Notes on Clifford Algebras and Spin Groups

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Chapter 1

Clifford Algebras

1.1 Elementary Properties

In this chapter we will introduce the Clifford algebra and discuss some of its elementary properties. The setting is the following:

Let V be a finite-dimensional vector space over the field $\mathbb{K}$ (predominantly $\mathbb{R}$ or $\mathbb{C}$) and $\varphi : V \times V \longrightarrow \mathbb{K}$ a symmetric bilinear form on V. φ is said to be *positive* (resp. *negative*) *definite* if for all $0 \neq v \in V$ we have $\varphi(v,v) > 0$ (resp. $\varphi(v,v) < 0$). φ is called *non-degenerate* if $\varphi(v,w) = 0$ for all $v \in V$ implies $w = 0$. From a bilinear form we construct a *quadratic form* $\Phi : V \longrightarrow \mathbb{K}$ given by $\Phi(v) := \varphi(v,v)$. We can recover the original bilinear form by the *polarization identity*:

$$\Phi(u+v) = \varphi(u+v,u+v) = \Phi(u) + \Phi(v) + 2\varphi(u,v),$$

hence

$$\varphi(u,v) = \frac{1}{2}(\Phi(u+v) - \Phi(u) - \Phi(v)). \tag{1.1}$$

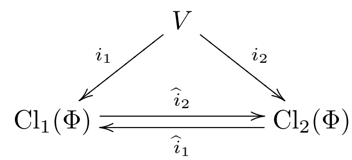
Thus we have a 1-1 correspondence between symmetric bilinear forms and quadratic forms. Thus a quadratic form Φ is called *positive definite/negative definite/non-degenerate* if φ is.

Definition 1.1. Let (V,Φ) be a vector space with a quadratic form Φ. The associated *Clifford algebra* $\mathrm{Cl}(V,\Phi)$ (abbreviated $\mathrm{Cl}(\Phi)$) is an associative, unital algebra over $\mathbb{K}$ with a linear map $i_\Phi : V \longrightarrow \mathrm{Cl}(\Phi)$ obeying the relation $i(v)^2 = \Phi(v)\cdot 1$ (1 is the unit element of $\mathrm{Cl}(\Phi)$). Furthermore $(\mathrm{Cl}(\Phi), i_\Phi)$ should have the property that for every unital algebra A, and every linear map $f : V \longrightarrow A$ satisfying $f(v)^2 = \Phi(v)\cdot 1$ there exists a unique algebra homomorphism $\widehat{f} : \mathrm{Cl}(\Phi) \longrightarrow A$, such that $f = \widehat{f} \circ i_\Phi$.

Two questions immediately arise: given (V,Φ), do such objects exist and if they do, are they unique? Fortunately, the answer to both questions is "yes":

Proposition 1.2. *For any vector space V with a quadratic form Φ let $\Im$ be the two-sided ideal in the tensor algebra $T(V)$ spanned by all elements of the form $a \otimes (v \otimes v - \Phi(v)\cdot 1) \otimes b$ (for $a,b \in T(V)$ and $v \in V$). Then $T(V)/\Im$, with the map $i_\Phi : V \longrightarrow T(V)/\Im$ being $\pi \circ \iota$ where $\iota : V \longrightarrow T(V)$ is the injection of V into $T(V)$, and $\pi : T(V) \longrightarrow T(V)/\Im$ is the quotient map, is a Clifford algebra, and any other Clifford algebra over (V,Φ) is isomorphic to this one.*

Proof. *Uniqueness.* Assume that $\mathrm{Cl}_1(\Phi)$ and $\mathrm{Cl}_2(\Phi)$ with linear maps $i_1 : V \longrightarrow \mathrm{Cl}_1(\Phi)$ and $i_2 : V \longrightarrow \mathrm{Cl}_2(\Phi)$ are Clifford algebras. Since $\mathrm{Cl}_1(\Phi)$ is a Clifford algebra, and i_2 is linear and satisfies $i_2(v)^2 = \Phi(v) \cdot 1$, it induces an algebra homomorphism $\widehat{i_2} : \mathrm{Cl}_1(\Phi) \longrightarrow \mathrm{Cl}_2(\Phi)$; likewise i_1 induces an algebra homomorphism $\widehat{i_1} : \mathrm{Cl}_2(\Phi) \longrightarrow \mathrm{Cl}_1(\Phi)$ such that the following diagram commutes:



We see that

$$i_2 = \widehat{i_2} \circ i_1 = \widehat{i_2} \circ \widehat{i_1} \circ i_2$$

and since $\widehat{i_2} \circ \widehat{i_1}$ is the unique map satisfying this, it must be $\mathrm{id}_{\mathrm{Cl}_2(\Phi)}$. Likewise $\widehat{i_1} \circ \widehat{i_2} = \mathrm{id}_{\mathrm{Cl}_1(\Phi)}$ which means that the two Clifford algebras are isomorphic. This proves uniqueness.

Existence. We now show that $\mathrm{Cl}(\Phi) := T(V)/\mathfrak{I}$ is indeed a Clifford algebra. i_Φ is easily seen to satisfy $i_\Phi(v)^2 = \Phi(v) \cdot 1$, where $1 \in \mathrm{Cl}(\Phi)$ is the coset containing the unit element of $T(V)$. Now let $f : V \longrightarrow A$ be linear with $f(v)^2 = \Phi(v) \cdot 1$. By the universal property of the tensor algebra this map factorizes uniquely through $T(V)$ to an algebra homomorphism $f' : T(V) \longrightarrow A$, such that $f = f' \circ \iota$. f' inherits the property $(f'(v))^2 = \Phi(v) \cdot 1$, and consequently

$$f'(v \otimes v - \Phi(v) \cdot 1) = (f'(v))^2 - \Phi(v) f'(1) = (f'(v))^2 - \Phi(v) \cdot 1$$

so f' vanishes on $\mathfrak{I}$. Therefore it factorizes uniquely through $\mathrm{Cl}(\Phi)$ to $\widehat{f} : \mathrm{Cl}(\Phi) \longrightarrow A$, such that $f = \widehat{f} \circ \pi \circ \iota = \widehat{f} \circ i_\Phi$. Thus $\mathrm{Cl}(\Phi) = T(V)/\mathfrak{I}$ is a Clifford algebra. □

We immediately see that if the quadratic form Φ on V is identically 0, the Clifford algebra $\mathrm{Cl}(\Phi)$ is nothing but the well-known exterior algebra $\Lambda^*(V)$.

From now on we will write i instead of i_Φ where no confusion is possible. A simple calculation reveals that for all $u, v \in V$: $i(u+v)^2 = i(u)^2 + i(v)^2 + i(u) \cdot i(v) + i(v) \cdot i(u)$. A comparison of this with the polarization identity using that $\Phi(u+v) \cdot 1 = i(u+v)^2$, $\Phi(u) \cdot 1 = i(u)^2$ and $\Phi(v) \cdot 1 = i(v)^2$ yields the following useful formula:

$$i(u) \cdot i(v) + i(v) \cdot i(u) = 2\varphi(u, v) \cdot 1. \tag{1.2}$$

It can be used to prove the following:

Proposition 1.3. *Assume Φ to be non-degenerate and let $\{e_1, \dots, e_n\}$ be an orthogonal basis for V. Then the set consisting of 1 and all products of the form $i(e_{j_1}) \cdots i(e_{j_k})$, where $j_1 < \cdots < j_k$ and $1 \leq k \leq n$ is a basis for $\mathrm{Cl}(\Phi)$. In particular $i : V \longrightarrow \mathrm{Cl}(\Phi)$ is injective, and the dimension of $\mathrm{Cl}(\Phi)$ is 2^n.*

Proof. First of all we will show that $\mathrm{Cl}(\Phi)$ is isomorphic to a subalgebra of $\mathrm{End}_{\mathbb{K}}(\Lambda^* V)$. For each $v \in V$ define endomorphisms $\varepsilon(v)$, $\iota(v)$ and $c(v)$ on $\Lambda^* V$ by

$$\varepsilon(v) : v_1 \wedge \cdots \wedge v_k \longmapsto v \wedge v_1 \wedge \cdots \wedge v_k.$$

$$\iota(v) : v_1 \wedge \cdots \wedge v_k \longmapsto \sum_{j=1}^{k} (-1)^{j+1} \varphi(v, v_j) v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_k.$$

$$c(v) = \varepsilon(v) + \iota(v).$$

It's a matter of calculations to show that $\varepsilon(v)^2 = \iota(v)^2 = 0$ and that we have

$$c(v)^2 = \Phi(v) \cdot 1$$
$$c(u)c(v) + c(v)c(u) = 2\varphi(u, v) \cdot 1.$$

The map $c : V \longrightarrow \mathrm{End}_{\mathbb{K}}(\Lambda^* V)$ is injective for if $c(v) = 0$ then $c(v)^2 = \Phi(v) \cdot 1 = 0$ i.e. $v = 0$ by non-degeneracy of Φ. By the universal property of Clifford algebras there exists an algebra homomorphism $\widehat{c} : \mathrm{Cl}(\Phi) \longrightarrow \mathrm{End}_{\mathbb{K}}(\Lambda^* V)$ extending c. By injectivity of c this map is an algebra isomorphism on the subalgebra of $\mathrm{End}_{\mathbb{K}}(\Lambda^* V)$ generated by the set $\{c(v) \mid c \in V\}$.

Now we consider the subalgebra of $\mathrm{End}_{\mathbb{K}}(\Lambda^* V)$. Any element of the form $c(v_1) \cdots c(v_m)$ can, by linearity of c and by the relations above be written as a linear combination of elements of the form $c(e_{i_1}) \cdots c(e_{i_k})$ where $i_1 < \cdots < i_k$. Let B denote the set of these element. Hence, these elements span the subalgebra. To see that they are linearly independent, observe that $c(e_{i_1}) \cdots c(e_{i_k})$ maps 1 to $e_{i_1} \wedge \cdots \wedge e_{i_k}$. This evaluation map is linear and since the set $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}$ is a basis for $\Lambda^* V$, B is a linearly independent set, hence a basis. By the isomorphism $\widehat{c}$ the set

$$\{i(e_{i_1}) \cdots i(e_{i_k}) \mid i_1 < \cdots < i_k\}$$

is a basis for $\mathrm{Cl}(\Phi)$. □

Since i is injective we can imagine V as sitting as a subspace of $\mathrm{Cl}(\Phi)$. Thus, henceforth we won't bother to write $i(v)$ but will write v instead.

The evaluation map gives, by composing it with the isomorphism mentioned in the proof above, a linear map $\sigma : \mathrm{Cl}(\Phi) \longrightarrow \Lambda^* V$ called the *symbol map*. If $\{e_1, \ldots, e_n\}$ is an orthogonal basis, then it maps $e_{i_1} \cdots e_{i_k}$ to $e_{i_1} \wedge \cdots \wedge e_{i_k}$, and is thus a linear isomorphism (although not necessarily an algebra isomorphism!) The inverse map $Q : \Lambda^* V \longrightarrow \mathrm{Cl}(\Phi)$, mapping $e_{i_1} \wedge \cdots \wedge e_{i_k}$ to $e_{i_1} \cdots e_{i_k}$ is called the *quantization map*.

If (V, φ) and (W, ψ) are two vector spaces with bilinear forms, a linear map $f : V \longrightarrow W$ is called *orthogonal* w.r.t. the bilinear forms if $\psi(f(v), f(w)) = \varphi(v, w)$ (or equivalently, if $\Psi(f(v)) = \Phi(v)$). We say that (V, φ) and (W, ψ) are *isomorphic* if there exists an orthogonal linear map $f : V \longrightarrow W$ which is also a vector space isomorphism and such a map we call an *orthogonal isomorphism*. If φ is non-degenerate, one can show that an orthogonal endomorphism $f : V \longrightarrow V$ is automatically an isomorphism. Thus the set of orthogonal endomorphisms of V is a group $\mathrm{O}(V, \Phi)$ (or just $\mathrm{O}(\Phi)$), called the *orthogonal group*. This is a closed subgroup of the Lie group $\mathrm{GL}(V)$, and thus itself a Lie group. Picking only those orthogonal endomorphisms having determinant 1 gives the *special orthogonal group* $\mathrm{SO}(V, \Phi)$ or just $\mathrm{SO}(\Phi)$. This is again a Lie group, being a closed subgroup of $\mathrm{O}(\Phi)$.

A curious fact is that there exists an isomorphism among the orthogonal groups. To see this, let $\varphi : (\mathbb{R}^{p+q}, \Phi_{p,q}) \longrightarrow (\mathbb{R}^{p+q}, \Phi_{q,p})$ be an *anti-orthogonal* linear map, i.e. satisfying

$$\Phi_{q,p}(\varphi(x)) = -\Phi_{p,q}(x)$$

and consider the map $\Psi : \mathrm{O}(p, q) \longrightarrow \mathrm{O}(q, p)$ given by $\Phi(A) = \varphi \circ A \circ \varphi^{-1}$. It is easily checked that this is a Lie group isomorphism. Restricting Ψ to $\mathrm{SO}(p, q)$, it maps into $\mathrm{SO}(q, p)$ since the determinant is unaltered by a conjugation. Thus we also have an isomorphism $\mathrm{SO}(p, q) \xrightarrow{\sim} \mathrm{SO}(q, p)$.

The next proposition shows that Cl is a covariant functor from the category of vector spaces with bilinear forms and orthogonal linear maps to the category of associated unital algebras and algebra homomorphisms.

Proposition 1.4. *An orthogonal linear map $f : V \longrightarrow W$ between (V, Φ) and (W, Ψ) induces a unique algebra homomorphism $\overline{f} : \mathrm{Cl}(\Phi) \longrightarrow \mathrm{Cl}(\Psi)$ satisfying $\overline{f}(i_\Phi(v)) = i_\Psi(f(v))$. If f is an orthogonal isomorphism, then $\overline{f}$ is an algebra isomorphism.*

If, furthermore, we have a vector space U with quadratic form Θ and linear maps $f : V \longrightarrow U$ and $g : U \longrightarrow W$ satisfying $\Theta(f(v)) = \Phi(v)$ and $\Psi(g(u)) = \Theta(u)$, then $\overline{g \circ f} = \overline{g} \circ \overline{f}$.

Exercise 1. Prove the preceding proposition.

The most interesting examples of Clifford algebras show up when the bilinear form is non-degenerate. For the time being we consider only real vector spaces and Clifford algebras. A prominent example of a vector space with non-degenerate bilinear form is $(\mathbb{R}^{p+q}, \varphi_{p,q})$ where $\varphi_{p,q}(e_i, e_j) = 0$ if $i \neq j$ and

$$\varphi_{p,q}(e_i, e_i) = \begin{cases} 1, & i \leq p \\ -1, & i > p \end{cases}$$

$(e_1, \ldots, e_{p+q}$ is the standard basis of $\mathbb{R}^{p+q})$. The associated quadratic form is denoted $\Phi_{p,q}$ and the corresponding Clifford algebra is denoted $\mathrm{Cl}_{p,q}$. In this case $\mathrm{O}(\Phi_{p,q})$ and $\mathrm{SO}(\Phi_{p,q})$ are the well-known orthogonal groups $\mathrm{O}(p,q)$ and $\mathrm{SO}(p,q)$.

Example 1.5. 1) Let us consider the vector space $\mathbb{R}$ with the single basis element $e_1 := 1$ and the quadratic form $\Phi_{0,1}(x_1 e_1) = -x_1^2$. By Proposition 1.3 $\{1, e_1\}$ (where 1 is now the unit element of $\mathrm{Cl}_{0,1}$) is a basis for $\mathrm{Cl}_{0,1}$. The fact that $e_1^2 = \Phi_{0,1}(e_1) \cdot 1 = -1$ shows that the linear map $\mathrm{Cl}_{0,1} \ni 1 \longmapsto 1 \in \mathbb{C}$, $e_1 \longmapsto i \in \mathbb{C}$ defines an algebra isomorphism $\mathrm{Cl}_{0,1} \xrightarrow{\sim} \mathbb{C}$ and that the injection $\mathbb{R} \hookrightarrow \mathrm{Cl}_{0,1}$ is given by $x \longmapsto ix$, i.e. $\mathbb{R}$ sits inside $\mathrm{Cl}_{0,1} \cong \mathbb{C}$ as the imaginary part. Thus, $\mathrm{Cl}_{0,1}$ is just the field of complex numbers.

Exercise 2. Consider $\mathbb{R}^2$ with the standard basis $\{e_1, e_2\}$ and the quadratic form $\Phi_{0,2}(x_1 e_1 + x_2 e_2) = -x_1^2 - x_2^2$. Show that the corresponding Clifford algebra can be identified with the algebra of quaternions $\mathbb{H}$. Because of this, Clifford algebras are sometimes called *generalized quaternions*. □

Now, let (V, φ) be any real vector space with non-degenerate bilinear form φ, and let $\{e_1, \ldots, e_n\}$ be a basis for V. We will consider the matrix (φ_{ij}) where $\varphi_{ij} = \varphi(e_i, e_j)$. Since φ is symmetric the matrix (φ_{ij}) is symmetric as well, i.e. it can be diagonalized. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues and $f_1, \ldots, f_n$ a basis of diagonalizing eigenvectors. This means that

$$\varphi(f_i, f_j) = 0 \text{ if } i \neq j \qquad \text{and} \qquad \varphi(f_i, f_i) = \lambda_i.$$

Observe, that none of the eigenvalues are 0 (a 0 eigenvalue would violate non-degeneracy of φ). Arrange the eigenvalues so that $\lambda_1, \ldots, \lambda_k$ are all strictly positive and $\lambda_{k+1}, \ldots, \lambda_n$ are all strictly negative. Define

$$\widetilde{f}_i = \frac{1}{\sqrt{|\lambda_i|}} f_i$$

then we see that

$$\varphi(\widetilde{f}_i, \widetilde{f}_j) = 0 \text{ if } i \neq j \qquad \text{and} \qquad \varphi(\widetilde{f}_i, \widetilde{f}_i) = \begin{cases} 1, & i \leq k \\ -1, & i > k \end{cases}.$$

A basis satisfying this is called a (real) *orthonormal basis* w.r.t. φ. Thus we have proven

Theorem 1.6 (Classification of Real Bilinear Forms). *Let (V, φ) be a real vector space with non-degenerate bilinear form. Then there exists an orthonormal basis for V and the map sending this basis to the standard basis for $\mathbb{R}^n$ is an orthogonal isomorphism $(V, \varphi) \longrightarrow (\mathbb{R}^n, \varphi_{k,n-k})$.*

In particular any real Clifford algebra originating from a non-degenerate quadratic form is isomorphic to $\mathrm{Cl}_{p,q}$ for a certain p and q (Proposition 1.4).

The complex case is a bit different. On $\mathbb{C}^n$ we have bilinear forms φ_n given by $\varphi_n(e_i, e_j) = \delta_{ij}$. The corresponding quadratic form is denoted Φ_n. For an arbitrary complex vector space (V, φ), chose a basis $\{e_1, \ldots, e_n\}$ and write φ in this basis (φ_{ij}). Again, since it is symmetric, it can be diagonalized. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues and $\{f_1, \ldots, f_n\}$ a diagonalizing basis of eigenvectors. None of the eigenvalues are 0 and hence we can define

$$\widetilde{f_i} = \frac{1}{\sqrt{\lambda_i}} f_i.$$

This basis satisfies $\varphi(\widetilde{f_i}, \widetilde{f_j}) = \delta_{ij}$, and is thus called a (complex) *orthonormal basis*. Thus we have shown

Theorem 1.7 (Classification of Complex Bilinear Forms). *Let (V, φ) be a complex vector space with non-degenerate bilinear form. Then there exists an orthonormal basis for V and the map sending this basis to the standard basis for $\mathbb{R}^n$ is an orthogonal isomorphism $(V, \varphi) \longrightarrow (\mathbb{C}^n, \varphi_n)$.*

Appealing to Proposition 1.4 we see that a complex Clifford algebra is isomorphic to $\mathrm{Cl}(\Phi_n)$ for some n.

Again we consider the general situation of Clifford algebras over $\mathbb{K}$. Now we want to equip the Clifford algebra with two involutions t and α which we will need later in the construction of various subgroups of $\mathrm{Cl}(\Phi)$.

Proposition 1.8. *Each Clifford algebra $\mathrm{Cl}(\Phi)$ admits a canonical anti-automorphism, i.e. a linear map $t : \mathrm{Cl}(\Phi) \longrightarrow \mathrm{Cl}(\Phi)$ that for all $x, y \in \mathrm{Cl}(\Phi)$ satisfies*

$$t(x \cdot y) = t(y) \cdot t(x) \quad , \quad t \circ t = \mathrm{id}_{\mathrm{Cl}(\Phi)} \quad , \quad t|_V = \mathrm{id}_V \,.$$

Proof. Consider the involution J of the tensor algebra given by $v_1 \otimes \cdots \otimes v_k \stackrel{J}{\longmapsto} v_k \otimes \cdots \otimes v_1$ (and extended by linearity). Now $\pi \circ J : T(V) \longrightarrow \mathrm{Cl}(\Phi)$ is an anti-homomorphism that vanishes on $\mathfrak{I}$ since

$$\begin{aligned} &J(a \otimes v \otimes v \otimes b - a \otimes (\Phi(v) \cdot 1) \otimes b) \\ &\qquad = J(b) \otimes v \otimes v \otimes J(a) - J(b) \otimes (\Phi(v) \cdot 1) \otimes J(a) \\ &\qquad = J(b) \otimes (v \otimes v - \Phi(v) \cdot 1) \otimes J(a) \in \mathfrak{I}. \end{aligned}$$

But then $\pi \circ J$ induces a unique anti-homomorphism $t : \mathrm{Cl}(\Phi) \longrightarrow \mathrm{Cl}(\Phi)$ determined by $t[x] = \pi \circ J(x)$ (where $[x] \in \mathrm{Cl}(\Phi)$ is the coset containing $x \in T(V)$). It is easy to see that $J(x \cdot y) = J(y) \cdot J(x)$, so we also have $t(x \cdot y) = t(y) \cdot t(x)$.

J is clearly an involution, i.e. $J \circ J = \mathrm{id}_{T(V)}$. On one hand $\mathrm{id}_{T(V)}$ induces the map $\mathrm{id}_{\mathrm{Cl}(\Phi)}$, and on the other hand $J \circ J$ induces the map $t \circ t$. By uniqueness we conclude $t \circ t = \mathrm{id}_{\mathrm{Cl}(\Phi)}$. The last property, $t|_V = \mathrm{id}_V$ follows from the fact that $J(v) = v$ for $v \in V$. □

Now for the construction of the second involution:

Proposition 1.9. *Each Clifford algebra $\mathrm{Cl}(\Phi)$ admits a canonical automorphism $\alpha : \mathrm{Cl}(\Phi) \longrightarrow \mathrm{Cl}(\Phi)$ which satisfies $\alpha \circ \alpha = \mathrm{id}_{\mathrm{Cl}(\Phi)}$ and $\alpha|_V = -\mathrm{id}_V$.*

Furthermore we have $\alpha(1) = 1$ and $\alpha(e_{i_1} \cdots e_{i_k}) = (-1)^k e_{i_1} \cdots e_{i_k}$.

Proof. Consider the linear bijection $\widetilde{\alpha} : V \longrightarrow V$given by $v \longmapsto -v$. By the functorial property of Cl it induces an automorphism $\alpha : \mathrm{Cl}(\Phi) \longrightarrow \mathrm{Cl}(\Phi)$, and we see that

$$\alpha \circ \alpha = \overline{\widetilde{\alpha} \circ \widetilde{\alpha}} = \overline{\mathrm{id}_V} = \mathrm{id}_{\mathrm{Cl}(\Phi)} \,.$$

The second property of α is obtained from the identity $\alpha \circ i = i \circ \widetilde{\alpha}$ which is seen from the commutative diagram in the proof of Proposition 1.4. This gives $\alpha(i(v)) = i(-v) = -i(v)$, and by considering v an element of $\mathrm{Cl}(\Phi)$ (by virtue of the injectivity of i) we have proven the claim. □

The the calculation of t and α on our model algebras $\mathbb{C}$ and $\mathbb{H}$ we refer to Example 2.6 in the next chapter.

Due to involutivity of α, we can split the the Clifford algebra in a direct sum of two subspaces:

$$\mathrm{Cl}(\Phi) = \mathrm{Cl}^0(\Phi) \oplus \mathrm{Cl}^1(\Phi) \tag{1.3}$$

where $\mathrm{Cl}^i(\Phi) = \{ x \in \mathrm{Cl}(\Phi) \mid \alpha(x) = (-1)^i x \}$ for $i = 0, 1$. It is easily seen that $\mathrm{Cl}^i(\Phi) \cdot \mathrm{Cl}^j(\Phi) \subseteq \mathrm{Cl}^{i+j \pmod 2}(\Phi)$ which says that the Clifford algebra is a $\mathbb{Z}_2$-*graded algebra* or a *super algebra*. $\mathrm{Cl}^0(\Phi)$ is called the *bosonic subalgebra* (note that it is actually a subalgebra), and $\mathrm{Cl}^1(\Phi)$ is called the *fermionic subspace*. We see that a product of the form $v_1 \cdots v_k$ for $v_i \in V$ is bosonic if k is even and fermionic if k is odd. Elements of $\mathrm{Cl}^0(\Phi) \cup \mathrm{Cl}^1(\Phi)$ are called homogenous elements and the degree of a homogenous element x is denoted by $|x|$.

