AMPLITUDE INEQUALITIES FOR LOCAL (CO)HOMOLOGY

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ABSTRACT. Peskine and Szpiro's Intersection Theorem for finitely generated modules was generalized by Foxby to all modules and bounded complexes of such. We strengthen Foxby's result and prove a dual result, which for complexes with finitely generated homology recovers Iversen's Amplitude Inequality but also applies to derived complete complexes.

INTRODUCTION

Let R be a commutative noetherian local ring. The Intersection Theorem says that for finitely generated R-modules M and N there is an inequality,

$$\mathrm{pd}_R M + \dim_R(N \otimes_R M) \geqslant \dim_R N .$$

It was first proved by Peskine and Szpiro [14] in the equicharacteristic case and later by Roberts [16, 17] in the general case. Foxby [6] proved that the inequality,

$$\mathrm{fd}_R M + \dim_R(N \otimes^{\mathbf{L}}_R M) \geqslant \dim_R N , \tag{0.0.1}$$

holds for R-complexes M and N with bounded homology, where only the homology of N is assumed to be finitely generated. Applied to finitely generated R-modules, Foxby's version recovers the original Intersection Theorem.

The Krull dimension of a complex with degreewise finitely generated homology is captured by vanishing of local cohomology, but in general only the inequality

$$\dim_R M \geqslant \sup\{n \in \mathbb{Z} \mid \mathrm{H}^n_{\mathfrak{m}}(M) \neq 0\}$$

holds, and it may be strict. In this paper, we revisit Foxby's work, switching focus from the Krull dimension to the supremum of nonvanishing local cohomology. This allows us to improve Foxby's result, see Theorem 3.1 and Remark 1.5; in particular, one can in (0.0.1) replace the Krull dimension by the supremum of nonvanishing local cohomology.

Even better, the focus on local cohomology suggests dual results related to nonvanishing of local homology. One such result, Theorem 2.8, is an inequality that, when specialized to complexes with degreewise finitely generated homology, recovers Iversen's Amplitude Inequality [12] but also applies to derived complete complexes. To be specific, let M be a complex with bounded homology and finite flat dimension; Corollary 2.9 says that if M is derived $\mathfrak{m}$-complete or has degreewise finitely

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generated homology, then the inequality

$$\operatorname{amp}(M \otimes^{\mathbf{L}}_R N) \;\geqslant\; \operatorname{amp} N$$

holds for complexes N with degreewise finitely generated homology.

In another direction we prove in Corollary 2.2 that derived complete modules of finite injective dimension only can be found over Cohen–Macaulay rings. This supplements the fact, first suggested by Bass, that finitely generated modules of finite injective dimension only exist over Cohen–Macaulay rings.

* * *

Throughout this paper, R is a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $\boldsymbol{k} = R/\mathfrak{m}$. The $\mathfrak{m}$-adic completion of R is denoted $\widehat{R}$; it is a local ring with maximal ideal $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{R}$ and residue field $\widehat{R}/\widehat{\mathfrak{m}} \cong \boldsymbol{k}$. We work in the derived category, $\mathsf{D}(R)$, over R and use homological notation for R-complexes; we refer the reader to [5] for any unexplained notation and terminology.

It is a well-established fact in local algebra that homological properties of finitely generated R-modules are detected by (non)vanishing of (co)homology with coefficients in $\boldsymbol{k}$. For example, a finitely generated R-module M with $\operatorname{Tor}_1^R(\boldsymbol{k}, M) = 0$ is free. In this paper we focus on a wide class of R-complexes, one that strictly includes complexes with degreewise finitely generated homology, and yet enjoys close homological resemblance to finitely generated modules, see 0.3 below. The importance of these complexes was already established in [6].

The left derived $\mathfrak{m}$-completion functor is denoted $\mathbf{L}\Lambda^{\mathfrak{m}}$, and an R-complex M with $\mathbf{L}\Lambda^{\mathfrak{m}}M$ canonically isomorphic to M is called *derived* $\mathfrak{m}$*-complete*. Similarly, the right derived $\mathfrak{m}$-torsion functor is denoted $\mathbf{R}\Gamma_{\mathfrak{m}}$, and an R-complex M with $\mathbf{R}\Gamma_{\mathfrak{m}}M$ canonically isomorphic to M is called *derived* $\mathfrak{m}$*-torsion*. The study of local cohomology, $\mathrm{H}^n_{\mathfrak{m}}(M) = \mathrm{H}_{-n}(\mathbf{R}\Gamma_{\mathfrak{m}}M)$, goes back to Grothendieck and Hartshorne [11]. The derived torsion and derived completion functors and their interactions in commutative algebra, and more broadly in ring theory, was developed by, primarily, Alonso Tarrío, Jeremías López, and Lipman [1], Greenlees and May [10], Porta, Shaul and Yekutieli [15], Schenzel [18], and Simon [19]. We recall a collection of key isomorphisms from these works; they can also be found in [5, 13.1.18, 13.3.19, 13.4.12, and 13.4.20].

0.1. For R-complexes M and N there are isomorphisms,

$$(\mathbf{R}\Gamma_{\mathfrak{m}}N) \otimes^{\mathbf{L}}_R M \;\simeq\; \mathbf{R}\Gamma_{\mathfrak{m}}(N \otimes^{\mathbf{L}}_R M) \;\simeq\; N \otimes^{\mathbf{L}}_R (\mathbf{R}\Gamma_{\mathfrak{m}}M) \tag{0.1.1}$$

and

$$\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{m}}N, M) \;\simeq\; \mathbf{L}\Lambda^{\mathfrak{m}}\,\mathbf{R}\mathrm{Hom}_R(N, M) \;\simeq\; \mathbf{R}\mathrm{Hom}_R(N, \mathbf{L}\Lambda^{\mathfrak{m}}M)\,. \tag{0.1.2}$$

If M is derived $\mathfrak{m}$-torsion, then there are isomorphisms,

$$\mathbf{R}\mathrm{Hom}_R(M, \mathbf{R}\Gamma_{\mathfrak{m}}N) \;\simeq\; \mathbf{R}\mathrm{Hom}_R(M, N) \;\simeq\; \mathbf{R}\mathrm{Hom}_R(M, \mathbf{L}\Lambda^{\mathfrak{m}}N) \tag{0.1.3}$$

and

$$M \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{m}}N \;\simeq\; M \otimes^{\mathbf{L}}_R N \;\simeq\; M \otimes^{\mathbf{L}}_R \mathbf{L}\Lambda^{\mathfrak{m}}N\,. \tag{0.1.4}$$

The depth is a classic invariant in local algebra. Through its cohomological characterization it generalizes naturally to complexes, and it is already present in [6] and developed further by Foxby and Iyengar in [8]. The dual notion of width made its first official appearance in Yassemi's [22]. For the equalities recalled below we refer to [5, 16.2.3, 16.2.5, 16.2.14, 16.2.16, and 16.2.34].

0.2. For an R-complex M the depth invariant can be computed in several ways:

$$\operatorname{depth}_R M \;=\; -\sup \mathbf{R}\mathrm{Hom}_R(\boldsymbol{k}, M) \;=\; -\sup \mathbf{R}\Gamma_{\mathfrak{m}} M \;\geqslant\; -\sup M \tag{0.2.1}$$

and further one has

$$\operatorname{depth}_R \mathbf{R}\Gamma_{\mathfrak{m}} M \;=\; \operatorname{depth}_R M \;=\; \operatorname{depth}_R \mathbf{L}\Lambda^{\mathfrak{m}} M\,. \tag{0.2.2}$$

Finally, the depth yields a bound on nonvanishing of local homology:

$$\sup \mathbf{L}\Lambda^{\mathfrak{m}} M \;\leqslant\; \dim R - \operatorname{depth}_R M\,. \tag{0.2.3}$$

The width invariant can similarly be computed as

$$\operatorname{width}_R M \;=\; \inf\,(\boldsymbol{k} \otimes^{\mathbf{L}}_R M) \;=\; \inf \mathbf{L}\Lambda^{\mathfrak{m}} M \;\geqslant\; \inf M \tag{0.2.4}$$

and further one has

$$\operatorname{width}_R \mathbf{L}\Lambda^{\mathfrak{m}} M \;=\; \operatorname{width}_R M \;=\; \operatorname{width}_R \mathbf{R}\Gamma_{\mathfrak{m}} M\,. \tag{0.2.5}$$

Finally, the width yields a bound on nonvanishing of local cohomology:

$$-\inf \mathbf{R}\Gamma_{\mathfrak{m}} M \;\leqslant\; \dim R - \operatorname{width}_R M\,. \tag{0.2.6}$$

We recall that the support and cosupport of an R-complex M are the sets

$$\begin{aligned} \operatorname{supp}_R M \;&=\; \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{H}(\boldsymbol{k}(\mathfrak{p}) \otimes^{\mathbf{L}}_R M) \neq 0\} \quad \text{and} \\ \operatorname{cosupp}_R M \;&=\; \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{H}(\mathbf{R}\mathrm{Hom}_R(\boldsymbol{k}(\mathfrak{p}), M)) \neq 0\}\,, \end{aligned}$$

where $\boldsymbol{k}(\mathfrak{p}) = (R/\mathfrak{p})_{\mathfrak{p}}$ is the residue field of $R_{\mathfrak{p}}$. In the context of commutative algebra, the support appeared in [6] under the name "small support" to emphasize that it is different from, actually a subset of, the classic notion of support defined in terms of localization. The importance of the dual notion of cosupport was recognized, still in the context of commutative algebra, by Benson, Iyengar, and Krause [4]. An elementary property of both support and cosupport is the ability to recognize acyclic complexes. Indeed, an R-complex M is *acyclic* if it satisfies the following equivalent conditions:

$$(i)\;\operatorname{supp}_R M \;=\; \varnothing \qquad (ii)\;\mathrm{H}(M) = 0 \qquad (iii)\;\operatorname{cosupp}_R M \;=\; \varnothing\,.$$

This is established in [4, 6]; for convenience we refer to [5, 15.1.15 and 15.2.8]. The facts recalled below can be found in [5, 13.2.5, 16.1.20, 16.1.34, 16.2.5, and 16.2.27].

0.3. Complexes that satisfy the equivalent conditions below have important homological properties in common with finitely generated R-modules.

$$\begin{array}{llll} (i) & \mathfrak{m} \in \operatorname{supp}_R M \qquad & (iii) & \operatorname{width}_R M < \infty \\ (ii) & \mathfrak{m} \in \operatorname{cosupp}_R M \qquad & (iv) & \operatorname{depth}_R M < \infty \end{array}$$

We denote by $\mathsf{D}^{\mathfrak{m}}(R)$ the full subcategory of $\mathsf{D}(R)$ whose objects satisfy the equivalent conditions above. Beware: $\mathsf{D}^{\mathfrak{m}}(R)$ is not triangulated subcategory of $\mathsf{D}(R)$; it is not even additive, as it does not contain the zero complex. It contains the non-acyclic complexes in the subcategories $\mathsf{D}^{\mathfrak{m}\text{-tor}}(R)$ of derived $\mathfrak{m}$-torsion complexes and $\mathsf{D}^{\mathfrak{m}\text{-com}}(R)$ of derived $\mathfrak{m}$-complete complexes. They can also be characterized in terms of support and cosupport:

$$\begin{aligned} \mathsf{D}^{\mathfrak{m}\text{-tor}}(R) \;&=\; \{M \in \mathsf{D}(R) \mid \operatorname{supp}_R M \subseteq \{\mathfrak{m}\}\} \quad \text{and} \\ \mathsf{D}^{\mathfrak{m}\text{-com}}(R) \;&=\; \{M \in \mathsf{D}(R) \mid \operatorname{cosupp}_R M \subseteq \{\mathfrak{m}\}\}\,. \end{aligned} \tag{0.3.1}$$

The category $\mathsf{D}^{\mathfrak{m}}(R)$ also contains the nonacyclic complexes from the subcategory $\mathsf{D}^{\mathsf{f}}(R)$ of complexes with degreewise finitely generated homology. One has

$$\text{(0.3.2)} \qquad \mathbf{L}\Lambda^{\mathfrak{m}}M \simeq \widehat{R} \otimes_R M \quad \text{and} \quad \operatorname{width}_R M = \inf M \quad \text{for } M \in \mathsf{D}^{\mathsf{f}}(R)\,.$$

We recall from [5, 15.1.16, 15.1.27, 15.2.9, 15.2.18, 16.2.9, and 16.2.24] some frequently used properties of complexes in $\mathsf{D}^{\mathfrak{m}}(R)$.

