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The tensor embedding for a Grothendieck cosmos

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Abstract While the Yoneda embedding and its generalizations have been studied extensively in the literature, the so-called tensor embedding has only received a little attention. In this paper, we study the tensor embedding for closed symmetric monoidal categories and show how it is connected to the notion of geometrically purity, which has recently been investigated in the works of Enochs et al. (2016) and Estrada et al. (2017). More precisely, for a Grothendieck cosmos, i.e., a bicomplete Grothendieck category $\mathcal{V}$ with a closed symmetric monoidal structure, we prove that the geometrically pure exact category $(\mathcal{V}, \mathscr{E}_{\otimes})$ has enough relative injectives; in fact, every object has a geometrically pure injective envelope. We also show that for some regular cardinal λ, the tensor embedding yields an exact equivalence between $(\mathcal{V}, \mathscr{E}_{\otimes})$ and the category of λ-cocontinuous $\mathcal{V}$-functors from $\mathrm{Pres}_{\lambda}(\mathcal{V})$ to $\mathcal{V}$, where the former is the full $\mathcal{V}$-subcategory of λ-presentable objects in $\mathcal{V}$. In many cases of interest, λ can be chosen to be $\aleph_0$ and the tensor embedding identifies the geometrically pure injective objects in $\mathcal{V}$ with the (categorically) injective objects in the abelian category of $\mathcal{V}$-functors from $\mathrm{fp}(\mathcal{V})$ to $\mathcal{V}$. As we explain, the developed theory applies, e.g., to the category $\mathsf{Ch}(R)$ of chain complexes of modules over a commutative ring R and to the category $\mathsf{Qcoh}(X)$ of quasi-coherent sheaves over a (suitably nice) scheme X.

Keywords enriched functor, exact category, (pre)envelope, (pure) injective object, purity, symmetric monoidal category, tensor embedding, Yoneda embedding

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1 Introduction

By the Gabriel-Quillen embedding theorem (see [47, Theorem A.7.1]), any small exact category admits an exact full embedding, which also reflects exactness, into some abelian category. Hence, any small exact category is equivalent, as an exact category, to an extension-closed subcategory of an abelian category. Actually, the same is true for many large exact categories of interest. Consider, for example the category R-Mod of left R-modules equipped with the *pure exact structure* $\mathscr{E}_{\mathrm{pure}}$, where the "exact sequences" (the conflations) are directed colimits of split exact sequences in R-Mod. The exact category $(R\text{-Mod}, \mathscr{E}_{\mathrm{pure}})$ admits two different exact full embeddings into abelian categories. One is the Yoneda embedding, i.e.,

$$(R\text{-Mod}, \mathscr{E}_{\mathrm{pure}}) \to [(R\text{-mod})^{\mathrm{op}}, \mathsf{Ab}]_0 \quad \text{given by } M \mapsto \mathrm{Hom}_R(-, M)\,|_{R\text{-mod}}; \tag{1.1}$$

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the other is the so-called tensor embedding, i.e.,

$$(R\text{-Mod}, \mathscr{E}_{\text{pure}}) \to [\text{mod-}R, \mathsf{Ab}]_0 \quad \text{given by } M \mapsto (- \otimes_R M)\,|_{\text{mod-}R}. \tag{1.2}$$

Here, "mod" means finitely presentable modules and $[\mathcal{X}, \mathsf{Ab}]_0$ denotes the category of additive functors from $\mathcal{X}$ to the category Ab of abelian groups. While the Yoneda embedding identifies pure projective left R-modules with projective objects in $[(R\text{-mod})^{\text{op}}, \mathsf{Ab}]_0$, the tensor embedding induces an equivalence between pure injective left R-modules and injective objects in $[\text{mod-}R, \mathsf{Ab}]_0$. For a detailed discussion and proofs of these embeddings, we refer to [29, Theorems B.11 and B.16].

There are several important and interesting generalizations of the Yoneda embedding (1.1). For example, any locally λ-presentable abelian category $\mathcal{V}$, where λ is a regular cardinal, can be equipped with the so-called *categorically pure exact structure* $\mathscr{E}_\lambda$ treated in [2] by Adámek and Rosický. Also in this case, the Yoneda functor

$$(\mathcal{V}, \mathscr{E}_\lambda) \to [\text{Pres}_\lambda(\mathcal{V})^{\text{op}}, \mathsf{Ab}]_0 \quad \text{given by } X \mapsto \text{Hom}_{\mathcal{V}}(-, X)\,|_{\text{Pres}_\lambda(\mathcal{V})} \tag{1.3}$$

is an exact full embedding, where $\text{Pres}_\lambda(\mathcal{V})$ is the category of λ-presentable objects in $\mathcal{V}$. For a locally finitely presentable category $\mathcal{V}$, the Yoneda embedding (1.3) with $\lambda = \aleph_0$ identifies pure projective objects in $\mathcal{V}$ (objects that are projective relative to the exact structure $\mathscr{E}_{\aleph_0}$) with projective objects in $[\text{fp}(\mathcal{V})^{\text{op}}, \mathsf{Ab}]_0$ (see Crawley and Boevey [13, (1.4) and Section 3] and Lenzing [33]). The Yoneda embedding (1.1) is the special case $\mathcal{V} = R\text{-Mod}$ and $\lambda = \aleph_0$.

In this paper, we study a generalization of the tensor embedding (1.2), where $(R\text{-Mod}, \mathscr{E}_{\text{pure}})$ is replaced by another exact category $(\mathcal{V}, \mathscr{E}_\otimes)$. Of course, to make sense of such a tensor embedding, one must require the existence of a suitable tensor product in $\mathcal{V}$. We consider the situation, where $(\mathcal{V}, \otimes, I, [-,-])$ is an *abelian cosmos*, as explained in Setup 3.3. The exact structure $\mathscr{E}_\otimes$ imposed on $\mathcal{V}$ is the so-called *geometrically pure exact structure*, where the admissible monomorphisms are the geometrically pure monomorphisms introduced by Fox [19] (see Definition 3.4). Before we present our main results about the exact category $(\mathcal{V}, \mathscr{E}_\otimes)$, let us mention a few concrete examples for the readers to have in mind.

(i) The category $\mathcal{V} = \mathsf{Ch}(R)$ of chain complexes of modules over a commutative ring R equipped with the *total tensor product* $\otimes = \otimes_R^\bullet$ is an abelian cosmos. In this case, a short exact sequence $0 \to C' \to C \to C'' \to 0$ is in $\mathscr{E}_\otimes$ if and only if it is degree-wise pure exact, meaning that $0 \to C'_n \to C_n \to C''_n \to 0$ is a pure exact sequence of R-modules for every $n \in \mathbb{Z}$ (see Example 3.5(a) for details).

(ii) The category $\mathcal{V} = \mathsf{Ch}(R)$ of chain complexes of modules over a commutative ring R equipped with the *modified* total tensor product $\otimes = \underline{\otimes}_R^\bullet$ is also an abelian cosmos. In this case, a short exact sequence $0 \to C' \to C \to C'' \to 0$ is in $\mathscr{E}_\otimes$ if and only if it is categorically pure in the sense discussed above. More precisely, the category $\mathsf{Ch}(R)$ is a locally $\aleph_0$-presentable (= locally finitely presentable) and the categorically pure exact structure $\mathscr{E}_{\aleph_0}$ agrees with the geometrically pure exact structure $\mathscr{E}_\otimes$ for the tensor product $\otimes = \underline{\otimes}_R^\bullet$ in question (see Example 3.5(b) for details).

(iii) The category $\mathcal{V} = \mathsf{Qcoh}(X)$ of quasi-coherent sheaves on a scheme $(X, \mathscr{O}_X)$ equipped with the usual tensor product $\otimes = \otimes_X$ is an abelian cosmos. If X is quasi-separated, then a short exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is in $\mathscr{E}_\otimes$ if and only if it is stalk-wise pure exact, meaning that $0 \to \mathscr{F}'_x \to \mathscr{F}_x \to \mathscr{F}''_x \to 0$ is a pure exact sequence of $\mathscr{O}_{X,x}$-modules for every $x \in X$ (see Example 3.6 for details). Note that for an affine scheme $X = \operatorname{Spec} R$, the exact category $(\mathcal{V}, \mathscr{E}_\otimes) = (\mathsf{Qcoh}(X), \mathscr{E}_{\otimes_X})$ coincides with the exact category $(R\text{-Mod}, \mathscr{E}_{\text{pure}})$ that appears in (1.1) and (1.2).

We now describe our main results about the exact category $(\mathcal{V}, \mathscr{E}_\otimes)$. In general, an exact category need not have enough relative injectives (= objects in that are injective relative to the exact structure); in fact, for a general exact category, the question about the existence of enough relative injectives is delicate. The first main result about the geometrically pure exact category, which we prove in Section 3 (see Proposition 3.12 and Theorem 3.13), is the following theorem.

Theorem A. *The exact category $(\mathcal{V}_0, \mathscr{E}_\otimes)$ has enough relative injectives. In the language of relative homological algebra, this means that every object in $\mathcal{V}_0$ has a geometrically pure injective preenvelope. If $\mathcal{V}_0$ is Grothendieck, then every object in $\mathcal{V}_0$ even has a geometrically pure injective envelope.*

The first part of this result can also be found in [27, Theorem 2.6]. To avoid confusion, we often write $\mathcal{V}_0$ when we consider $\mathcal{V}$ as an ordinary category (as in the theorem above), and we use the symbol $\mathcal{V}$ when it is viewed as a $\mathcal{V}$-category (see 2.6).

Next, we turn to the construction of the tensor embedding for the exact category $(\mathcal{V}_0, \mathscr{E}_\otimes)$. We begin with a quite general version of this functor. The main result in Section 4 is Theorem 4.6, which contains the following theorem.

Theorem B. *Let $\mathcal{A}$ be any small full $\mathcal{V}$-subcategory of $\mathcal{V}$ containing the unit object I for the tensor product $\otimes$. The tensor embedding yields a fully faithful exact functor, i.e.,*

$$\Theta_0 \colon (\mathcal{V}_0, \mathscr{E}_\otimes) \to ([\mathcal{A}, \mathcal{V}]_0, \mathscr{E}_\star) \quad \text{given by } X \mapsto (X \otimes -)\,|_{\mathcal{A}},$$

which induces an equivalence of exact categories $(\mathcal{V}_0, \mathscr{E}_\otimes) \simeq (\mathrm{Ess.\,Im}\,\Theta, \mathscr{E}_\star\,|_{\mathrm{Ess.Im}\,\Theta})$.

Here, $[\mathcal{A}, \mathcal{V}]_0$ denotes the ordinary category of $\mathcal{V}$-functors $\mathcal{A} \to \mathcal{V}$, which is abelian by Al Hwaeer and Garkusha [3, Theorem 4.2]. Being an abelian category, $[\mathcal{A}, \mathcal{V}]_0$ has a canonical exact structure $\mathscr{E}_{\mathrm{ab}}$; but Theorem B is *not* true (for all the choices of $\mathcal{A}$) if we consider $[\mathcal{A}, \mathcal{V}]_0$ as an exact category in this way. Instead one has to equip $[\mathcal{A}, \mathcal{V}]_0$ with the so-called *$\star$-pure exact structure $\mathscr{E}_\star$*. This exact structure is introduced in Definition 3.16 and it is usually strictly coarser than the exact structure induced by the abelian structure on $[\mathcal{A}, \mathcal{V}]_0$.

So far (in Sections 3 and 4) $\mathcal{V}$ has been an abelian cosmos and $\mathcal{A} \subseteq \mathcal{V}$ has been any small full $\mathcal{V}$-subcategory containing the unit object. In Section 5, we require $\mathcal{V}$ to be a *Grothendieck* cosmos. We show in Proposition 5.2 that there exists some regular cardinal λ for which $\mathcal{V}$ is a locally λ-presentable base (see 2.5), and we focus now only on the case where $\mathcal{A} = \mathrm{Pres}_\lambda(\mathcal{V})$ is the the $\mathcal{V}$-subcategory of λ-presentable objects in $\mathcal{V}$ (in the ordinary categorical sense, or in the enriched sense; it makes no difference by [6, Corollary 3.3]). In this special situation, Theorem B simplifies a great deal because it is possible to get the following:

- Explicitly describe the essential image $\mathrm{Ess.\,Im}\,\Theta_0$ of the tensor embedding Θ_0.
- Show that the (rather strange) $\star$-pure exact structure $\mathscr{E}_\star$ and the (canonical) abelian exact structure $\mathscr{E}_{\mathrm{ab}}$ on $[\mathcal{A}, \mathcal{V}]_0 = [\mathrm{Pres}_\lambda(\mathcal{V}), \mathcal{V}]_0$ coincide on $\mathrm{Ess.\,Im}\,\Theta_0$.

The precise statements are given below; they are contained in Theorem 5.9, which is the main result of Section 5.

Theorem C. *The essential image of the fully faithful tensor embedding*

$$\Theta_0 \colon \mathcal{V}_0 \to [\mathrm{Pres}_\lambda(\mathcal{V}), \mathcal{V}]_0 \quad \text{given by } X \mapsto (X \otimes -)\,|_{\mathrm{Pres}_\lambda(\mathcal{V})}$$

is precisely $\mathrm{Ess.\,Im}\,\Theta = \lambda\text{-Cocont}(\mathrm{Pres}_\lambda(\mathcal{V}), \mathcal{V})$*, i.e., the subcategory of λ-cocontinuous $\mathcal{V}$-functors from* $\mathrm{Pres}_\lambda(\mathcal{V})$ *to $\mathcal{V}$. Furthermore, Θ_0 induces an equivalence of exact categories, i.e.,*

$$(\mathcal{V}_0, \mathscr{E}_\otimes) \simeq \lambda\text{-Cocont}(\mathrm{Pres}_\lambda(\mathcal{V}), \mathcal{V});$$

the exact structure on the right-hand side is induced by the abelian structure on $[\mathrm{Pres}_\lambda(\mathcal{V}), \mathcal{V}]_0$.

The definition of λ-cocontinuous $\mathcal{V}$-functors is given in Definition 5.6. The Grothendieck cosmos $\mathcal{V} = R$-Mod, where R is a commutative ring, is locally finitely presentable, so Theorem C applies with $\lambda = \aleph_0$ to recover the classical tensor embedding (1.2). As mentioned in the beginning of Section 1, it is known that this tensor embedding restricts to an equivalence between pure injective R-modules and (categorically) injective objects in the functor category. The purpose of the final Section 6 is to establish such a result in the much more general context of Theorem C.

In Section 6, we specialize the setup even further: $\mathcal{V}$ is required to be a Grothendieck cosmos (as in Section 5) generated by a set of *dualizable* objects and the unit object I is assumed to be finitely presentable (see Setup 6.1). This specialized setup still applies to several examples; indeed, the category $\mathsf{Ch}(R)$ from (i) and (ii) above always satisfies these requirements, and so does the category $\mathsf{Qcoh}(X)$ from (iii) for most schemes X (see Examples 6.2 and 6.3). We prove in Proposition 6.9 that such a category $\mathcal{V}$ is a locally finitely presentable base, which means that we can apply Theorem C with $\lambda = \aleph_0$. In this case, $\mathrm{Pres}_{\aleph_0}(\mathcal{V}) =: \mathrm{fp}(\mathcal{V})$ is the class of finitely presentable objects in $\mathcal{V}$. The main result in the last section is Theorem 6.13, of which the following is a special case.

Theorem D. *For $\mathcal{V}$ as in Setup* 6.1*, the tensor embedding from Theorem* C *with $\lambda = \aleph_0$ restricts to an equivalence between the geometrically pure injective objects in $\mathcal{V}_0$ and the* (*categorically*) *injective objects in* $[\mathrm{fp}(\mathcal{V}), \mathcal{V}]_0$. *In symbols,* $\mathrm{PureInj}_{\otimes}(\mathcal{V}_0) \simeq \mathrm{Inj}([\mathrm{fp}(\mathcal{V}), \mathcal{V}]_0)$.

The paper ends with Remark 6.14, where we explain how many of our results can be extended to an even more general (but also more technical) setting.

