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Hermitian Symmetric Spaces and Their Unitary Highest Weight Modules

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The purpose of this article is to determine the set of unitarizable highest weight modules corresponding to Hermitian symmetric spaces of the noncompact type. The major step is that of proving unitarity at the "last possible place." With this established the description of the full set of unitarizable highest weight modules follows by a straightforward tensor product argument combined with the main ingredients of the proof of the key theorem: Bernstein–Gelfand–Gelfand, and a diagramatic representation of the set of positive noncompact roots.

INTRODUCTION

The purpose of this article is to determine the set of unitarizable highest weight modules corresponding to Hermitian symmetric spaces of the noncompact type. Specifically let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{R}$ and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition. By assumption $\mathfrak{k}$ has a nontrivial center $\eta = \mathbb{R} \cdot h_0$ and $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathbb{R} \cdot h_0$, where $\mathfrak{k}_1 = [\mathfrak{k}, \mathfrak{k}]$. The modules W_Λ considered are determined by a pair (Λ_0, λ), where Λ_0 is $\mathfrak{k}_1$-dominant and integral and $\lambda \in \mathbb{R}$. That is, $\Lambda = (\Lambda_0, \lambda)$ determines a finite-dimensional $\mathscr{U}(\mathfrak{k}^{\mathbb{C}})$-module V_Λ and W_Λ is the irreducible quotient of $\mathscr{U}(\mathfrak{g}^{\mathbb{C}}) \otimes_{\mathscr{U}(\mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^-)} V_\Lambda$, where $\mathfrak{p}^+ = \{z \in \mathfrak{p}^{\mathbb{C}} \mid [h_0, z] = iz\}$.

W_Λ may be represented as a space of V_Λ-valued polynomials on $\mathfrak{p}^+$ and the $\mathfrak{g}$-invariant Hermitian form on W_Λ, restricted to a $\mathfrak{k}_1$-invariant subspace of dth order polynomials is a dth order polynomial in λ. By considering the set of first order polynomials on $\mathfrak{p}^+$ this leads to the idea of "the last possible place of unitariry;" explicitly defined and determined in [6]. The main theorem we prove here is that the module at this last possible place indeed *is* unitarizable. From this the picture is completed by forming tensor products, along the lines of [4], of the unitary modules with the most singular, nontrivial, unitarizable module W_{λ_s} corresponding to $\Lambda_0 = 0$.

The main ingredients in the proof are Bernstein–Gelfand–Gelfand and a

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diagrammatic representation of Δ_n^+; the set of positive noncompact roots, which we develop here. The final steps in the proof then consists of describing certain subsets of Δ_n^+ in terms of its diagram. For some of the classical groups the combinatorics connected with this has so far been too involved, and for those groups we have to rely on the proof of the Kashiwara–Vergne conjecture [3, 5–7, 13]. After this work was completed we have learned that Enright, Howe, and Wallach [10] have obtained the same result, see also Garland and Zuckerman [11], and [12]. Finally, the significant contribution by Parthasarathy [14] should be mentioned.

1. Notation

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{R}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$. We assume that $\mathfrak{k}$ has a nonempty center η; in this case $\eta = \mathbb{R} \cdot h_0$ for an $h_0 \in \eta$ whose eigenvalues under the adjoint action on $\mathfrak{p}^{\mathbb{C}}$ are $\pm i$. Let

$$\mathfrak{p}^+ = \{z \in \mathfrak{p}^{\mathbb{C}} \mid [h_0, z] = iz\}$$

and

$$\mathfrak{p}^- = \{z \in \mathfrak{p}^{\mathbb{C}} \mid [h_0, z] = -iz\}.$$

Let $\mathfrak{k}_1 = [\mathfrak{k}, \mathfrak{k}]$ and let $\mathfrak{h}$ be a maximal Abelian subalgebra of $\mathfrak{k}$. Then $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathbb{R} \cdot h_0$, $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}_1) \oplus \mathbb{R} \cdot h_0$, $(\mathfrak{h} \cap \mathfrak{k}_1)^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{k}_1^{\mathbb{C}}$, and $\mathfrak{h}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. We let σ denote the conjugation in $\mathfrak{g}^{\mathbb{C}}$ relative to the real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$. The sets of compact and noncompact roots of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h}^{\mathbb{C}}$ are denoted Δ_c and Δ_n, respectively. $\Delta = \Delta_c \cup \Delta_n$. We choose an ordering of Δ such that

