Advances in Mathematics 357 (2019) 106826

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Model categories of quiver representations

Henrik Holm [a,*], Peter Jørgensen [b]

[a] *Department of Mathematical Sciences, Universitetsparken 5, University of Copenhagen, 2100 Copenhagen Ø, Denmark*
[b] *School of Mathematics and Statistics, Newcastle University, Newcastle upon Tyne NE1 7RU, United Kingdom*

ARTICLE INFO

Article history:
Received 29 April 2019
Received in revised form 19 August 2019
Accepted 11 September 2019
Available online xxxx
Communicated by Ross Street

MSC:
18E30
18E35
18G55

Keywords:
Abelian model categories
Chain complexes
Cotorsion pairs
Gillespie's and Hovey's Theorems
N-complexes
Periodic chain complexes

ABSTRACT

Gillespie's Theorem gives a systematic way to construct model category structures on $\mathscr{C}(\mathscr{M})$, the category of chain complexes over an abelian category $\mathscr{M}$.
We can view $\mathscr{C}(\mathscr{M})$ as the category of representations of the quiver $\cdots \to 2 \to 1 \to 0 \to -1 \to -2 \to \cdots$ with the relations that two consecutive arrows compose to 0. This is a self-injective quiver with relations, and we generalise Gillespie's Theorem to other such quivers with relations. There is a large family of these, and following Iyama and Minamoto, their representations can be viewed as generalised chain complexes. Our result gives a systematic way to construct model category structures on many categories. This includes the category of N-periodic chain complexes, the category of N-complexes where $\partial^N = 0$, and the category of representations of the repetitive quiver $\mathbb{Z}A_n$ with mesh relations.
© 2019 Elsevier Inc. All rights reserved.

* Corresponding author.
E-mail addresses: holm@math.ku.dk (H. Holm), peter.jorgensen@ncl.ac.uk (P. Jørgensen).
URLs: http://www.math.ku.dk/~holm/ (H. Holm), http://www.staff.ncl.ac.uk/peter.jorgensen (P. Jørgensen).

https://doi.org/10.1016/j.aim.2019.106826
0001-8708/© 2019 Elsevier Inc. All rights reserved.

0. Introduction

Gillespie's Theorem permits the construction of model category structures on categories of chain complexes. We will generalise it to representations of self-injective quivers with relations, which can be viewed as generalised chain complexes by the work of Iyama and Minamoto, see [17] and [18, sec. 2].

0.i. Outline

Let $\mathscr{M}$ be an abelian category. An abelian model category structure on $\mathscr{C}(\mathscr{M})$, the category of chain complexes over $\mathscr{M}$, consists of three classes of morphisms, $(\mathrm{fib}, \mathrm{cof}, \mathrm{weq})$, known as fibrations, cofibrations, and weak equivalences, subject to several axioms, see [15, def. 2.1] and [26, sec. I.1]. It provides an extensive framework for the construction and manipulation of the localisation $\mathrm{weq}^{-1}\mathscr{C}(\mathscr{M})$, where the morphisms in weq have been inverted formally. Some of the localisations thus obtained are of considerable interest, not least the derived category $\mathscr{D}(\mathscr{M})$.

Hovey's Theorem says that each abelian model category structure can be constructed from two so-called complete, compatible cotorsion pairs, see Theorem 0.2. This motivates Gillespie's Theorem, which takes a hereditary cotorsion pair in $\mathscr{M}$ and produces two compatible cotorsion pairs in $\mathscr{C}(\mathscr{M})$, see Theorem 0.3.

Gillespie's Theorem can be viewed as a result on quiver representations since $\mathscr{C}(\mathscr{M})$ is the category of representations of Q with values in $\mathscr{M}$, where Q is the following self-injective quiver with relations.

$$\begin{cases} \text{Quiver:} & \cdots \to 2 \to 1 \to 0 \to -1 \to -2 \to \cdots \\ \text{Relations:} & \text{Two consecutive arrows compose to 0.} \end{cases} \tag{0.1}$$

The notion of self-injectivity is made precise in Paragraph 2.4. This paper will generalise Gillespie's Theorem to other self-injective quivers with relations. They form a large family, see for example Equations (0.3) and (0.4) and Section 0.viii.

Let k be a field, R a k-algebra, Q a self-injective quiver with relations over k, and let $\mathscr{X}$ be the category of representations of Q with values in ${}_R\mathrm{Mod}$, the category of R-left-modules. Our main theorem, Theorem A, takes a hereditary cotorsion pair in ${}_R\mathrm{Mod}$ and produces two compatible cotorsion pairs in $\mathscr{X}$. It specialises to Gillespie's Theorem for $\mathscr{M} = {}_R\mathrm{Mod}$ if Q is the quiver with relations from (0.1).

0.ii. Cotorsion pairs

Let $\mathscr{Y}$ be an abelian category. If Γ and Δ are classes of objects of $\mathscr{Y}$, then we write

$$\Gamma^{\perp} = \{\, Y \in \mathscr{Y} \mid \mathrm{Ext}^1_{\mathscr{Y}}(C, Y) = 0 \text{ for } C \in \Gamma \,\}$$
$${}^{\perp}\Delta = \{\, Y \in \mathscr{Y} \mid \mathrm{Ext}^1_{\mathscr{Y}}(Y, D) = 0 \text{ for } D \in \Delta \,\}.$$

Definition 0.1. Recall the following from the literature.

(i) A *cotorsion pair* in $\mathscr{Y}$ is a pair (Γ, Δ) of classes of objects of $\mathscr{Y}$ such that $\Gamma = {}^{\perp}\Delta$ and $\Gamma^{\perp} = \Delta$, see [28, p. 12]. A cotorsion pair (Γ, Δ) is determined by each of the classes Γ and Δ, because it is equal to $(\Gamma, \Gamma^{\perp})$ and to $({}^{\perp}\Delta, \Delta)$.

(ii) The cotorsion pair (Γ, Δ) in $\mathscr{Y}$ is *complete* if each $Y \in \mathscr{Y}$ permits short exact sequences $0 \to D \to C \to Y \to 0$ and $0 \to Y \to D' \to C' \to 0$ with $C, C' \in \Gamma$ and $D, D' \in \Delta$, see [12, lem. 5.20].

(iii) The cotorsion pair (Γ, Δ) is *hereditary* if Γ is closed under kernels of epimorphisms and Δ is closed under cokernels of monomorphisms, see [12, lem. 5.24].

(iv) The cotorsion pairs $(\Phi, \Phi^{\perp})$ and $({}^{\perp}\Psi, \Psi)$ in $\mathscr{Y}$ are *compatible* if they satisfy the following conditions, see [10, sec. 1].
(Comp1) $\operatorname{Ext}^1_{\mathscr{Y}}(\Phi, \Psi) = 0$.

(Comp2) $\Phi \cap \Phi^{\perp} = {}^{\perp}\Psi \cap \Psi$.
Condition (Comp1) is equivalent to $\Phi \subseteq {}^{\perp}\Psi$ and to $\Phi^{\perp} \supseteq \Psi$. It is not symmetric in the two cotorsion pairs; their order matters.

Note that our definition of compatibility is weaker than Gillespie's from [11, def. 3.7], and that his cotorsion pairs $(\tilde{\mathscr{A}}, \operatorname{dg}\tilde{\mathscr{B}})$ and $(\operatorname{dg}\tilde{\mathscr{A}}, \tilde{\mathscr{B}})$ in $\mathscr{C}(\mathscr{M})$ from [11, prop. 3.6] are always compatible in our sense. Indeed, $\tilde{\mathscr{A}} \cap \operatorname{dg}\tilde{\mathscr{B}}$ and $\operatorname{dg}\tilde{\mathscr{A}} \cap \tilde{\mathscr{B}}$ are both equal to the class of split exact complexes with terms in $\mathscr{A} \cap \mathscr{B}$.

(v) Let (Γ, Δ) be a cotorsion pair in $\mathscr{Y}$, and let $\mathscr{C}$ be a class of objects in $\mathscr{Y}$. If $\Delta = \mathscr{C}^{\perp}$, then we say that (Γ, Δ) is *generated* by $\mathscr{C}$. If $\Gamma = {}^{\perp}\mathscr{C}$, then we say that (Γ, Δ) is *cogenerated* by $\mathscr{C}$. See [12, def. 5.15].

For example, if $\mathscr{Y}$ has enough projective objects, then $(\text{projective objects}, \mathscr{Y})$ is called the *projective cotorsion pair*. If $\mathscr{Y}$ has enough injective objects, then $(\mathscr{Y}, \text{injective objects})$ is called the *injective cotorsion pair*. These cotorsion pairs are complete and hereditary. Note that the triangulated version of compatible cotorsion pairs was investigated by Nakaoka under the name concentric twin cotorsion pair, see [24, def. 3.3].

0.iii. Hovey's Theorem: abelian model category structures

We will not reproduce Hovey's Theorem in full, but rather state the following result, which motivates the interest in compatible cotorsion pairs and dovetails with Gillespie's Theorem.

Theorem 0.2 *([9, prop. 2.3 and sec. 4.2], [10, thm. 1.1], [15, thm. 2.2]). Let $(\Phi, \Phi^{\perp})$ and $({}^{\perp}\Psi, \Psi)$ be complete, hereditary, compatible cotorsion pairs in the abelian category $\mathscr{Y}$.*

There is a class $\mathscr{W}$ of objects, often referred to as trivial, *characterised by*

$$\begin{aligned}\mathscr{W} &= \{\, Y \in \mathscr{Y} \mid \textit{there is a short exact sequence } 0 \to P \to F \to Y \to 0 \\ &\qquad \textit{with } P \in \Psi,\ F \in \Phi \,\} \\ &= \{\, Y \in \mathscr{Y} \mid \textit{there is a short exact sequence } 0 \to Y \to P' \to F' \to 0 \\ &\qquad \textit{with } P' \in \Psi,\ F' \in \Phi \,\}.\end{aligned}$$

Moreover, there is a model category structure on $\mathscr{Y}$ with

$$\begin{aligned}\mathrm{fib} &= \{\, \textit{epimorphisms with kernel in } \Phi^{\perp} \,\}, \\ \mathrm{cof} &= \{\, \textit{monomorphisms with cokernel in } {}^{\perp}\Psi \,\}, \\ \mathrm{weq} &= \left\{ \begin{array}{l} \textit{morphisms which factor as a monomorphism with cokernel in } \mathscr{W} \\ \textit{followed by an epimorphism with kernel in } \mathscr{W} \end{array} \right\},\end{aligned}$$

and the localisation $\mathrm{weq}^{-1}\mathscr{Y}$ *is triangulated.*

We will give an example after recalling Gillespie's Theorem.

0.iv. Gillespie's Theorem: chain complexes

Gillespie's Theorem gives a systematic way to construct compatible cotorsion pairs in the category of chain complexes. It requires the following setup.

- $\mathscr{M}$ is an abelian category with enough projective and enough injective objects.
- $\mathscr{C}(\mathscr{M})$ is the category of chain complexes over $\mathscr{M}$.
- For $q \in \mathbb{Z}$, consider the functors

$$\begin{tikzcd} \mathscr{M} \arrow[r, "S_q"] & \mathscr{C}(\mathscr{M}), \arrow[l, bend right, "C_q"'] \arrow[l, bend left, "K_q"] \end{tikzcd}$$

  where S_q sends M to the chain complex with M in degree q and zero everywhere else, and C_q and K_q are given by

$$C_q(X) = \mathrm{Coker}(\partial^X_{q+1}) \ \ , \ \ K_q(X) = \mathrm{Ker}(\partial^X_q).$$

  Here ∂^X_q is the qth differential of the chain complex X. There are adjoint pairs (C_q, S_q) and (S_q, K_q).

The following is Gillespie's Theorem.

Theorem 0.3 *([11, thm. 3.12 and cor. 3.13]). If $(\mathscr{A}, \mathscr{B})$ is a hereditary cotorsion pair in $\mathscr{M}$, then there are hereditary, compatible cotorsion pairs $\big(\Phi(\mathscr{A}), \Phi(\mathscr{A})^{\perp}\big)$ and $\big({}^{\perp}\Psi(\mathscr{B}), \Psi(\mathscr{B})\big)$ in $\mathscr{C}(\mathscr{M})$, where*

$$\Phi(\mathscr{A}) = \{\, X \in \mathscr{C}(\mathscr{M}) \mid \textit{If } q \in \mathbb{Z} \textit{ then } C_q(X) \in \mathscr{A} \textit{ and } \mathrm{H}_q(X) = 0 \,\},$$

$$\Psi(\mathscr{B}) = \{\, X \in \mathscr{C}(\mathscr{M}) \mid \textit{If } q \in \mathbb{Z} \textit{ then } K_q(X) \in \mathscr{B} \textit{ and } \mathrm{H}_q(X) = 0 \,\}.$$

For instance, the projective cotorsion pair $(\mathscr{A}, \mathscr{B}) = (\text{projective objects}, \mathscr{M})$ gives

$$\big(\Phi(\mathscr{A}), \Phi(\mathscr{A})^{\perp}\big) = (\mathscr{P}, \mathscr{P}^{\perp}) \ , \ \big({}^{\perp}\Psi(\mathscr{B}), \Psi(\mathscr{B})\big) = ({}^{\perp}\mathscr{E}, \mathscr{E}), \tag{0.2}$$

where $\mathscr{P}$ is the class of projective objects in $\mathscr{C}(\mathscr{M})$ and $\mathscr{E}$ is the class of exact chain complexes. Note that $\mathscr{P}^{\perp} = \mathscr{C}(\mathscr{M})$. The cotorsion pairs (0.2) are hereditary and compatible by Gillespie's Theorem. If $\mathscr{M}$ is a complete and cocomplete category, then the cotorsion pairs (0.2) are complete, and then Theorem 0.2 says that they determine an abelian model category structure on $\mathscr{C}(\mathscr{M})$. The associated localisation $\mathrm{weq}^{-1}\mathscr{C}(\mathscr{M})$ is the derived category $\mathscr{D}(\mathscr{M})$, see [9, thm. 5.3].

0.v. The main theorem: quiver representations

Our main theorem is a generalisation of Gillespie's Theorem to quiver representations. It requires the following setup, which we keep in the rest of the introduction.

- k is a field, R is a k-algebra, ${}_R\mathrm{Mod}$ is the category of R-left-modules.
- Q is a self-injective quiver with relations over k, see Paragraph 2.4.
- $\mathscr{X}$ is the category of representations of Q with values in ${}_R\mathrm{Mod}$. If $p \xrightarrow{\pi} q$ is an arrow in Q, then the corresponding homomorphism in $X \in \mathscr{X}$ is $X_p \xrightarrow{X_\pi} X_q$.
- For q an element of Q_0, the set of vertices of Q, consider the functors

$$\begin{tikzcd} {}_R\mathrm{Mod} \arrow[r, "S_q"] & \mathscr{X} \arrow[l, bend right, "C_q"'] \arrow[l, bend left, "K_q"] \end{tikzcd}$$

defined by:

$$C_q(-) = \mathrm{D}S\langle q\rangle \underset{Q}{\otimes} - \quad , \quad S_q(-) = S\langle q\rangle \underset{k}{\otimes} - \quad , \quad K_q(-) = \mathrm{Hom}_Q(S\langle q\rangle, -).$$

Here $S\langle q\rangle$ is the simple representation of Q supported at q. Its dual $\mathrm{D}S\langle q\rangle = \mathrm{Hom}_k(S\langle q\rangle, k)$ is the simple representation of the opposite quiver Q^{o} supported at q. The symbols $\underset{Q}{\otimes}$ and Hom_Q denote the tensor product and homomorphism functors of representations of Q. Note that S_q sends M to the representation with M at vertex q and zero everywhere else. There are adjoint pairs (C_q, S_q) and (S_q, K_q).

Our main theorem is the following.

Theorem A. *If $(\mathscr{A}, \mathscr{B})$ is a hereditary cotorsion pair in ${}_R\mathrm{Mod}$, then there are hereditary, compatible cotorsion pairs $\big(\Phi(\mathscr{A}), \Phi(\mathscr{A})^{\perp}\big)$ and $\big({}^{\perp}\Psi(\mathscr{B}), \Psi(\mathscr{B})\big)$ in $\mathscr{X}$, the category of representations of Q with values in ${}_R\mathrm{Mod}$, where*

$$\Phi(\mathscr{A}) = \{\, X \in \mathscr{X} \mid \textit{If } q \in Q_0 \textit{ then } C_q(X) \in \mathscr{A} \textit{ and } L_1C_q(X) = 0 \,\},$$

$$\Psi(\mathscr{B}) = \{\, X \in \mathscr{X} \mid \textit{If } q \in Q_0 \textit{ then } K_q(X) \in \mathscr{B} \textit{ and } R^1K_q(X) = 0 \,\}.$$

In the body of the paper, we prove the more general Theorem 3.2 where Q is a small k-preadditive category, and $\mathscr{X}$ is the functor category of k-linear functors $Q \to {}_R\mathrm{Mod}$. Paragraph 2.4 explains how a quiver can be viewed as a category, whence Theorem 3.2 specialises to Theorem A.

Theorem A specialises to Gillespie's Theorem for $\mathscr{M} = {}_R\mathrm{Mod}$ if Q is the quiver with relations from (0.1). Then $\mathscr{X}$ is the category of chain complexes over ${}_R\mathrm{Mod}$. A computation shows that the functors C_q, S_q, K_q specialise to those of Section 0.iv, and that

$$L_1C_q = \mathrm{H}_{q+1} \quad , \quad R^1K_q = \mathrm{H}_{q-1},$$

whence the formulae in Theorem A specialise to those in Gillespie's Theorem. However, Theorem A applies to many other quivers with relations. Following Iyama and Minamoto [18, def. 8], we then think of L_1C_q and R^1K_q as generalised homology functors.