0.4. Let M and N be complexes in $\mathsf{D}^{\mathfrak{m}}(R)$. The complexes

$$\text{(0.4.1)} \qquad \mathbf{L}\Lambda^{\mathfrak{m}}M\,, \quad \mathbf{R}\Gamma_{\mathfrak{m}}M\,, \quad \mathbf{R}\mathrm{Hom}_R(N,M)\,, \quad \text{and} \quad N \otimes^{\mathbf{L}}_R M$$

also belong to $\mathsf{D}^{\mathfrak{m}}(R)$, and there are equalities:

$$\text{(0.4.2)} \qquad \operatorname{depth}_R \mathbf{R}\mathrm{Hom}_R(N,M) = \operatorname{width}_R N + \operatorname{depth}_R M$$

and

$$\text{(0.4.3)} \qquad \operatorname{width}_R(N \otimes^{\mathbf{L}}_R M) = \operatorname{width}_R N + \operatorname{width}_R M\,.$$

0.5. In [6] Foxby defined the Krull dimension of an R-complex M and characterized it as follows, see also [5, 14.2.1]:

$$\text{(0.5.1)} \qquad \dim_R M = \sup\{\dim_R \mathrm{H}_n(M) - n \mid n \in \mathbb{Z}\}\,.$$

Foxby also extended Grothendieck Vanishing to complexes, see also [5, 18.3.22]:

$$\text{(0.5.2)} \qquad \dim_R M \geqslant -\inf \mathbf{R}\Gamma_{\mathfrak{m}}M \quad \text{with equality for } M \in \mathsf{D}^{\mathsf{f}}(R)\,.$$

The difference between the Krull dimension and the depth,

$$\text{(0.5.3)} \qquad \operatorname{cmd}_R M = \dim_R M - \operatorname{depth}_R M\,,$$

is called the *Cohen–Macaulay defect* of M. For $M \in \mathsf{D}^{\mathfrak{m}}(R)$ the Cohen–Macaulay defect is nonnegative, see [5, 18.3.31], and (0.2.1) and (0.5.2) combine to yield

$$\text{(0.5.4)} \qquad \operatorname{cmd}_R M \geqslant \operatorname{amp} \mathbf{R}\Gamma_{\mathfrak{m}}M \quad \text{with equality for } M \in \mathsf{D}^{\mathsf{f}}(R)\,.$$

0.6. We use the notation $\mathsf{D}^{\ell}(R)$ for the subcategory of complexes with degreewise finite length homology, and we recall from [5, 16.1.46] that one has

$$\text{(0.6.1)} \qquad \mathsf{D}^{\ell}(R) \subseteq \mathsf{D}^{\mathfrak{m}\text{-com}}(R) \cap \mathsf{D}^{\mathfrak{m}\text{-tor}}(R)\,.$$

For a derived $\mathfrak{m}$-torsion complex M one has $\sup M = -\operatorname{depth}_R M$ by (0.2.1) and

$$\text{(0.6.2)} \qquad \inf M = -\dim_R M$$

by (0.5.1), see also [5, 16.1.34(b)]; thus the next equality holds,

$$\text{(0.6.3)} \qquad \operatorname{cmd}_R M = \operatorname{amp} M\,.$$

By (0.6.1) this equality holds, in particular, for M in $\mathsf{D}^{\ell}(R)$.

1. Depth and width

We start by collecting a smattering of homological properties of derived $\mathfrak{m}$-torsion complexes and derived $\mathfrak{m}$-complete complexes; they will be used throughout.

1.1 Proposition. *Let M be an R-complex; the following assertions hold.*

(a) *The complex $\mathbf{R}\Gamma_{\mathfrak{m}}M$ is an $\widehat{R}$-complex and derived $\widehat{\mathfrak{m}}$-torsion.*

(b) *If M belongs to $\mathsf{D}^{\mathfrak{m}}(R)$, then $\mathbf{R}\Gamma_{\mathfrak{m}}M$ belongs to $\mathsf{D}^{\widehat{\mathfrak{m}}}(\widehat{R})$.*

(c) *There are isomorphisms,*

$$\boldsymbol{k} \otimes^{\mathbf{L}}_{\widehat{R}} \mathbf{R}\Gamma_{\mathfrak{m}}M \;\simeq\; \boldsymbol{k} \otimes^{\mathbf{L}}_{R} \mathbf{R}\Gamma_{\mathfrak{m}}M \;\simeq\; \boldsymbol{k} \otimes^{\mathbf{L}}_{R} M$$

and

$$\mathbf{R}\mathrm{Hom}_{\widehat{R}}(\boldsymbol{k}, \mathbf{R}\Gamma_{\mathfrak{m}}M) \;\simeq\; \mathbf{R}\mathrm{Hom}_{R}(\boldsymbol{k}, \mathbf{R}\Gamma_{\mathfrak{m}}M) \;\simeq\; \mathbf{R}\mathrm{Hom}_{R}(\boldsymbol{k}, M)\,.$$

(d) *There are (in)equalities,*

$$\sup \mathbf{R}\Gamma_{\mathfrak{m}}M \;\leqslant\; \mathrm{fd}_{\widehat{R}} \mathbf{R}\Gamma_{\mathfrak{m}}M \;=\; \mathrm{fd}_{R} \mathbf{R}\Gamma_{\mathfrak{m}}M \;=\; \sup\,(\boldsymbol{k} \otimes^{\mathbf{L}}_{R} M)\,.$$

(e) *If $\boldsymbol{k} \otimes^{\mathbf{L}}_{R} M$ is in $\mathsf{D}_{\square}(R)$, then so is $\mathbf{R}\Gamma_{\mathfrak{m}}M$ and $\mathrm{fd}_R\, \mathbf{R}\Gamma_{\mathfrak{m}}M$ is finite.*

(f) *If the complex $\mathbf{R}\mathrm{Hom}_R(\boldsymbol{k}, M)$ belongs to $\mathsf{D}_{\sqsubset}(R)$, then there are (in)equalities,*

$$-\inf \mathbf{R}\Gamma_{\mathfrak{m}}M \;\leqslant\; \mathrm{id}_{\widehat{R}} \mathbf{R}\Gamma_{\mathfrak{m}}M \;=\; \mathrm{id}_{R} \mathbf{R}\Gamma_{\mathfrak{m}}M \;=\; -\inf \mathbf{R}\mathrm{Hom}_{R}(\boldsymbol{k}, M)\,.$$

(g) *If $\mathbf{R}\mathrm{Hom}_R(\boldsymbol{k}, M)$ is in $\mathsf{D}_{\square}(R)$, then so is $\mathbf{R}\Gamma_{\mathfrak{m}}M$ and $\mathrm{id}_R\, \mathbf{R}\Gamma_{\mathfrak{m}}M$ is finite.*

Proof. The assertions in part (a) hold by [5, 11.3.18 and 13.3.23(a)].

(c): In the first display, the first isomorphism holds by [5, 13.4.18(c)], and as $\boldsymbol{k}$ per (0.6.1) is derived $\mathfrak{m}$-torsion, the second isomorphism holds by (0.1.4). Similarly, the isomorphisms in second display hold by [5, 13.4.18(b)] and (0.1.3).

(b): For M in $\mathsf{D}^{\mathfrak{m}}(R)$ the complexes in (c) are not acyclic, so $\mathbf{R}\Gamma_{\mathfrak{m}}M$ is in $\mathsf{D}^{\widehat{\mathfrak{m}}}(\widehat{R})$.

(d): The inequality is standard and the equalities hold by [5, 15.4.25 and 17.3.7].

(e): The assertions follow from part (d) and the (in)equalities,

$$\inf \mathbf{R}\Gamma_{\mathfrak{m}}M \;\geqslant\; \mathrm{width}_R\, M - \dim R \;=\; \inf\,(\boldsymbol{k} \otimes^{\mathbf{L}}_{R} M) - \dim R\,,$$

which hold by (0.2.6) and (0.2.4).

(f): The inequality is standard and the equalities hold in view of (0.2.1) by [5, 16.1.23 and 16.3.13].

(g): The assertions follow from part (f) and (0.2.1). □

1.2 Proposition. *Let M be an R-complex; the following assertions hold.*

(a) *The complex $\mathbf{L}\Lambda^{\mathfrak{m}}M$ is an $\widehat{R}$-complex and derived $\widehat{\mathfrak{m}}$-complete.*

(b) *If M belongs to $\mathsf{D}^{\mathfrak{m}}(R)$, then $\mathbf{L}\Lambda^{\mathfrak{m}}M$ belongs to $\mathsf{D}^{\widehat{\mathfrak{m}}}(\widehat{R})$.*

(c) *There are isomorphisms,*

$$\boldsymbol{k} \otimes^{\mathbf{L}}_{\widehat{R}} \mathbf{L}\Lambda^{\mathfrak{m}}M \;\simeq\; \boldsymbol{k} \otimes^{\mathbf{L}}_{R} \mathbf{L}\Lambda^{\mathfrak{m}}M \;\simeq\; \boldsymbol{k} \otimes^{\mathbf{L}}_{R} M$$

and

$$\mathbf{R}\mathrm{Hom}_{\widehat{R}}(\boldsymbol{k}, \mathbf{L}\Lambda^{\mathfrak{m}}M) \;\simeq\; \mathbf{R}\mathrm{Hom}_{R}(\boldsymbol{k}, \mathbf{L}\Lambda^{\mathfrak{m}}M) \;\simeq\; \mathbf{R}\mathrm{Hom}_{R}(\boldsymbol{k}, M)\,.$$

(d) *There are (in)equalities,*

$$-\inf \mathbf{L}\Lambda^{\mathfrak{m}}M \;\leqslant\; \mathrm{id}_{\widehat{R}} \mathbf{L}\Lambda^{\mathfrak{m}}M \;=\; \mathrm{id}_{R} \mathbf{L}\Lambda^{\mathfrak{m}}M \;=\; -\inf \mathbf{R}\mathrm{Hom}_{R}(\boldsymbol{k}, M)\,.$$

(e) *If $\mathbf{R}\mathrm{Hom}_R(\boldsymbol{k}, M)$ is in $\mathsf{D}_{\square}(R)$, then so is $\mathbf{L}\Lambda^{\mathfrak{m}}M$ and $\mathrm{id}_R\, \mathbf{L}\Lambda^{\mathfrak{m}}M$ is finite.*

(f) *If $\boldsymbol{k} \otimes^{\mathbf{L}}_R M$ belongs to $\mathsf{D}_{\sqsupset}(R)$, then there are (in)equalities,*

$$\sup \mathbf{L}\Lambda^{\mathfrak{m}} M \;\leqslant\; \operatorname{fd}_{\widehat{R}} \mathbf{L}\Lambda^{\mathfrak{m}} M \;=\; \operatorname{fd}_R \mathbf{L}\Lambda^{\mathfrak{m}} M \;=\; \sup\,(\boldsymbol{k} \otimes^{\mathbf{L}}_R M)\,.$$

(g) *If $\boldsymbol{k} \otimes^{\mathbf{L}}_R M$ is in $\mathsf{D}_{\square}(R)$, then so is $\mathbf{L}\Lambda^{\mathfrak{m}} M$ and $\operatorname{fd}_R \mathbf{L}\Lambda^{\mathfrak{m}} M$ is finite.*

Proof. The assertions in part (a) hold by [5, 11.3.4 and 13.1.21(a)].

(c): In the first display, the first isomorphism holds by [5, 13.4.18(c)], and as $\boldsymbol{k}$ per (0.6.1) is derived $\mathfrak{m}$-torsion, the second isomorphism holds by (0.1.4). Similarly, the isomorphisms in second display hold by [5, 13.4.18(b)] and (0.1.3).

(b): For M in $\mathsf{D}^{\mathfrak{m}}(R)$ the complexes in (c) are not acyclic, so $\mathbf{L}\Lambda^{\mathfrak{m}} M$ is in $\mathsf{D}^{\widehat{\mathfrak{m}}}(\widehat{R})$.

(d): The inequality is standard and the equalities hold by [5, 17.3.20 and 17.3.23].

(e): The assertions follow from part (d) and the (in)equalities,

$$\sup \mathbf{L}\Lambda^{\mathfrak{m}} M \;\leqslant\; \dim R - \operatorname{depth}_R M \;=\; \dim R + \sup \mathbf{R}\mathrm{Hom}_R(\boldsymbol{k}, M)\,,$$

which hold by (0.2.3) and (0.2.1).

(f): The inequality is standard and the equalities hold in view of (0.2.4) by [5, 16.1.24 and 16.3.18].

(g): The assertions follow from part (f) and (0.2.4). □

In the balance of this section, we establish some formulas for computing depth and width of derived Hom and tensor product complexes. The first result below extends [5, 16.3.1(a)] and its corollary [5, 16.3.3]. In the terminology of Frankild and Jørgensen [9], the assumption on M is that it is a complex of finite $\boldsymbol{k}$-projective dimension. Indeed, by Matlis duality one has

$$-\inf \mathbf{R}\mathrm{Hom}_R(M, \boldsymbol{k}) \;=\; \sup\,(\boldsymbol{k} \otimes^{\mathbf{L}}_R M)\,,$$

see [5, 16.3.8], and that quantity is assumed to be finite.

