariant-covariant version. We will need the classical notion:

Definition 4.1. The *left finitistic projective dimension* $\mathsf{Left}\mathrm{FPD}(R)$ of R is defined as

$$\mathsf{Left}\mathrm{FPD}(R) = \sup\{\mathrm{pd}_R M \mid M \text{ is a } \textit{left } R\text{-module with } \mathrm{pd}_R M < \infty\}.$$

The right finitistic projective dimension $\mathsf{Right}\mathrm{FPD}(R)$ of R is defined similarly.

Remark 4.2. When R is commutative and Noetherian, the dimensions $\mathsf{Left}\mathrm{FPD}(R)$ and $\mathsf{Right}\mathrm{FPD}(R)$ coincide and are equal to the Krull dimension of R, by [10, Théorème (3.2.6) (Seconde partie)].

We will need the following three results, [12, Proposition 3.3], [12, Theorem 3.5] and [12, Proposition 3.18], respectively:

Proposition 4.3. *If R is right coherent with finite $\mathsf{Left}\mathrm{FPD}(R)$, then every Gorenstein projective left R-module is also Gorenstein flat. That is, there is an inclusion ${}_R\mathcal{GP} \subseteq {}_R\mathcal{GF}$.* □

Theorem 4.4. *For any left R-module M, we consider the following three conditions:*

- (i) *The left R-module M is G-flat.*
- (ii) *The Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$ (which is a right R-module) is G-injective.*
- (iii) *M has an augmented proper right resolution $0 \to M \to F^0 \to F^1 \to \cdots$ consisting of flat left R-modules, and $\mathrm{Tor}_i^R(I,M) = 0$ for all injective right R-modules I, and all $i > 0$.*

The implication (i) ⇒ (ii) always holds. If R is right coherent, then also (ii) ⇒ (iii) ⇒ (i), and hence all three conditions are equivalent. □

Proposition 4.5. *Assume that R is right coherent. If M is a left R-module with $\mathrm{Gfd}_R M < \infty$, then there exists a short exact sequence $0 \to K \to G \to M \to 0$, where $G \to M$ is an ${}_R\mathcal{GF}$-precover of M, and $\mathrm{fd}_R K = \mathrm{Gfd}_R M - 1$ (in the case where M is Gorenstein flat, this should be interpreted as $K = 0$).*

In particular, every left R-module with finite Gorenstein flat dimension has a proper left ${}_R\mathcal{GF}$-resolution (that is, there is an inclusion ${}_R\widetilde{\mathcal{GF}} \subseteq \mathsf{LeftRes}_{{}_R\mathcal{M}}({}_R\mathcal{GF})$). □

Our first result is:

Lemma 4.6. *Let M be a left R-module with $\mathrm{Gpd}_R M < \infty$, and let $\boldsymbol{G}^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left ${}_R\mathcal{GP}$-resolution of M (which exists by Proposition 3.1). Then the following conclusions hold:*

(*i*) *$T \otimes_R \boldsymbol{G}^+$ is exact for all Gorenstein flat right R-modules T.*

(*ii*) *If R is left coherent with finite $\mathsf{RightFPD}(R)$, then $T \otimes_R \boldsymbol{G}^+$ is exact for all Gorenstein projective right R-modules T.*

Proof. (*i*) By Theorem 4.4 above, the Pontryagin dual $H = \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{Q}/\mathbb{Z})$ is a Gorenstein injective left R-module. Hence $\mathrm{Hom}_R(\boldsymbol{G}^+, H) \cong \mathrm{Hom}_{\mathbb{Z}}(T \otimes_R \boldsymbol{G}^+, \mathbb{Q}/\mathbb{Z})$ is exact by Proposition 3.4. Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, $T \otimes_R \boldsymbol{G}^+$ is exact too.