If A and B are two $\mathbb{Z}_2$-graded algebras we could form the tensor product $A \otimes B$ with the usual product $(a \otimes b)(a' \otimes b') = (aa') \otimes (bb')$. However, this is usually not of great interest since the resulting algebra is not $\mathbb{Z}_2$-graded (at least not non-trivially). Therefore we define the so-called *graded tensor product* or *super tensor product* $A \widehat{\otimes} B$ in the following way: As a vector space it is just the ordinary tensor product $A \otimes B$ but the product is given on homogenous elements $a, a' \in A$ and $b, b' \in B$ by

$$(a \otimes b)(a' \otimes b') = (-1)^{|a'||b|}(aa') \otimes (bb').$$

This gives $A \widehat{\otimes} B$ a natural grading by defining

$$(A \widehat{\otimes} B)^0 = (A^0 \otimes B^0) \oplus (A^1 \otimes B^1)$$
$$(A \widehat{\otimes} B)^1 = (A^0 \otimes B^1) \oplus (A^1 \otimes B^0).$$

With this at hand we can accomplish out final task of this section, showing how Cl reacts to a direct sum of vector spaces. By an *orthogonal decomposition* of (V, Φ), we understand a decomposition $V = V_1 \oplus V_2$ such that if $v = v_1 + v_2$ we have $\Phi(v) = \Phi_1(v_1) + \Phi_2(v_2)$ (equivalently if $\varphi(V_1, V_2) = 0$).

Proposition 1.10. *Assume we have an orthogonal decomposition $V = V_1 \oplus V_2$ of (V, Φ), then $\mathrm{Cl}(V, \Phi) \cong \mathrm{Cl}(V_1, \Phi_1) \widehat{\otimes} \mathrm{Cl}(V_2, \Phi_2)$ as algebras.*

PROOF. We use the universal property of Clifford algebras to cook up a map. First, define $g : V \longrightarrow \mathrm{Cl}(V, \Phi) \cong \mathrm{Cl}(V_1, \Phi_1) \widehat{\otimes} \mathrm{Cl}(V_2, \Phi_2)$ by

$$g(v) = v_1 \otimes 1 + 1 \otimes v_2.$$

A quick calculation shows that $g(v)^2 = \Phi(v)(1 \otimes 1)$ and thus by the universal property of Cl there exists an algebra homomorphism $\widehat{g} : \mathrm{Cl}(V, \Phi) \longrightarrow \mathrm{Cl}(V_1, \Phi_1) \widehat{\otimes} \mathrm{Cl}(V_2, \Phi_2)$ extending g. To see that this is indeed an isomorphism choose a basis $\{e_1, \ldots, e_m\}$ for V_1 and a basis $\{f_1, \ldots, f_n\}$ for V_2 and put them together to a basis $\{e_1, \ldots, f_n\}$ for V. A basis for $\mathrm{Cl}(V, \Phi)$ consists of elements of the form $e_{i_1} \cdots e_{i_k} f_{j_1} \cdots f_{j_l}$ where $i_1 < \cdots < i_k$, $k \leq m$ and $j_1 < \cdots < j_l$, $l \leq n$. Similarly a basis for $\mathrm{Cl}(V_1, \Phi_1) \widehat{\otimes} \mathrm{Cl}(V_2, \Phi_2)$ is given by $e_{i_1} \cdots e_{i_k} \otimes f_{j_1} \cdots f_{j_l}$ (with the same restrictions on the indices as above). One can verify that

$$\widehat{g}(e_{i_1} \cdots e_{i_k} f_{j_1} \cdots f_{j_l}) = e_{i_1} \cdots e_{i_k} \otimes f_{j_1} \cdots f_{j_l}$$

i.e. it maps basis to basis. Thus it is an algebra isomorphism. □

1.2 Classification of Clifford Algebras

In this section we set out to classify Clifford algebras originating from non-degenerate bilinear forms. At first we concentrate on real Clifford algebras. We have taken the first step in the classification in that we have shown in the previous section, that it is enough to concentrate our treatment of Clifford algebras to the case where the vector space is $\mathbb{R}^{p+q}$ equipped with the quadratic form

$$\Phi_{p,q}(x_1e_1 + \cdots + x_{p+q}e_{p+q}) := x_1^2 + \cdots + x_p^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2)$$

(where $\{e_1, \ldots, e_{p+q}\}$ denotes the usual standard basis for $\mathbb{R}^{p+q}$). The associated Clifford algebra was denoted $\mathrm{Cl}_{p,q}$. As we saw in Example 1.5, we have

$$\mathrm{Cl}_{0,1} \cong \mathbb{C} \qquad \text{and} \qquad \mathrm{Cl}_{0,2} \cong \mathbb{H}. \tag{1.4}$$

Likewise, one can show that the following isomorphisms hold

$$\mathrm{Cl}_{1,0} \cong \mathbb{R} \oplus \mathbb{R} \qquad \text{and} \qquad \mathrm{Cl}_{2,0} \cong \mathrm{Cl}_{1,1} \cong \mathbb{R}(2) \tag{1.5}$$

($\mathbb{K}(n)$ is the algebra of $n \times n$ matrices over $\mathbb{K}$, where $\mathbb{K}$ can be either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$). As is apparent, all four Clifford algebras are isomorphic to either an algebra of matrices over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, or to a direct sum of two such algebras. The goal of this section is to show that this is no coincidence. Actually, it is a consequence of the *Cartan-Bott Periodicity Theorem* – which we will prove at the end of this section – that this holds for every Clifford algebra $\mathrm{Cl}_{p,q}$.

Before proving it, we need two lemmas:

Lemma 1.11. *We have the following algebra isomorphisms:*

$$\mathbb{R}(m) \otimes \mathbb{R}(n) \cong \mathbb{R}(mn) \quad \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C} \quad \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{C}(2) \quad \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{R}(4).$$

If $\mathbb{K}$ denotes either $\mathbb{C}$ or $\mathbb{H}$, then

$$\mathbb{R}(n) \otimes_{\mathbb{R}} \mathbb{K} \cong \mathbb{K}(n).$$

This is well-known so we won't prove it here.[1] Instead we show the next lemma which really does most of the work:

Lemma 1.12. *We have the following three algebra isomorphisms:*

$$\mathrm{Cl}_{0,n+2} \cong \mathrm{Cl}_{n,0} \otimes \mathrm{Cl}_{0,2} \qquad , \qquad \mathrm{Cl}_{n+2,0} \cong \mathrm{Cl}_{0,n} \otimes \mathrm{Cl}_{2,0} \qquad \textit{and}$$

$$\mathrm{Cl}_{p+1,q+1} \cong \mathrm{Cl}_{p,q} \otimes \mathrm{Cl}_{1,1}\,.$$

for all $p, q, n \in \mathbb{N} \cup \{0\}$.

Proof. To prove the first isomorphism, the strategy is the following: we will construct a linear map $f : \mathbb{R}^{n+2} \longrightarrow \mathrm{Cl}_{n,0} \otimes \mathrm{Cl}_{0,2}$, show that f factorizes through $\mathrm{Cl}_{0,n+2}$, and that the induced map $\widehat{f} : \mathrm{Cl}_{0,n+2} \longrightarrow \mathrm{Cl}_{n,0} \otimes \mathrm{Cl}_{0,2}$ is an isomorphism. Letting $\{e_1, \ldots, e_{n+2}\}$ denote the standard basis for $\mathbb{R}^{n+2}$, $\{e'_1, \ldots, e'_n\}$ the usual basis for $\mathbb{R}^n$, and $\{e''_1, e''_2\}$ the usual basis for $\mathbb{R}^2$, and thereby generators for the Clifford algebras $\mathrm{Cl}_{0,n+2}$, $\mathrm{Cl}_{n,0}$ and $\mathrm{Cl}_{0,2}$ respectively, we define f by

$$f(e_i) = \begin{cases} e'_i \otimes e''_1 e''_2 & \text{if } 1 \le i \le n \\ 1 \otimes e''_{i-n} & \text{if } n+1 \le i \le n+2 \end{cases}$$

[1] For a proof consult for instance [Lawson and Michelson], pp 26-27, Proposition 4.2.

For $1 \leq i, j \leq n$ we compute, using the rules for multiplication in a Clifford algebra, that

$$\begin{aligned} f(e_i)\cdot f(e_j) + f(e_j)\cdot f(e_i) &= (e_i' \otimes e_1''e_2'')\cdot(e_j' \otimes e_1''e_2'') + (e_j' \otimes e_1''e_2'')\cdot(e_i' \otimes e_1''e_2'') \\ &= (e_i'e_j') \otimes (e_1''e_2''e_1''e_2'') + (e_j'e_i') \otimes (e_1''e_2''e_1''e_2'') \\ &= -(e_i'e_j' + e_j'e_i') \otimes (e_1''e_1''e_2''e_2'') = -2\delta_{ij} 1 \otimes 1 \end{aligned}$$

because $e_i'e_j' + e_j'e_i' = 2\delta_{ij} \cdot 1$, as $\{e_1', \dots, e_n'\}$ is basis for $\mathbb{R}^n$ orthonormal w.r.t. $\Phi_{n,0}$. For $n+1 \leq i, j \leq n+2$ we have

$$\begin{aligned} f(e_i)\cdot f(e_j) + f(e_j)\cdot f(e_i) &= (1 \otimes e_{i-n}'')\cdot(1 \otimes e_{j-n}'') + (1 \otimes e_{j-n}'')\cdot(1 \otimes e_{i-n}'') \\ &= 1 \otimes e_{i-n}''e_{j-n}'' + 1 \otimes e_{j-n}''e_{i-n}'' \\ &= 1 \otimes (e_{i-n}''e_{j-n}'' + e_{j-n}''e_{i-n}'') = -2\delta_{ij} 1 \otimes 1 \end{aligned}$$

where the last minus is due to $\{e_1'', e_2''\}$ being a basis for $\mathbb{R}^2$ orthonormal w.r.t. $\Phi_{0,2}$. A similar computation shows that $f(e_i)f(e_j) + f(e_j)f(e_i) = 0$ if $1 \leq i \leq n$ and $n+1 \leq j \leq n+2$. But then for $x = x_1e_1 + \dots + x_{n+2}e_{n+2}$ we have by linearity of f that

$$f(x)^2 = -(x_1^2 + \dots + x_{n+2}^2) 1 \otimes 1 = \Phi_{0,n+2}(x) \cdot 1 \otimes 1.$$

Therefore f factorizes uniquely through $\mathrm{Cl}_{0,n+2}$ to an algebra homomorphism $\widehat{f} : \mathrm{Cl}_{0,n+2} \longrightarrow \mathrm{Cl}_{n,0} \otimes \mathrm{Cl}_{0,2}$. f maps to a set of generators for $\mathrm{Cl}_{n,0} \otimes \mathrm{Cl}_{0,2}$; thus $\widehat{f}$ maps to a set of generators for $\mathrm{Cl}_{n,0} \otimes \mathrm{Cl}_{0,2}$. Since $\widehat{f}$ is an algebra homomorphism, $\widehat{f}$ must then be surjective. Since

$$\dim \mathrm{Cl}_{0,n+2} = 2^{n+2} = 2^n \cdot 2^2 = (\dim \mathrm{Cl}_{n,0})(\dim \mathrm{Cl}_{0,2}) = \dim(\mathrm{Cl}_{n,0} \otimes \mathrm{Cl}_{0,2}),$$

the Dimension Theorem from linear algebra tells us that $\widehat{f}$ is also injective. Thus $\widehat{f}$ is the desired isomorphism.

The second isomorphism is proved in exactly the same way, and we avoid repeating ourselves.

The proof of the third isomorphism is essentially the same as the two first. Let $\{e_1, \dots, e_{p+1}, \varepsilon_1, \dots, \varepsilon_{q+1}\}$ be an orthogonal basis for $\mathbb{R}^{p+q+2}$ (i.e. $\varphi(v, w) = 0$ when $v \neq w$) with the quadratic form $\Phi_{p+1,q+1}$, such that $\Phi_{p+1,q+1}(e_i) = 1$, $\Phi_{p+1,q+1}(\varepsilon_i) = -1$, and let $\{e_1', \dots, e_p', \varepsilon_1', \dots, \varepsilon_q'\}$ and $\{e_1'', \varepsilon_1''\}$ be similar bases for $\mathbb{R}^{p+q}$ and $\mathbb{R}^{1+1}$ (and thereby generators for the Clifford algebras $\mathrm{Cl}_{p+1,q+1}$, $\mathrm{Cl}_{p,q}$ and $\mathrm{Cl}_{1,1}$ respectively). We now define a linear map $f : \mathbb{R}^{p+q+2} \longrightarrow \mathrm{Cl}_{p,q} \otimes \mathrm{Cl}_{1,1}$ by

$$f(e_i) = \begin{cases} e_i' \otimes e_1''\varepsilon_1'' & \text{if } 1 \leq i \leq p \\ 1 \otimes e_1'' & \text{if } i = p+1 \end{cases} \quad , \quad f(\varepsilon_j) = \begin{cases} \varepsilon_j' \otimes e_1''\varepsilon_1'' & \text{if } 1 \leq j \leq q \\ 1 \otimes \varepsilon_1'' & \text{if } j = q+1 \end{cases}.$$

Just like before it can be shown that

$$f(x)^2 = \Phi_{p+1,q+1}(x) \cdot 1 \otimes 1,$$

and thus f induces an isomorphism $\widehat{f} : \mathrm{Cl}_{p+1,q+1} \xrightarrow{\sim} \mathrm{Cl}_{p,q} \otimes \mathrm{Cl}_{1,1}$. □

Now we are ready to state and prove the Cartan-Bott Theorem:

Theorem 1.13 (Cartan-Bott I). *We have the following isomorphisms*:

$$\mathrm{Cl}_{0,n+8} \cong \mathrm{Cl}_{0,n} \otimes \mathbb{R}(16) \qquad \text{and} \qquad \mathrm{Cl}_{n+8,0} \cong \mathrm{Cl}_{n,0} \otimes \mathbb{R}(16).$$

PROOF. Using the two first isomorphisms from Lemma 1.12 a couple of times yields

$$\begin{aligned}\mathrm{Cl}_{0,n+8} &\cong \mathrm{Cl}_{n+6,0}\otimes \mathrm{Cl}_{0,2} \cong \cdots \cong \mathrm{Cl}_{0,n}\otimes \mathrm{Cl}_{2,0}\otimes \mathrm{Cl}_{0,2}\otimes \mathrm{Cl}_{2,0}\otimes \mathrm{Cl}_{0,2}\\ &\cong \mathrm{Cl}_{0,n}\otimes \mathrm{Cl}_{2,0}\otimes \mathrm{Cl}_{2,0}\otimes \mathrm{Cl}_{0,2}\otimes \mathrm{Cl}_{0,2}\end{aligned}$$

where the last isomorphism follows from the fact that for arbitrary real algebras A and B we have $A\otimes B\cong B\otimes A$. From (1.4) we have $\mathrm{Cl}_{0,2}\cong \mathbb{H}$, and from (1.5) that $\mathrm{Cl}_{2,0}\cong \mathbb{R}(2)$. Thus, using $\mathbb{H}\otimes_{\mathbb{R}}\mathbb{H}\cong\mathbb{R}(4)$ from Lemma 1.11, we get

$$\begin{aligned}\mathrm{Cl}_{2,0}\otimes \mathrm{Cl}_{2,0}\otimes \mathrm{Cl}_{0,2}\otimes \mathrm{Cl}_{0,2} &\cong \mathbb{R}(2)\otimes\mathbb{R}(2)\otimes(\mathbb{H}\otimes_{\mathbb{R}}\mathbb{H})\cong\mathbb{R}(4)\otimes\mathbb{R}(4)\\ &\cong \mathbb{R}(16).\end{aligned}$$

This completes the proof of the first isomorphism. The proof of the second isomorphism is identical to this. □

Now it's evident that once we know the Clifford algebras $\mathrm{Cl}_{0,0},\mathrm{Cl}_{0,1},\ldots,\mathrm{Cl}_{0,7}$ and $\mathrm{Cl}_{0,0},\mathrm{Cl}_{1,0},\ldots,\mathrm{Cl}_{7,0}$ we know all of them: Consider $\mathrm{Cl}_{p,q}$ and assume that $p\geq q$. Then by the third isomorphism in Lemma 1.12 we have that $\mathrm{Cl}_{p,q}$ is isomorphic to $\mathrm{Cl}_{p-q,0}\otimes(\mathrm{Cl}_{1,1})^{\otimes q}\cong \mathrm{Cl}_{p-q,0}\otimes\mathbb{R}(2^q)$, and $\mathrm{Cl}_{p-q,0}$ can be expressed as a tensor product of $\mathrm{Cl}_{k,0}$ (with $0\leq k\leq 7$) and some copies of $\mathbb{R}(16)$. We can do the same if $q\geq p$. Using the isomorphisms from Lemma 1.11 one obtains the table of Clifford algebras in Appendix A. From this table we see that any Clifford algebra is either a matrix algebra or a sum of two such algebras, as we pointed out in the beginning of this section.

Example 1.14. As an example, let us show that $\mathrm{Cl}_{2,11}\cong\mathbb{C}(64)$:

$$\begin{aligned}\mathrm{Cl}_{2,11} &\cong \mathrm{Cl}_{0,9}\otimes \mathrm{Cl}_{1,1}\otimes \mathrm{Cl}_{1,1}\cong \mathrm{Cl}_{0,1}\otimes\mathbb{R}(16)\otimes\mathbb{R}(2)\otimes\mathbb{R}(2)\\ &\cong \mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}(64)\cong\mathbb{C}(64)\,.\end{aligned}$$

Exercise 3. Show that $\mathrm{Cl}_{3,2}\cong\mathbb{R}(4)\oplus\mathbb{R}(4)$.

By repeating the arguments of Example 1.14 in a more general setting we obtain:

Corollary 1.15. $\mathrm{Cl}_{p,q}$ *is a direct sum of two matrix algebras exactly when* $p-q\equiv 1 \pmod 4$.

We conclude this treatment by proving

Proposition 1.16. *We have the following algebra isomorphism* $\mathrm{Cl}_{p,q}\cong \mathrm{Cl}^0_{p,q+1}$.

PROOF. Let $\{e_1,\ldots,e_{p+q+1}\}$ denote an orthogonal basis for $\mathbb{R}^{p+q+1}$ which satisfies $\Phi(e_i)=1$ for $i=1,\ldots,p$ and $\Phi(e_j)=-1$ for $j=p+1,\ldots,p+q+1$. Assume the basis has been chosen so that $\{e_1,\ldots,e_{p+q}\}$ is a basis for $\mathbb{R}^{p+q}$. Now define a linear map $f:\mathbb{R}^{p+q}\longrightarrow \mathrm{Cl}^0_{p,q+1}$ by

$$f(e_i)=e_{p+q+1}e_i$$

for $i\leq p+q$. Like in the proof of Lemma 1.12 one checks that f satisfies $f(x)^2=\Phi(x)\cdot 1$ and thus factorizes to an algebra homomorphism $\widehat{f}:\mathrm{Cl}_{p,q}\longrightarrow \mathrm{Cl}^0_{p,q+1}$. By inspection, this is the desired isomorphism. □

For the rest of this section we will consider complex Clifford algebras. It turns out that complex Clifford algebras behave even nicer than their real counterparts. As we have already seen a complex Clifford algebra associated with a non-degenerate bilinear form is isomorphic to $\mathrm{Cl}(\Phi_n)$. The fact that there is only one index on Φ and not two as in the real case, indicates some sort of simplification.

But first we introduce a way of turning real vector spaces/algebras into complex ones:

Definition 1.17. By the *complexification* of a real vector space V we mean the real tensor product $V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. If If the vector space V carries a bilinear form ϕ, then the *complexification* of the bilinear form is $\varphi^{\mathbb{C}}(v \otimes \lambda, v' \otimes \lambda') := \lambda\lambda'\varphi(v, v')$. The *complexification* of the corresponding quadratic form Φ is then $\Phi^{\mathbb{C}}(v \otimes \lambda) = \lambda^2\Phi(v)$.

In the same way we define the *complexification* of a real algebra A by $A^{\mathbb{C}} := A \otimes_{\mathbb{R}} \mathbb{C}$ carrying the product

$$(a \otimes \lambda)(a' \otimes \lambda') = (aa') \otimes (\lambda\lambda').$$

Assume (V, φ) to be a real vector space with φ a non-degenerate bilinear form. Then $\varphi^{\mathbb{C}}$ is a non-degenerate bilinear form on $V^{\mathbb{C}}$ for assume $v_0 \otimes \lambda_0$ to satisfy $\varphi^{\mathbb{C}}(v_0 \otimes \lambda_0, v \otimes \lambda) = \lambda_0\lambda\varphi(v_0, v) \neq 0$ for all $v \otimes \lambda$. Then $\lambda_0 \neq 0$ and $\varphi(v_0, v) \neq 0$ for all v and by non-degeneracy of φ this implies that $v_0 \neq 0$. Thus $v_0 \otimes \lambda_0 \neq 0$. In particular, if $p + q = n$ we have that $\varphi^{\mathbb{C}}_{p,q}$ is equivalent to φ_n and thus $\Phi^{\mathbb{C}}_{p,q}$ is equivalent to Φ_n.

Now, one can pose the question: is the complexification of a real Clifford algebra a complex Clifford algebra? By the following lemma the answer is yes.

Lemma 1.18. *For* $p + q = n$ *we have* $\mathrm{Cl}(\Phi^{\mathbb{C}}_{p,q}) \cong \mathrm{Cl}(\Phi_n) \cong \mathrm{Cl}^{\mathbb{C}}_{0,n}$.

Proof. The first isomorphism is due to the fact that the complexification of $\Phi_{p,q}$ is equivalent to Φ_n and thus the corresponding Clifford algebras are isomorphic.

To verify the second isomorphism we construct a linear map $\varphi : \mathbb{C}^n \longrightarrow \mathrm{Cl}^{\mathbb{C}}_{0,n}$ and show that it factorizes to an isomorphism $\widehat{\varphi} : \mathrm{Cl}(\Phi_n) \xrightarrow{\sim} \mathrm{Cl}^{\mathbb{C}}_{0,n}$. At first, we remark that $\mathbb{C}^n \cong \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C}$. We then define φ by $\varphi(v \otimes z) = i(v) \otimes z$ where $i : \mathbb{R}^n \longrightarrow \mathrm{Cl}_{0,n}$ denotes the usual embedding. Since

$$\begin{aligned}\varphi(v \otimes z)^2 &= (i(v) \otimes z)^2 = i(v)^2 \otimes z^2 \\ &= \Phi_{0,n}(v)z^2 \cdot 1 \otimes 1 = \Phi^{\mathbb{C}}_n(v \otimes z) \cdot 1 \otimes 1,\end{aligned}$$

φ factorizes uniquely to an algebra homomorphism $\widehat{\varphi} : \mathrm{Cl}(\Phi_n) \longrightarrow \mathrm{Cl}^{\mathbb{C}}_{0,n}$. Both algebras have complex dimension 2^n so it's enough to show that $\widehat{\varphi}$ is surjective. But $\widehat{\varphi}$ is surjective since $\widehat{\varphi}$ is an algebra homomorphism and φ maps onto a set of generators of $\mathrm{Cl}^{\mathbb{C}}_{0,n}$. Namely, the set of elements of the form $i(v)$ generate $\mathrm{Cl}_{0,n}$, and $1 \in \mathbb{C}$ generates $\mathbb{C}$. □

Thus, henceforth we will stick to the notation $\mathrm{Cl}^{\mathbb{C}}_n$ for the complex Clifford algebra over $\mathbb{C}^n$ equipped with any non-degenerate quadratic form, since the preceding lemma guarantees that they are all isomorphic.

This result, in combination with the classification results for real Clifford algebras, we get the complex version of the Cartan-Bott Theorem:

Theorem 1.19 (Cartan-Bott II). *We have the following*2-"*periodicity*": $\mathrm{Cl}^{\mathbb{C}}_{n+2} \cong \mathrm{Cl}^{\mathbb{C}}_n \otimes_{\mathbb{C}} \mathrm{Cl}^{\mathbb{C}}_2$, *and furthermore that* $\mathrm{Cl}^{\mathbb{C}}_2 \cong \mathbb{C}(2)$.

Proof. Invoking Lemma 1.12 and Lemma 1.18 we obtain the following chain of isomorphisms:

$$\begin{aligned}\mathrm{Cl}^{\mathbb{C}}_{n+2} &\cong \mathrm{Cl}_{0,n+2} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{Cl}_{n,0} \otimes_{\mathbb{R}} \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}_{0,2} \\ &\cong \mathrm{Cl}_{n,0} \otimes_{\mathbb{R}} (\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}) \otimes_{\mathbb{R}} \mathrm{Cl}_{0,2} \cong (\mathrm{Cl}_{n,0} \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}_{0,2}) \\ &\cong \mathrm{Cl}^{\mathbb{C}}_n \otimes_{\mathbb{C}} \mathrm{Cl}^{\mathbb{C}}_2 .\end{aligned}$$

For the second isomorphism, just recall that $\mathrm{Cl}_{0,2} \cong \mathbb{H}$ and $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}(2)$. □

Remembering that $\mathrm{Cl}_{1,0} \cong \mathbb{R} \oplus \mathbb{R}$ so that $\mathrm{Cl}^{\mathbb{C}}_1 \cong \mathbb{C} \oplus \mathbb{C}$, we obtain:

Corollary 1.20. *If* $n = 2k$, *then* $\mathrm{Cl}^{\mathbb{C}}_n \cong \mathbb{C}(2^k)$. *If* $n = 2k + 1$, *then* $\mathrm{Cl}^{\mathbb{C}}_n \cong \mathbb{C}(2^k) \oplus \mathbb{C}(2^k)$.

1.3 Representation Theory

In this section we will turn our attention to the representation theory of Clifford algebras. For the following definition, let us denote by $\mathbb{K}$ either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$:

Definition 1.21. Let A be an algebra over $\mathbb{K}$ and V a vector space over $\mathbb{K}$. A $\mathbb{K}$-*representation* of A is an algebra homomorphism $\rho : A \longrightarrow \mathrm{End}_{\mathbb{K}}(V)$. A subspace U of V is called *invariant under* ρ if $\rho(x)U \subseteq U$ for all $x \in A$. The representation ρ is called *irreducible* if the only invariant subspaces are $\{0\}$ and V.

By an *intertwiner* of two representations ρ and ρ' of A on V and V' we understand a linear map $f : V \longrightarrow V'$ satisfying

$$\rho'(x) \circ f = f \circ \rho(x)$$

for all $x \in A$. Two representations are called *equivalent* if there exists an intertwiner between them which is also an isomorphism of vector spaces.