$$\mathfrak{p}^+ = \sum_{\alpha \in \Delta_n^+} \mathfrak{g}^\alpha,$$

and set

$$\mathfrak{g}^+ = \sum_{\alpha \in \Delta^+} \mathfrak{g}^\alpha, \qquad \mathfrak{g}^- = \sum_{\alpha \in \Delta^-} \mathfrak{g}^\alpha,$$

and

$$\rho = \tfrac{1}{2} \sum_{\alpha \in \Delta^+} \alpha.$$

Throughout β denotes the unique simple noncompact root. For $\gamma \in \Delta$ let H_γ be the unique element of $i\mathfrak{h} \cap [(\mathfrak{g}^{\mathbb{C}})^\gamma, (\mathfrak{g}^{\mathbb{C}})^{-\gamma}]$ for which $\gamma(H_\gamma) = 2$. Then for all γ_1 in Δ

$$\langle \gamma_1, \gamma \rangle = \frac{2(\gamma_1, \gamma)}{(\gamma, \gamma)} = \gamma_1(H_\gamma), \tag{1.1}$$

where $(\cdot, \cdot)$ is the bilinear form on $(\mathfrak{h}^{\mathbb{C}})^*$ obtained from the Killing form of $\mathfrak{g}^{\mathbb{C}}$. The reflexion corresponding to $\gamma \in \Delta$ is denoted by σ_γ,

$$\sigma_\gamma(\gamma_1) = \gamma_1 - \langle \gamma_1, \gamma \rangle \gamma. \tag{1.2}$$

For $\alpha \in \Delta_n^+$ choose $z_\alpha \in (\mathfrak{g}^{\mathbb{C}})^\alpha$ such that

$$[z_\alpha, z_\alpha^\sigma] = H_\alpha, \tag{1.3}$$

and let $z_{-\alpha} = z_\alpha^\sigma$. Following the notation of [8] we let γ_r denote the highest root. Then $\gamma_r \in \Delta_n^+$, and $H_{\gamma_r} \notin [\mathfrak{h} \cap \mathfrak{k}_1]^{\mathbb{C}}$. Finally we let $u \to u^*$ be the antilinear antiautomorphism of $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})$ that extends the map $x \to -x^\sigma$ of $\mathfrak{g}^{\mathbb{C}}$.

2. Modules

Corresponding to the decomposition $\mathscr{U}(\mathfrak{g}^{\mathbb{C}}) = (\mathscr{U}(\mathfrak{g}^{\mathbb{C}})\, \mathfrak{g}^+ \oplus \mathfrak{g}^- \mathscr{U}(\mathfrak{g}^{\mathbb{C}})) \oplus \mathscr{U}(\mathfrak{h}^{\mathbb{C}})$ we let, for $u \in \mathscr{U}(\mathfrak{g}^{\mathbb{C}})$, $\gamma(u)$ denote the unique element of $\mathscr{U}(\mathfrak{h}^{\mathbb{C}})$ for which $u - \gamma(u)$ is in $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})\, \mathfrak{g}^+ \oplus \mathfrak{g}^- \mathscr{U}(\mathfrak{g}^{\mathbb{C}})$.

Let $\chi \in (\mathfrak{h}^{\mathbb{C}})^*$. The Verma module M_χ of highest weight $\chi - \rho$ is defined to be $M_\chi = \mathscr{U}(\mathfrak{g}^{\mathbb{C}})/I_{\chi-\rho}$, where $I_{\chi-\rho}$ is the left ideal generated by the elements $(H - \chi(H)) + \rho(H))$, $H \in \mathfrak{h}^{\mathbb{C}}$, and $\mathfrak{g}^+$. We denote the image of 1 in M_χ by $1_{\chi-\rho}$, and the unique irreducible quotient is denoted by L_χ.