We close Section 1 with some perspective remarks and an explanation of how our work is related to the existing literature.

We should start by pointing out that we are not the first to study the geometrically pure exact structure; however, its connections with the tensor embedding uncovered in this paper seem to be new. In fact, the investigation of geometrically purity was initiated by Fox [19] and was recently continued by Enochs et al. [14] and Estrada et al. [17].

Notice that if $\mathcal{V}$ is a Grothendieck cosmos, then by Proposition 5.2, there is a regular cardinal λ for which $\mathcal{V}$ is a locally λ-presentable base; in particular, $\mathcal{V}$ is a locally λ-presentable category (see 2.1 and 2.5), so it makes sense to consider both the Yoneda embedding (1.3) and the tensor embedding from Theorem C. Even though these embeddings are akin, there is a fundamental difference: while the Yoneda embedding is related to the categorically pure exact structure $\mathscr{E}_\lambda$, the tensor embedding pertains the geometrically pure exact structure $\mathscr{E}_\otimes$. The former exact structure is coarser than the latter, i.e., one has $\mathscr{E}_\lambda \subseteq \mathscr{E}_\otimes$, and in general this containment is strict. In the discussion preceding Setup 3.3, we compare these two exact structures.

The category $\mathsf{Qcoh}(X)$ of quasi-coherent sheaves over a quasi-separated scheme X is particularly interesting in our study. As already mentioned in the example (iii), earlier in Section 1, the category $\mathsf{Qcoh}(X)$ equipped with the stalk-wise (= geometrically) pure exact structure $\mathscr{E}_{\mathrm{stalk}}$ ($= \mathscr{E}_\otimes$), which is an algebro-geometric generalization of the (representation theoretical) exact category (R-Mod, $\mathscr{E}_{\mathrm{pure}}$), is just one example of a geometrically pure exact category $(\mathcal{V}, \mathscr{E}_\otimes)$. This special case has already been studied in detail in [14,17]. From Theorem A, it follows that every quasi-coherent sheaf over X has a (geometrically) pure injective envelope; this was already proved in a different way in [14, Theorem 4.10]. If the scheme X is concentrated (i.e., quasi-compact and quasi-separated), then the category $\mathsf{Qcoh}(X)$ is locally finitely presentable by [22, Proposition 7], so Theorem D yields a well-behaved tensor embedding of the exact category $(\mathsf{Qcoh}(X), \mathscr{E}_{\mathrm{stalk}})$ into a Grothendieck category. Besides, it should be mentioned that recently, Estrada and Virili [18, Theorem 4.13 and Corollary 4.16] have successfully introduced and studied a "local" version of the Yoneda embedding for $(\mathsf{Qcoh}(X), \mathscr{E}_{\mathrm{stalk}})$. On one hand, if X is concentrated, the categorically pure exact category $(\mathsf{Qcoh}(X), \mathscr{E}_{\aleph_0})$ has not only enough relative projectives but also enough relative injectives (see Herzog [26, Theorem 6]). On the other hand, one can study the categorically pure exact category $(\mathsf{Qcoh}(X), \mathscr{E}_{\aleph_0})$ via the Yoneda embedding (1.3). Besides, by [13, pp. 1658 and 1660], the exact category $(\mathsf{Qcoh}(X), \mathscr{E}_{\aleph_0})$ can be embedded in an abelian categoy $\mathcal{D}$, which identifies relative injectives with absolute injectives in $\mathcal{D}$. However, the deficiency of the categorically pure exact structure $\mathscr{E}_{\aleph_0}$ on $\mathsf{Qcoh}(X)$ is that it interacts poorly with flat quasi-coherent sheaves while the stalk-wise pure exact structure $\mathscr{E}_{\mathrm{stalk}}$ is closely related to such sheaves. For this reason, the latter notion of purity is widely accepted to be the "correct" one for $\mathsf{Qcoh}(X)$, and this favours the tensor embedding.

As a final remark, the Yoneda embedding has also been studied in the context of enriched categories: let $\mathcal{V}$ be a locally λ-presentable base (2.5) and $\mathcal{C}$ be a locally λ-presentable $\mathcal{V}$-category in the sense of Borceux et al. [6, Definitions 1.1 and 6.1]. Denote by $\mathrm{Pres}_\lambda(\mathcal{C})$ the full $\mathcal{V}$-subcategory of λ-presentable objects in $\mathcal{C}$ in the enriched sense [6, Definition 3.1], and let $[\mathrm{Pres}_\lambda(\mathcal{C})^{\mathrm{op}}, \mathcal{V}]$ be the $\mathcal{V}$-category of $\mathcal{V}$-functors from $\mathrm{Pres}_\lambda(\mathcal{C})^{\mathrm{op}}$ to $\mathcal{V}$. In the proof of [6, Theorem 6.3], it is shown that the Yoneda $\mathcal{V}$-functor

$$\Upsilon \colon \mathcal{C} \to [\mathrm{Pres}_\lambda(\mathcal{C})^{\mathrm{op}}, \mathcal{V}] \quad \text{given by } X \mapsto \mathcal{C}(-, X)\,|_{\mathrm{Pres}_\lambda(\mathcal{C})}$$

is fully faithful with the essential image

$$\mathrm{Ess.\,Im}\,\Upsilon = \lambda\text{-}\mathrm{Flat}(\mathrm{Pres}_\lambda(\mathcal{C})^{\mathrm{op}}, \mathcal{V}) = \lambda\text{-}\mathrm{Cont}(\mathrm{Pres}_\lambda(\mathcal{C})^{\mathrm{op}}, \mathcal{V}).$$

Here, λ-Flat$(\mathrm{Pres}_\lambda(\mathcal{C})^{\mathrm{op}}, \mathcal{V})$ is the $\mathcal{V}$-subcategory of λ-flat $\mathcal{V}$-functors in the enriched sense, and λ-Cont$(\mathrm{Pres}_\lambda(\mathcal{C})^{\mathrm{op}}, \mathcal{V})$ is the $\mathcal{V}$-subcategory of λ-continuous $\mathcal{V}$-functors. Note that our Theorems C and D are, in some sense, dual to this result.

2 Preliminaries

We recall some definitions and terminologies from the enriched category theory that are important in this paper. We also give some examples, to which we repeatedly return.

2.1 (Locally presentable categories [1])**.** Let λ be a regular cardinal. An object A in a category $\mathcal{V}$ is said to be *λ-presentable* if the functor $\mathcal{V}_0(A, -)\colon \mathcal{V} \to \mathsf{Set}$ preserves λ-directed colimits. One says that $\mathcal{V}$ is *locally λ-presentable* if it is cocomplete and there is a *set* $\mathcal{S}$ of λ-presentable objects such that every object in $\mathcal{V}$ is a λ-directed colimit of objects in $\mathcal{S}$.

It is customary to say *finitely presentable* instead of "$\aleph_0$-presentable"; thus $\aleph_0$-presentable objects are called *finitely presentable objects* and locally $\aleph_0$-presentable categories are called *locally finitely presentable categories*. Moreover, "$\aleph_0$-directed colimits" are simply called *directed colimits*.

2.2 (Monoidal categories [31])**.** A *monoidal category* consists of a category $\mathcal{V}$, a bifunctor $\otimes\colon \mathcal{V}\times\mathcal{V} \to \mathcal{V}$ (tensor product), a unit object $I \in \mathcal{V}$, and natural isomorphisms a (associator), l (left unitor) and r (right unitor) subject to the coherence axioms found in [31, Subsection 1.1, (1.1) and (1.2)]. A monoidal category $\mathcal{V}$ is said to be *symmetric* if there is a natural isomorphism c (symmetry) subject to further coherence axioms that express the compatibility of c with a, l and r (see [31, Subsection 1.4, (1.14)–(1.16)]). In particular, the symmetry c identifies l and r, so there is no need to distinguish between them. Due to Mac Lane's *coherence theorem* (see [36] or [37, Subsection VII.2]), it is customary to suppress a, l, r and c, and we simply write $(\mathcal{V}, \otimes, I)$ when referring to a (symmetric) monoidal category. A symmetric monoidal category is said to be *closed* if for every $X \in \mathcal{V}$, the functor $-\otimes X\colon \mathcal{V} \to \mathcal{V}$ has a right adjoint $[X, -]\colon \mathcal{V} \to \mathcal{V}$ (see [31, Subsection 1.5]). It turns out that $[-,-]$ is a bifunctor $\mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}$, and we write a closed symmetric monoidal category as a quadruple $(\mathcal{V}, \otimes, I, [-,-])$.

Example 2.3. The category $\mathsf{Ch}(R)$ of chain complexes of modules over a commutative ring R is a Grothendieck category with two different closed symmetric monoidal structures.

(a) $(\mathsf{Ch}(R), \otimes_R^\bullet, S(R), \mathrm{Hom}_R^\bullet)$, where $\otimes_R^\bullet$ is the *total tensor product*, $\mathrm{Hom}_R^\bullet$ the *total Hom*, and $S(R) = 0 \to R \to 0$ is the *stalk* complex with R in degree 0 (see [11, Appendix A.2]).

(b) $(\mathsf{Ch}(R), \underline{\otimes}_R^\bullet, D(R), \underline{\mathrm{Hom}}_R^\bullet)$, where $\underline{\otimes}_R^\bullet$ is the *modified* total tensor product, $\underline{\mathrm{Hom}}_R^\bullet$ the *modified* total Hom, and $D(R) = 0 \to R \to R \to 0$ is the *disc* complex concentrated in homological degrees 0 and -1 (see [15, Section 2] or [21, Subsection 4.2]).

Example 2.4. Some important examples of closed symmetric monoidal categories, which are also Grothendieck, come from algebraic geometry. Let X be any scheme.

(a) $(\mathsf{Mod}(X), \otimes_X, \mathscr{O}_X, \mathscr{H}om_X)$ is a closed symmetric monoidal category, where $\mathsf{Mod}(X)$ is the abelian category of all the sheaves (of $\mathscr{O}_X$-modules) on X (see [25, Chapter II, Section 5]). It is well known that this is a Grothendieck category (see [23, Proposition 3.1.1]).

(b) The category $\mathsf{Qcoh}(X)$ of quasi-coherent sheaves on X is an abelian subcategory of $\mathsf{Mod}(X)$ (see [24, (i) and (ii) of Corollary (2.2.2)] or [25, Chapter II, Proposition 5.7]). As $I = \mathscr{O}_X$ is quasi-coherent and quasi-coherent sheaves are closed under tensor products by [24, Corollary (2.2.2)(v)], it follows that $(\mathsf{Qcoh}(X), \otimes_X, \mathscr{O}_X)$ is a monoidal subcategory of $(\mathsf{Mod}(X), \otimes_X, \mathscr{O}_X)$. In general, $\mathscr{H}om_X$ is not an internal hom in $\mathsf{Qcoh}(X)$. However, the inclusion functor $\mathsf{Qcoh}(X) \to \mathsf{Mod}(X)$ admits a right adjoint $Q_X\colon \mathsf{Mod}(X) \to \mathsf{Qcoh}(X)$, called the *coherator*, and the counit $Q_X(\mathscr{F}) \to \mathscr{F}$ is an isomorphism for every quasi-coherent sheaf $\mathscr{F}$ (see [46, Tag 08D6]). It is well known, and completely formal, that the functor $\mathscr{H}om_X^{\mathrm{qc}} := Q_X \mathscr{H}om_X$ yields a closed structure on $(\mathsf{Qcoh}(X), \otimes_X, \mathscr{O}_X)$. The category $\mathsf{Qcoh}(X)$ is Grothendieck by [44, Lemma 1.3].

A category can be locally presentable (as in 2.1) and closed symmetric monoidal (as in 2.2) at the same time, but in general one cannot expect any compatibility between the two structures. This is the reason for the next definition, which comes from [6].

2.5 (Locally presentable bases [6, Definition 1.1]). Let λ be a regular cardinal. A closed symmetric monoidal category $(\mathcal{V}, \otimes, I, [-,-])$ is said to be a *locally λ-presentable base* if it satisfies the following condition:

(1) the category $\mathcal{V}$ is locally λ-presentable;

(2) the unit object I is λ-presentable;

(3) the class of λ-presentable objects is closed under the tensor product $\otimes$.

A locally $\aleph_0$-presentable base is simply called a *locally finitely presentable base*.

2.6 (Enriched category theory). We assume the familiarity with basic notions and results from the enriched category theory as presented in [31, Chapters 1 and 2, and parts of Chapter 3]. In particular, for a closed symmetric monoidal category $(\mathcal{V}, \otimes, I, [-,-])$, the definitions and properties of *$\mathcal{V}$-categories* and their *underlying ordinary categories*, *$\mathcal{V}$-functors*, *$\mathcal{V}$-natural transformations* and *weighted limits and colimits* will be important. When we use specific results from the enriched category theory, we give the appropriate references to [31], but a few general points are mentioned below.

To avoid confusion, we often write $\mathcal{V}_0$ when we think of $\mathcal{V}$ as an ordinary category, and we use the symbol $\mathcal{V}$ when it is viewed as a $\mathcal{V}$-category.

If the category $\mathcal{V}_0$ is complete, $\mathcal{K}$ is a small $\mathcal{V}$-category and $\mathcal{C}$ is any $\mathcal{V}$-category, then there is a $\mathcal{V}$-category $[\mathcal{K}, \mathcal{C}]$, whose objects are $\mathcal{V}$-functors $\mathcal{K} \to \mathcal{C}$. The underlying ordinary category $[\mathcal{K}, \mathcal{C}]_0$ has $\mathcal{V}$-natural transformations as morphisms (see [31, Subsections 2.1 and 2.2]).

In the proof of Proposition 5.8, we use the *unit $\mathcal{V}$-category $\mathcal{I}$*. It has one object $*$ and $\mathcal{I}(*,*) = I$. The composition law is given by the isomorphism $I \otimes I \to I$ (see [31, Subsection 1.3]).

In Definition 2.7, 2.8 and 2.9 below, $(\mathcal{V}, \otimes, I, [-,-])$ denotes a *cosmos*, i.e., a closed symmetric monoidal category for which $\mathcal{V}_0$ is bicomplete. Examples 2.3 and 2.4 are all the cosmoses.

The next notion of smallness for a $\mathcal{V}$-functor with values in $\mathcal{V}$ will be important to us.

Definition 2.7 (See [6, Definition 2.1]). Let λ be a regular cardinal. A $\mathcal{V}$-functor $T\colon \mathcal{K} \to \mathcal{V}$ is said to be *λ-small* if the following conditions are satisfied:

(1) the class $\operatorname{Ob}\mathcal{K}$ is a set of cardinality strictly less than λ;

(2) for all the objects $X, Y \in \mathcal{K}$, the hom-object $\mathcal{K}(X, Y)$ is λ-presentable in $\mathcal{V}_0$;

(3) for every object $X \in \mathcal{K}$, the object $T(X)$ is λ-presentable in $\mathcal{V}_0$.

