

COVERS, PRECOVERS, AND PURITY

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ABSTRACT. We show that if a class of modules is closed under pure quotients, then it is precovering if and only if it is covering, and this happens if and only if it is closed under direct sums. This is inspired by a dual result by Rada and Saorín.

We also show that if a class of modules contains the ground ring and is closed under extensions, direct sums, pure submodules, and pure quotients, then it forms the first half of a so-called perfect cotorsion pair as introduced by Salce; this is stronger than being covering.

Some applications are given to concrete classes of modules such as kernels of homological functors and torsion free modules in a torsion pair.

0. INTRODUCTION

Covers. The main topic of this paper is the notion of covering classes. To explain what that means, observe that the classical homological algebra of a ring can be phrased in terms of the class of projective modules. This class permits the construction of projective resolutions which again enable the computation of derived functors.

In relative homological algebra, the class of projective modules is replaced by another, suitably chosen class of modules. This replaces projective resolutions by resolutions in terms of modules in the chosen class, and derived functors by relative derived functors. A classical example of this is pure homological algebra where the projective modules are replaced by the so-called pure projective modules; these are the direct summands in direct sums of finitely presented modules. Pure homological algebra is a useful tool with a number of applications; see for instance [12].

Some conditions have to be imposed on a class if it is to be a suitable replacement for the projectives, and this leads to the notion of precovering classes. These can be used instead of the projective modules for doing homological algebra. A class \mathbb{F} of modules over a ring is precovering (or, as it is also called, contravariantly finite) if each module M has an \mathbb{F} -precover, that

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is, a homomorphism $F \rightarrow M$ with F in \mathbb{F} , which has the property that each homomorphism $F' \rightarrow M$ with F' in \mathbb{F} has a factorization $F' \rightarrow F \rightarrow M$. A precovering class enables the construction of well behaved resolutions: Pick a precover $F_0 \rightarrow M$, let K_0 be the kernel, pick a precover $F_1 \rightarrow K_0$, let K_1 be the kernel, and so on. This results in a complex $\cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ which is called a proper \mathbb{F} -resolution of M . It has the property that it becomes exact when the functor $\text{Hom}(F, -)$ is applied to it for each F in \mathbb{F} . This implies that it is unique up to homotopy, and hence well suited for homological tasks such as the computation of relative derived functors.

Covering classes arise as a sharpening of the notion of precovering classes. A class \mathbb{F} is covering if each module M has an \mathbb{F} -cover $F \rightarrow M$, that is, an \mathbb{F} -precover with the additional property that if $F \rightarrow F$ is an endomorphism for which $F \rightarrow F \rightarrow M$ equals $F \rightarrow M$, then $F \rightarrow F$ is in fact an automorphism.

The notions of precovering and covering classes can be dualized, and this results in the notions of preenveloping and enveloping classes.

Considerable energy has gone into proving that concrete classes are (pre)-covering or (pre)enveloping under suitable conditions on the ground ring. Examples include the classes of modules which are projective, flat, injective, Gorenstein projective, Gorenstein flat, Gorenstein injective, pure projective, pure injective, of projective dimension $\leq n$, torsion free, and cotorsion. A number of these results can be found in Enochs and Jenda's pivotal book [10], but see also [1], [2], [3], [4], [5], [7], [8], [9], [11], [13], [14], [15], and [17].

This paper shows that classes possessing some simple properties from pure homological algebra are covering, as we shall now describe.

Purity. Consider a short exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ where M' is a submodule of M and M'' is the corresponding quotient module. The sequence is called *pure exact* if it stays exact when tensored with any module, and then M' is called a pure submodule and M'' is called a pure quotient module of M .

Recall the clever result [15, cor. 3.5(c)] by Rada and Saorín, that if a class \mathbb{G} of modules over a ring is closed under pure submodules, then \mathbb{G} is preenveloping if and only if it is closed under direct products.

Our first main result (Theorem 2.5) is the dual of this. In fact, we prove more than the dual, namely, if a class \mathbb{F} is closed under pure quotient modules, then \mathbb{F} is precovering if and only if it is covering, and this happens if and only if \mathbb{F} is closed under direct sums.

The proof does not use the methods of Rada and Saorín which do not dualize. Instead, it is based on the deep result [3, thm. 5] which was used by Bican, El Bashir, and Enochs to prove the Flat Cover Conjecture. To be precise, we use a special case of [3, thm. 5] which is reproduced as Theorem 2.1 below. We also take inspiration from the proof of the Flat Cover Conjecture itself, see the proof of Proposition 2.3.