(*ii*) With the given assumptions on R, the dual of Proposition 4.3 implies that every Gorenstein projective right R-module also is Gorenstein flat. □

Lemma 4.7. *Assume that R is right coherent with finite $\mathsf{LeftFPD}(R)$. Let M be a left R-module with $\mathrm{Gfd}_R M < \infty$, and let $\boldsymbol{G}^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left ${}_R\mathcal{GF}$-resolution of M (which exists by Proposition 4.5, since R is right coherent). Then the following conclusions hold:*

(*i*) *$\mathrm{Hom}_R(\boldsymbol{G}^+, H)$ is exact for all Gorenstein injective left R-modules H.*

(*ii*) *$T \otimes_R \boldsymbol{G}^+$ is exact for all Gorenstein flat right R-modules T.*

(*iii*) *If R is also left coherent with finite $\mathsf{RightFPD}(R)$, then $T \otimes_R \boldsymbol{G}^+$ is exact for all Gorenstein projective right R-modules T.*

Proof. (*i*) Since $\mathrm{Gfd}_R M < \infty$ and R is right coherent, Proposition 4.5 gives a special short exact sequence $0 \to K' \to G' \to M \to 0$, where $G' \to M$ is an ${}_R\mathcal{GF}$-precover of M, and $\mathrm{fd}_R K' < \infty$. Since R has $\mathsf{LeftFPD}(R) < \infty$, [13, Proposition 6] implies that also $\mathrm{pd}_R K' < \infty$. Now the proof of Lemma 3.4 applies.

(*ii*) If T is a Gorenstein flat right R-module, then the left R-module $H = \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{Q}/\mathbb{Z})$ is Gorenstein injective, by (the dual of) Theorem 4.4 above. By the result (*i*), just proved, we have exactness of

$$\mathrm{Hom}_R(\boldsymbol{G}^+, H) \;\cong\; \mathrm{Hom}_{\mathbb{Z}}(T \otimes_R \boldsymbol{G}^+, \mathbb{Q}/\mathbb{Z}).$$

Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, we also have exactness of $T \otimes_R \boldsymbol{G}^+$, as desired.

(*iii*) Under the extra assumptions on R, the dual of Proposition 4.3 implies that every Gorenstein projective right R-module is also Gorenstein flat. Thus (*iii*) follows from (*ii*). □

Theorem 4.8. *Assume that R is both left and right coherent, and that both $\mathsf{LeftFPD}(R)$ and $\mathsf{RightFPD}(R)$ are finite. For every right R-module M, and every left R-module N, the following conclusions hold:*

(*i*) *If* $\mathrm{Gfd}_R M < \infty$ *and* $\mathrm{Gfd}_R N < \infty$*, then*

$$\mathrm{Tor}_n^{\mathcal{GF}_R}(M,N) \cong \mathrm{Tor}_n^{{}_R\mathcal{GF}}(M,N).$$

(*ii*) *If* $\mathrm{Gpd}_R M < \infty$ *and* $\mathrm{Gfd}_R N < \infty$*, then*

$$\mathrm{Tor}_n^{\mathcal{GP}_R}(M,N) \cong \mathrm{Tor}_n^{\mathcal{GF}_R}(M,N) \cong \mathrm{Tor}_n^{{}_R\mathcal{GF}}(M,N).$$

(*iii*) *If* $\mathrm{Gfd}_R M < \infty$ *and* $\mathrm{Gpd}_R N < \infty$*, then*

$$\mathrm{Tor}_n^{\mathcal{GF}_R}(M,N) \cong \mathrm{Tor}_n^{{}_R\mathcal{GP}}(M,N) \cong \mathrm{Tor}_n^{{}_R\mathcal{GF}}(M,N).$$

(*iv*) *If* $\mathrm{Gpd}_R M < \infty$ *and* $\mathrm{Gpd}_R N < \infty$*, then*

$$\mathrm{Tor}_n^{\mathcal{GP}_R}(M,N) \cong \mathrm{Tor}_n^{\mathcal{GF}_R}(M,N) \cong \mathrm{Tor}_n^{{}_R\mathcal{GP}}(M,N) \cong \mathrm{Tor}_n^{{}_R\mathcal{GF}}(M,N).$$