We also need the "enriched versions" of limits and colimits:

2.8 (Weighted limits and colimits [31, Chapter 3]). Let $F\colon \mathcal{K} \to \mathcal{V}$ and $G\colon \mathcal{K} \to \mathcal{A}$ be $\mathcal{V}$-functors. The *$\mathcal{V}$-limit of G weighted by F*, if it exists, is an object $\{F, G\} \in \mathcal{A}$ for which there is a $\mathcal{V}$-natural isomorphism in $A \in \mathcal{A}$:

$$\mathcal{A}(A, \{F, G\}) \cong [\mathcal{K}, \mathcal{V}](F, \mathcal{A}(A, G(-))).$$

Given $\mathcal{V}$-functors $G\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ and $F\colon \mathcal{K} \to \mathcal{A}$, the *$\mathcal{V}$-colimit of F weighted by G*, if it exists, is an object $G \star F \in \mathcal{A}$ for which there is a $\mathcal{V}$-natural isomorphism in $A \in \mathcal{A}$:

$$\mathcal{A}(G \star F, A) \cong [\mathcal{K}^{\mathrm{op}}, \mathcal{V}](G, \mathcal{A}(F(-), A)). \tag{2.1}$$

2.9 (Tensors and cotensors [5, Proposition 6.5.7]). Let $\mathcal{K}$ be a small $\mathcal{V}$-category. The $\mathcal{V}$-category $[\mathcal{K}, \mathcal{V}]$ is both *tensored* and *cotensored*. By [5, Definition 6.5.1], this means that for every $V \in \mathcal{V}$ and $F \in [\mathcal{K}, \mathcal{V}]$, there exist objects $V \otimes F$ and $[V, F]$ in $[\mathcal{K}, \mathcal{V}]$ and $\mathcal{V}$-natural isomorphisms

$$[\mathcal{K}, \mathcal{V}](V \otimes F, \cdot) \cong [V, [\mathcal{K}, \mathcal{V}](F, \cdot)] \quad \text{and} \quad [\mathcal{K}, \mathcal{V}](\cdot, [V, F]) \cong [V, [\mathcal{K}, \mathcal{V}](\cdot, F)]. \tag{2.2}$$

The proof of [5, Proposition 6.5.7] reveals that the $\mathcal{V}$-functors $V \otimes F$ and $[V, F]$ are just the compositions $V \otimes F = (V \otimes -) \circ F$ and $[V, F] = [V, -] \circ F$.

3 Exact categories and purity

We demonstrate (see Proposition 3.2) a general procedure to construct exact structures on an abelian category, and apply it to establish the so-called geometrically pure exact structure on $\mathcal{V}_0$ (see Definition 3.4) and the $\star$-pure exact structure on $[\mathcal{K},\mathcal{V}]_0$ (see Definition 3.16).

3.1 (Exact categories [40]). Let $\mathcal{X}$ be an additive category and $\mathscr{E}$ be a class of kernel-cokernel pairs (i,p) in $\mathcal{X}$:

$$X \rightarrowtail^{i} Y \twoheadrightarrow^{p} Z,$$

i.e., i is the kernel of p, and p is the cokernel of i. The morphism i is called an *admissible monic* and p is called an *admissible epic* in $\mathscr{E}$. The class $\mathscr{E}$ is said to form an *exact structure* on $\mathcal{X}$ if it is closed under isomorphisms and satisfies the following axioms:

(E0) For every object X in $\mathcal{X}$, the identity morphism id_X is both an admissible monic and an admissible epic in $\mathscr{E}$.

(E1) The classes of admissible monics and admissible epics in $\mathscr{E}$ are closed under compositions.

(E2) The pushout (resp. pullback) of an admissible monic (resp. admissible epic) along an arbitrary morphism exists and yields an admissible monic (resp. admissible epic).

In this situation, the pair $(\mathcal{X},\mathscr{E})$ is called an *exact category*. An object in $J \in \mathcal{X}$ is said to be *injective relative to* $\mathscr{E}$ if the functor $\mathrm{Hom}_{\mathcal{X}}(-,J)$ maps sequences in $\mathscr{E}$ to short exact sequences in Ab. For a detailed treatment on the subject, see [10].

We begin with a general result, potentially of independent interest, which shows how to construct an exact structure $\mathscr{E}_{\mathfrak{T}}$ on an abelian category $\mathcal{C}$ from a collection $\mathfrak{T}$ of functors. Inspired by terminologies from topology, we call $\mathscr{E}_{\mathfrak{T}}$ the *initial exact structure on* $\mathcal{C}$ *with respect to* $\mathfrak{T}$.

Proposition 3.2. *Let $\mathcal{C}$ be an abelian category and $\mathfrak{T}$ be a collection of additive functors $T\colon \mathcal{C}\to\mathcal{D}_T$, where each category $\mathcal{D}_T$ is abelian and each functor T is left exact or right exact. Denote by $\mathscr{E}_{\mathfrak{T}}$ the class of all the short exact sequences $0\to X\to Y\to Z\to 0$ in $\mathcal{C}$ such that $0\to TX\to TY\to TZ\to 0$ is exact in $\mathcal{D}_T$ for every T in $\mathfrak{T}$. Then $\mathscr{E}_{\mathfrak{T}}$ is an exact structure on $\mathcal{C}$; in fact, it is the finest (i.e., the largest with respect to inclusion) exact structure $\mathscr{E}$ on $\mathcal{C}$, which satisfies the condition that $T\colon(\mathcal{C},\mathscr{E})\to\mathcal{D}_T$ is an exact functor for every T in $\mathfrak{T}$.*

Proof. Once we have proved that $\mathscr{E}_{\mathfrak{T}}$ is, in fact, an exact structure on $\mathcal{C}$, then certainly $T\colon(\mathcal{C},\mathscr{E}_{\mathfrak{T}})\to\mathcal{D}_T$ is an exact functor for every T in $\mathfrak{T}$. Moreover, if $\mathscr{E}$ is any exact structure on $\mathcal{C}$ for which every T in $\mathfrak{T}$ is an exact functor $T\colon(\mathcal{C},\mathscr{E})\to\mathcal{D}_T$, then evidently $\mathscr{E}\subseteq\mathscr{E}_{\mathfrak{T}}$.

We now show that $\mathscr{E}_{\mathfrak{T}}$ satisfies the axioms in 3.1. The condition (E0) is immediate from the definition of $\mathscr{E}_{\mathfrak{T}}$. To show (E1), let $f\colon X\to Y$ and $g\colon Y\to Z$ be composable morphisms in $\mathcal{C}$. We prove that if f and g are admissible monics in $\mathscr{E}_{\mathfrak{T}}$, then so is gf. The case where f and g are admissible epics in $\mathscr{E}_{\mathfrak{T}}$ is proved similarly. If f and g are admissible monics in $\mathscr{E}_{\mathfrak{T}}$, then by definition, f and g are monics in $\mathcal{C}$ and the short exact sequences

$$0\longrightarrow X\xrightarrow{f} Y\longrightarrow \mathrm{Cok}\,f\longrightarrow 0 \quad\text{and}\quad 0\longrightarrow Y\xrightarrow{g} Z\longrightarrow \mathrm{Cok}\,g\longrightarrow 0$$

stay exact under every functor T in $\mathfrak{T}$. The composition gf is certainly a monic in $\mathcal{C}$, so it remains to prove that the short exact sequence

$$0\longrightarrow X\xrightarrow{gf} Z\longrightarrow \mathrm{Cok}(gf)\longrightarrow 0 \tag{3.1}$$

stays exact under every functor T in $\mathfrak{T}$. Let T in $\mathfrak{T}$ be given and recall that T is assumed to be left exact or right exact. As the composition of two monics, $T(gf)=T(g)T(f)$ is a monic. Thus, if T is right exact, the sequence (3.1) certainly stays exact under T. Assume that T is left exact. In the leftmost commutative diagram below, the lower row is exact by the Snake lemma; the remaining rows and all the columns are trivially exact. The rightmost commutative diagram is obtained by applying the functor T to the leftmost one. In the right diagram, the 1st column and the 2nd row are exact by the assumption, and the 1st row and the 3rd column are trivially exact. The epimorphism $T(Z) \twoheadrightarrow T(\operatorname{Cok} g)$ in the 2nd row factorizes as $T(Z) \to T(\operatorname{Cok}(gf)) \to T(\operatorname{Cok} g)$, and hence the last morphism $T(\operatorname{Cok}(gf)) \to T(\operatorname{Cok} g)$ in the 3rd row is epic too. Since T is left exact, the entire 3rd row is exact. Consequently, in the rightmost diagram below, all three rows and the 1st and 3rd columns are exact:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & X & \xrightarrow{=} & X & \to & 0 & \to & 0 \\
 & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle gf} & & \downarrow & & \\
0 & \to & Y & \xrightarrow{g} & Z & \to & \operatorname{Cok} g & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle =} & & \\
0 & \to & \operatorname{Cok} f & \to & \operatorname{Cok}(gf) & \to & \operatorname{Cok} g & \to & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & T(X) & \xrightarrow{=} & T(X) & \to & 0 & \to & 0 \\
 & & {\scriptstyle T(f)}\downarrow & & \downarrow{\scriptstyle T(gf)} & & \downarrow & & \\
0 & \to & T(Y) & \xrightarrow{T(g)} & T(Z) & \to & T(\operatorname{Cok} g) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle =} & & \\
0 & \to & T(\operatorname{Cok} f) & \to & T(\operatorname{Cok}(gf)) & \to & T(\operatorname{Cok} g) & \to & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Thus, we can consider the rightmost diagram as an exact sequence $0 \to C_1 \to C_2 \to C_3 \to 0$ of complexes, where C_i is the i-th column in the diagram. As C_1 and C_3 are exact, so is C_2. Hence the sequence (3.1) stays exact under the functor T, as desired.

It remains to show (E2). We show that the pushout of an admissible monic in $\mathscr{E}_{\mathfrak{T}}$ along an arbitrary morphism yields an admissible monic. A similar argument shows that the pullback of an admissible epic in $\mathscr{E}_{\mathfrak{T}}$ along an arbitrary morphism is an admissible epic. Thus, consider a pushout diagram in $\mathcal{C}$:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & \text{(pushout)} & \vdots \\
X' & \xrightarrow{f'} & Y',
\end{array}
$$

where f is an admissible monic in $\mathscr{E}_{\mathfrak{T}}$ and $X \to X'$ is any morphism. As f is, in particular, a monomorphism, so is f' by [20, Theorem 2.54*], and hence there is a short exact sequence

$$0 \longrightarrow X' \xrightarrow{f'} Y' \longrightarrow \operatorname{Cok} f' \longrightarrow 0.$$

We must argue that this sequence stays exact under every T in $\mathfrak{T}$.

First, assume that T is right exact. In this case, $T(X') \to T(Y') \to T(\operatorname{Cok} f') \to 0$ is exact, and it remains to see that $T(f')$ is monic. As T preserves pushouts, $T(f')$ is a pushout of the monic $T(f)$, so another application of [20, Theorem 2.54*] yields that $T(f')$ is monic.

Next, assume that T is left exact. In this case, $0 \to T(X') \to T(Y') \to T(\operatorname{Cok} f')$ is exact, and it remains to see that $T(Y') \to T(\operatorname{Cok} f')$ is epic. As f' is a pushout of f, the canonical morphism $\operatorname{Cok} f \to \operatorname{Cok} f'$ is an isomorphism (see the dual of [20, Theorem 2.52]), and hence so is $T(\operatorname{Cok} f) \to T(\operatorname{Cok} f')$. By the assumption, $T(Y) \to T(\operatorname{Cok} f)$ is epic, so the composite morphism $T(Y) \twoheadrightarrow T(\operatorname{Cok} f) \xrightarrow{\cong} T(\operatorname{Cok} f')$ is epic. But this composite is the same as the composite $T(Y) \to T(Y') \to T(\operatorname{Cok} f')$, which is therefore an epimorphism, and it follows that $T(Y') \to T(\operatorname{Cok} f')$ is an epimorphism. □

Any locally λ-presentable abelian category $\mathcal{V}$ (see 2.1) can be equipped with an exact structure (see 3.1) which is called the *categorically pure exact structure* and denoted by $\mathscr{E}_\lambda$. In this exact structure, the admissible monomorphisms are precisely the λ-*pure subobjects* and the admissible epimorphisms are precisely the λ-*pure quotients* in the sense of [2]. That these classes of morphisms do, in fact, yield an exact structure follows from [2, Proposition 5, observation 11 and Proposition 15]. Alternatively, it follows directly from Proposition 3.2 with $\mathcal{C} = \mathcal{V}$ and $\mathfrak{T}$ the collection of functors $\mathcal{V}(A, -)\colon \mathcal{V} \to \mathsf{Ab}$, where A ranges over the λ-presentable objects in $\mathcal{V}$. In the special case $\lambda = \aleph_0$, this kind of purity was studied in [13, Section 3].

If $\mathcal{V}$ is a closed symmetric monoidal abelian category, there is also a notion of purity in $\mathcal{V}_0$ based on the tensor product (see Definition 3.4). In the literature, this kind of purity is often called *geometrically purity* (as opposed to categorically purity, mentioned above). The study of geometrical purity was initiated in [19] and was recently continued in [14, 17]. Below we establish the *geometrically pure exact structure* $\mathscr{E}_{\otimes}$ on $\mathcal{V}_0$, and show that the exact category $(\mathcal{V}_0, \mathscr{E}_{\otimes})$ has enough relative injectives (see Propositions 3.7 and 3.12).

As mentioned in [17, Remark 2.8], when both the categorically and the geometrically pure exact structures are available, the former is coarser than the latter, i.e., one has $\mathscr{E}_{\lambda} \subseteq \mathscr{E}_{\otimes}$. In general, this is a strict containment, however, in the locally finitely presentable category $\mathcal{V} = \mathsf{Mod}(R)$, where R is a commutative ring, one has $\mathscr{E}_{\aleph_0} = \mathscr{E}_{\otimes}$ (see, e.g., [29, Theorem 6.4]). As mentioned in Example 3.5(b) below, this equality also holds for $\mathcal{V} = \mathsf{Ch}(R)$ with the modified total tensor product $\underline{\otimes}^{\bullet}_R$.

Note that Examples 2.3 and 2.4 all satisfy the following setup.

Setup 3.3. In the rest of this section, $(\mathcal{V}, \otimes, I, [-,-])$ denotes a *cosmos*, i.e., a closed symmetric monoidal category which is bicomplete[1]. We also assume that $\mathcal{V}_0$ is abelian[2] and the category $\mathcal{V}_0$ has an injective cogenerator E.

Following Fox [19], a morphism $f\colon X \to Y$ in $\mathcal{V}_0$ is said to be *geometrically pure* if $f \otimes V\colon X \otimes V \to Y \otimes V$ is a monomorphism for every $V \in \mathcal{V}$. Note that a geometrically pure morphism is necessarily a monomorphism (take $V = I$).

Definition 3.4. Let $\mathscr{E}_{\otimes}$ be the class of all the short exact sequences in $\mathcal{V}_0$ which remain exact under the functor $- \otimes V$ for every $V \in \mathcal{V}$. We call $\mathscr{E}_{\otimes}$ the *geometrically pure exact structure* on $\mathcal{V}_0$ (see Proposition 3.7 below). Sequences in $\mathscr{E}_{\otimes}$ are called *geometrically pure* (*short*) *exact sequences*. An object $J \in \mathcal{V}_0$, which is injective relative to $\mathscr{E}_{\otimes}$, is called a *geometrically pure injective* object. We set

$$\mathrm{PureInj}_{\otimes}(\mathcal{V}_0) = \{J \in \mathcal{V}_0 \mid J \text{ is geometrically pure injective}\}.$$

Example 3.5. Consider the abelian cosmos from Example 2.3.

(a) It is easy to see that a short exact sequence $\mathbb{S}$ in $\mathsf{Ch}(R)$ is geometrically pure exact in $(\mathsf{Ch}(R), \otimes^{\bullet}_R)$ if and only if $\mathbb{S}_n$ is a pure exact sequence of R-modules in each degree n. Therefore, geometrically pure injective objects in $(\mathsf{Ch}(R), \otimes^{\bullet}_R)$ are precisely contractible chain complexes of pure injective R-modules (see [43, Corollary 5.7]).

(b) The geometrically pure exact sequences in $(\mathsf{Ch}(R), \underline{\otimes}^{\bullet}_R)$ have been characterized in several ways in [15, Theorem 2.5] and [21, Theorem 5.1.3]. Namely, a short exact sequence $\mathbb{S}$ in $\mathsf{Ch}(R)$ is geometrically pure exact in $(\mathsf{Ch}(R), \underline{\otimes}^{\bullet}_R)$ if and only if $\mathbb{S}$ is a categorically pure exact sequence in $\mathsf{Ch}(R)$. Furthermore, if a chain complex J of R-modules is a geometrically pure injective object in $(\mathsf{Ch}(R), \underline{\otimes}^{\bullet}_R)$, then J_n and $\operatorname{Ker} \partial^J_n$ are pure injective R-modules for every integer n (see [21, Proposition 5.1.4]).