Just as we had complexification of a real algebra, we can complexify complex representations: If $\rho : A \longrightarrow \mathrm{End}(V)$ is a representation of a real algebra on a complex vector space V, we define the *complexification* $\rho^{\mathbb{C}} : A^{\mathbb{C}} \longrightarrow \mathrm{End}(V)$ by $\rho^{\mathbb{C}}(x \otimes \lambda) = \lambda\rho(v)$. The notions of invariant subspaces and irreducibility of a representation and its complexified are closely related as the following proposition shows

Proposition 1.22. *Let A be a real unital algebra, and ρ an algebra representations on the complex V. Let $A^{\mathbb{C}}$ and $\rho^{\mathbb{C}}$ denote the associated complexifications. Then the following hold:*

1) *A subspace $W \subseteq V$ is ρ-invariant if and only if it is $\rho^{\mathbb{C}}$-invariant.*

2) *ρ is irreducible if and only if $\rho^{\mathbb{C}}$ is irreducible.*

Proof. 1) If W is ρ-invariant then

$$\rho_{\mathbb{C}}(x \otimes \lambda)W = \lambda\rho(x)W \subseteq W.$$

Conversely, if W is $\rho^{\mathbb{C}}$-invariant then for $x \in A$ we have $\rho(x)W = \rho^{\mathbb{C}}(x \otimes 1)W \subseteq W$. This proves 1).

2) Follows immediately from 1). □

A representation ρ of A on V gives V the structure of a left A-module simply by defining $a \cdot v := \rho(a)v$. This is compatible with addition in V.

The next proposition actually contains all the information we need to determine all the irreducible representations (up to equivalence) of the Clifford algebras:

Proposition 1.23. *The matrix algebra $\mathbb{K}(n)$ has only one irreducible $\mathbb{K}$-representation, namely the* defining representation *i.e. the natural isomorphism $\pi_n : \mathbb{K}(n) \xrightarrow{\sim} \mathrm{End}_{\mathbb{K}}(\mathbb{K}^n)$. The algebra $\mathbb{K}(n) \oplus \mathbb{K}(n)$ has exactly 2 inequivalent irreducible $\mathbb{K}$-representations, namely:*

$$\pi_n^0(x_1, x_2) := \pi_n(x_1) \qquad \textit{and} \qquad \pi_n^1(x_1, x_2) := \pi_n(x_2). \tag{1.6}$$

We saw in the previous section that $\mathrm{Cl}_{p,q}$ is of the form $\mathbb{K}(n) \oplus \mathbb{K}(n)$ iff $p - q \equiv 1 \pmod 4$ and a matrix algebra otherwise. This observation along side with the preceding proposition yields the number of irreducible representations of the real Clifford algebras.

But there is a slight problem here, in that the real Clifford algebra is not always a real matrix algebra or a sum of real matrix algebras. Thus the irreducible representations of Proposition 1.23 need not be real! For instance $\mathrm{Cl}_{1,4} \cong \mathbb{H}(2) \oplus \mathbb{H}(2)$, and Proposition 1.23 gives us two irreducible $\mathbb{H}$-representations over $\mathbb{H}^2$! But fortunately we can always turn a complex or quaternionic representation into a real representation if we just remember to adjust the dimension.[2] Without going further into details with keeping track of the dimensions we have shown:

Theorem 1.24. *Consider the real Clifford algebra $\mathrm{Cl}_{p,q}$. If $p - q \equiv 1 \pmod 4$ $\mathrm{Cl}_{p,q}$ has up to equivalence two real irreducible representations and up to equivalence exactly one real irreducible representation otherwise.*

In particular if $n \equiv 1 \pmod 4$ $\mathrm{Cl}_{0,n}$ has two irreducible representations, ρ_n^0 and ρ_n^1, and otherwise only one ρ_n. If $n \equiv 1 \pmod 4$ we define $\rho_n := \rho_n^0$ such that to each real Clifford algebra $\mathrm{Cl}_{0,n}$ we associate a real irreducible representation ρ_n called the *real spin representation*. The elements of the corresponding vector spaces are called *spinors*.

The similar situation for complex Clifford algebras is simpler due to the fact that each complex Clifford algebra decomposes into *complex* matrix algebras (cf. Corollary 1.20). Thus all irreducible representations are complex. This will make it a lot easier to keep track of the dimensions.

Theorem 1.25. *Consider the complex Clifford algebra $\mathrm{Cl}_n^{\mathbb{C}}$. If $n = 2k$ we have (up to equivalence) exactly one irreducible complex representation κ_n on $\mathbb{C}^{2^k}$, namely the isomorphism $\mathrm{Cl}_{2k}^{\mathbb{C}} \xrightarrow{\sim} \mathrm{End}(\mathbb{C}^{2^k})$.*

If $n = 2k + 1$ we have (up to equivalence) exactly two irreducible representations κ_n^0 and κ_n^1 on $\mathbb{C}^{2^k}$.

For $n = 2k$ or $n = 2k + 1$ like above, define $\Delta_n := \mathbb{C}^{2^k}$. The elements of Δ_n are called *Dirac spinors* or *complex n-spinors*. The irreducible representations of $\mathrm{Cl}_n^{\mathbb{C}}$ are representations on Δ_n.

In the case where n is odd we want to single out κ_n^0 and define $\kappa_n := \kappa_n^0$ which is just the composition

$$\kappa_n = \kappa_n^0 : \mathrm{Cl}_n^{\mathbb{C}} \xrightarrow{\sim} \mathrm{End}_{\mathbb{C}}(\Delta_n) \oplus \mathrm{End}_{\mathbb{C}}(\Delta_n) \xrightarrow{\pi_1} \mathrm{End}_{\mathbb{C}}(\Delta_n)$$

of the isomorphism with the projection π_1 onto the first component. Hence, for each n we have an irreducible complex representation on Δ_n called the *complex spin representation*.

Finally, let's try to break up the action of the Clifford algebra into smaller pieces and see how they act on the spinors. This requires introduction of the so-called volume element. It is well-known that, $\Lambda^*(\mathbb{R}^n)$ has a unique *volume element* Ω given unambiguously, in any orthonormal basis $\{e_1, \dots, e_n\}$, by $\Omega = e_1 \wedge \cdots \wedge e_n$. Applying the quantization map to this yields an element $\omega := Q(\Omega) \in \mathrm{Cl}_{0,n}$, also called the *volume element*, given by $\omega = e_1 \cdots e_n$. For the complex Clifford algebra $\mathrm{Cl}_n^{\mathbb{C}}$ we define the volume element by

$$\omega_{\mathbb{C}} := i^{\lfloor \frac{n+1}{2} \rfloor} \omega.$$

In the case $n = 2k$ we note that $\omega^2 = (-1)^k$ and that ω commutes with every element of $\mathrm{Cl}_{0,2k}^0$, while ω anti-commutes $\mathrm{Cl}_{0,2k}^1$, for instance we see that

$$\omega e_1 = (e_1 \cdots e_{2k}) e_1 = (-1)^{2k-1} e_1^2 e_2 \cdots e_{2k} = -e_1 \omega.$$

[2] $\mathbb{C}^n$ is naturally isomorphic to $\mathbb{R}^{2n}$ via the isomorphism $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}^{2n}$, $(\lambda_1, \dots, \lambda_n) \longmapsto (\mathrm{Re}\,\lambda_1, \mathrm{Im}\,\lambda_1, \dots, \mathrm{Re}\,\lambda_n, \mathrm{Im}\,\lambda_n)$. If π is a complex representation of an algebra A, then we just define a real representation on $\mathbb{R}^{2n}$ by $\widetilde{\pi}(x)(v) = \varphi(\pi(x)\varphi^{-1}(v))$. It's easy to see that $\widetilde{\pi}$ is irreducible iff π is. Likewise with a quaternionic representation we exploit the natural isomorphism $\mathbb{H}^n \cong \mathbb{R}^{4n}$.

If $n = 2k + 1$ then ω commutes with everything.

We are only interested in the even case, so assume $n = 2k$, define $\omega_{\mathbb{C}} := i^k$ and consider the map

$$f := i^k \kappa_{2k}(\omega) : \Delta_{2k} \longrightarrow \Delta_{2k}.$$

From this we see that f commutes with $\kappa_{2k}(\xi)$:

$$\begin{aligned} f(\kappa_{2k}(\xi)\psi) &= i^k \kappa_{2k}(\omega)(\kappa_{2k}(\xi)\psi) = i^k \kappa_{2k}(\omega\xi)\psi = i^k \kappa_{2k}(\xi\omega)\psi \\ &= \kappa_{2k}(\xi) i^k \kappa_{2k}(\omega)\psi = \kappa_{2k}(\xi) f(\psi) \end{aligned}$$

for $\xi \in \mathrm{Cl}^0_{0,2k}$ and $\psi \in \Delta_{2k}$. Furthermore f is an involution:

$$f \circ f = i^{2k} \kappa_{2k}(\omega^2) = (-1)^k \kappa_{2k}((-1)^k) = (-1)^{2k} \,\mathrm{id}_{\Delta_{2k}} = \mathrm{id}_{\Delta_{2k}} .$$

Then it is well-known that f has the eigenvalues ± 1 and corresponding eigenspaces $\Delta^{\pm}_{2k}$ (of equal dimension) such that $\Delta_{2k} \cong \Delta^+_{2k} \oplus \Delta^-_{2k}$. The elements of $\Delta^{\pm}_{2k}$ are called *positive* and *negative Weyl spinors* or *even* and *odd chiral spinors*, respectively.

We can use the map f to induce another splitting of the complexified Clifford algebra (in even dimension!), for left multiplication by $\omega_{\mathbb{C}}$ on $\mathrm{Cl}^{\mathbb{C}}_{2k}$ is an involution, hence the algebra splits into eigenspaces

$$\mathrm{Cl}^{\mathbb{C}}_{2k} = (\mathrm{Cl}^{\mathbb{C}}_{2k})_+ \oplus (\mathrm{Cl}^{\mathbb{C}}_{2k})_-$$

where, in fact

$$(\mathrm{Cl}^{\mathbb{C}}_{2k})_{\pm} = \tfrac{1}{2}(1 \pm \omega_{\mathbb{C}}) \,\mathrm{Cl}^{\mathbb{C}}_{2k} .$$

We can combine this splitting with the splitting given by the involution α to obtain the spaces

$$(\mathrm{Cl}^{\mathbb{C}}_{2k})^0_{\pm} := \tfrac{1}{2}(1 \pm \omega_{\mathbb{C}})(\mathrm{Cl}^{\mathbb{C}}_{2k})^0 \qquad \text{and} \qquad (\mathrm{Cl}^{\mathbb{C}}_{2k})^1_{\pm} := \tfrac{1}{2}(1 \pm \omega_{\mathbb{C}})(\mathrm{Cl}^{\mathbb{C}}_{2k})^1.$$

For the next proposition we will identify $\mathrm{End}(\Delta^+_{2k})$ as the subspace of $\mathrm{End}(\Delta_{2k})$ consisting of maps $\Delta_{2k} \longrightarrow \Delta_{2k}$ which map Δ^+_{2k} to itself and which map Δ^-_{2k} to 0, and in a similar way we identify $\mathrm{End}(\Delta^-_{2k})$ and $\mathrm{Hom}(\Delta^{\pm}_{2k}, \Delta^{\mp}_{2k})$ as subspaces of $\mathrm{End}(\Delta_{2k})$.

Proposition 1.26. *The spin representation* $\kappa_{2k} : \mathrm{Cl}^{\mathbb{C}}_{2k} \xrightarrow{\sim} \mathrm{End}(\Delta_{2k})$ *restricts to the following isomorphisms*

$$\begin{aligned} (\mathrm{Cl}^{\mathbb{C}}_{2k})^0_+ &\cong \mathrm{End}(\Delta^+_{2k}), & (\mathrm{Cl}^{\mathbb{C}}_{2k})^0_- &\cong \mathrm{End}(\Delta^-_{2k}) \\ (\mathrm{Cl}^{\mathbb{C}}_{2k})^1_+ &\cong \mathrm{Hom}(\Delta^-_{2k}, \Delta^+_{2k}), & (\mathrm{Cl}^{\mathbb{C}}_{2k})^1_- &\cong \mathrm{Hom}(\Delta^+_{2k}, \Delta^-_{2k}). \end{aligned}$$

PROOF. First assume $\psi \in \Delta^+_{2k}$, i.e. $f(\psi) = \psi$, then for $\xi \in (\mathrm{Cl}^{\mathbb{C}}_{2k})^0_+$:

$$\begin{aligned} f(\kappa_{2k}(\xi)\psi) &= \kappa_{2k}(\omega_{\mathbb{C}}\xi)\psi = \kappa_{2k}(\xi\omega_{\mathbb{C}})\psi \\ &= \kappa_{2k}(\xi) f(\psi) = \kappa_{2k}(\xi)\psi, \end{aligned}$$

i.e. $\kappa_{2k}(\xi)\psi \in \Delta^+_{2k}$. For this result we only used that $(\xi \in \mathrm{Cl}^{\mathbb{C}}_{2k})^0$, but to show that $\kappa_{2k}(\xi)$ is 0 on Δ^-_{2k} we need that $\xi = \frac{1}{2}(1 + \omega_{\mathbb{C}})\overline{\xi}$. Let $\psi \in \Delta^-_{2k}$, i.e. $\kappa_{2k}(1 + \omega_{\mathbb{C}})\psi = 0$, then

$$\kappa_{2k}(\xi)\psi = \tfrac{1}{2}\kappa_{2k}((1 + \omega_{\mathbb{C}})\overline{\xi})\psi = \tfrac{1}{2}\kappa_{2k}(\overline{\xi}(1 + \omega_{\mathbb{C}})) = 0.$$

This shows that κ_{2k} maps $(\mathrm{Cl}^{\mathbb{C}}_{2k})^0_+$ into $\mathrm{End}(\Delta^+_{2k})$. The reasoning is the same in the other 4 cases. But then, since κ_{2k} is an isomorphism, the restricted maps must be isomorphisms as well, and this proves the proposition. □

Chapter 2

Spin Groups

2.1 The Clifford Group

Having introduced the Clifford algebra $\mathrm{Cl}(\Phi)$, we proceed to define its Clifford group $\Gamma(\Phi)$. The point of doing this is that the Clifford group has two particularly interesting subgroups, the pin and spin groups.

Let V be a finite-dimensional real vector space, and let Φ be a quadratic form on V.

Definition 2.1. Let $\mathrm{Cl}^*(\Phi)$ denote the multiplicative group of invertible elements of $\mathrm{Cl}(\Phi)$. The *Clifford group* (by some also called the *Lipschitz group*) of $\mathrm{Cl}(\Phi)$ is the group

$$\Gamma(\Phi) := \{x \in \mathrm{Cl}^*(\Phi) \mid \alpha(x)vx^{-1} \in V \quad \text{for all } \ v \in V\}.$$

One mechanically verifies that $\Gamma(\Phi)$ is truly a group. The group $\mathrm{Cl}^*(\Phi)$ is an open subgroup of $\mathrm{Cl}(\Phi)$, just as $\mathrm{Aut}(V)$ is an open subgroup of $\mathrm{End}(V)$ (at least when V is finite-dimensional). In the latter case the Lie algebra of $\mathrm{Aut}(V)$ is just $\mathrm{End}(V)$ with the commutator bracket. In the same way the Lie algebra $\mathfrak{cl}^*(\Phi)$ of the group $\mathrm{Cl}^*(\Phi)$ is just $\mathrm{Cl}(\Phi)$ with the commutator bracket.

It is very conspicuous from the definition that we are interested in a particular representation $\Lambda : \Gamma(\Phi) \longrightarrow \mathrm{Aut}(V)$, namely $\Gamma(\Phi) \ni x \longmapsto \Lambda_x$ where $\Lambda_x : V \longrightarrow V$ is given by $\Lambda_x(v) = \alpha(x)vx^{-1}$ and called the *twisted adjoint representation* (indeed, the form of Λ_x is reminiscent of the adjoint representation of a Lie group).

One reason for considering $\Lambda_x(v) = \alpha(x)vx^{-1}$ instead of $\mathrm{Ad}_x(v) = xvx^{-1}$ is that the twisted adjoint representation keeps track of an otherwise annoying sign. W.r.t. the bilinear form φ we define, for $x \in V$ with $\Phi(x) \neq 0$ the *reflection* s_x through the hyperplane orthogonal to x by

$$s_x(v) := v - 2\frac{\varphi(v,x)}{\Phi(x)}x.$$

We then have the following geometric interpretation of Λ_x which in addition to being a pretty fact, is crucial in the proof of Lemma 2.10.

Proposition 2.2. *For any $x \in V$ with $\Phi(x) \neq 0$ we have $x \in \Gamma(\Phi)$, and the map $\Lambda_x : V \longrightarrow V$ given by $\Lambda_x(v) = \alpha(x)vx^{-1}$ is the reflection through the hyperplane orthogonal to x.*

Proof. First, since $x^2 = \Phi(x) \cdot 1 \neq 0$, x is invertible in $\mathrm{Cl}(\Phi)$ with inverse

$x^{-1} = \frac{1}{\Phi(x)} x \in V$. Using this and Eq. (1.2) we see that

$$\begin{aligned}\Phi(x)\alpha(x)vx^{-1} &= -xv(\Phi(x)x^{-1}) = xvx \\ &= -2\varphi(v,x)x + \Phi(x)v \in V\end{aligned}$$

i.e. we have $x \in \Gamma(\Phi)$ and

$$\Lambda_x v = v - 2\frac{\varphi(v,x)}{\Phi(x)}x = s_x(v) \in V.$$ □

The following proposition is extremely useful:

Proposition 2.3. *Let φ be a non-degenerate bilinear form on V. Then for the twisted adjoint representation* $\ker \Lambda = \mathbb{R}^* \cdot 1$ (*where* $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$).

PROOF. Since φ is non-degenerate, we can choose an orthonormal basis $e_1, \dots, e_n$ for V such that $\Phi(e_i) = \pm 1$ and $\varphi(e_i, e_j) = 0$ when $i \neq j$. Let $x \in \ker \Lambda$; this means $\alpha(x)v = vx$ for all $v \in V$. Because of the $\mathbb{Z}_2$-grading of $\mathrm{Cl}(\Phi)$ we can write $x = x_0 + x_1$ where x_0 and x_1 belong to $\mathrm{Cl}^0(\Phi)$ and $\mathrm{Cl}^1(\Phi)$, respectively. This gives us the equations

$$vx_0 = x_0 v \tag{2.1}$$

$$-vx_1 = x_1 v. \tag{2.2}$$

The terms x_0 and x_1 can be written as polynomials in $e_1, \dots, e_n$. By successively applying the identity $e_i e_j + e_j e_i = 2\varphi(e_i, e_j)$ we can express x_0 in the form $x_0 = a_0 + e_1 a_1$ where a_0 and a_1 are both polynomial expressions in $e_2, \dots, e_n$. Applying α to this equality shows that $a_0 \in \mathrm{Cl}^0(\Phi)$ and $a_1 \in \mathrm{Cl}^1(\Phi)$. Setting $v = e_1$ in Eq. (2.1) we get

$$e_1 a_0 + e_1^2 a_1 = a_0 e_1 + e_1 a_1 e_1 = e_1 a_0 - e_1^2 a_1$$

where the last equality follows from $a_0 \in \mathrm{Cl}^0(\Phi)$ and $a_1 \in \mathrm{Cl}^1(\Phi)$. We deduce $0 = e_1^2 a_1 = \Phi(e_1)a_1$; since $\Phi(e_1) \neq 0$, we have $a_1 = 0$. So, the polynomial expression for x_0 does not contain e_1. Proceeding inductively, we realize that x_0 does not contain any of the terms $e_1, \dots, e_n$ and so must have the form $x_0 = t \cdot 1$ where $t \in \mathbb{R}$.

We can apply an analogous argument to x_1 to conclude that neither does the polynomial expression for x_1 contain any of the terms $e_1, \dots, e_n$. However, $x_1 \in \mathrm{Cl}^1(\Phi)$, so $x_1 = 0$.

Thus, $x = x_0 + x_1 = t \cdot 1$. Since $x \neq 0$, we must have $t \in \mathbb{R}^*$. This shows $\ker \Lambda \subseteq \mathbb{R}^* \cdot 1$; the reverse inclusion is obvious. □

The assumption that Φ is non-degenerate is not redundant. Consider a real vector space V with $\dim V \geq 2$. If $\Phi \equiv 0$, then $\mathrm{Cl}(V, \Phi) = \Lambda^* V$, the exterior algebra of V. Consider the element $x = 1 + e_1 e_2$. Clearly, $x^{-1} = 1 - e_1 e_2$, and we have

$$\alpha(1 + e_1 e_2)\, v\, (1 + e_1 e_2)^{-1} = (1 + e_1 e_2)\, v\, (1 - e_1 e_2) = v,$$

i.e. $1 + e_1 e_2 \in \ker \Lambda$, yet $1 + e_1 e_2$ is not a scalar multiple of 1. Thus, since the following propositions use Proposition 2.3 in their proofs we will, from this point on, *always assume Φ to be non-degenerate*. This is not a severe restriction since practically all interesting Clifford algebras originate from non-degenerate bilinear forms.

We now introduce the important notions of *conjugation* and *norm*.

Definition 2.4. For any $x \in \mathrm{Cl}(\Phi)$, the *conjugate* of x is defined as $\overline{x} := t(\alpha(x))$. Moreover, the *norm* of x is defined as $N(x) := x\overline{x}$.

Note that $t \circ \alpha = \alpha \circ t$ (it clearly holds on $e_{i_1} \cdots e_{i_r}$, and by linearity on any element of $\mathrm{Cl}(\Phi)$). Also note that $\overline{\overline{x}} = x$. The term norm is justified in the following lemma, displaying some elementary properties of the norm

Lemma 2.5. *When Φ is non-degenerate, the norm possesses the following properties*:

1) *If $v \in V$ then $N(v) = \Phi(v) \cdot 1$, i.e. N is an extension of Φ to the algebra.*
2) *If $x \in \Gamma(\Phi)$, then $N(x) \in \mathbb{R}^* \cdot 1$.*
3) *When restricted to $\Gamma(\Phi)$, the norm $N : \Gamma(\Phi) \longrightarrow \mathbb{R}^* \cdot 1$ is a homomorphism. Moreover, $N(\alpha(x)) = N(x)$.*

Proof. 1) This is just a simple calculation

$$N(v) = v(t \circ \alpha(v)) = -v^2 = -\Phi(v) \cdot 1.$$

2) According to Proposition 2.3, it's enough to show that $N(x) \in \ker \Lambda$. By definition of the Clifford group, $x \in \Gamma(\Phi)$ implies

$$\alpha(x)vx^{-1} \in V \quad \text{for all } v \in V.$$

As $t|_V = \mathrm{id}_V$, we thus have

$$\alpha(x)vx^{-1} = t(\alpha(x)vx^{-1}) = t(x^{-1})vt(\alpha(x)).$$

Isolating the v on the right-hand-side we get, using $t \circ \alpha = \alpha \circ t$,

$$v = t(x)\alpha(x)v(t(\alpha(x))x)^{-1} = \alpha(\overline{x}x)v(\overline{x}x)^{-1},$$

i.e. $\overline{x}x \in \ker \Lambda$. But then $x\overline{x} = \overline{\overline{x}}\,\overline{x} \in \ker \Lambda$.

3) Two simple calculations:

$$N(xy) = xy\overline{y}\,\overline{x} = xN(y)\overline{x} = x\overline{x}N(y) = N(x)N(y)$$

and, since $\overline{\alpha(x)} = \alpha(\overline{x})$, (following from $t \circ \alpha = \alpha \circ t$)

$$N(\alpha(x)) = \alpha(x)\alpha(\overline{x}) = \alpha(x\overline{x}) = \alpha(N(x)) = N(x).$$ □

Example 2.6. Let's calculate the conjugate and the norm on the two model Clifford algebras. First $\mathrm{Cl}_{0,1} \cong \mathbb{C}$. Recall that $\mathbb{R}$ sits inside $\mathbb{C}$ as the imaginary line and that α is the involution satisfying $\alpha(1) = 1$ and $\alpha(\lambda) = -\lambda$ for $\lambda \in \mathbb{R}$. Thus α is just conjugation: $\alpha(z) = \overline{z}$. Since t is just the identity on $\mathrm{Cl}_{0,1}$ it follows that conjugation in the Clifford sense is just usual conjugation

$$t(\alpha(z)) = \overline{z}.$$

Therefore the norm becomes $N(z) = z\overline{z} = |z|^2$, i.e. the square of the usual norm.

Exercise 4. Carry out a similar calculation for $\mathrm{Cl}_{0,2} \cong \mathbb{H}$.

For the next proposition recall the definition of the orthogonal group $\mathrm{O}(\Phi)$ (when Φ is non-degenerate!) as the endomorphisms $f : V \longrightarrow V$ satisfying $\Phi(f(v)) = \Phi(v)$.