We go on to show that if \mathbb{F} contains the ground ring and is closed under extensions, direct sums, pure submodules, and pure quotients, then \mathbb{F} is the first half of a so-called perfect cotorsion pair (Theorem 3.4); this is a stronger property than being covering. Cotorsion pairs go back to Salce [16], and have gained popularity as a framework for relative homological algebra. The formal definition is stated in Definition 3.3; the book [10] is a useful reference, but see also [2], [3], [7], [11] and [13].

Applications. As an application of these results, we investigate classes of the form $\text{Ker Ext}^1(\mathbb{A}, -)$, $\text{Ker Tor}_1(\mathbb{B}, -)$, and $\text{Ker Ext}^1(-, \mathbb{C})$, where \mathbb{A} is a class of finitely presented modules, \mathbb{B} is any class of modules, and \mathbb{C} is a class of pure injective modules. The notation is straightforward; for instance, $\text{Ker Tor}_1(\mathbb{B}, -)$ is $\{M \mid \text{Tor}_1(B, M) = 0 \text{ for each } B \text{ in } \mathbb{B}\}$. Such classes had been studied previously; for instance, it was proved by Eklof and Trlifaj in [7, cor. 10 and thm. 12(i)] that $\text{Ker Tor}_1(\mathbb{B}, -)$ and $\text{Ker Ext}^1(-, \mathbb{C})$ are both covering, and when the ground ring is left-coherent, El Bashir's result [8, thm. 3.3] implies that $\text{Ker Ext}^1(\mathbb{A}, -)$ is also covering if one is willing to assume Vopenka's Principle on high cardinal numbers.

However, we prove some new results; for instance, if the ground ring is left-coherent, then $\text{Ker Ext}^1(\mathbb{A}, -) \cap \text{Ker Ext}^1(-, \mathbb{C})$ is covering (Theorem 4.3). We also give some concrete examples of classes of this form (Example 4.4), including the class of left-modules of flat dimension $\leq m$ and injective dimension $\leq n$ over a left-noetherian ring, and the class of fp-injective left-modules.

The fp-injective left-modules had already been proved to be preenveloping by Adams [1], and over a left-coherent ring, they had been proved to be covering by Pinzon [14], but we recover their results with new proofs. We also use our theory to give new proofs of the following: In a so-called hereditary torsion pair of finite type, the torsion free modules form a covering class (Theorem 4.8), and if, moreover, the ground ring is itself torsion free, then the torsion free modules form the first half of a cotorsion pair (Theorem 4.9). These results were first proved by different methods by Bican and Torrecillas in [5, cor. 4.1], and Angeleri-Hügel, Tonolo, and Trlifaj in [2, exa. 2.7].

Notation. Our notation is standard and should not require explanations, but we do wish to introduce the following blanket items.

Throughout the paper, R is a ring and the word *class* means *class of R -left-modules closed under isomorphisms*.

The cardinality of a module M is denoted by $|M|$.

1. CARDINALITY AND CO-CARDINALITY CLASSES

In this preliminary section, we introduce the notions of cardinality and co-cardinality classes. They are inspired by [10, Props. 5.2.2 and 6.2.1], and can be used to prove that classes of modules are precovering and preenveloping.

Definition 1.1. A class \mathbb{F} is called a *cardinality class* when, for each R -left-module M , there is a cardinal number \mathfrak{f} such that each homomorphism $F \rightarrow M$ with F in \mathbb{F} can be factored as $F \rightarrow F' \rightarrow M$ with F' in \mathbb{F} satisfying $|F'| \leq \mathfrak{f}$.

A class \mathbb{G} is called a *co-cardinality class* when, for each R -left-module N , there is a cardinal number \mathfrak{g} such that each homomorphism $N \rightarrow G$ with G in \mathbb{G} can be factored as $N \rightarrow G' \rightarrow G$ with G' in \mathbb{G} satisfying $|G'| \leq \mathfrak{g}$.

The next proposition is very close to being in [10], but we think it is worth to state it explicitly.

Proposition 1.2. *Let \mathbb{F} be a class which is closed under direct summands. Then \mathbb{F} is precovering if and only if it is a cardinality class which is closed under set indexed direct sums.*

Let \mathbb{G} be a class which is closed under direct summands. Then \mathbb{G} is preenveloping if and only if it is a co-cardinality class which is closed under set indexed direct products.

Proof. It is enough to prove the first statement since the second one is dual.

If \mathbb{F} is a cardinality class which is closed under set indexed direct sums, then it is precovering by [10, prop. 5.2.2]. Conversely, if \mathbb{F} is precovering, then it is a cardinality class, also by [10, prop. 5.2.2].