Example 3.6. Consider the abelian cosmos from Example 2.4(b). For a quasi-separated scheme X, a short exact sequence $\mathbb{S}$ is geometrically pure exact in $(\mathsf{Qcoh}(X), \otimes_X)$ if and only if $\mathbb{S}_x$ is a pure exact sequence of $\mathscr{O}_{X,x}$-modules for every $x \in X$. This is proved in [14, Proposition 3.4 and Remark 3.5].

Proposition 3.7. *The pair* $(\mathcal{V}_0, \mathscr{E}_{\otimes})$ *is an exact category.*

Proof. This is known and is implicit in the proof of [17, Lemma 3.6]. It is also a special case of Proposition 3.2 with $\mathcal{C} = \mathcal{V}_0$ and $\mathfrak{T}$ the class of functors $- \otimes V\colon \mathcal{V}_0 \to \mathcal{V}_0$ where $V \in \mathcal{V}$. □

Lemma 3.8. *For every* $X \in \mathcal{V}$, *the object* $[X, E]$ *is geometrically pure injective.*

Proof. For any geometrically pure exact sequence $\mathbb{S}$, the sequence $\mathbb{S} \otimes X$ is exact, and hence so is $\mathcal{V}_0(\mathbb{S} \otimes X, E)$, as E is injective. The isomorphism $\mathcal{V}_0(\mathbb{S}, [X, E]) \cong \mathcal{V}_0(\mathbb{S} \otimes X, E)$ shows that $\mathcal{V}_0(\mathbb{S}, [X, E])$ is exact, which means that $[X, E]$ is geometrically pure injective. □

---

[1] Actually, we do not use the bicompleteness of $\mathcal{V}_0$ until we get to Lemma 3.14 and the subsequent results.

[2] When we talk about an *abelian* closed symmetric monoidal category, we tacitly assume that the tensor product $- \otimes -$ and the internal hom $[-,-]$ are additive functors in each variable.

Lemma 3.9. *A short exact sequence $\mathbb{S}$ in $\mathcal{V}_0$ is geometrically pure exact if and only if $[\mathbb{S}, E]$ is a split short exact sequence in $\mathcal{V}_0$.*

Proof. As E is an injective cogenerator in $\mathcal{V}_0$, the sequence $\mathbb{S}$ is geometrically pure exact if and only if $\mathcal{V}_0(\mathbb{S} \otimes V, E)$ is a short exact sequence in Ab for every $V \in \mathcal{V}$. $[\mathbb{S}, E]$ is a split short exact sequence in $\mathcal{V}_0$ if and only if $\mathcal{V}_0(V, [\mathbb{S}, E])$ is a short exact sequence in Ab for every $V \in \mathcal{V}$. The isomorphism $\mathcal{V}_0(\mathbb{S} \otimes V, E) \cong \mathcal{V}_0(V, [\mathbb{S}, E])$ yields the conclusion. □

Lemma 3.10. *The functor $[-, E] \colon \mathcal{V}_0^{\mathrm{op}} \to \mathcal{V}_0$ is faithful.*

Proof. There is a natural isomorphism $\mathcal{V}_0(-, E) \cong \mathcal{V}_0(I, [-, E])$. If $f \neq 0$ is a morphism, then $\mathcal{V}_0(f, E) \neq 0$, as E is a cogenerator in $\mathcal{V}_0$, so $\mathcal{V}_0(I, [f, E]) \neq 0$ and thus $[f, E] \neq 0$. □

Observation 3.11. There is a pair of adjoint functors (F, G) as follows:

$$\mathcal{V}_0 \underset{G=[-,E]}{\overset{F=[-,E]}{\rightleftarrows}} \mathcal{V}_0^{\mathrm{op}}.$$

Indeed, for all $X \in \mathcal{V}$ and $Y \in \mathcal{V}^{\mathrm{op}}$ (equivalently, $Y \in \mathcal{V}$), one has

$$\begin{aligned}\mathcal{V}_0^{\mathrm{op}}(F(X), Y) &= \mathcal{V}_0(Y, FX) = \mathcal{V}_0(Y, [X, E]) \cong \mathcal{V}_0(Y \otimes X, E) \\ &\cong \mathcal{V}_0(X \otimes Y, E) \cong \mathcal{V}_0(X, [Y, E]) = \mathcal{V}_0(X, G(Y)).\end{aligned}$$

Write ε for the counit of the adjunction. For every object Y in $\mathcal{V}$, note that ε_Y is an element in $\mathcal{V}_0^{\mathrm{op}}(FG(Y), Y) = \mathcal{V}_0(Y, FG(Y))$, so ε_Y is a morphism $Y \to FG(Y) = [[Y, E], E]$ in $\mathcal{V}_0$.

Proposition 3.12. *For every $Y \in \mathcal{V}$, the morphism $\varepsilon_Y \colon Y \to [[Y, E], E]$ from Observation 3.11 is a geometrically pure monomorphism. In particular, the exact category $(\mathcal{V}_0, \mathscr{E}_\otimes)$ has enough relative injectives (= enough geometrically pure injectives).*

Proof. First, we show that ε_Y is monic. Let f be a morphism in $\mathcal{V}_0$ with $\varepsilon_Y \circ f = 0$. It follows that $[f, E] \circ [\varepsilon_Y, E] = 0$ in $\mathcal{V}_0$. By the adjoint functor theory (see [37, Subsection IV.1, Theorem 1]), the morphism $G(\varepsilon_Y) = [\varepsilon_Y, E]$ has a right-inverse (which is actually $\varepsilon_{[Y,E]}$, but this is not important), and hence $[f, E] = 0$. Now Lemma 3.10 implies $f = 0$, so ε_Y is a monomorphism. To show that ε_Y is a geometrically pure monomorphism, consider the short exact sequence

$$\mathbb{S} = \ 0 \longrightarrow Y \xrightarrow{\varepsilon_Y} [[Y, E], E] \longrightarrow \operatorname{Cok} \varepsilon_Y \longrightarrow 0.$$

By Lemma 3.9, we need to prove that $[\mathbb{S}, E]$ splits, but as already argued above, $[\varepsilon_Y, E]$ is a split epimorphism, so we are done. To see that $(\mathcal{V}_0, \mathscr{E}_\otimes)$ has enough relative injectives, it remains to note that $[[Y, E], E]$ is a geometrically pure injective object by Lemma 3.8. □

From [16, Definition 6.1.1], recall the notions of *preenvelopes* and *envelopes*.

Theorem 3.13. *Assume that $\mathcal{V}_0$ is Grothendieck (i.e., $\mathcal{V}$ is a Grothendieck cosmos). Every object in $\mathcal{V}_0$ has a geometrically pure injective envelope, i.e., an envelope with respect to the class* $\mathrm{PureInj}_\otimes(\mathcal{V}_0)$.

Proof. Let $\mathbb{A}$ be the class of geometrically pure monomorphisms and $\mathbb{J} = \mathrm{PureInj}_\otimes(\mathcal{V}_0)$ be the class of geometrically pure injective objects in $\mathcal{V}_0$. The following conditions hold:

(1) An object $J \in \mathcal{V}_0$ belongs to $\mathbb{J}$ if and only if $\mathcal{V}_0(Y, J) \to \mathcal{V}_0(X, J) \to 0$ is exact in Ab for every $X \to Y$ in $\mathbb{A}$.

(2) A morphism $X \to Y$ in $\mathcal{V}_0$ belongs to $\mathbb{A}$ if and only if $\mathcal{V}_0(Y, J) \to \mathcal{V}_0(X, J) \to 0$ is exact in Ab for every $J \in \mathbb{J}$.

(3) Every object in $\mathcal{V}_0$ has a $\mathbb{J}$-preenvelope.

Indeed, the "only if" part of (1) holds by the definition of geometrically pure injective objects. For the "if" part, by Proposition 3.12 take a morphism $J \to J'$ in $\mathbb{A}$ with $J' \in \mathbb{J}$. By the assumption, $\mathcal{V}_0(J', J) \to \mathcal{V}_0(J, J) \to 0$ is exact, so id_J has a left-inverse $J' \to J$. Thus J is a direct summand in $J' \in \mathbb{J}$ and it follows that $J \in \mathbb{J}$. The "only if" part of (2) holds by the definition of geometrically pure injective objects. For the "if" part, let $f\colon X \to Y$ be any morphism in $\mathcal{V}_0$. For every $V \in \mathcal{V}_0$, one has $[V,E] \in \mathbb{J}$ by Lemma 3.8, so $\mathcal{V}_0(f,[V,E])$ is surjective by the assumption. As in the proof of Lemma 3.8, this means that $f \otimes V$ is monic, and hence f is in $\mathbb{A}$. Condition (3) holds by Proposition 3.12.

These arguments show that $(\mathbb{A},\mathbb{J})$ is an *injective structure* in the sense of [16, Definition 6.6.2]. Even though the definitions and results (with proofs) about such structures found in [16] are formulated for the category of modules over a ring, they carry over to any Grothendieck cosmos. Since the injective structure $(\mathbb{A},\mathbb{J})$ is *determined* by the class $\mathcal{G} := \mathcal{V}_0$ in the sense of [16, Definition 6.6.3], the desired conclusion follows from [16, Theorem 6.6.4(1)]. □

It is well known that if $\mathcal{K}$ is a small ordinary category, then the category of functors $\mathcal{K} \to \mathsf{Ab}$ is abelian, and even Grothendieck. In [3, Theorem 4.2], it is shown that if $\mathcal{K}$ is a small $\mathcal{V}$-category, then the ordinary category $[\mathcal{K},\mathcal{V}]_0$ of $\mathcal{V}$*-functors* $\mathcal{K} \to \mathcal{V}$ is abelian too, and even Grothendieck if $\mathcal{V}$ is[3]. Moreover, (co)limits, in particular, (co)kernels, in the category $[\mathcal{K},\mathcal{V}]_0$ are formed object-wise. Below we construct a certain exact structure on $[\mathcal{K},\mathcal{V}]_0$.

Lemma 3.14. *Let $\mathcal{K}$ be a small $\mathcal{V}$-category and $0 \to F' \to F \to F'' \to 0$ be an exact sequence in the abelian category $[\mathcal{K},\mathcal{V}]_0$. For every $\mathcal{V}$-functor $G\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$, the sequence $G \star F' \to G \star F \to G \star F'' \to 0$ is exact in $\mathcal{V}_0$.*

Proof. It follows immediately from the fact that $G \star F \cong F \star G$ (see [31, (3.9)]), and from the axiom (2.1) in the definition of weighted colimits, that $(- \star G, [G,-])$ is an adjoint pair. This implies that the functor $G \star - \cong - \star G$ is right exact. □

Proposition 3.15. *Let $\mathcal{K}$ be a small $\mathcal{V}$-category and $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $[\mathcal{K},\mathcal{V}]_0$. The following conditions are equivalent*:

(i) $0 \to G \star F' \to G \star F \to G \star F'' \to 0$ *is an exact sequence in $\mathcal{V}_0$ for every $G \in [\mathcal{K}^{\mathrm{op}},\mathcal{V}]$;*

(ii) $0 \to [F'',E] \to [F,E] \to [F',E] \to 0$ *is a split short exact sequence in $[\mathcal{K}^{\mathrm{op}},\mathcal{V}]_0$.*

Proof. Let $\mathbb{S}$ be the given exact sequence. By the definition of weighted colimits (see (2.1)), there is an isomorphism $[G \star \mathbb{S}, E] \cong [\mathcal{K}^{\mathrm{op}},\mathcal{V}](G,[\mathbb{S},E])$ of sequences in $\mathcal{V}_0$ and thus an induced isomorphism of sequences in Ab:

$$\mathcal{V}_0(G \star \mathbb{S}, E) \cong [\mathcal{K}^{\mathrm{op}},\mathcal{V}]_0(G,[\mathbb{S},E]). \tag{3.2}$$

As E is an injective cogenerator in $\mathcal{V}_0$, the condition (i) holds if and only if the left-hand side of (3.2) is exact for every $G \in [\mathcal{K}^{\mathrm{op}},\mathcal{V}]$. Evidently, (ii) holds if and only if the right-hand side of (3.2) is exact for every $G \in [\mathcal{K}^{\mathrm{op}},\mathcal{V}]$. Hence, (i) and (ii) are equivalent. □

Definition 3.16. Let $\mathscr{E}_\star$ denote the class of all the short exact sequences in $[\mathcal{K},\mathcal{V}]_0$ that satisfy the equivalent conditions in Proposition 3.15. We call $\mathscr{E}_\star$ the $\star$*-pure exact structure* on $[\mathcal{K},\mathcal{V}]_0$ (see the next result). Sequences in $\mathscr{E}_\star$ are called $\star$*-pure* (*short*) *exact sequences*.

Proposition 3.17. *The pair $([\mathcal{K},\mathcal{V}]_0, \mathscr{E}_\star)$ is an exact category.*

Proof. Apply Proposition 3.2 with $\mathcal{C} = [\mathcal{K},\mathcal{V}]_0$ and $\mathfrak{T}$ the class of functors $G \star -\colon [\mathcal{K},\mathcal{V}]_0 \to \mathcal{V}_0$ where $G \in [\mathcal{K}^{\mathrm{op}},\mathcal{V}]$. Note that every functor $G \star -$ is right exact by Lemma 3.14. □

Recall that a left R-module M is *absolutely pure* (or *FP-injective*) if it is a pure submodule of every R-module that contains it (see [29, Definition A.17]). Equivalently, every short exact sequence $0 \to M \to K \to K' \to 0$ is pure exact, i.e., $0 \to X \otimes M \to X \otimes K \to X \otimes K' \to 0$ is exact for every right R-module X. The definition of absolutely pure $\mathcal{V}$-functors $\mathcal{K} \to \mathcal{V}$ given below is completely analogous to this.

Definition 3.18. Let $\mathcal{K}$ be a small $\mathcal{V}$-category and H be an object in $[\mathcal{K},\mathcal{V}]_0$. Recall that

$$H \text{ is injective} \Leftrightarrow \begin{cases} \text{every exact sequence } 0 \to H \to F \to F' \to 0 \\ \text{in the abelian category } [\mathcal{K},\mathcal{V}]_0 \text{ is split exact.} \end{cases}$$

[3] Note that in [3, Theorem 4.2], the symbol $[\mathcal{K},\mathcal{V}]$ is used for the *ordinary* category of $\mathcal{V}$-functors $\mathcal{K} \to \mathcal{V}$ (but we use the symbol $[\mathcal{K},\mathcal{V}]_0$) whereas the $\mathcal{V}$*-category* of such functors is denoted by $\mathcal{F}(\mathcal{K})$ (but we use the symbol $[\mathcal{K},\mathcal{V}]$).

Inspired by the remarks above, we define

$$H \text{ is } \textit{absolutely pure} \Leftrightarrow \begin{cases} \text{every exact sequence } 0 \to H \to F \to F' \to 0 \\ \text{in the abelian category } [\mathcal{K}, \mathcal{V}]_0 \text{ is } \star\text{-pure exact.} \end{cases}$$

We also set

$$\begin{aligned} &\mathrm{Inj}([\mathcal{K}, \mathcal{V}]_0) = \{H \in [\mathcal{K}, \mathcal{V}]_0 \mid H \text{ is injective}\}, \\ &\mathrm{AbsPure}([\mathcal{K}, \mathcal{V}]_0) = \{H \in [\mathcal{K}, \mathcal{V}]_0 \mid H \text{ is absolutely pure}\}. \end{aligned}$$

These categories will appear in Theorem 6.13, the final result of this paper.

4 The tensor embedding for an abelian cosmos

We establish some general properties of the tensor embedding defined in Definition 4.2 below. The main result is Theorem 4.6, which shows that the tensor embedding identifies the geometrically pure exact category $(\mathcal{V}_0, \mathscr{E}_\otimes)$ from Proposition 3.7 with a certain exact subcategory of $([\mathcal{A}, \mathcal{V}]_0, \mathscr{E}_\star)$ from Proposition 3.17.

Setup 4.1. The setup for this section is the same as in Setup 3.3, i.e., $(\mathcal{V}, \otimes, I, [-,-])$ is an abelian[4] cosmos with an injective cogenerator E.