If Λ_0 is a dominant integral weight of $\mathfrak{k}_1$ and if $\lambda \in \mathbb{R}$ we denote by $\Lambda = (\Lambda_0, \lambda)$ the linear functional on $\mathfrak{h}^{\mathbb{C}}$ given by

$$\Lambda\,|_{(\mathfrak{h}\cap\mathfrak{k}_1)^{\mathbb{C}}} = \Lambda_0, \qquad \Lambda(H_{\gamma_r}) = \lambda. \tag{2.1}$$

Further we let V_Λ denote the irreducible finite-dimensional $\mathscr{U}(\mathfrak{k}^{\mathbb{C}})$-module of highest weight Λ. As $\mathscr{U}(\mathfrak{k}_1^{\mathbb{C}})$-modules, clearly $V_\Lambda = V_{\Lambda_0}$.

The sesquilinear form B_Λ on $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})$,

$$B_\Lambda(u, v) = \Lambda(\gamma(v^* u)) \tag{2.2}$$

is $\mathfrak{g}$-invariant. We let N_Λ denote the kernel of B_Λ,

$$N_\Lambda = \{u \in \mathscr{U}(\mathfrak{g}^{\mathbb{C}}) \mid \forall v \in \mathscr{U}(\mathfrak{g}^{\mathbb{C}}) : \Lambda(\gamma(v^* u)) = 0\}, \tag{2.3}$$

and set

$$N_\Lambda(\mathfrak{k}) = N_\Lambda \cap \mathscr{U}(\mathfrak{k}^{\mathbb{C}}). \tag{2.4}$$

Clearly,

$$I_\Lambda \subseteq N_\Lambda \qquad \text{and} \qquad I_\Lambda(\mathfrak{k}) \subseteq N_\Lambda(\mathfrak{k}). \tag{2.5}$$

Let $J_\Lambda = I_\Lambda + \mathscr{U}(\mathfrak{g}^{\mathbb{C}})\, N_\Lambda(\mathfrak{k})$. Since $\mathscr{U}(\mathfrak{g}^{\mathbb{C}}) = \mathscr{U}(\mathfrak{p}^-)\,\mathscr{U}(\mathfrak{k}^{\mathbb{C}})\,\mathscr{U}(\mathfrak{p}^+)$ and $V_\Lambda = \mathscr{U}(\mathfrak{k}^{\mathbb{C}})/N_\Lambda(\mathfrak{k})$ we have

LEMMA 2.1. $\mathscr{U}(\mathfrak{g}^{\mathbb{C}}) \otimes_{\mathscr{U}(\mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^+)} V_\Lambda = \mathscr{U}(\mathfrak{g}^{\mathbb{C}})/J_\Lambda$.

Since $J_\Lambda \subseteq N_\Lambda$, B_Λ gives rise to a $\mathfrak{g}$-invariant Hermitian form, also denoted by B_Λ, on $\mathscr{U}(\mathfrak{g}^{\mathbb{C}}) \otimes_{\mathscr{U}(\mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^+)} V_\Lambda$.

The unique irreducible quotient W_Λ of $\mathscr{U}(\mathfrak{g}^{\mathbb{C}}) \otimes_{\mathscr{U}(\mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^+)} V_\Lambda$ is given as

$$W_\Lambda = \mathscr{U}(\mathfrak{g}^{\mathbb{C}})/N_\Lambda. \tag{2.5}$$

Any $\mathfrak{g}$-invariant Hermitian form on W_Λ is proportional to B_Λ.

For further background information we refer to [8, Sect. 1; 4, Sect. 2].

3. BERNSTEIN, GELFAND, AND GELFAND

The only major theorem we shall be using describes the exact circumstances under which the irreducible quotient L_χ of one Verma module can occur in the Jordan–Hölder series $\mathrm{JH}(M_\psi)$ of another. First some terminology.