1.3 Proposition. *Let M be an R-complex with $\boldsymbol{k} \otimes^{\mathbf{L}}_R M$ in $\mathsf{D}_{\sqsubset}(R)$. One has*

(a) $$\operatorname{depth}_R M \;=\; \operatorname{depth} R - \sup\,(\boldsymbol{k} \otimes^{\mathbf{L}}_R M) \;>\; -\infty\,.$$

Moreover, for every R-complex N with $\mathbf{R}\mathrm{Hom}_R(\boldsymbol{k}, N)$ in $\mathsf{D}_{\sqsubset}(R)$ there is an equality,

(b) $$\operatorname{depth}_R(N \otimes^{\mathbf{L}}_R M) \;=\; \operatorname{depth}_R N + \operatorname{depth}_R M - \operatorname{depth} R\,.$$

Proof. The R-module $\boldsymbol{k}$ is derived $\mathfrak{m}$-torsion, see (0.6.1), so the first equality below follows from (0.1.4). By Proposition 1.1(d) the R-complex $\mathbf{R}\Gamma_{\mathfrak{m}} M$ has finite flat dimension, whence [5, 16.3.3] yields the second equality. The third and final equality holds by (0.2.2).

$$\begin{aligned}\sup\,(\boldsymbol{k} \otimes^{\mathbf{L}}_R M) \;&=\; \sup\,(\boldsymbol{k} \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{m}} M)\\ &=\; \operatorname{depth} R - \operatorname{depth}_R \mathbf{R}\Gamma_{\mathfrak{m}} M\\ &=\; \operatorname{depth} R - \operatorname{depth}_R M\,.\end{aligned}$$

Rearranging the terms yields the equality asserted in part (a), and the inequality holds by the assumption on M.

For part (b), notice that $\operatorname{depth}_R N > -\infty$ holds by (0.2.1). Now the asserted equality follows from the next computation where the equalities hold, in order, by

(0.2.2), (0.1.1), [5, 16.3.1(a)], and (0.2.2).

$$\begin{aligned}\operatorname{depth}_R(N \otimes^{\mathbf{L}}_R M) &= \operatorname{depth}_R \mathbf{R}\Gamma_{\mathfrak{m}}(N \otimes^{\mathbf{L}}_R M)\\ &= \operatorname{depth}_R(N \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{m}} M)\\ &= \operatorname{depth}_R N + \operatorname{depth}_R \mathbf{R}\Gamma_{\mathfrak{m}} M - \operatorname{depth} R\\ &= \operatorname{depth}_R N + \operatorname{depth}_R M - \operatorname{depth} R\,. \qquad \square\end{aligned}$$

Every complex M in $\mathsf{D}^{\mathfrak{m}}(R)$ satisfies the inequality

$$\operatorname{depth}_R M + \operatorname{width}_R M \leqslant \dim R\,,$$

which Strooker [21, Corollary 6.1.10] credits to Bartijn; see also Schenzel and Simon [20, Proposition 5.3.9] or [5, 16.2.35]. The next result and its partner, Corollary 1.7, improve the bound under additional assumptions on M.

1.4 Corollary. *Let M be an R-complex. If $\boldsymbol{k} \otimes^{\mathbf{L}}_R M$ is in $\mathsf{D}_{\sqsubset}(R)$, then one has*

$$\operatorname{depth}_R M + \operatorname{width}_R M = \operatorname{depth} R - \operatorname{amp}(\boldsymbol{k} \otimes^{\mathbf{L}}_R M)\,.$$

Proof. One has $\operatorname{width}_R M = \inf\,(\boldsymbol{k} \otimes^{\mathbf{L}}_R M)$, see (0.2.4), and adding this quantity to the equality in Proposition 1.3(a) yields the asserted equality. $\square$

1.5 Remark. In [7] Foxby showed that in the inequality (0.0.1) one can replace $\operatorname{fd}_R M$ with the smaller quantity $\operatorname{depth} R - \operatorname{depth}_R M$; see [5, 16.3.4]. Doing so and subtracting the equality from Proposition 1.3, on the form

$$\operatorname{depth} R - \operatorname{depth}_R M + \operatorname{depth}_R(N \otimes^{\mathbf{L}}_R M) = \operatorname{depth}_R N\,,$$

one arrives at

$$\operatorname{cmd}_R(N \otimes^{\mathbf{L}}_R M) \geqslant \operatorname{cmd}_R N\,,$$

which is how Foxby's version of the Intersection Theorem is stated in [5, 7]. It serves as a template for the main results in this paper, including Theorem 3.1, which improves Foxby's version by replacing $\operatorname{cmd}_R(N \otimes^{\mathbf{L}}_R M)$ by $\operatorname{amp} \mathbf{R}\Gamma_{\mathfrak{m}}(N \otimes^{\mathbf{L}}_R M)$, cf. (0.5.3), and by adding an upper bound on this quantity based on Corollary 1.4.

The next result extends [5, 16.3.9(a)] and its corollary [5, 16.3.11]. The condition on M is, in the terminology of [9], that it is a complex of finite $\boldsymbol{k}$-injective dimension.

1.6 Proposition. *Let M be an R-complex with $\mathbf{R}\mathrm{Hom}_R(\boldsymbol{k}, M)$ in $\mathsf{D}_{\sqsupset}(R)$. One has*

(a) $$\operatorname{width}_R M = \operatorname{depth} R + \inf \mathbf{R}\mathrm{Hom}_R(\boldsymbol{k}, M) > -\infty\,.$$

Moreover, for every R-complex K with $\mathbf{R}\mathrm{Hom}_R(\boldsymbol{k}, K)$ in $\mathsf{D}_{\sqsubset}(R)$ there is an equality,

(b) $$\operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(K, M) = \operatorname{depth}_R K + \operatorname{width}_R M - \operatorname{depth} R\,.$$

Proof. The R-module $\boldsymbol{k}$ is derived $\mathfrak{m}$-torsion, see (0.6.1), so the first equality below follows from (0.1.3). The R-complex $\mathbf{L}\Lambda^{\mathfrak{m}} M$ has finite injective dimension by Proposition 1.2(d), and hence [5, 16.3.11] yields the second equality. The third and final equality holds by (0.2.5).

$$\begin{aligned}-\inf \mathbf{R}\mathrm{Hom}_R(\boldsymbol{k}, M) &= -\inf \mathbf{R}\mathrm{Hom}_R(\boldsymbol{k}, \mathbf{L}\Lambda^{\mathfrak{m}} M)\\ &= \operatorname{depth} R - \operatorname{width}_R \mathbf{L}\Lambda^{\mathfrak{m}} M\\ &= \operatorname{depth} R - \operatorname{width}_R M\,.\end{aligned}$$

Rearranging the terms yields the equality asserted in part (a), and the inequality holds by the assumption on M.

For part (b), notice that $\operatorname{depth}_R K > -\infty$ holds by (0.2.1). Now the asserted equality follows from the next computation where the equalities hold, in order, by (0.2.5), (0.1.2), [5, 16.3.9(a)], and (0.2.5).

$$\begin{aligned}
\operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(K,M) &= \operatorname{width}_R \mathbf{L}\Lambda^{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(K,M)\\
&= \operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(K,\mathbf{L}\Lambda^{\mathfrak{m}} M)\\
&= \operatorname{depth}_R K + \operatorname{width}_R \mathbf{L}\Lambda^{\mathfrak{m}} M - \operatorname{depth} R\\
&= \operatorname{depth}_R K + \operatorname{width}_R M - \operatorname{depth} R\,. \qquad \square
\end{aligned}$$

1.7 Corollary. *Let M be an R-complex. If $\mathbf{R}\mathrm{Hom}_R(\boldsymbol{k},M)$ is in $\mathsf{D}_{\sqsupset}(R)$, then one has*

$$\operatorname{depth}_R M + \operatorname{width}_R M \;=\; \operatorname{depth} R - \operatorname{amp} \mathbf{R}\mathrm{Hom}_R(\boldsymbol{k},M)\,.$$

Proof. One has $\operatorname{depth}_R M = -\sup \mathbf{R}\mathrm{Hom}_R(\boldsymbol{k},M)$, see (0.2.1), and adding this quantity to the equality in Proposition 1.6(a) yields the asserted equality. $\square$

The next result, which extends [5, 16.3.5(a)], supplements parts (b) in Propositions 1.3 and 1.6. There is no part (a) stated, as it would only duplicate part (a) in Proposition 1.3, and for the same reason no corollary is stated; it would just rephrase Corollary 1.4.

1.8 Proposition. *Let M be an R-complex such that $\boldsymbol{k} \otimes^{\mathbf{L}}_R M$ belongs to $\mathsf{D}_{\sqsubset}(R)$. For every R-complex K with $\boldsymbol{k} \otimes^{\mathbf{L}}_R K$ in $\mathsf{D}_{\sqsupset}(R)$ there is an equality,*

$$\operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(M,K) \;=\; \operatorname{depth}_R M + \operatorname{width}_R K - \operatorname{depth} R\,.$$

Proof. By Proposition 1.1(d) the complex $\mathbf{R}\Gamma_{\mathfrak{m}} M$ has finite flat dimension and, thus, finite projective dimension as R is local. Further, $\operatorname{width}_R K > -\infty$ holds by (0.2.4). Now the asserted equality follows from the next computation where the equalities hold, in order, by (0.2.5), (0.1.2), [5, 16.3.5(a)], and (0.2.2).

$$\begin{aligned}
\operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(M,K) &= \operatorname{width}_R \mathbf{L}\Lambda^{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(M,K)\\
&= \operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{m}} M,K)\\
&= \operatorname{depth}_R \mathbf{R}\Gamma_{\mathfrak{m}} M + \operatorname{width}_R K - \operatorname{depth} R\\
&= \operatorname{depth}_R M + \operatorname{width}_R K - \operatorname{depth} R\,. \qquad \square
\end{aligned}$$

2. Nonvanishing of local homology

For complexes in $\mathsf{D}^{\mathsf{f}}(R)$ the next result is, in view of (0.3.2), known from [5, 18.5.7].

2.1 Proposition. *Let M a complex in $\mathsf{D}^{\mathfrak{m}}(R)$. If the complex $\mathbf{R}\mathrm{Hom}_R(\boldsymbol{k},M)$ belongs to $\mathsf{D}_{\sqsupset}(R)$, then there is an inequality,*

$$\operatorname{amp} \mathbf{L}\Lambda^{\mathfrak{m}} M \;\geqslant\; \operatorname{cmd} R\,.$$

Proof. Let H be an R-module of maximal depth, i.e. $\operatorname{depth}_R H = \dim R$; such modules exist by work of André [2], see [5, 18.4.9]. The complex $\mathbf{R}\mathrm{Hom}_R(H,\mathbf{L}\Lambda^{\mathfrak{m}} M)$ belongs to $\mathsf{D}^{\mathfrak{m}}(R)$, see (0.4.1); in particular, it is not acyclic which explains the second inequality in the computation below. The first inequality holds by [5, 7.6.7]. The first equality is trivial, and the second follows from (0.1.2). The fourth equality comes from Proposition 1.6, and the penultimate equality holds by (0.5.3). The remaining equalities hold by (0.2.4).

$$\begin{aligned}
\sup \mathbf{L}\Lambda^{\mathfrak{m}} M &= \sup \mathbf{L}\Lambda^{\mathfrak{m}} M - \inf H \\
&\geqslant \sup \mathbf{R}\mathrm{Hom}_R(H, \mathbf{L}\Lambda^{\mathfrak{m}} M) \\
&\geqslant \inf \mathbf{R}\mathrm{Hom}_R(H, \mathbf{L}\Lambda^{\mathfrak{m}} M) \\
&= \inf \mathbf{L}\Lambda^{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(H, M) \\
&= \operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(H, M) \\
&= \operatorname{depth}_R H + \operatorname{width}_R M - \operatorname{depth} R \\
&= \operatorname{cmd} R + \operatorname{width}_R M \\
&= \operatorname{cmd} R + \inf \mathbf{L}\Lambda^{\mathfrak{m}} M\,.
\end{aligned}$$

The asserted inequality now follows by rearranging the terms. □

It was suggested by Bass [3] and proved by Peskine and Szpiro [14] in the equicharacteristic case, and in general by Roberts [16, 17], that existence of a nonzero finitely generated R-module of finite injective dimension forces R to be Cohen–Macaulay. The next corollary elaborates on this classic result. For perspective, we recall from [5, 13.1.33] that an $\mathfrak{m}$-complete module is derived $\mathfrak{m}$-complete.

2.2 Corollary. *If there exists a nonzero derived $\mathfrak{m}$-complete or finitely generated R-module of finite injective dimension, then R is Cohen–Macaulay.*

Proof. Let $M \neq 0$ be an R-module of finite injective dimension. If M is derived $\mathfrak{m}$-complete, then one has $\operatorname{amp} \mathbf{L}\Lambda^{\mathfrak{m}} M = \operatorname{amp} M = 0$. If M is finitely generated, then there is an isomorphism $\mathbf{L}\Lambda^{\mathfrak{m}} M \simeq \widehat{R} \otimes_R M$, see (0.3.2), so again $\operatorname{amp} \mathbf{L}\Lambda^{\mathfrak{m}} M = 0$ holds. Now invoke Proposition 2.1. □

The proof of the next theorem is inspired by Foxby's proof of [6, Theorem 4.1].