To serve as the input for Theorem 0.2, the cotorsion pairs $\big(\Phi(\mathscr{A}), \Phi(\mathscr{A})^{\perp}\big)$ and $\big({}^{\perp}\Psi(\mathscr{B}), \Psi(\mathscr{B})\big)$ must be complete. In the setup of Theorem 0.3, this is indeed true under the conditions that $\mathscr{M}$ is a complete and cocomplete category and $(\mathscr{A}, \mathscr{B})$ is a complete cotorsion pair, see [7, thm. 2.4]. In the more complicated setup of Theorem A, we do not have an equally neat result, but we do prove completeness in certain cases, see Theorem 3.3.

0.vi. Application: N-periodic chain complexes

Let $N \geqslant 1$ be an integer.

- In Section 0.vi only, Q is the following quiver with relations.

$$\begin{cases} \text{Quiver:} & \begin{tikzcd} N-1 \arrow[r] & N-2 \arrow[r] & \cdots \arrow[r] & 1 \arrow[r] & 0 \arrow[lllll, bend left] \end{tikzcd} \\ \text{Relations:} & \text{Two consecutive arrows compose to } 0 \end{cases} \tag{0.3}$$

This is a self-injective quiver with relations, see Paragraph 2.4.

An object $X \in \mathscr{X}$ has the form

$$\begin{tikzcd} X_{N-1} \arrow[r, "\partial^X_{N-1}"] & X_{N-2} \arrow[r, "\partial^X_{N-2}"] & \cdots \arrow[r] & X_1 \arrow[r, "\partial^X_1"] & X_0, \arrow[llll, bend left, "\partial^X_0"'] \end{tikzcd}$$

where two consecutive morphisms compose to 0. Hence $\mathscr{X}$ is the category of N-periodic chain complexes over ${}_R\mathrm{Mod}$. This even makes sense for $N = 1$, in which case X is a so-called module with differentiation in the sense of [6, sec. IV.1], consisting of an object $X_0 \in {}_R\mathrm{Mod}$ and a morphism $X_0 \xrightarrow{\partial^X_0} X_0$ squaring to 0.

For $0 \leqslant q \leqslant N-1$ there is a homology functor $\mathscr{X} \xrightarrow{\mathrm{H}_q} {}_R\mathrm{Mod}$ defined in an obvious fashion. We will use our theory to prove the following.

Theorem B. *Let $(\mathscr{A}, \mathscr{B})$ be a hereditary cotorsion pair in ${}_R\mathrm{Mod}$.*

(i) *There are hereditary, compatible cotorsion pairs $\big(\Phi(\mathscr{A}), \Phi(\mathscr{A})^{\perp}\big)$ and $\big({}^{\perp}\Psi(\mathscr{B}), \Psi(\mathscr{B})\big)$ in $\mathscr{X}$, the category of N-periodic chain complexes over ${}_R\mathrm{Mod}$, where*

$$\Phi(\mathscr{A}) = \{\, X \in \mathscr{X} \mid \textit{If } 0 \leqslant q \leqslant N-1 \textit{ then } \mathrm{Coker}(\partial^X_q) \in \mathscr{A} \textit{ and } \mathrm{H}_q(X) = 0 \,\},$$

$$\Psi(\mathscr{B}) = \{\, X \in \mathscr{X} \mid \textit{If } 0 \leqslant q \leqslant N-1 \textit{ then } \mathrm{Ker}(\partial^X_q) \in \mathscr{B} \textit{ and } \mathrm{H}_q(X) = 0 \,\}.$$

(ii) *If $\mathscr{A}$ is closed under pure quotients and $(\mathscr{A}, \mathscr{B})$ is generated by a set, then the cotorsion pairs in part (i) are complete.*

This applies to the so-called flat cotorsion pair $(\mathscr{A}, \mathscr{B}) = (\text{flat modules}, \text{cotorsion modules})$: Heredity holds by [12, thm. 8.1(a)], the class of flat modules is easily seen to be closed under pure quotients, and generation by a set holds by [5, prop. 2] (in which "cogenerated" means the same as our "generated"). Hence Theorem B provides an N-periodic version of Gillespie's result for chain complexes from [11] (see theorem 3.12 and corollaries 3.13, 4.10, 4.18 in that paper). Theorem B also applies to the injective cotorsion pair $(\mathscr{A}, \mathscr{B}) = ({}_R\mathrm{Mod}, \text{injective modules})$.

0.vii. Application: $\mathbb{Z}A_3$ with mesh relations

The following is a slightly more complicated example.

- In Section 0.vii only, Q is the repetitive quiver $\mathbb{Z}A_3$ modulo the mesh relations. That is, Q is

$$\begin{tikzcd}
 & (3,2) \arrow[dr] & & (2,2) \arrow[dr] & & (1,2) \arrow[dr] & & (0,2) \arrow[dr] & & (-1,2) \arrow[dr] & & \\
\cdots & & (2,1) \arrow[ur] \arrow[dr] & & (1,1) \arrow[ur] \arrow[dr] & & (0,1) \arrow[ur] \arrow[dr] & & (-1,1) \arrow[ur] \arrow[dr] & & \cdots \\
 & (2,0) \arrow[ur] & & (1,0) \arrow[ur] & & (0,0) \arrow[ur] & & (-1,0) \arrow[ur] & & (-2,0) \arrow[ur] &
\end{tikzcd} \tag{0.4}$$

modulo the relations that each composition of the form $\circ \nearrow \circ \searrow \circ$ or $\circ \searrow \circ \nearrow \circ$, which starts and ends on the edge of the quiver, is zero, and that each square of the form $\begin{tikzcd} & \circ \arrow[dr] & \\ \circ \arrow[ur] \arrow[dr] & & \circ \\ & \circ \arrow[ur] & \end{tikzcd}$ is anticommutative. This is a self-injective quiver with relations, see Paragraph 2.4.

For $j \in \mathbb{Z}$, the mesh relations imply that there are short chain complexes

$$\begin{aligned}
&X_{(j,0)} \to X_{(j,1)} \to X_{(j-1,0)}, \\
&X_{(j,1)} \to X_{(j-1,0)} \oplus X_{(j,2)} \to X_{(j-1,1)}, \\
&X_{(j,2)} \to X_{(j-1,1)} \to X_{(j-1,2)}.
\end{aligned} \tag{0.5}$$

We will use our theory to prove the following.

Theorem C. *If $(\mathscr{A}, \mathscr{B})$ is a hereditary cotorsion pair in ${}_R\mathrm{Mod}$, then there are hereditary, compatible cotorsion pairs $\big(\Phi(\mathscr{A}), \Phi(\mathscr{A})^{\perp}\big)$ and $\big({}^{\perp}\Psi(\mathscr{B}), \Psi(\mathscr{B})\big)$ in $\mathscr{X}$, the category of representations of Q with values in ${}_R\mathrm{Mod}$, where*

$$\Phi(\mathscr{A}) = \left\{ X \in \mathscr{X} \;\middle|\; \begin{array}{l} \textit{If } j \in \mathbb{Z} \textit{ then each of the following cokernels is in } \mathscr{A}: \\ \mathrm{Coker}(X_{(j,1)} \to X_{(j-1,0)}), \\ \mathrm{Coker}(X_{(j,0)} \oplus X_{(j+1,2)} \to X_{(j,1)}), \\ \mathrm{Coker}(X_{(j,1)} \to X_{(j,2)}), \\ \textit{and each of the short chain complexes (0.5) is exact} \end{array} \right\},$$

$$\Psi(\mathscr{B}) = \left\{ X \in \mathscr{X} \;\middle|\; \begin{array}{l} \textit{If } j \in \mathbb{Z} \textit{ then each of the following kernels is in } \mathscr{B}\textit{:} \\ \mathrm{Ker}(X_{(j,0)} \to X_{(j,1)}), \\ \mathrm{Ker}(X_{(j,1)} \to X_{(j-1,0)} \oplus X_{(j,2)}), \\ \mathrm{Ker}(X_{(j,2)} \to X_{(j-1,1)}), \\ \textit{and each of the short chain complexes } (0.5) \textit{ is exact} \end{array} \right\}.$$

0.viii. Other self-injective quivers with relations

There are many other self-injective quivers with relations to which Theorem A can be applied, for instance $\cdots \to 2 \to 1 \to 0 \to -1 \to -2 \to \cdots$ with the relations that N consecutive arrows compose to 0. Then $\mathscr{X}$ is the category of N-complexes over ${}_R\mathrm{Mod}$ in the sense of [20, def. 0.1]. Other possibilities are $\mathbb{Z}A_n$ with mesh relations, the quiver with relations of a finite dimensional self-injective k-algebra, and quivers with relations of repetitive algebras, see [22, sec. 3.1] and [29, sec. 2].

0.ix. An observation on the model category literature

Observe that Theorem A does not assume the existence of a model category structure on ${}_R\mathrm{Mod}$. This is in contrast to several results from the literature, where a model category structure on a functor category $\mathrm{Fun}(\mathscr{I}, \mathscr{M})$ is induced by a model category structure on $\mathscr{M}$. If $\mathscr{I}$ is a small category, then such results exist when $\mathscr{M}$ has a cofibrantly generated or combinatorial model category structure, see [13, thm. 11.6.1] and [23, prop. A.2.8.2], and when $\mathscr{M}$ has an arbitrary model category structure and $\mathscr{I}$ is a direct, an inverse, or a Reedy category, see [16, thms. 5.1.3 and 5.2.5].

0.x. Contents of the paper

Section 1 defines the cotorsion pairs $\big(\Phi(\mathscr{A}), \Phi(\mathscr{A})^{\perp}\big)$ and $\big({}^{\perp}\Psi(\mathscr{B}), \Psi(\mathscr{B})\big)$ in an abstract setup, and shows that they are hereditary and compatible under certain assumptions. Section 2 introduces functor categories. Section 3 proves Theorem 3.2, which has Theorem A as a special case. Sections 4, 5, and 6 provide several results used in the proof of Theorem 3.2. Section 7 proves Theorem B. Section 8 proves Theorem C. Appendix A provides additional background on functor categories.

1. The cotorsion pairs $(\Phi(\mathscr{A}), \Phi(\mathscr{A})^{\perp})$ and $({}^{\perp}\Psi(\mathscr{B}), \Psi(\mathscr{B}))$ in an abstract setup

This section defines the cotorsion pairs $\big(\Phi(\mathscr{A}), \Phi(\mathscr{A})^{\perp}\big)$ and $\big({}^{\perp}\Psi(\mathscr{B}), \Psi(\mathscr{B})\big)$ in an abstract setup, and shows that they are hereditary and compatible under certain assumptions.

Setup 1.1. Section 1 uses the following setup.

- $\mathscr{M}$ and $\mathscr{X}$ are abelian categories with enough projective and enough injective objects.
- $(\mathscr{A}, \mathscr{B})$ is a cotorsion pair in $\mathscr{M}$.
- J is an index set.
- For each $j \in J$ there are adjoint pairs of functors (C_j, S_j) and (S_j, K_j) as follows.

$$\begin{tikzcd} \mathscr{M} \arrow[rr, "S_j"] & & \mathscr{X} \arrow[ll, bend right, "C_j"'] \arrow[ll, bend left, "K_j"] \end{tikzcd}$$

Note that this implies that S_j is exact.

The following lemma provides a so-called "five term exact sequence". It is classic, but we show the proof because we do not have a reference for the precise statement.

Lemma 1.2. *Let (C, S) be an adjoint pair of functors as follows: $\mathscr{M} \underset{S}{\overset{C}{\rightleftarrows}} \mathscr{X}$. Assume that S is exact. For $N \in \mathscr{M}$ and $X \in \mathscr{X}$ there is an exact sequence*

$$\begin{aligned} 0 &\to \operatorname{Ext}^1_{\mathscr{M}}(CX, N) \to \operatorname{Ext}^1_{\mathscr{X}}(X, SN) \to \operatorname{Hom}_{\mathscr{M}}(L_1 CX, N) \to \operatorname{Ext}^2_{\mathscr{M}}(CX, N) \\ &\to \operatorname{Ext}^2_{\mathscr{X}}(X, SN). \end{aligned}$$

Proof. Consider the functors $\mathscr{X} \xrightarrow{C} \mathscr{M} \xrightarrow{B} \mathscr{A}b$ where $\mathscr{A}b$ is the category of abelian groups and $B(-) = \operatorname{Hom}_{\mathscr{M}}(-, N)$.

The contravariant functor B is left exact. If $P \in \mathscr{X}$ is projective, then $C(P) \in \mathscr{M}$ is projective because $\operatorname{Hom}_{\mathscr{M}}(CP, -) \simeq \operatorname{Hom}_{\mathscr{X}}\big(P, S(-)\big)$ is an exact functor since S is exact. In particular, the functor C maps projective objects to right B-acyclic objects, that is, objects on which the derived functors $R^{\geqslant 1}B$ vanish.

By [27, thm. 10.49] there is a Grothendieck third quadrant spectral sequence

$$E_2^{i\ell} = (R^i B)(L_\ell C)X \underset{i}{\Rightarrow} R^n(BC)X.$$

If P is a projective resolution of X, then

$$R^n(BC)X \cong \mathrm{H}^n(BCP) = \mathrm{H}^n \operatorname{Hom}_{\mathscr{M}}(CP, N) \cong \mathrm{H}^n \operatorname{Hom}_{\mathscr{X}}(P, SN) \cong \operatorname{Ext}^n_{\mathscr{X}}(X, SN).$$

Hence the spectral sequence is

$$E_2^{i\ell} = \operatorname{Ext}^i_{\mathscr{M}}(L_\ell CX, N) \underset{i}{\Rightarrow} \operatorname{Ext}^n_{\mathscr{X}}(X, SN).$$

By [27, thm. 10.33] there is an associated exact sequence, which gives the sequence in the lemma. □

We record the dual without a proof:

Lemma 1.3. *Let (S, K) be an adjoint pair of functors as follows: $\mathscr{M} \underset{K}{\overset{S}{\rightleftarrows}} \mathscr{X}$. Assume that S is exact. For $N \in \mathscr{M}$ and $X \in \mathscr{X}$ there is an exact sequence*

$$0 \to \operatorname{Ext}^1_{\mathscr{M}}(N, KX) \to \operatorname{Ext}^1_{\mathscr{X}}(SN, X) \to \operatorname{Hom}_{\mathscr{M}}(N, R^1KX) \to \operatorname{Ext}^2_{\mathscr{M}}(N, KX) \\ \to \operatorname{Ext}^2_{\mathscr{X}}(SN, X).$$

The following is well known.

Lemma 1.4. *A cotorsion pair $(\mathscr{A}, \mathscr{B})$ in $\mathscr{M}$ is hereditary if and only if $\mathscr{A}$ is resolving, that is, contains the projective objects and is closed under kernels of epimorphisms.*

Proof. See [12, lem. 5.24], the proof of which works in the present generality. □

Definition 1.5. Let

$$\mathscr{E}_L = \{\, X \in \mathscr{X} \mid \text{If } j \in J \text{ then } L_1C_j(X) = 0 \,\},$$
$$\mathscr{E}_R = \{\, X \in \mathscr{X} \mid \text{If } j \in J \text{ then } R^1K_j(X) = 0 \,\}.$$

If $\mathscr{C}$ is a class of objects in $\mathscr{M}$, then let

$$\Phi(\mathscr{C}) = \{\, X \in \mathscr{X} \mid \text{If } j \in J \text{ then } C_j(X) \in \mathscr{C} \text{ and } L_1C_j(X) = 0 \,\},$$
$$\Psi(\mathscr{C}) = \{\, X \in \mathscr{X} \mid \text{If } j \in J \text{ then } K_j(X) \in \mathscr{C} \text{ and } R^1K_j(X) = 0 \,\}.$$

Note that $\Phi(\mathscr{C}) \subseteq \mathscr{E}_L$ and $\Psi(\mathscr{C}) \subseteq \mathscr{E}_R$.

If $\mathscr{C}$ is a class of objects in $\mathscr{M}$, then we use the shorthand $\{\, S_*(\mathscr{C}) \,\} = \{\, S_j(C) \mid j \in J, C \in \mathscr{C} \,\}$.

Lemma 1.6. *Let $\mathscr{C}$ be a class of objects in $\mathscr{M}$.*

(i) *Assume that for each non-zero $M \in \mathscr{M}$ there is an injective object I which is in $\mathscr{C}$ and satisfies $\operatorname{Hom}_{\mathscr{M}}(M, I) \neq 0$. Then $\Phi({}^{\perp}\mathscr{C}) = {}^{\perp}\{\, S_*(\mathscr{C}) \,\}$.*

(ii) *Assume that for each non-zero $M \in \mathscr{M}$ there is a projective object P which is in $\mathscr{C}$ and satisfies $\operatorname{Hom}_{\mathscr{M}}(P, M) \neq 0$. Then $\Psi(\mathscr{C}^{\perp}) = \{\, S_*(\mathscr{C}) \,\}^{\perp}$.*

Proof. First note that for $N \in \mathscr{C}$, $X \in \mathscr{X}$, $j \in J$, there is an exact sequence

$$0 \to \operatorname{Ext}^1_{\mathscr{M}}(C_j X, N) \to \operatorname{Ext}^1_{\mathscr{X}}(X, S_j N) \to \operatorname{Hom}_{\mathscr{M}}(L_1 C_j X, N) \to \operatorname{Ext}^2_{\mathscr{M}}(C_j X, N) \quad (1.1)$$

by Lemma 1.2.

Part (i), the inclusion $\subseteq$: Let $X \in \Phi({}^{\perp}\mathscr{C})$ and $j \in J$ be given. Then $C_j(X) \in {}^{\perp}\mathscr{C}$ and $L_1 C_j(X) = 0$ by the definition of Φ. It follows that for $N \in \mathscr{C}$, the terms in (1.1) which involve $\operatorname{Ext}^1_{\mathscr{M}}$ and $\operatorname{Hom}_{\mathscr{M}}$ are 0, so (1.1) implies $\operatorname{Ext}^1_{\mathscr{X}}(X, S_j N) = 0$. Hence $X \in {}^{\perp}\{\, S_*(\mathscr{C})\,\}$.

Part (i), the inclusion $\supseteq$: Let $X \in {}^{\perp}\{\, S_*(\mathscr{C})\,\}$ and $j \in J$ be given.