DEFINITION 3.1. Let $\chi, \psi \in (\mathfrak{h}^{\mathbb{C}})^*$. A sequence of roots $\gamma_1, ..., \gamma_k \in \Delta^+$ is said to satisfy condition (A) for the pair (χ, ψ) if

(i) $\chi = \phi_{\gamma_k}, ..., \sigma_{\gamma_1} \psi$.

(ii) Put $\chi_0 = \psi$, $\chi_i = \sigma_{\gamma_i}, ..., \sigma_{\gamma_1} \psi$. Then $\chi_{i-1} - \chi_i = n_i \gamma_i$, where $n_i \in \mathbb{N}$.

Observe that $n_i = \langle \chi_{i-1}, \gamma_i \rangle$.

THEOREM 3.2 [1, p. 42]. *Let $\chi, \psi \in (\mathfrak{h}^{\mathbb{C}})^*$. Then $L_\chi \in \mathrm{JH}(M_\psi)$ if and only if there exists a sequence $\gamma_1, ..., \gamma_k \in \Delta^+$ satisfying condition* (A) *for the pair (χ, ψ).*

In the present situation, $\Delta^+ = \Delta_n^+ \cup \Delta_c^+$. Through a series of elementary lemmas it will now be proved that if $\chi, \psi \in (\mathfrak{h}^{\mathbb{C}})^*$, if a sequence of roots from Δ^+ satisfies condition (A) for the pair (χ, ψ), and if moreover χ is $\mathfrak{k}_1$-dominant, then there also exists a sequence of roots from Δ_n^+ satisfying the condition for this pair.

The starting point is

LEMMA 3.3. *If $\forall \mu \in \Delta_c^+$, $\langle \chi, \mu \rangle \geqslant 0$, and if $\gamma_1, ..., \gamma_k \in \Delta^+$ satisfies condition* (A) *for the pair (χ, ψ), then $\gamma_k \in \Delta_n^+$.*

Proof. Let $\omega = n_1\gamma_1 + \cdots + n_k\gamma_k$, and $\omega' = \omega - n_k\gamma_k$. By assumption,

$$n_k = \langle \psi - \omega', \gamma_k \rangle = \langle \chi + n_k\gamma_k, \gamma_k \rangle = \langle \chi, \gamma_k \rangle + 2n_k.$$

Thus,

$$\langle \chi, \gamma_k \rangle = -n_k < 0. \quad \blacksquare$$

We now invoke the basic structure of the given situation through the following well-known fact:

LEMMA 3.4. *Let $\gamma \in \Delta$. The coefficient of β in γ is -1, 0, or 1.*

In particular, if $\gamma \in \Delta_n^+$, the coefficient is 1.

COROLLARY 3.5. *Let $\alpha \in \Delta_n$ and $\mu \in \Delta_c$. Then*

$$\langle \mu, \alpha \rangle \in \{-1, 0, 1\} \qquad \textit{and} \qquad \langle \alpha, \mu \rangle \in \{-2, -1, 0, 1, 2\}.$$

Proof. $\sigma_\alpha(\mu) = \mu - \langle \mu, \alpha \rangle \alpha$ and thus the first assertion is clear. Let $\langle \alpha, \mu \rangle = n$ and observe that $\alpha - n\mu \in \Delta$. Since $\langle \alpha - n\mu, \alpha \rangle = 2 - |n|$ it follows that $\sigma_\alpha(\alpha - n\mu) = (|n| - 1)\alpha - n\mu$. $\blacksquare$

Let $\gamma_1, \gamma_2, \ldots, \gamma_k$ be a sequence of roots in Δ^+ that satisfies condition (A) for a pair (χ, ψ) of elements of $(\mathfrak{h}^{\mathbb{C}})^*$, let $i \leqslant k$, $k \geqslant 2$, and assume that $\gamma_{i-1} \in \Delta_c^+$ and $\gamma_i \in \Delta_n^+$. Thus

$$\begin{gathered} \chi_i = \sigma_{\gamma_i}\sigma_{\gamma_{i-1}}\chi_{i-2}, \\ n_{i-1} = \langle \chi_{i-2}, \gamma_{i-1} \rangle \qquad \text{and} \qquad n_i = \langle \chi_{i-1}, \gamma_i \rangle = \langle \sigma_{\gamma_{i-1}}\chi_{i-2}, \gamma_i \rangle. \end{gathered} \tag{3.1}$$