2.3 Theorem. *Let M and K be complexes with $M \in \mathsf{D}^{\mathfrak{m}}(R)$ and $K \in \mathsf{D}^{\mathsf{f}}_{\sqsubset}(R)$. If the complex $\mathbf{R}\mathrm{Hom}_R(\boldsymbol{k}, M)$ belongs to $\mathsf{D}_{\sqsupset}(R)$, then there are inequalities,*

$$\operatorname{amp} \mathbf{R}\mathrm{Hom}_R(\boldsymbol{k}, M) + \operatorname{cmd}_R K \;\geqslant\; \operatorname{amp} \mathbf{L}\Lambda^{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(K, M) \;\geqslant\; \operatorname{cmd}_R K\,.$$

Proof. Both inequalities are trivial if K is acyclic, so assume that is not the case. That is, assume that K belongs to $\mathsf{D}^{\mathfrak{m}}(R)$, see 0.3. As K belongs to $\mathsf{D}_{\sqsubset}(R)$ so does $\mathbf{R}\mathrm{Hom}_R(\boldsymbol{k}, K)$, see [5, 7.6.7]. Now (0.2.4) and Proposition 1.6 yield

$$\begin{aligned}
-\inf \mathbf{L}\Lambda^{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(K, M) &= -\operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(K, M) \\
&= -\operatorname{depth}_R K - \operatorname{width}_R M + \operatorname{depth} R\,.
\end{aligned}$$

Thus, it follows from Corollary 1.7 and (0.5.3) that it is sufficient to establish the inequalities

$$(*)\qquad \begin{aligned}
\dim_R K - \operatorname{depth}_R M &\geqslant \sup \mathbf{L}\Lambda^{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(K, M) \\
&\geqslant \dim_R K + \operatorname{width}_R M - \operatorname{depth} R\,.
\end{aligned}$$

By (0.1.2) and (0.1.3) there are isomorphisms

$$\mathbf{L}\Lambda^{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(K, M) \;\simeq\; \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{m}} K, M) \;\simeq\; \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{m}} K, \mathbf{R}\Gamma_{\mathfrak{m}} M)\,.$$

The left-hand inequality in $(*)$ now follows from (0.5.2), (0.2.1), and [5, 7.6.7]:

$$\dim_R K - \operatorname{depth}_R M \;=\; -\inf \mathbf{R}\Gamma_{\mathfrak{m}} K + \sup \mathbf{R}\Gamma_{\mathfrak{m}} M \;\geqslant\; \sup \mathbf{L}\Lambda^{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(K, M)\,.$$

In the next computation, the first inequality holds by (0.2.1). The equalities follow from (0.2.2) and (0.4.2). The final inequality holds by (0.2.3).

$$\begin{aligned}\sup \mathbf{L}\Lambda^{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(K,M) &\geqslant -\operatorname{depth}_R \mathbf{L}\Lambda^{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(K,M)\\ &= -\operatorname{depth}_R \mathbf{R}\mathrm{Hom}_R(K,M)\\ &= -\operatorname{width}_R K - \operatorname{depth}_R M\\ &\geqslant -\operatorname{width}_R K + \sup \mathbf{L}\Lambda^{\mathfrak{m}} M - \dim R\,.\end{aligned}$$

Per (0.3.2) one has $\operatorname{width}_R K = \inf K$, and the computation above shows that the right-hand inequality in $(*)$ holds if one has $\inf K = -\infty$ or $\sup \mathbf{L}\Lambda^{\mathfrak{m}} M = \infty$. One can henceforth assume that K belongs to $\mathsf{D}^{\mathsf{f}}_{\square}(R)$ and $\mathbf{L}\Lambda^{\mathfrak{m}} M$ to $\mathsf{D}_{\square}(R)$ as $\inf \mathbf{L}\Lambda^{\mathfrak{m}} M = \operatorname{width}_R M > -\infty$ holds by (0.2.4) and Proposition 1.6.

To prove the right-hand inequality in $(*)$ we first assume that K is a cyclic module $R/\mathfrak{a}$. The $R/\mathfrak{a}$-complex $X = \mathbf{R}\mathrm{Hom}_R(K,M)$ meets the assumptions in Proposition 2.1; indeed $\boldsymbol{k}$ is also the residue field of the local ring $R/\mathfrak{a}$ and one has

$$\mathbf{R}\mathrm{Hom}_{R/\mathfrak{a}}(\boldsymbol{k},X) \simeq \mathbf{R}\mathrm{Hom}_R(\boldsymbol{k},M)\,.$$

As M is in $\mathsf{D}^{\mathfrak{m}}(R)$, the complex $\mathbf{R}\mathrm{Hom}_R(\boldsymbol{k},M)$ is not acyclic, so X is in $\mathsf{D}^{\mathfrak{m}/\mathfrak{a}}(R/\mathfrak{a})$ and $\mathbf{R}\mathrm{Hom}_{R/\mathfrak{a}}(\boldsymbol{k},X)$ belongs to $\mathsf{D}_{\sqsupset}(R/\mathfrak{a})$. By (0.2.1) one has

$$\sup \mathbf{R}\mathrm{Hom}_R(\boldsymbol{k},K) = -\operatorname{depth}_R R/\mathfrak{a} < \infty\,,$$

so the R-complexes K and M satisfy the assumptions in Proposition 1.6. In the computation below the first equality is trivial and the second holds by (0.2.4) and independence of base for local homology, see [5, 13.1.21(a)]. The inequality follows from Proposition 2.1 and the penultimate equality from Proposition 1.6. The final equality follows from (0.5.3).

$$\begin{aligned}\sup \mathbf{L}\Lambda^{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(K,M) &= \operatorname{amp} \mathbf{L}\Lambda^{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(K,M) + \inf \mathbf{L}\Lambda^{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(K,M)\\ &= \operatorname{amp} \mathbf{L}\Lambda^{\mathfrak{m}/\mathfrak{a}} X + \operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(K,M)\\ &\geqslant \operatorname{cmd} R/\mathfrak{a} + \operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(K,M)\\ &= \operatorname{cmd}_R K + \operatorname{depth}_R K + \operatorname{width}_R M - \operatorname{depth} R\\ &= \dim_R K + \operatorname{width}_R M - \operatorname{depth} R\,.\end{aligned}$$

Next we prove the right-hand inequality in $(*)$ under the assumption that K is a finitely generated R-module. Set $\mathfrak{a} = (0 :_R K)$; the second equality below holds by [5, 17.6.13]. The first and third equalities follow from (0.1.2). The inequality was established above and the last equality is standard, see [5, 14.1.1].

$$\begin{aligned}\sup \mathbf{L}\Lambda^{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(K,M) &= \sup \mathbf{R}\mathrm{Hom}_R(K,\mathbf{L}\Lambda^{\mathfrak{m}} M)\\ &= \sup \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a},\mathbf{L}\Lambda^{\mathfrak{m}} M)\\ &= \sup \mathbf{L}\Lambda^{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a},M)\\ &\geqslant \dim_R R/\mathfrak{a} + \operatorname{width}_R M - \operatorname{depth} R\\ &= \dim_R K + \operatorname{width}_R M - \operatorname{depth} R\,.\end{aligned}$$

Finally we prove the right-hand inequality in $(*)$ for any complex K in $\mathsf{D}^{\mathsf{f}}_{\square}(R)$. The first and third equalities below follow from (0.1.2). The second equality holds by

[5, 17.6.14]. The inequality was established above, and the final equality is (0.5.1).

$$\begin{aligned}
&\sup \mathbf{L}\Lambda^{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(K,M)\\
&\qquad = \sup \mathbf{R}\mathrm{Hom}_R(K, \mathbf{L}\Lambda^{\mathfrak{m}} M)\\
&\qquad = \sup\{\sup \mathbf{R}\mathrm{Hom}_R(\mathrm{H}_n(K), \mathbf{L}\Lambda^{\mathfrak{m}} M) - n \mid n \in \mathbb{Z}\}\\
&\qquad = \sup\{\sup \mathbf{L}\Lambda^{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(\mathrm{H}_n(K), M) - n \mid n \in \mathbb{Z}\}\\
&\qquad \geqslant \sup\{\dim_R \mathrm{H}_n(K) - n \mid n \in \mathbb{Z}\} + \mathrm{width}_R M - \mathrm{depth}\, R\\
&\qquad = \dim_R K + \mathrm{width}_R M - \mathrm{depth}\, R\,. \qquad \square
\end{aligned}$$

The next result has a counterpart in Corollary 3.8.

2.4 Corollary. *Let M and K be R-complexes with $K \in \mathsf{D}^{\mathsf{f}}_{\sqsubset}(R)$. Assume that M is of finite injective dimension and not acyclic. If M is derived $\mathfrak{m}$-complete or belongs to $\mathsf{D}^{\mathsf{f}}_{\square}(R)$, then there are inequalities,*

$$\mathrm{id}_R M - \mathrm{depth}_R M + \mathrm{cmd}_R K \;\geqslant\; \mathrm{amp}\, \mathbf{R}\mathrm{Hom}_R(K,M) \;\geqslant\; \mathrm{cmd}_R K\,.$$

Proof. The complex M belongs to $\mathsf{D}^{\mathfrak{m}}(R)$, see 0.3. If M is derived $\mathfrak{m}$-complete, then (0.2.1) and Proposition 1.2(d) yield

$$(*)\quad \sup \mathbf{R}\mathrm{Hom}_R(\boldsymbol{k},M) \;=\; -\,\mathrm{depth}_R M \quad\text{and}\quad \inf \mathbf{R}\mathrm{Hom}_R(\boldsymbol{k},M) \;=\; -\,\mathrm{id}_R M\,;$$

in particular, the complex $\mathbf{R}\mathrm{Hom}_R(\boldsymbol{k},M)$ belongs to $\mathsf{D}_{\sqsupset}(R)$. Further, the complex $\mathbf{R}\mathrm{Hom}_R(K,M)$ is derived $\mathfrak{m}$-complete, see (0.1.2), so the asserted inequalities follow from Theorem 2.3 and the equalities $(*)$.

If M belongs to $\mathsf{D}^{\mathsf{f}}_{\square}(R)$, then the equalities $(*)$ hold by (0.2.1) and [5, 16.4.8]. The complex $\mathbf{R}\mathrm{Hom}_R(K,M)$ is in $\mathsf{D}^{\mathsf{f}}_{\sqsupset}(R)$, see [5, 15.4.9], so there are by (0.3.2), and faithful flatness of $\widehat{R}$ as an R-module, equalities

$$\mathrm{amp}\, \mathbf{L}\Lambda^{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(K,M) \;=\; \mathrm{amp}(\widehat{R} \otimes_R \mathbf{R}\mathrm{Hom}_R(K,M)) \;=\; \mathrm{amp}\, \mathbf{R}\mathrm{Hom}_R(K,M)\,.$$

Now the asserted inequalities follow from Theorem 2.3 and the equalities $(*)$. $\square$

2.5 Example. The complex $D = \mathbf{L}\Lambda^{\mathfrak{m}}\, \mathrm{E}_R(\boldsymbol{k})$ is dualizing for $\widehat{R}$, see [5, 18.2.9]. The equalities $\mathrm{id}_R D = \mathrm{id}_{\widehat{R}} D = \mathrm{depth}_{\widehat{R}} D = \mathrm{depth}_R D$ hold by [5, 17.3.20, 18.2.11, and 18.3.1], so for a complex K in $\mathsf{D}^{\mathsf{f}}_{\sqsubset}(R)$ Corollary 2.4 yields

$$\mathrm{amp}\, \mathbf{R}\mathrm{Hom}_R(K,D) \;=\; \mathrm{cmd}_R K\,.$$

Further, by (0.1.2), [5, 16.1.25(6)], and (0.2.1) one has

$$\inf \mathbf{R}\mathrm{Hom}_R(K,D) \;=\; \inf \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{m}} K, \mathrm{E}_R(\boldsymbol{k})) = -\sup \mathbf{R}\Gamma_{\mathfrak{m}} K \;=\; \mathrm{depth}_R K$$

and, therefore, per (0.5.3) also

$$\sup \mathbf{R}\mathrm{Hom}_R(K,D) \;=\; \dim_R K\,.$$

These interpretations of supremum and infimum of $\mathbf{R}\mathrm{Hom}_R(K,D)$ are standard if one replaces D by a dualizing R-complex, see [5, 18.2.31], and like in the proof of that statement one can verify that they hold for all complexes K in $\mathsf{D}^{\mathsf{f}}(R)$.

The inequalities in the next result also hold under slightly different conditions on the complexes, see Corollary 3.9.