Proposition 2.7. *For any $x \in \Gamma(\Phi)$, the map Λ_x is an orthogonal transformation of V. That is, $\Lambda(\Gamma(\Phi)) \subseteq \mathrm{O}(\Phi)$.*

Proof. Let $x \in \Gamma(\Phi)$ and use the fact that N is a homomorphism:

$$N(\Lambda_x v) = N(\alpha(x)vx^{-1}) = N(\alpha(x))N(v)N(x^{-1}) = N(x)N(v)N(x)^{-1} = N(v),$$

Since $N(v) = -\Phi(v) \cdot 1$, this shows that Λ_x is Φ-preserving. Along with the linearity, this exactly shows that $\Lambda_x \in \mathrm{O}(\Phi)$. □

2.2 Pin and Spin Groups

Definition 2.8. The *pin group* is defined as

$$\mathrm{Pin}(\Phi) := \{x \in \Gamma(\Phi) \,|\, N(x) = \pm 1\}.$$

The *spin group* consists of those elements of $\mathrm{Pin}(\Phi)$ that are linear combinations of even-degree elements:

$$\mathrm{Spin}(\Phi) := \mathrm{Pin}(\Phi) \cap \mathrm{Cl}^0(\Phi).$$

It's not too difficult to verify that $\mathrm{Pin}(\Phi)$ and $\mathrm{Spin}(\Phi)$ really are groups. We will write $\mathrm{Pin}(p, q)$ and $\mathrm{Spin}(p, q)$ for the pin and spin groups associated with $\mathrm{Cl}_{p,q}$ and likewise $\mathrm{Pin}(n) := \mathrm{Pin}(0, n)$ and $\mathrm{Spin}(n) := \mathrm{Spin}(0, n)$ (not to be confused with the complex pin and spin groups $\mathrm{Pin}(\Phi_n)$ and $\mathrm{Spin}(\Phi_n)$ sitting inside $\mathrm{Cl}_n^{\mathbb{C}}$!). Recall that the algebra structure in $\mathrm{Cl}_{0,n}$ is that generated by the relations $v \cdot v = -\|v\|^2 \cdot 1$.

Since Φ is assumed to be non-degenerate, any real Clifford algebra is (up to isomorphism) of the form $\mathrm{Cl}_{p,q}$ and thus we can in fact always assume the real pin and spin groups to be of the form $\mathrm{Pin}(p, q)$ and $\mathrm{Spin}(p, q)$.

In addition to the complex pin and spin groups there are also "complexifications" of the real pin/spin groups, namely let (V, Φ) be a real quadratic vector space and define $\mathrm{Pin}^c(\Phi) \subseteq \mathrm{Cl}(\Phi)^{\mathbb{C}} = \mathrm{Cl}(\Phi) \otimes \mathbb{C}$ to be the subgroup of invertible elements in $\mathrm{Cl}(\Phi) \otimes \mathbb{C}$ generated by $\mathrm{Pin}(\Phi) \otimes 1$ and $1 \otimes \mathrm{U}(1)$. Similarly, $\mathrm{Spin}^c(\Phi)$ is defined as the subgroup generated by $\mathrm{Spin}(\Phi) \otimes 1$ and $1 \otimes \mathrm{U}(1)$. These are *not* to be confused with $\mathrm{Spin}(\Phi^{\mathbb{C}})$!

Proposition 2.9. *There is a Lie group isomorphism*

$$\mathrm{Spin}^c(\Phi) \xrightarrow{\sim} \mathrm{Spin}(\Phi) \times_{\pm 1} \mathrm{U}(1)$$

i.e. the $\mathrm{Spin}^c(\Phi)$ *is quotient of* $\mathrm{Spin}(\Phi) \times \mathrm{U}(1)$ *where we identify* $(1, 1)$ *and* $(-1, -1)$. *In particular* $\mathrm{Spin}^c(\Phi)$ *is connected if* $\mathrm{Spin}(\Phi)$ *is connected.*

PROOF. We consider the smooth map $\mathrm{Spin}(\Phi) \times \mathrm{U}(1) \longrightarrow \mathrm{Spin}^c(\Phi)$ given by $(g, z) \longmapsto gz$. This is a surjective Lie group homomorphism. The kernel consists of elements (g, z) such that $gz = 1$ i.e. $g = z^{-1} \in \mathrm{U}(1)$. But these elements are rare, there are only ± 1. Thus the kernel equals $\{\pm(1, 1)\}$, and the map above descends to an isomorphism. $\square$

The group $\mathrm{Spin}^c(\Phi)$ is important in the study of complex manifolds.

We have the following important lemma, to be used in the proof of Theorem 2.16.

Lemma 2.10. *The map* $\Lambda : \mathrm{Pin}(\Phi) \longrightarrow \mathrm{O}(\Phi)$ *is a surjective homomorphism with kernel* $\{-1, 1\}$. *Similarly, the map* $\Lambda : \mathrm{Spin}(\Phi) \longrightarrow \mathrm{SO}(\Phi)$ *is a surjective homomorphism with kernel* $\{-1, 1\}$.

PROOF. First, Proposition 2.7 guarantees $\Lambda(\mathrm{Pin}(\Phi)) \subseteq \mathrm{O}(\Phi)$. It is trivial to verify that Λ is a homomorphism. To prove that Λ is surjective we use the Cartan-Dieudonne Theorem[1] which states that any element T of $\mathrm{O}(\Phi)$ can be written as the composition $T = s_1 \circ \cdots \circ s_p$ of reflections where s_j is the reflection through the hyperplane orthogonal to some vector $u_j \in V$ But according to Proposition 2.2 $s_j = \Lambda_{u_j}$. Since $N(u_j) = -\Phi(u_j) \cdot 1$, replacing u_j by $\frac{u_j}{\sqrt{|\Phi(u_j)|}}$ (this doesn't change the reflection) leaves $N(u_j) = -1$ and thus $N(u_1 \cdots u_p) =$

[1]See for instance [GALLIER] ([5]), Theorem 7.2.1.

± 1, i.e. $u_1 \cdots u_p \in \mathrm{Pin}(\Phi)$. Hence $T = \Lambda_{u_1} \cdots \Lambda_{u_p} = \Lambda_{u_1 \cdots u_p}$ which proves that $\Lambda|_{\mathrm{Pin}(\Phi)}$ is surjective. The kernel is easily calculated:

$$\ker \Lambda|_{\mathrm{Pin}(\Phi)} = \ker \Lambda \cap \mathrm{Pin}(\Phi) = \{t \in \mathbb{R}^* \cdot 1 \,|\, N(t) = \pm 1\} = \{-1, 1\}.$$

To prove the analogous statement for $\mathrm{Spin}(\Phi)$ we need first to show that Λ maps $\mathrm{Spin}(\Phi)$ to $\mathrm{SO}(\Phi)$. Assume, for contradiction, that this is not the case, i.e. that an element $f \in \mathrm{O}(\Phi) \setminus \mathrm{SO}(\Phi)$ exists such that $\Lambda_x = f$ for some $x \in \mathrm{Spin}(\Phi)$. By the Cartan-Dieudonne Theorem f can be written as an odd number of reflections $f = s_1 \circ \cdots \circ s_{2k+1}$, and to each such reflection s_j corresponds a vector u_j so that $s_j = \Lambda_{u_j}$. In other words we have

$$\Lambda_x = \Lambda_{u_1 \cdots u_{2k+1}} \qquad \text{or} \qquad \mathrm{id} = \Lambda_{x^{-1} u_1 \cdots u_{2k+1}}.$$

By Proposition 2.3 $x^{-1} u_1 \cdots u_{2k+1} = \lambda \cdot 1$ for some $\lambda \in \mathbb{R}^*$, i.e. $x = \lambda^{-1} u_1 \cdots u_{2k+1}$ and this is a contradiction, since $u_1 \cdots u_{2k+1} \in \mathrm{Cl}^1(\Phi)$. Thus Λ maps $\mathrm{Spin}(\Phi)$ to $\mathrm{SO}(\Phi)$ and since $\{\pm 1\} \subseteq \mathrm{Spin}(\Phi)$ the kernel is still $\mathbb{Z}_2$. □

As a consequence of the proof of this lemma we note, that the set $\{v \in V \mid \Phi(v) = 1\}$ of unit vectors generate $\mathrm{Pin}(\Phi)$ as a subgroup of $\Gamma(\Phi)$, i.e. any element of $\mathrm{Pin}(\Phi)$ can be written as a product of unit vectors in V. Similarly elements of $\mathrm{Spin}(\Phi)$ can be written as a product of an even number of unit vectors in V.

Let's warm up by doing a few simple examples

Example 2.11. Let's calculate $\mathrm{Pin}(1)$ and $\mathrm{Spin}(1)$. They are subgroups of $\mathrm{Cl}_{0,1} \cong \mathbb{C}$ and the vector space, from which the Clifford algebra originates, is $\mathbb{R}$ which sits inside $\mathbb{C}$ as the imaginary line (cf. Example 1.5). In $\mathbb{R}$ we have just two unit vectors, namely ± 1. They sit in $\mathbb{C}$ as $\pm i$ and they generate $\mathrm{Pin}(1)$ which are thus seen to be isomorphic to $\mathbb{Z}_4$ (the fourth roots of unity). $\mathrm{Spin}(1)$ is then generated by products of two such unit vectors, i.e. $\mathrm{Spin}(1) = \mathbb{Z}_2$.

Exercise 5. Calculate $\mathrm{Pin}(2)$ and $\mathrm{Spin}(2)$.

At this point, it is probably not very clear why there should be anything particularly interesting about pin and spin groups. The explanation is that they are double coverings of $\mathrm{O}(\Phi)$ and $\mathrm{SO}(\Phi)$ (Theorem 2.16). When $n \geq 3$ then $\mathrm{Spin}(n)$ is even the *universal* double covering of $\mathrm{SO}(n)$ (Corollary 2.20). The following more tedious example can be thought of as a special case of this fact: it shows that $\mathrm{Spin}(3)$ is isomorphic to $\mathrm{SU}(2)$ which is known to be the universal double covering of $\mathrm{SO}(3)$.

Example 2.12. *Calculation of* $\mathrm{Spin}(3)$. Choose an orthonormal basis e_1, e_2, e_3 of $\mathbb{R}^3$. By Proposition 1.3, the list of elements

$$1,\ e_1,\ e_2,\ e_3,\ e_1e_2,\ e_1e_3,\ e_2e_3,\ e_1e_2e_3$$

forms a basis for the Clifford algebra $\mathrm{Cl}_{0,3}$. The group $\mathrm{Spin}(3)$ can be written

$$\mathrm{Spin}(3) = \{x \in \mathrm{Cl}^0_{0,3} \mid \forall v \in V : \alpha(x) v x^{-1} \in V \text{ and } N(x) = 1\},$$

as $N(x) = 1$ implies $x \in \mathrm{Cl}^*_{0,3}$ (explicitly, $x^{-1} = \overline{x}$). The first thing we want to show is that $x \in \mathrm{Spin}(3)$ if and only if $x \in \mathrm{Cl}^0_{0,3}$ and $N(x) = 1$. The only non-trivial statement to be proven is that the conditions $x \in \mathrm{Cl}^0_{0,3}$ and $N(x) = 1$ imply that $\alpha(x) v x^{-1} \in V$ for any $v \in V$.

Since $x \in \mathrm{Cl}^0_{0,3}$ and $v \in \mathrm{Cl}^1_{0,3}$, we have $x v x^{-1} \in \mathrm{Cl}^1_{0,3}$. Thereby,

$$x v x^{-1} = u + \lambda e_1 e_2 e_3$$

with $u \in V$ and $\lambda \in \mathbb{R}$. Moreover, observe that $\overline{v} = -v$ for all $v \in V$, and $\overline{x} = x^{-1}$ since $N(x) = 1$, so

$$\overline{xvx^{-1}} = \overline{x^{-1}}\,\overline{v}\,\overline{x} = -xvx^{-1}.$$

But $\overline{e_1e_2e_3} = -e_3e_2e_1 = e_1e_2e_3$, so

$$-u + \lambda e_1e_2e_3 = \overline{xvx^{-1}} = -xvx^{-1} = -u - \lambda e_1e_2e_3;$$

hence $\lambda = 0$, and so $xvx^{-1} \in V$. This is equivalent to $\alpha(x)vx^{-1} \in V$, as desired.

Knowing that $x \in \mathrm{Spin}(3)$ iff $x \in \mathrm{Cl}^0_{0,3}$ and $N(x) = 1$, we can characterize the elements x of $\mathrm{Spin}(3)$ in a very handy way. Namely, since $x \in \mathrm{Cl}^0_{3,0}$, x must have the form

$$x = a1 + be_1e_2 + ce_1e_3 + de_2e_3$$

where $a^2 + b^2 + c^2 + d^2 = N(x) = 1$. Thus, $\mathrm{Spin}(3)$ consists of all elements x of the form $x = a1 + be_1e_2 + ce_1e_3 + de_2e_3$ with $a^2 + b^2 + c^2 + d^2 = 1$. This allows us to establish an isomorphism $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$ like follows:

$$e_1e_2 \longmapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_1e_3 \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_2e_3 \longmapsto \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$

i.e. $x \longmapsto \begin{pmatrix} a+ib & -c-id \\ c-id & a-ib \end{pmatrix}$. Thus, $\mathrm{Spin}(3)$ is isomorphic to $\mathrm{SU}(2)$. One can use similar arguments to prove $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$ and $\mathrm{Spin}(3,1)_0 \cong \mathrm{SL}(2,\mathbb{C})$ (cf. [Lawson and Michelson], p. 56, Theorem 8.4). □

Example 2.13. Let's calculate some of the spin^c groups also. This is rather easy, given Proposition 2.9 and the calculations above.

First we show that $\mathrm{Spin}^c(3) \cong \mathrm{U}(2)$. Define the map

$$\mathrm{Spin}(3) \times \mathrm{U}(1) \cong \mathrm{SU}(2) \times \mathrm{U}(1) \longrightarrow \mathrm{U}(2)$$

by $(A, z) \longmapsto zA$. It is well-known that this map is a surjective Lie group homomorphism, and it is easily seen that the kernel is $(I_2, 1)$ (I_2 is just the 2×2 identity matrix). Thus a Lie group isomorphism is induced on the quotient, i.e. an isomorphism $\mathrm{Spin}^c(3) \xrightarrow{\sim} \mathrm{U}(2)$.

We can argue almost similarly to calculate $\mathrm{Spin}^c(4)$. For brevity, put

$$\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2)) := \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \mid A_1, A_2 \in \mathrm{U}(2),\ \det A_1 \det A_2 = 1 \right\}$$

Define a map

$$\mathrm{Spin}(4) \times \mathrm{U}(1) \cong \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1) \longrightarrow \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2))$$

$$(A_1, A_2, z) \longmapsto \begin{pmatrix} zA_1 & 0 \\ 0 & zA_2 \end{pmatrix}.$$

This map is again a surjective Lie group homomorphism, as above, and the kernel is $\pm(I_2, I_2, 1)$, and thus it induces an isomorphism $\mathrm{Spin}^c(4) \xrightarrow{\sim} \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2))$. □

Note that $\mathrm{Pin}(\Phi)$ and $\mathrm{Spin}(\Phi)$ are Lie groups. This is because the multiplicative group $\mathrm{Cl}^*(\Phi)$ of invertible elements is an open subset of $\mathrm{Cl}(\Phi)$ (this is a general result for algebras) which is a finite-dimensional linear space, hence a manifold. Thus, $\mathrm{Cl}^*(\Phi)$ is a manifold, and since multiplication and inversion are smooth maps, it is a Lie group. As $\mathrm{Pin}(\Phi)$ is a closed subgroup of $\mathrm{Cl}^*(\Phi)$ (since N is continuous) and $\mathrm{Spin}(\Phi)$ is a closed subgroup of $\mathrm{Pin}(\Phi)$ (since $\mathrm{Cl}^0(\Phi)$ is a closed subspace of $\mathrm{Cl}(\Phi)$), they are Lie groups.

2.3 Double Coverings

In this section we will prove that Pin(Φ) and Spin(Φ) are double coverings of O(Φ) and SO(Φ), respectively. This will allow us to prove furthermore that Spin(n) is the *universal* double covering of SO(n) which is our main result. We first recall the notion of a covering space in the general setting of two topological spaces:

Definition 2.14. Let Y and X be topological spaces. A *covering map* is a continuous and surjective map $p : Y \longrightarrow X$ with the property that for any $x \in X$ there is an open neighborhood U of x so that $p^{-1}(U)$ can be written as the disjoint union of open subsets V_α (called the *sheets*) with the restriction $p|_{V_\alpha} : V_\alpha \longrightarrow U$ a homeomorphism. We say that U is *evenly covered* by p and call Y a *covering space* of X. When all the fibers $p^{-1}(x)$ have the same finite cardinality n, we call p an n*-covering.* If Y is simply connected, the covering is called a *universal covering.*

When X is pathwise connected, all fibers $p^{-1}(x)$ will have the same cardinality. If the covering is the universal covering this is precisely the cardinality of the fundamental group $\pi_1(X)$.

Quite often, covering maps between groups arise from *group actions* $G \times Y \longrightarrow Y$. We now introduce the notion of an *even action* or *covering action* as it allows an elegant proof of Theorem 2.16. A group G is said to *act evenly* on the topological space Y if each point $y \in Y$ has a neighborhood U such that $g \cdot U \cap h \cdot U = \emptyset$ if $g \neq h$. As usual, Y/G denotes the orbit space under the action of G, equipped with the quotient topology.

Lemma 2.15. *Let G be a finite group acting evenly on a topological space Y. Then the canonical map $p : Y \longrightarrow Y/G$ is a $|G|$-covering.*

Proof. p is obviously continuous and surjective. Let $[y] = p(y) \in Y/G$ for some $y \in Y$. We shall produce a neighborhood of $[y]$ which is evenly covered by p. As G is acting evenly there exists a neighborhood $V \subseteq Y$ of y such that $g \cdot V \cap h \cdot V = \emptyset$ if $g \neq h$; define $U := p(V)$. U is open, for we have

$$p^{-1}(U) = \bigcup_{g \in G} g \cdot V,$$

where $g \cdot V$ are all open (as the map $Y \ni x \longmapsto g \cdot x$ is a homeomorphism for all $g \in G$). Consequently, $p^{-1}(U)$ is open and by definition of the quotient topology, U is open. Thus, we have that U is a neighborhood of $[y]$ and that $p^{-1}(U)$ is a disjoint union of sets homeomorphic to V, and that the number of sheets is equal to the order of the group G. The only thing we need to show is that $p|_V : V \longrightarrow U$ is a homeomorphism. The map is obviously surjective. To show injectivity assume that $p(x) = p(y)$ (for $x, y \in V$) that is there exists a $g \in G$ such that $y = g \cdot x$. But then $y \in g \cdot V$. From the fact that $V \cap g \cdot V = \emptyset$ if $g \neq e$, we deduce $g = e$ and consequently $x = y$.

Continuity of $p|_V$ is obvious, since it's a restriction of a continuous map. We now only need to show that $p|_V$ is a open map. But p is itself an open map, for let $O \subseteq Y$ be any open subset. By definition of the quotient topology, $p(O)$ is open if and only if $p^{-1}(p(O))$ is open. But

$$p^{-1}(p(O)) = \bigcup_{g \in G} g \cdot O$$

is open, being the union of open sets $g \cdot O$. Being a restriction of an open map p to an open set V, $p|_V$ is an open map, and $p|_V$ is therefore a homeomorphism. □

Theorem 2.16. *The map* $\Lambda : \mathrm{Pin}(\Phi) \longrightarrow \mathrm{O}(\Phi)$ *is a double covering. Moreover,* $\Lambda : \mathrm{Spin}(\Phi) \longrightarrow \mathrm{SO}(\Phi)$ *is a double covering.*

PROOF. By Lemma 2.10, the homomorphism $\Lambda : \mathrm{Pin}(\Phi) \longrightarrow \mathrm{O}(\Phi)$ is surjective and has the kernel $\{1, -1\} = \mathbb{Z}_2$. By standard results we have that $\mathrm{O}(\Phi)$ and $\mathrm{Pin}(\Phi)/\mathbb{Z}_2$ are isomorphic as Lie groups. We let $\mathbb{Z}_2$ act on $\mathrm{Pin}(\Phi)$ by multiplication which is obviously an even action. By the preceding Lemma 2.15, the quotient map $\mathrm{Pin}(\Phi) \longrightarrow \mathrm{Pin}(\Phi)/\mathbb{Z}_2$ is a double covering. Since $\Lambda : \mathrm{Pin}(\Phi) \longrightarrow \mathrm{O}(\Phi)$ can be identified with this map via the above isomorphism of Lie groups, Λ is a double covering. The proof for $\Lambda : \mathrm{Spin}(\Phi) \longrightarrow \mathrm{SO}(\Phi)$ is completely analogous. □

Corollary 2.17. *The groups* $\mathrm{Pin}(n)$ *and* $\mathrm{Spin}(n)$ *are compact groups.*

PROOF. Since $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are compact and $\mathrm{Pin}(n)$ and $\mathrm{Spin}(n)$ are finite coverings, the result follows from standard covering space theory. □

This does not hold in general, for instance $\mathrm{Spin}(3,1)_0 = \mathrm{SL}(2,\mathbb{C})$ which is definitely *not* compact. If the identity component in non-compact, the entire group must be non-compact.

In Chapter 1 we constructed explicit isomorphisms $\Psi : \mathrm{O}(p,q) \xrightarrow{\sim} \mathrm{O}(q,p)$ and $\Psi : \mathrm{SO}(p,q) \xrightarrow{\sim} \mathrm{SO}(q,p)$. A natural question would be if these isomorphisms lift to isomorphisms on the level of pin and spin groups? For the pin groups the answer is no, in general. But for the spin groups the answer is affirmative:

Proposition 2.18. *There exists a Lie group isomorphism* $\overline{\Psi} : \mathrm{Spin}(p,q) \xrightarrow{\sim} \mathrm{Spin}(q,p)$ *such that the following diagram commutes*

$$\begin{array}{ccc} \mathrm{Spin}(p,q) & \xrightarrow[\overline{\Psi}]{\sim} & \mathrm{Spin}(q,p) \\ {\scriptstyle\Lambda}\downarrow & & {\scriptstyle\Lambda}\downarrow \\ \mathrm{SO}(p,q) & \xrightarrow[\Psi]{\sim} & \mathrm{SO}(q,p) \end{array}$$

PROOF. Let φ be an anti-orthogonal linear map as mentioned in Chapter 1. By the same sort of argument as in the proof of Proposition 1.4, it is seen that φ extends to a map on the corresponding Clifford algebras. This is also denoted φ. Define $\overline{\Psi} : \mathrm{Spin}(p,q) \longrightarrow \mathrm{Spin}(q,p)$ by (recall that $\mathrm{Spin}(p,q)$ is generated by an even number of unit vectors)

$$\overline{\Psi}(v_1 \cdots v_{2k}) = (-1)^k \varphi(v_1) \cdots \varphi(v_{2k})$$

where $v_1, \ldots, v_{2k} \in \mathbb{R}^{p+q}$. This is easily seen to be a Lie group isomorphism. Thus we only need to check commutativity of the diagram. First we observe

$$\begin{aligned} \Psi(\Lambda_{v_i})v = \varphi \circ \Lambda_{v_i} \circ \varphi^{-1}(v) &= \varphi(-v_i \varphi^{-1}(v) v_i^{-1}) \\ &= -\varphi(v_i) v \varphi(v_i^{-1}) = -\varphi(v_i) v \varphi(v_i)^{-1} \\ &= \Lambda_{\varphi(v_i)} v. \end{aligned}$$

Then we immediately get

$$\begin{aligned} \Lambda(\overline{\Psi}(v_1 \cdots v_{2k})) &= \Lambda((-1)^k \varphi(v_1) \cdots \varphi(v_{2k})) \\ &= \Lambda_{\varphi(v_1)} \cdots \Lambda_{\varphi(v_{2k})} = \Psi(\Lambda_{v_1}) \cdots \Psi(\Lambda_{v_{2k}}) \\ &= \Psi(\Lambda_{v_1} \cdots \Lambda_{v_{2k}}) = \Psi(\Lambda(v_1 \cdots v_{2k})) \end{aligned}$$ □

Restricting attention to $\mathrm{Spin}(n)$ we can prove another delicious fact concerning its topology:

Theorem 2.19. Spin(n) *is path-connected when* $n \geq 2$ *and simply connected when* $n \geq 3$.

PROOF. We first remark that Spin(n) is pathwise connected. Consider the element $v_1 \cdots v_{2k}$ where $v_j \in \mathbb{R}^n$ and $\Phi_{0,n}(v_j) = 1$ and note that each v_j can be connected to v_1 by a continuous path running on the unit sphere $S^{n-1} = \{v \in \mathbb{R}^n \mid \Phi_{0,n}(v) = 1\}$ in $\mathbb{R}^n$. Thus there is a continuous path from $v_1 \cdots v_{2k}$ to $v_1 \cdots v_1 = 1$ and thereby Spin(n) is path-connected.

Next we need to show, that the fundamental group at each point of Spin(n) is the trivial group. Since Spin(n) is path-connected it suffices to show this for just a single point. So let $x_0 \in \mathrm{Spin}(n)$. By standard covering space theory $\rho_* : \pi_1(\mathrm{Spin}(n), x_0) \longrightarrow \pi_1(\mathrm{SO}(n), \rho(x_0))$ is injective, and the index of the subgroup $\rho_*(\pi_1(\mathrm{Spin}(n), x_0))$ in $\pi_1(\mathrm{SO}(n), \rho(x_0))$ is equal to the number of sheets of the covering ρ, which we have just showed was 2. For $n \geq 3$ the fundamental group $\pi_1(\mathrm{SO}(n), \rho(x_0))$ is $\mathbb{Z}_2$, so as $\rho_* : \pi_1(\mathrm{Spin}(n), x_0)$ has index 2, it must be the trivial subgroup. Since ρ_* was injective also $\pi_1(\mathrm{Spin}(n), x_0)$ is trivial, proving that Spin(n) is simply connected. □

Putting $p = 0$ in Theorem 2.16 and combining with Theorem 2.19 we get the main result:

Corollary 2.20. *For* $n \geq 3$*, the group* Spin(n) *is the universal* (*double*) *covering of* SO(n).

It's a classical fact from differential geometry that for connected Lie groups G and H, if $F : G \longrightarrow H$ is a covering map, then the induced map $F_* : \mathfrak{g} \longrightarrow \mathfrak{h}$ is an isomorphism.[2] Spin(n) is connected and simply connected (Theorem 2.19), and SO(n) is connected. By Theorem 2.16, the homomorphism $\Lambda : \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$ is a covering map, and thus we have an isomorphism

$$\Lambda_* : \mathfrak{spin}(n) \xrightarrow{\sim} \mathfrak{so}(n),$$

in particular, $\dim \mathfrak{spin}(n) = \dim \mathfrak{so}(n) = \frac{n(n-1)}{2}$.