For $N \in \mathscr{C}$, the term in (1.1) which involves $\operatorname{Ext}^1_{\mathscr{X}}$ is 0, so (1.1) implies $\operatorname{Ext}^1_{\mathscr{M}}(C_j X, N) = 0$. Hence $C_j(X) \in {}^{\perp}\mathscr{C}$.

Assume that $L_1 C_j(X) \neq 0$. Pick an injective object N which is in $\mathscr{C}$ and satisfies $\operatorname{Hom}_{\mathscr{M}}(L_1 C_j X, N) \neq 0$. By the previous paragraph, the term in (1.1) which involves $\operatorname{Ext}^1_{\mathscr{X}}$ is 0. However, the term involving $\operatorname{Ext}^2_{\mathscr{M}}$ is also 0 since N is injective, so (1.1) implies $\operatorname{Hom}_{\mathscr{M}}(L_1 C_j X, N) = 0$. This is a contradiction, so we conclude $L_1 C_j(X) = 0$. Combining with the previous paragraph shows $X \in \Phi({}^{\perp}\mathscr{C})$.

Part (ii): Proved dually to part (i). $\square$

Theorem 1.7. *There are cotorsion pairs* $\big(\Phi(\mathscr{A}), \Phi(\mathscr{A})^{\perp}\big)$ *and* $\big({}^{\perp}\Psi(\mathscr{B}), \Psi(\mathscr{B})\big)$ *in* $\mathscr{X}$.

Proof. The class $\mathscr{A}$ contains the projective objects of $\mathscr{M}$, and the class $\mathscr{B}$ contains the injective objects of $\mathscr{M}$. Since we also have $\mathscr{A} = {}^{\perp}\mathscr{B}$ and $\mathscr{B} = \mathscr{A}^{\perp}$, Lemma 1.6 implies

$$\Phi(\mathscr{A}) = {}^{\perp}\{\, S_*(\mathscr{B})\,\} \ , \ \ \Psi(\mathscr{B}) = \{\, S_*(\mathscr{A})\,\}^{\perp}.$$

Hence there are the following cotorsion pairs, see [12, def. 5.15]:

$$\big(\Phi(\mathscr{A}), \Phi(\mathscr{A})^{\perp}\big) = \big({}^{\perp}\{\, S_*(\mathscr{B})\,\}, ({}^{\perp}\{\, S_*(\mathscr{B})\,\})^{\perp}\big),$$
$$\big({}^{\perp}\Psi(\mathscr{B}), \Psi(\mathscr{B})\big) = \big({}^{\perp}(\{\, S_*(\mathscr{A})\,\}^{\perp}), \{\, S_*(\mathscr{A})\,\}^{\perp}\big). \quad \square$$

Theorem 1.8. *Assume that* $(\mathscr{A}, \mathscr{B})$ *is a hereditary cotorsion pair in* $\mathscr{M}$.

(i) *If* $L_2 C_j(\mathscr{E}_L) = 0$ *for* $j \in J$, *then there is a hereditary cotorsion pair* $\big(\Phi(\mathscr{A}), \Phi(\mathscr{A})^{\perp}\big)$ *in* $\mathscr{X}$.

(ii) *If* $R^2 K_j(\mathscr{E}_R) = 0$ *for* $j \in J$, *then there is a hereditary cotorsion pair* $\big({}^{\perp}\Psi(\mathscr{B}), \Psi(\mathscr{B})\big)$ *in* $\mathscr{X}$.

Proof. The cotorsion pairs exist by Theorem 1.7, and we must prove heredity under the given assumptions.

(i): Lemma 1.4 implies that $\mathscr{A}$ is resolving, and that it is enough to prove that so is $\Phi(\mathscr{A})$. Let $0 \to X' \to X \to X'' \to 0$ be a short exact sequence in $\mathscr{X}$ with

$X, X'' \in \Phi(\mathscr{A})$, and let $j \in J$ be given. By definition we have $C_j(X), C_j(X'') \in \mathscr{A}$ and $L_1C_j(X) = L_1C_j(X'') = 0$. In particular, $X'' \in \mathscr{E}_L$, so the assumption in part (i) says $L_2C_j(X'') = 0$. Hence the long exact sequence

$$\cdots \to L_2C_j(X'') \to L_1C_j(X') \to L_1C_j(X) \to L_1C_j(X'') \to C_j(X') \to C_j(X) \to C_j(X'') \to 0$$

reads

$$\cdots \to 0 \to L_1C_j(X') \to 0 \to 0 \to C_j(X') \to C_j(X) \to C_j(X'') \to 0.$$

This implies $L_1C_j(X') = 0$. It also implies $C_j(X') \in \mathscr{A}$ because $\mathscr{A}$ is resolving and $C_j(X), C_j(X'') \in \mathscr{A}$. Hence $X' \in \Phi(\mathscr{A})$ as desired.

(ii): Proved dually to (i). □

Definition 1.9. Consider the following conditions on the classes $\mathscr{E}_L$, $\mathscr{E}_R$, $\Phi(\mathscr{A})$, $\Psi(\mathscr{B})$ from Definition 1.5.

(Ex) $\mathscr{E}_L = \mathscr{E}_R$.

(Seq) If $j \in J$ is given, then:

(i) Each $A \in \mathscr{A}$ permits a short exact sequence in $\mathscr{X}$,

$$0 \to S_j(A) \to R \to U \to 0,$$

with $R \in \Phi(\mathscr{A})$ and $\mathrm{Ext}^2_{\mathscr{X}}(U, \mathscr{E}_R) = 0$.

(ii) Each $B \in \mathscr{B}$ permits a short exact sequence in $\mathscr{X}$,

$$0 \to W \to T \to S_j(B) \to 0,$$

with $T \in \Psi(\mathscr{B})$ and $\mathrm{Ext}^2_{\mathscr{X}}(\mathscr{E}_L, W) = 0$.

Remark 1.10. It is not obvious that the sequences in condition (Seq) of the definition exist. Their construction in the category of representations of a self-injective quiver is a key technical part of the paper, see Section 6.

Theorem 1.11. *Assume that conditions* (Comp1), (Ex), *and* (Seq) *hold (see Definitions 0.1 and 1.9). Then there are compatible cotorsion pairs* $(\Phi(\mathscr{A}), \Phi(\mathscr{A})^{\perp})$ *and* $({}^{\perp}\Psi(\mathscr{B}), \Psi(\mathscr{B}))$ *in* $\mathscr{X}$.

Proof. The cotorsion pairs exist by Theorem 1.7, and we must prove that they are compatible under the given assumptions, which amounts to proving that condition (Comp2) holds. We have assumed condition (Ex), so write $\mathscr{E} = \mathscr{E}_L = \mathscr{E}_R$. It is enough to prove

$$^{\perp}\Psi(\mathscr{B}) \cap \mathscr{E} = \Phi(\mathscr{A}), \tag{1.2}$$

$$\Phi(\mathscr{A})^{\perp} \cap \mathscr{E} = \Psi(\mathscr{B}), \tag{1.3}$$

since then

$$\Phi(\mathscr{A}) \cap \Phi(\mathscr{A})^{\perp} \cap \mathscr{E} = \Phi(\mathscr{A}) \cap \Psi(\mathscr{B}) = {}^{\perp}\Psi(\mathscr{B}) \cap \Psi(\mathscr{B}) \cap \mathscr{E},$$

and this shows (Comp2) since $\mathscr{E}$ can be removed from the displayed formula because $\Phi(\mathscr{A}), \Psi(\mathscr{B}) \subseteq \mathscr{E}$.

We prove Equation (1.2) by establishing the two inclusions.

The inclusion $\subseteq$: Let $X \in {}^{\perp}\Psi(\mathscr{B}) \cap \mathscr{E}$ be given. Given $j \in J$ and $B \in \mathscr{B}$, condition (Seq)(ii) provides a short exact sequence in $\mathscr{X}$,

$$0 \to W \to T \to S_j(B) \to 0,$$

with $T \in \Psi(\mathscr{B})$ and $\operatorname{Ext}^2_{\mathscr{X}}(\mathscr{E}, W) = 0$. There is a long exact sequence containing

$$\operatorname{Ext}^1_{\mathscr{X}}(X, T) \to \operatorname{Ext}^1_{\mathscr{X}}(X, S_j B) \to \operatorname{Ext}^2_{\mathscr{X}}(X, W).$$

The first term is zero since $X \in {}^{\perp}\Psi(\mathscr{B})$ and $T \in \Psi(\mathscr{B})$. The last term is zero since $X \in \mathscr{E}$ and $\operatorname{Ext}^2_{\mathscr{X}}(\mathscr{E}, W) = 0$. Hence the middle term is zero: $\operatorname{Ext}^1_{\mathscr{X}}(X, S_j B) = 0$. By Lemma 1.2 this implies $\operatorname{Ext}^1_{\mathscr{M}}(C_j X, B) = 0$ whence $C_j(X) \in {}^{\perp}\mathscr{B} = \mathscr{A}$. We also know $X \in \mathscr{E}$, so $L_1 C_j(X) = 0$. It follows that $X \in \Phi(\mathscr{A})$.

The inclusion $\supseteq$: This follows because $\Phi(\mathscr{A}) \subseteq \mathscr{E}$, while condition (Comp1) implies $\Phi(\mathscr{A}) \subseteq {}^{\perp}\Psi(\mathscr{B})$.

Equation (1.3) is proved dually to Equation (1.2). □

We end by recording a lemma which has almost the same proof as Theorem 1.7.

Lemma 1.12. *Let $\mathscr{C}$ be a class of objects in $\mathscr{M}$.*

(i) *Assume that for each non-zero $M \in \mathscr{M}$ there is an injective object I which is in $\mathscr{C}$ and satisfies $\operatorname{Hom}_{\mathscr{M}}(M, I) \neq 0$.*

If $(\mathscr{A}, \mathscr{B})$ is the cotorsion pair in $\mathscr{M}$ cogenerated by $\mathscr{C}$, then $\big(\Phi(\mathscr{A}), \Phi(\mathscr{A})^{\perp}\big)$ is the cotorsion pair in $\mathscr{X}$ cogenerated by $\{\, S_*(\mathscr{C}) \,\}$.

(ii) *Assume that for each non-zero $M \in \mathscr{M}$ there is a projective object P which is in $\mathscr{C}$ and satisfies $\operatorname{Hom}_{\mathscr{M}}(P, M) \neq 0$.*

If $(\mathscr{A}, \mathscr{B})$ is the cotorsion pair in $\mathscr{M}$ generated by $\mathscr{C}$, then $\big({}^{\perp}\Psi(\mathscr{B}), \Psi(\mathscr{B})\big)$ is the cotorsion pair in $\mathscr{X}$ generated by $\{\, S_*(\mathscr{C}) \,\}$.

Proof. (i): If $(\mathscr{A}, \mathscr{B})$ is cogenerated by $\mathscr{C}$, then $\mathscr{A} = {}^{\perp}\mathscr{C}$, so Lemma 1.6 implies $\Phi(\mathscr{A}) = {}^{\perp}\{\, S_*(\mathscr{C}) \,\}$. Hence

$$\big(\Phi(\mathscr{A}), \Phi(\mathscr{A})^{\perp}\big) = \big({}^{\perp}\{\, S_*(\mathscr{C})\,\}, ({}^{\perp}\{\, S_*(\mathscr{C})\,\})^{\perp}\big)$$

is the cotorsion pair cogenerated by $\{\, S_*(\mathscr{C})\,\}$.

(ii) is proved dually to (i). $\square$

2. Functor categories

This section introduces functor categories. In particular, Paragraph 2.4 explains how a category of quiver representations can be viewed as a functor category, whence Theorem 3.2 has Theorem A as a special case.

Setup 2.1. Section 2 uses the following setup.

- k is a field.
- R is a k-algebra.

2.2 ***(Functor categories).*** Let Q be a small k-preadditive category; that is, each Hom-space is a k-vector space and composition of morphisms is k-bilinear. The homomorphism functor and the radical of Q will be denoted $Q(-,-)$ and $\mathrm{rad}_Q(-,-)$, see [1, sec. A.3], [8, sec. 3.2], and [21, p. 303].

Let Q^{o} denote the opposite category, and let ${}_k\mathrm{Mod}$ and ${}_R\mathrm{Mod}$ denote, respectively, the categories of k-vector spaces and R-left-modules. There are the following functor categories.

$$\begin{aligned}
{}_Q\mathrm{Mod} &= \text{the category of } k\text{-linear functors } Q \to {}_k\mathrm{Mod}\\
\mathrm{Mod}_Q &= \text{the category of } k\text{-linear functors } Q^{\mathrm{o}} \to {}_k\mathrm{Mod}\\
{}_{Q,R}\mathrm{Mod} &= \text{the category of } k\text{-linear functors } Q \to {}_R\mathrm{Mod}\\
{}_Q\mathrm{Mod}_Q &= \text{the category of } k\text{-linear functors } Q^{\mathrm{o}} \times Q \to {}_k\mathrm{Mod}
\end{aligned}$$

Their homomorphism functors are denoted Hom_Q, $\mathrm{Hom}_{Q^{\mathrm{o}}}$, $\mathrm{Hom}_{Q,R}$, and Hom_{Q^e}.

We think of them as the categories of Q-left-modules, Q-right-modules, Q-left-R-left-modules, and Q-bi-modules. They are abelian categories with enough projective and injective objects, which are in fact Grothendieck categories. In each of the categories ${}_Q\mathrm{Mod}$, Mod_Q, and ${}_{Q,R}\mathrm{Mod}$, a sequence of functors $L' \xrightarrow{\lambda'} L \xrightarrow{\lambda} L''$ is short exact if $0 \to L'(q) \xrightarrow{\lambda'_q} L(q) \xrightarrow{\lambda_q} L''(q) \to 0$ is a short exact sequence in ${}_k\mathrm{Mod}$ or ${}_R\mathrm{Mod}$ for each $q \in Q$. An object X of ${}_{Q,R}\mathrm{Mod}$ can be viewed as an object of ${}_Q\mathrm{Mod}$ by forgetting the R-structure on each $X(q)$ for $q \in Q$. We refer to Appendix A for additional information.

Definition 2.3. The following are conditions we can impose on a small k-preadditive category Q.

(Fin) Each Hom-space of Q is finite dimensional over k. If $q \in Q$ is fixed then $Q(p,q) = Q(q,p) = 0$ except for finitely many $p \in Q$. There is an integer N such that $\mathrm{rad}_Q^N = 0$.

(Rad) If $q \in Q$ then $Q(q,q)$ is a local k-algebra, and the canonical map $k \to Q(q,q)/\operatorname{rad} Q(q,q)$ is an isomorphism of k-algebras. If $p \neq q$ are in Q then $Q(p,q) = \mathrm{rad}_Q(p,q)$.

(SelfInj) The category Q has a Serre functor, that is, a k-linear autoequivalence $W : Q \to Q$ such that there are natural isomorphisms $Q(p,q) \cong \mathrm{D}Q(q,Wp)$ where $\mathrm{D}(-) = \mathrm{Hom}_k(-,k)$.

Note that the last part of condition (Rad) implies that different objects of Q are non-isomorphic. Conditions (Fin) and (Rad) imply that Q is a locally bounded spectroid in the terminology of [8, secs. 3.5 and 8.3], whence the functor categories over Q share many properties of module categories over a finite dimensional algebra, see Appendix A. If condition (SelfInj) also holds, then projective, injective, and flat objects coincide in each of ${}_Q\mathrm{Mod}$ and Mod_Q, see Paragraph A.6.

2.4 ***(Special case: Quivers with relations)***. Let Q be a quiver with relations over k in the sense of [1, def. II.2.3]. Then Q can be viewed as a small k-preadditive category: The objects are the vertices, and the morphism spaces are the k-linear combinations of paths modulo relations. Composition of morphisms is induced by concatenation of paths.

Viewed as a quiver with relations, Q has a category $\mathscr{X}$ of representations with values in ${}_R\mathrm{Mod}$. Viewed as a small k-preadditive category, Q has the functor category ${}_{Q,R}\mathrm{Mod}$. The categories $\mathscr{X}$ and ${}_{Q,R}\mathrm{Mod}$ can be identified.

We say that Q is a *self-injective quiver with relations* if Q, viewed as a small k-preadditive category, satisfies conditions (Fin), (Rad), and (SelfInj) of Definition 2.3.

The quivers with relations from the introduction are self-injective. In particular, the Serre functors are given as follows: For (0.1) by the shift $q \mapsto q-1$, for (0.3) by the shift $q \mapsto q-1$ where q is taken modulo N, and for (0.4) by reflecting in a horizontal line through the vertices $(1,j)$, then shifting one vertex to the right.

2.5 ***(Special case: Finite quivers with relations)***. Let Q be a self-injective quiver with relations over k. Assume that Q is finite and connected, and that its relations are given by an admissible ideal $\mathfrak{a}$ in the path algebra A over k, see [1, def. II.2.1].

On the one hand, Q can be viewed as a small k-preadditive category, which has the functor category ${}_{Q,R}\mathrm{Mod}$. On the other hand, there is a finite dimensional algebra $\Lambda = A/\mathfrak{a}$, which has the category ${}_{\Lambda,R}\mathrm{Mod}$ of Λ-left-R-left-modules. The categories ${}_{Q,R}\mathrm{Mod}$ and ${}_{\Lambda,R}\mathrm{Mod}$ can be identified.

A more extensive list of identifications is given in Fig. 1, where the entries in the first column are explained in Paragraph 2.2 and Appendix A. The list can be extended with Ext- and Tor-functors.