2.6 Corollary. *Let M and K be R-complexes with $M \in \mathsf{D}^{\mathfrak{m}}(R)$ and $K \in \mathsf{D}^{\ell}_{\sqsubset}(R)$. If the complex $\mathbf{R}\mathrm{Hom}_R(\boldsymbol{k},M)$ belongs to $\mathsf{D}_{\sqsupset}(R)$, then there are inequalities,*

$$\mathrm{amp}\, \mathbf{R}\mathrm{Hom}_R(\boldsymbol{k},M) + \mathrm{amp}\, K \;\geqslant\; \mathrm{amp}\, \mathbf{R}\mathrm{Hom}_R(K,M) \;\geqslant\; \mathrm{amp}\, K\,.$$

Proof. Recall from (0.6.1) that K is derived $\mathfrak{m}$-torsion, so $\mathbf{R}\mathrm{Hom}_R(K,M)$ is per (0.1.2) derived $\mathfrak{m}$-complete. Now apply Theorem 2.3 and (0.6.3). $\square$

The next result extends [5, 13.1.19(c)] for the ideal $\mathfrak{a} = \mathfrak{m}$.

2.7 Proposition. *Let M and N be R-complexes with $N \in \mathsf{D}^{\mathrm{f}}(R)$. If the complex $\boldsymbol{k} \otimes^{\mathbf{L}}_R M$ belongs to $\mathsf{D}_{\square}(R)$, then there is an isomorphism,*

$$\mathbf{L}\Lambda^{\mathfrak{m}}(N \otimes^{\mathbf{L}}_R M) \simeq N \otimes^{\mathbf{L}}_R \mathbf{L}\Lambda^{\mathfrak{m}} M .$$

Proof. The first two isomorphisms in the display below follow from [5, 13.4.1(c)] and (0.1.1). By Proposition 1.1(e) the R-complex $\mathbf{R}\Gamma_{\mathfrak{m}} M$ has bounded homology and finite flat dimension. As N is in $\mathsf{D}^{\mathrm{f}}(R)$ the third isomorphism now holds by [5, 13.1.19(c) and 13.4.1(c)].

$$\begin{aligned}\mathbf{L}\Lambda^{\mathfrak{m}}(N \otimes^{\mathbf{L}}_R M) &\simeq \mathbf{L}\Lambda^{\mathfrak{m}}\mathbf{R}\Gamma_{\mathfrak{m}}(N \otimes^{\mathbf{L}}_R M)\\ &\simeq \mathbf{L}\Lambda^{\mathfrak{m}}(N \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{m}} M)\\ &\simeq N \otimes^{\mathbf{L}}_R \mathbf{L}\Lambda^{\mathfrak{m}} M . \end{aligned}$$ $\square$

We derive the next result from Theorem 2.3 by way of Grothendieck Duality.

2.8 Theorem. *Let M and N be complexes with $M \in \mathsf{D}^{\mathfrak{m}}(R)$ and $N \in \mathsf{D}^{\mathrm{f}}(R)$. If the complex $\boldsymbol{k} \otimes^{\mathbf{L}}_R M$ belongs to $\mathsf{D}_{\square}(R)$, then there are inequalities,*

$$\operatorname{amp}(\boldsymbol{k} \otimes^{\mathbf{L}}_R M) + \operatorname{amp} N \geqslant \operatorname{amp} \mathbf{L}\Lambda^{\mathfrak{m}}(N \otimes^{\mathbf{L}}_R M) \geqslant \operatorname{amp} N .$$

Proof. Both inequalities are trivial if N is acyclic, so assume that is not the case. That is, assume that N belongs to $\mathsf{D}^{\mathfrak{m}}(R)$, see 0.3. The equalities below hold by (0.2.4) and (0.4.3).

$$\inf \mathbf{L}\Lambda^{\mathfrak{m}}(N \otimes^{\mathbf{L}}_R M) = \operatorname{width}_R(N \otimes^{\mathbf{L}}_R M) = \operatorname{width}_R N + \operatorname{width}_R M .$$

Per (0.3.2) one has $\operatorname{width}_R N = \inf N$, so the equalities show that both of the asserted inequalities are trivial if $\inf N = -\infty$ holds.

One can now assume that N belongs to $\mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$. Set $\widehat{N} = \widehat{R} \otimes_R N$; it is by [5, 18.3.2(c)] a complex in $\mathsf{D}^{\mathrm{f}}_{\sqsupset}(\widehat{R})$. In the display below, the first isomorphism holds by Proposition 1.2(a), and the second follows from Proposition 2.7. As $\mathbf{L}\Lambda^{\mathfrak{m}} M$ by another application of Proposition 1.2(a) is a complex in $\mathsf{D}^{\widehat{\mathfrak{m}}}(\widehat{R})$, the third isomorphism is standard, see [5, 12.3.28]. The ring $\widehat{R}$ has a dualizing complex, D, and the fourth isomorphism is Grothendieck Duality, see [5, 18.2.3]. The last isomorphism holds by [5, 12.3.23(c)], as the $\widehat{R}$-complex $\mathbf{L}\Lambda^{\mathfrak{m}} M$ has bounded homology and finite flat dimension by Proposition 1.2(f,g).

$$\begin{aligned}\mathbf{L}\Lambda^{\mathfrak{m}}(N \otimes^{\mathbf{L}}_R M) &\simeq \mathbf{L}\Lambda^{\widehat{\mathfrak{m}}}\mathbf{L}\Lambda^{\mathfrak{m}}(N \otimes^{\mathbf{L}}_R M)\\ &\simeq \mathbf{L}\Lambda^{\widehat{\mathfrak{m}}}(N \otimes^{\mathbf{L}}_R \mathbf{L}\Lambda^{\mathfrak{m}} M)\\ &\simeq \mathbf{L}\Lambda^{\widehat{\mathfrak{m}}}(\widehat{N} \otimes^{\mathbf{L}}_{\widehat{R}} \mathbf{L}\Lambda^{\mathfrak{m}} M)\\ &\simeq \mathbf{L}\Lambda^{\widehat{\mathfrak{m}}}(\mathbf{R}\mathrm{Hom}_{\widehat{R}}(\mathbf{R}\mathrm{Hom}_{\widehat{R}}(\widehat{N}, D), D) \otimes^{\mathbf{L}}_{\widehat{R}} \mathbf{L}\Lambda^{\mathfrak{m}} M)\\ &\simeq \mathbf{L}\Lambda^{\widehat{\mathfrak{m}}}\, \mathbf{R}\mathrm{Hom}_{\widehat{R}}(\mathbf{R}\mathrm{Hom}_{\widehat{R}}(\widehat{N}, D), D \otimes^{\mathbf{L}}_{\widehat{R}} \mathbf{L}\Lambda^{\mathfrak{m}} M) . \end{aligned}$$

The complex $D \otimes^{\mathbf{L}}_{\widehat{R}} \mathbf{L}\Lambda^{\mathfrak{m}} M$ belongs to $\mathsf{D}^{\widehat{\mathfrak{m}}}(\widehat{R})$, see (0.4.1). The first isomorphism below holds by [5, 12.3.23(c)]. The second isomorphism holds as one can assume that D is normalized, see [5, 18.2.33 and 18.2.24], and the third comes from Proposition 1.2(c).

$$(*)\qquad \begin{aligned} \mathbf{R}\mathrm{Hom}_{\widehat{R}}(\boldsymbol{k}, D\otimes^{\mathbf{L}}_{\widehat{R}} \mathbf{L}\Lambda^{\mathfrak{m}} M) &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}}(\boldsymbol{k}, D)\otimes^{\mathbf{L}}_{\widehat{R}} \mathbf{L}\Lambda^{\mathfrak{m}} M \\ &\simeq \boldsymbol{k}\otimes^{\mathbf{L}}_{\widehat{R}} \mathbf{L}\Lambda^{\mathfrak{m}} M \\ &\simeq \boldsymbol{k}\otimes^{\mathbf{L}}_{R} M\,. \end{aligned}$$

Thus, $\mathbf{R}\mathrm{Hom}_{\widehat{R}}(\boldsymbol{k}, D\otimes^{\mathbf{L}}_{\widehat{R}} \mathbf{L}\Lambda^{\mathfrak{m}} M)$ belongs to $\mathsf{D}_{\square}(\widehat{R})$ by the assumption on M. By [5, 18.2.3] the complex $\mathbf{R}\mathrm{Hom}_{\widehat{R}}(\widehat{N}, D)$ belongs to $\mathsf{D}^{\mathsf{f}}_{\sqsubset}(\widehat{R})$, so Theorem 2.3 yields in view of $(*)$ the inequalities,

$$\begin{aligned} &\mathrm{amp}(\boldsymbol{k}\otimes^{\mathbf{L}}_{R} M) + \mathrm{cmd}_{\widehat{R}}\, \mathbf{R}\mathrm{Hom}_{\widehat{R}}(\widehat{N}, D) \\ &\qquad\qquad \geqslant \mathrm{amp}\, \mathbf{L}\Lambda^{\mathfrak{m}}(N\otimes^{\mathbf{L}}_{R} M) \;\geqslant\; \mathrm{cmd}_{\widehat{R}}\, \mathbf{R}\mathrm{Hom}_{\widehat{R}}(\widehat{N}, D)\,. \end{aligned}$$

Finally, from [5, 18.2.31] and faithful flatness of $\widehat{R}$ as an R-module one gets:

$$\mathrm{cmd}_{\widehat{R}}\, \mathbf{R}\mathrm{Hom}_{\widehat{R}}(\widehat{N}, D) \;=\; \mathrm{amp}\, \widehat{N} \;=\; \mathrm{amp}\, N\,. \qquad \square$$

Nested in the next result, which has a counterpart in Corollary 3.3, is Iversen's Amplitude Inequality [12, Theorem 3.2].

2.9 Corollary. *Let M and N be complexes with $M \in \mathsf{D}_{\square}(R)$ and $N \in \mathsf{D}^{\mathsf{f}}(R)$. Assume that M is of finite flat dimension and not acyclic. If M is derived $\mathfrak{m}$-complete or belongs to $\mathsf{D}^{\mathsf{f}}_{\square}(R)$, then there are inequalities,*

$$\mathrm{fd}_R\, M - \inf M + \mathrm{amp}\, N \;\geqslant\; \mathrm{amp}(N\otimes^{\mathbf{L}}_{R} M) \;\geqslant\; \mathrm{amp}\, N\,.$$

Proof. The complex M belongs to $\mathsf{D}^{\mathfrak{m}}(R)$, see 0.3. If M is derived $\mathfrak{m}$-complete, then Proposition 1.2(f) and (0.2.4) yield

$$(*)\qquad \sup(\boldsymbol{k}\otimes^{\mathbf{L}}_{R} M) \;=\; \mathrm{fd}_R\, M \qquad \text{and} \qquad \inf(\boldsymbol{k}\otimes^{\mathbf{L}}_{R} M) \;=\; \inf M\,;$$

in particular, the complex $\boldsymbol{k}\otimes^{\mathbf{L}}_{R} M$ belongs to $\mathsf{D}_{\square}(R)$. It follows from Proposition 2.7 that the complex $N\otimes^{\mathbf{L}}_{R} M$ is derived $\mathfrak{m}$-complete, so the asserted inequalities follow from Theorem 2.8 and the equalities $(*)$.

If M belongs to $\mathsf{D}^{\mathsf{f}}_{\square}(R)$, then the equalities $(*)$ hold by [5, 16.4.1], (0.2.4), and (0.3.2). The complex $N\otimes^{\mathbf{L}}_{R} M$ is in $\mathsf{D}^{\mathsf{f}}(R)$, see [5, 15.4.3], so by (0.3.2) and faithful flatness of $\widehat{R}$ as an R-module there are equalities,

$$\mathrm{amp}\, \mathbf{L}\Lambda^{\mathfrak{m}}(N\otimes^{\mathbf{L}}_{R} M) \;=\; \mathrm{amp}(\widehat{R}\otimes_R (N\otimes^{\mathbf{L}}_{R} M)) \;=\; \mathrm{amp}(N\otimes^{\mathbf{L}}_{R} M)\,.$$

Now the asserted inequalities follow from Theorem 2.8 and the equalities $(*)$. $\square$

Iversen's Amplitude Inequality yields a short proof of Auslander's zero-divisor conjecture, which is the finitely generated case of the next example. The other case shows that derived $\mathfrak{m}$-complete modules of finite flat dimension are "Auslander modules" in the sense of Nasehpour [13].

2.10 Example. Let M be an R-module and $\boldsymbol{x} = x_1, \ldots, x_n$ an M-regular sequence in $\mathfrak{m}$. If M has finite flat dimension and is finitely generated or derived $\mathfrak{m}$-complete, then $\boldsymbol{x}$ is R-regular. Indeed, with K denoting the Koszul complex on $\boldsymbol{x}$ one has $\mathrm{amp}(M\otimes^{\mathbf{L}}_{R} K) \geqslant \mathrm{amp}\, K$, so the conclusion follows from [5, 16.2.31].

The inequalities in the next result also hold under slightly different conditions on the complexes, see Corollary 3.4.