Finally, let \mathbb{F} be precovering and let $\{F_i\}$ be a set indexed system from \mathbb{F} . Pick an \mathbb{F} -precover $F \rightarrow \bigoplus F_i$. Each F_j has an inclusion into $\bigoplus F_i$, and since F_j is in \mathbb{F} , each inclusion lifts through $F \rightarrow \bigoplus F_i$. Taken together, this gives a splitting of $F \rightarrow \bigoplus F_i$, so $\bigoplus F_i$ is a direct summand of F . But \mathbb{F} is closed under direct summands, so $\bigoplus F_i$ is in \mathbb{F} . \square

Example 1.3. If \mathbb{B} is a set of R -left-modules, then it is easy to verify that $\text{Add } \mathbb{B}$, the class of modules which are isomorphic to a direct summand of a set indexed direct sum of modules from \mathbb{B} , is a co-cardinality class.

Hence, if $\text{Add } \mathbb{B}$ is closed under set indexed direct products, then it is preenveloping by Proposition 1.2. Note that in some cases of interest, $\text{Add } \mathbb{B}$ is indeed closed under set indexed direct products, for example if $\mathbb{B} = \{B\}$ for a finitely generated module B over an Artin algebra, cf. [13, lem. 1.2].

2. PURITY

This section shows that if a class \mathbb{F} is closed under pure quotient modules, then \mathbb{F} is precovering if and only if it is covering, and this happens if and only if \mathbb{F} is closed under direct sums (Theorem 2.5).

To begin, let us recall the following deep result which is a special case of [3, thm. 5].

Theorem 2.1. *For each cardinal number \mathfrak{m} there exists a cardinal number \mathfrak{f} , depending only on \mathfrak{m} and the ground ring R , such that if an inclusion*

of R -left-modules $K \subseteq F$ has $|F/K| \leq \mathfrak{m}$ and $|F| \geq \mathfrak{f}$ then there exists $0 \neq K'' \subseteq K \subseteq F$ such that K'' is a pure submodule of F .

The following lemma is a standard application of Zorn's lemma.

Lemma 2.2. *Given an inclusion of modules $K \subseteq F$, there exists a K' which is maximal with the properties that $K' \subseteq K \subseteq F$ and that K' is a pure submodule of F .*

The proof of the following proposition is inspired by the proof of [3, thm. 6].

Proposition 2.3. *If a class \mathbb{F} is closed under pure quotient modules, then it is a cardinality class (cf. Definition 1.1).*

Proof. Let M be an R -left-module of cardinality \mathfrak{m} and let \mathfrak{f} be the cardinal number from Theorem 2.1. Let $F \rightarrow M$ be a homomorphism with F in \mathbb{F} . We will construct a factorization as required by Definition 1.1.

If $|F| \leq \mathfrak{f}$, then consider the factorization of $F \rightarrow M$ as $F \rightarrow F \rightarrow M$, where the first arrow is the identity. This meets the requirements of Definition 1.1.

If $|F| > \mathfrak{f}$, then let $K = \text{Ker}(F \rightarrow M)$ and use Lemma 2.2 to find K' maximal with the properties that $K' \subseteq K \subseteq F$ and that K' is a pure submodule of F . Then $F \rightarrow M$ has the factorization $F \rightarrow F/K' \rightarrow M$, and we will show that this meets the requirements of Definition 1.1.

First, \mathbb{F} is closed under pure quotients, so F/K' is in \mathbb{F} .

Secondly, we must show $|F/K'| \leq \mathfrak{f}$. Assume to the contrary that $|F/K'| > \mathfrak{f}$. Consider the inclusion $K/K' \subseteq F/K'$. Since F/K is isomorphic to a submodule of M , we have $\left| \frac{F/K'}{K/K'} \right| = |F/K| \leq |M| = \mathfrak{m}$, and hence Theorem 2.1 says that there exists $0 \neq K''/K' \subseteq K/K' \subseteq F/K'$ such that K''/K' is a pure submodule of F/K' . We now have $K' \subsetneq K'' \subseteq K \subseteq F$, and we claim that K'' is in fact a pure submodule of F , contradicting the maximality of K' .

For this, consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K' & \longrightarrow & K'' & \longrightarrow & K''/K' & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K' & \longrightarrow & F & \longrightarrow & F/K' & \longrightarrow & 0 \end{array}$$

The lower row is pure exact and the inclusion $K''/K' \subseteq F/K'$ is pure, both by construction. Hence, if we tensor the diagram with an arbitrary R -right-module Q it follows from the snake lemma that $Q \otimes K'' \rightarrow Q \otimes F$ is injective. So K'' is a pure submodule of F as desired. \square

The following lemma is due to Angeleri-Hügel, Mantese, Tonolo, and Trlifaj; cf. [7, proof of lem. 9].