Let's investigate this map a little further. Recall that the Lie algebra of $\mathrm{Cl}_{0,n}^*$ is just the Clifford algebra $\mathrm{Cl}_{0,n}$ itself with the commutator bracket. Spin(n) is a Lie subgroup of $\mathrm{Cl}_{0,n}^*$ and hence the Lie algebra $\mathfrak{spin}(n)$ is a Lie subalgebra of $\mathrm{Cl}_{0,n}$.

Proposition 2.21. *Let* $\{e_1, \ldots, e_n\}$ *be an orthonormal basis for* $\mathbb{R}^n$*, then* $\mathfrak{spin}(n) \subseteq \mathrm{Cl}_{0,n}$ *is spanned by elements of the form* $e_i e_j$ *where* $1 \leq i < j \leq n$*. Furthermore* Λ_* *maps* $e_i e_j$ *to the matrix* $2B_{ij} \in \mathfrak{so}(n)$ *where* B_{ij} *is the* $n \times n$*-matrix which is* -1 *in its* ij*'th entry and* 1 *in its* ji*'th entry.*

PROOF. Consider the curve

$$t \longmapsto (e_i \cos t + e_j \sin t)(-e_i \cos t + e_j \sin t) = \cos(2t) + \sin(2t) e_i e_j.$$

It is a curve in Spin(n) since it is the product of two unit vectors, and its value at $t = 0$ is the neutral element 1. Upon differentiating at $t = 0$ we get $2e_i e_j$, which is then an element of $T_1 \mathrm{Spin}(n) \cong \mathfrak{spin}(n)$. They are all linearly independent i $\mathrm{Cl}_{0,n}$ hence also in $\mathfrak{spin}(n)$, and there are exactly $\frac{n(n-1)}{2}$ of them, i.e. they span $\mathfrak{spin}(n)$.

Now, Λ is the restriction of the twisted adjoint representation to Spin(n), and since $\mathrm{Spin}(n) \subseteq \mathrm{Cl}_{0,n}^0$ we get $\Lambda(g)v = gvg^{-1}$ for $g \in \mathrm{Spin}(n)$ and $v \in \mathbb{R}^n$. As for the usual adjoint representation one can calculate

$$(\Lambda_* X)v = Xv - vX \tag{2.3}$$

[2]See for instance [WARNER], Proposition 3.26.

in particular we get

$$\Lambda_*(e_ie_j)e_k = e_ie_je_k - e_ke_ie_j = \begin{cases} 0, & k \neq i,j \\ 2e_j, & k = i \\ -2e_i, & k = j \end{cases}$$

We see that $\Lambda_*(e_ie_j)$ acts in the same way on $\mathbb{R}^n$ as the matrix $2B_{ij}$, thus we may identify $\Lambda_*(e_ie_j) = 2B_{ij}$. □

We can rephrase the first part of this proposition by saying that under the symbol map $\sigma : \mathrm{Cl}_{0,n} \xrightarrow{\sim} \Lambda^*\mathbb{R}^n$, the Lie algebra $\mathfrak{spin}(n)$ gets mapped to $\Lambda^2\mathbb{R}^n$.

We end the section by a short description of some covering properties of $\mathrm{Spin}^c(\Phi)$.

Proposition 2.22. *The map* $\Lambda^c : \mathrm{Spin}^c(\Phi) \longrightarrow \mathrm{SO}(\Phi)\times\mathrm{U}(1)$ *given by* $[g,z] \longmapsto (\Lambda(g), z^2)$ *is a double covering.*

Proof. It is easy to see that it is well-defined (since $\Lambda(-g) = \Lambda(g)$ and $(-z)^2 = z^2$). It is a covering map because it is the quotient map of the even $\mathbb{Z}_2$-action on $\mathrm{Spin}^c(\Phi)$ given by $(-1,[g,z]) \longmapsto [-g,z] = [g,-z]$. Thus it follows from Lemma 2.15. □

Since $\mathrm{SO}(n) \times \mathrm{U}(1)$ is compact, $\mathrm{Spin}^c(n)$ is also compact. Furthermore, for $n \geq 2$ $\mathrm{Spin}^c(n)$ is connected (cf. Proposition 2.9 and connectivity of $\mathrm{Spin}(n)$). Thus $\pi_1(\mathrm{Spin}^c(n))$ can be identified with a subgroup of $\pi_1(\mathrm{SO}(n) \times \mathrm{U}(1)) = \mathbb{Z}_2 \times \mathbb{Z}$ of index 2, i.e. $\pi_1(\mathrm{Spin}^c(n)) \cong \mathbb{Z}$.

2.4 Spin Group Representations

In this section we will treat the basics of the representation theory of spin groups. In this section we will restrict our attention to a particular complex representation of $\mathrm{Spin}(n)$, the spinor representation, defined as follows:

Definition 2.23. By the *complex spinor representation* κ_n of $\mathrm{Spin}(n)$ we understand the restriction $\kappa_n|_{\mathrm{Spin}(n)} \longrightarrow \mathrm{Aut}_{\mathbb{C}}(\Delta_n)$ of the complex spin representation κ_n of $\mathrm{Cl}_n^{\mathbb{C}}$ to $\mathrm{Spin}(n)$.

Similarly, we define the *spin^c-representation* κ_n^c of $\mathrm{Spin}^c(n)$ by restricting the spin representation to $\mathrm{Spin}^c(n) \subseteq \mathrm{Cl}_n^{\mathbb{C}}$.

We stress that we use the term *spin representation* for the irreducible Clifford algebra representations and *spinor representation* for the associated spin group representations.

Of course we can in a similar way define the real spinor representation of $\mathrm{Spin}(n)$ by restricting ρ_n to $\mathrm{Spin}(n)$, but we will not consider them here.

Theorem 2.24. *For each n the complex spinor representation κ_n is a faithful representation of* $\mathrm{Spin}(n)$.

Proof. If $n = 2k$ is even, then κ_n is a restriction of the isomorphism $\kappa_n : \mathrm{Cl}_n^{\mathbb{C}} \xrightarrow{\sim} \mathrm{End}_{\mathbb{C}}(\Delta_{2k})$ and therefore injective.

So let's assume that $n = 2k+1$. By definition we have $\Delta_{2k} = \Delta_{2k+1}$ and consequently $\mathrm{Aut}(\Delta_{2k}) = \mathrm{Aut}(\Delta_{2k+1})$. We can think of $\mathrm{Spin}(2k)$ as sitting inside $\mathrm{Spin}(2k+1)$.[3] Denoting the injection ι the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Spin}(2k) & \xrightarrow{\kappa_{2k}} & \mathrm{Aut}(\Delta_{2k}) \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathrm{Spin}(2k+1) & \xrightarrow[\kappa_{2k+1}]{} & \mathrm{Aut}(\Delta_{2k+1}) \end{array}$$

Now, put $H := \ker \kappa_{2k+1} \subseteq \mathrm{Spin}(2k+1)$. The goal is to verify $H = \{1\}$, but first we show that $H \cap \mathrm{Spin}(2k) = \{1\}$. The inclusion $\supseteq$ follows since 1 clearly sits in $\ker \kappa_{2k+1}$. Now, assume that $h \in H \cap \mathrm{Spin}(2k)$. In particular, $h \in H$ and $h = \iota(\widetilde{h})$ for some $\widetilde{h} \in \mathrm{Spin}(2k)$. Since h sits in H, $\kappa_{2k+1}(h) = \mathrm{id}_{\Delta_{2k+1}}$. From the commutativity of the diagram it follows that $\kappa_{2k}(\widetilde{h}) = \mathrm{id}_{\Delta_{2k+1}}$. But since κ_{2k} is injective, $\widetilde{h}$ must be 1, and so must h. This shows $H \cap \mathrm{Spin}(2k) \subseteq \{1\}$.

Identifying elements $A \in \mathrm{SO}(n)$ with elements in $\mathrm{SO}(2k+1)$ of the form $\mathrm{diag}(A, 1)$, we obtain $\mathrm{SO}(2k) \subseteq \mathrm{SO}(2k+1)$ like the spin groups. Recall that $\Lambda : \mathrm{Spin}(2k+1) \longrightarrow \mathrm{SO}(2k+1)$ is a surjective homomorphism. Thus $\Lambda(H)$ is a normal subgroup of $\mathrm{SO}(2k+1)$, since H as a kernel is normal in $\mathrm{Spin}(2k+1)$. Now we claim

$$\Lambda(H \cap \mathrm{Spin}(2k)) = \Lambda(H) \cap \mathrm{SO}(2k).$$

The inclusion "$\subseteq$" is obvious, and "$\supseteq$" follows from the surjectivity of Λ. Hence we have $\Lambda(H) \cap \mathrm{SO}(2k) = \{I\}$ (here, I denotes the identity matrix). We want to show that $\Lambda(H) = \{I\}$, so let $A \in \Lambda(H) \subseteq \mathrm{SO}(2k+1)$. Its characteristic polynomial is of odd degree, and it thus has a real root. As $A \in \mathrm{SO}(2k+1)$, all eigenvalues have modulus 1. Moreover, A preserves orientation, so this root must be 1. Denote the corresponding eigenvector by v_0 and choose an ordered, positively oriented orthonormal basis for $\mathbb{R}^{2k+1}$ containing v_0 as the last vector. If B denotes the change-of-basis matrix, then we have the block diagonal matrix $BAB^{-1} = \mathrm{diag}(\widetilde{A}, 1)$, where $\widetilde{A} \in \mathrm{SO}(2k)$ and 1 is the unit of $\mathbb{R}$. We can now identify $\widetilde{A}$ with BAB^{-1}. Hence, $BAB^{-1} \in \mathrm{SO}(2k)$, and since $\Lambda(H)$ was normal, we also have $BAB^{-1} \in \Lambda(H)$. All together we have $BAB^{-1} \in \Lambda(H) \cap \mathrm{SO}(2k) = \{I\}$ and so $A = I$.

Now we have $\Lambda(H) = \{I\}$. We have two possibilities: $H = \{1\}$ or $H = \{\pm 1\}$. But -1 cannot be in the kernel of the spinor representation (because it's not in the kernel of the spin representation, from which it came). Therefore, $H = \{1\}$, and κ_{2k+1} is injective. □

This theorem is not as innocent as it might look. It actually tells us that the spinor representations do *not* arise as lifts of $\mathrm{SO}(n)$-representations, since a lift of an $\mathrm{SO}(n)$-representation necessarily contains $\{\pm 1\}$ in its kernel.

We now want to decompose the spinor representations into irreducible representations. To this end we need:

Lemma 2.25. *For any complex vector space V the endomorphism algebra $\mathrm{End}(V)$ is a simple algebra, i.e. the only ideals are the trivial ones. In particular if $\dim W < \dim V$, then any homomorphism $\varphi : \mathrm{End}(V) \longrightarrow \mathrm{End}(W)$ is trivial: $\varphi \equiv 0$.*

Proof. Let n be the dimension of V and fix a basis for V. Then we can think of $\mathrm{End}(V)$ as the algebra of complex $n \times n$-matrices. Now let $\mathcal{I} \subseteq \mathrm{End}(V)$ be any non-zero ideal, and let $0 \neq a \in \mathcal{I}$. Then a has an eigenvalue $\lambda \neq 0$ (because $\mathbb{C}$ is algebraically closed). By a suitable basis transformation, given by

[3] If $\mathrm{Cl}_{0,2k}$ is generated by $\{e_1, \dots, e_{2k}\}$ and $\mathrm{Cl}_{0,2k+1}$ is generated by $\{e'_1, \dots, e'_{2k+1}\}$ then we have a linear injection $\iota : \mathrm{Cl}_{0,2k} \hookrightarrow \mathrm{Cl}_{0,2k+1}$ by defining $\iota(e_j) = e'_j$. This restricts to an injection $\iota : \mathrm{Spin}(2k) \hookrightarrow \mathrm{Spin}(2k+1)$.

a change-of-basis-matrix b, and subsequently multiplying $\mathrm{diag}(1/\lambda, 0, \dots, 0)$ on $b^{-1}ab$ from the left we obtain the matrix $\mathrm{diag}(1, 0 \dots, 0)$. It has been constructed from a just by multiplication, so it's in $\mathcal{I}$. By a similar argument we obtain $\mathrm{diag}(0, \dots, 0, 1, 0 \dots, 0) \in \mathcal{I}$. The sum of all these is just the identity matrix, which therefore is also in $\mathcal{I}$. Thus, $\mathcal{I} = \mathrm{End}(V)$.

If $\varphi : \mathrm{End}(V) \longrightarrow \mathrm{End}(W)$ is a homomorphism, $\ker \varphi$ is an ideal in $\mathrm{End}(V)$, thus $\ker \varphi = \{0\}$ or $\ker \varphi = \mathrm{End}(V)$. But since $\dim W < \dim V$ injectivity of φ is impossible. Therefore $\ker \varphi = \mathrm{End}(V)$ and $\varphi \equiv 0$. □

Decomposing κ_{2k+1} into irreducibles is easy:

Theorem 2.26. *The spinor representation κ_{2k+1} of $\mathrm{Spin}(2k+1)$ is irreducible.*

Proof. Let's assume that $\{0\} \neq W \subsetneq \Delta_{2k+1}$ is a $\mathrm{Spin}(2k+1)$-invariant subspace, i.e. for each $g \in \mathrm{Spin}(2k+1)$ $\kappa_{2k+1}(g)W \subseteq W$. Consider an element of the form $e_i e_j$, $i < j$ ($\{e_1, \dots, e_{2k+1}\}$ is an orthonormal basis for the vector space $\mathbb{R}^{2k+1}$ underlying $\mathrm{Cl}_{0,2k+1}$). It is an element of $\mathrm{Spin}(2k+1)$ and therefore $\kappa_{2k+1}(e_i e_j)W \subseteq W$. I.e. κ_{2k+1} is actually defined on all elements $\{e_{i_1} \cdots e_{i_m}\}$ where $i_1 < \cdots < i_m$ and m is even. On the other hand these elements constitute a complex basis for $(\mathrm{Cl}^{\mathbb{C}}_{2k+1})^0$. Hence we get an algebra representation $\varphi : (\mathrm{Cl}^{\mathbb{C}}_{2k+1})^0 \longrightarrow \mathrm{End}(W)$ by extending κ_{2k+1} linearly. But recall that $(\mathrm{Cl}^{\mathbb{C}}_{2k+1})^0 \cong \mathrm{Cl}^{\mathbb{C}}_{2k} \cong \mathrm{End}(\Delta_{2k})$ (Proposition 1.16) so that we get an algebra homomorphism

$$\varphi : \mathrm{End}(\Delta_{2k}) \longrightarrow \mathrm{End}(W).$$

W was a proper subspace of $\Delta_{2k+1} = \Delta_{2k}$, so $\dim W < \dim \Delta_{2k}$. Lemma 2.25 now guarantees that $\varphi \equiv 0$. Since φ is an extension of κ_{2k+1}, this should also be zero. As κ_{2k+1} is injective by Theorem 2.24 this is a contradiction, so W cannot be invariant. □

Example 2.27. Again we consider our favorite spin group $\mathrm{Spin}(3)$. What is the spinor representation of $\mathrm{Spin}(3)$? Recall that $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$ and that for each n $\mathrm{SU}(2)$ has exactly one irreducible representation π_n of dimension $n+1$ on the space of homogenous n-degree polynomials in two variables. The spinor representation κ_3 is a 2-dimensional irreducible representation, thus it must be equivalent to π_1. □

κ_{2k} is *not* an irreducible representation, but it can quite easily be decomposed into such. To do this recall the *volume element*, the unique element ω in $\mathrm{Cl}_{0,2k}$ given by $e_1 \cdots e_{2k}$. It commutes with everything in the even part of $\mathrm{Cl}_{0,2k}$ and anti-commutes with the odd part. The map $f = i^k \kappa_{2k}(\omega)$, which was an involution, gave rise to a splitting $\Delta_{2k} \cong \Delta^+_{2k} \oplus \Delta^-_{2k}$.

Lemma 2.28. *Δ^+_{2k} and Δ^-_{2k} are κ_{2k}-invariant subspaces. Thus, κ_{2k} induces representations $\kappa^\pm_{2k}$ on $\Delta^\pm_{2k}$ such that $\kappa_{2k} = \kappa^+_{2k} \oplus \kappa^-_{2k}$.*

Proof. We want to show $\kappa_{2k}(g)\Delta^\pm_{2k} \subseteq \Delta^\pm_{2k}$ for any $g \in \mathrm{Spin}(2k)$, so let ψ be a positive Weyl spinor. Then

$$f(\kappa_{2k}(g)\psi) = \kappa_{2k}(g)f(\psi) = \kappa_{2k}(g)\psi$$

so $\kappa_{2k}(g)\psi \in \Delta^+_{2k}$. Likewise if ψ is a negative Weyl spinor. □

Theorem 2.29. *$\kappa^\pm_{2k}$ are irreducible representations of $\mathrm{Spin}(2k)$.*

Proof. Like in the proof of Theorem 2.26 a κ_{2k}^+-invariant subspace $\{0\} \neq W \subsetneq \Delta_{2k}^+$ gives rise to a representation

$$\varphi : (\mathrm{Cl}_{2k}^{\mathbb{C}})^0 \longrightarrow \mathrm{End}(W).$$

Again, by Proposition 1.16 and Corollary 1.20:

$$(\mathrm{Cl}_{2k}^{\mathbb{C}})^0 \cong \mathrm{Cl}_{2k-1}^{\mathbb{C}} \cong \mathrm{End}(\Delta_{2k-1}) \oplus \mathrm{End}(\Delta_{2k-1}).$$

We get homomorphisms $\varphi_1, \varphi_2 : \mathrm{End}(\Delta_{2k-1}) \longrightarrow \mathrm{End}(W)$ simply by

$$\varphi_1(x) = \varphi(x,1) \qquad \text{and} \qquad \varphi_2(y) = \varphi(1,y).$$

By assumption $\dim W < \dim \Delta_{2k}^+ = \dim \Delta_{2k-1}$ and so by Lemma 2.25 $\varphi_1, \varphi_2 \equiv 0$. This means $\varphi(x,y) = \varphi((x,1)(1,y)) = \varphi_1(x)\varphi_2(y) = 0$, hence $\varphi \equiv 0$ and thus $\kappa_{2k}^+ \equiv 0$ which is a contradiction. □

The covering space results in the previous section yields the following result

Corollary 2.30. *Let $\kappa : \mathrm{Spin}(n) \longrightarrow \mathrm{Aut}(V)$ be a finite-dimensional representation of $\mathrm{Spin}(n)$ which is the restriction of an algebra representation $\rho : \mathrm{Cl}_{0,n} \longrightarrow \mathrm{End}(V)$ (e.g. the spinor representation κ_n). Then for the induced representation $\kappa_* : \mathfrak{spin}(n) \longrightarrow \mathrm{End}(V)$ it holds that*

$$\kappa_*(X)v = \rho(X)v$$

for $X \in \mathfrak{spin}(n) \subseteq \mathrm{Cl}_{0,n}$ and $v \in V$.

Proof. Note that ρ is a linear map, hence the induced representation of the restriction $\rho|_{\mathrm{Cl}_{0,n}^*}$ is just ρ itself (where we have identified $\mathfrak{cl}_{0,n}^* \cong \mathrm{Cl}_{0,n}$). The induced representation of $\kappa = \rho|_{\mathrm{Spin}(n)}$ is just the restriction of $\rho|_{\mathrm{Cl}_{0,n}^*}$, hence the formula follows. □

A close examination of the proofs above will reveal that nothing is used which does not hold for $\mathrm{Spin}^c(n)$ as well. We may therefore summarize the results above in the following statement about the spin^c-representations:

Theorem 2.31. *For each n the spin^c-representation κ_n^c is faithful. If n is odd, the representation is irreducible and if n is even it splits in a direct sum of two irreducible representations $\kappa_n^c = (\kappa_n^c)^+ \oplus (\kappa_n^c)^-$ where $(\kappa_n^c)^\pm$ are representations on the space $\Delta_n^\pm$.*

2.5 Spin Structures

In this section let us recall/introduce the notions of spin and spin^c structures and fix some notation.

Definition 2.32 (Spin Structure). Let $E \longrightarrow M$ be a real oriented Riemannian vector bundle of rank $n \geq 3$ and $\pi : P_{\mathrm{SO}}(E) \longrightarrow M$ its oriented orthonormal frame bundle. A *spin structure* on E is a "lift" of $P_{\mathrm{SO}}(E)$ to a principal $\mathrm{Spin}(n)$-bundle. More precisely, a spin structure is a pair $(P_{\mathrm{Spin}}(E), \Phi)$ of a principal $\mathrm{Spin}(n)$-bundle $\widetilde{\pi} : P_{\mathrm{Spin}}(E) \longrightarrow M$ and a bundle map $\Phi : P_{\mathrm{Spin}}(E) \longrightarrow P_{\mathrm{SO}}(E)$ such that

$$\Phi(p \cdot g) = \Phi(p) \cdot \Lambda(g)$$

where $\Lambda : \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$ is the double covering.

Often, the vector bundle in question will be the tangent bundle of some manifold M (provided of course that this manifold is oriented and has been given a metric). Then we will write $P_{\mathrm{SO}}(M)$ and $P_{\mathrm{Spin}}(M)$ for the oriented frame bundle resp. the spin bundle. If TM has a spin structure we say that M is a *spin manifold*.

Let's try and see this from a local perspective. It is well-known that a principal G-bundle over a manifold M can be (uniquely) described by the following data: a cover $(U_\alpha)_{\alpha\in A}$ of open sets and for each pair $(\alpha,\beta)\in A\times A$ a smooth map (the *transition function*) $g_{\alpha\beta}: U_{\alpha\beta}\longrightarrow G$ (where $U_{\alpha\beta}:=U_\alpha\cap U_\beta$) satisfying that $g_{\alpha\alpha}(x)=1$ for all $x\in U_\alpha$ and satisfying the *cocycle condition*,

$$g_{\alpha\beta}(x)g_{\beta\gamma}(x)g_{\gamma\alpha}(x)=1$$

for all triples $(\alpha,\beta,\gamma)\in A\times A\times A$ and for all $x\in U_{\alpha\beta\gamma}$. This collection of data is called a *gluing cocycle*. So if we consider our principal $\mathrm{SO}(n)$-bundle $P_{\mathrm{SO}}(E)$, then there exists a cover (U_α) and transition functions $g_{\alpha\beta}: U_{\alpha\beta}\longrightarrow \mathrm{SO}(n)$ satisfying the cocycle condition. The existence of a spin structure is then equivalent to the existence of lifts $\widetilde{g}_{\alpha\beta}: U_{\alpha\beta}\longrightarrow \mathrm{Spin}(n)$ over Λ satisfying the cocycle condition. It is well-known that such a set of lifts exists if and only if the second Stiefel-Whitney class is zero. In fact this follows more or less by definition of the second Stiefel-Whitney class (in the setting of Cech cohomology). In the affirmative case the possible spin structures on M are parametrized by elements of the first Cech cohomology group $\check{H}^1(M;\mathbb{Z}_2)$ which is, of course, isomorphic to the singular cohomology group $H^1(M;\mathbb{Z}_2)$ with coefficients in $\mathbb{Z}_2$. For instance S^n for $n\geq 3$ admits a spin structure, since $H^2(S^n;\mathbb{Z}_2)=0$, so the second Stiefel-Whitney class can be nothing but 0. Since $H^1(S^n;\mathbb{Z}_2)=0$ the spin structure must be unique.

The spin group $\mathrm{Spin}(n)$ has a distinguished complex representation, called the *spinor representation* $\kappa_n:\mathrm{Spin}(n)\longrightarrow \mathrm{Aut}(\Delta_n)$ where $\Delta_n=\mathbb{C}^{2^k}$ and where $k=\lfloor\frac{n}{2}\rfloor$, the integer part of $\frac{n}{2}$. This is a faithful representation (hence does *not* descend to a representation of $\mathrm{SO}(n)$) and when n is odd it is irreducible. For n even, it decomposes into a direct sum of two irreducible representations $\kappa_n=\kappa_n^+\oplus\kappa_n^-$ and the corresponding representation spaces are denoted Δ_n^+ and Δ_n^- respectively.

Given a principal $\mathrm{Spin}(n)$-bundle $\pi: Q\longrightarrow M$ (originating from a spin structure, say) we can form the associated complex vector bundle w.r.t. the spinor representation, namely $S:=Q\times_{\kappa_n}\Delta_n$ which is the quotient of the direct product $Q\times\Delta_n$ under the equivalence relation $(p,v)=(p\cdot g,\kappa_n(g^{-1})v)$ and with projection $q: S\longrightarrow M$ given by $q([p,v])=\pi(p)$. This is called the *spinor bundle* and sections of this bundle are called *spinors* or *Dirac spinors*. If n is even this bundle splits, in the same way as the representation, into two bundles $S=S^+\oplus S^-$ where in fact $S^\pm$ is the associated bundle $Q\times_{\kappa_n^\pm}\Delta_n^\pm$. Sections of these vector bundles are called positive resp. negative *Weyl spinors* or even resp. odd *chiral spinors*. We will discuss these vector bundles and some of their properties in more detail in the next section when we define the Dirac operator.

Next, recall how the Lie group $\mathrm{Spin}^c(n)$ is defined: It is the group inside $\mathrm{Cl}_{0,n}\otimes\mathbb{C}$ generated by $\mathrm{Spin}(n)\otimes 1$ and $1\otimes \mathrm{U}(1)$. Equivalently, $\mathrm{Spin}^c(n)=\mathrm{Spin}(n)\times_{\pm 1}\mathrm{U}(1)$, the quotient where we collapse the subgroup $\{\pm(1,1)\}$. Thus, in $\mathrm{Spin}(n)\times\mathrm{U}(1)$ we identify (g,z) with $(-g,-z)$. The equivalence class containing (g,z) will be denoted $[g,z]$. This is usually how we will view $\mathrm{Spin}^c(n)$.