2.11 Corollary. *Let M and N be complexes with $M \in \mathsf{D}^{\mathfrak{m}}(R)$ and $N \in \mathsf{D}^{\ell}(R)$. If the complex $\boldsymbol{k} \otimes^{\mathbf{L}}_R M$ belongs to $\mathsf{D}_{\square}(R)$, then there are inequalities,*

$$\operatorname{amp}(\boldsymbol{k} \otimes^{\mathbf{L}}_R M) + \operatorname{amp} N \;\geqslant\; \operatorname{amp}(N \otimes^{\mathbf{L}}_R M) \;\geqslant\; \operatorname{amp} N\,.$$

Proof. Recall from (0.6.1) that the complex N is derived $\mathfrak{m}$-torsion, so by (0.1.4) and Proposition 2.7 there are isomorphisms

$$N \otimes^{\mathbf{L}}_R M \;\simeq\; N \otimes^{\mathbf{L}}_R \mathbf{L}\Lambda^{\mathfrak{m}} M \;\simeq\; \mathbf{L}\Lambda^{\mathfrak{m}}(N \otimes^{\mathbf{L}}_R M)\,.$$

Now the asserted inequalities follow from Theorem 2.8. □

3. Nonvanishing of local cohomology

The first result of this section strengthens [5, 18.5.2], see Example 3.5. It could be proved in a manner dual to Theorem 2.3, but we obtain it instead by reducing to a situation where results from [5, 18.5] apply.

3.1 Theorem. *Let M and N be complexes with $M \in \mathsf{D}^{\mathfrak{m}}(R)$ and $N \in \mathsf{D}^{\mathsf{f}}_{\sqsubset}(R)$. If the complex $\boldsymbol{k} \otimes^{\mathbf{L}}_R M$ belongs to $\mathsf{D}_{\sqsubset}(R)$, then there are inequalities,*

$$\operatorname{amp}(\boldsymbol{k} \otimes^{\mathbf{L}}_R M) + \operatorname{cmd}_R N \;\geqslant\; \operatorname{amp} \mathbf{R}\Gamma_{\mathfrak{m}}(N \otimes^{\mathbf{L}}_R M) \;\geqslant\; \operatorname{cmd}_R N\,.$$

Proof. Both inequalities are trivial if N is acyclic, so assume that is not the case. That is, assume that N belongs to $\mathsf{D}^{\mathfrak{m}}(R)$, see 0.3. As N is in $\mathsf{D}_{\sqsubset}(R)$ also $\mathbf{R}\mathrm{Hom}_R(\boldsymbol{k}, N)$ belongs to $\mathsf{D}_{\sqsubset}(R)$, see [5, 7.6.7], so (0.2.1) and Proposition 1.3 yield

$$\begin{aligned}\sup \mathbf{R}\Gamma_{\mathfrak{m}}(N \otimes^{\mathbf{L}}_R M) \;&=\; -\operatorname{depth}_R(N \otimes^{\mathbf{L}}_R M)\\ &=\; -\operatorname{depth}_R N - \operatorname{depth}_R M + \operatorname{depth} R\,.\end{aligned}$$

Thus, it follows from Corollary 1.4 and (0.5.3) that it is sufficient to establish the inequalities

$$(*) \qquad \begin{aligned}\dim_R N - \operatorname{width}_R M \;&\geqslant\; -\inf \mathbf{R}\Gamma_{\mathfrak{m}}(N \otimes^{\mathbf{L}}_R M)\\ &\geqslant\; \dim_R N + \operatorname{depth}_R M - \operatorname{depth} R\,.\end{aligned}$$

By (0.1.1) and (0.1.4) there are isomorphisms,

$$\mathbf{R}\Gamma_{\mathfrak{m}}(N \otimes^{\mathbf{L}}_R M) \;\simeq\; \mathbf{R}\Gamma_{\mathfrak{m}} N \otimes^{\mathbf{L}}_R M \;\simeq\; \mathbf{R}\Gamma_{\mathfrak{m}} N \otimes^{\mathbf{L}}_R \mathbf{L}\Lambda^{\mathfrak{m}} M\,.$$

The left-hand inequality in $(*)$ now follows from (0.5.2), (0.2.4), and [5, 7.6.8]:

$$\dim_R N - \operatorname{width}_R M \;=\; -\inf \mathbf{R}\Gamma_{\mathfrak{m}} N - \inf \mathbf{L}\Lambda^{\mathfrak{m}} M \;\geqslant\; -\inf \mathbf{R}\Gamma_{\mathfrak{m}}(N \otimes^{\mathbf{L}}_R M)\,.$$

To prove the right-hand inequality, first note that one has

$$\begin{aligned}-\inf \mathbf{R}\Gamma_{\mathfrak{m}}(N \otimes^{\mathbf{L}}_R M) \;&\geqslant\; -\operatorname{width}_R \mathbf{R}\Gamma_{\mathfrak{m}}(N \otimes^{\mathbf{L}}_R M)\\ &=\; -\operatorname{width}_R(N \otimes^{\mathbf{L}}_R M)\\ &=\; -\operatorname{width}_R N - \operatorname{width}_R M\\ &\geqslant\; -\operatorname{width}_R N - \inf \mathbf{R}\Gamma_{\mathfrak{m}} M - \dim R\,,\end{aligned}$$

where the first inequality holds by (0.2.4) and the second by (0.2.6); the equalities hold by (0.2.5) and (0.4.3). It follows that the right-hand inequality in $(*)$ holds if one has $\inf \mathbf{R}\Gamma_{\mathfrak{m}} M = -\infty$, so we can assume that $\mathbf{R}\Gamma_{\mathfrak{m}} M$ belongs to $\mathsf{D}_{\square}(R)$ as $\sup \mathbf{R}\Gamma_{\mathfrak{m}} M = -\operatorname{depth}_R M < \infty$ holds by (0.2.1) and Proposition 1.3. By Proposition 1.1(d) the complex $\mathbf{R}\Gamma_{\mathfrak{m}} M$ has finite flat dimension, so one can apply

[5, 18.5.1] to get the inequality in the computation below. The equalities hold by (0.6.2), (0.1.1), and (0.2.2).

$$\begin{aligned}
-\inf \mathbf{R}\Gamma_{\mathfrak{m}}(N \otimes^{\mathbf{L}}_R M) &= \dim_R \mathbf{R}\Gamma_{\mathfrak{m}}(N \otimes^{\mathbf{L}}_R M)\\
&= \dim_R(N \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{m}} M)\\
&\geqslant \dim_R N + \operatorname{depth}_R \mathbf{R}\Gamma_{\mathfrak{m}} M - \operatorname{depth} R\\
&= \dim_R N + \operatorname{depth}_R M - \operatorname{depth} R\,. \qquad \square
\end{aligned}$$

A consequence of Peskine and Szpiro's Intersection Theorem is that finite length modules of finite projective dimension only exist over Cohen–Macaulay rings; see [17, Proposition 6.2.4]. The next corollary soups up this classic result, cf. (0.6.1). The statement is parallel to Corollary 2.2, but it is worth recalling that a module is derived $\mathfrak{m}$-torsion if and only if it is $\mathfrak{m}$-torsion, see [5, 13.4.9].

3.2 Corollary. *If there exists a nonzero derived $\mathfrak{m}$-torsion R-module of finite flat dimension, then R is Cohen–Macaulay.*

Proof. If $M \neq 0$ a derived $\mathfrak{m}$-torsion R-module of finite flat dimension, then M is in $\mathsf{D}^{\mathfrak{m}}(R)$, see (0.3.1), and $\boldsymbol{k} \otimes^{\mathbf{L}}_R M$ is in $\mathsf{D}_{\sqsubset}(R)$. As $\operatorname{amp}(R \otimes^{\mathbf{L}}_R M) = \operatorname{amp} M = 0$ holds, Theorem 3.1 yields $\operatorname{cmd} R = 0$. $\square$

The next result should be compared with Corollary 2.9.

3.3 Corollary. *Let M and N be R-complexes with $N \in \mathsf{D}^{\mathsf{f}}_{\sqsubset}(R)$. Assume that M is of finite flat dimension and not acyclic. If M is derived $\mathfrak{m}$-torsion or belongs to $\mathsf{D}^{\mathsf{f}}_{\square}(R)$, then there are inequalities,*

$$\operatorname{fd}_R M - \operatorname{width}_R M + \operatorname{cmd}_R N \;\geqslant\; \operatorname{cmd}_R(N \otimes^{\mathbf{L}}_R M) \;\geqslant\; \operatorname{cmd}_R N\,.$$

Proof. The complex M belongs to $\mathsf{D}^{\mathfrak{m}}(R)$, see 0.3. If M is derived $\mathfrak{m}$-torsion, then Proposition 1.1(d) and (0.2.4) yield

$$(*) \qquad \sup(\boldsymbol{k} \otimes^{\mathbf{L}}_R M) \;=\; \operatorname{fd}_R M \qquad \text{and} \qquad \inf(\boldsymbol{k} \otimes^{\mathbf{L}}_R M) \;=\; \operatorname{width}_R M\,;$$

in particular, $\boldsymbol{k} \otimes^{\mathbf{L}}_R M$ belongs to $\mathsf{D}_{\sqsubset}(R)$. Further, the complex $N \otimes^{\mathbf{L}}_R M$ is derived $\mathfrak{m}$-torsion, see (0.1.2), so the asserted inequalities follow from Theorem 3.1, (0.6.3), and the equalities $(*)$.

If M belongs to $\mathsf{D}^{\mathsf{f}}_{\square}(R)$, then the equalities $(*)$ hold by [5, 16.4.1] and (0.2.4). The complex $N \otimes^{\mathbf{L}}_R M$ belongs to $\mathsf{D}^{\mathsf{f}}_{\sqsubset}(R)$, see [5, 15.4.3], so the asserted inequalities follow from Theorem 3.1, (0.5.4), and the equalities $(*)$. $\square$

Notice that the conclusion in the next result is identical to the conclusion in Corollary 2.11 but the boundedness conditions on the complexes are different.

3.4 Corollary. *Let M and N be R-complexes with $M \in \mathsf{D}^{\mathfrak{m}}(R)$ and $N \in \mathsf{D}^{\ell}_{\sqsubset}(R)$. If the complex $\boldsymbol{k} \otimes^{\mathbf{L}}_R M$ belongs to $\mathsf{D}_{\sqsubset}(R)$, then there are inequalities,*

$$\operatorname{amp}(\boldsymbol{k} \otimes^{\mathbf{L}}_R M) + \operatorname{amp} N \;\geqslant\; \operatorname{amp}(N \otimes^{\mathbf{L}}_R M) \;\geqslant\; \operatorname{amp} N\,.$$

Proof. Recall from (0.6.1) that the complex N is derived $\mathfrak{m}$-torsion, so $N \otimes^{\mathbf{L}}_R M$ is per (0.1.1) derived $\mathfrak{m}$-torsion. Now apply Theorem 3.1 and (0.6.3). $\square$

3.5 Example. Let R be Gorenstein of positive Krull dimension and $\mathfrak{p}$ be a minimal prime ideal in R. Consider the injective R-module $M = \mathrm{E}_R(R/\mathfrak{p}) \oplus \mathrm{E}_R(\boldsymbol{k})$. As R is Gorenstein, one has $\sup(\boldsymbol{k} \otimes^{\mathbf{L}}_R M) \leqslant \mathrm{fd}_R M < \infty$, so Theorem 3.1 and [5, 18.5.2] both apply to the complex $R \otimes^{\mathbf{L}}_R M \simeq M$.

One has $\mathbf{R}\Gamma_{\mathfrak{m}} M = \Gamma_{\mathfrak{m}} M = \mathrm{E}_R(\boldsymbol{k})$, so $\operatorname{amp} \mathbf{R}\Gamma_{\mathfrak{m}} M = 0 = \operatorname{depth}_R M$ holds. Since the module $\mathrm{E}_R(R/\mathfrak{p})_{\mathfrak{q}} \cong \mathrm{E}_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}/\mathfrak{p}_{\mathfrak{q}})$ is nonzero for every prime ideal $\mathfrak{q}$ in R that contains $\mathfrak{p}$, one has $\dim_R M = \dim R$. Therefore, [5, 18.5.2] yields

$$\dim R \;=\; \mathrm{cmd}_R(R \otimes^{\mathbf{L}}_R M) \;\geqslant\; \mathrm{cmd}\, R \;=\; 0\,,$$

and by the assumption on R the inequality is strict. Yet, the right-hand inequality in Theorem 3.1 reads $0 \geqslant 0$ and, in fact, so does the left-hand inequality as one has

$$\dim R \;\geqslant\; \mathrm{fd}_R M \;\geqslant\; \sup(\boldsymbol{k} \otimes^{\mathbf{L}}_R M) \;\geqslant\; \mathrm{width}_R M \;=\; \operatorname{depth} R \;=\; \dim R\,,$$

by standard (in)equalities, (0.2.4), and [5, 17.4.11 and 16.2.29].