Structures over the category Q	Structures over the algebra Λ
${}_{Q,R}\mathrm{Mod}$	${}_{\Lambda,R}\mathrm{Mod}$ = Λ-left-R-left-modules
${}_{Q}\mathrm{Mod}$	${}_{\Lambda}\mathrm{Mod}$ = Λ-left-modules
Mod_{Q}	Mod_{Λ} = Λ-right-modules
${}_{Q}\mathrm{Mod}_{Q}$	${}_{\Lambda}\mathrm{Mod}_{\Lambda}$ = Λ-bimodules
$\mathrm{Hom}_{Q,R}$	$\mathrm{Hom}_{\Lambda,R}$ = homomorphisms of Λ-left-R-left-modules
Hom_{Q}	Hom_{Λ} = homomorphisms of Λ-left-modules
$\mathrm{Hom}_{Q^{\circ}}$	$\mathrm{Hom}_{\Lambda^{\circ}}$ = homomorphisms of Λ-right-modules
$\mathrm{Hom}_{Q^{e}}$	$\mathrm{Hom}_{\Lambda^{e}}$ = homomorphisms of Λ-bimodules
$\underset{Q}{\otimes}$	$\underset{\Lambda}{\otimes}$ = tensor product of Λ-modules
$\underset{k}{\otimes}$	$\underset{k}{\otimes}$ = tensor product of k-vector spaces
$S\langle q\rangle$	$S\langle q\rangle$ = the simple Λ-left-module supported at vertex q
$\mathrm{D}S\langle q\rangle$	$\mathrm{D}S\langle q\rangle$ = the simple Λ-right-module supported at vertex q

Fig. 1. A finite self-injective quiver with relations Q can be viewed as a small k-preadditive category. On the other hand, it gives a finite dimensional algebra Λ, and structures over Q and Λ can be identified as shown.

Note that since Q is a self-injective quiver with relations, Λ is a self-injective algebra.

3. Proof of Theorem A

This section proves Theorem 3.2, which has Theorem A as a special case, see Paragraph 2.4.

Sections 3 through 6 are phrased in the language of functor categories over a small k-preadditive category Q. A reader who prefers modules instead of functors can use Fig. 1 to specialise everything to the case of modules over a finite dimensional self-injective algebra Λ.

Setup 3.1. Sections 3 through 6 use the following setup, which dovetails with Setups 1.1 and 2.1 so the results of Sections 1 and 2 can be used verbatim. We refer to Appendix A for additional information, in particular on several functors which will be used extensively: $\underset{k}{\otimes}$, Hom_k, $\underset{Q}{\otimes}$, Tor_i^Q, Hom_Q, Ext_Q^i, $\mathrm{Hom}_{Q,R}$, $\mathrm{Ext}_{Q,R}^i$.

- k is a field.
- R is a k-algebra.
- Q is a small k-preadditive category satisfying conditions (Fin), (Rad), and (SelfInj) of Definition 2.3.
- $\mathscr{M} = {}_R\mathrm{Mod}$ is the category of R-left-modules.
- $\mathscr{X} = {}_{Q,R}\mathrm{Mod}$ is the category of k-linear functors $Q \to {}_R\mathrm{Mod}$.

  The categories $\mathscr{M}$ and $\mathscr{X}$ have enough projective and enough injective objects by Paragraph A.4.

- $(\mathscr{A}, \mathscr{B})$ is a cotorsion pair in $\mathscr{M}$.
- $J = \operatorname{obj} Q$. The statement $q \in \operatorname{obj} Q$ will be abbreviated $q \in Q$.
- For each $q \in Q$, there is a simple object $\mathrm{D}S\langle q\rangle \in \mathrm{Mod}_Q$ and a simple object $S\langle q\rangle \in {}_Q\mathrm{Mod}$, see Paragraph A.4(i). The functors

$$\begin{tikzcd} {}_R\mathrm{Mod} = \mathscr{M} \arrow[rr, "S_q"] & & \mathscr{X} = {}_{Q,R}\mathrm{Mod} \arrow[ll, bend right, "C_q"'] \arrow[ll, bend left, "K_q"] \end{tikzcd}$$

are defined by:

$$\begin{aligned} C_q(-) &= \mathrm{D}S\langle q\rangle \underset{Q}{\otimes} -, \\ S_q(-) &= S\langle q\rangle \underset{k}{\otimes} -, \\ K_q(-) &= \mathrm{Hom}_Q(S\langle q\rangle, -). \end{aligned} \tag{3.1}$$

There is an adjoint pair (C_q, S_q) by Paragraph A.2(ii) and the observation that we have $S_q(-) = \mathrm{Hom}_k(\mathrm{D}S\langle q\rangle, -)$ by Paragraph A.2(vi). There is an adjoint pair (S_q, K_q) by Paragraph A.2(i).

- $\mathscr{E}$ denotes either $\mathscr{E}_L$ or $\mathscr{E}_R$; these classes are equal because condition (Ex) of Definition 1.9 holds by Proposition 4.2.

Theorem 3.2. *If $(\mathscr{A}, \mathscr{B})$ is a hereditary cotorsion pair in $\mathscr{M} = {}_R\mathrm{Mod}$, then there are hereditary, compatible cotorsion pairs $\big(\Phi(\mathscr{A}), \Phi(\mathscr{A})^{\perp}\big)$ and $\big({}^{\perp}\Psi(\mathscr{B}), \Psi(\mathscr{B})\big)$ in $\mathscr{X} = {}_{Q,R}\mathrm{Mod}$, where*

$$\Phi(\mathscr{A}) = \{\, X \in \mathscr{X} \mid \textit{If } q \in Q \textit{ then } C_q(X) \in \mathscr{A} \textit{ and } L_1 C_q(X) = 0 \,\},$$

$$\Psi(\mathscr{B}) = \{\, X \in \mathscr{X} \mid \textit{If } q \in Q \textit{ then } K_q(X) \in \mathscr{B} \textit{ and } R^1 K_q(X) = 0 \,\}.$$

Proof. The formulae for $\Phi(\mathscr{A})$ and $\Psi(\mathscr{B})$ are those of Definition 1.5 adapted to the present setup, so the results of Section 1 apply. In particular, the cotorsion pairs exist by Theorem 1.7. They are hereditary by Theorem 1.8 combined with Proposition 4.2 below. They are compatible by Theorem 1.11 combined with Propositions 4.2, 5.2, and 6.18 below. □

Theorem 3.3. *We have:*

(i) *If the cotorsion pair* $(\mathscr{A}, \mathscr{B})$ *is generated by a set, then the cotorsion pair* $\big({}^{\perp}\Psi(\mathscr{B}), \Psi(\mathscr{B})\big)$ *is complete.*

(ii) *Assume that* Q *arises from the special case described in Paragraph 2.5, so corresponds to a finite dimensional self-injective* k*-algebra* Λ*. If* $\mathscr{A}$ *is closed under pure quotients, then the cotorsion pair* $\big(\Phi(\mathscr{A}), \Phi(\mathscr{A})^{\perp}\big)$ *is complete.*

Proof. (i) Suppose that $\mathscr{C}$ is a set of objects of $\mathscr{M}$ which generates $(\mathscr{A}, \mathscr{B})$. This is still the case after adding the projective R-left-module ${}_RR$ to $\mathscr{C}$. Then Lemma 1.12(ii) says that $\{S_*(\mathscr{C})\}$ is a set of objects of $\mathscr{X}$ which generates $\big({}^{\perp}\Psi(\mathscr{B}), \Psi(\mathscr{B})\big)$. This cotorsion pair is hence complete by [12, lem. 5.20] and [30, def. 3.11, prop. 3.13, and prop. 5.8]. Note that the proof of [12, lem. 5.20] goes through for $\mathscr{X}$ because it has enough projective objects and enough injective objects by Paragraph A.4.

(ii) As explained in Paragraph 2.5, we can view $\big(\Phi(\mathscr{A}), \Phi(\mathscr{A})^{\perp}\big)$ as a cotorsion pair in ${}_{\Lambda,R}\mathrm{Mod}$, hence as a cotorsion pair in ${}_{\Lambda\otimes R}\mathrm{Mod}$, the category of left-modules over the k-algebra $\Lambda \underset{k}{\otimes} R$, which will be denoted simply by $\Lambda \otimes R$. By [12, lem. 5.13(b) and lem. 5.20] and [14, thm. 2.5] it is enough to show that $\Phi(\mathscr{A})$ is closed under pure quotients.

Let

$$\sigma \;=\; 0 \to X' \to X \to X'' \to 0$$

be pure short exact in ${}_{\Lambda\otimes R}\mathrm{Mod}$ with $X \in \Phi(\mathscr{A})$. We will show $X'' \in \Phi(\mathscr{A})$, that is, $C_q(X'') \in \mathscr{A}$ and $L_1C_q(X'') = 0$ for each $q \in Q$.

Given $q \in Q$ and $B \in \mathrm{Mod}_R$ we have that

$$B \underset{R}{\otimes} C_q(\sigma) = B \underset{R}{\otimes} (\mathrm{D}S\langle q\rangle \underset{\Lambda}{\otimes} \sigma) \cong (B \underset{k}{\otimes} \mathrm{D}S\langle q\rangle) \underset{\Lambda\otimes R}{\otimes} \sigma$$

is exact, because σ is pure short exact in ${}_{\Lambda\otimes R}\mathrm{Mod}$. The isomorphism is by [6, prop. IX.2.1]. Hence

$$C_q(\sigma) \;=\; 0 \to C_q(X') \to C_q(X) \to C_q(X'') \to 0$$

is pure short exact in ${}_R\mathrm{Mod}$. But $X \in \Phi(\mathscr{A})$ implies $C_q(X) \in \mathscr{A}$ whence $C_q(X'') \in \mathscr{A}$ since $\mathscr{A}$ is closed under pure quotients.

Given $M \in \mathrm{Mod}_\Lambda$ we have that

$$M \underset{\Lambda}{\otimes} \sigma \cong M \underset{\Lambda}{\otimes} \big((\Lambda \otimes R) \underset{\Lambda\otimes R}{\otimes} \sigma\big) \cong \big(M \underset{\Lambda}{\otimes} (\Lambda \otimes R)\big) \underset{\Lambda\otimes R}{\otimes} \sigma$$

is exact, because σ is pure short exact in ${}_{\Lambda\otimes R}\mathrm{Mod}$. Hence σ, viewed as a sequence of Λ-left-modules, is pure short exact. But $X \in \Phi(\mathscr{A})$ implies $X \in \mathscr{E}$, so X, viewed as a

Λ-left-module, is flat by Proposition 4.2. It follows that X'', viewed as a Λ-left-module, is flat, so $L_1C_q(X'') = 0$. $\square$

We end this section with a description of the trivial objects in the model category structure on $\mathscr{X}$ provided by Theorems 0.2 and 3.2. We thank an anonymous referee for drawing attention to this question.

Theorem 3.4. *If $(\mathscr{A}, \mathscr{B})$ is a hereditary cotorsion pair in $\mathscr{M} = {}_R\mathrm{Mod}$, and either of the cotorsion pairs $\big(\Phi(\mathscr{A}), \Phi(\mathscr{A})^{\perp}\big)$ and $\big({}^{\perp}\Psi(\mathscr{B}), \Psi(\mathscr{B})\big)$ of Theorem 3.2 is complete (see Theorem 3.3), then the class $\mathscr{W}$ of trivial objects (defined in Theorem 0.2) satisfies $\mathscr{W} = \mathscr{E}$.*

Proof. The inclusion $\mathscr{W} \subseteq \mathscr{E}$: By the first part of the first formula of Theorem 0.2, each $W \in \mathscr{W}$ sits in a short exact sequence $0 \to P \to F \to W \to 0$ in $\mathscr{X}$ with $P \in \Psi(\mathscr{B})$, $F \in \Phi(\mathscr{A})$. We have $P, F \in \mathscr{E}$ by Definition 1.5. By Proposition 4.2 this means that P and F are injective when viewed as objects of ${}_Q\mathrm{Mod}$. The same is hence true for W, so $W \in \mathscr{E}$ by Proposition 4.2 again.

The inclusion $\mathscr{W} \supseteq \mathscr{E}$: First, suppose $\big(\Phi(\mathscr{A}), \Phi(\mathscr{A})^{\perp}\big)$ is complete. Given $E \in \mathscr{E}$ there is a short exact sequence $0 \to Q \to F \to E \to 0$ in $\mathscr{X}$ with $Q \in \Phi(\mathscr{A})^{\perp}$, $F \in \Phi(\mathscr{A})$. By the first part of the first formula of Theorem 0.2 it is enough to show $Q \in \Psi(\mathscr{B})$. We have $F \in \mathscr{E}$ by Definition 1.5. Proposition 4.2 says that F and E are projective when viewed as objects of ${}_Q\mathrm{Mod}$. The same is hence true for Q, so $Q \in \mathscr{E}$ by Proposition 4.2 again. Hence $Q \in \Phi(\mathscr{A})^{\perp} \cap \mathscr{E}$. It follows that, as desired, $Q \in \Psi(\mathscr{B})$, see Equation (1.3) in the proof of Theorem 1.11. The theorem applies because conditions (Comp1), (Ex), and (Seq) (see Definitions 0.1(iv) and 1.9) hold by Propositions 4.2, 5.2, and 6.18 below.

Secondly, suppose $\big({}^{\perp}\Psi(\mathscr{B}), \Psi(\mathscr{B})\big)$ is complete. Then $\mathscr{W} \supseteq \mathscr{E}$ is proved similarly using the second part of the first formula of Theorem 0.2 and Equation (1.2). $\square$

4. Condition (Ex)

Section 4 continues to use Setup 3.1. The aim is to prove Proposition 4.2, by which condition (Ex) of Definition 1.9 holds. This is used in the proof of Theorem 3.2. We also establish some other properties of the class $\mathscr{E}$.

Lemma 4.1. *If $i \geqslant 0$ and $q \in Q$ then*

(i) $L_iC_q(-) = \mathrm{Tor}_i^Q(\mathrm{D}S\langle q\rangle, -)$,

(ii) $R^iK_q(-) = \mathrm{Ext}_Q^i(S\langle q\rangle, -)$,

and there are isomorphisms in ${}_R\mathrm{Mod}$,

(iii) $L_iC_q(M \underset{k}{\otimes} B) \cong \mathrm{Tor}_i^Q(\mathrm{D}S\langle q\rangle, M) \underset{k}{\otimes} B$,

(iv) $R^i K_q(M \underset{k}{\otimes} B) \cong \operatorname{Ext}^i_Q(S\langle q\rangle, M) \underset{k}{\otimes} B$,

natural in $M \in {}_Q\mathrm{Mod}$ *and* $B \in {}_R\mathrm{Mod}$.

Proof. Parts (i) and (ii) follow from Equation (3.1) and Paragraph A.5. Parts (iii) and (iv) follow from parts (i) and (ii) combined with Paragraph A.5, parts (iii) and (ii). □

Proposition 4.2. *In the situation of Setup 3.1, condition (Ex) of Definition 1.9 holds, that is* $\mathscr{E}_L = \mathscr{E}_R$. *We write* $\mathscr{E} = \mathscr{E}_L = \mathscr{E}_R$ *and have*

$$\begin{aligned}\mathscr{E} &= \{\, X \in \mathscr{X} \mid X \textit{ is projective when viewed as an object of } {}_Q\mathrm{Mod} \,\}\\ &= \{\, X \in \mathscr{X} \mid X \textit{ is flat when viewed as an object of } {}_Q\mathrm{Mod} \,\}\\ &= \{\, X \in \mathscr{X} \mid X \textit{ is injective when viewed as an object of } {}_Q\mathrm{Mod} \,\}.\end{aligned}$$

Proof. Combining Definition 1.5 and Lemma 4.1(i) shows

$$\mathscr{E}_L = \{\, X \in \mathscr{X} \mid \text{If } q \in Q \text{ then } \operatorname{Tor}^Q_1(\mathrm{D}S\langle q\rangle, X) = 0 \,\}.$$

Combining this with Equation (A.3) proves

$$\mathscr{E}_L = \{\, X \in \mathscr{X} \mid X \text{ is flat when viewed as an object of } {}_Q\mathrm{Mod} \,\}.$$

Similarly, combining Definition 1.5, Lemma 4.1(ii), and Equation (A.2) proves

$$\mathscr{E}_R = \{\, X \in \mathscr{X} \mid X \text{ is injective when viewed as an object of } {}_Q\mathrm{Mod} \,\}.$$

The proposition now follows from Equation (A.4). □

Lemma 4.3. *If* $M \in {}_Q\mathrm{Mod}$ *has finite length and* $I \in {}_R\mathrm{Mod}$ *is injective, then* $M \underset{k}{\otimes} I \in \mathscr{E}^{\perp}$.

Proof. Let $E \in \mathscr{E}$ have the projective resolution P in $\mathscr{X} = {}_{Q,R}\mathrm{Mod}$. Then

$$\begin{aligned}\operatorname{Ext}^1_{Q,R}\big(E, \operatorname{Hom}_k(\mathrm{D}S\langle q\rangle, I)\big) &\cong \mathrm{H}^1 \operatorname{Hom}_{Q,R}\big(P, \operatorname{Hom}_k(\mathrm{D}S\langle q\rangle, I)\big)\\ &\overset{(a)}{\cong} \mathrm{H}^1 \operatorname{Hom}_R(\mathrm{D}S\langle q\rangle \underset{Q}{\otimes} P, I)\\ &\overset{(b)}{\cong} \operatorname{Hom}_R\big(\mathrm{H}_1(\mathrm{D}S\langle q\rangle \underset{Q}{\otimes} P), I\big)\\ &\overset{(c)}{\cong} \operatorname{Hom}_R\big(\operatorname{Tor}^Q_1(\mathrm{D}S\langle q\rangle, E), I\big)\\ &\overset{(d)}{\cong} \operatorname{Hom}_R\big(0, I\big)\\ &= 0.\end{aligned}$$

Here (a) is by Paragraph A.2(ii) and (b) is because I is injective. The isomorphism (c) is because P consists of projective objects in $\mathscr{X}$, which are also projective when viewed as objects of ${}_Q\mathrm{Mod}$ by Paragraph A.4(iv). Finally, (d) is by Proposition 4.2.