Lemma 2.4. *If a class \mathbb{F} is closed under set indexed direct sums and pure quotients, then it is also closed under colimits indexed by partially ordered sets.*

Proof. Let $\{F_i\}$ be a system in \mathbb{F} indexed by a partially ordered set. Then it is easy to see that the canonical surjection $\bigoplus F_i \rightarrow \varinjlim F_i$ is a pure epimorphism, and the lemma follows. \square

Theorem 2.5. *If a class \mathbb{F} is closed under pure quotient modules, then the following conditions are equivalent.*

- (i) \mathbb{F} is closed under set indexed direct sums.
- (ii) \mathbb{F} is precovering.
- (iii) \mathbb{F} is covering.

Proof. Observe that \mathbb{F} is closed under direct summands, since the projection onto a direct summand turns it into a pure quotient. Moreover, \mathbb{F} is a cardinality class by Proposition 2.3.

Proposition 1.2 gives that (i) and (ii) are equivalent. By definition, (iii) implies (ii).

Finally, suppose that (ii) holds. Then (i) also holds by the above, and so Lemma 2.4 says that \mathbb{F} is closed under colimits indexed by partially ordered sets, and in particular under well ordered colimits. But then \mathbb{F} is covering by [10, thm. 5.2.3], proving (iii). \square

Remark 2.6. The dual of Proposition 2.3 and the dual of Theorem 2.5 except the covering part were proved, up to differences of terminology, by Rada and Saorín in [15, prop. 2.8 and cor. 3.5(c)]. Namely, if a class \mathbb{G} is closed under pure submodules, then it is a co-cardinality class (cf. Definition 1.1), and if a class \mathbb{G} is closed under pure submodules, then \mathbb{G} is preenveloping if and only if it is closed under set indexed direct products.

The covering part of Theorem 2.5 cannot be dualized. For example, if a ring is right-coherent, then the class of flat left-modules is closed under set indexed products, and it is easy to see that this class is also closed under pure submodules. But flat envelopes of left-modules do not necessarily exist, see [9, thm. 6.1].

3. COTORSION PAIRS

This section shows that if \mathbb{F} is a class which contains the ground ring and is closed under extensions, direct sums, pure submodules, and pure quotients, then \mathbb{F} is the first half of a so-called perfect cotorsion pair (see Definition 3.3).

Recall the following important notion from Enochs and López-Ramos [11, def. 2.1].

Definition 3.1. A class \mathbb{F} is called a *Kaplansky class* if there is a cardinal number \mathfrak{f} such that, when F is in \mathbb{F} and f is an element of F , we have

$f \in F' \subseteq F$ for some submodule F' where F' and F/F' are in \mathbb{F} and where $|F'| \leq \mathfrak{f}$.

Proposition 3.2. *If a class \mathbb{F} is closed under pure submodules and pure quotient modules, then it is a Kaplansky class.*

Proof. Given F in \mathbb{F} and f in F , the submodule Rf has $|Rf| \leq |R|$.

By [10, lem. 5.3.12], there is a cardinal number \mathfrak{f} depending only on $|R|$ such that we can enlarge Rf to a pure submodule F' of F with $|F'| \leq \mathfrak{f}$. So $f \in F' \subseteq F$, and F' and F/F' are both in \mathbb{F} since \mathbb{F} is closed under pure submodules and pure quotients. \square

Let us recall the definition of a cotorsion pair. This goes back to Salce [16], and has gained popularity as a framework for relative homological algebra; among our references we could mention [2], [3], [7], [10], [11], and [13].

Definition 3.3. Let \mathbb{F} and \mathbb{G} be classes. Then $\mathbb{F}^\perp = \text{Ker Ext}^1(\mathbb{F}, -)$ is the class $\{N \in \text{Mod } R \mid \text{Ext}^1(F, N) = 0 \text{ for } F \in \mathbb{F}\}$ and ${}^\perp\mathbb{G} = \text{Ker Ext}^1(-, \mathbb{G})$ is the class $\{M \in \text{Mod } R \mid \text{Ext}^1(M, G) = 0 \text{ for } G \in \mathbb{G}\}$.

The pair (\mathbb{F}, \mathbb{G}) is called a *cotorsion pair* if $\mathbb{F}^\perp = \mathbb{G}$ and $\mathbb{F} = {}^\perp\mathbb{G}$.

The cotorsion pair is called *perfect* if \mathbb{F} is covering and \mathbb{G} is enveloping.