The next result improves [5, 13.3.20(c)] for the ideal $\mathfrak{a} = \mathfrak{m}$.

3.6 Proposition. *Let M and K be R-complexes with $K \in \mathsf{D}^{\mathrm{f}}(R)$. If the complex $\mathbf{R}\mathrm{Hom}_R(\boldsymbol{k}, M)$ belongs to $\mathsf{D}_{\square}(R)$, then there is an isomorphism,*

$$\mathbf{R}\Gamma_{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(K, M) \;\simeq\; \mathbf{R}\mathrm{Hom}_R(K, \mathbf{R}\Gamma_{\mathfrak{m}} M)\,.$$

Proof. The first two isomorphisms in the display below follow from [5, 13.4.1(d)] and (0.1.2). By Proposition 1.2(e) the R-complex $\mathbf{L}\Lambda^{\mathfrak{m}} M$ has bounded homology and finite injective dimension. As K belongs to $\mathsf{D}^{\mathrm{f}}(R)$, the third isomorphism holds by [5, 13.3.20(c) and 13.4.1(d)].

$$\begin{aligned}
\mathbf{R}\Gamma_{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(K, M) \;&\simeq\; \mathbf{R}\Gamma_{\mathfrak{m}} \mathbf{L}\Lambda^{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(K, M)\\
&\simeq\; \mathbf{R}\Gamma_{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(K, \mathbf{L}\Lambda^{\mathfrak{m}} M)\\
&\simeq\; \mathbf{R}\mathrm{Hom}_R(K, \mathbf{R}\Gamma_{\mathfrak{m}} M)\,. \qquad \square
\end{aligned}$$

We derive the next result from Theorem 3.1 by way of Grothendieck Duality.

3.7 Theorem. *Let M and K be complexes with $M \in \mathsf{D}^{\mathfrak{m}}(R)$ and $K \in \mathsf{D}^{\mathrm{f}}(R)$. If the complex $\mathbf{R}\mathrm{Hom}_R(\boldsymbol{k}, M)$ belongs to $\mathsf{D}_{\square}(R)$, then there are inequalities,*

$$\operatorname{amp} \mathbf{R}\mathrm{Hom}_R(\boldsymbol{k}, M) + \operatorname{amp} K \;\geqslant\; \operatorname{amp} \mathbf{R}\Gamma_{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(K, M) \;\geqslant\; \operatorname{amp} K\,.$$

Proof. Both inequalities are trivial if K is acyclic, so assume that is not the case. That is, assume that K belongs to $\mathsf{D}^{\mathfrak{m}}(R)$, see 0.3. The equalities below hold by (0.2.1) and (0.4.2).

$$\sup \mathbf{R}\Gamma_{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(K, M) \;=\; -\operatorname{depth}_R \mathbf{R}\mathrm{Hom}_R(K, M) \;=\; -\,\mathrm{width}_R K - \operatorname{depth}_R M\,.$$

Per (0.3.2) one has $\mathrm{width}_R K = \inf K$, so the equalities show that both of the asserted inequalities are trivial if $\inf K = -\infty$ holds.

One can now assume that K belongs to $\mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$. Set $\widehat{K} = \widehat{R} \otimes_R K$; it is by [5, 18.3.2(c)] a complex in $\mathsf{D}^{\mathrm{f}}_{\sqsupset}(\widehat{R})$. The first isomorphism in the display below holds by Proposition 1.1(a), and the second follows from Proposition 3.6. As $\mathbf{R}\Gamma_{\mathfrak{m}} M$ by another application of Proposition 1.1(a) is a complex in $\mathsf{D}^{\widehat{\mathfrak{m}}}(\widehat{R})$, the third isomorphism is standard, see [5, 12.3.30]. The ring $\widehat{R}$ has a dualizing complex, D, and the fourth isomorphism is Grothendieck Duality, see [5, 18.2.3]. The last isomorphism holds by [5, 12.3.26(c)], as the $\widehat{R}$-complex $\mathbf{R}\Gamma_{\mathfrak{m}} M$ has bounded homology and finite injective dimension by Proposition 1.1(f,g).

$$\begin{aligned}
\mathbf{R}\Gamma_{\mathfrak{m}}\,\mathbf{R}\mathrm{Hom}_R(K,M) &\simeq \mathbf{R}\Gamma_{\widehat{\mathfrak{m}}}\mathbf{R}\Gamma_{\mathfrak{m}}\,\mathbf{R}\mathrm{Hom}_R(K,M)\\
&\simeq \mathbf{R}\Gamma_{\widehat{\mathfrak{m}}}\,\mathbf{R}\mathrm{Hom}_R(K,\mathbf{R}\Gamma_{\mathfrak{m}} M)\\
&\simeq \mathbf{R}\Gamma_{\widehat{\mathfrak{m}}}\,\mathbf{R}\mathrm{Hom}_{\widehat{R}}(\widehat{K},\mathbf{R}\Gamma_{\mathfrak{m}} M)\\
&\simeq \mathbf{R}\Gamma_{\widehat{\mathfrak{m}}}\,\mathbf{R}\mathrm{Hom}_{\widehat{R}}(\mathbf{R}\mathrm{Hom}_{\widehat{R}}(\mathbf{R}\mathrm{Hom}_R(\widehat{K},D),D),\mathbf{R}\Gamma_{\mathfrak{m}} M)\\
&\simeq \mathbf{R}\Gamma_{\widehat{\mathfrak{m}}}(\mathbf{R}\mathrm{Hom}_{\widehat{R}}(\widehat{K},D)\otimes^{\mathbf{L}}_{\widehat{R}}\mathbf{R}\mathrm{Hom}_{\widehat{R}}(D,\mathbf{R}\Gamma_{\mathfrak{m}} M))\,.
\end{aligned}$$

The complex $\mathbf{R}\mathrm{Hom}_{\widehat{R}}(D,\mathbf{R}\Gamma_{\mathfrak{m}} M)$ belongs to $\mathsf{D}^{\widehat{\mathfrak{m}}}(\widehat{R})$, see (0.4.1). The first isomorphism below holds by [5, 12.3.26(c)]. The second isomorphism holds as one can assume that D is normalized, see [5, 18.2.23 and 18.2.24], and the third comes from Proposition 1.1(c).

$$(*)\qquad\begin{aligned}
\boldsymbol{k}\otimes^{\mathbf{L}}_{\widehat{R}}\mathbf{R}\mathrm{Hom}_{\widehat{R}}(D,\mathbf{R}\Gamma_{\mathfrak{m}} M) &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}}(\mathbf{R}\mathrm{Hom}_{\widehat{R}}(\boldsymbol{k},D),\mathbf{R}\Gamma_{\mathfrak{m}} M)\\
&\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}}(\boldsymbol{k},\mathbf{R}\Gamma_{\mathfrak{m}} M)\\
&\simeq \mathbf{R}\mathrm{Hom}_R(\boldsymbol{k},M)\,.
\end{aligned}$$

Thus, $\boldsymbol{k}\otimes^{\mathbf{L}}_{\widehat{R}}\mathbf{R}\mathrm{Hom}_{\widehat{R}}(D,\mathbf{R}\Gamma_{\mathfrak{m}} M)$ belongs to $\mathsf{D}_{\Box}(\widehat{R})$ by the assumption on M. By [5, 18.2.3] the complex $\mathbf{R}\mathrm{Hom}_{\widehat{R}}(\widehat{K},D)$ belongs to $\mathsf{D}^{\mathsf{f}}_{\sqsubset}(\widehat{R})$, so Theorem 3.1 yields in view of $(*)$ the inequalities,

$$\begin{aligned}
&\operatorname{amp}\mathbf{R}\mathrm{Hom}_R(\boldsymbol{k},M)+\operatorname{cmd}_{\widehat{R}}\mathbf{R}\mathrm{Hom}_{\widehat{R}}(\widehat{K},D)\\
&\qquad\geqslant \operatorname{amp}\mathbf{R}\Gamma_{\mathfrak{m}}\,\mathbf{R}\mathrm{Hom}_R(K,M)\geqslant \operatorname{cmd}_{\widehat{R}}\mathbf{R}\mathrm{Hom}_{\widehat{R}}(\widehat{K},D)\,.
\end{aligned}$$

Finally, from [5, 18.2.31] and faithful flatness of $\widehat{R}$ as an R-module one gets

$$\operatorname{cmd}_{\widehat{R}}\mathbf{R}\mathrm{Hom}_{\widehat{R}}(\widehat{K},D) = \operatorname{amp}\widehat{K} = \operatorname{amp}K\,. \qquad \Box$$

A consequence of Grothendieck's Local Duality Theorem is the formula

$$\operatorname{cmd}_R\mathbf{R}\mathrm{Hom}_R(K,D) = \operatorname{amp}K\,,$$

used in the last line of the proof above. The inequalities in the next corollary can be seen as a generalization, as they collapse to the formula above for $M = D$ a dualizing R-complex. The corollary should be compared with Corollary 2.4.

3.8 Corollary. *Let M and K be complexes with $M\in\mathsf{D}_{\Box}(R)$ and $K\in\mathsf{D}^{\mathsf{f}}(R)$. Assume that M is of finite injective dimension and not acyclic. If M is derived $\mathfrak{m}$-torsion or belongs to $\mathsf{D}^{\mathsf{f}}_{\Box}(R)$, then there are inequalities,*

$$\operatorname{id}_R M-\operatorname{depth}_R M+\operatorname{amp}K\geqslant \operatorname{cmd}_R\mathbf{R}\mathrm{Hom}_R(K,M)\geqslant \operatorname{amp}K\,.$$

Proof. The complex M belongs to $\mathsf{D}^{\mathfrak{m}}(R)$, see 0.3. If M is derived $\mathfrak{m}$-torsion, then (0.2.1) and Proposition 1.1(f) yield

$$(*)\quad \sup\mathbf{R}\mathrm{Hom}_R(\boldsymbol{k},M) = -\operatorname{depth}_R M \quad\text{and}\quad \inf\mathbf{R}\mathrm{Hom}_R(\boldsymbol{k},M) = -\operatorname{id}_R M\,;$$

thus the complex $\mathbf{R}\mathrm{Hom}_R(\boldsymbol{k},M)$ belongs to $\mathsf{D}_{\Box}(R)$ as $-\operatorname{depth}_R M\leqslant\sup M$ holds by (0.2.1). It follows from Proposition 3.6 that $\mathbf{R}\mathrm{Hom}_R(K,M)$ is derived $\mathfrak{m}$-torsion, so the asserted inequalities follow from Theorem 3.7, (0.6.3), and the equalities $(*)$.

If M belongs to $\mathsf{D}^{\mathrm{f}}_{\square}(R)$, then the equalities $(*)$ hold by (0.2.1) and [5, 16.4.8]. The complex $\mathbf{R}\mathrm{Hom}_R(K,M)$ belongs to $\mathsf{D}^{\mathrm{f}}(R)$, see [5, 15.4.9]. Now the asserted inequalities follow from Theorem 3.7, (0.5.4), and the equalities $(*)$. □

Note that the conclusion in the next corollary matches the conclusion in Corollary 2.6 but the boundedness conditions on the complexes are different.

3.9 Corollary. *Let M and K be complexes with $M \in \mathsf{D}^{\mathfrak{m}}(R)$ and $K \in \mathsf{D}^{\ell}(R)$. If the complex $\mathbf{R}\mathrm{Hom}_R(\boldsymbol{k},M)$ belongs to $\mathsf{D}_{\square}(R)$, then there are inequalities,*

$$\operatorname{amp}\mathbf{R}\mathrm{Hom}_R(\boldsymbol{k},M) + \operatorname{amp}K \;\geqslant\; \operatorname{amp}\mathbf{R}\mathrm{Hom}_R(K,M) \;\geqslant\; \operatorname{amp}K\,.$$

Proof. Recall from (0.6.1) that the complex K is derived $\mathfrak{m}$-torsion, so by (0.1.3) and Proposition 3.6 there are isomorphisms

$$\mathbf{R}\mathrm{Hom}_R(K,M) \;\simeq\; \mathbf{R}\mathrm{Hom}_R(K,\mathbf{R}\Gamma_{\mathfrak{m}}M) \;\simeq\; \mathbf{R}\Gamma_{\mathfrak{m}}\,\mathbf{R}\mathrm{Hom}_R(K,M)\,.$$

Now the asserted inequalities follow from Theorem 3.7. □

3.10 Example. Let D be a dualizing R-complex and K a complex in $\mathsf{D}^{\ell}(R)$. Corollary 3.9 yields

$$\operatorname{amp}\mathbf{R}\mathrm{Hom}_R(K,D) \;=\; \operatorname{amp}K\,.$$

This equality is well-known, for example from [5, 18.2.40 and 16.1.25(6)].

4. Additional nonvanishing of local (co)homology

In a derived Hom complex one can place conditions of vanishing of (co)homology with coefficients in $\boldsymbol{k}$ on either one of the two arguments. In Theorem 2.3 the assumption is that the second complex is of finite $\boldsymbol{k}$-injective dimension. In the next result, the assumption is instead that the first complex is of finite $\boldsymbol{k}$-flat dimension. We have placed the result in this short closing section because we obtain it from Theorem 3.1 by way of Matlis Duality.