Hence $\mathrm{Hom}_k(\mathrm{D}S\langle q\rangle, I) \in \mathscr{E}^{\perp}$, and Paragraph A.2(vi) gives $S\langle q\rangle \underset{k}{\otimes} I \in \mathscr{E}^{\perp}$. However, by Paragraph A.4(i) the object M has a finite filtration with quotients of the form $S\langle q\rangle$ for $q \in Q$, so by Paragraph A.1(iv) the object $M \underset{k}{\otimes} I$ has a finite filtration with quotients of the form $S\langle q\rangle \underset{k}{\otimes} I$ for $q \in Q$, and it follows that $M \underset{k}{\otimes} I \in \mathscr{E}^{\perp}$ as claimed. □

Lemma 4.4. *We have $\mathscr{E}^{\perp} = \mathscr{E}^{\perp\infty}$ as subcategories of $\mathscr{X}$ where $\mathscr{E}^{\perp\infty} = \{\, X \in \mathscr{X} \mid \mathrm{Ext}^{\geqslant 1}_{\mathscr{X}}(\mathscr{E}, X) = 0 \,\}$.*

Proof. The proof of [12, cor. 5.25] goes through for $\mathscr{X} = {}_{Q,R}\mathrm{Mod}$, so it is enough to see that $\mathscr{E}$ is closed under syzygies. Let $0 \to \Omega E \to P \to E \to 0$ be a short exact sequence in $\mathscr{X}$ with P projective and $E \in \mathscr{E}$. Then P and E are projective when viewed in ${}_Q\mathrm{Mod}$, see Proposition 4.2 and Paragraph A.4(iv). Hence ΩE is projective when viewed in ${}_Q\mathrm{Mod}$, so $\Omega E \in \mathscr{E}$ by Proposition 4.2. □

5. Condition (Comp1)

Section 5 continues to use Setup 3.1. The aim is to prove Proposition 5.2, by which condition (Comp1) of Definition 0.1(iv) holds. This is used in the proof of Theorem 3.2.

Lemma 5.1. *If $q \in Q$ and $A \in \mathscr{A}$, then $S_q(A) \in {}^{\perp}\Psi(\mathscr{B})$.*

Proof. The categories $\mathscr{M}$ and $\mathscr{X}$ have enough projective and injective objects by Paragraph A.4, and the functor $S_q(-) = S\langle q\rangle \underset{k}{\otimes} -$ is exact by Paragraph A.1(iv). Given $Y \in \Psi(\mathscr{B})$ we have $R^1K_q(Y) = 0$ by definition of $\Psi(\mathscr{B})$, so Lemma 1.3 gives an isomorphism $\mathrm{Ext}^1_{\mathscr{M}}(A, K_qY) \cong \mathrm{Ext}^1_{\mathscr{X}}(S_qA, Y)$. The first Ext is zero since $K_q(Y) \in \mathscr{B}$ by definition of $\Psi(\mathscr{B})$, so the lemma follows. □

Proposition 5.2. *In the situation of Setup 3.1, condition (Comp1) of Definition 0.1 holds.*

Proof. If $X \in \Phi(\mathscr{A})$ is given, then $X \in \mathscr{E}_L = \mathscr{E}$ by definition of $\Phi(\mathscr{A})$, so Proposition 4.2 says that X is flat when viewed as an object of ${}_Q\mathrm{Mod}$. This means that $- \underset{Q}{\otimes} X$ is exact, so the filtration (A.6) induces a filtration in $\mathscr{X} = {}_{Q,R}\mathrm{Mod}$:

$$0 = \mathrm{rad}^N_Q \underset{Q}{\otimes} X \subseteq \cdots \subseteq \mathrm{rad}^1_Q \underset{Q}{\otimes} X \subseteq \mathrm{rad}^0_Q \underset{Q}{\otimes} X = X.$$

The final equality is by Equation (A.1). The quotients are

$$
\begin{aligned}
(\operatorname{rad}_Q^i / \operatorname{rad}_Q^{i-1}) \underset{Q}{\otimes} X &\overset{\text{(a)}}{\cong} \left(\coprod_{p,q \in Q} (\mathrm{D}S\langle p\rangle \underset{k}{\otimes} S\langle q\rangle)^{n_i(p,q)} \right) \underset{Q}{\otimes} X \\
&\overset{\text{(b)}}{\cong} \coprod_{p,q \in Q} \left(S\langle q\rangle \underset{k}{\otimes} (\mathrm{D}S\langle p\rangle \underset{Q}{\otimes} X)\right)^{n_i(p,q)} \\
&= \coprod_{p,q \in Q} S_q C_p(X)^{n_i(p,q)} \\
&= (*),
\end{aligned}
$$

where (a) is by Equation (A.7), while (b) uses that $- \underset{Q}{\otimes} X$ preserves coproducts, followed by Paragraph A.2(iv). However, $C_p(X) \in \mathscr{A}$ by definition of $\Phi(\mathscr{A})$, so Lemma 5.1 implies $(*) \in {}^{\perp}\Psi(\mathscr{B})$, whence also $X \in {}^{\perp}\Psi(\mathscr{B})$ as desired. $\square$

6. Condition (Seq)

Section 6 continues to use Setup 3.1. The aim is to prove Proposition 6.18 by which condition (Seq) of Definition 1.9 holds. This is used in the proof of Theorem 3.2.

Setup 6.1. In addition to Setup 3.1, Section 6 uses the following setup.

- $M_0 \in {}_Q\mathrm{Mod}$ is an object of finite length. By Paragraph A.4(ii) it has an augmented minimal projective resolution, which we break into short exact sequences as follows.

$$
\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[rr, "\partial_2^P"] \arrow[dr, two heads, "\pi_2"'] & & P_1 \arrow[rr, "\partial_1^P"] \arrow[dr, two heads, "\pi_1"'] & & P_0 \arrow[r, two heads, "\pi_0"] & M_0 \\
& & M_2 \arrow[ur, hook, "\mu_2"'] & & M_1 \arrow[ur, hook, "\mu_1"'] & &
\end{tikzcd}
$$

  Each P_i and each M_i has finite length, and for each $q \in Q$, the functors $\mathrm{D}S\langle q\rangle \underset{Q}{\otimes} -$ and $\mathrm{Hom}_Q(S\langle q\rangle, -)$ vanish on the ∂_i^P.

- $B^0 \in {}_R\mathrm{Mod}$ is a module with an augmented injective resolution, which we break into short exact sequences as follows.

$$
\begin{tikzcd}
B^0 \arrow[r, hook, "\beta^0"] & I^0 \arrow[rr, "\partial_I^0"] \arrow[dr, two heads, "\alpha^0"'] & & I^1 \arrow[rr, "\partial_I^1"] \arrow[dr, two heads, "\alpha^1"'] & & I^2 \arrow[r] & \cdots \\
& & B^1 \arrow[ur, hook, "\beta^1"'] & & B^2 \arrow[ur, hook, "\beta^2"'] & &
\end{tikzcd}
$$

Construction 6.2. This construction consists of two parts labelled (i) and (ii).

(i) For each $i \geqslant 0$ we define a short exact sequence

$$0 \to E^i \xrightarrow{\varepsilon^i} T^i \xrightarrow{\tau^i} M_i \underset{k}{\otimes} B^i \to 0 \tag{6.1}$$

in $\mathscr{X}$ as follows:

If $i = 0$ then (6.1) is defined to be

$$0 \to 0 \to M_0 \underset{k}{\otimes} B^0 \xrightarrow{\mathrm{id}} M_0 \underset{k}{\otimes} B^0 \to 0.$$

If (6.1) has already been defined for a given value $i \geqslant 0$, then we use it as the last non-trivial column of the following diagram. The lower right square is a pullback, and the rows and columns are exact, see Paragraph A.1(iv).

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & E^i \arrow[r, equal] \arrow[d, "\zeta^i"'] & E^i \arrow[d, "\varepsilon^i"] & \\
0 \arrow[r] & M_{i+1} \underset{k}{\otimes} B^i \arrow[r, "\theta^{i+1}"] \arrow[d, equal] & E^{i+1} \arrow[r, "\kappa^{i+1}"] \arrow[d, "\eta^i"'] & T^i \arrow[r] \arrow[d, "\tau^i"] & 0 \\
0 \arrow[r] & M_{i+1} \underset{k}{\otimes} B^i \arrow[r, "\mu_{i+1} \otimes B^i"'] & P_i \underset{k}{\otimes} B^i \arrow[r, "\pi_i \otimes B^i"'] \arrow[d] & M_i \underset{k}{\otimes} B^i \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd} \tag{6.2}$$

The row which contains E^{i+1} is used as the first non-trivial row of the following diagram. The upper left square is a pushout, and the rows and columns are exact.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M_{i+1} \underset{k}{\otimes} B^i \arrow[r, "\theta^{i+1}"] \arrow[d, "M_{i+1}\otimes\beta^i"'] & E^{i+1} \arrow[r, "\kappa^{i+1}"] \arrow[d, "\varepsilon^{i+1}"] & T^i \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_{i+1} \underset{k}{\otimes} I^i \arrow[r, "\gamma^{i+1}"'] \arrow[d, "M_{i+1}\otimes\alpha^i"'] & T^{i+1} \arrow[r, "\delta^{i+1}"'] \arrow[d, "\tau^{i+1}"] & T^i \arrow[r] & 0 \\
 & M_{i+1} \underset{k}{\otimes} B^{i+1} \arrow[r, equal] \arrow[d] & M_{i+1} \underset{k}{\otimes} B^{i+1} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd} \tag{6.3}$$

The column which contains E^{i+1} defines (6.1) for $i+1$. Note that Diagrams (6.2) and (6.3) define a number of morphisms in addition to those in (6.1). The first steps of the construction give

$$E^0 = 0 \ , \ \ E^1 \cong P_0 \underset{k}{\otimes} B^0 \ \ , \ \ T^0 \cong M_0 \underset{k}{\otimes} B^0. \tag{6.4}$$

(ii) Part (i) permits us to construct a short exact sequence of inverse systems as follows: Set

$$\Delta^i = \begin{cases} \mathrm{id}_{T^0} & \text{for } i = 0, \\ \delta^1 \circ \cdots \circ \delta^i & \text{for } i \geqslant 1 \end{cases}$$

and

$$W^i = \operatorname{Ker} \Delta^i.$$

Each Δ^i is an epimorphism because each δ^i is an epimorphism by Diagram (6.3). Hence there is a short exact sequence of inverse systems, where it is easy to check that the induced morphisms ω^i are also epimorphisms:

$$\begin{tikzcd}
\cdots \arrow[r] & W^2 \arrow[r, two heads, "\omega^2"] \arrow[d, hook] & W^1 \arrow[r, two heads, "\omega^1"] \arrow[d, hook] & W^0 \arrow[d, hook] \\
\cdots \arrow[r] & T^2 \arrow[r, two heads, "\delta^2"] \arrow[d, two heads, "\Delta^2"'] & T^1 \arrow[r, two heads, "\delta^1"] \arrow[d, two heads, "\Delta^1"] & T^0 \arrow[d, two heads, "\Delta^0"] \\
\cdots \arrow[r] & T^0 \arrow[r, equal] & T^0 \arrow[r, equal] & T^0.
\end{tikzcd} \tag{6.5}$$

The inverse limits of the two first systems will be denoted

$$W = \varprojlim W^i \ , \ \ T = \varprojlim T^i. \tag{6.6}$$

The inverse limit of the third system is

$$\varprojlim T^0 \cong \varprojlim M_0 \underset{k}{\otimes} B^0 \cong M_0 \underset{k}{\otimes} B^0 \tag{6.7}$$

by Equation (6.4).

Remark 6.3. If $q \in Q$ and $B \in \mathscr{B}$ are given, then we can set $M_0 = S\langle q \rangle$ and $B^0 = B$ in Setup 6.1. We will prove that if $(\mathscr{A}, \mathscr{B})$ is hereditary, then the inverse limit of (6.5) is a short exact sequence

$$0 \to W \to T \to S_q(B) \to 0, \tag{6.8}$$

which can be used as the sequence in condition (Seq)(ii) of Definition 1.9. This will be accomplished in Proposition 6.18.

As an example, suppose that Q is the quiver with relations (0.1), viewed as a k-preadditive category. Then $\mathscr{X}$ is the category of chain complexes over ${}_R\mathrm{Mod}$. If $q = 1$ and we write $I^j = I_{-j}$, then

$$T^i = \cdots \to 0 \to B \to I_0 \to I_{-1} \to \cdots \to I_{-i+1} \to 0 \to \cdots$$

with B in degree 1, and

$$W^i = \cdots \to 0 \to I_0 \to I_{-1} \to \cdots \to I_{-i+1} \to 0 \to \cdots .$$

The inverse limits become the augmented injective resolution

$$T = \cdots \to 0 \to B \to I_0 \to I_{-1} \to I_{-2} \to \cdots$$

with B in degree 1, and the injective resolution

$$W = \cdots \to 0 \to I_0 \to I_{-1} \to I_{-2} \to \cdots .$$

With these T and W, the short exact sequence (6.8) is dual to a sequence with projective objects, which was used indirectly by Gillespie in his proof of compatibility, see the proof of [11, thm. 3.12].

Lemma 6.4. *If $i \geqslant 0$ then $E^i \in \mathscr{E}$.*

Proof. It follows from Definition 1.5 that $\mathscr{E}$ contains $E^0 = 0$ and is closed under extensions. Diagram (6.2) contains the short exact sequence $0 \to E^i \to E^{i+1} \to P_i \underset{k}{\otimes} B^i \to 0$, so it is enough to show $P_i \underset{k}{\otimes} B^i \in \mathscr{E}$ for each $i \geqslant 0$. However, for $q \in Q$ we have $L_1 C_q(P_i \underset{k}{\otimes} B^i) \cong \operatorname{Tor}_1^Q(\mathrm{D}S\langle q\rangle, P_i) \underset{k}{\otimes} B^i = (*)$ by Lemma 4.1(iii). Since P_i is projective, this is $(*) = 0 \underset{k}{\otimes} B^i = 0$. This shows $P_i \underset{k}{\otimes} B^i \in \mathscr{E}_L = \mathscr{E}$. $\square$

Lemma 6.5. *If $i \geqslant 0$ and $q \in Q$ then there is a short exact sequence*

$$0 \to K_q(M_{i+1} \underset{k}{\otimes} B^i) \xrightarrow{K_q(M_{i+1} \otimes \beta^i)} K_q(M_{i+1} \underset{k}{\otimes} I^i) \xrightarrow{K_q(M_{i+1} \otimes \alpha^i)} K_q(M_{i+1} \underset{k}{\otimes} B^{i+1}) \to 0.$$

Proof. By Paragraph A.1(iv) the functor $\operatorname{Hom}_Q(S\langle q\rangle, M_{i+1}) \underset{k}{\otimes} -$ is exact. Applying it to the short exact sequence $0 \to B^i \xrightarrow{\beta^i} I^i \xrightarrow{\alpha^i} B^{i+1} \to 0$ gives the sequence in the lemma by Lemma 4.1(iv). $\square$

Lemma 6.6. *If $i \geqslant 0$ and $q \in Q$ then:*

(i) *There is a short exact sequence*

$$0 \to K_q(E^i) \xrightarrow{K_q(\varepsilon^i)} K_q(T^i) \xrightarrow{K_q(\tau^i)} K_q(M_i \underset{k}{\otimes} B^i) \to 0.$$

(ii) *There is an isomorphism*

$$R^1 K_q(T^i) \xrightarrow{R^1 K_q(\tau^i)} R^1 K_q(M_i \underset{k}{\otimes} B^i).$$

Proof. The functor $K_q(-) = \operatorname{Hom}_Q(S\langle q\rangle, -)$ is left exact, so applying it to the short exact sequence (6.1) gives a long exact sequence

$$\begin{aligned} 0 &\to K_q(E^i) \xrightarrow{K_q(\varepsilon^i)} K_q(T^i) \xrightarrow{K_q(\tau^i)} K_q(M_i \underset{k}{\otimes} B^i) \\ &\to R^1 K_q(E^i) \xrightarrow{R^1 K_q(\varepsilon^i)} R^1 K_q(T^i) \xrightarrow{R^1 K_q(\tau^i)} R^1 K_q(M_i \underset{k}{\otimes} B^i) \to R^2 K_q(E^i) \to \cdots . \end{aligned}$$

This implies both parts of the lemma because $R^{\geqslant 1}K_q(E^i) = 0$ by Proposition 4.2 and Lemma 6.4. □

Lemma 6.7. *If $i \geqslant 1$ and $q \in Q$ then there is an exact sequence*

$$K_q(E^{i+1}) \xrightarrow{K_q(\kappa^{i+1})} K_q(T^i) \xrightarrow{K_q(\tau^i)} K_q(M_i \underset{k}{\otimes} B^i).$$

Proof. Since $\mathrm{Hom}_Q(S\langle q\rangle, -)$ is left-exact and μ_i a monomorphism, $\mathrm{Hom}_Q(S\langle q\rangle, \mu_i)$ is injective. But Setup 6.1 implies

$$0 = \mathrm{Hom}_Q(S\langle q\rangle, \partial_i^P) = \mathrm{Hom}_Q(S\langle q\rangle, \mu_i) \circ \mathrm{Hom}_Q(S\langle q\rangle, \pi_i),$$

so we conclude $\mathrm{Hom}_Q(S\langle q\rangle, \pi_i) = 0$. By Lemma 4.1(iv) this implies

$$K_q(\pi_i \underset{k}{\otimes} B^i) = \mathrm{Hom}_Q(S\langle q\rangle, \pi_i) \underset{k}{\otimes} B^i = 0. \tag{6.9}$$

Now observe that the left exact functor K_q preserves the pullback in Diagram (6.2), so there is the following pullback square.

$$\begin{tikzcd}
K_q(E^{i+1}) \arrow[r, "K_q(\kappa^{i+1})"] \arrow[d, "K_q(\eta^i)"'] & K_q(T^i) \arrow[d, "K_q(\tau^i)"] \\
K_q(P_i \underset{k}{\otimes} B^i) \arrow[r, "K_q(\pi_i \otimes B^i)"'] & K_q(M_i \underset{k}{\otimes} B^i)
\end{tikzcd}$$

Combining with Equation (6.9) proves the lemma. □

Lemma 6.8. *If $i \geqslant 1$ and $q \in Q$ then* $\mathrm{Im}\, K_q(\kappa^i) = \mathrm{Im}\, K_q(\delta^i)$.