Theorem 3.4. *If a class \mathbb{F} contains the ground ring R and is closed under extensions, set indexed direct sums, pure submodules, and pure quotient modules, then $(\mathbb{F}, \mathbb{F}^\perp)$ is a perfect cotorsion pair.*

In particular, \mathbb{F} is covering and \mathbb{F}^\perp is enveloping.

Proof. We shall use the powerful result [11, thm. 2.9] to prove this.

To verify that the conditions of [11, thm. 2.9] are satisfied, first note that \mathbb{F} is a Kaplansky class by Proposition 3.2.

Since \mathbb{F} contains R and is closed under set indexed direct sums, it follows that \mathbb{F} contains all free modules. But \mathbb{F} is closed under pure quotients and so in particular under direct summands, and so in fact, \mathbb{F} contains all projective modules.

Finally, since \mathbb{F} is closed under set indexed direct sums and pure quotients, it is closed under all colimits indexed by partially ordered sets by Lemma 2.4.

This shows that the conditions of [11, thm. 2.9] are satisfied, and the present theorem follows. \square

4. APPLICATIONS

This section gives a number of applications of the theory developed above.

Remark 4.1. Recall the “ \perp ” notation from Definition 3.3. In the following results, note that $\text{Ker Tor}_1(\mathbb{B}, -)$ can be obtained as ${}^\perp\mathbb{C}$ by setting \mathbb{C} equal to the set of all Pontryagin duals $\text{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})$ for B in \mathbb{B} . This holds by [7, proof of cor. 11] and also follows from computation (2) below.

Lemma 4.2. *Let \mathbb{C} be a class of pure injective R -left-modules, \mathbb{B} a class of R -right-modules, and \mathbb{A} a class of finitely presented R -left-modules.*

- (i) *The class ${}^{\perp}\mathbb{C}$ is closed under set indexed direct sums and pure quotients.*
- (ii) *The class $\text{Ker Tor}_1(\mathbb{B}, -)$ is closed under set indexed direct sums, pure quotients, and pure submodules.*

If R is right-coherent and \mathbb{B} consists of finitely presented modules, then $\text{Ker Tor}_1(\mathbb{B}, -)$ is closed under set indexed direct products.

- (iii) *The class \mathbb{A}^{\perp} is closed under set indexed direct products and pure submodules.*

If R is left-coherent, then \mathbb{A}^{\perp} is closed under set indexed direct sums and pure quotients.

Proof. (i). This is easy to prove, using the observation that for C in \mathbb{C} , the functor $\text{Hom}(-, C)$ sends pure exact sequences to exact sequences.

(ii). Since $\text{Ker Tor}_1(\mathbb{B}, -)$ has the form ${}^{\perp}\mathbb{C}$ for a suitable set \mathbb{C} by Remark 4.1, the statements about set indexed direct sums and pure quotients follow from part (i).

If $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ is a pure exact sequence, then by [12, thm. 6.4], the Pontryagin duality functor $(-)^{\vee} = \text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ gives a split exact sequence $0 \rightarrow (X'')^{\vee} \rightarrow X^{\vee} \rightarrow (X')^{\vee} \rightarrow 0$, so if B is in \mathbb{B} then there is a split exact sequence

$$0 \rightarrow \text{Ext}_{R^{\circ}}^1(B, (X'')^{\vee}) \rightarrow \text{Ext}_{R^{\circ}}^1(B, X^{\vee}) \rightarrow \text{Ext}_{R^{\circ}}^1(B, (X')^{\vee}) \rightarrow 0. \quad (1)$$

Moreover, a standard computation shows

$$\begin{aligned} \text{Ext}_{R^{\circ}}^1(B, (-)^{\vee}) &= \text{Ext}_{R^{\circ}}^1(B, \text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})) \\ &\simeq \text{Hom}_{\mathbb{Z}}(\text{Tor}_1^R(B, -), \mathbb{Q}/\mathbb{Z}) \\ &= \text{Tor}_1^R(B, -)^{\vee}. \end{aligned} \quad (2)$$

Now let X be in $\text{Ker Tor}_1(\mathbb{B}, -)$ so $\text{Tor}_1^R(B, X) = 0$. The last computation implies $\text{Ext}_{R^{\circ}}^1(B, X^{\vee}) = 0$. The sequence (1) shows $\text{Ext}_{R^{\circ}}^1(B, (X')^{\vee}) = 0$, and then the last computation again implies that $\text{Tor}_1^R(B, X') = 0$. So X' is in $\text{Ker Tor}_1(\mathbb{B}, -)$.