We wish to replace the pair (γ_{i-1}, γ_i) in the sequence by either a pair (γ_a, γ_b) of positive roots such that $\gamma_a \in \Delta_n^+$ or by a single noncompact positive root γ_c in such a way that the new sequence also satisfies condition (A) for (χ, ψ). This is in fact possible even in a more general context. In the present situation, however, it follows from Corollary 3.5 that only a few cases need to be described. We do this, and omit the simple verifications:

(i) $n_{i-1} + \langle \gamma_i, \gamma_{i-1} \rangle n_i > 0$:

$$(\gamma_a, \gamma_b) = (\sigma_{\gamma_{i-1}}\gamma_i, \gamma_{i-1}).$$

(ii) $n_{i-1} + \langle \gamma_i, \gamma_{i-1} \rangle n_i = 0$:

$$\gamma_c = \sigma_{\gamma_{i-1}}\gamma_i.$$

(iii) $n_i + \langle \gamma_{i-1}, \gamma_i \rangle n_{i-1} > 0$:

$$(\gamma_a, \gamma_b) = (\gamma_i, \sigma_{\gamma_i}\gamma_{i-1}).$$

(iv) $n_i + \langle \gamma_{i-1}, \gamma_i \rangle n_{i-1} = 0$:

$$\gamma_c = \sigma_{\gamma_i} \gamma_{i-1}.$$

(v) $n_{i-1} + \langle \gamma_i, \gamma_{i-1} \rangle n_i < 0$ and $n_i + \langle \gamma_{i-1}, \gamma_i \rangle n_{i-1} < 0$:

$$(\gamma_a, \gamma_b) = (\sigma_{\gamma_i} \gamma_{i-1}, \sigma_{\gamma_{i-1}} \gamma_i).$$

Observe that in the last case, $\langle \gamma_i, \gamma_{i-1} \rangle = -2$ and $\langle \gamma_{i-1}, \gamma_i \rangle = -1$.

The net effect of this is that we may rearrange, perhaps even shorten, our sequence $\gamma_1, \ldots, \gamma_k$ in such a way that compact roots participating in it move towards γ_k or disappear. If we are in the situation of Lemma 3.3 they can thus be made to disappear completely. Hence we may state

PROPOSITION 3.6. *Let χ, $\psi \in (\mathfrak{h}^{\mathbb{C}})^*$ and assume that the sequence $\gamma_1, \ldots, \gamma_k$ satisfies condition* (A) *for the pair (χ, ψ). If χ is $\mathfrak{k}_1$-dominant we may assume that $\gamma_i \in \Delta_n^+$, $i = 1, \ldots, k$.*

4. CONCERNING Δ_n^+

In the following we shall consider sets built up of elements from Δ_n^+. There is a pictorial way of representing these subsets which stems from a 2-dimensional diagram of Δ_n^+. This construction, which we present here, is quite analogous to, and easily derived from, the Dynkin diagram of Δ.

We stress that besides elementary facts about root systems, everything follows from Lemma 3.4.

Let Σ_c denote the set of simple compact roots.

LEMMA 4.1. *Let $\alpha \in \Delta_n^+$, let $\mu_1, \ldots, \mu_i$ be distinct elements of Σ_c, and assume that $\alpha + \mu_j \in \Delta_n^+$ for all $j = 1, \ldots, i$. Then $i \leqslant 2$. If $i = 2$, $\alpha + \mu_1 + \mu_2 \in \Delta_n^+$.*

Proof. (i) Assume $i \geqslant 2$, and $\forall j$: $(\alpha, \mu_j) = 0$. Since in this case $(\alpha + \mu_j, \alpha + \mu_j) = (\alpha, \alpha) + (\mu_j, \mu_j)$, $(\alpha + \mu_j, \alpha + \mu_j) = 2(\alpha, \alpha) = 2(\mu_j, \mu_j)$. Consider two distinct elements, μ_{j_1} and μ_{j_2}, from the set $\{\mu_1, \ldots, \mu_i\}$, and let $\langle \alpha + \mu_{j_1}, \alpha + \mu_{j_2} \rangle = n$.