All the isomorphisms are functorial in M *and* N*.*

Proof. Use Lemmas 4.6 and 4.7 as input in the covariant-covariant version of Theorem 2.6. □

4.9 (Definition of gTor and GTor). Assume that R is both left and right coherent, and that both $\mathsf{LeftFPD}(R)$ and $\mathsf{RightFPD}(R)$ are finite. Furthermore, let M be a right R-module, and let N be a left R-module. If $\mathrm{Gfd}_R M < \infty$ and $\mathrm{Gfd}_R N < \infty$, then we write

$$g\mathrm{Tor}_n^R(M,N) \;:=\; \mathrm{Tor}_n^{\mathcal{GF}_R}(M,N) \;\cong\; \mathrm{Tor}_n^{{}_R\mathcal{GF}}(M,N)$$

for the isomorphic abelian groups in Theorem 4.8(i). If $\mathrm{Gpd}_R M < \infty$ and $\mathrm{Gpd}_R N < \infty$, then we write

$$\mathrm{GTor}_n^R(M,N) \;:=\; \mathrm{Tor}_n^{\mathcal{GP}_R}(M,N) \;\cong\; \mathrm{Tor}_n^{{}_R\mathcal{GP}}(M,N)$$

for the isomorphic abelian groups in Theorem 4.8(iv).

We can now reformulate some of the content of Theorem 4.8:

Theorem 4.10. *Assume that* R *is both left and right coherent, and that both* $\mathsf{LeftFPD}(R)$ *and* $\mathsf{RightFPD}(R)$ *are finite. For every right* R*-module* M *with finite* $\mathrm{Gpd}_R M$*, and for every left* R*-module* N *with* $\mathrm{Gpd}_R N < \infty$*, we have isomorphisms:*

$$g\mathrm{Tor}_n^R(M,N) \;\cong\; \mathrm{GTor}_n^R(M,N)$$

that are functorial in M *and* N*.*

Finally we compare gTor (and hence GTor) with the usual Tor.

Theorem 4.11. *Assume that* R *is both left and right coherent, and that both* $\mathsf{LeftFPD}(R)$ *and* $\mathsf{RightFPD}(R)$ *are finite. Furthermore, let* M *be a right* R*-module with* $\mathrm{Gfd}_R M < \infty$*, and let* N *be a left* R*-module with* $\mathrm{Gfd}_R N < \infty$*. If either* $\mathrm{fd}_R M < \infty$ *or* $\mathrm{fd}_R N < \infty$*, then there are isomorphisms*

$$g\mathrm{Tor}_n^R(M,N) \cong \mathrm{Tor}_n^R(M,N)$$

that are functorial in M *and* N*.*

Proof. If $\mathrm{fd}_R M < \infty$, then we also have $\mathrm{pd}_R M < \infty$ by [13, Proposition 6] (since $\mathsf{RightFPD}(R) < \infty$). Let $\boldsymbol{P}$ be any projective resolution of M. As noted in Remark 3.3, $\boldsymbol{P}$ is also a proper left $\mathcal{GP}_R$-resolution of M. Hence, Theorem 4.8(ii) and the definitions give:

$$g\mathrm{Tor}_n^R(M,N) \;=\; \mathrm{Tor}_n^{\mathcal{GP}_R}(M,N) \;=\; \mathrm{H}_n(\boldsymbol{P} \otimes_R N) \;=\; \mathrm{Tor}_n^R(M,N),$$

as desired. □

Acknowledgments

I would like to express my gratitude to my Ph.D. advisor Hans-Bjørn Foxby for his support and our helpful discussions. Furthermore, I would like to thank the referee for correcting many of my misprints.

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