Finally, if R is right-coherent and \mathbb{B} consists of finitely presented modules, then each B in \mathbb{B} has a projective resolution consisting of finitely generated modules, so $\text{Ker Tor}_1(\mathbb{B}, -)$ is closed under set indexed direct products because these are preserved by the functor $\text{Tor}_1(B, -)$.

(iii). It is clear that \mathbb{A}^{\perp} is closed under set indexed direct products because these are preserved by the functor $\text{Ext}^1(A, -)$.

If

$$0 \rightarrow Y' \rightarrow Y \rightarrow Y'' \rightarrow 0 \quad (3)$$

is a pure exact sequence and A is in \mathbb{A} , then there is an exact sequence $\text{Hom}(A, Y) \rightarrow \text{Hom}(A, Y'') \rightarrow \text{Ext}^1(A, Y') \rightarrow \text{Ext}^1(A, Y)$. The first arrow is surjective because A is finitely presented, so the second arrow is zero. If Y is in \mathbb{A}^\perp then $\text{Ext}^1(A, Y) = 0$, but then the sequence shows $\text{Ext}^1(A, Y') = 0$ whence Y' is in \mathbb{A}^\perp .

Now suppose that R is left-coherent. Then each A in \mathbb{A} has a projective resolution consisting of finitely generated modules, so \mathbb{A}^\perp is closed under set indexed direct sums because these are preserved by the functor $\text{Ext}^1(A, -)$.

By [12, thm. 6.4], Pontryagin duality $(-)^{\vee} = \text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ sends the pure exact sequence (3) to a split exact sequence $0 \rightarrow (Y'')^{\vee} \rightarrow Y^{\vee} \rightarrow (Y')^{\vee} \rightarrow 0$, so if A is in \mathbb{A} then there is a split exact sequence

$$0 \rightarrow \text{Tor}_1((Y'')^{\vee}, A) \rightarrow \text{Tor}_1(Y^{\vee}, A) \rightarrow \text{Tor}_1((Y')^{\vee}, A) \rightarrow 0. \quad (4)$$

However, A has a projective resolution consisting of finitely generated modules, so a standard computation shows

$$\begin{aligned} \text{Tor}_1^R((-)^{\vee}, A) &= \text{Tor}_1^R(\text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}), A) \\ &\simeq \text{Hom}_{\mathbb{Z}}(\text{Ext}_R^1(A, -), \mathbb{Q}/\mathbb{Z}) \\ &= \text{Ext}_R^1(A, -)^{\vee}. \end{aligned}$$

Now let Y be in \mathbb{A}^\perp so $\text{Ext}^1(A, Y) = 0$. The last computation implies $\text{Tor}_1(Y^{\vee}, A) = 0$. The sequence (4) shows that $\text{Tor}_1((Y'')^{\vee}, A) = 0$, and then the last computation again implies that $\text{Ext}^1(A, Y'') = 0$. So Y'' is in \mathbb{A}^\perp . \square

Some parts of the following theorem were already known; for instance, it was proved by Eklof and Trlifaj in [7, cor. 10 and thm. 12(i)] that the classes $\text{Ker Tor}_1(\mathbb{B}, -)$ and ${}^\perp\mathbb{C}$ are both covering, and when the ground ring is left-coherent, El Bashir's result [8, thm. 3.3] implies that \mathbb{A}^\perp is also covering if one is willing to assume Vopenka's Principle on high cardinal numbers. However, it is new that we are able to work with the intersections of such classes.

Theorem 4.3. *Let \mathbb{C} be a class of pure injective R -left-modules, \mathbb{B} a class of R -right-modules, and \mathbb{A} a class of finitely presented R -left-modules.*

- (i) *The classes ${}^\perp\mathbb{C}$ and $\text{Ker Tor}_1(\mathbb{B}, -)$ are covering.
If R is left-coherent, then $\mathbb{A}^\perp \cap {}^\perp\mathbb{C}$ and $\mathbb{A}^\perp \cap \text{Ker Tor}_1(\mathbb{B}, -)$ are covering.*
- (ii) *The class $\text{Ker Tor}_1(\mathbb{B}, -)$ is the first half of a perfect cotorsion pair.
If R is left-coherent and is an element of \mathbb{A}^\perp , then the class*

$$\mathbb{A}^\perp \cap \text{Ker Tor}_1(\mathbb{B}, -)$$

is the first half of a perfect cotorsion pair.