Definition 4.2. Recall from [31, Subsection 1.6] that $\otimes$ is a $\mathcal{V}$-functor[5] $\mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$. For a small full $\mathcal{V}$-subcategory $\mathcal{A}$ of $\mathcal{V}$, restriction yields a $\mathcal{V}$-functor $\otimes \colon \mathcal{V} \otimes \mathcal{A} \to \mathcal{V}$. Via the isomorphism

$$\mathcal{V}\text{-CAT}(\mathcal{V} \otimes \mathcal{A}, \mathcal{V}) \cong \mathcal{V}\text{-CAT}(\mathcal{V}, [\mathcal{A}, \mathcal{V}])$$

from [31, Subsection 2.3, (2.20)], the latter $\mathcal{V}$-functor corresponds to the $\mathcal{V}$-functor

$$\Theta \colon \mathcal{V} \to [\mathcal{A}, \mathcal{V}] \quad \text{given by } X \mapsto (X \otimes -)\,|_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{V}.$$

We refer to this $\mathcal{V}$-functor as the *tensor embedding*. Note that it induces an additive functor $\Theta_0 \colon \mathcal{V}_0 \to [\mathcal{A}, \mathcal{V}]_0$ of the underlying abelian categories.

Remark 4.3. If $I \in \mathcal{A}$, then the $\mathcal{V}$-functor $\mathcal{A}(I, -) = [I, -]$ exists and it is clearly naturally isomorphic to the inclusion $\mathcal{V}$-functor, inc: $\mathcal{A} \to \mathcal{V}$. Thus, in the notation of 2.9, one has

$$\Theta(X) = (X \otimes -)\,|_{\mathcal{A}} = X \otimes \mathrm{inc} \cong X \otimes \mathcal{A}(I, -). \tag{4.1}$$

For the next result, recall the notions of geometrically pure exact sequences and $\star$-pure exact sequences from Definitions 3.4 and 3.16.

Lemma 4.4. *Let $\mathcal{A}$ be a small full $\mathcal{V}$-subcategory of $\mathcal{V}$ and $\mathbb{S}$ be a short exact sequence in $\mathcal{V}_0$. The following two conditions are equivalent*:

(i) $\mathbb{S} \otimes A$ *is a short exact sequence in* $\mathcal{V}_0$ *for every* $A \in \mathcal{A}$.

(ii) $\Theta_0(\mathbb{S})$ *is a short exact sequence in* $[\mathcal{A}, \mathcal{V}]_0$.

If I belongs to $\mathcal{A}$, then the following two conditions are equivalent:

(i′) $\mathbb{S}$ *is a geometrically pure exact sequence in* $\mathcal{V}_0$.

(ii′) $\Theta_0(\mathbb{S})$ *is a* $\star$*-pure exact sequence in* $[\mathcal{A}, \mathcal{V}]_0$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is evident from the definitions. Now assume that $I \in \mathcal{A}$. For every $\mathcal{V}$-functor $G \colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$, there is an equivalence of endofunctors on $\mathcal{V}_0$:

$$G \star \Theta_0(-) \cong - \otimes G(I). \tag{4.2}$$

[4] Note that the abelianness of $\mathcal{V}_0$ is not used, neither is it important for the part (a) in Theorem 4.6.

[5] Note that in [31, Subsection 1.6], the symbol "Ten" is used for this $\mathcal{V}$-functor whereas "$\otimes$" is reserved for the ordinary functor $\mathcal{V}_0 \times \mathcal{V}_0 \to \mathcal{V}_0$, however, we abuse the notation and use the latter symbol for both functors.

Indeed, for $X \in \mathcal{V}$, one has the next isomorphisms, where the 1st is by (4.1), the 2nd follows as the $\mathcal{V}$-functor $X \otimes \cdot \colon \mathcal{V} \to \mathcal{V}$ preserves weighted colimits (this follows from, e.g., [5, Proposition 6.6.12]), and the 3rd is by [31, (3.10)], i.e.,

$$G \star \Theta(X) \cong G \star (X \otimes \mathcal{A}(I,-)) \cong X \otimes (G \star \mathcal{A}(I,-)) \cong X \otimes G(I).$$

It is clear from (4.2) that (i′) implies (ii′). Conversely, assume (ii′) and let $V \in \mathcal{V}$ be given. As $G = [-,V]$ is a $\mathcal{V}$-functor $\mathcal{A}^{\mathrm{op}} \to \mathcal{V}$, the sequence $[-,V] \star \Theta(\mathbb{S})$ is exact by the assumption. Another application of (4.2) shows that $\mathbb{S} \otimes [I,V] \cong \mathbb{S} \otimes V$ is exact, so (i′) holds. □

Recall that the *essential image* of a $\mathcal{V}$-functor $T \colon \mathcal{C} \to \mathcal{D}$, denoted by $\mathrm{Ess.\,Im}\,T$, is just the essential image of the underlying ordinary functor $T_0 \colon \mathcal{C}_0 \to \mathcal{D}_0$. Thus $\mathrm{Ess.\,Im}\,T$ is the collection of all the objects $D \in \mathcal{D}$ such that $D \cong T(C)$ in $\mathcal{D}_0$ for some object $C \in \mathcal{C}$. We may consider $\mathrm{Ess.\,Im}\,T$ as a full $\mathcal{V}$-subcategory of $\mathcal{D}$ or as a full subcategory of $\mathcal{D}_0$.

Lemma 4.5. *Let $\mathcal{A}$ be a small full $\mathcal{V}$-subcategory of $\mathcal{V}$ with $I \in \mathcal{A}$. For any short exact sequence $0 \to F' \to F \to F'' \to 0$ in the abelian category $[\mathcal{A},\mathcal{V}]_0$, one has*

$$F', F'' \in \mathrm{Ess.\,Im}\,\Theta \Rightarrow F \in \mathrm{Ess.\,Im}\,\Theta.$$

Consequently, $\mathrm{Ess.\,Im}\,\Theta$ *is an extension-closed subcategory of both of the exact categories* $([\mathcal{A},\mathcal{V}]_0, \mathscr{E}_{\mathrm{ab}})$ *and* $([\mathcal{A},\mathcal{V}]_0, \mathscr{E}_\star)$*, where $\mathscr{E}_{\mathrm{ab}}$ is the exact structure induced by the abelian structure, and $\mathscr{E}_\star$ is the* (*coarser*) *exact structure from Proposition* 3.17*. It follows that the sequences in* $\mathrm{Ess.\,Im}\,\Theta$ *which are exact, respectively, $\star$-pure exact, in $[\mathcal{A},\mathcal{V}]_0$ form an exact structure on* $\mathrm{Ess.\,Im}\,\Theta$*, which we denote by $\mathscr{E}_{\mathrm{ab}}|_{\mathrm{Ess.Im}\,\Theta}$, respectively, $\mathscr{E}_\star|_{\mathrm{Ess.Im}\,\Theta}$. In this way, we obtain exact categories* $(\mathrm{Ess.\,Im}\,\Theta, \mathscr{E}_{\mathrm{ab}}|_{\mathrm{Ess.Im}\,\Theta})$ *and* $(\mathrm{Ess.\,Im}\,\Theta, \mathscr{E}_\star|_{\mathrm{Ess.Im}\,\Theta})$.

Proof. For all the objects $X \in \mathcal{V}$ and $F \in [\mathcal{A},\mathcal{V}]$, there are isomorphisms

$$[\mathcal{A},\mathcal{V}](\Theta(X),F) \cong [\mathcal{A},\mathcal{V}](X \otimes \mathcal{A}(I,-),F) \cong [X,[\mathcal{A},\mathcal{V}](\mathcal{A}(I,-),F)] \cong [X,F(I)], \tag{4.3}$$

which follow from (4.1), (2.2) and the strong Yoneda lemma [31, Subsection 2.4, (2.31)]. Consequently, there is also an isomorphism $[\mathcal{A},\mathcal{V}]_0(\Theta(X),F) \cong \mathcal{V}_0(X,F(I))$ which shows that there is a pair of adjoint functors $(\Theta_0, (\mathrm{Ev}_I)_0)$, where Ev_I is the $\mathcal{V}$-functor given by evaluation at the unit object I (see [31, Subsection 2.2]):

$$\mathcal{V}_0 \underset{(\mathrm{Ev}_I)_0}{\overset{\Theta_0}{\rightleftarrows}} [\mathcal{A},\mathcal{V}]_0.$$

Write θ for the counit of this adjunction; thus for $F \in [\mathcal{A},\mathcal{V}]_0$, we have the $\mathcal{V}$-natural transformation $\theta_F \colon \Theta_0(F(I)) \to F$. Clearly, θ_F is an isomorphism if and only if $F \in \mathrm{Ess.\,Im}\,\Theta$.

Now, let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $[\mathcal{A},\mathcal{V}]_0$. It induces an exact sequence $0 \to F'(I) \to F(I) \to F''(I) \to 0$ in $\mathcal{V}_0$ and hence the exact sequence in the upper row of the next commutative diagram (by right exactness of the tensor product), i.e.,

$$\begin{array}{ccccccccc} & & \Theta_0(F'(I)) & \longrightarrow & \Theta_0(F(I)) & \longrightarrow & \Theta_0(F''(I)) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_{F'}} & & \downarrow{\scriptstyle \theta_F} & & \downarrow{\scriptstyle \theta_{F''}} & & \\ 0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' & \longrightarrow & 0. \end{array}$$

If F' and F'' are in $\mathrm{Ess.\,Im}\,\Theta$, then $\theta_{F'}$ and $\theta_{F''}$ are isomorphisms in $[\mathcal{A},\mathcal{V}]_0$; whence θ_F is an isomorphism by the Five Lemma, so F belongs to $\mathrm{Ess.\,Im}\,\Theta$.

The last assertion follows directly from [10, Lemma 10.20]. □

Theorem 4.6. *Let $\mathcal{V}$ be as in Setup* 4.1 *and $\mathcal{A}$ be any small full $\mathcal{V}$-subcategory of $\mathcal{V}$. The tensor embedding*

$$\Theta \colon \mathcal{V} \to [\mathcal{A},\mathcal{V}] \quad \text{given by } X \mapsto (X \otimes -)|_{\mathcal{A}}$$

from Definition 4.2 *is cocontinuous, i.e., it preserves all the small weighted colimits. If I belongs to $\mathcal{A}$, then Θ is a fully faithful functor and it induces two equivalences*

(a) *an equivalence of* $\mathcal{V}$*-categories* $\Theta\colon \mathcal{V} \xrightarrow{\simeq} \mathrm{Ess.Im}\,\Theta$;

(b) *an equivalence of exact categories* $\Theta_0\colon (\mathcal{V}_0, \mathscr{E}_\otimes) \xrightarrow{\simeq} (\mathrm{Ess.Im}\,\Theta, \mathscr{E}_\star|_{\mathrm{Ess.Im}\,\Theta})$.

Proof. First, note that the $\mathcal{V}$-categories $\mathcal{V}$ and $[\mathcal{A},\mathcal{V}]$ are bicomplete, i.e., they have small weighted limits and colimits. This follows from [31, Subsections 3.1 and 3.3] as the ordinary category $\mathcal{V}_0$ is assumed to be bicomplete. We now show that Θ is cocontinuous.

Let $\mathcal{K}$ be a small $\mathcal{V}$-category and $G\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ and $F\colon \mathcal{K} \to \mathcal{V}$ be $\mathcal{V}$-functors. We must show $G\star(\Theta\circ F) \cong \Theta(G\star F)$, where the weighted colimit on the left-hand side is computed in $[\mathcal{A},\mathcal{V}]$ and the one on the right-hand side in $\mathcal{V}$. Via the isomorphism $\mathcal{V}\text{-CAT}(\mathcal{K},[\mathcal{A},\mathcal{V}]) \cong \mathcal{V}\text{-CAT}(\mathcal{K}\otimes\mathcal{A},\mathcal{V})$ from [31, Subsection 2.3, (2.20)], the $\mathcal{V}$-functor $\Theta\circ F\colon \mathcal{K} \to [\mathcal{A},\mathcal{V}]$ corresponds to the $\mathcal{V}$-functor $P\colon \mathcal{K}\otimes\mathcal{A} \to \mathcal{V}$ given by $P(K,A) = F(K)\otimes A$. So $P(-,A)$ is $F\otimes A$ in the notation of 2.9. In the computation below, the 1st isomorphism holds as weighted colimits in $[\mathcal{A},\mathcal{V}]$ are computed object-wise (more precisely, we use the weighted colimit counterpart of [31, Subsection 3.3, (3.16)]); the 2nd and 3rd isomorphisms hold by the definitions of P and Θ; the 4th isomorphism holds as the $\mathcal{V}$-functor $-\otimes A\colon \mathcal{V}\to\mathcal{V}$ preserves weighted colimits

$$(G\star(\Theta\circ F))(A) \cong G\star P(-,A) \cong G\star(F\otimes A) \cong (G\star F)\otimes A \cong \Theta(G\star F)(A).$$

Consequently, $G\star(\Theta\circ F)\cong\Theta(G\star F)$, as claimed.

Now assume that $I\in\mathcal{A}$. As Θ is a $\mathcal{V}$-functor, it comes equipped with a natural morphism, i.e.,

$$\Theta_{XY}\colon [X,Y] \to [\mathcal{A},\mathcal{V}](\Theta(X),\Theta(Y))$$

for every pair of objects $X,Y\in\mathcal{V}$. The claim is that Θ_{XY} is an isomorphism in $\mathcal{V}$. However, the morphism Θ_{XY} is precisely the following composite, where the second isomorphism comes from (4.3) with $F=\Theta(Y)$:

$$[X,Y] \cong [X,\Theta(Y)(I)] \cong [\mathcal{A},\mathcal{V}](\Theta(X),\Theta(Y)).$$

(a) The asserted $\mathcal{V}$-equivalence is a formal consequence of the fact that Θ is fully faithful (see [31, Subsection 1.11, p. 24]).

(b) By the part (a), we have an equivalence of additive categories $\Theta_0\colon \mathcal{V}_0 \xrightarrow{\simeq} \mathrm{Ess.Im}\,\Theta$. The assertion about exact categories now follows from the equivalence (i′) ⇔ (ii′) in Lemma 4.4 and from Lemma 4.5. □

We end this section with two results that show how to construct a (co)generating set of objects in the category $[\mathcal{K},\mathcal{V}]_0$ of $\mathcal{V}$-functors from a (co)generating set of objects in $\mathcal{V}_0$. This will be used in the proof of Proposition 6.12.

Lemma 4.7. *Let* $\mathcal{K}$ *be a small* $\mathcal{V}$*-category. The following hold*:

(a) *If* $\mathcal{S}$ *is a cogenerating set of objects in* $\mathcal{V}_0$*, then* $\{[\mathcal{K}(-,K),S]\}_{K\in\mathcal{K},S\in\mathcal{S}}$ *is a cogenerating set of objects in* $[\mathcal{K},\mathcal{V}]_0$.

(b) *If* $S\in\mathcal{V}_0$ *is injective, then* $[\mathcal{K}(-,K),S]$ *is injective in* $[\mathcal{K},\mathcal{V}]_0$ *for every* $K\in\mathcal{K}$.

Proof. For K in $\mathcal{K}$, the functor $[\mathcal{K},\mathcal{V}]_0\to\mathcal{V}_0$ given by $\cdot\mapsto\mathcal{K}(-,K)\star\cdot$ is just the evaluation functor $\mathrm{Ev}_K(\cdot)$ at K (see [31, Subsection 3.1, (3.10)]). This fact, the defining property (2.1) of weighted colimits, and [31, Subsection 3.1, (3.9)] yield isomorphisms in $\mathcal{V}$:

$$[\mathcal{K},\mathcal{V}](\cdot,[\mathcal{K}(-,K),S]) \cong [\mathcal{K}(-,K)\star\cdot,S] \cong [\mathrm{Ev}_K(\cdot),S].$$

In particular, $[\mathcal{K},\mathcal{V}]_0(\cdot,[\mathcal{K}(-,K),S]) \cong \mathcal{V}_0(\mathrm{Ev}_K(\cdot),S)$, which yields both assertions. □

Lemma 4.7 has the next dual of which the part (a) can also be found in [3, Theorem 4.2].