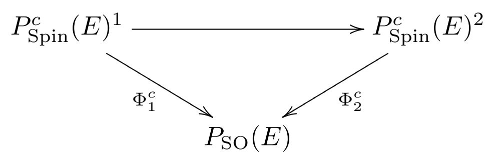
We can define a Lie group homomorphism $\rho^c:\mathrm{Spin}^c(n)\longrightarrow\mathrm{SO}(n)$ by $[g,z]\longmapsto\Lambda(g)$ (again, $\Lambda:\mathrm{Spin}(n)\longrightarrow\mathrm{SO}(n)$ is the double covering). This is well-defined, since $\Lambda(-g)=\Lambda(g)$, however, contrary to Λ, this is no longer a covering map, since its fibers are not discrete. Instead we may view this map as an n-dimensional representation of $\mathrm{Spin}^c(n)$.

Similarly, we define a Lie group homomorphism $\lambda : \mathrm{Spin}^c(n) \longrightarrow \mathrm{U}(1)$ by $[g, z] \longmapsto z^2$. Again, this is well-defined since $(-z)^2 = z^2$. We may view λ as a 1-dimensional unitary representation of $\mathrm{Spin}^c(n)$.

Finally, $\mathrm{Spin}^c(n)$ *is* a covering space, not of $\mathrm{SO}(n)$ as we saw above, but of $\mathrm{SO}(n) \times \mathrm{U}(1)$. We simply define the homomorphism $\Lambda^c : \mathrm{Spin}^c(n) \longrightarrow \mathrm{SO}(n) \times \mathrm{U}(1)$ by $\Lambda^c([g, z]) = (\Lambda(g), z^2)$. One can then check that this is a smooth double covering of $\mathrm{SO}(n) \times \mathrm{U}(1)$.

Definition 2.33 (Spin^c-structure). Let $E \longrightarrow M$ be a real oriented Riemannian vector bundle of rank $n \geq 3$ with oriented orthonormal frame bundle $\pi : P_{\mathrm{SO}}(E) \longrightarrow M$. A *spin^c-structure* on E is a principal $\mathrm{Spin}^c(n)$-bundle $\widetilde{\pi} : P^c_{\mathrm{Spin}}(E) \longrightarrow M$ and a bundle map $\Phi^c : P^c_{\mathrm{Spin}}(E) \longrightarrow P_{\mathrm{SO}}(E)$ such that $\Phi^c(p \cdot g) = \Phi^c(p) \cdot \rho^c(g)$ where $\rho^c : \mathrm{Spin}^c(n) \longrightarrow \mathrm{SO}(n)$ is the map defined above.

Two spin^c-structures $(P^c_{\mathrm{Spin}}(E)^1, \Phi^c_1)$ and $(P^c_{\mathrm{Spin}}(E)^2, \Phi^c_2)$ are said to be *isomorphic* if there exists a bundle isomorphism $P^c_{\mathrm{Spin}}(E)^1 \longrightarrow P^c_{\mathrm{Spin}}(E)^2$ making the following diagram commutative:



The set of isomorphism classes of spin^c-structures on E is denoted $\mathrm{Spin}^c(E)$. In the case where E happens to be the tangent bundle of an oriented Riemannian manifold a manifold equipped with a spin^c-structure is called a *spin^c-manifold*. The set of isomorphism classes of spin^c-structures on M is denoted $\mathrm{Spin}^c(M)$.

Assume that E is an oriented Riemannian vector bundle with a spin structure $\pi : P_{\mathrm{Spin}}(E) \longrightarrow M$ and bundle map $\Phi : P_{\mathrm{Spin}}(E) \longrightarrow P_{\mathrm{SO}}(E)$. Then E has a canonical spin^c-structure given in the following way: Define

$$P^c_{\mathrm{Spin}}(E) := P_{\mathrm{Spin}}(E) \times_{\pm 1} \mathrm{U}(1)$$

more precisely we take the product $P_{\mathrm{Spin}}(E) \times \mathrm{U}(1)$ and mod out by the equivalence relation $\sim$ given by $(p, z) \sim (p', z')$ iff $\pi(p) = \pi(p')$ and $(p', z') = \pm(p, z)$. The space $P^c_{\mathrm{Spin}}(E)$ can then be equipped with a right $\mathrm{Spin}^c(n)$ action

$$[p, z] \cdot [g, z'] = [p \cdot g, zz']$$

which in combination with the projection map $\widetilde{\pi} : P^c_{\mathrm{Spin}}(E) \longrightarrow M$ given by $\widetilde{\pi}([p, z]) = \pi(p)$ turns $P^c_{\mathrm{Spin}}(E)$ into a principal $\mathrm{Spin}^c(n)$-bundle. Finally, define $\Phi^c : P^c_{\mathrm{Spin}}(E) \longrightarrow P_{\mathrm{SO}}(E)$ by $\Phi^c([p, z]) = \Phi(p)$. We see that

$$\begin{aligned} \Phi^c([p, z] \cdot [g, z']) &= \Phi^c([p \cdot g, zz']) = \Phi(p \cdot g) = \Phi(p) \cdot \Lambda(g) \\ &= \Phi^c([p, z]) \cdot \rho^c([g, z']) \end{aligned}$$

and thus that $(P^c_{\mathrm{Spin}}(E), \Phi^c)$ is a spin^c-structure on E.

This shows that the concept of a spin^c-structure is more general than that of a spin structure. I will not go into a topological discussion of when spin^c-structures exist, except mentioning the following result

Theorem 2.34 (Hirzebruch-Hopf). *Any oriented Riemannian 4-manifold is a spin^c-manifold.*

A proof of this statement can be found in [**?**], Lemma 3.1.2.

Definition 2.35 (Determinant Line Bundle). Given a principal $\mathrm{Spin}^c(n)$-bundle $Q \longrightarrow M$ we can form the complex line bundle $L := Q \times_\lambda \mathbb{C}$ associated to the 1-dimensional representation λ defined above. This bundle is called the *determinant line bundle*, $\det(Q)$ for the $\mathrm{Spin}^c(n)$-bundle Q. If the $\mathrm{Spin}^c(n)$-bundle originates from a spin^c-structure σ, we will often write $\det(\sigma)$ for the determinant line bundle.

The determinant line bundle can be given a hermitian fiber metric by defining $\langle [p,w],[p,w'] \rangle := \langle w, w' \rangle_{\mathbb{C}} = w\overline{w}'$, we simply transfer the usual inner product on $\mathbb{C}$ to L. This is well-defined since

$$\langle [p\cdot g, \lambda(g)w], [p \cdot g, \lambda(g)w'] \rangle = \langle \lambda(g)w, \lambda(g)w' \rangle_{\mathbb{C}} = \langle w, w' \rangle_{\mathbb{C}}$$

since $\lambda(g) \in \mathrm{U}(1)$ and we may therefore form the unitary frame bundle $L^0 := P_{\mathrm{U}}(L)$, a principal $\mathrm{U}(1)$-bundle over M.

Lemma 2.36. *There is a bundle isomorphism $L^0 \cong P^c_{\mathrm{Spin}}(E) \times_\lambda \mathrm{U}(1)$.*

Proof. We define the bundle map $P^c_{\mathrm{Spin}}(E) \times_\lambda \mathrm{U}(1) \longrightarrow L^0$ in the following way: the element $[p,z] \in P^c_{\mathrm{Spin}}(E) \times_\lambda \mathrm{U}(1)$ should be mapped to the frame (i.e. the isometric isomorphism $\mathbb{C} \longrightarrow L_{\widetilde{\pi}(p)}$) given by $w \longmapsto [p, zw]$. Since $z \in \mathrm{U}(1)$, this is an isometric isomorphism, i.e. a frame at $\widetilde{\pi}(p)$. Since any frame at $\widetilde{\pi}(p)$ must be of the form $w \longmapsto w[p,z] = [p,zw]$ for a unit vector $[p,z] \in L_{\widetilde{\pi}(p)}$, i.e. for $z \in \mathrm{U}(1)$, we see that the bundle map is surjective. Its not hard to see that it is injective also and hence a bundle isomorphism. □

Let's put a local perspective on this as we did for the spin structures. Our starting point is again a principal $\mathrm{SO}(n)$-bundle which is given in terms of a cover (U_α) and transition functions $g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathrm{SO}(n)$. Picking a spin^c-structure (if it exists) is then equivalent to picking lifts $\widetilde{g}_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathrm{Spin}^c(n)$ along ρ^c such that the cocycle condition is satisfied. We see that $\widetilde{g}_{\alpha\beta}$ must be of the form $[h_{\alpha\beta}, z_{\alpha\beta}]$ where $h_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathrm{Spin}(n)$ and $z_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathrm{U}(1)$ are such that $\Lambda \circ h_{\alpha\beta} = g_{\alpha\beta}$ and such that the pair

$$(h_{\alpha\beta}(x)h_{\beta\gamma}(x)h_{\gamma\alpha}(x), z_{\alpha\beta}(x)z_{\beta\gamma}(x)z_{\gamma\alpha}(x))$$

is either $(-1,-1)$ or $(1,1)$, thus $(h_{\alpha\beta})$ and $(z_{\alpha\beta})$ need only satisfy the cocycle condition up to a sign. However if the bundle admits a spin structure, we may pick $h_{\alpha\beta}$ such that it does satisfy the cocycle condition, and then we can pick $z_{\alpha\beta} = 1$, this is the canonical spin^c-structure of a spin structure.

Given the families of maps $(h_{\alpha\beta})$ and $(z_{\alpha\beta})$ we can define $\lambda_{\alpha\beta} := z^2_{\alpha\beta}$ which maps $U_{\alpha\beta}$ into $\mathrm{U}(1)$. This family of maps satisfies the cocycle condition and thus represents a principal $\mathrm{U}(1)$-bundle. This bundle is nothing but the unitary frame bundle of the determinant line bundle. By the remarks above we then conclude that the determinant line bundle of the canonical spin^c-structure induced by a spin structure has trivial determinant line bundle.

Let M be a spin^c-manifold. How many different spin^c-structures does this manifold have? To shed some light on this question, let $\mathrm{Pic}^\infty(M)$ denote the set of complex Riemannian line bundles (i.e. bundles carrying a sesquilinear, conjugate symmetric, positive definit 2-form). This is a group under tensor product, known as the *Picard group*. This group is in 1-1 correspondence with the set of principal $\mathrm{U}(1)$-bundles over M - the map from $\mathrm{Pic}^\infty(M)$ to the set of principal $\mathrm{U}(1)$-bundles is simply given by forming the unitary frame bundle. Recall that the first Chern class is a bijection

$$c_1 : \mathrm{Pic}^\infty(M) \longrightarrow H^2(M;\mathbb{Z}). \tag{2.4}$$

We can let $\mathrm{Pic}^\infty(M)$ act on $\mathrm{Spin}^c(M)$ in the following way: If $\sigma \in \mathrm{Spin}^c(M)$ is a spin^c-structure given by the gluing cocycle $[h_{\alpha\beta}, z_{\alpha\beta}]$ and if $\mathcal{L} \in \mathrm{Pic}^\infty(M)$ is given by the gluing cocycle $(\zeta_{\alpha\beta})$ then we define the spin^c-structure $\sigma \otimes \mathcal{L}$ by the gluing cocycle $[h_{\alpha\beta}, z_{\alpha\beta}\zeta_{\alpha\beta}]$. Since $(z_{\alpha\beta}\zeta_{\alpha\beta})^2 = \lambda_{\alpha\beta}\zeta_{\alpha\beta}^2$ we see that

$$\det(\sigma \otimes \mathcal{L}) = \det(\sigma) \otimes \mathcal{L}^{\otimes 2}.$$

Proposition 2.37. *The action of* $\mathrm{Pic}^\infty(M)$ *on* $\mathrm{Spin}^c(M)$ *is free and transitive. Thus for a fixed spin^c-structure σ_0 the map $\mathcal{L} \longmapsto \sigma_0 \otimes \mathcal{L}$ is a bijection* $\mathrm{Pic}^\infty(M) \longrightarrow \mathrm{Spin}^c(M)$. *Composing with* (2.4) *we obtain a bijection*

$$\mathrm{Spin}^c(M) \xrightarrow{\sim} H^2(M;\mathbb{Z}).$$

Moreover, at most finitely many spin^c-structures have the same determinant line bundle.

PROOF. First, the action is free: if $\sigma \otimes \mathcal{L} = \sigma$, i.e. if $[h_{\alpha\beta}, z_{\alpha\beta}\zeta_{\alpha\beta}] = [h_{\alpha\beta}, z_{\alpha\beta}]$, then we must have $z_{\alpha\beta}\zeta_{\alpha\beta} = z_{\alpha\beta}$, i.e. $\zeta_{\alpha\beta} = 1$ and since this is the gluing cocycle for $\mathcal{L}$, this bundle must be trivial.

The action is transitive: Assume we have two spin^c-structures σ_1 and σ_2 given by gluing cocycles $[h^{(i)}_{\alpha\beta}, z^{(i)}_{\alpha\beta}]$. Since $\Lambda(h^{(1)}_{\alpha\beta}) = \Lambda(h^{(2)}_{\alpha\beta}) = g_{\alpha\beta}$ we must have $h^{(1)}_{\alpha\beta} = \pm h^{(2)}_{\alpha\beta}$. By a change of sign if necessary we can thus assume $h^{(1)}_{\alpha\beta} = h^{(2)}_{\alpha\beta}$. Now put $\zeta_{\alpha\beta} := z^{(2)}_{\alpha\beta}/z^{(1)}_{\alpha\beta}$. Clearly $\zeta_{\alpha\beta}$ maps into $\mathrm{U}(1)$ and we see that

$$[h^{(2)}_{\alpha\beta}, z^{(2)}_{\alpha\beta}] = \left[h^{(1)}_{\alpha\beta}, z^{(1)}_{\alpha\beta}\frac{z^{(2)}_{\alpha\beta}}{z^{(1)}_{\alpha\beta}}\right] = [h^{(1)}_{\alpha\beta}, z^{(1)}_{\alpha\beta}\zeta_{\alpha\beta}]$$

and hence that $\sigma_2 = \sigma_1 \otimes \mathcal{L}$.

At last, assume σ_1 and σ_2 are two spin^c-structures having the same determinant line bundle. By the first part of the proof, there exists a unique line bundle $\mathcal{L}$ such that $\sigma_2 = \sigma_1 \otimes \mathcal{L}$. If $\mathcal{L}$ is given by the gluing cocycle $(\zeta_{\alpha\beta})$, then the requirement $\det(\sigma_1) = \det(\sigma_2)$ implies that $\zeta^2_{\alpha\beta} = 1$, i.e. $\zeta_{\alpha\beta}$ maps into $\mathbb{Z}_2$. Thus $(\zeta_{\alpha\beta})$ determines an element of the Cech cohomology group $\check{H}^1(M;\mathbb{Z}_2)$ which is isomorphic to the singular cohomology group $H^1(M;\mathbb{Z}_2)$. Since this is finite, $\mathcal{L}$ belongs to a finite set, hence the conclusion. □

In the same way we defined the spinor bundles associated to a principal $\mathrm{Spin}(n)$-bundle, we can form spinor bundles associated to a $\mathrm{Spin}^c(n)$-bundle. $\mathrm{Spin}^c(n)$ sits inside $\mathrm{Cl}^{\mathbb{C}}_n$ and the fundamental representation of this algebra on the space Δ_n of Dirac spinors restricts to a group representation $\kappa^c_n : \mathrm{Spin}^c(n) \longrightarrow \mathrm{Aut}(\Delta_n)$. Thus if E is a real vector bundle carrying a spin^c-structure $P^c_{\mathrm{Spin}}(E)$ we define the *complex spinor bundle*:

$$S^c(E) := P^c_{\mathrm{Spin}}(E) \times_{\kappa^c_n} \Delta_n.$$

If the principal $\mathrm{Spin}^c(n)$-bundle is given by the gluing cocycle $[h_{\alpha\beta}, z_{\alpha\beta}]$ then $S^c(E)$ is given by the gluing cocycle $\kappa^c_n([h_{\alpha\beta}, z_{\alpha\beta}])$. If we change spin^c-structure from σ to $\sigma \otimes \mathcal{L}$ where $\mathcal{L}$ is a line bundle given by the gluing cocycle $(\zeta_{\alpha\beta})$ then the spinor bundle is given by the gluing cocycle

$$\kappa^c_n([h_{\alpha\beta}, z_{\alpha\beta}\zeta_{\alpha\beta}] = \kappa^c_n([h_{\alpha\beta}, z_{\alpha\beta}])\zeta_{\alpha\beta}$$

and hence the “new” spinor bundle is just $S^c(E) \otimes \mathcal{L}$.

As observed above $k := \dim_{\mathbb{C}} \Delta_n = \mathrm{rank}_{\mathbb{C}} S^c(E)$ is an even number. In the representation theory of spin groups it is shown that the two representations $\kappa_n^{\wedge k} = \kappa_n \wedge \cdots \wedge \kappa_n$ and $\lambda^{\otimes k/2}$ are equivalent. This means that the associated vector bundles are isomorphic, giving us the following relations between the spinor bundles and determinant line bundle L:

$$\Lambda^k S^c(E) \cong L^{\otimes k/2}. \tag{2.5}$$

The same result holds for $S^c(E)^\pm$:

$$\Lambda^{k/2}(S^c(E)^\pm) \cong L^{\otimes k/4}. \tag{2.6}$$

The formula (2.5) also explains the name determinant line bundle, since the top exterior product of a vector space or a vector bundle traditionally is called the *determinant*.

We also need to discuss the notion of *connections* on vector bundles and principal bundles. First we recall the definitions

Definition 2.38 (**Connection on a Vector Bundle**). Let E be a smooth $\mathbb{K}$-vector bundle ($\mathbb{K}$ being either $\mathbb{R}$ or $\mathbb{C}$) over M and let $\Omega^1(M,E)$ denote the E-valued 1-forms, i.e. sections of $T^*M \otimes_{\mathbb{K}} E$. By a *connection* on E we understand a $\mathbb{K}$-linear map

$$\nabla : \Gamma(E) \longrightarrow \Omega^1(M,E)$$

satisfying the "Leibniz rule"

$$\nabla(fs) = df \otimes s + f\nabla s \tag{2.7}$$

for $f \in C^\infty(M)$ and $s \in \Gamma(E)$. ∇s is called the *covariant derivative* of s.

If E is a Riemannian vector bundle, we say that a connection ∇ is *compatible with the metric* (or just *metric*) if

$$X\langle s, s'\rangle = \langle \nabla_X s, s'\rangle + \langle s, \nabla_X s'\rangle$$

for all vector fields X and all sections $s, s' \in \Gamma(E)$.

One can show that any vector bundle can be equipped with a connection, and that the space of connections is an affine space modeled on $\Omega^1(M, \mathrm{End}(E))$, i.e. any two connections differ by an element in $\Omega^1(M, \mathrm{End}(E))$.

Given a connection on TM we define its *torsion* by

$$\tau_\nabla(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y].$$

This is an anti-symmetric 2-tensor. The Fundamental Theorem of Riemannian Geometry states that on a Riemannian manifold, there exists a unique metric connection whose torsion tensor vanishes identically. This is called the *Levi-Civita connection* on M.

Let $G \hookrightarrow P \xrightarrow{\pi} M$ be a smooth principal G-bundle over M. For each $p \in P$ we have the so-called *vertical subspace* $V_pP \subseteq T_pP$, namely the kernel of the differential $d\pi_p : T_pP \longrightarrow T_{\pi(p)}M$. A connection on P is then loosely speaking a smooth choice of an algebraic complement over each point. Formally

Definition 2.39 (**Connection on a Principal Bundle**). For the principal G-bundle $G \hookrightarrow P \longrightarrow M$ a *connection* is a smooth tangent distribution HP on P such that for each $p \in P$ we have $T_pP = H_pP \oplus V_pP$ (H_pP is called the *horizontal subspace*) and such that $H_{p\cdot g}P = (d\sigma_g)_p H_pP$ where $\sigma_g : P \longrightarrow P$ is the map $p \longmapsto p \cdot g$.

There are several equivalent definitions of a connection on a principal G-bundle. One of them is given in terms of a connection 1-form: If $\mathfrak{g}$ denotes the Lie algebra of G, then a *connection 1-form* is a smooth $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$

satisfying the following two axioms: $(\sigma_g)^*\omega = \mathrm{Ad}_{g^{-1}} \circ \omega$ and $\omega_p(A^\sharp(p)) = A$ for all $A \in \mathfrak{g}$ where $A^\sharp$ is the *fundamental vector field* on P determined by A. Letting σ_p be the map $G \longrightarrow P$, $g \longmapsto p \cdot g$, then $A^\sharp$ is given by

$$A^\sharp(p) = d\sigma_p(A) = \frac{d}{dt}\bigg|_{t=0} (p \cdot \exp(tA)$$

so the second axiom could be phrased as $\omega_p \circ d\sigma_p = \mathrm{id}$.

A connection 1-form induces a connection on the principal bundle, simply by $H_pP = \ker \omega_p$, and vice versa.

Yet a third definition of a connection is via local gauge potentials satisfying a compatibility conditions on the overlap of their local domains.

Now assume that $\pi : P \longrightarrow M$ is a principal $\mathrm{SO}(n)$-bundle over M and assume it has a connection. Assume furthermore that the bundle lifts to a spin-bundle $\widetilde{\pi} : S(P) \longrightarrow M$. The map $\Phi : S(P) \longrightarrow P$ is in fact a double covering map: it is not hard to see that we can pick local trivializations of $S(P)$ and P on a common neighborhood $U \subseteq M$ such that Φ locally takes the form $(x, g) \longmapsto (x, \Lambda(g))$. Since Λ is a double covering map, $\Phi|_U$ is a double covering map, and since being a covering map is a local property, Φ itself is a double covering map. In particular it is a local diffeomorphism and its differential $d\Phi_p : T_pS(P) \longrightarrow T_{\Phi(p)}P$ is an isomorphism for all $p \in S(P)$. But then we can lift the connection on P to a connection on $S(P)$, simply by defining the horizontal subspace $H_pS(P) := (d\Phi_p)^{-1}(H_{\Phi(p)}P)$. Defined in this way from a local diffeomorphism, it is obviously a smooth tangent distribution. The additional requirement is also satisfied, as can be seen directly as follows:

$$\begin{aligned} H_{p\cdot g}S(P) &= (d\Phi_p)^{-1}(H_{\Phi(p\cdot g)}P) = (d\Phi_p)^{-1}(H_{\Phi(p)\cdot\Lambda(g)}P) \\ &= (d\Phi_p)^{-1}\left(d\sigma_{\Lambda(g)}H_{\Phi(p)}P\right) = d\widetilde{\sigma}_g(d\Phi_p)^{-1}(H_{\Phi(p)}P) \\ &= d\widetilde{\sigma}_g H_pS(P), \end{aligned}$$

the fourth identity follows from the requirement $\Phi \circ (\widetilde{\sigma}_g) = \sigma_{\Lambda(g)} \circ \Phi$. Thus, the connection on P lifts to a connection on $S(P)$. If P is the oriented orthonormal frame bundle and the connection is the Levi-Civita connection, the lifted connection is called the *spin connection*.

The situation for spin^c-structures is somewhat more complicated. Assume again $P_{\mathrm{SO}}(E)$ is the frame bundle of a vector bundle E and assume it carries a spin^c-structure as well as a connection. Since the map $\Phi^c : P^c_{\mathrm{Spin}}(E) \longrightarrow P_{\mathrm{SO}}(E)$ is *not* a covering map (because $\rho^c : \mathrm{Spin}^c(n) \longrightarrow \mathrm{SO}(n)$ is not a covering map), we cannot simply lift a connection from $P_{\mathrm{SO}}(E)$ to $P^c_{\mathrm{Spin}}(E)$ as we did before. To fix a connection on the $\mathrm{Spin}^c(n)$-bundle we need not only a connection on the $\mathrm{SO}(n)$-bundle but also a connection $\mathcal{A}$ on the frame bundle L^0 of the determinant line bundle L. Let's spend a few moments to describe how this works out. Consider the product bundle $\pi \times \pi^0 : P_{\mathrm{SO}}(E) \times L^0 \longrightarrow M \times M$. This is an $\mathrm{SO}(n) \times \mathrm{U}(1)$-principal bundle over $M \times M$. The connections on $P_{\mathrm{SO}}(E)$ and L^0 give a natural connection on $P_{\mathrm{SO}}(E) \times L^0$: namely choose in the tangent space $T_{(p,q)}(P_{\mathrm{SO}}(E) \times L^0) \cong T_pP_{\mathrm{SO}}(E) \times T_qL^0$ the horizontal subspace $H_pP_{\mathrm{SO}}(E) \times H_qL^0$. Then we get a decomposition

$$T_{(p,q)}(P_{\mathrm{SO}}(E) \times L^0) = (H_pP_{\mathrm{SO}}(E) \times H_qL^0) \oplus (V_pP_{\mathrm{SO}}(E) \times V_qL^0).$$

It shouldn't be hard to check that this is a connection on the product bundle.

Let $\Delta : M \longrightarrow M \times M$ be the diagonal map $x \longmapsto (x, x)$ and consider the pullback bundle $Q := \Delta^*(P_{\mathrm{SO}}(E) \times L^0)$ (i.e. the restriction to M viewed as the diagonal in $M \times M$). Q is what we will call the *fibered product* or the *spliced bundle* of P and L^0. Restrict the connection on $P_{\mathrm{SO}}(E) \times L^0$ to this new bundle (i.e. pull the connection back along Δ). Thus we have a principal $\mathrm{SO}(n) \times \mathrm{U}(1)$-bundle $Q \longrightarrow M$ carrying a connection determined by the connections on $P_{\mathrm{SO}}(E)$ and L^0. If $\pi_1 : Q \longrightarrow P_{\mathrm{SO}}(E)$ and $\pi_2 : Q \longrightarrow L^0$ denote the obvious projection maps, and if ω is the connection 1-form for the connection on P and $\mathcal{A}$ is the connection form for the connection on L^0 one can show that the connection on Q is given by the connection 1-form

$$\omega^{\mathcal{A}} := (\pi_1^* \omega) \oplus (\pi_2^* \mathcal{A}) \tag{2.8}$$

which takes values in the Lie algebra $\mathfrak{spin}^c(n) \cong \mathfrak{spin}(n) \oplus i\mathbb{R}$.