- (iii) *The class \mathbb{A}^\perp is preenveloping.
If R is right-coherent and \mathbb{B} consists of finitely presented modules, then the class $\mathbb{A}^\perp \cap \text{Ker Tor}_1(\mathbb{B}, -)$ is preenveloping.*

Proof. (i). It is enough to prove the statements involving ${}^{\perp}\mathbb{C}$, since the class $\text{Ker Tor}_1(\mathbb{B}, -)$ has the form ${}^{\perp}\mathbb{C}$ by Remark 4.1.

The class ${}^{\perp}\mathbb{C}$ is closed under set indexed direct sums and pure quotients by Lemma 4.2(i). If R is left-coherent, then \mathbb{A}^{\perp} has the same properties by Lemma 4.2(iii), and so $\mathbb{A}^{\perp} \cap {}^{\perp}\mathbb{C}$ also has the same properties. Now use Theorem 2.5.

(ii). The class $\text{Ker Tor}_1(\mathbb{B}, -)$ clearly contains R and is closed under extensions, and by Lemma 4.2(ii) it is also closed under set indexed direct sums, pure quotients, and pure submodules. If R is left-coherent and is contained in \mathbb{A}^{\perp} , then \mathbb{A}^{\perp} has the same properties by Lemma 4.2(iii), and so $\mathbb{A}^{\perp} \cap \text{Ker Tor}_1(\mathbb{B}, -)$ also has the same properties. Now use Theorem 3.4.

(iii). The class \mathbb{A}^{\perp} is closed under set indexed direct products and pure submodules by Lemma 4.2(iii). If R is right-coherent and \mathbb{B} consists of finitely presented modules, then $\text{Ker Tor}_1(\mathbb{B}, -)$ has the same properties by Lemma 4.2(ii), and so $\mathbb{A}^{\perp} \cap \text{Ker Tor}_1(\mathbb{B}, -)$ has the same properties. Now use Remark 2.6. \square

Example 4.4. (i) Let m be a non-negative integer and consider the class

$$\mathbb{F}_{\leq m} = \{ F \mid F \text{ is an } R\text{-left-module with flat dimension } \leq m \}.$$

Then $(\mathbb{F}_{\leq m}, (\mathbb{F}_{\leq m})^{\perp})$ is a perfect cotorsion pair. In particular, $\mathbb{F}_{\leq m}$ is covering.

Moreover, if R is right-coherent, then $\mathbb{F}_{\leq m}$ is also preenveloping.

This follows from Theorem 4.3, (ii) and (iii), by setting \mathbb{B} equal to the m 'th syzygies in projective resolutions of finitely presented modules; cf. [12, thm. A.8].

(ii) Suppose that R is left-noetherian and let n be a non-negative integer. Then the class

$$\mathbb{I}_{\leq n} = \{ I \mid I \text{ is an } R\text{-left-module with injective dimension } \leq n \}$$

is covering and preenveloping.

This follows from Theorem 4.3, (i) and (iii), by setting \mathbb{A} equal to the n 'th syzygies in projective resolutions of finitely generated modules; cf. [12, thm. A.6].

(iii) Suppose that R is left-noetherian and let m and n be non-negative integers. Then the class

$$\mathbb{F}_{\leq m} \cap \mathbb{I}_{\leq n} = \left\{ X \mid \begin{array}{l} X \text{ is an } R\text{-left module with flat dimen-} \\ \text{sion } \leq m \text{ and injective dimension } \leq n \end{array} \right\}$$

is covering.

Moreover, if R is right-coherent, then $\mathbb{F}_{\leq m} \cap \mathbb{I}_{\leq n}$ is also preenveloping.

This follows from Theorem 4.3, (i) and (iii), using the same \mathbb{A} and \mathbb{B} as above.

(iv) The class of fp-injective R -left-modules,

$$\mathbb{J} = \{ J \mid \text{Ext}^1(A, J) = 0 \text{ for } A \text{ a finitely presented } R\text{-left-module} \},$$

is preenveloping.

Moreover, if R is left-coherent, then \mathbb{J} is also covering.

This follows from Theorem 4.3, (i) and (iii), by setting \mathbb{A} equal to all the finitely presented modules.

These results on \mathbb{J} were known from [1] (see also [10, prop. 6.2.4]) and [14] with different proofs.

Finally, we use our methods to give new proofs of some known results about the torsion free modules in a torsion pair.

Definition 4.5. Recall from [6] that a pair of classes (\mathbb{T}, \mathbb{F}) is called a *torsion pair* if $\mathbb{T} \cap \mathbb{F}$ contains only modules isomorphic to 0, the class \mathbb{T} is closed under quotient modules, the class \mathbb{F} is closed under submodules, and each module M permits a short exact sequence of the form $0 \rightarrow T \rightarrow M \rightarrow F \rightarrow 0$ with T in \mathbb{T} and F in \mathbb{F} .