Lemma 4.8. *Let* $\mathcal{K}$ *be a small* $\mathcal{V}$*-category. The following hold*:

(a) *If* $\mathcal{S}$ *is a generating set of objects in* $\mathcal{V}_0$*, then* $\{S\otimes\mathcal{K}(K,-)\}_{K\in\mathcal{K},S\in\mathcal{S}}$ *is a generating set of objects in* $[\mathcal{K},\mathcal{V}]_0$.

(b) *If* $S\in\mathcal{V}_0$ *is projective, then* $S\otimes\mathcal{K}(K,-)$ *is projective in* $[\mathcal{K},\mathcal{V}]$ *for every* $K\in\mathcal{K}$.

Proof. For any objects $S \in \mathcal{V}$ and $K \in \mathcal{K}$, the first isomorphism below follows from (2.2) and the second follows from the strong Yoneda lemma [31, Subsection 2.4, (2.31)], i.e.,

$$[\mathcal{K},\mathcal{V}](S \otimes \mathcal{K}(K,-),\cdot) \cong [S,[\mathcal{K},\mathcal{V}](\mathcal{K}(K,-),\cdot)] \cong [S,\mathrm{Ev}_K(\cdot)].$$

In particular, $[\mathcal{K},\mathcal{V}]_0(S \otimes \mathcal{K}(K,-),\cdot) \cong \mathcal{V}_0(S,\mathrm{Ev}_K(\cdot))$, which yields both assertions. □

5 The tensor embedding for a Grothendieck cosmos

In this section, we work with Setup 5.1 below and we consider the tensor embedding

$$\Theta\colon \mathcal{V} \to [\mathrm{Pres}_\lambda(\mathcal{V}),\mathcal{V}] \tag{5.1}$$

from Definition 4.2 in the special case where $\mathcal{A} = \mathrm{Pres}_\lambda(\mathcal{V})$. The goal is to strengthen Theorem 4.6 and make it more explicit in this situation; this is achieved in Theorem 5.9 below. Note that the examples found in Examples 2.3 and 2.4 all satisfy the following setup.

Setup 5.1. In this section, $(\mathcal{V},\otimes,I,[-,-])$ is a cosmos and $\mathcal{V}$ is a Grothendieck category.

- We fix a regular cardinal λ such that $\mathcal{V}$ is a locally λ-presentable base[6] in the sense of 2.5; such a choice is possible by Proposition 5.2 below.
- We let $\mathrm{Pres}_\lambda(\mathcal{V})$ be the (small) collection of all the λ-presentable objects in $\mathcal{V}$.
- We fix an injective cogenerator E in $\mathcal{V}_0$; the existence is guaranteed by [30, Theorem 9.6.3].

Proposition 5.2. *There is a regular cardinal λ for which $\mathcal{V}$ is a locally λ-presentable base.*

Proof. As $\mathcal{V}_0$ is Grothendieck, it follows from [4, Proposition 3.10] that it is a locally γ-presentable category for some regular cardinal γ. Note that for every regular cardinal $\lambda \geqslant \gamma$, the category $\mathcal{V}_0$ is also locally λ-presentable by [1, Remark after Theorem 1.20], so the condition (1) in 2.5 holds for all such λ. Let $\mathcal{S}$ be a set of representatives for the isomorphism classes of γ-presentable objects in $\mathcal{V}_0$. Let $\mathcal{S}'$ be the set consisting of the unit object I and all finite tensor products of objects in $\mathcal{S}$. As every object in $\mathcal{V}_0$ is presentable (i.e., μ-presentable for some regular cardinal μ) (see again [1, Remark after Theorem 1.20]), since $\mathcal{S}'$ is a set, there exists some regular cardinal $\lambda \geqslant \gamma$ such that every object in $\mathcal{S}'$ is λ-presentable. In particular, the condition (2) in 2.5 holds. If necessary we can replace λ with its successor λ^+ (every successor cardinal is regular) and thus by [1, Example 2.13(2)] assume that γ is *sharply smaller* than λ (in symbols: $\gamma \lhd \lambda$) in the sense of [1, Definition 2.12]. This will play a role in the following argument, which shows that the condition (3) in 2.5 holds.

Let X and Y be λ-presentable objects in $\mathcal{V}_0$. As the category $\mathcal{V}_0$ is locally γ-presentable (and hence also γ-accessible) and $\lambda \rhd \gamma$, it follows from [1, Remark 2.15] that X and Y are direct summands of a λ-small directed colimit of γ-presentable objects in $\mathcal{V}_0$, i.e., we have

$$X \oplus X' \cong \mathrm{colim}_{p\in P} X_p \quad \text{and} \quad Y \oplus Y' \cong \mathrm{colim}_{q\in Q} Y_q,$$

where $X_p, Y_q \in \mathcal{S}$ and $|P|,|Q| < \lambda$. As $\otimes$ preserves all the colimits, one has

$$(X \oplus X') \otimes (Y \oplus Y') \cong \mathrm{colim}_{(p,q)\in P\times Q}\, X_p \otimes Y_q. \tag{5.2}$$

By construction, each $X_p \otimes Y_q$ is in $\mathcal{S}'$, so it is a λ-presentable object. Furthermore, the category $P \times Q$ is λ-small. Hence, [1, Proposition 1.16] and (5.2) imply that $(X \oplus X') \otimes (Y \oplus Y')$ is λ-presentable. Since $X \otimes Y$ is a direct summand of $(X \oplus X') \otimes (Y \oplus Y')$, the object $X \otimes Y$ is λ-presentable too by [1, Remark after Proposition 1.16][7]. □

[6] Thus the blanket setup at the end of the introduction in [6] is satisfied, and we can apply the theory herein.

[7] According to [1, Remark 1.30(2)], it follows from [38] that if λ is any regular cardinal greater than or equal to γ, then every λ-presentable object is a λ-small colimit of γ-presentable objects. If this is true, then a couple of simplifications can be made in the proof of Proposition 5.2. Indeed, in this case, we would not have to worry about γ being *sharply smaller* than λ, and we could simply take both X' and Y' to be zero. However, as there seems to be some doubt about the correctness of the claim in [1, Remark 1.30(2)] (https://mathoverflow.net/questions/325278/mu-presentable-object-as-mu-small-colimit-of-lambda-presentable-objects), we have chosen to give a (slightly more complicated) proof of Proposition 5.2 based on [1, Remark 2.15] instead.

Under Setup 5.1, one can improve some of the statements about purity from Sections 3 and 4. This will be our first goal.

Lemma 5.3. *For a short exact sequence $\mathbb{S}$ in $\mathcal{V}_0$, the following conditions are equivalent*:

(i) $\mathbb{S} \otimes C$ *is a short exact sequence in* $\mathcal{V}_0$ *for every* $C \in \mathrm{Pres}_\lambda(\mathcal{V})$.

(ii) $\mathcal{V}_0(\mathbb{S}, [C, E])$ *is a short exact sequence in* Ab *for every* $C \in \mathrm{Pres}_\lambda(\mathcal{V})$.

(iii) $\mathbb{S}$ *is a geometrically pure exact sequence in* $\mathcal{V}_0$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows as $\mathcal{V}_0(\mathbb{S} \otimes C, E) \cong \mathcal{V}_0(\mathbb{S}, [C, E])$ and E is an injective cogenerator in $\mathcal{V}_0$. Evidently, (iii) $\Rightarrow$ (i). Finally, assume that (i) holds. Every object $V \in \mathcal{V}$ is a directed colimit of λ-presentable objects, i.e., $V \cong \mathrm{colim}_{q \in Q} C_q$. The sequence $\mathbb{S} \otimes V \cong \mathrm{colim}_{q \in Q}(\mathbb{S} \otimes C_q)$ is exact as each $\mathbb{S} \otimes C_q$ is exact (by the assumption) and any directed colimit of exact sequences is again exact because $\mathcal{V}_0$ is Grothendieck. So (iii) holds. □

Corollary 5.4. *Consider the tensor embedding* (5.1)*. The exact structures* $\mathscr{E}_{\mathrm{ab}}$ *and* $\mathscr{E}_\star$ *on the category* $[\mathrm{Pres}_\lambda(\mathcal{V}), \mathcal{V}]_0$ *from Lemma* 4.5 *agree on the subcategory* $\mathrm{Ess.\,Im}\,\Theta$, *i.e.*, $\mathscr{E}_{\mathrm{ab}}|_{\mathrm{Ess.Im}\,\Theta} = \mathscr{E}_\star|_{\mathrm{Ess.Im}\,\Theta}$.

Proof. Every short exact sequence $0 \to F' \to F \to F'' \to 0$ in $\mathrm{Ess.\,Im}\,\Theta$ has the form (up to isomorphism) $\Theta_0(\mathbb{S})$ for some short exact sequence $\mathbb{S} = 0 \to X' \to X \to X'' \to 0$ in $\mathcal{V}_0$. Lemma 5.3 shows that conditions (i) and (i′) in Lemma 4.4 (with $\mathcal{A} = \mathrm{Pres}_\lambda(\mathcal{V})$) are equivalent. Thus (ii) and (ii′) in Lemma 4.4 are equivalent too, as asserted. □

We know from Proposition 3.12 that the exact category $(\mathcal{V}_0, \mathscr{E}_\otimes)$ has enough relative injectives. An alternative demonstration of this fact is contained in the next proof.

Proposition 5.5. *An object* $X \in \mathcal{V}_0$ *is geometrically pure injective if and only if it is a direct summand of an object* $\prod_{q \in Q} [B_q, E]$ *for some family* $\{B_q\}_{q \in Q}$ *of* λ*-presentable objects.*

Proof. The "if" part is clear since each object $[B_q, E]$ is geometrically pure injective by Lemma 3.8. Conversely, let X be any object in $\mathcal{V}_0$. Choose a set $\mathcal{C}$ of representatives for the isomorphism classes of λ-presentable objects in $\mathcal{V}_0$ and consider the canonical morphism

$$\alpha\colon X \to \prod_{C \in \mathcal{C}} [C, E]^{J_C}, \quad \text{where } J_C = \mathcal{V}_0(X, [C, E]).$$

We show that α is a monomorphism. Let $\beta\colon Y \to X$ be a morphism with $\alpha\beta = 0$. This implies that for every $C \in \mathcal{C}$, the map $\mathcal{V}_0(\beta, [C, E])\colon \mathcal{V}_0(X, [C, E]) \to \mathcal{V}_0(Y, [C, E])$ is zero. By adjunction this means that $\mathcal{V}_0(\beta \otimes C, E) = 0$ and thus $\beta \otimes C = 0$ since E is an injective cogenerator. As every object in $\mathcal{V}_0$, in particular the unit object I, is a directed colimit of objects from $\mathcal{C}$, since $\beta \otimes -$ preserves colimits, it follows that $\beta \cong \beta \otimes I = 0$.

By what we have just proved, there is a short exact sequence

$$\mathbb{S} = \; 0 \longrightarrow X \xrightarrow{\alpha} \prod_{C \in \mathcal{C}} [C, E]^{J_C} \longrightarrow \mathrm{Cok}\,\alpha \longrightarrow 0.$$

By construction of α, every morphism $X \to [C, E]$ with $C \in \mathcal{C}$ factors through α; thus for every $C \in \mathcal{C}$, the morphism $\mathcal{V}_0(\alpha, [C, E])$ is surjective and therefore $\mathcal{V}_0(\mathbb{S}, [C, E])$ is a short exact sequence. It follows from Lemma 5.3 that $\mathbb{S}$ is geometrically pure exact. Thus, if X is geometrically pure injective, then $\mathbb{S}$ splits and X is a direct summand in $\prod_{C \in \mathcal{C}} [C, E]^{J_C}$. □

Definition 5.6 (See [6, Definition 2.1])**.** Let λ be a regular cardinal (in our applications, λ will be the fixed cardinal from Setup 5.1) and $\mathcal{A}$ be a $\mathcal{V}$-category. Let $G\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ and $F\colon \mathcal{K} \to \mathcal{A}$ be $\mathcal{V}$-functors and assume that the weighted colimit $G \star F \in \mathcal{A}$ exists. If the weight G is λ-small in the sense of Definition 2.7, then $G \star F$ is called a λ-*small weighted colimit*.

A $\mathcal{V}$-functor $T\colon \mathcal{A} \to \mathcal{B}$ is said to be λ-*cocontinuous* if it preserves all the λ-small weighted colimits, i.e., for every λ-small weight G, one has $G \star (T \circ F) \cong T(G \star F)$. We set

$$\lambda\text{-Cocont}(\mathcal{A}, \mathcal{B}) = \{\text{the collection of all the } \lambda\text{-cocontinuous } \mathcal{V}\text{-functors } \mathcal{A} \to \mathcal{B}\}.$$

Remark 5.7. By [6, Proposition 3.2 and Corollary 3.3], the full $\mathcal{V}$-subcategory $\mathcal{A} = \mathrm{Pres}_\lambda(\mathcal{V})$ is closed under λ-small weighted colimits in $\mathcal{V}$. Thus $\mathrm{Pres}_\lambda(\mathcal{V})$ has all the λ-small weighted colimits.

Proposition 5.8. *For the tensor embedding* (5.1)*, the essential image is*

$$\mathrm{Ess.\,Im}\,\Theta = \lambda\text{-Cocont}(\mathrm{Pres}_\lambda(\mathcal{V}), \mathcal{V}).$$

Proof. For every object $X \in \mathcal{V}$, the $\mathcal{V}$-functor $\Theta(X) = (X \otimes -)\,|_{\mathrm{Pres}_\lambda(\mathcal{V})}$ is λ-cocontinuous. Indeed, $X \otimes -\colon \mathcal{V} \to \mathcal{V}$ preserves all the weighted colimits, and by Remark 5.7, any λ-small weighted colimit in $\mathrm{Pres}_\lambda(\mathcal{V})$ is, in fact, a (λ-small) weighted colimit in $\mathcal{V}$. Thus "$\subseteq$" holds.

Conversely, let $T\colon \mathrm{Pres}_\lambda(\mathcal{V}) \to \mathcal{V}$ be any λ-cocontinuous $\mathcal{V}$-functor. For any $A \in \mathrm{Pres}_\lambda(\mathcal{V})$, consider the $\mathcal{V}$-functors $G\colon \mathcal{I}^{\mathrm{op}} \to \mathcal{V}$ and $F\colon \mathcal{I} \to \mathrm{Pres}_\lambda(\mathcal{V})$ given by $G(*) = A$ and $F(*) = I$. Here, $\mathcal{I}$ is the unit $\mathcal{V}$-category from 2.6. Note that the weight G is λ-small as the objects $\mathcal{I}^{\mathrm{op}}(*,*) = I$ and $G(*) = A$ are λ-presentable. Evidently, one has

$$G \star F = G(*) \otimes F(*) = A \otimes I,$$

and since T is assumed to preserve λ-small weighted colimits, it follows that

$$T(A) \cong T(A \otimes I) \cong T(G \star F) \cong G \star (T \circ F) \cong G(*) \otimes (T \circ F)(*) \cong A \otimes T(I).$$

Hence, T is $\mathcal{V}$-naturally isomorphic to $\Theta(T(I)) = (T(I) \otimes -)\,|_{\mathrm{Pres}_\lambda(\mathcal{V})}$, so $T \in \mathrm{Ess.\,Im}\,\Theta$. □

Theorem 5.9. *Let* $\mathcal{V}$ *be as in Setup* 5.1*. The tensor embedding*

$$\Theta\colon \mathcal{V} \to [\mathrm{Pres}_\lambda(\mathcal{V}), \mathcal{V}] \quad \textit{given by } X \mapsto (X \otimes -)\,|_{\mathrm{Pres}_\lambda(\mathcal{V})}$$

is cocontinuous and it induces two equivalences:

(a) *an equivalence of* $\mathcal{V}$*-categories* $\Theta\colon \mathcal{V} \xrightarrow{\simeq} \lambda\text{-Cocont}(\mathrm{Pres}_\lambda(\mathcal{V}), \mathcal{V})$;

(b) *an equivalence of exact categories* $\Theta_0\colon (\mathcal{V}_0, \mathscr{E}_\otimes) \xrightarrow{\simeq} \lambda\text{-Cocont}(\mathrm{Pres}_\lambda(\mathcal{V}), \mathcal{V})$*, where the latter is an extension-closed subcategory of the abelian category* $[\mathrm{Pres}_\lambda(\mathcal{V}), \mathcal{V}]_0$.