4.1 Theorem. *Let M and K be complexes with $M \in \mathsf{D}^{\mathfrak{m}}(R)$ and $K \in \mathsf{D}^{\mathrm{art}}_{\sqsupset}(R)$. If the complex $\boldsymbol{k}\otimes^{\mathbf{L}}_R M$ belongs to $\mathsf{D}_{\sqsubset}(R)$, then there are inequalities,*

$$\operatorname{amp}(\boldsymbol{k}\otimes^{\mathbf{L}}_R M) + \operatorname{amp}\mathbf{L}\Lambda^{\mathfrak{m}}K \;\geqslant\; \operatorname{amp}\mathbf{L}\Lambda^{\mathfrak{m}}\,\mathbf{R}\mathrm{Hom}_R(M,K) \;\geqslant\; \operatorname{amp}\mathbf{L}\Lambda^{\mathfrak{m}}K\,.$$

Proof. Assume first that R is complete. By Matlis duality [5, 18.1.9] the complex $N = \mathrm{Hom}_R(K,\mathrm{E}_R(\boldsymbol{k}))$ belongs to $\mathsf{D}^{\mathrm{f}}_{\sqsubset}(R)$ and one has

$$(*) \qquad K \;\simeq\; \mathbf{R}\mathrm{Hom}_R(N,\mathrm{E}_R(\boldsymbol{k}))\,.$$

Thus, by $(*)$, Hom–tensor adjunction, (0.1.2), and [5, 16.1.25(6)] there are equalities

$$\begin{aligned}
\operatorname{amp}\mathbf{L}\Lambda^{\mathfrak{m}}\,\mathbf{R}\mathrm{Hom}_R(M,K) &= \operatorname{amp}\mathbf{L}\Lambda^{\mathfrak{m}}\,\mathbf{R}\mathrm{Hom}_R(M,\mathbf{R}\mathrm{Hom}_R(N,\mathrm{E}_R(\boldsymbol{k})))\\
&= \operatorname{amp}\mathbf{L}\Lambda^{\mathfrak{m}}\,\mathbf{R}\mathrm{Hom}_R(N\otimes^{\mathbf{L}}_R M,\mathrm{E}_R(\boldsymbol{k}))\\
&= \operatorname{amp}\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{m}}(N\otimes^{\mathbf{L}}_R M),\mathrm{E}_R(\boldsymbol{k}))\\
&= \operatorname{amp}\mathbf{R}\Gamma_{\mathfrak{m}}(N\otimes^{\mathbf{L}}_R M)\,.
\end{aligned}$$

The asserted inequalities now follow from Theorem 3.1 as one has

$$\operatorname{cmd}_R N \;=\; \operatorname{amp}\mathbf{R}\Gamma_{\mathfrak{m}}N \;=\; \operatorname{amp}\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{m}}N,\mathrm{E}_R(\boldsymbol{k})) \;=\; \operatorname{amp}\mathbf{L}\Lambda^{\mathfrak{m}}K$$

by (0.5.4), (0.1.2), and $(*)$.

Now consider the general case. Per Proposition 1.1(b,c) and the assumptions on M, the complex $\mathbf{R}\Gamma_{\mathfrak{m}}M$ belongs to $\mathsf{D}^{\widehat{\mathfrak{m}}}(\widehat{R})$ and $\boldsymbol{k} \otimes^{\mathbf{L}}_{\widehat{R}} \mathbf{R}\Gamma_{\mathfrak{m}}M$ is in $\mathsf{D}_{\sqsubset}(\widehat{R})$. Further, as $K \in \mathsf{D}^{\mathrm{art}}(R)$ one has $\mathbf{R}\Gamma_{\mathfrak{m}}K \in \mathsf{D}^{\mathrm{art}}(\widehat{R})$, i.e. each $\widehat{R}$-module $\mathrm{H}_n(\mathbf{R}\Gamma_{\mathfrak{m}}K)$ is artinian. To see this, it suffices to argue that $\mathrm{H}_n(\mathbf{R}\Gamma_{\mathfrak{m}}K)$ is artinian as an R-module; however, in $\mathsf{D}(R)$ one has $\mathbf{R}\Gamma_{\mathfrak{m}}K \simeq K$ as $\mathsf{D}^{\mathrm{art}}(R) \subseteq \mathsf{D}^{\mathfrak{m}\text{-tor}}(R)$ by [5, 16.1.33], and hence $\mathrm{H}_n(\mathbf{R}\Gamma_{\mathfrak{m}}K) \cong \mathrm{H}_n(K)$, which is an artinian R-module by assumption.

The first isomorphism in the computation below holds by Proposition 1.2(a). The remaining isomorphisms hold by (0.1.2), (0.1.3), and [5, 13.4.16(b)].

$$\begin{aligned}
\mathbf{L}\Lambda^{\mathfrak{m}}\,\mathbf{R}\mathrm{Hom}_R(M,K) &\simeq \mathbf{L}\Lambda^{\widehat{\mathfrak{m}}}\mathbf{L}\Lambda^{\mathfrak{m}}\,\mathbf{R}\mathrm{Hom}_R(M,K)\\
&\simeq \mathbf{L}\Lambda^{\widehat{\mathfrak{m}}}\,\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{m}}M,K)\\
&\simeq \mathbf{L}\Lambda^{\widehat{\mathfrak{m}}}\,\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{m}}M,\mathbf{R}\Gamma_{\mathfrak{m}}K)\\
&\simeq \mathbf{L}\Lambda^{\widehat{\mathfrak{m}}}\,\mathbf{R}\mathrm{Hom}_{\widehat{R}}(\mathbf{R}\Gamma_{\mathfrak{m}}M,\mathbf{R}\Gamma_{\mathfrak{m}}K)
\end{aligned}$$

As we have established the desired inequalities over complete local rings, we get

$$\begin{aligned}
&\mathrm{amp}(\boldsymbol{k} \otimes^{\mathbf{L}}_{\widehat{R}} \mathbf{R}\Gamma_{\mathfrak{m}}M) + \mathrm{amp}\,\mathbf{L}\Lambda^{\widehat{\mathfrak{m}}}\mathbf{R}\Gamma_{\mathfrak{m}}K\\
&\qquad\qquad \geqslant\ \mathrm{amp}\,\mathbf{L}\Lambda^{\mathfrak{m}}\,\mathbf{R}\mathrm{Hom}_R(M,K)\ \geqslant\ \mathrm{amp}\,\mathbf{L}\Lambda^{\widehat{\mathfrak{m}}}\mathbf{R}\Gamma_{\mathfrak{m}}K\,.
\end{aligned}$$

By Proposition 1.1(c) one has $\mathrm{amp}(\boldsymbol{k} \otimes^{\mathbf{L}}_{\widehat{R}} \mathbf{R}\Gamma_{\mathfrak{m}}M) = \mathrm{amp}(\boldsymbol{k} \otimes^{\mathbf{L}}_{R} M)$ and, finally,

$$\mathbf{L}\Lambda^{\widehat{\mathfrak{m}}}\mathbf{R}\Gamma_{\mathfrak{m}}K \ \simeq\ \mathbf{L}\Lambda^{\mathfrak{m}}\mathbf{R}\Gamma_{\mathfrak{m}}K \ \simeq\ \mathbf{L}\Lambda^{\mathfrak{m}}K$$

by independence of base for local homology, see [5, 13.1.21(a), and 13.4.1(c)]. $\square$

4.2 Corollary. *Let M and K be R-complexes with $K \in \mathsf{D}^{\mathrm{art}}_{\sqsupset}(R)$. If M is derived $\mathfrak{m}$-torsion of finite flat dimension and not acyclic, then there are inequalities,*

$$\mathrm{fd}_R\,M - \mathrm{width}_R\,M + \mathrm{amp}\,\mathbf{L}\Lambda^{\mathfrak{m}}K\ \geqslant\ \mathrm{amp}\,\mathbf{R}\mathrm{Hom}_R(M,K)\ \geqslant\ \mathrm{amp}\,\mathbf{L}\Lambda^{\mathfrak{m}}K\,.$$

Proof. The complex M belongs to $\mathsf{D}^{\mathfrak{m}}(R)$, see 0.3, and Proposition 1.1(d) and (0.2.4) yield

$$\sup\,(\boldsymbol{k} \otimes^{\mathbf{L}}_{R} M)\ =\ \mathrm{fd}_R\,M \qquad \text{and} \qquad \inf\,(\boldsymbol{k} \otimes^{\mathbf{L}}_{R} M)\ =\ \mathrm{width}_R\,M\,;$$

in particular, the complex $\boldsymbol{k} \otimes^{\mathbf{L}}_{R} M$ belongs to $\mathsf{D}_{\sqsubset}(R)$. Further, per (0.1.2) the complex $\mathbf{R}\mathrm{Hom}_R(M,K)$ is derived $\mathfrak{m}$-complete, so the asserted inequalities follow from Theorem 4.1 and the equalities displayed above. $\square$

4.3 Corollary. *Let M and K be complexes with $M \in \mathsf{D}^{\mathfrak{m}}(R)$ and $K \in \mathsf{D}^{\ell}_{\sqsupset}(R)$. If the complex $\boldsymbol{k} \otimes^{\mathbf{L}}_{R} M$ belongs to $\mathsf{D}_{\sqsubset}(R)$, then there are inequalities,*

$$\mathrm{amp}(\boldsymbol{k} \otimes^{\mathbf{L}}_{R} M) + \mathrm{amp}\,K\ \geqslant\ \mathrm{amp}\,\mathbf{R}\mathrm{Hom}_R(M,K)\ \geqslant\ \mathrm{amp}\,K\,.$$

Proof. The complex K is derived $\mathfrak{m}$-complete by (0.6.1) and belongs to $\mathsf{D}^{\mathrm{art}}_{\sqsupset}(R)$. In particular, the complex $\mathbf{R}\mathrm{Hom}_R(M,K)$ is derived $\mathfrak{m}$-complete by (0.1.2), so the asserted inequalities follow from Theorem 4.1. $\square$

The next result is dual to Theorem 3.7; it applies to show that the inequalities in Corollary 4.3 also hold under the slightly different assumptions $M \in \mathsf{D}^{\mathfrak{m}}(R)$ with $\boldsymbol{k} \otimes^{\mathbf{L}}_{R} M \in \mathsf{D}_{\square}(R)$ and $K \in \mathsf{D}^{\ell}(R)$. This is the phenomenon observed in Corollaries 2.6 and 3.9 as well as in Corollaries 2.11 and 3.4.

4.4 Theorem. *Let M and K be complexes with $M \in \mathsf{D}^{\mathfrak{m}}(R)$ and $K \in \mathsf{D}^{\mathrm{art}}(R)$. If the complex $\boldsymbol{k} \otimes^{\mathbf{L}}_R M$ belongs to $\mathsf{D}_{\Box}(R)$, then there are inequalities,*

$$\operatorname{amp}(\boldsymbol{k} \otimes^{\mathbf{L}}_R M) + \operatorname{amp} K \;\geqslant\; \operatorname{amp} \mathbf{R}\Gamma_{\mathfrak{m}}\, \mathbf{R}\mathrm{Hom}_R(M,K) \;\geqslant\; \operatorname{amp} K\,.$$

Proof. Using the same technique as in the proof of Theorem 4.1, this result is derived from Theorem 3.7 by way of Matlis Duality. □

4.5 Corollary. *Let M and K be complexes with $M \in \mathsf{D}^{\mathrm{f}}_{\Box}(R)$ and $K \in \mathsf{D}^{\mathrm{art}}(R)$. If M has finite projective dimension and is not acyclic, then there are inequalities,*

$$\operatorname{pd}_R M - \inf M + \operatorname{amp} K \;\geqslant\; \operatorname{amp} \mathbf{R}\mathrm{Hom}_R(M,K) \;\geqslant\; \operatorname{amp} K\,.$$

Proof. As M belongs to $\mathsf{D}^{\mathrm{f}}_{\Box}(R)$, there are by [5, 16.4.1] and (0.3.2) equalities,

$$\sup\,(\boldsymbol{k} \otimes^{\mathbf{L}}_R M) \;=\; \operatorname{pd}_R M \qquad \text{and} \qquad \inf\,(\boldsymbol{k} \otimes^{\mathbf{L}}_R M) \;=\; \inf M\,.$$

One has $\operatorname{supp}_R \mathbf{R}\mathrm{Hom}_R(M,K) = \operatorname{supp}_R M \cap \operatorname{supp}_R K \subseteq \{\mathfrak{m}\}$ by [5, 17.1.11(a) and 16.1.33], so the complex $\mathbf{R}\mathrm{Hom}_R(M,K)$ is derived $\mathfrak{m}$-torsion, see (0.3.1). Now the asserted inequalities hold by Theorem 4.4 and the equalities displayed above. □

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