In order to lift this connection to the $\mathrm{Spin}^c(n)$-bundle we need a covering of Q. We know that $\Lambda^c : \mathrm{Spin}^c(n) \longrightarrow \mathrm{SO}(n) \times \mathrm{U}(1)$ is a double covering, and so inspired by this we seek a bundle map $P^c_{\mathrm{Spin}}(E) \longrightarrow Q$ which locally looks like Λ^c. Our candidate: $\Xi(p) := (\Phi^c(p), [p, 1])$. To see that it locally looks like Λ^c, pick trivializations Ψ^c and Ψ for $P^c_{\mathrm{Spin}}(E)$ resp. $P_{\mathrm{SO}}(E)$ over a common domain $U \subseteq M$ such that $\Psi \circ \Phi^c \circ (\Psi^c)^{-1}(x, [g, z]) = (x, \Lambda(g))$ (remember that $[g, z] \in \mathrm{Spin}^c(n)$ for $g \in \mathrm{Spin}(n)$). The trivialization Ψ^c for $P_{\mathrm{Spin}}(E)$ gives a trivialization $\overline{\Psi}$ of L^0 over U by (using the isomorphism from Lemma 2.36)

$$\overline{\Psi}([s(x), z]) = (x, z)$$

(where $s(x) := (\Psi^c)^{-1}(x, e)$ is the local section of $P^c_{\mathrm{Spin}}(E)$ corresponding to the trivialization Ψ and $e \in \mathrm{Spin}^c(n)$ is the neutral element), and further Ψ and $\overline{\Psi}$ give a trivialization of Q over U, denoted $\Psi \times \overline{\Psi}$ (with a slight abuse of notation, since the trivialization is only a "fiberwise" product). We want to show that

$$(\Psi \times \overline{\Psi}) \circ \Xi \circ (\Psi^c)^{-1}(x, [g, z]) = (x, \Lambda(g), z^2), \tag{2.9}$$

and to see this put $p := (\Psi^c)^{-1}(x, [g, z])$. This is mapped to $(\Phi^c(p), [p, 1])$ by Ξ and Ψ maps the first component to $(x, \Lambda(g))$ as it should. Note that

$$[p, 1] = [s(x) \cdot [g, z], 1] = [s(x), \lambda([g, z])1] = [s(x), z^2]$$

which by $\overline{\Psi}$ is mapped to (x, z^2), thus we have verified (2.9).

Now we have a double covering map $\Xi : P^c_{\mathrm{Spin}}(E) \longrightarrow Q$ and thus we can repeat what we did before, lifting the connection on Q to a connection on $P^c_{\mathrm{Spin}}(E)$. If E is the tangent bundle for a spin^c-manifold M, the connection is called the *spin^c-connection*.

2.6 The Dirac Operator

In this section we let M denote an oriented Riemannian manifold. No compactness condition is imposed unless specified.

Let E be a real oriented Riemannian vector bundle over M of rank n (often we will take it to be the tangent bundle). Thus we can construct its oriented frame bundle $P_{\mathrm{SO}}(E)$. We want to construct the so-called *Clifford bundle* over E, i.e. an algebra bundle over M whose fiber at x is isomorphic to the Clifford algebra $\mathrm{Cl}(E_x)$. The construction is accomplished as an associated bundle in the following way: Consider $\mathrm{Cl}_{0,n}$, the Clifford algebra over $\mathbb{R}^n$ with the usual negative definit inner product. We have a representation ρ of $\mathrm{SO}(n)$ on $\mathrm{Cl}_{0,n}$: any $A \in \mathrm{SO}(n)$ viewed as a linear map $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ preserves the inner product, and hence induces an algebra homomorphism $\widetilde{A} : \mathrm{Cl}_{0,n} \longrightarrow \mathrm{Cl}_{0,n}$, so the representation is given as $\rho(A) = \widetilde{A}$.

Definition 2.40 (Clifford Bundle). The *Clifford bundle* of E is the associated bundle

$$\mathrm{Cl}(E) = P_{\mathrm{SO}}(E) \times_\rho \mathrm{Cl}_{0,n} .$$

Elements in $\mathrm{Cl}(E)$ are equivalence classes $[p, \xi]$, where $p \in P_{\mathrm{SO}}(E)$ and $\xi \in \mathrm{Cl}_{0,n}$ and the equivalence relation on $P_{\mathrm{SO}}(E) \times \mathrm{Cl}_{0,n}$ is $(p, \xi) \sim (p \cdot A, \rho(A^{-1})\xi)$. The projection map is $\widetilde{\pi} : \mathrm{Cl}(E) \longrightarrow M$, $[p, \xi] \longmapsto \pi(p)$ where $\pi : P_{\mathrm{SO}}(E) \longrightarrow M$ is the projection in the frame bundle. The vector space structure on the fibers is given by

$$a[p, \xi] + b[p, \xi'] = [p, a\xi + b\xi']$$

(note, by transitivity of the right $\mathrm{SO}(n)$-action on each fiber in $P_{\mathrm{SO}}(E)$ we can always assume the p's to be equal). Similarly, the algebra structure is given by

$$[p, \xi] \cdot [p, \xi'] = [p, \xi\xi'],$$

and the identity element is $[p, 1]$. It is easy to check that these operations are well-defined. Since $\xi^2 = -\|\xi\|^2 \cdot 1$, we get

$$[p, \xi][p, \xi] = [p, \xi \cdot \xi] = [p, -\|\xi\|^2 \cdot 1] = -\|\xi\|^2 [p, 1],$$

thus each fiber is indeed a Clifford algebra of type $(0, n)$. Thus $\mathrm{Cl}(E)_x$ (the fiber in the Clifford bundle) is isomorphic to $\mathrm{Cl}(E_x)$ (the Clifford algebra of the vector space E_x).

Observe that we have $\mathbb{R}^n \subseteq \mathrm{Cl}_{0,n}$ and that $\mathbb{R}^n$ is a ρ-invariant subspace and that $\rho(A)|_{\mathbb{R}^n} = A$, so ρ restricted to this invariant subspace is just the defining representation of $\mathrm{SO}(n)$ on $\mathbb{R}^n$. We write it as id. But this means that we have the subbundle $P_{\mathrm{SO}}(E) \times_{\mathrm{id}} \mathbb{R}^n \cong E$ sitting inside $\mathrm{Cl}(E)$. Elements in $E \subseteq \mathrm{Cl}(E)$ are characterized by being of the form $[p, v]$ where $v \in \mathbb{R}^n$. In particular we may view $\Gamma(E)$ as sitting inside $\Gamma(\mathrm{Cl}(E))$.

For the purpose of studying spinor bundles, as we will do later in this section, we need another description of the Clifford bundle. Assume that the oriented Riemannian vector bundle E has a spin structure $\overline{\pi} : P_{\mathrm{Spin}}(E) \longrightarrow M$ with double covering bundle map $\Phi : P_{\mathrm{Spin}}(E) \longrightarrow P_{\mathrm{SO}}(E)$. Consider the representation $\mathrm{Ad} : \mathrm{Spin}(n) \longrightarrow \mathrm{Aut}(\mathrm{Cl}_{0,n})$ given by

$$\mathrm{Ad}(g)\xi = g\xi g^{-1}$$

(recall that $\mathrm{Spin}(n)$ sits inside $\mathrm{Cl}_{0,n}$, so multiplication makes sense), then the following diagram commutes (simply because $\mathrm{Ad}(g)$ is the unique extension of $\Lambda(g)$ to $\mathrm{Cl}_{0,n}$):

$$\begin{array}{ccc} \mathrm{Spin}(n) & \xrightarrow{\ \mathrm{Ad}\ } & \mathrm{Aut}(\mathrm{Cl}_{0,n}) \\ {\scriptstyle \Lambda}\big\downarrow & \nearrow_{\rho} & \\ \mathrm{SO}(n) & & \end{array}$$

From the principal $\mathrm{Spin}(n)$-bundle $P_{\mathrm{Spin}}(E)$ and the representation Ad, we can form the associated bundle $P_{\mathrm{Spin}}(E) \times_{\mathrm{Ad}} \mathrm{Cl}_{0,n}$.

Analogously, define $\mathrm{Ad}^c : \mathrm{Spin}^c(n) \longrightarrow \mathrm{Aut}(\mathrm{Cl}_{0,n})$ by $\mathrm{Ad}^c([g, z]) = \mathrm{Ad}(g)$. This is well-defined since $\mathrm{Ad}(-g) = \mathrm{Ad}(g)$.

Lemma 2.41. *The map* $\Psi : P_{\mathrm{Spin}}(E) \times_{\mathrm{Ad}} \mathrm{Cl}_{0,n} \longrightarrow \mathrm{Cl}(E) = P_{\mathrm{SO}}(E) \times_\rho \mathrm{Cl}_{0,n}$ *given by* $[p, \xi] \longmapsto [\Phi(p), \xi]$ *is a well-defined smooth algebra bundle isomorphism.*

Similarly, the map $\Psi^c : P^c_{\mathrm{Spin}}(E) \times_{\mathrm{Ad}^c} \mathrm{Cl}_{0,n} \longrightarrow \mathrm{Cl}(E)$ *given by* $[p, \xi] \longmapsto [\Phi^c(p), \xi]$ *is a well-defined smooth algebra bundle isomorphism.*

PROOF. It is well-defined, since

$$\begin{aligned}\Psi([p\cdot g^{-1},\mathrm{Ad}(g)\xi]) &= [\Phi(p\cdot g^{-1}),\mathrm{Ad}(g)\xi] = [\Phi(p)\cdot\Lambda(g)^{-1},\rho(\Lambda(g))\xi]\\ &= [\Phi(p),\xi] = \Psi([p,\xi]),\end{aligned}$$

the third identity is a consequence of equivariance of Ψ and of the commuting diagram just above. Restricted to the fiber over x, the map is an algebra homomorphism (we skip checking linearity):

$$\begin{aligned}\Psi_x([p,\xi][p,\xi']) &= \Psi_x([p,\xi\cdot\xi']) = [\Phi(p),\xi\cdot\xi'] = [\Phi(p),\xi]\cdot[\Phi(p),\xi']\\ &= \Psi_x([p,\xi])\Psi_x([p,\xi']).\end{aligned}$$

It is injective, for if $0=\Psi_x([p,\xi])=[\Phi(p),\xi]$, then ξ must be 0, hence $[p,\xi]=0$. Moreover Ψ_x is surjective, since Φ is. Thus Ψ is an algebra bundle isomorphism. The verification that Ψ^c is an algebra bundle isomorphism is completely similar and so we skip it. □

Definition 2.42 (Dirac Bundle). Let E be a real Riemannian vector bundle over M and ∇ a metric connection. A complex vector bundle S over M, is called a $\mathrm{Cl}(E)$-*module* or a left *Clifford module* if for each $x\in M$ there is a representation of the algebra $\mathrm{Cl}(E)_x$ on S_x.

A left $\mathrm{Cl}(E)$-module is called a *Dirac bundle* over E, provided it is equipped with a fiber metric $\langle\ ,\ \rangle$ and a compatible connection $\widetilde{\nabla}$ satisfying the two additional conditions:

1) Clifford multiplication is skew-adjoint, i.e. for each $x\in M$ and each $V_x\in E_x$ and $\psi_1,\psi_2\in S_x$:
$$\langle V_x\cdot\psi_1,\psi_2\rangle+\langle\psi_1,V_x\cdot\psi_2\rangle=0. \tag{2.10}$$

2) The connection on S is compatible with the connection on E in the following sense:
$$\widetilde{\nabla}_X(V\cdot\psi)=(\nabla_XV)\cdot\psi+V\cdot(\widetilde{\nabla}_X\psi) \tag{2.11}$$

for $X\in\mathfrak{X}(M)$, $V\in\Gamma(E)$ and $\psi\in\Gamma(S)$.

The single most important example of a Dirac bundle is the *spinor bundle*

$$S(E):=P_{\mathrm{Spin}}(E)\times_{\kappa_n}\Delta_n$$

as defined in the previous section. To show that it is a Dirac bundle we first equip it with an action of the Clifford bundle $\mathrm{Cl}(E)=P_{\mathrm{Spin}}(E)\times_{\mathrm{Ad}}\mathrm{Cl}_{0,n}$. Consider on the threefold product $P_{\mathrm{Spin}}(E)\times\mathrm{Cl}_{0,n}\times\Delta_n$ the equivalence relation $\sim$

$$(p,\xi,v)\sim(p\cdot g^{-1},\mathrm{Ad}(g)\xi,\kappa_n(g)v)$$

for any $g\in\mathrm{Spin}(n)$. Elements in the quotient space $P_{\mathrm{Spin}}(E)\times\mathrm{Cl}_{0,n}\times\Delta_n/\sim$ are denoted $[p,\xi,v]$. As in the proof of Lemma 2.41 one can show that the map

$$\mathrm{Cl}(E)\times S(E)\longrightarrow P_{\mathrm{Spin}}(E)\times\mathrm{Cl}_{0,n}\times\Delta_n/\sim$$

given by $([p,\xi],[p,v])\longmapsto[p,\xi,v]$ is a well-defined bundle isomorphism. This allows us to define the Clifford action in the following way: Define

$$\widetilde{\mu}:P_{\mathrm{Spin}}(E)\times\mathrm{Cl}_{0,n}\times\Delta_n\longrightarrow P_{\mathrm{Spin}}(E)\times\Delta_n$$

by $\widetilde{\mu}(q,\xi,v) := (q, \rho_n(\xi)v)$ (where $\rho_n : \mathrm{Cl}_{0,n} \longrightarrow \mathrm{Aut}(\Delta_n)$ is the *spin representation* of the Clifford algebra) and note that the following diagram is commutative

$$\begin{array}{ccc} P_{\mathrm{Spin}}(E) \times \mathrm{Cl}_{0,n} \times \Delta_n & \xrightarrow{\ \widetilde{\mu}\ } & P_{\mathrm{Spin}}(E) \times \Delta_n \\ \downarrow{\cdot g} & & \downarrow{\cdot g} \\ P_{\mathrm{Spin}}(E) \times \mathrm{Cl}_{0,n} \times \Delta_n & \xrightarrow[\ \widetilde{\mu}\ ]{} & P_{\mathrm{Spin}}(E) \times \Delta_n \end{array}$$

where the first vertical map is $(p,\xi,v) \longmapsto (p \cdot g^{-1}, \mathrm{Ad}(g)\xi, \kappa_n(g)v)$ and the second is $(p,v) \longmapsto (p\cdot g^{-1}, \kappa_n(g)v)$. Thus $\widetilde{\mu}$ induces a map $\mu : \mathrm{Cl}(E) \times S(E) \longrightarrow S(E)$ given explicitly by the formula

$$[p,\xi] \cdot [p,v] := \mu([p,\xi],[p,v]) = [p, \rho_n(\xi)v].$$

This is the desired Clifford action, turning $S(E)$ into a left $\mathrm{Cl}(E)$-module.

Next we want to give $S(E)$ a metric. Inside $\mathrm{Cl}_{0,n}$ we have the finite group

$$G_n := \{e_{i_1} \cdots e_{i_k} \mid 1 \le k \le n,\ 1 \le i_1 < \cdots < i_k \le n\}$$

(where $\{e_1, \ldots, e_n\}$ is some orthonormal basis for $\mathbb{R}^n$). Restricting the spin representation ρ_n of $\mathrm{Cl}_{0,n}$ to G_n gives a representation of G_n on Δ_n, also denoted ρ_n. By a well-known result from representation theory, there exists an inner product $\langle\ ,\ \rangle_{\Delta_n}$ on Δ_n relative to which ρ_n is a unitary representation, i.e.

$$\langle \rho_n(e_{i_1} \cdots e_{i_k})v, \rho_n(e_{i_1} \cdots e_{i_k})w \rangle_{\Delta_n} = \langle v, w \rangle_{\Delta_n}.$$

(To make the notation in the following less cumbersome, we will simply write the action of $\rho_n(\xi)$ on v as $\xi \cdot v$.) If $\xi = \sum_{i=1}^n a_i e_i$ is a unit vector in $\mathbb{R}^n \subseteq \mathrm{Cl}_{0,n}$ then $\rho_n(e)$ is a unitary operator as well: First we observe

$$\begin{aligned} \langle e_i \cdot v, e_j \cdot w \rangle_{\Delta_n} &= \langle e_j \cdot (e_i \cdot v), e_j^2 \cdot w \rangle_{\Delta_n} = -\langle (e_j e_i) \cdot v, w \rangle_{\Delta_n} \\ &= \langle (e_i e_j) \cdot v, w \rangle_{\Delta_n} = \langle (e_i^2 e_j) \cdot v, e_i \cdot w \rangle_{\Delta_n} \\ &= -\langle e_j \cdot v, e_i \cdot w \rangle_{\Delta_n}, \end{aligned}$$

and from this we get

$$\begin{aligned} \langle \xi \cdot v, \xi \cdot w \rangle_{\Delta_n} &= \Big\langle \sum_{i=1}^n a_i e_i \cdot v, \sum_{j=1}^n a_j e_j \cdot w \Big\rangle_{\Delta_n} \\ &= \sum_{i=1}^n a_i^2 \langle e_i \cdot v, e_i \cdot w \rangle_{\Delta_n} + \sum_{i \ne j} a_i a_j \langle e_i \cdot v, e_j \cdot w \rangle_{\Delta_n} \\ &= \sum_{i=1}^n a_i^2 \langle v, w \rangle_{\Delta_n} + \sum_{i<j} a_i a_j \big(\langle e_i \cdot v, e_j \cdot w \rangle_{\Delta_n} + \langle e_j \cdot v, e_i \cdot w \rangle_{\Delta_n} \big) \\ &= \langle v, w \rangle_{\Delta_n}. \end{aligned}$$

Since $\mathrm{Spin}(n)$ is generated by unit vectors, we see immediately that κ_n is a unitary representation w.r.t. this inner product.

Furthermore, for any $\xi \in \mathbb{R}^n$, unit vector or not, $\rho_n(\xi)$ is a skew-adjoint map:

$$\langle \xi \cdot v, w \rangle_{\Delta_n} = \langle \tfrac{\xi}{\|\xi\|} \cdot (\xi \cdot v), \tfrac{\xi}{\|\xi\|} \cdot w \rangle_{\Delta_n} = \frac{1}{\|\xi\|^2} \langle \xi^2 \cdot v, \xi \cdot w \rangle_{\Delta_n} = -\langle v, \xi \cdot w \rangle_{\Delta_n}.$$

Note that this implies that $\rho_n(\xi)$ is skew-adjoint, when ξ is in $\mathrm{Cl}^1_{0,n}$ (the odd part of $\mathrm{Cl}_{0,n}$) and that $\rho_n(\xi)$ is self-adjoint when $\xi \in \mathrm{Cl}^0_{0,n}$ (the even part of $\mathrm{Cl}_{0,n}$). In particular, $\kappa_n = \rho_n|_{\mathrm{Spin}(n)}$ is self-adjoint.

We can readily extend this inner product to a fiber metric on $S(E)$, simply by defining

$$\langle [p,v],[p,w]\rangle := \langle v,w\rangle_{\Delta_n}.$$

This is well-defined by unitarity of $\kappa_n(g)$:

$$\begin{aligned}\langle [p\cdot g^{-1},\kappa_n(g)v],[p\cdot g^{-1},\kappa_n(g)w]\rangle &= \langle \kappa_n(g)v,\kappa_n(g)w\rangle_{\Delta_n} = \langle v,w\rangle_{\Delta_n}\\ &= \langle [p,v],[p,w]\rangle.\end{aligned}$$

Checking condition 1 in the definition of a Dirac bundle is not hard: Let $V_x \in E_x \subseteq \mathrm{Cl}(E)_x$ and $\psi_1,\psi_2 \in S_x(E)$. We have presentations $V_x = [p,v]$ and $\psi_i = [p,w_i]$ where $v\in\mathbb{R}^n\subseteq \mathrm{Cl}_{0,n}$, $p\in P_{\mathrm{Spin}}(E)$ and $w_i\in\Delta_n$, and hence:

$$\begin{aligned}\langle V_x\cdot\psi_1,\psi_2\rangle &= \langle [p,v]\cdot[p,w_1],[p,w_2]\rangle = \langle [p,v\cdot w_1],[p,w_2]\rangle\\ &= \langle v\cdot w_1,w_2\rangle_{\Delta_n} = -\langle w_1, v\cdot w_2\rangle_{\Delta_n} = -\langle [p,w],[p,v]\cdot[p,w_2]\rangle\\ &= -\langle \psi_1, V_x\cdot\psi_2\rangle.\end{aligned}$$

In the previous section we equipped $S(E)$ with a connection, namely the lift of some connection on $P_{\mathrm{SO}}(E)$. If ω denotes the connection 1-form for the connection on $P_{\mathrm{SO}}(E)$, the connection 1-form of the lifted connection can be constructed as follows: $\widetilde{\omega} := (d\Lambda)^{-1}\circ\Phi^*\omega$. This is a $\mathfrak{spin}(n)$-valued 1-form, and to see that it is a connection form, we only have to check that the two axioms are satisfied. The first one:

$$\begin{aligned}\widetilde{\sigma}_g^*\widetilde{\omega} &= (d\Lambda)^{-1}\sigma_g^*\Phi^*\omega = (d\Lambda)^{-1}\Phi^*\sigma_{\Lambda(g)}^*\omega\\ &= (d\Lambda)^{-1}\Phi^*\,\mathrm{Ad}(\Lambda(g^{-1}))\circ\omega = (d\Lambda)^{-1}\circ\mathrm{Ad}(\Lambda(g^{-1}))(\Phi^*\omega)\\ &= \mathrm{Ad}(g^{-1})\circ(d\Lambda)^{-1}\Phi^*\omega = \mathrm{Ad}(g^{-1})\circ\widetilde{\omega}.\end{aligned}$$

For the second one let $p\in P_{\mathrm{Spin}}(E)$ and recall $\widetilde{\sigma}_p : \mathrm{Spin}(n)\longrightarrow P_{\mathrm{Spin}}(E)$ given by $g\longmapsto p\cdot g$ and note that $\Phi\circ\widetilde{\sigma}_p = \sigma_{\Phi(p)}\circ\Lambda$, where $\sigma_{\Phi(p)}$ is the similar map on the bundle $P_{\mathrm{SO}}(E)$. Then:

$$\begin{aligned}\widetilde{\omega}_p\circ(\widetilde{\sigma}_p)_* &= (d\Lambda)^{-1}\circ(\Phi^*\omega)_p\circ d\widetilde{\sigma}_p = (d\Lambda)^{-1}\omega_{\Phi(p)}\circ d\Phi\circ d\widetilde{\sigma}_p\\ &= (d\Lambda)^{-1}\omega_{\Phi(p)}\circ d\sigma_{\Phi(p)}\circ d\Lambda = (d\Lambda)^{-1}\circ d\Lambda = \mathrm{id}_{\mathfrak{spin}(n)}.\end{aligned}$$

This is the connection 1-form of the lifted connection, since one can easily check that $\ker(\widetilde{\omega})_p = H_pP_{\mathrm{Spin}}(E)$.

There is a standard procedure for transforming connections on principal bundles to connections on the associated bundles. In general let $\pi : P\longrightarrow M$ be a principal G-bundle, $\rho : G\longrightarrow \mathrm{Aut}(V)$ a finite-dimensional representation of G on V and $E := P\times_\rho V$ the associated vector bundle. Recall that there is a 1-1 correspondence between sections of E and functions $f : P\longrightarrow V$ satisfying $f(p\cdot g) = \rho(g^{-1})f(p)$ (the so-called *equivariant functions*). If $\psi\in\Gamma(E)$ we write $\widehat{\psi}$ for the associated equivariant function.

The map $\nabla : \mathfrak{X}(M)\times\Gamma(E)\longrightarrow\Gamma(E)$ given by $(X,\psi)\longmapsto\nabla_X\psi$ where $\nabla_X\psi$ is the section of E corresponding to the equivariant function $p\longmapsto\overline{X}_p(\widehat{\psi})$ (here $\overline{X}$ is the unique lift of X to a horizontal vector field on P) defines a connection on E. In short

$$\widehat{\nabla_X\psi}(p) = \overline{X}_p(\widehat{\psi}). \tag{2.12}$$

We also have a local description of the situation. Given a set of trivializations $(U_\alpha,\Phi_\alpha)_{\alpha\in I}$ and transition functions $g_{\alpha\beta} : U_{\alpha\beta}\longrightarrow \mathrm{GL}(n,\mathbb{K})$ there is a 1-1 correspondence between smooth sections of E and collections $(\psi_\alpha)_{\alpha\in I}$ of smooth functions $\psi_\alpha : U_\alpha\longrightarrow\mathbb{R}^n$ satisfying $\psi_\alpha(x) = g_{\alpha\beta}(x)\psi_\beta(x)$ for $x\in U_{\alpha\beta}$: Given a section $\psi \in \Gamma(E)$, $\psi_\alpha : U_\alpha \longrightarrow \mathbb{R}^n$ is the unique function satisfying $\Phi_\alpha \circ \psi(x) = (x, \psi_\alpha(x))$, i.e.

$$\psi_\alpha(x) = \mathrm{pr}_2 \circ \Phi_\alpha(\psi(x)).$$

So the question arises: how does the induced connection on E look locally? Well, given a local section $s_\alpha : U_\alpha \longrightarrow P$ of the principal bundle, the local function of $\nabla_X \psi$ relative to the corresponding trivialization of E is given by (see L. Claessens "Field Theory from a Bundle Point of View", Section 6.2, in [?] Section 6.2)

$$(\nabla_X \psi)_\alpha(x) = X_x \psi_\alpha - \rho_*(A^\alpha(X_x))\psi_\alpha(x) \tag{2.13}$$

where $\rho_* : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ is the induced Lie algebra representation of ρ and $A^\alpha = s_\alpha^* \omega$ is the so-called *local gauge potential*.