By Lemma 3.4, $n = 0$, 1, or 2, but $n = 1$ is excluded since $\mu_{j_1} - \mu_{j_2}$ is not a root. It follows that $\langle \mu_{j_1}, \mu_{j_2} \rangle = \pm 2$ and thus, since the roots are simple, $\langle \mu_{j_1}, \mu_{j_2} \rangle = -2$. This, however, is not possible since by symmetry, $\langle \mu_{j_2}, \mu_{j_1} \rangle = -2$ and μ_{j_1} is not proportional to μ_{j_2}.

Thus there can be at most one μ_j such that $(\alpha, \mu_j) = 0$. In this case α is short, and $\alpha + \mu_j$ is long.

(ii) Assume $i \geqslant 3$, $(\alpha, \mu_{j_1}) = 0$, $(\alpha, \mu_{j_2}) \neq 0$, and $(\alpha, \mu_{j_3}) \neq 0$. Let $n_2 =$

$\langle\alpha,\mu_{j_2}\rangle$ and $n_3=\langle\alpha,\mu_{j_3}\rangle$. By Corollary 3.5 it follows easily that root strings are of length at most 3 (cf. Proposition 6.2 in [6]). By (i), α is short and hence $n_2=n_3=-1$. Thus, by Corollary 3.5, $\langle\mu_{j_2},\alpha\rangle=\langle\mu_{j_3},\alpha\rangle=-1$, and, in particular, μ_{j_2} and μ_{j_3} are short. $\alpha+\mu_{j_2}$ and $\alpha+\mu_{j_3}$ are clearly long. It follows that $\langle\alpha+\mu_{j_1},\alpha+\mu_{j_2}\rangle=1+\langle\mu_{j_1},\mu_{j_2}\rangle$, and as in (i) this implies that $\langle\mu_{j_1},\mu_{j_2}\rangle=\langle\mu_{j_1},\mu_{j_3}\rangle=-1$.

Consider $\langle\alpha+\mu_{j_2},\alpha+\mu_{j_3}\rangle$. By the previous observations this equals $\frac{1}{2}\langle\mu_{j_2},\mu_{j_3}\rangle$, and since it can only take the values 0 or 2, $\langle\mu_{j_2},\mu_{j_3}\rangle=0$. However, this leads to a contradiction of Lemma 3.4 since $\alpha+\mu_{j_2}+\mu_{j_3}\in\Delta_n^+$ and $\langle\alpha+\mu_{j_2}+\mu_{j_3},\alpha+\mu_{j_1}\rangle=-1$. In other words: if μ_{j_1} is present either μ_{j_2} or μ_{j_3} (or both) is not. If, say, μ_{j_2} *is* present the results concerning it and μ_{j_1} remain valid. In particular, $\langle\mu_{j_1},\mu_{j_2}\rangle=-1$ and $\alpha+\mu_{j_1}+\mu_{j_2}$ as well as $\alpha+\mu_{j_1}+2\mu_{j_2}$ are elements of Δ_n^+.

(iii) Assume $i\geqslant 2$ and $\langle\alpha,\mu_1\rangle=-2$. It follows from (ii) that $\langle\alpha,\mu_2\rangle=-n<0$, and Corollary 3.5 implies that α is long and μ_1 is short. An easy computation now gives that $\langle\alpha+n\mu_2,\alpha+2\mu_1\rangle<0$ which contradicts Lemma 3.4. Thus, if $\langle\alpha,\mu_1\rangle=-2$, $i=1$.