Proof. If $i \geqslant 0$, then K_q can be applied to Diagram (6.3). Replacing the third non-trivial column by the images of the relevant morphisms gives the following commutative diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_q(M_{i+1} \underset{k}{\otimes} B^i) \arrow[r, "K_q(\theta^{i+1})"] \arrow[d, "K_q(M_{i+1}\otimes\beta^i)"'] & K_q(E^{i+1}) \arrow[r] \arrow[d, "K_q(\varepsilon^{i+1})"] & \operatorname{Im} K_q(\kappa^{i+1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_q(M_{i+1} \underset{k}{\otimes} I^i) \arrow[r, "K_q(\gamma^{i+1})"'] \arrow[d, "K_q(M_{i+1}\otimes\alpha^i)"'] & K_q(T^{i+1}) \arrow[r] \arrow[d, "K_q(\tau^{i+1})"] & \operatorname{Im} K_q(\delta^{i+1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_q(M_{i+1} \underset{k}{\otimes} B^{i+1}) \arrow[r, equal] \arrow[d] & K_q(M_{i+1} \underset{k}{\otimes} B^{i+1}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

It is enough to show that the third non-trivial column is a short exact sequence. We use the Nine Lemma, so have to show that the rows and the first two non-trivial columns are short exact. The row which contains an identity morphism is trivially short exact. Since K_q is left-exact, the other rows are short exact by construction. The first non-trivial column is short exact by Lemma 6.5 and the second by Lemma 6.6(i). $\square$

Lemma 6.9. *If $i \geqslant 1$ and $q \in Q$ then* $\operatorname{Im} K_q(\varepsilon^i) = \operatorname{Im} K_q(\delta^{i+1})$.

Proof. Using Lemmas 6.6(i), 6.7, and 6.8 gives the equalities

$$\operatorname{Im} K_q(\varepsilon^i) = \operatorname{Ker} K_q(\tau^i) = \operatorname{Im} K_q(\kappa^{i+1}) = \operatorname{Im} K_q(\delta^{i+1}). \quad \square$$

Lemma 6.10. *If $i \geqslant 1$ and $q \in Q$ then there are short exact sequences:*

(i) $0 \to \operatorname{Im} K_q(\delta^{i+1}) \xrightarrow{\psi'} K_q(T^i) \xrightarrow{K_q(\tau^i)} K_q(M_i \underset{k}{\otimes} B^i) \to 0,$

(ii) $0 \to K_q(M_{i+1} \underset{k}{\otimes} I^i) \xrightarrow{K_q(\gamma^{i+1})} K_q(T^{i+1}) \xrightarrow{\psi''} \operatorname{Im} K_q(\delta^{i+1}) \to 0,$

where ψ' is the canonical inclusion and ψ'' is induced by $K_q(\delta^{i+1})$.

Proof. Applying the left exact functor K_q to the short exact sequence (6.1) gives a long exact sequence containing

$$K_q(E^i) \xrightarrow{K_q(\varepsilon^i)} K_q(T^i) \xrightarrow{K_q(\tau^i)} K_q(M_i \underset{k}{\otimes} B^i) \to R^1 K_q(E^i).$$

The last term is zero by Proposition 4.2 and Lemma 6.4, and $\operatorname{Im} K_q(\varepsilon^i) = \operatorname{Im} K_q(\delta^{i+1})$ by Lemma 6.9, so we get the sequence (i).

Diagram (6.3) contains the short exact sequence $0 \to M_{i+1} \underset{k}{\otimes} I^i \xrightarrow{\gamma^{i+1}} T^{i+1} \xrightarrow{\delta^{i+1}} T^i \to 0$. Applying the left exact functor K_q gives the sequence (ii). $\square$

Definition 6.11. If $i \geqslant 0$ and $q \in Q$, then φ^{i+2} is the unique morphism which makes the following square commutative, where the vertical morphisms are the canonical inclusions.

$$\begin{tikzcd}
\operatorname{Im} K_q(\delta^{i+2}) \arrow[r, "\varphi^{i+2}"] \arrow[d, hook] & \operatorname{Im} K_q(\delta^{i+1}) \arrow[d, hook] \\
K_q(T^{i+1}) \arrow[r, "K_q(\delta^{i+1})"'] & K_q(T^i)
\end{tikzcd}$$

Lemma 6.12. *If $i \geqslant 1$ and $q \in Q$ then there is a short exact sequence*

$$0 \to K_q(M_{i+1}) \underset{k}{\otimes} B^i \to \operatorname{Im} K_q(\delta^{i+2}) \xrightarrow{\varphi^{i+2}} \operatorname{Im} K_q(\delta^{i+1}) \to 0.$$

Proof. In view of Lemma 4.1(iv), it is enough to show that there is a commutative diagram as follows, in which the first non-trivial row is a short exact sequence.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_q(M_{i+1} \underset{k}{\otimes} B^i) \arrow[r, "\psi'''"] \arrow[d, "K_q(M_{i+1}\otimes\beta^i)"'] & \operatorname{Im} K_q(\delta^{i+2}) \arrow[r, "\varphi^{i+2}"] \arrow[d, "\psi'"] & \operatorname{Im} K_q(\delta^{i+1}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K_q(M_{i+1} \underset{k}{\otimes} I^i) \arrow[r, "K_q(\gamma^{i+1})"] \arrow[d, "K_q(M_{i+1}\otimes\alpha^i)"'] & K_q(T^{i+1}) \arrow[r, "\psi''"] \arrow[d, "K_q(\tau^{i+1})"] & \operatorname{Im} K_q(\delta^{i+1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_q(M_{i+1} \underset{k}{\otimes} B^{i+1}) \arrow[r, equal] \arrow[d] & K_q(M_{i+1} \underset{k}{\otimes} B^{i+1}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

To construct the diagram, observe that it has three non-trivial columns, each of which is short exact. The first comes from Lemma 6.5, the second from Lemma 6.10(i), and the

third is trivial. As for the morphisms between the columns, $K_q(\gamma^{i+1})$ is obtained from Diagram (6.3), which also shows that the lower left square is commutative. There is a unique induced morphism ψ''' making the upper left square commutative. The morphism ψ'' is induced by $K_q(\delta^{i+1})$, and the upper right square is commutative by Definition 6.11. The lower right square is trivially commutative.

We use the Nine Lemma, so it remains to show that the last two non-trivial rows are short exact. The row which contains an identity morphism is trivially short exact, and the row above it is short exact by Lemma 6.10(ii). $\square$

Lemma 6.13. *If $i \geqslant 1$ and $q \in Q$ then $R^1K_q(\delta^i) = 0$.*

Proof. If $i \geqslant 0$ then Diagram (6.3) contains a short exact sequence $0 \to M_{i+1} \underset{k}{\otimes} I^i \xrightarrow{\gamma^{i+1}} T^{i+1} \xrightarrow{\delta^{i+1}} T^i \to 0$. It induces a long exact sequence containing

$$R^1K_q(M_{i+1} \underset{k}{\otimes} I^i) \xrightarrow{R^1K_q(\gamma^{i+1})} R^1K_q(T^{i+1}) \xrightarrow{R^1K_q(\delta^{i+1})} R^1K_q(T^i),$$

so it is enough to see that $R^1K_q(\gamma^{i+1})$ is an epimorphism.

Setup 6.1 gives an epimorphism $I^i \xrightarrow{\alpha^i} B^{i+1}$, and

$$R^1K_q(M_{i+1} \underset{k}{\otimes} I^i) \xrightarrow{R^1K_q(M_{i+1}\otimes\alpha^i)} R^1K_q(M_{i+1} \underset{k}{\otimes} B^{i+1})$$

is an epimorphism because Lemma 4.1(iv) says it can be identified with the morphism obtained by applying the exact functor $\mathrm{Ext}^1_Q(S\langle q\rangle, M_{i+1}) \underset{k}{\otimes} -$ to α^i. Combining this with Lemma 6.6(ii) shows that applying R^1K_q to the lower square in Diagram (6.3) gives the following commutative square with an epimorphism on the left and an isomorphism on the right.

$$\begin{tikzcd}
R^1K_q(M_{i+1} \underset{k}{\otimes} I^i) \arrow[r, "R^1K_q(\gamma^{i+1})"] \arrow[d, two heads, "R^1K_q(M_{i+1}\otimes\alpha^i)"'] & R^1K_q(T^{i+1}) \arrow[d, "R^1K_q(\tau^{i+1})", "\wr"'] \\
R^1K_q(M_{i+1} \underset{k}{\otimes} B^{i+1}) \arrow[r, equal] & R^1K_q(M_{i+1} \underset{k}{\otimes} B^{i+1})
\end{tikzcd}$$

Hence $R^1K_q(\gamma^{i+1})$ is an epimorphism as desired. $\square$

Lemma 6.14. *For each $q \in Q$ we have:*

(i) $K_q(T) \cong \varprojlim K_q(T^i)$,

(ii) *There is a short exact sequence*

$$0 \to R^1 \varprojlim K_q(T^i) \to R^1 K_q(T) \to \varprojlim R^1 K_q(T^i) \to 0.$$

Proof. Recall from Construction 6.2(ii) that T is the inverse limit of $\cdots \xrightarrow{\delta^2} T^1 \xrightarrow{\delta^1} T^0$. Each δ^i is an epimorphism by Diagram (6.3), so this system satisfies the Mittag-Leffler condition and there is a short exact sequence

$$0 \to T \to \prod_i T^i \xrightarrow{\mathrm{id} - \mathrm{shift}} \prod_i T^i \to 0,$$

see Paragraph A.7. It gives a long exact sequence containing

$$\begin{aligned} 0 \to K_q(T) \to K_q\Big(\prod_i T^i\Big) &\xrightarrow{K_q(\mathrm{id} - \mathrm{shift})} K_q\Big(\prod_i T^i\Big) \\ \to R^1 K_q(T) \to R^1 K_q\Big(\prod_i T^i\Big) &\xrightarrow{R^1 K_q(\mathrm{id} - \mathrm{shift})} R^1 K_q\Big(\prod_i T^i\Big). \end{aligned} \tag{6.10}$$

Combining Equation (3.1) and Paragraph A.5(i) shows that there are natural isomorphisms

$$R^\ell K_q\Big(\prod T^i\Big) \xrightarrow{\cong} \prod R^\ell K_q(T^i),$$

so (6.10) can be identified with

$$\begin{aligned} 0 \to K_q(T) \to \prod_i K_q(T^i) &\xrightarrow{\mathrm{id} - \mathrm{shift}} \prod_i K_q(T^i) \\ \to R^1 K_q(T) \to \prod_i R^1 K_q(T^i) &\xrightarrow{\mathrm{id} - \mathrm{shift}} \prod R^1 K_q(T^i), \end{aligned}$$

which implies both parts of the lemma. □

Lemma 6.15. *Assume that $(\mathscr{A}, \mathscr{B})$ is hereditary, that $B^0 \in \mathscr{B}$, and that $M \in {}_Q\mathrm{Mod}$ has finite length. If $i \geqslant 0$ and $q \in Q$ then $K_q(M) \underset{k}{\otimes} B^i \in \mathscr{B}$.*

Proof. Since $(\mathscr{A}, \mathscr{B})$ is hereditary, $B^i \in \mathscr{B}$ for each $i \geqslant 0$ by [12, lem. 5.24]. Since $M \in {}_Q\mathrm{Mod}$ has finite length, $\dim_k K_q(M) < \infty$ by Paragraph A.4(v). Hence $K_q(M) \underset{k}{\otimes} B^i$ is in $\mathscr{B}$ because it is a finite coproduct of copies of B^i. □

Lemma 6.16. *Assume that $(\mathscr{A}, \mathscr{B})$ is hereditary and that $B^0 \in \mathscr{B}$. Then*

$$\varprojlim \big(\cdots \to \mathrm{Im}\, K_q(\delta^4) \xrightarrow{\varphi^4} \mathrm{Im}\, K_q(\delta^3) \xrightarrow{\varphi^3} \mathrm{Im}\, K_q(\delta^2) \big)$$

is in $\mathscr{B}$.

Proof. Since $\mathscr{B} = \mathscr{A}^{\perp}$, the Lukas Lemma implies that it is enough to show the following, see Paragraph A.7.

(i) φ^i is an epimorphism for $i \geqslant 3$.

(ii) $\operatorname{Im} K_q(\delta^2) \in \mathscr{B}$.

(iii) $\operatorname{Ker} \varphi^i \in \mathscr{B}$ for $i \geqslant 3$.

Lemma 6.12 gives (i). It also gives $\operatorname{Ker} \varphi^{i+2} \cong K_q(M_{i+1}) \underset{k}{\otimes} B^i$ for $i \geqslant 1$, and this is in $\mathscr{B}$ by Lemma 6.15 since M_{i+1} has finite length by Setup 6.1. This shows (iii).

To show (ii), we compute:

$$\operatorname{Im} K_q(\delta^2) \overset{\text{(a)}}{\cong} \operatorname{Im} K_q(\varepsilon^1) \overset{\text{(b)}}{\cong} K_q(E^1) \overset{\text{(c)}}{\cong} K_q(P_0 \underset{k}{\otimes} B^0) \overset{\text{(d)}}{\cong} K_q(P_0) \underset{k}{\otimes} B^0.$$

Here (a) is by Lemma 6.9. For (b), apply the left exact functor K_q to the short exact sequence (6.1) for $i = 1$. Equation (6.4) implies (c), and Lemma 4.1(iv) implies (d). But $K_q(P_0) \underset{k}{\otimes} B^0 \in \mathscr{B}$ by Lemma 6.15 since P_0 has finite length by Setup 6.1. $\square$

Lemma 6.17. *Assume that $(\mathscr{A}, \mathscr{B})$ is hereditary and that $B^0 \in \mathscr{B}$. Then $T \in \Psi(\mathscr{B})$.*

Proof. Let $q \in Q$ be given. By Definition 1.5 we must show $K_q(T) \in \mathscr{B}$ and $R^1 K_q(T) = 0$.

$K_q(T) \in \mathscr{B}$: Lemma 6.10(i) gives the vertical short exact sequences in the following diagram.

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{Im} K_q(\delta^4) \arrow[r, "\varphi^4"] \arrow[d, hook] & \operatorname{Im} K_q(\delta^3) \arrow[r, "\varphi^3"] \arrow[d, hook] & \operatorname{Im} K_q(\delta^2) \arrow[d, hook] \\
\cdots \arrow[r] & K_q(T^3) \arrow[r, "K_q(\delta^3)"] \arrow[d, two heads, "K_q(\tau^3)"'] & K_q(T^2) \arrow[r, "K_q(\delta^2)"] \arrow[d, two heads, "K_q(\tau^2)"] & K_q(T^1) \arrow[d, two heads, "K_q(\tau^1)"] \\
\cdots \arrow[r] & K_q(M_3 \underset{k}{\otimes} B^3) \arrow[r, "0"'] & K_q(M_2 \underset{k}{\otimes} B^2) \arrow[r, "0"'] & K_q(M_1 \underset{k}{\otimes} B^1)
\end{tikzcd}$$

The upper squares are commutative by Definition 6.11, and the lower squares are obviously commutative, so the diagram constitutes a short exact sequence of inverse systems. The long exact $\varprojlim$ sequence contains

$$0 \to \varprojlim \operatorname{Im} K_q(\delta^i) \to \varprojlim K_q(T^i) \to \varprojlim K_q(M_i \underset{k}{\otimes} B^i),$$

where the last term is zero because all morphisms in the third inverse system are zero. This gives the first of the following isomorphisms,

$$\varprojlim \operatorname{Im} K_q(\delta^i) \cong \varprojlim K_q(T^i) \cong K_q(T),$$

and the second isomorphism is by Lemma 6.14(i). However, the left hand side is in $\mathscr{B}$ by Lemma 6.16.

$R^1K_q(T) = 0$: We show this without using the assumption $B^0 \in \mathscr{B}$. If $i \geqslant 2$ and $q \in Q$ then the epimorphism in the short exact sequence of Lemma 6.12 shows

$$\operatorname{Im}\left(K_q(\delta^i) \circ K_q(\delta^{i+1})\right) = \operatorname{Im} K_q(\delta^i).$$

Hence the system

$$\cdots \xrightarrow{K_q(\delta^2)} K_q(T^1) \xrightarrow{K_q(\delta^1)} K_q(T^0)$$

satisfies the Mittag-Leffler condition, so the first term of the exact sequence in Lemma 6.14(ii) is zero, see Paragraph A.7. The last term is zero because Lemma 6.13 says that the morphisms vanish in the inverse system

$$\cdots \xrightarrow{R^1K_q(\delta^2)} R^1K_q(T^1) \xrightarrow{R^1K_q(\delta^1)} R^1K_q(T^0).$$

Hence $R^1K_q(T) = 0$ as desired. □

Proposition 6.18. *In the situation of Setup 3.1, if $(\mathscr{A}, \mathscr{B})$ is hereditary then condition (Seq) of Definition 1.9 holds.*

Proof. We show condition (Seq)(ii). Condition (Seq)(i) follows by a dual argument, parts of which are simplified by exactness of direct limits.

Let $q \in Q$ and $B \in \mathscr{B}$ be given. Set $M_0 = S\langle q\rangle$ and $B^0 = B$ in Setup 6.1. Consider the short exact sequence of inverse systems (6.5). The morphisms in the inverse systems are epimorphisms, so Paragraph A.7 says there is an induced short exact sequence

$$0 \to \varprojlim W^i \to \varprojlim T^i \to \varprojlim T^0 \to 0,$$

which by Equations (6.6) and (6.7) reads

$$0 \to W \to T \to S_q(B) \to 0,$$

where we have used $M_0 \underset{k}{\otimes} B^0 = S\langle q\rangle \underset{k}{\otimes} B = S_q(B)$ by Equation (3.1). We claim this sequence can be used as the sequence in condition (Seq)(ii).