The torsion pair is called *hereditary* if \mathbb{T} is also closed under submodules, see [17, p. 441].

The torsion pair is said to be of *finite type* if each left-ideal \mathfrak{a} for which R/\mathfrak{a} is in \mathbb{T} contains a finitely generated left-ideal \mathfrak{b} for which R/\mathfrak{b} is in \mathbb{T} , see [4, p. 649]. Note that if R is left-noetherian, then the torsion pair is automatically of finite type.

Lemma 4.6. *Let (\mathbb{T}, \mathbb{F}) be a hereditary torsion pair. A module F is in \mathbb{F} if and only if $\text{Hom}(R/\mathfrak{a}, F) = 0$ for each ideal \mathfrak{a} such that R/\mathfrak{a} is in \mathbb{T} .*

If (\mathbb{T}, \mathbb{F}) is of finite type, then F is in \mathbb{F} if and only if $\text{Hom}(R/\mathfrak{b}, F) = 0$ for each finitely generated left-ideal \mathfrak{b} such that R/\mathfrak{b} is in \mathbb{T} .

Proof. The module F is in \mathbb{F} if and only if $\text{Hom}(T, F) = 0$ for each T in \mathbb{T} , see [6]. It is a small computation to see that this implies the lemma's statements. \square

Lemma 4.7. *Let (\mathbb{T}, \mathbb{F}) be a torsion pair. Then*

- (i) \mathbb{F} is closed under extensions.
- (ii) If (\mathbb{T}, \mathbb{F}) is hereditary, then \mathbb{F} is closed under set indexed direct sums.
- (iii) If (\mathbb{T}, \mathbb{F}) is hereditary and of finite type, then \mathbb{F} is closed under pure submodules and pure quotient modules.

Proof. (i). This holds because F is in \mathbb{F} if and only if $\text{Hom}(T, F) = 0$ for each T in \mathbb{T} .

(ii). Let $\{F_i\}$ be a set indexed system in \mathbb{F} . Let \mathfrak{a} be a left-ideal in R with R/\mathfrak{a} in \mathbb{T} . Since R/\mathfrak{a} is finitely generated,

$$\mathrm{Hom}(R/\mathfrak{a}, \bigoplus F_i) \cong \bigoplus \mathrm{Hom}(R/\mathfrak{a}, F_i) = 0$$

where $=$ is because each F_i is in \mathbb{F} . By Lemma 4.6 this shows that $\bigoplus F_i$ is in \mathbb{F} .

(iii). Let F be in \mathbb{F} and let $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ be a pure exact sequence. As \mathbb{F} is closed under submodules, F' is in \mathbb{F} as desired.

Let \mathfrak{b} be a finitely generated left-ideal in R with R/\mathfrak{b} in \mathbb{T} , and let $R/\mathfrak{b} \rightarrow F''$ be a homomorphism. Since R/\mathfrak{b} is finitely presented, $R/\mathfrak{b} \rightarrow F''$ factors through the pure epimorphism $F \rightarrow F''$. But F is in \mathbb{F} so each homomorphism $R/\mathfrak{b} \rightarrow F$ is zero, and it follows that $R/\mathfrak{b} \rightarrow F''$ is zero. Hence F'' is in \mathbb{F} by Lemma 4.6, as desired. \square

The following result was first proved by Bican and Torrecillas in [5, cor. 4.1].

Theorem 4.8. *Let (\mathbb{T}, \mathbb{F}) be a hereditary torsion pair of finite type.*

Then \mathbb{F} is covering.

Proof. Lemma 4.7(ii) says that \mathbb{F} is closed under set indexed direct sums, and Lemma 4.7(iii) says that \mathbb{F} is closed under pure quotients, so \mathbb{F} is covering by Theorem 2.5. \square

The following result was first proved by Angeleri-Hügel, Tonolo, and Trlifaj in [2, exa. 2.7].

Theorem 4.9. *Let (\mathbb{T}, \mathbb{F}) be a hereditary torsion pair of finite type where the ground ring R is in \mathbb{F} .*

Then $(\mathbb{F}, \mathbb{F}^\perp)$ is a perfect cotorsion pair.

In particular, \mathbb{F} is covering and \mathbb{F}^\perp is enveloping.

Proof. Lemma 4.7 says that \mathbb{F} is closed under extensions, set indexed direct sums, pure submodules, and pure quotients. As R is in \mathbb{F} , it follows that $(\mathbb{F}, \mathbb{F}^\perp)$ is a perfect cotorsion pair by Theorem 3.4. \square

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