Proof. Combine Theorem 4.6, Proposition 5.8 and Corollary 5.4. □

6 The case where $\mathcal{V}$ is generated by dualizable objects

In this final section, we work with Setup 6.1 below. We prove in Proposition 6.9(a) that under the assumptions in Setup 6.1, $\mathcal{V}$ is a locally finitely presentable base. Thus in Setup 5.1 and in all of the results from Section 5, we can set $\lambda = \aleph_0$. As it is customary, we write $\mathrm{fp}(\mathcal{V}) := \mathrm{Pres}_{\aleph_0}(\mathcal{V})$ for the class of finitely presentable objects in $\mathcal{V}$, so the tensor embedding (5.1) becomes $\Theta\colon \mathcal{V} \to [\mathrm{fp}(\mathcal{V}), \mathcal{V}]$. We improve and make Theorem 5.9 more explicit in this situation. Let us explain the two main insights that we obtain in this section.

- We know from Theorem 5.9(b) that the geometrically pure exact category $(\mathcal{V}_0, \mathscr{E}_\otimes)$ is equivalent, as an exact category, to $\mathrm{Ess.\,Im}\,\Theta = \aleph_0\text{-Cocont}(\mathrm{fp}(\mathcal{V}), \mathcal{V})$. We prove in Theorem 6.13 that $\mathrm{Ess.\,Im}\,\Theta$ also coincides with the class of absolutely pure objects in $[\mathrm{fp}(\mathcal{V}), \mathcal{V}]_0$ in the sense of Definition 3.18.
- As mentioned in the remarks preceding Lemma 3.14, it follows from [3, Theorem 4.2] that $[\mathrm{fp}(\mathcal{V}), \mathcal{V}]_0$ is a Grothendieck category; in particular, it has enough injectives. Theorem 6.13 gives a very concrete description of the injective objects in $[\mathrm{fp}(\mathcal{V}), \mathcal{V}]_0$: they are precisely the $\mathcal{V}$-functors of the form $(X \otimes -)\,|_{\mathrm{fp}(\mathcal{V})}$, where X is a geometrically pure injective object in $\mathcal{V}_0$ in the sense of Definition 3.4.

Setup 6.1. In this section, $(\mathcal{V}, \otimes, I, [-,-])$ is a cosmos and $\mathcal{V}$ is a Grothendieck category (just as in Setup 5.1) subject to the following requirements:

- The unit object I is finitely presentable in $\mathcal{V}_0$.
- The category $\mathcal{V}_0$ is generated by a set of dualizable objects (defined in 6.5 below).

Example 6.2. The two Grothendieck cosmoses from Example 2.3, i.e.,

$$(\mathsf{Ch}(R), \otimes_R^\bullet, S(R), \mathrm{Hom}_R^\bullet) \quad \text{and} \quad (\mathsf{Ch}(R), \underline{\otimes}_R^\bullet, D(R), \underline{\mathrm{Hom}}_R^\bullet) \tag{6.1}$$

satisfy Setup 6.1. Indeed, it is not hard to see that

$$\mathrm{fp}(\mathsf{Ch}(R)) = \{X \in \mathsf{Ch}(R) \mid X \text{ is bounded and each module } X_n \text{ is finitely presentable}\},$$

in particular, the unit objects $S(R)$ and $D(R)$ of the two cosmoses are finitely presentable. Thus the first condition in Setup 6.1 holds. Evidently,

$$\{\Sigma^n D(R) \mid n \in \mathbb{Z}\} \ \text{(where } \Sigma^n \text{ is the } n\text{-th shift of a complex)}$$

is a generating set of objects in $\mathsf{Ch}(R)$. Each object $\Sigma^n D(R)$ is dualizable in the rightmost cosmos in (6.1) as $I = D(R)$ is the unit object. The object $\Sigma^n D(R)$ is also dualizable in the leftmost cosmos in (6.1); indeed, it is not hard to see that every *perfect* R-complex (i.e., a bounded complex of finitely generated projective R-modules) is dualizable in this cosmos. Hence, the second condition in Setup 6.1 holds as well.

Example 6.3. Consider the Grothendieck cosmos from Example 2.4(b), i.e.,

$$(\mathsf{Qcoh}(X), \otimes_X, \mathscr{O}_X, \mathscr{H}om_X^{\mathrm{qc}}).$$

If X is *concentrated* (e.g., if X is quasi-compact and quasi-separated), then $I = \mathscr{O}_X$ is finitely presentable in $\mathsf{Qcoh}(X)$ by [14, Proposition 3.7], so the first condition in Setup 6.1 holds.

Note by the way that for any noetherian scheme X, it follows from [12, Lemma B.3] (and the fact that in a locally noetherian Grothendieck category, finitely presentable objects and noetherian objects are the same [42, Chapter V, Section 4]) that

$$\mathrm{fp}(\mathsf{Qcoh}(X)) = \mathsf{Coh}(X) := \{X \in \mathsf{Qcoh}(X) \mid X \text{ is coherent}\}.$$

The dualizable objects in $\mathsf{Qcoh}(X)$ are precisely the locally free sheaves of finite rank (see [8, Proposition 4.7.5]), and hence the second condition in Setup 6.1 holds if and only if X has the *strong resolution property* in the sense of [8, Definition 2.2.7]. Many types of schemes—for example, any projective scheme and any separated noetherian locally factorial scheme—do have the strong resolution property (see the remarks after [8, Definition 2.2.7]). We refer to [45, Subsection 3.4] for further remarks and insights about the strong resolution property.

6.4 (Evaluation morphisms)**.** For any closed symmetric monoidal category, there are canonical natural morphisms in $\mathcal{V}_0$ (see, e.g., [35, Chapter III, Section 1, p. 120] for the construction of the first map):

$$\vartheta_{XYZ}\colon [X,Y]\otimes Z \to [X, Y\otimes Z] \quad \text{and} \quad \eta_{XYZ}\colon X\otimes [Y,Z] \to [[X,Y],Z].$$

In commutative algebra, i.e., in the case where $\mathcal{V} = \mathsf{Mod}(R)$ for a commutative ring R, people sometimes refer to ϑ as the *tensor evaluation* and to η as the *homomorphism evaluation* (see, for example, [11, (A.2.10) and (A.2.11)]).

We now consider *dualizable* objects as defined in [28, 35].

6.5 (Dualizable objects)**.** For every object P in $\mathcal{V}$, the following conditions are equivalent:

(i) The canonical morphism $[P,I]\otimes P \to [P,P]$ is an isomorphism.

(ii) The canonical morphism $[P,I]\otimes Z \to [P,Z]$ is an isomorphism for all the objects $Z \in \mathcal{V}$.

(iii) $\vartheta_{PYZ}\colon [P,Y]\otimes Z \to [P, Y\otimes Z]$ is an isomorphism for all the objects $Y, Z \in \mathcal{V}$.

Evidently, (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). Objects P satisfying (i) are in [35, Chapter III, Section 1, Definition 1.1] called *finite*, and in [35, Chapter III, Section 1, Proposition 1.3(ii)], it is proved that (i) implies (iii). Hence, all three conditions are equivalent. Objects P satisfying (ii) are called (*strongly*) *dualizable* in [28, Definition 1.1.2]; we adopt this terminology, "dualizable", for objects satisfying the conditions (i)–(iii).

Remark 6.6. If $P \in \mathcal{V}$ is dualizable, then so is $[P,I]$, and the natural map $P \to [[P,I],I]$ is an isomorphism (see [35, Chapter III, Section 1, Propositions 1.2 and 1.3(i)]). By the condition 6.5(ii), there is a natural isomorphism $[P,I]\otimes - \cong [P,-]$. By replacing P with $[P,I]$, one also gets $P\otimes - \cong [[P,I],-]$.

Lemma 6.7. *If P and Q are dualizable, then so are $P \oplus Q$, $P \otimes Q$, and $[P, Q]$.*

Proof. Using the condition 6.5(ii) and additivity of the functors $- \otimes -$ and $[-,-]$, it is easy to see that $P \oplus Q$ is dualizable. By the condition 6.5(iii), the computation

$$[P \otimes Q, Y] \otimes Z \cong [P, [Q, Y]] \otimes Z \cong [P, [Q, Y] \otimes Z] \cong [P, [Q, Y \otimes Z]] \cong [P \otimes Q, Y \otimes Z]$$

shows that $P \otimes Q$ is dualizable. To see that $[P, Q]$ is dualizable, note that $[P, Q] \cong [P, I] \otimes Q$ (as P is dualizable) and apply what we have just proved combined with Remark 6.6. □

Definition 6.8. Following [34, Definition 1.1], we use the next terms for an object J in $\mathcal{V}$:

J is injective $\Leftrightarrow$ the functor $\mathcal{V}_0(-, J)\colon \mathcal{V}_0^{\mathrm{op}} \to \mathsf{Ab}$ is exact.

J is *internally injective* $\Leftrightarrow$ the functor $[-, J]\colon \mathcal{V}_0^{\mathrm{op}} \to \mathcal{V}_0$ is exact.

A question of interest in [34] is when every injective object in $\mathcal{V}$ is internally injective; in this case Lewis would say that $\mathcal{V}$ satisfies the condition **IiII** (**I**njective **i**mplies **I**nternally **I**njective). As we prove next that this condition does hold under Setup 6.1.

Proposition 6.9. *For $\mathcal{V}$ as in Setup* 6.1*, the following assertions hold*:

(a) *$\mathcal{V}$ is a locally finitely presentable base* (*see* 2.5).

(b) *$\mathcal{V}$ satisfies Lewis' condition* **IiII***, i.e., every injective object is internally injective.*

(c) *For objects X, Y and J in $\mathcal{V}$ where X is finitely presentable and J is injective, the morphism $\eta_{XYJ}\colon X \otimes [Y, J] \to [[X, Y], J]$ from* 6.4 *is an isomorphism.*

Proof. (a) As I is finitely presentable, the condition 2.5(2) holds with $\lambda = \aleph_0$. Note that every dualizable object P is finitely presentable. Indeed, $[P, -]\colon \mathcal{V}_0 \to \mathcal{V}_0$ preserves colimits as it is naturally isomorphic to $[P, I] \otimes -$ by Remark 6.6, and $\mathcal{V}_0(I, -)$ preserves directed colimits as I is finitely presentable. Thus $\mathcal{V}_0(P, -) \cong \mathcal{V}_0(I, [P, -])$ preserves directed colimits.

As $\mathcal{V}_0$ is a Grothendieck category generated by a set of finitely presentable (even dualizable) objects, it is a locally finitely presentable Grothendieck category in the sense of Breitsprecher [9, Definition (1.1)]. Hence, $\mathcal{V}_0$ is also a locally finitely presentable category in the ordinary sense (see [9, Satz (1.5)] and [13, (2.4)]), so the condition 2.5(1) holds with $\lambda = \aleph_0$.

To see that 2.5(3) holds, i.e., that the class of finitely presentable objects is closed under $\otimes$, note that as $\mathcal{V}_0$ is generated by a set of dualizable objects, [9, Satz (1.11)] and Lemma 6.7 yield that an object X is finitely presentable if and only if there is an exact sequence

$$P_1 \to P_0 \to X \to 0 \tag{6.2}$$

with P_0 and P_1 dualizable. Now let Y be yet a finitely presentable object and choose an exact sequence $Q_1 \to Q_0 \to Y \to 0$ with Q_0 and Q_1 dualizable. It is not hard to see that there is an exact sequence $(P_1 \otimes Q_0) \oplus (P_0 \otimes Q_1) \to P_0 \otimes Q_0 \to X \otimes Y \to 0$ (see [7, Chapter II, Subsection 3.6, Proposition 6]), so it follows from Lemma 6.7 that $X \otimes Y$ is finitely presentable.

(b) As $\mathcal{V}_0$ is generated by a set, i.e., $\mathcal{P}$ of dualizable objects, the category $\mathcal{V}_0$ has a $\otimes$-flat generator, namely, $\bigoplus_{P \in \mathcal{P}} P$. The conclusion now follows from [41, Lemma 3.1].

(c) As noted in the proof of (a), if X is a finitely presentable object, then there exist dualizable objects P_0 and P_1 and an exact sequence (6.2). It induces a commutative diagram in $\mathcal{V}_0$:

$$\begin{array}{ccccccc} P_1 \otimes [Y, J] & \longrightarrow & P_0 \otimes [Y, J] & \longrightarrow & X \otimes [Y, J] & \longrightarrow & 0 \\ \downarrow{\scriptstyle \eta_{P_1 Y J}} & & \downarrow{\scriptstyle \eta_{P_0 Y J}} & & \downarrow{\scriptstyle \eta_{X Y J}} & & \\ [[P_1, Y], J] & \longrightarrow & [[P_0, Y], J] & \longrightarrow & [[X, Y], J] & \longrightarrow & 0. \end{array} \tag{6.3}$$

In this diagram, the upper row is exact by right exactness of the functor $- \otimes [Y, J]$. The sequence $0 \to [X, Y] \to [P_0, Y] \to [P_1, Y]$ is exact by left exactness of the functor $[-, Y]$. Thus, if J is injective, and hence also internally injective by the part (b), then we get exactness of the lower row in (6.3). Thus, to prove that η_{XYJ} is an isomorphism, it suffices by the Five Lemma to show that η_{P_0YJ} and η_{P_1YJ} are isomorphisms. However, for every dualizable object P and arbitrary objects Y and Z, the map η_{PYZ} is an isomorphism. Indeed, Remark 6.6, the adjunction $(-\otimes Y) \dashv [Y,-]$ and the condition 6.5(ii) yield isomorphisms

$$P\otimes [Y,Z] \cong [[P,I],[Y,Z]] \cong [[P,I]\otimes Y, Z] \cong [[P,Y],Z].$$

It is straightforward to verify that the composite of these isomorphisms is η_{PYZ}. □

Lemma 6.10. *For $\mathcal{V}$ as in Setup* 6.1*, the following assertions hold*:

(a) *For any family $\{B_q\}_{q\in Q}$ of objects and every finitely presentable object A in $\mathcal{V}_0$, the next canonical morphism is an isomorphism, i.e.,*

$$\left(\prod_{q\in Q}[B_q,E]\right)\otimes A \xrightarrow{\cong} \prod_{q\in Q}([B_q,E]\otimes A).$$

(b) *Assume that $\mathcal{V}_0$ satisfies Grothendieck's axiom* (AB4*)*, i.e., the product of a family of epimorphisms is an epimorphism. For any family $\{C_q\}_{q\in Q}$ of objects and every finitely presentable object A in $\mathcal{V}_0$, the next canonical morphism is an isomorphism, i.e.,*

$$\left(\prod_{q\in Q}C_q\right)\otimes A \xrightarrow{\cong} \prod_{q\in Q}(C_q\otimes A).$$

Proof. (a) In the computation below, the 1st and 4th isomorphisms follow as $[-,E]$ converts coproducts to products; the 2nd and 5th isomorphisms follow from Proposition 6.9(c); the 3rd isomorphism holds as $[A,-]$ preserves directed colimits by [6, Lemma 2.6 and Corollary 3.3] since A is finitely presentable (as mentioned in the footnote of Setup 5.1, we can apply the theory of [6] as $\mathcal{V}$ is a locally presentable base).

$$\begin{aligned}\left(\prod_{q\in Q}[B_q,E]\right)\otimes A &\cong \left[\bigoplus_{q\in Q}B_q,E\right]\otimes A \cong \left[\left[A,\bigoplus_{q\in Q}B_q\right],E\right] \cong \left[\bigoplus_{q\in Q}[A,B_q],E\right]\\ &\cong \prod_{q\in Q}[[A,B_q],E] \cong \prod_{q\in Q}([B_q,E]\otimes A).\end{aligned}$$

(b) Assume that $\mathcal{V}_0$ satisfies (AB4*). We must show that $-\otimes A$ preserves *all* the products. By the proof of Proposition 6.9(a), there is an exact sequence $P_1\to P_0\to A\to 0$ with P_0 and P_1 dualizable. In the induced commutative diagram below, the upper row is exact by right exactness of the functor $(\prod_{q\in Q}C_q)\otimes -$, and the lower row is exact as $\mathcal{V}_0$ satisfies (AB4*), i.e.,

$$\begin{array}{ccccccc}
\left(\prod_{q\in Q}C_q\right)\otimes P_1 & \longrightarrow & \left(\prod_{q\in Q}C_q\right)\otimes P_0 & \longrightarrow & \left(\prod_{q\in Q}C_q\right)\otimes A & \longrightarrow & 0\\
\downarrow\cong & & \downarrow\cong & & \downarrow & & \\
\prod_{q\in Q}(C_q\otimes P_1) & \longrightarrow & \prod_{q\in Q}(C_q\otimes P_0) & \longrightarrow & \prod_{q\in Q}(C_q\otimes A) & \longrightarrow & 0.
\end{array}$$

The two leftmost vertical morphisms are isomorphisms as the functor $-\otimes P_n$ preserves products; indeed, by Remark 6.6 this functor is naturally isomorphic to $[[P_n,I],-]$, which is a right adjoint. By the Five Lemma, the rightmost morphism is an isomorphism too. □

Some important abelian categories fail to satisfy Grothendieck's axiom (AB4*). For example, this is often the case for the category of quasi-coherent sheaves on a scheme (see [32, Example 4.9]). Fortunately, we shall not need the strong conclusion in Lemma 6.10(b) (we have only included it for completeness), as the weaker part (a) is sufficient for our purpose (the proof of Proposition 6.12 below). We shall also need the next general lemma.