We have $T \in \Psi(\mathscr{B})$ by Lemma 6.17, so it remains to show $\operatorname{Ext}^2_{Q,R}(\mathscr{E}, W) = 0$. By Lemma 4.4 it is enough to show $W \in \mathscr{E}^{\perp}$. The Lukas Lemma can be applied to the first inverse system in (6.5) because ω^i is an epimorphism for $i \geqslant 1$. Hence it is sufficient to show the following, see Paragraph A.7.

(i) $W^0 \in \mathscr{E}^{\perp}$.

(ii) $\operatorname{Ker} \omega^i \in \mathscr{E}^{\perp}$ for $i \geqslant 1$.

But (i) is trivially true because $W^0 = \operatorname{Ker} \Delta^0 = \operatorname{Ker} \mathrm{id}_{T^0} = 0$. For (ii), let $i \geqslant 1$ be given. From Diagram (6.5) it is easy to prove the first of the isomorphisms

$$\operatorname{Ker} \omega^i \cong \operatorname{Ker} \delta^i \cong M_i \underset{k}{\otimes} I^{i-1} = (*),$$

and the second isomorphism is by Diagram (6.3). But M_i has finite length and I^{i-1} is injective by Setup 6.1, so $(*) \in \mathscr{E}^{\perp}$ by Lemma 4.3. $\square$

7. Proof of Theorem B

Section 7 continues to use Setup 3.1, except that:

- Q is the quiver with relations (0.3), viewed as a k-preadditive category; see Section 0.vi.

We think of objects of ${}_Q\mathrm{Mod}$ and $\mathscr{X}$ as quiver representations. In particular, the value of X at q is denoted X_q, instead of $X(q)$ which would be used if we thought of X as a functor. Recall from Section 0.vi that each $X \in \mathscr{X}$ has the form

$$\begin{tikzcd} X_{N-1} \arrow[r, "\partial^X_{N-1}"] & X_{N-2} \arrow[r, "\partial^X_{N-2}"] & \cdots \arrow[r] & X_1 \arrow[r, "\partial^X_1"] & X_0, \arrow[llll, bend left, "\partial^X_0"] \end{tikzcd}$$

where two consecutive morphisms compose to 0. For $0 \leqslant q \leqslant N-1$ there is a homology functor $\mathscr{X} \xrightarrow{\mathrm{H}_q} {}_R\mathrm{Mod}$ defined in an obvious fashion.

Lemma 7.1. *For $0 \leqslant q \leqslant N-1$ and $X \in \mathscr{X}$ we have*

$$C_q(X) = \operatorname{Coker}(\partial^X_{q+1}) \ , \ K_q(X) = \operatorname{Ker}(\partial^X_q) \ , \ R^1 K_q = \mathrm{H}_{q-1} \ , \ L_1 C_q = \mathrm{H}_{q+1},$$

with subscripts taken modulo N.

Proof. The functor $S_q : \mathscr{M} \to \mathscr{X}$ sends an object M to an object $S_q(M)$ which has M at vertex q and 0 at all other vertices. The two first formulae in the lemma are easily verified to define left and right adjoint functors to S_q, hence define C_q and K_q.

The simple object $S\langle q\rangle$ has k at vertex q and 0 at every other vertex. There is an indecomposable projective object $P\langle q\rangle$, see Paragraph A.4(i). It has copies of k at vertices q and $q-1$ and 0 at every other vertex. The homomorphism between the copies of k is the identity map, and vertices are taken modulo N. This permits to determine the minimal augmented projective resolution of $S\langle q\rangle$ in ${}_Q\mathrm{Mod}$. The first terms are the following, with indices taken modulo N.

$$\cdots \to P\langle q-2\rangle \to P\langle q-1\rangle \to P\langle q\rangle \to S\langle q\rangle \to 0 \to \cdots$$

Each morphism of projective objects is induced by an arrow in Q. We can now compute $R^1K_q(X) = \mathrm{Ext}^1_Q(S\langle q\rangle, X)$ by using the projective resolution and Paragraph A.4(iii). This gives the third formula in the lemma, and the fourth formula is proved similarly. □

Proof of Theorem B. Paragraph 2.4 means that Theorems 3.2 and 3.3 apply to the setup of Theorem B. The formulae for $\Phi(\mathscr{A})$ and $\Psi(\mathscr{B})$ in Theorem 3.2 can be converted into the formulae in Theorem B, part (i) by using Lemma 7.1, and Theorem 3.3 implies Theorem B, part (ii). □

8. Proof of Theorem C

Section 8 continues to use Setup 3.1, except that:

- Q is the repetitive quiver $\mathbb{Z}A_3$ modulo the mesh relations, viewed as a k-preadditive category; see Section 0.vii.

As in Section 7 we think of objects of ${}_Q\mathrm{Mod}$ and $\mathscr{X}$ as quiver representations. For $j \in \mathbb{Z}$ there is an arrow $(j,0) \to (j,1)$ in Q, so a corresponding homomorphism $X_{(j,0)} \to X_{(j,1)}$ for each $X \in \mathscr{X}$. This and similar homomorphisms are used in the following two lemmas.

Lemma 8.1. *For $j \in \mathbb{Z}$ and $X \in \mathscr{X}$ we have:*

(i) *The functors C_q are given by*

$$C_{(j,0)}(X) = \mathrm{Coker}(X_{(j+1,1)} \to X_{(j,0)}),$$

$$C_{(j,1)}(X) = \mathrm{Coker}(X_{(j,0)} \oplus X_{(j+1,2)} \to X_{(j,1)}),$$

$$C_{(j,2)}(X) = \mathrm{Coker}(X_{(j,1)} \to X_{(j,2)}).$$

(ii) *The functors K_q are given by*

$$K_{(j,0)}(X) = \mathrm{Ker}(X_{(j,0)} \to X_{(j,1)}),$$

$$K_{(j,1)}(X) = \mathrm{Ker}(X_{(j,1)} \to X_{(j-1,0)} \oplus X_{(j,2)}),$$

$$K_{(j,2)}(X) = \mathrm{Ker}(X_{(j,2)} \to X_{(j-1,1)}).$$

Proof. The functor $S_q : \mathscr{M} \to \mathscr{X}$ sends an object M to an object $S_q(M)$ which has M at vertex q and 0 at all the other vertices. The formulae in the lemma are easily verified to define left and right adjoint functors to S_q, hence define C_q and K_q. $\square$

Lemma 8.2. *For $j \in \mathbb{Z}$ and $X \in \mathscr{X}$ we have:*

$$R^1K_{(j,0)}(X) = \mathrm{H}(X_{(j,0)} \to X_{(j,1)} \to X_{(j-1,0)}),$$

$$R^1K_{(j,1)}(X) = \mathrm{H}(X_{(j,1)} \to X_{(j-1,0)} \oplus X_{(j,2)} \to X_{(j-1,1)}),$$

$$R^1K_{(j,2)}(X) = \mathrm{H}(X_{(j,2)} \to X_{(j-1,1)} \to X_{(j-1,2)}).$$

Here H *denotes the homology of a three term chain complex, taken at the middle term. The mesh relations imply that the arguments of* H *are indeed chain complexes.*

Proof. For readability, the simple object $S\langle(j,\ell)\rangle$ of ${}_Q\mathrm{Mod}$ is denoted $S\langle j,\ell\rangle$. It has k at vertex (j,ℓ) and 0 at every other vertex. The indecomposable projective object $P\langle(j,\ell)\rangle$ of ${}_Q\mathrm{Mod}$ is denoted $P\langle j,\ell\rangle$. It is one of the following, where in each case, one of the vertices is identified by a superscript.

$$P\langle j,0\rangle = \cdots \quad \begin{tikzcd}[column sep=small]
0 \arrow[dr] & & 0 \arrow[dr] & & k \arrow[dr] & & 0 \arrow[dr] & & 0 \arrow[dr] & & \\
& 0 \arrow[ur] \arrow[dr] & & k \arrow[ur, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & & 0 \arrow[ur] \arrow[dr] & & 0 \arrow[ur] \arrow[dr] & & 0 \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & k^{(j,0)} \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0
\end{tikzcd} \quad \cdots$$

$$P\langle j,1\rangle = \cdots \quad \begin{tikzcd}[column sep=small]
0 \arrow[dr] & & 0 \arrow[dr] & & k \arrow[dr] & & 0 \arrow[dr] & & 0 \arrow[dr] & & \\
& 0 \arrow[ur] \arrow[dr] & & k^{(j,1)} \arrow[ur, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & & k \arrow[ur, "\mathrm{id}"] \arrow[dr, "-\mathrm{id}"'] & & 0 \arrow[ur] \arrow[dr] & & 0 \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & 0 \arrow[ur] & & k \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0
\end{tikzcd} \quad \cdots$$

$$P\langle j,2\rangle = \quad \cdots \quad \begin{tikzcd}[column sep=small] & 0 \arrow[dr] & & 0 \arrow[dr] & & k^{(j,2)} \arrow[dr, "\mathrm{id}"] & & 0 \arrow[dr] & & 0 \arrow[dr] & \\ 0 \arrow[ur] & & 0 \arrow[ur] \arrow[dr] & & 0 \arrow[ur] \arrow[dr] & & k \arrow[ur] \arrow[dr, "\mathrm{id}"] & & 0 \arrow[ur] \arrow[dr] & & 0 \\ & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & k \arrow[ur] & & 0 \arrow[ur] & \end{tikzcd} \quad \cdots$$

This permits to determine the minimal augmented projective resolutions of the simple objects in ${}_Q\mathrm{Mod}$. In each case, the first terms are the following.

$$\cdots \to P\langle j-1,0\rangle \to P\langle j,1\rangle \to P\langle j,0\rangle \to S\langle j,0\rangle \to 0 \to \cdots$$

$$\cdots \to P\langle j-1,1\rangle \to P\langle j-1,0\rangle \oplus P\langle j,2\rangle \to P\langle j,1\rangle \to S\langle j,1\rangle \to 0 \to \cdots$$

$$\cdots \to P\langle j-1,2\rangle \to P\langle j-1,1\rangle \to P\langle j,2\rangle \to S\langle j,2\rangle \to 0 \to \cdots$$

Each morphism of projective objects is induced by arrows in Q. We can now compute $R^1K_{(j,\ell)}(X) = \mathrm{Ext}^1_Q(S\langle j,\ell\rangle, X)$ by using the projective resolutions and Paragraph A.4(iii), and this gives the formulae in the lemma. □

Proof of Theorem C. Paragraph 2.4 means that Theorem 3.2 applies to the setup of Theorem C. The formulae for $\Phi(\mathscr{A})$ and $\Psi(\mathscr{B})$ in Theorem 3.2 can be converted into the formulae in Theorem C by combining Definition 1.5, Proposition 4.2, and Lemmas 8.1 and 8.2. □

Acknowledgment

We thank an anonymous referee for reading the paper carefully and making a number of useful suggestions.

We thank Jim Gillespie, Osamu Iyama, Berhard Keller, Sondre Kvamme, and Hiroyuki Minamoto for a number of illuminating comments, and Bernhard Keller for pointing out references [8] and [22].

This work was supported by EPSRC grant EP/P016014/1 "Higher Dimensional Homological Algebra" and LMS Scheme 4 Grant 41664.

Appendix A. Compendium on functor categories

In this appendix, k, Q, and R are as in Setup 3.1: k is a field, R a k-algebra, Q a small k-preadditive category satisfying conditions (Fin), (Rad), and (SelfInj) of Definition 2.3. The homomorphism functor and the radical of Q will be denoted $Q(-,-)$ and $\mathrm{rad}_Q(-,-)$, see [1, sec. A.3], [8, sec. 3.2], and [21, p. 303].

The appendix explains some properties of the functor categories ${}_Q\mathrm{Mod}$, Mod_Q, ${}_{Q,R}\mathrm{Mod}$, and ${}_Q\mathrm{Mod}_Q$ from Paragraph 2.2, which are used extensively in Sections 3 through 6. They share many properties of the module categories ${}_\Lambda\mathrm{Mod}$, Mod_Λ, ${}_{\Lambda,R}\mathrm{Mod}$,

and ${}_\Lambda\mathrm{Mod}_\Lambda$, where Λ is a finite dimensional k-algebra. This follows from conditions (Fin) and (Rad), which imply that Q is a locally bounded spectroid in the terminology of [8, secs. 3.5 and 8.3]. We can even think of Λ as self-injective because condition (SelfInj) implies that projective, injective, and flat objects coincide in each of ${}_Q\mathrm{Mod}$ and Mod_Q, see Paragraph A.6.

Note that each statement in the appendix for ${}_Q\mathrm{Mod}$ has an analogue for Mod_Q.

The appendix has been included because we do not have references for all the results we need on functor categories. Some hold by [8] as we shall point out along the way. The rest follow by amending the proofs in the following references: [2, chp. 1], [3, secs. 1-4], [4, secs. 1 and 2], [19, app. B], [25].

A.1 ***(Hom and tensor functors).*** The following functors are used extensively in this paper.

(i) The homomorphism functor of ${}_Q\mathrm{Mod}$ is

$$\mathrm{Hom}_Q(-,-) : ({}_Q\mathrm{Mod})^{\mathrm{o}} \times {}_Q\mathrm{Mod} \to {}_k\mathrm{Mod}\,.$$

It is defined by $\mathrm{Hom}_Q(M,N)$ being the set of k-linear natural transformations $M \to N$ for $M, N \in {}_Q\mathrm{Mod}$.

If $X \in {}_{Q,R}\mathrm{Mod}$, then X is a k-linear functor $Q \to {}_R\mathrm{Mod}$, so R acts on $\mathrm{Hom}_Q(M,X)$. Hence Hom_Q can also be viewed as a functor

$$\mathrm{Hom}_Q(-,-) : ({}_Q\mathrm{Mod})^{\mathrm{o}} \times {}_{Q,R}\mathrm{Mod} \to {}_R\mathrm{Mod}\,.$$

(ii) There are functors

$$-\underset{Q}{\otimes}- : \mathrm{Mod}_Q \times {}_Q\mathrm{Mod} \to {}_k\mathrm{Mod},$$

$$-\underset{Q}{\otimes}- : \mathrm{Mod}_Q \times {}_{Q,R}\mathrm{Mod} \to {}_R\mathrm{Mod},$$

$$-\underset{Q}{\otimes}- : {}_Q\mathrm{Mod}_Q \times {}_{Q,R}\mathrm{Mod} \to {}_{Q,R}\mathrm{Mod},$$

see [25, p. 93]. They are right exact in each variable, and the last of them satisfies

$$Q(-,-) \underset{Q}{\otimes} X \cong X \tag{A.1}$$

naturally in $X \in {}_{Q,R}\mathrm{Mod}$. This makes sense because $Q(-,-)$ is an object of ${}_Q\mathrm{Mod}_Q$.

(iii) There is a functor

$$\mathrm{Hom}_k(-,-) : (\mathrm{Mod}_Q)^{\mathrm{o}} \times {}_R\mathrm{Mod} \to {}_{Q,R}\mathrm{Mod}$$

defined by

$$\mathrm{Hom}_k(N, B)(q) = \mathrm{Hom}_k\big(N(q), B\big),$$

where Hom_k on the right hand side is Hom of k-vector spaces. It is exact in both variables.

(iv) There is a functor

$$- \underset{k}{\otimes} - : {}_Q\mathrm{Mod} \times {}_R\mathrm{Mod} \to {}_{Q,R}\mathrm{Mod}$$

defined by

$$(M \underset{k}{\otimes} B)(q) = M(q) \underset{k}{\otimes} B,$$

where $\underset{k}{\otimes}$ on the right hand side is tensor of k-vector spaces. It is exact in both variables.

(v) There is a functor

$$- \underset{k}{\otimes} - : \mathrm{Mod}_Q \times {}_Q\mathrm{Mod} \to {}_Q\mathrm{Mod}_Q$$

defined by

$$(M \underset{k}{\otimes} N)(q', q'') = M(q') \underset{k}{\otimes} N(q''),$$

where $\underset{k}{\otimes}$ on the right hand side is tensor of k-vector spaces. It is exact in both variables.

(vi) We can view $\mathrm{D}(-) = \mathrm{Hom}_k(-, k)$ as a functor ${}_Q\mathrm{Mod} \xrightarrow{\mathrm{D}} \mathrm{Mod}_Q$.

A.2 ***(Standard isomorphisms)***. The functors from Paragraph A.1 permit the following standard isomorphisms, among others.

(i) There is an adjunction isomorphism in ${}_k\mathrm{Mod}$,

$$\mathrm{Hom}_{Q,R}(M \underset{k}{\otimes} B, X) \xrightarrow{\cong} \mathrm{Hom}_R\big(B, \mathrm{Hom}_Q(M, X)\big),$$

natural in $M \in {}_Q\mathrm{Mod}$, $B \in {}_R\mathrm{Mod}$, $X \in {}_{Q,R}\mathrm{Mod}$.

(ii) There is an adjunction isomorphism in ${}_k\mathrm{Mod}$,

$$\mathrm{Hom}_R(M \underset{Q}{\otimes} X, B) \xrightarrow{\cong} \mathrm{Hom}_{Q,R}\big(X, \mathrm{Hom}_k(M, B)\big),$$

natural in $M \in \mathrm{Mod}_Q$, $X \in {}_{Q,R}\mathrm{Mod}$, $B \in {}_R\mathrm{Mod}$.

(iii) There is an associativity isomorphism in ${}_R\mathrm{Mod}$,

$$(M \underset{Q}{\otimes} N) \underset{k}{\otimes} B \xrightarrow{\cong} M \underset{Q}{\otimes} (N \underset{k}{\otimes} B),$$

natural in $M \in \mathrm{Mod}_Q$, $N \in {}_Q\mathrm{Mod}$, $B \in {}_R\mathrm{Mod}$, where $\underset{k}{\otimes}$ on the left hand side is tensor of k-vector spaces.

(iv) There is an associativity isomorphism in ${}_{Q,R}\mathrm{Mod}$,

$$(M \underset{k}{\otimes} N) \underset{Q}{\otimes} X \xrightarrow{\cong} N \underset{k}{\otimes} (M \underset{Q}{\otimes} X),$$

natural in $M \in \mathrm{Mod}_Q$, $N \in {}_Q\mathrm{Mod}$, $X \in {}_{Q,R}\mathrm{Mod}$.