(iv) Assume $\langle\alpha,\mu_1\rangle=\langle\alpha,\mu_2\rangle=\langle\alpha,\mu_3\rangle=-1$. Obviously, α, μ_1, μ_2, and μ_3 are of equal length. By considering $\langle\alpha+\mu_i,\alpha+\mu_j\rangle$ it is seen that the compact roots are pairwise orthogonal. Since this implies that, e.g., $\alpha+\mu_1+\mu_2\in\Delta_n^+$ and since $\langle\alpha+\mu_1+\mu_2,\alpha+\mu_3\rangle=-1$, we conclude that there can be at most two roots in Σ_c, μ_1 and μ_2, say, such that $\langle\alpha,\mu_1\rangle=\langle\alpha,\mu_2\rangle=-1$. In this case $\langle\mu_1,\mu_2\rangle=0$ and $\alpha+\mu_1+\mu_2\in\Delta_n^+$. ∎

By considering the basis of Δ consisting of the negatives of the elements in Σ_c together with the previously highest root γ_r we naturally obtain an analogous result concerning the possibilities of subtracting simple compact roots from a given $\alpha\in\Delta_n^+$.

Now, $\mathfrak{p}^+$ is a highest weight module for $\mathfrak{k}_1^{\mathbb{C}}$ and each root space is one-dimensional, hence any $\alpha\in\Delta_n^+$ can be written as $\alpha=\alpha_1+\mu$, where $\alpha_1\in\Delta_n^+$ and $\mu\in\Sigma_c$. This observation together with Lemma 4.1 leads directly to the construction of the diagram of Δ_n^+:

One begins with (say) β and draws an arrow originating at β for each simple root μ_i such that $\beta+\mu_i\in\Delta_n^+$. Suppose for simplicity that $i=2$ and consider $\beta+\mu_1$. By Lemma 4.1 $(\beta+\mu_1)+\mu_2\in\Delta_n^+$ so there can be at most one more $\mu_3\in\Sigma_c$ such that $(\beta+\mu_1)+\mu_3\in\Delta_n^+$.

It follows again from Lemma 4.1 that μ_3, if it exists, is different from μ_1, and $\beta+\mu_1+\mu_2+\mu_3\in\Delta_n^+$. In this case one draws two arrows originating at $\beta+\mu_1$; one parallel to μ_2 and with the same label, and another parallel to μ_1 and labelled μ_3. Similarly μ_1 is drawn from $\beta+\mu_2$ in such a way that its endpoint coincides with the endpoint of the μ_2 drawn from $\beta+\mu_1$. Continuing along these lines the diagram may easily be completed. In fact, the only situation which a priori might ruin this simple picture, that in which we reach an $\alpha \in \Delta_n^+$ in the diagram and for reasons of the structure of the previously constructed part of the diagram are forced to make the same simple compact root μ_0 point out from α in two different directions, is excluded by assumption. This observation is rather useful in the limitations it puts on the diagrams. We formulate it as a lemma which eliminates a situation that obviously would have to occur if the described phenomenon could take place.

LEMMA 4.2. *For no $\alpha \in \Delta_n^+$ does there exist three distinct roots, μ_0, μ_1, and $\mu_2 \in \Sigma_c$ such that $\alpha - \mu_1$, $\alpha - \mu_2$, $\alpha - \mu_1 + \mu_0$, and $\alpha - \mu_2 + \mu_0$ all are elements of Δ_n^+.*

Proof. We may assume that for no α' smaller than α does such a phenomenon occur. ($\alpha = \beta$ is excluded by assumption.) By excluding all other possibilities it follows from Lemma 4.1 that $\langle \alpha, \mu_1 \rangle = \langle \alpha, \mu_2 \rangle = 1$. (See Fig. 1.) Suppose $(\mu_0, \mu_1) < 0$, and consider $\alpha - \mu_1$. It follows that we must be in case (ii) of the proof of Lemma 4.1. Thus $\alpha - \mu_1 - \mu_0$ is a root and the same case then gives that $(\mu_2, \mu_0) < 0$. However, as this implies that $\alpha - \mu_2 - \mu_0$ is a root we quickly reach an α' smaller than α at which a phenomenon similar to the one in Fig. 1 takes place. This is contradictory to the original assumption and hence $(\mu_0, \mu_1) = (\mu_0, \mu_2) = 0$. But clearly $(\alpha, \mu_0) < 0$ and thus $(\alpha - \mu_1 - \mu_2, \mu_0) < 0$, which implies that three different roots originate at $\alpha - \mu_1 - \mu_2$, and this is impossible. ∎

We remark that the assumption that leads to Lemma 4.2; essentially that each element of Δ_n^+ should occur exactly once in the diagram, has been made in order to avoid having to deal with degenerate situations in the subsequent proofs. In other situations (e.g., for $sp(n, \mathbb{R})$) it may well be natural to allow roots to occur more than once.