In the case of the Levi-Civita connection on $E = TM$, the formula reads

$$(\nabla_X Y)_\alpha(x) = X_x Y_\alpha - \Big(\sum_{i \in I} A_i^\alpha(X_x) B_i\Big) Y_\alpha(x) \tag{2.14}$$

where $(B_i)_{i\in I}$ is some basis for the Lie algebra $\mathfrak{so}(n)$ where $m = \dim M$.

Getting back to the spin bundle case, the connection $\widetilde{\omega}$ on $P_{\mathrm{Spin}}(E)$ induces a connection $\widetilde{\nabla}$ on the spinor bundle $S(E)$. Let's try to unveil (2.13) in this particular setting. Let $t_\alpha : U_\alpha \longrightarrow P_{\mathrm{Spin}}(E)$ be a local section of the spin bundle and put $s_\alpha := \Phi \circ t_\alpha$. This is a local section of the frame bundle $P_{\mathrm{SO}}(E)$. Let

$$\widetilde{A}^\alpha := t_\alpha^* \widetilde{\omega} \qquad \text{and} \qquad A^\alpha := s_\alpha^* \omega$$

denote the local gauge potentials of the connection ω, resp. the connection $\widetilde{\omega}$. Then we have

$$\begin{aligned} \widetilde{A}^\alpha &= t_\alpha^* \widetilde{\omega} = t_\alpha^*((d\Lambda)^{-1} \circ \Phi^* \omega) = (d\Lambda)^{-1} \circ t_\alpha^* \Phi^* \omega \\ &= (d\Lambda)^{-1} \circ s_\alpha^* \omega = (d\Lambda)^{-1} \circ A^\alpha, \end{aligned}$$

so the gauge potentials are related in the nicest possible way.

Putting this into (2.13) we obtain

$$\begin{aligned} (\widetilde{\nabla}_X \psi)_\alpha(x) &= X_x \psi_\alpha - (\kappa_n)_*(\widetilde{A}^\alpha(X_x))\psi_\alpha(x) \\ &= X_x \psi_\alpha - \rho_n((d\Lambda)^{-1}(A^\alpha(X_x)))\psi_\alpha(x) \end{aligned}$$

for $x \in U_\alpha$ (recall that the action of $(\kappa_n)_*$ is just ρ_n itself). Now we pick the usual basis $(B_{ij})_{i<j}$ for $\mathfrak{so}(n)$ (where B_{ij} is the $n \times n$-matrix whose ij'th entry is -1, the ji'th entry is 1 and all other entries are 0) and write the $\mathfrak{so}(n)$-valued 1-form A^α in terms of this basis: $A^\alpha = \sum_{i<j} A_{ij}^\alpha B_{ij}$. Then (remembering that $d\Lambda : \mathfrak{spin}(n) \longrightarrow \mathfrak{so}(n)$ maps $e_i e_j$ to $2B_{ij}$, where (e_i) is the standard basis for $\mathbb{R}^n$) we get

$$\begin{aligned} (\widetilde{\nabla}_X \psi)_\alpha(x) &= X_x \psi_\alpha - \rho_n\bigg((d\Lambda)^{-1}\Big(\sum_{i<j} A_{ij}^\alpha(X_x) B_{ij}\Big)\bigg)\psi_\alpha(x) \\ &= X_x \psi_\alpha - \rho_n\Big(\frac{1}{2}\sum_{i<j} A_{ij}^\alpha(X_x) e_i e_j\Big)\psi_\alpha(x) \\ &= X_x \psi_\alpha - \frac{1}{2}\sum_{i<j} A_{ij}^\alpha(X_x) e_i e_j \cdot \psi_\alpha(x). \end{aligned} \tag{2.15}$$

Now, let us show compatibility of the connection $\widetilde{\nabla}$ with the fiber metric. We will use the local expression above. Locally $\psi(x) = [s_\alpha(x), \psi_\alpha(x)]$ (for some section $s_\alpha : U_\alpha \longrightarrow P_{\mathrm{Spin}}(E)$ defined around x) we get by definition of the fiber metric

$$\langle\psi(x), \psi'(x)\rangle = \langle\psi_\alpha(x), \psi'_\alpha(x)\rangle_{\Delta_n}$$

for $x \in U_\alpha$. Thus

$$\begin{aligned}\langle\widetilde{\nabla}_X\psi(x), \psi'(x)\rangle &= \Big\langle X_x\psi_\alpha - \frac{1}{2}\sum_{i<j} A^\alpha_{ij}(X_x)e_ie_j\cdot\psi_\alpha(x)\ ,\ \psi'_\alpha(x)\Big\rangle_{\Delta_n} \\ &= \langle X_x\psi, \psi'(x)\rangle_{\Delta_n} - \frac{1}{2}\sum_{i<j} A^\alpha_{ij}(X_x)\langle e_ie_j\cdot\psi_\alpha(x), \psi'_\alpha(x)\rangle_{\Delta_n}.\end{aligned}$$

Note that

$$\begin{aligned}\langle e_ie_j\cdot\psi_\alpha(x), \psi'_\alpha(x)\rangle_{\Delta_n} &= -\langle e_j\cdot\psi_\alpha(x), e_i\cdot\psi'_\alpha(x)\rangle_{\Delta_n} = \langle\psi_\alpha(x), e_je_i\cdot\psi'_\alpha(x)\rangle_{\Delta_n} \\ &= -\langle\psi_\alpha(x), e_ie_j\cdot\psi'_\alpha(x)\rangle_{\Delta_n},\end{aligned}$$

and therefore

$$\langle\widetilde{\nabla}_X\psi(x), \psi'(x)\rangle + \langle\psi(x), \widetilde{\nabla}_X\psi'(x)\rangle = \langle X_x\psi, \psi'(x)\rangle_{\Delta_n} + \langle\psi(x), X_x\psi'\rangle_{\Delta_n}.$$

As mentioned $X_x\psi_\alpha$ should be interpreted componentwise, i.e. pick a complex basis $\{v_1, \ldots, v_N\}$ for Δ_n (of course $N = 2^{\lfloor\frac{n}{2}\rfloor}$) and write

$$\psi_\alpha = \sum_{i=1}^N \psi_{\alpha,i}v_i$$

where $\psi_{\alpha,i}$, the i'th component of ψ_α, is a complex-valued function on U_α, then

$$X_x\psi_\alpha = \sum_{i=1}^N (X_x\psi_{\alpha,i})v_i \in \Delta_n.$$

Since we have a metric in play, it would be wise of us to assume the basis $\{v_1, \ldots, v_N\}$ to be orthonormal. Then

$$\begin{aligned}\langle X_x\psi_\alpha, \psi'_\alpha(x)\rangle_{\Delta_n} + \langle\psi_\alpha(x), X_x\psi'_\alpha\rangle_{\Delta_n} &= \sum_{i=1}^N (X_x\psi_{\alpha,i})\psi'_{\alpha,i}(x) + \sum_{i=1}^N \psi_{\alpha,i}(x)X_x\psi'_{\alpha,i} \\ &= X_x\Big(\sum_{i=1}^N \psi_{\alpha,i}\psi'_{\alpha,i}\Big) = X_x\langle\psi_\alpha, \psi'_\alpha\rangle_{\Delta_n}.\end{aligned}$$

Thus we have proved that the spin connection is compatible with the metric.

Finally we need to check condition 2 in the definition of a Dirac bundle, that is

$$\widetilde{\nabla}_X(Y\cdot\psi)(x) = (\nabla_XY)(x) + Y_x\cdot(\widetilde{\nabla}_X\psi(x)) \tag{2.16}$$

for each $x \in M$. Again we use the local expressions, i.e. we consider a cover (U_α) which are domains of trivializations of both TM and E. Let Φ_α denote the trivializations of the tangent bundle (we may assume it to preserve the metric on M, i.e. $\Phi_x : T_xM \xrightarrow{\sim} \mathbb{R}^m$ is an isometry). Since $\widetilde{\nabla}_X$ is $\mathbb{C}$-linear and satisfies the Leibniz rule, it is sufficient to verify the above condition for $Y = E_k$ where $E_k(x) = \Phi_\alpha^{-1}(x, e_k)$ are local orthonormal vector fields.

But first, recall the following formula for the differential of the double covering

$$d\Lambda(X)v = Xv - vX$$

for $X \in \mathfrak{spin}(n)$ and $v \in \mathbb{R}^n \subseteq \mathrm{Cl}_{0,n}$ and replace in that X by $\sum_{i<j} A^\alpha_{ij}(X_x)e_ie_j \in \mathfrak{spin}(n)$ and v by e_k to get

$$\Big(\sum_{i<j} A^\alpha_{ij}(X_x)e_ie_j\Big)e_k = e_k\Big(\sum_{i<j} A^\alpha_{ij}(X_x)e_ie_j\Big) + d\Lambda\Big(\sum_{i<j} A^\alpha_{ij}(X_x)e_ie_j\Big)e_k$$
$$= e_k\Big(\sum_{i<j} A^\alpha_{ij}(X_x)e_ie_j\Big) + 2\sum_{i<j} A^\alpha_{ij}(X_x)B_{ij}e_k. \qquad (2.17)$$

Note that concatenation here means multiplication inside the Clifford algebra, and *not* the Clifford action. Recall also formula (2.14) for the local form of the Levi-Civita connection. It will be used in the following calculations (for explanations see below):

$$\begin{aligned}(\widetilde{\nabla}_X(E_k\cdot\psi))_\alpha(x) &= X_x(e_k\cdot\psi_\alpha) - \Big(\frac{1}{2}\sum_{i<j} A^\alpha_{ij}(X_x)e_ie_j\Big)e_k\cdot\psi_\alpha(x)\\ &= e_k\cdot(X_x\psi_\alpha) - e_k\Big(\frac{1}{2}\sum_{i<j} A^\alpha_{ij}(X_x)e_ie_j\Big)\cdot\psi_\alpha(x)\\ &\quad - \sum_{i<j}(A^\alpha_{ij}(X_x)B_{ij}e_k)\cdot\psi_\alpha(x)\\ &= e_k\cdot(\widetilde{\nabla}_X\psi)_\alpha(x) + (\nabla_X E_k)_\alpha(x)\cdot\psi_\alpha(x)\end{aligned}$$

and this is precisely the local form of the right-hand side of (2.16). For the first identity we used that $(E_k\cdot\psi)_\alpha = e_k\cdot\psi_\alpha$ and in the second we used (2.17) as well as the fact, that $e_k\cdot$ is a linear map, and thus commutes with X_x. This verifies condition 2 and hence we have shown that the spinor bundle $S(E)$ is a Dirac bundle.

The second most important example of a Dirac bundle is the complex spinor bundle $S^c(E)$. We can proceed in almost the same way we did before and we begin by giving $S^c(E)$ a metric: On Δ_n we can, as before, find an inner product $\langle\,,\,\rangle_{\Delta_n}$ such that $\kappa^c_n(g)$ is unitary for each $g \in \mathrm{Spin}^c(n)$ and such that $\rho_n(v)$ is skew-adjoint for any $v \in \mathbb{R}^n \subseteq \mathrm{Cl}_{0,n}$. We transfer this inner product to a fiber metric on $S^c(E)$ in the usual way by defining

$$\langle [p,v],[p,w]\rangle := \langle v,w\rangle_{\Delta_n}.$$

Unitarity of $\kappa^c_n(g)$ guarantees that this is well-defined.

Thanks to the isomorphism $\mathrm{Cl}(E) \cong P^c_{\mathrm{Spin}}(E)\times_{\mathrm{Ad}^c}\mathrm{Cl}_{0,n}$ we can also define a Clifford action $\mathrm{Cl}(E)\times S^c(E) \longrightarrow S^c(E)$ by

$$([p,\xi],[p,v]) \longmapsto [p,\rho_n(\xi)v].$$

Checking condition 1 in the definition of a Dirac bundle is done as above.

In the previous section we gave $P^c_{\mathrm{Spin}}(E)$ a connection depending on the choice of connection $\mathcal{A}$ on L^0. The corresponding connection 1-form is given in the same way as for the spin bundle

$$\widetilde{\omega}^{\mathcal{A}} = (d\Lambda^c)^{-1}\circ(\Xi^*\omega^{\mathcal{A}})$$

where $\Xi : P^c_{\mathrm{Spin}}(E) \longrightarrow Q$ is the double covering and Q is the bundle is as defined in the previous section. This connection induces a connection $\widetilde{\nabla}^{\mathcal{A}}$ on $S^c(E)$. We are interested in calculating its local expression from (2.13). Note first that $d\Lambda^c : \mathfrak{spin}^c(n) \cong \mathfrak{spin}(n)\oplus i\mathbb{R} \longrightarrow \mathfrak{so}(n)\oplus i\mathbb{R}$ (we identify the Lie algebra of $\mathrm{U}(1)$ with $i\mathbb{R}$) is simply given by

$$(t,i\theta) \longmapsto (d\Lambda(t),2i\theta). \qquad (2.18)$$

Consider trivializations of $P^c_{\text{Spin}}(E)$ and Q over U_α, let $t_\alpha : U_\alpha \longrightarrow P^c_{\text{Spin}}(E)$ denote the corresponding local section and put $s_\alpha := \Xi \circ t_\alpha$. If

$$A^\alpha := s_\alpha^* \omega^{\mathcal{A}} \qquad \text{and} \qquad \widetilde{A}^\alpha := t_\alpha^* \widetilde{\omega}^{\mathcal{A}}$$

denote the local gauge potentials (local connection 1-forms), then we have $\widetilde{A}^\alpha = (d\Lambda^c)^{-1} \circ A^\alpha$. Note also that since A^α is an $\mathfrak{so}(n) \oplus i\mathbb{R}$-valued 1-form, we may split it: $A^\alpha = A^\alpha_\omega \oplus A^\alpha_{\mathcal{A}}$ where A^α_ω is the gauge potential for the connection ω on $P_{\text{SO}}(E)$ and $A^\alpha_{\mathcal{A}}$ is the gauge potential for the connection $\mathcal{A}$ on L^0. From (2.18) we get

$$(d\Lambda^c)^{-1} A^\alpha = (d\Lambda^{-1}(A^\alpha_\omega), \tfrac{1}{2} A^\alpha_{\mathcal{A}})$$

and the induced Lie algebra representation of κ^c_n is just ρ_n restricted to the Lie algebra. $\kappa_n(\frac{1}{2} A^\alpha_{\mathcal{A}}(X_x))$ is just multiplication with the imaginary number $\frac{1}{2} A^\alpha_{\mathcal{A}}(X_x)$ and $(d\Lambda)^{-1} A^\alpha_\omega$ is already known to us from our discussion above. Hence we get:

$$(\widetilde{\nabla}^{\mathcal{A}}_X \psi)_\alpha(x) = X_x \psi_a - \frac{1}{2} \sum_{i<j} (A^\alpha_\omega)_{ij}(X_x) e_i e_j \cdot \psi_\alpha(x) + \frac{1}{2} A^\alpha_{\mathcal{A}}(X_x) \psi_\alpha(x). \quad (2.19)$$

With this local expression at our disposal we can show that the requirements of Definition 2.42 are satisfied, and hence that $S^c(E)$ is a Dirac bundle. The arguments are identical to the ones above for the spin bundle and so we skip them.

Definition 2.43 (**Dirac Operator**). Let S be a Dirac bundle. The *Dirac operator* $\partial\!\!\!/$ is then defined as the composition

$$\Gamma(S) \xrightarrow{\widetilde{\nabla}} \Gamma(T^*M \otimes S) \xrightarrow{\sim} \Gamma(TM \otimes S) \longrightarrow \Gamma(S) \qquad (2.20)$$

where the last map is Clifford multiplication $\mathfrak{X}(M) \otimes \Gamma(S) \longrightarrow \Gamma(S)$.

Next follow three elementary facts about general Dirac operators. Proofs are not included here, they can be found in any standard treatment of the subject, for instance in [**?**] Part II.5.

Lemma 2.44. *The Dirac operator is a first order differential operator, and given a local orthonormal frame $\{E_1, \ldots, E_m\}$ for TM over U, the Dirac operator takes the local form*

$$\partial\!\!\!/ \psi | U = \sum_{j=1}^m E_j \cdot (\widetilde{\nabla}_{E_j} \psi). \qquad (2.21)$$

As a differential operator, we can compute its symbol:

Proposition 2.45. *For $\xi_x \in T^*_x M$, let $\xi^\sharp_x \cdot : S_x \longrightarrow S_x$ be Clifford multiplication with the metric dual $\xi^\sharp_x$ of ξ_x. Then*

$$\sigma(\partial\!\!\!/)(\xi_x) = i\xi^\sharp_x \cdot \qquad \textit{and} \qquad \sigma(\partial\!\!\!/^2)(\xi_x) = \|\xi_x\|^2.$$

Thus both $\partial\!\!\!/$ and $\partial\!\!\!/^2$ (called the Dirac Laplacian*) are elliptic.*

Proposition 2.46. *The Dirac operator is formally self-adjoint, i.e. for ψ_1 and ψ_2 in $\Gamma_c(S)$ (the set of sections of S with compact support) we have*

$$(\partial\!\!\!/ \psi_1 | \psi_2) = (\psi_1 | \partial\!\!\!/ \psi_2). \qquad (2.22)$$

We augment this list of elementary properties of the Dirac operator with a perhaps less renowned result. The proof may be found in [**?**] Theorem 8.2:

Theorem 2.47 (Unique Continuation Property). *If $\psi \in \Gamma(S)$ is in the kernel of the Dirac operator $\not{\partial}$ and ψ is 0 on some open set, then ψ is identically equal to 0.*

In the next example we present two of the most important Dirac operators.

Example 2.48. (**Spin-Dirac operator**). We consider the spinor bundle $S(E)$ associated to some oriented Riemannian vector bundle E carrying a spin structure. We saw earlier that this vector bundle is a Dirac bundle and thus it carries a Dirac operator, $\not{D}$ called the *spin-Dirac operator* or just *the* Dirac operator (in [?] it is called the *Atiyah-Singer operator*). Thanks to Lemma 2.44 and the calculations done in the previous example, we arrive at the following local description

$$
\begin{aligned}
(\not{D}\,\psi)_\alpha(x) &= \Big(\sum_{k=1}^m E_k \cdot \widetilde{\nabla}_{E_k}\psi\Big)_\alpha(x) = \sum_{k=1}^m e_k \cdot (\widetilde{\nabla}_{E_k}\psi)_\alpha(x) \\
&= \sum_{k=1}^m e_k \cdot \Big((E_k)_x\psi_\alpha - \frac{1}{2}\sum_{i<j}^n A^\alpha_{ij}((E_k)_x)e_ie_j \cdot \psi_\alpha(x)\Big) \\
&= \sum_{k=1}^m e_k \cdot (E_k)_x\psi_\alpha - \frac{1}{2}\sum_{i<j} A^\alpha_{ij}((E_k)_x)e_ke_ie_j \cdot \psi_\alpha(x) \qquad (2.23)
\end{aligned}
$$

where $\{E_1, \ldots, E_m\}$ is a local tangent frame ($m = \dim M$) such that $E_k(x) = \Phi_\alpha^{-1}(x, e_k)$ [4].

(**Geometric Dirac operator**). The Dirac operator associated to a complex spinor bundle of a principal $\mathrm{Spin}^c(n)$-bundle is called the *geometric Dirac operator*. Inserting the local expression (2.19) of the connection into (2.21) we get

$$
\begin{aligned}
(\not{D}_{\mathcal{A}}\psi)_\alpha(x) = \sum_{k=1}^m \Big(& e_k \cdot (E_k)_x\psi_\alpha - \frac{1}{2}\sum_{i<j}(A^\alpha_\omega)_{ij}((E_k)_x)e_ke_ie_j \cdot \psi_\alpha \\
&+ \frac{1}{2}A^\alpha_{\mathcal{A}}((E_k)_x)e_k \cdot \psi_\alpha(x)\Big). \qquad (2.24)
\end{aligned}
$$

where $\mathcal{A}$ is a choice of a connection on the determinant line bundle, and where m is the dimension of the base manifold. If we replace the connection $\mathcal{A}$ on the determinant line bundle by $\mathcal{A} + \beta$ for some $\beta \in i\Omega^1(M)$ the local connection 1-forms change to from $A^\alpha_{\mathcal{A}}$ to $A^\alpha_{\mathcal{A}} + \beta$ and from the local expression for the Dirac operator we immediately deduce

$$
\not{D}_{\mathcal{A}+\beta}\psi = \not{D}_{\mathcal{A}}\psi + \frac{1}{2}\beta \cdot \psi. \qquad (2.25)
$$

We will make extensively use of this result. □

In the Clifford algebra $\mathrm{Cl}_n^{\mathbb{C}}$ we have the *volume form* given unambiguously by $\omega_{\mathbb{C}} = i^{\lfloor \frac{n+1}{2} \rfloor} e_1 \cdots e_n$ whenever $\{e_1, \ldots, e_n\}$ is an orthonormal basis for $\mathbb{R}^n$. In the same manner we may define a *volume section* (also denoted $\omega_{\mathbb{C}}$) of the *complexified* Clifford bundle $\mathrm{Cl}(M) \otimes \mathbb{C}$ by the local formula $\omega_{\mathbb{C}}|_U = i^{\lfloor \frac{m+1}{2} \rfloor} E_1 \cdots E_m$ when $(E_1, \ldots, E_m)$ is a local orthonormal tangent frame over U.

Assume now that $m = 2k$, then $\omega_{\mathbb{C}} = i^k E_1 \cdots E_{2k}$. In this dimension it is well-known that the volume form commutes with everything in $(\mathrm{Cl}_m^{\mathbb{C}})^0$ - the even part of the Clifford algebra. For a Dirac bundle S over E, $\widetilde{\nabla}$ the connection and ψ a section of S, we get since $X_x(\omega_{\mathbb{C}} \cdot \psi)_\alpha = X_x(\omega_{\mathbb{C}} \cdot \psi_\alpha) = \omega_{\mathbb{C}} \cdot (X_x\psi_\alpha)$ (in the

[4] Note that often $(E_k)_x\psi_\alpha$ is written as $\frac{\partial \psi_\alpha}{\partial e_k}(x)$.

first expression $\omega_{\mathbb{C}}$ denotes the volume *section* and in the two last expressions it denotes the volume *form*) that

$$\begin{aligned}[\widetilde{\nabla}_X(\omega_{\mathbb{C}}\cdot\psi)]_\alpha(x) &= X_x(\omega_{\mathbb{C}}\cdot\psi)_\alpha - \frac{1}{2}\sum_{i<j}A^\alpha_{ij}(X_x)e_ie_j\cdot(\omega_{\mathbb{C}}\cdot\psi_\alpha(x))\\ &= \omega_{\mathbb{C}}\cdot(X_x\psi_\alpha) - \omega_{\mathbb{C}}\cdot\frac{1}{2}\sum_{i<j}A^\alpha_{ij}(X_x)e_ie_j\cdot\psi_\alpha(x)\\ &= \omega_{\mathbb{C}}\cdot(\widetilde{\nabla}_X\psi)_\alpha(x).\end{aligned}$$

If E has a spin structure and $S(E)$ is the associated spinor bundle, then we have the splitting $S=S^+\oplus S^-$ induced by the decomposition $\Delta_{2k}=\Delta^+_{2k}\oplus\Delta^-_{2k}$. The subspaces $\Delta^\pm_{2k}$ are the ±1-eigenspaces of the action of $\omega_{\mathbb{C}}$ on Δ_{2k}. Therefore, sections of $S(E)^\pm$ are exactly the spinor fields satisfying $\omega_{\mathbb{C}}\cdot\psi=\pm\psi$. Therefore the formula above implies that $\widetilde{\nabla}_X$ maps $\Gamma(S^\pm)\longrightarrow\Gamma(S^\pm)$. To see how the Dirac operator reacts to this splitting, note the following:

$$\begin{aligned}\not{D}(\omega_{\mathbb{C}}\cdot\psi) &= \sum_{i=1}^m E_i\cdot\widetilde{\nabla}_{E_i}(\omega_{\mathbb{C}}\cdot\psi) = \sum_{i=1}^m E_i\cdot\omega_{\mathbb{C}}\cdot\widetilde{\nabla}_{E_i}(\psi)\\ &= -\omega_{\mathbb{C}}\cdot\sum_{i=1}^m E_i\cdot\widetilde{\nabla}_{E_i}(\psi) = -\omega_{\mathbb{C}}\cdot\not{D}\psi\end{aligned}$$

($m=\dim M$). From this it is apparent, that $\not{D}$ maps $\Gamma(S(E)^\pm)\longrightarrow\Gamma(S(E)^\mp)$. If $\not{D}^\pm$ denotes the restriction of $\not{D}$ to $\Gamma(S(E)^\pm)$ we may write the Dirac operator relative to the splitting as a matrix

$$\not{D}=\begin{pmatrix}0 & \not{D}^-\\ \not{D}^+ & 0\end{pmatrix}.$$

Exactly the same holds true for the geometric Dirac operator. Also the spin^c-representation $\kappa^c_{2k}:\mathrm{Spin}^c(2k)\longrightarrow\mathrm{Aut}(\Delta_{2k})$ decomposes into irreducible representation spaces $\Delta_{2k}=\Delta^+_{2k}\oplus\Delta^-_{2k}$ and hence also the complex spinor bundle $S^c(E)$ exhibits a splitting $S^c(E)=S^c(E)^+\oplus S^c(E)^-$, relative to which the geometric Dirac operator $\not{D}_{\mathcal{A}}$ takes the form

$$\not{D}_{\mathcal{A}}=\begin{pmatrix}0 & \not{D}^-_{\mathcal{A}}\\ \not{D}^+_{\mathcal{A}} & 0\end{pmatrix}$$

where $\not{D}^\pm_{\mathcal{A}}:\Gamma(S^c(E)^\pm)\longrightarrow\Gamma(S^c(E)^\mp)$.