Lemma 6.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be additive and idempotent complete categories, $\Phi\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful additive functor, and $C \in \mathcal{C}$ and $D \in \mathcal{D}$ be objects. If D is a direct summand in $\Phi(C)$, then D has the form $D \cong \Phi(C')$ for some direct summand C' in C.*

Proof. Since the functor Φ is fully faithful and additive, it induces a ring isomorphism, i.e.,

$$\mathrm{End}_{\mathcal{C}}(C,C) \xrightarrow{\cong} \mathrm{End}_{\mathcal{D}}(\Phi(C),\Phi(C)) \quad \text{given by } f \mapsto \Phi(f). \tag{6.4}$$

As D is a direct summand in $\Phi(C)$, there exist morphisms $h\colon D \to \Phi(C)$ and $k\colon \Phi(C) \to D$ in $\mathcal{D}$ such that $kh = \mathrm{id}_D$, and thus hk is an idempotent element in $\mathrm{End}_{\mathcal{D}}(\Phi(C),\Phi(C))$. By the ring isomorphism (6.4), there is an idempotent e in $\mathrm{End}_{\mathcal{C}}(C,C)$ with $\Phi(e) = hk$. As $\mathcal{C}$ is idempotent complete, there is an object $C' \in \mathcal{C}$ and morphisms $f\colon C' \to C$ and $g\colon C \to C'$ such that $gf = \mathrm{id}_{C'}$ and $fg = e$; in particular, C' is a direct summand in C. The morphisms

$$\Phi(C') \xrightarrow{\Phi(f)} \Phi(C) \xrightarrow{k} D \quad \text{and} \quad \Phi(C') \xleftarrow{\Phi(g)} \Phi(C) \xleftarrow{h} D$$

satisfy $k\Phi(f) \circ \Phi(g)h = \mathrm{id}_D$ and $\Phi(g)h \circ k\Phi(f) = \mathrm{id}_{\Phi(C')}$, so $\Phi(C') \cong D$ as claimed. □

Proposition 6.12. *A $\mathcal{V}$-functor $H\colon \mathrm{fp}(\mathcal{V}) \to \mathcal{V}$ is an injective object in $[\mathrm{fp}(\mathcal{V}),\mathcal{V}]_0$ if and only if it has the form $H \cong \Theta_0(X)$ for some geometrically pure injective object $X \in \mathcal{V}_0$.*

Proof. Necessity. We start by showing that $\Theta_0(X)$ is injective in $[\mathrm{fp}(\mathcal{V}),\mathcal{V}]_0$ for every geometrically pure injective object X in $\mathcal{V}$. By Proposition 5.5 (and the additivity of the functor Θ_0), we may assume that $X = \prod_{q\in Q}[B_q,E]$ for some family $\{B_q\}_{q\in Q}$ of finitely presentable objects. It follows from Lemma 6.10(a) (notice that this isomorphism is $\mathcal{V}$-natural in A) that

$$\Theta_0(X) = \Theta_0\biggl(\prod_{q\in Q}[B_q,E]\biggr) \cong \prod_{q\in Q}\Theta_0([B_q,E]).$$

Thus we may further reduce the case where $X = [B,E]$ for a single finitely presentable object B. Now Proposition 6.9(c) yields the isomorphism in the following computation:

$$\Theta_0(X) = ([B,E]\otimes -)\,|_{\mathrm{fp}(\mathcal{V})} \cong [[-,B],E]\,|_{\mathrm{fp}(\mathcal{V})} = [\mathrm{fp}(\mathcal{V})(-,B),E].$$

The latter is an injective object in $[\mathrm{fp}(\mathcal{V}),\mathcal{V}]_0$ by Lemma 4.7(b) with $\mathcal{K} = \mathrm{fp}(\mathcal{V})$.

Sufficiency. By Lemma 4.7 with $\mathcal{K} = \mathrm{fp}(\mathcal{V})$ and $\mathcal{S} = \{E\}$, we see that $\{[[-,B],E]\}_{B\in\mathrm{fp}(\mathcal{V})}$ is a cogenerating set of (injective) objects in $[\mathrm{fp}(\mathcal{V}),\mathcal{V}]_0$. So every H in $[\mathrm{fp}(\mathcal{V}),\mathcal{V}]_0$ can be embedded into a product $F = \prod_{q\in Q}[[-,B_q],E]$, where each B_q is finitely presentable. Set $X = \prod_{q\in Q}[B_q,E] \in \mathcal{V}$; this is a geometrically pure injective object by Lemma 3.8, and the arguments above show that $F \cong \Theta_0(X)$. Thus we have an embedding $H \rightarrowtail \Theta_0(X)$. Consequently, if H is injective, then it is a direct summand in $\Theta_0(X)$, and it follows from Lemma 6.11 that $H \cong \Theta_0(X')$ for some direct summand X' in X. As X is geometrically pure injective, so is X'. □

We can now give the result that is explained in the beginning of this section.

Theorem 6.13. *Let $\mathcal{V}$ be as in Setup* 6.1. *The underlying tensor embedding functor*

$$\Theta_0\colon \mathcal{V}_0 \to [\mathrm{fp}(\mathcal{V}),\mathcal{V}]_0 \quad \textit{given by } X \mapsto (X\otimes -)\,|_{\mathrm{fp}(\mathcal{V})}$$

induces a commutative diagram of exact categories and exact functors, i.e.,

$$\begin{array}{ccc} (\mathcal{V}_0,\mathscr{E}_\otimes) & \xrightarrow[\simeq]{\Theta_0} & (\mathrm{AbsPure}([\mathrm{fp}(\mathcal{V}),\mathcal{V}]_0),\mathscr{E}_{\mathrm{ab}}) \\ \uparrow{\scriptstyle \mathrm{inc}} & & \uparrow{\scriptstyle \mathrm{inc}} \\ (\mathrm{PureInj}_\otimes(\mathcal{V}_0),\mathscr{E}_{\mathrm{split}}) & \xrightarrow[\simeq]{\Theta_0} & (\mathrm{Inj}([\mathrm{fp}(\mathcal{V}),\mathcal{V}]_0),\mathscr{E}_{\mathrm{split}}), \end{array} \tag{6.5}$$

where $\mathscr{E}_\otimes$ is the geometrically pure exact structure (*see Definition* 3.4)*, $\mathscr{E}_{\mathrm{ab}}$ denotes the exact structure on* $\mathrm{AbsPure}([\mathrm{fp}(\mathcal{V}),\mathcal{V}]_0)$ *induced by the abelian structure on* $[\mathrm{fp}(\mathcal{V}),\mathcal{V}]_0$*, and $\mathscr{E}_{\mathrm{split}}$ is the* (*trivial*) *split exact structure. Furthermore,* "inc" *denotes the inclusion functor.*

In this diagram, the vertical functors are equivalences of exact categories.

Proof. The asserted equivalence of exact categories in the top of the diagram (6.5) will follow from Theorem 4.6(b) and Corollary 5.4, once we have proved the equality, i.e.,

$$\mathrm{Ess.\,Im}\,\Theta_0 = \mathrm{AbsPure}([\mathrm{fp}(\mathcal{V}), \mathcal{V}]_0). \tag{6.6}$$

It then follows from Proposition 6.12 that the equivalence in the top of (6.5) restricts to the one in the bottom. We now prove the equality (6.6).

"$\subseteq$": Let H be in $\mathrm{Ess.\,Im}\,\Theta_0$, i.e., $H \cong \Theta_0(X)$ for some $X \in \mathcal{V}$. We must argue that every short exact sequence $0 \to \Theta_0(X) \to F \to F' \to 0$ in $[\mathrm{fp}(\mathcal{V}), \mathcal{V}]_0$ is $\star$-pure exact. By Lemma 3.14, it suffices to prove that $\Theta_0(X) \rightarrowtail F$ is a *$\star$-pure monomorphism*, meaning that $G \star \Theta_0(X) \to G \star F$ is monic for every $\mathcal{V}$-functor $G\colon \mathrm{fp}(\mathcal{V})^{\mathrm{op}} \to \mathcal{V}$. By Proposition 3.12, there are a geometrically pure injective object J and a geometrically pure monomorphism $X \rightarrowtail J$ in $\mathcal{V}_0$. Thus $\Theta_0(X) \rightarrowtail \Theta_0(J)$ is a $\star$-pure monomorphism by Lemma 4.4. As $\Theta_0(J)$ is injective in $[\mathrm{fp}(\mathcal{V}), \mathcal{V}]_0$ by Proposition 6.12, the morphism $\Theta_0(X) \to \Theta_0(J)$ admits a lift, i.e.,

$$\begin{array}{ccc} \Theta_0(X) & \rightarrowtail & F. \\ \downarrow & \swarrow & \\ \Theta_0(J) & & \end{array}$$

As $\Theta_0(X) \rightarrowtail \Theta_0(J)$ is a $\star$-pure monomorphism, so is $\Theta_0(X) \rightarrowtail F$.

"$\supseteq$": Let H be an absolutely pure object in $[\mathrm{fp}(\mathcal{V}), \mathcal{V}]_0$. As this category has enough injectives, it follows from Proposition 6.12 that there exists a monomorphism $H \rightarrowtail \Theta_0(J)$ for some geometrically pure injective object $J \in \mathcal{V}$. By the assumption on H, this is even a $\star$-pure monomorphism, and thus $[H(-), E]$ is a direct summand in $[\Theta_0(J)(-), E] \cong [-, [J, E]]$ by the equivalent conditions in Proposition 3.15. By [31, Subsection 2.4], the Yoneda embedding

$$\Upsilon_0\colon \mathcal{V}_0 \to [\mathrm{fp}(\mathcal{V})^{\mathrm{op}}, \mathcal{V}]_0 \quad \text{given by } X \mapsto [-, X]\,|_{\mathrm{fp}(V)}$$

is fully faithful, so because $[H(-), E]$ is a direct summand in $\Upsilon_0([J, E]) = [-, [J, E]]$, it follows from Lemma 6.11 that $[H(-), E] \cong [-, Y]$ for some direct summand Y in $[J, E]$. By evaluating this isomorphism on the unit object, it follows that $Y \cong [I, Y] \cong [H(I), E]$, so

$$[H(-), E] \cong [-, [H(I), E]] \cong [H(I) \otimes -, E].$$

It can be verified that this $\mathcal{V}$-natural isomorphism is $[\theta_H, E]$, where $\theta_H\colon \Theta_0(H(I)) = H(I) \otimes - \to H(-)$ is the $\mathcal{V}$-natural transformation from the proof of Lemma 4.5. By Lemma 3.10, the functor $[-, E]$ is faithful, and hence it reflects isomorphisms. We conclude that θ_H is a $\mathcal{V}$-natural isomorphism, and hence H belongs to $\mathrm{Ess.\,Im}\,\Theta_0$. □

We end this paper with a follow-up on our comment at the end of Section 1.

Remark 6.14. This work has been developed in the setting of an abelian cosmos $\mathcal{V}$. This setting excludes applications to the "non-commutative realm"; in particular, it does not cover the original tensor embedding (1.2). However, it is possible to develop much of the theory, not just for the category $\mathcal{V}$, but for the category R-Mod of *R-left-objects* in the sense of Pareigis [39], where R is any *monoid* in $\mathcal{V}$. Notice that $\mathcal{V}$ is a special case of R-Mod as the unit object I is a commutative monoid in $\mathcal{V}$ with I-Mod $= \mathcal{V}$. To develop the theory found in this paper for R-Mod instead of just $\mathcal{V}$, one basically uses the same proofs, but things become much more technical. A reader who wants to carry out this program should be able to do so with the information given below.

Let $(\mathcal{V}, \otimes, I, [-,-])$ be a closed symmetric moniodal category and R be a *monoid* in $\mathcal{V}$. Write R-Mod (resp. Mod-R) for the category of *R-left-objects* (resp. *R-right-objects*) in $\mathcal{V}$ (see [39, Section 2]). Note that R-Mod and Mod-R are complete, cocomplete, abelian, or Grothendieck if $\mathcal{V}$ is so. Moreover, there are functors

$${}_R[-,-]\colon (R\text{-Mod})^{\mathrm{op}} \times R\text{-Mod} \to \mathcal{V},$$

$$\begin{aligned} &[-,-]_R\colon (\text{Mod-}R)^{\text{op}} \times \text{Mod-}R \to \mathcal{V}, \\ &-\otimes_R -\colon \text{Mod-}R \times R\text{-Mod} \to \mathcal{V}, \end{aligned} \tag{6.7}$$

which behave in expected ways. For example, for $X \in \text{Mod-}R$ the functor $[X,-]$ (which *a priori* is a functor from $\mathcal{V}$ to $\mathcal{V}$) takes values in R-Mod and yields a right adjoint of $X \otimes_R -$.

For any full subcategory $\mathcal{A}$ of R-Mod, one can now consider the tensor embedding

$$\Theta_0\colon \text{Mod-}R \to [\mathcal{A},\mathcal{V}]_0 \quad \text{given by } X \mapsto (X \otimes_R -)\,|_{\mathcal{A}}.$$

With these functors at hand, we leave it to the readers to formulate appropriate versions of, for example, Theorems A–D from Section 1 and check how the existing proofs can be modified to show these. Concerning Theorem C, one can use the adjunctions associated with the functors in (6.7) to show that if $\mathcal{V}$ is locally λ-presentable, then so is Mod-R. To prove that geometrically pure injective objects in Mod-R correspond to injective objects in $[\text{fp}(R\text{-Mod}),\mathcal{V}]_0$ (as in Theorem D), a crucial input is the hypothesis that for all the objects $X, Y \in \text{Mod-}R$, where X is finitely presentable and $J \in \mathcal{V}$ is injective, the following canonical morphism is an isomorphism:

$$X \otimes_R [Y,J] \to [[X,Y]_R, J].$$

We briefly mention a few examples. A monoid in Ab is nothing but a ring. For any ring R, the stalk complex $S(R)$ and the disc complex $D(R)$ from Example 2.3 are monoids in $(\mathsf{Ch}(\mathbb{Z}), \otimes^{\bullet}_{\mathbb{Z}})$ and in $(\mathsf{Ch}(\mathbb{Z}), \underline{\otimes}^{\bullet}_{\mathbb{Z}})$, respectively. This viewpoint allows one to deal with Example 2.3 also in the case where R is non-commutative.

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