(v) There is a morphism in ${}_R\mathrm{Mod}$,

$$\mathrm{Hom}_Q(M, N) \underset{k}{\otimes} B \to \mathrm{Hom}_Q(M, N \underset{k}{\otimes} B),$$

natural in $M, N \in {}_Q\mathrm{Mod}$, $B \in {}_R\mathrm{Mod}$. It is an isomorphism if M has finite length. Note that $\underset{k}{\otimes}$ on the left hand side is tensor of k-vector spaces.

(vi) There is a morphism in ${}_{Q,R}\mathrm{Mod}$,

$$M \underset{k}{\otimes} B \to \mathrm{Hom}_k(\mathrm{D}M, B),$$

natural in $M \in {}_Q\mathrm{Mod}$, $B \in {}_R\mathrm{Mod}$. It is an isomorphism if M has finite length.

A.3 ***(Products and coproducts)***. We will explain products and coproducts in ${}_Q\mathrm{Mod}$. What we say applies equally to Mod_Q and ${}_{Q,R}\mathrm{Mod}$.

Let $\{M_\alpha\}$ be a family of objects of ${}_Q\mathrm{Mod}$. The product of the M_α in ${}_Q\mathrm{Mod}$ is given by

$$\Big(\prod_\alpha M_\alpha\Big)(-) = \prod_\alpha M_\alpha(-),$$

where the second $\prod$ is in ${}_k\mathrm{Mod}$. There is a similar formula for $\coprod$. This implies that ${}_Q\mathrm{Mod}$ inherits the following properties from ${}_k\mathrm{Mod}$: It is complete and cocomplete, and products, coproducts, and filtered colimits preserve exact sequences.

Each of the tensor product functors from Paragraph A.1 preserves coproducts in each variable.

A.4 ***(Projective, injective, and simple objects)***. Each of the categories ${}_Q\mathrm{Mod}$, Mod_Q, ${}_{Q,R}\mathrm{Mod}$, and ${}_Q\mathrm{Mod}_Q$ has enough projective objects and enough injective objects. We list some additional properties.

(i) By [8, sec. 3.7] we have the following: For each $q \in Q$ there is an indecomposable projective object

$$P\langle q\rangle = Q(q,-)$$

in ${}_Q\mathrm{Mod}$. By conditions (Fin) and (Rad) of Definition 2.3, it has finite length and there is a unique maximal subobject $\mathfrak{r}P\langle q\rangle \subsetneq P\langle q\rangle$ given by $\mathfrak{r}P\langle q\rangle = \mathrm{rad}_Q(q,-)$. The quotient

$$S\langle q\rangle = P\langle q\rangle/\mathfrak{r}P\langle q\rangle$$

is a simple object in ${}_Q\mathrm{Mod}$, which satisfies

$$S\langle q\rangle(p) \cong \begin{cases} k & \text{for } p = q, \\ 0 & \text{otherwise.} \end{cases}$$

The simple objects of ${}_Q\mathrm{Mod}$ are precisely the $S\langle q\rangle$ for $q \in Q$. The simple objects of Mod_Q are precisely the duals $\mathrm{D}S\langle q\rangle$ for $q \in Q$.

(ii) By [8, p. 85, exa. 2] we have the following: Each $M \in {}_Q\mathrm{Mod}$ has an augmented projective resolution

$$\cdots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} M \to 0 \to \cdots,$$

which can be constructed by choosing an epimorphism $P_0 \overset{\partial_0}{\twoheadrightarrow} M$ with P_0 projective, then, when ∂_{i-1} has been defined, choosing an epimorphism $P_i \twoheadrightarrow \operatorname{Ker}\partial_{i-1}$ and defining ∂_i to be the composition $P_i \twoheadrightarrow \operatorname{Ker}\partial_{i-1} \hookrightarrow P_{i-1}$.

If M has finite length, then condition (Fin) of Definition 2.3 implies that each P_i can be chosen as a coproduct of finitely many objects of the form $P\langle q\rangle$, and then each P_i and each $\operatorname{Ker}\partial_i$ has finite length. Moreover, by choosing each of the epimorphisms $P_0 \overset{\partial_0}{\twoheadrightarrow} M$ and $P_i \twoheadrightarrow \operatorname{Ker}\partial_{i-1}$ as a projective cover, we can even make the resolution *minimal*, that is, if $i \geqslant 1$ then ∂_i is in the radical of ${}_Q\mathrm{Mod}$. This implies that the functors $\mathrm{D}S\langle q\rangle \underset{Q}{\otimes} -$ and $\mathrm{Hom}_Q(S\langle q\rangle,-)$ vanish on ∂_i.

(iii) A morphism $p \xrightarrow{\pi} q$ in Q induces a natural transformation $Q(q,-) \to Q(p,-)$, that is, a morphism $P\langle q\rangle \to P\langle p\rangle$. By Yoneda's Lemma, this in turn induces a commutative square

$$\begin{tikzcd}
\mathrm{Hom}_Q(P\langle p\rangle, X) \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}_Q(P\langle q\rangle, X) \arrow[d, "\wr"] \\
X(p) \arrow[r, "X(\pi)"'] & X(q),
\end{tikzcd}$$

natural in $X \in {}_{Q,R}\mathrm{Mod}$, where the vertical arrows are isomorphisms.

(iv) If X is a projective object of ${}_{Q,R}\mathrm{Mod}$, then X is projective when viewed as an object of ${}_Q\mathrm{Mod}$.

(v) $M, N \in {}_Q\mathrm{Mod}$ have finite length $\Rightarrow \dim_k \mathrm{Hom}_Q(M,N) < \infty$.

A.5 *(*Ext ***and*** Tor ***functors****).* The functors Hom_Q and $\underset{Q}{\otimes}$ of Paragraph A.1 have right and left derived functors,

$$\mathrm{Ext}^i_Q(-,-) : ({}_Q\mathrm{Mod})^{\mathrm{o}} \times {}_Q\mathrm{Mod} \to {}_k\mathrm{Mod},$$
$$\mathrm{Tor}^Q_i(-,-) : \mathrm{Mod}_Q \times {}_Q\mathrm{Mod} \to {}_k\mathrm{Mod}$$

for $i \geqslant 0$. Like Hom_Q and $\underset{Q}{\otimes}$ they can also be viewed as functors

$$\mathrm{Ext}^i_Q(-,-) : ({}_Q\mathrm{Mod})^{\mathrm{o}} \times {}_{Q,R}\mathrm{Mod} \to {}_R\mathrm{Mod},$$
$$\mathrm{Tor}^Q_i(-,-) : \mathrm{Mod}_Q \times {}_{Q,R}\mathrm{Mod} \to {}_R\mathrm{Mod}\,.$$

We list some additional properties.

(i) Since products preserve exact sequences, there are isomorphisms in ${}_R\mathrm{Mod}$,

$$\mathrm{Ext}^i_Q(N, \prod_\alpha M_\alpha) \xrightarrow{\cong} \prod_\alpha \mathrm{Ext}^i_Q(N, M_\alpha),$$

natural in $N \in {}_Q\mathrm{Mod}$ and $M_\alpha \in {}_{Q,R}\mathrm{Mod}$.

(ii) The morphism in Paragraph A.2(v) induces standard morphisms in ${}_R\mathrm{Mod}$,

$$\mathrm{Ext}^i_Q(M,N) \underset{k}{\otimes} B \to \mathrm{Ext}^i_Q(M, N \underset{k}{\otimes} B),$$

natural in $M, N \in {}_Q\mathrm{Mod}$, $B \in {}_R\mathrm{Mod}$. They are isomorphisms if M has finite length.

(iii) The isomorphism in Paragraph A.2(iii) induces standard isomorphisms in ${}_R\mathrm{Mod}$,

$$\mathrm{Tor}^Q_i(M,N) \underset{k}{\otimes} B \xrightarrow{\cong} \mathrm{Tor}^Q_i(M, N \underset{k}{\otimes} B),$$

natural in $M \in \mathrm{Mod}_Q$, $N \in {}_Q\mathrm{Mod}$, $B \in {}_R\mathrm{Mod}$.

A.6 ***(Criteria for injectivity and flatness)****.* Condition (Fin) of Definition 2.3 implies that $M \in {}_Q\mathrm{Mod}$ satisfies

$$M \text{ is injective } \Leftrightarrow \text{ If } q \in Q \text{ then } \mathrm{Ext}^1_Q(S\langle q\rangle, M) = 0. \tag{A.2}$$

Similarly, M is *flat* if the functor $- \underset{Q}{\otimes} M$ is exact, and

$$M \text{ is flat } \Leftrightarrow \text{ If } q \in Q \text{ then } \mathrm{Tor}_1^Q(\mathrm{D}S\langle q\rangle, M) = 0. \tag{A.3}$$

Conditions (Fin) and (SelfInj) of Definition 2.3 imply that $M \in {}_Q\mathrm{Mod}$ satisfies

$$M \text{ is projective } \Leftrightarrow M \text{ is flat } \Leftrightarrow M \text{ is injective.} \tag{A.4}$$

A.7 ***(Inverse limits)***. We will explain inverse limits in ${}_Q\mathrm{Mod}$. What we say applies equally to Mod_Q and ${}_{Q,R}\mathrm{Mod}$.

Since products exist and preserve exact sequences, the results on (derived) inverse limits in [31, sec. 3.5] apply. In particular, if there is an inverse system

$$\cdots \to M^2 \xrightarrow{\mu^2} M^1 \xrightarrow{\mu^1} M^0, \tag{A.5}$$

then there is an exact sequence

$$0 \to \varprojlim M^i \to \prod_i M^i \xrightarrow{\mathrm{id} - \mathrm{shift}} \prod_i M^i \to R^1 \varprojlim M^i \to 0,$$

where $\mathrm{id} - \mathrm{shift}$ is the difference between the identity morphism and the shift morphism induced by the μ^i.

The inverse system is said to satisfy the *Mittag-Leffler condition* if, for each $i \geqslant 0$, the images of the maps $M^\ell \to M^i$ for $\ell \geqslant i$ satisfy the descending chain condition. In this case we have $R^1 \varprojlim M^i = 0$. This holds in particular if each morphism in (A.5) is an epimorphism.

If there is a short exact sequence

$$\begin{tikzcd}
\cdots \arrow[r, two heads] & M'^2 \arrow[r, two heads] \arrow[d, hook] & M'^1 \arrow[r, two heads] \arrow[d, hook] & M'^0 \arrow[d, hook] \\
\cdots \arrow[r] & M^2 \arrow[r] \arrow[d, two heads] & M^1 \arrow[r] \arrow[d, two heads] & M^0 \arrow[d, two heads] \\
\cdots \arrow[r] & M''^2 \arrow[r] & M''^1 \arrow[r] & M''^0
\end{tikzcd}$$

of inverse systems, then there is an induced long exact sequence

$$0 \to \varprojlim M'_i \to \varprojlim M_i \to \varprojlim M''_i \to R^1 \varprojlim M'_i \to R^1 \varprojlim M_i \to R^1 \varprojlim M''_i \to 0.$$

If each morphism in the M'-system is an epimorphism, then $R^1 \varprojlim M'_i = 0$ and there is a short exact sequence

$$0 \to \varprojlim M'_i \to \varprojlim M_i \to \varprojlim M''_i \to 0.$$

Some of the results on inverse limits in [12, sec. 6] also apply. In particular, the Lukas Lemma says that if N is fixed, then in order to conclude $\varprojlim M^i \in N^\perp$, it is enough to verify the following for the inverse system (A.5), see [12, lem. 6.37].

(i) μ^i is an epimorphism for $i \geqslant 1$.
(ii) $M^0 \in N^\perp$.
(iii) $\operatorname{Ker} \mu^i \in N^\perp$ for $i \geqslant 1$.

A.8 ***(The radical filtration)***. If $i \geqslant 0$ then the i'th power of the radical, $\operatorname{rad}^i_Q(-,-)$, is an object of ${}_Q\mathrm{Mod}_Q$. Because of condition (Rad) of Definition 2.3, there is a finite filtration in ${}_Q\mathrm{Mod}_Q$,

$$0 = \operatorname{rad}^N_Q \subsetneq \cdots \subsetneq \operatorname{rad}^1_Q \subsetneq \operatorname{rad}^0_Q = Q(-,-), \tag{A.6}$$

where $N \geqslant 0$ is chosen minimal such that $\operatorname{rad}^N_Q = 0$. Each quotient $\operatorname{rad}^i_Q / \operatorname{rad}^{i-1}_Q$ is annihilated on both sides by rad_Q, and this implies

$$\operatorname{rad}^i_Q / \operatorname{rad}^{i-1}_Q \cong \coprod_{p,q \in Q} (\mathrm{D}S\langle p\rangle \underset{k}{\otimes} S\langle q\rangle)^{n_i(p,q)} \tag{A.7}$$

for certain integers $n_i(p,q)$.

References

[1] I. Assem, D. Simson, A. Skowroński, Elements of the Representation Theory of Associative Algebras, London Math. Soc. Stud. Texts, vol. 1 vol. 65, Cambridge University Press, Cambridge, 2006.
[2] M. Auslander, Representation Dimension of Artin Algebras, Queen Mary College Mathematics Notes, Queen Mary College, London, 1971, Reprinted in: in: Selected Works of Maurice Auslander, vol. 1, Reiten, Smalø, Solberg (Eds.), American Mathematical Society, Providence, 1999, pp. 505–574.
[3] M. Auslander, Representation theory of Artin algebras I, Comm. Algebra 1 (1974) 177–268.
[4] M. Auslander, Representation theory of Artin algebras II, Comm. Algebra 1 (1974) 269–310.
[5] L. Bican, R. El Bashir, E.E. Enochs, All modules have flat covers, Bull. Lond. Math. Soc. 33 (2001) 385–390.
[6] H. Cartan, S. Eilenberg, Homological Algebra, Princeton University Press, Princeton, 1956, Reprinted in: Princeton Landmarks Math., Princeton University Press, Princeton, 1999.
[7] N. Ding, X. Yang, On a question of Gillespie, Forum Math. 27 (2015) 3205–3231.
[8] P. Gabriel, A.V. Roiter, Algebra VIII: Representations of Finite-Dimensional Algebras, with a chapter by B. Keller, Encyclopaedia Math. Sci., vol. 73, Springer-Verlag, Berlin, 1992.
[9] J. Gillespie, Hereditary abelian model categories, Bull. Lond. Math. Soc. 48 (2016) 895–922.
[10] J. Gillespie, How to construct a Hovey triple from two cotorsion pairs, Fund. Math. 230 (2015) 281–289.
[11] J. Gillespie, The flat model structure on $\mathrm{Ch}(R)$, Trans. Amer. Math. Soc. 356 (2004) 3369–3390.
[12] R. Göbel, J. Trlifaj, Approximations and Endomorphism Algebras of Modules, 2nd revised and extended edition, de Gruyter Exp. Math., vol. 41, Walter de Gruyter, Berlin/Boston, 2012.
[13] P.S. Hirschhorn, Model Categories and Their Localizations, Math. Surveys Monogr., vol. 99, American Mathematical Society, Providence, 2003.

[14] H. Holm, P. Jørgensen, Covers, precovers, and purity, Illinois J. Math. 52 (2008) 691–703.
[15] M. Hovey, Cotorsion pairs, model category structures, and representation theory, Math. Z. 241 (2002) 553–592.
[16] M. Hovey, Model Categories, Math. Surveys Monogr., vol. 63, American Mathematical Society, Providence, 1999.
[17] O. Iyama, H. Minamoto, $\mathscr{A}$-derived categories, in preparation.
[18] O. Iyama, H. Minamoto, On a generalization of complexes and their derived categories, in: 47th Ring and Representation Theory Symposium, Osaka City University, 2014, extended abstract of a talk.
[19] C.U. Jensen, H. Lenzing, Model-Theoretic Algebra with Particular Emphasis on Fields, Rings, Modules, Algebra Logic Appl., vol. 2, Gordon and Breach, New York, 1989.
[20] M.M. Kapranov, On the q-analog of homological algebra, preprint, arXiv:q-alg/9611005, 1996.
[21] G.M. Kelly, On the radical of a category, J. Aust. Math. Soc. 4 (1964) 299–307.
[22] B. Leclerc, P.-G. Plamondon, Nakajima varieties and repetitive algebras, Publ. Res. Inst. Math. Sci. 49 (2013) 531–561.
[23] J. Lurie, Higher Topos Theory, Ann. of Math. Stud., vol. 170, Princeton University Press, Princeton, 2009.
[24] H. Nakaoka, A simultaneous generalization of mutation and recollement of cotorsion pairs on a triangulated category, Appl. Categ. Structures 26 (2018) 491–544.
[25] U. Oberst, H. Rohrl, Flat and coherent functors, J. Algebra 14 (1970) 91–105.
[26] D. Quillen, Homotopical Algebra, Lecture Notes in Math., vol. 43, Springer, Berlin–Heidelberg, 1967.
[27] J.J. Rotman, An Introduction to Homological Algebra, second edition, Springer, New York, 2009.
[28] L. Salce, Cotorsion theories for abelian groups, in: Symposia Mathematica, Vol. XXIII, Conf. Abelian Groups and Their Relationship to the Theory of Modules, INDAM, Rome, 1977, Academic Press, London–New York, 1979, pp. 11–32.
[29] J. Schröer, On the quiver with relations of a repetitive algebra, Arch. Math. (Basel) 72 (1999) 426–432.
[30] J. Šťovíček, Exact model categories, approximation theory, and cohomology of quasi-coherent sheaves, in: Benson, Krause, Skowroński (Eds.), Advances in Representation Theory of Algebras, Proceedings of the International Conference on Representations of Algebras, Bielefeld, 2015, in: EMS Series of Congress Reports, European Mathematical Society, Zürich, 2013, pp. 297–367.
[31] C. Weibel, An Introduction to Homological Algebra, Cambridge Stud. Adv. Math., vol. 38, Cambridge University Press, Cambridge, 1994.