To further illustrate the simplicity of the construction and for future reference we present the resulting diagrams in the Appendix. That there are no more Hermitian symmetric spaces of the noncompact type than those listed is of course a classical result due to Cartan [2]. However, the criterion that one should be able to pick a root β in the Dynkin diagram such than the

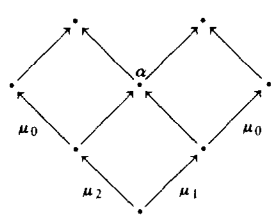


FIGURE 1

set of roots bigger than, or equal to, β, can be represented by a 2-dimensional diagram as above also excludes all other Dynkin diagrams and all other choices of β.

We now begin to collect some of the technical lemmas that will be needed in the following.

First we consider the $\mathscr{U}(\mathfrak{k}_1^{\mathbb{C}})$-module $\mathfrak{p}^- \otimes V_{\Lambda_0}$, where V_{Λ_0} is a finite-dimensional irreducible module of highest weight Λ_0. The highest weights of $\mathfrak{p}^- \otimes V_{\Lambda_0}$ are of the form $\Lambda_0 - \alpha$ for certain $\alpha \in \Delta_n^+$. We wish to describe these in terms of our diagrams. It follows from Proposition 7.3 in [6] and its proof (notably the last part) that

LEMMA 4.3. *Let $\alpha \in \Delta_n^+$. $\Lambda_0 - \alpha$ is a highest weight for the $\mathscr{U}(\mathfrak{k}_1^{\mathbb{C}})$-module $\mathfrak{p}^- \otimes V_{\Lambda_0}$ if and only if*

(i) *$\Lambda_0 - \alpha$ is $\mathfrak{k}_1$-dominant, and*

(ii) *if $\alpha = \alpha_1 + \mu$ with $\mu \in \Sigma_c$ and $\alpha_1 \in \Delta_n^+$, then $\Lambda_0(H_\mu) > 0$.*

LEMMA 4.4. *Let $\alpha \in \Delta_n^+$ and assume $\alpha - \mu_j \in \Delta_n^+$ for $\mu_j \in \Sigma_c$; $j = 1,..., i$, and $i \leqslant 2$.*

Then $\Lambda_0 - \alpha$ is a highest weight for the $\mathscr{U}(\mathfrak{k}_1^{\mathbb{C}})$-module $\mathfrak{p}^- \otimes V_{\Lambda_0}$ if and only if for all $j = 1,..., i$,

$$\langle \Lambda_0, \mu_i \rangle \geqslant \max\{1, \langle \alpha, \mu_i \rangle\}.$$

Proof. $\Lambda_0 - \alpha$ is a highest weight if and only if $\forall \mu \in \Delta_c^+$: $\langle \Lambda_0 - \alpha, \mu \rangle \geqslant 0$. In view of Lemma 4.3 the necessity is thus clear. As for the sufficiency, observe that since Λ_0 is dominant we need only consider those $\mu \in \Delta_c^+$ for which $(\alpha, \mu) > 0$. Let μ be as such: $\mu = \sum_{\mu_j \in \Sigma_c} r_j \mu_j$, $r_j \in \mathbb{N} \cup \{0\}$.

It follows that there must be at least one $\mu_{j_0} \in \Sigma_c$ such that $r_{j_0} > 0$ and $(\alpha, \mu_{j_0}) > 0$. Hence for this simple root, $\alpha - \mu_{j_0} \in \Delta_n^+$. Let $n_0 = \langle \alpha, \mu_{j_0} \rangle$ and $n = \langle \alpha, \mu \rangle$. Then, by assumption,

$$\langle \Lambda_0, \mu \rangle \geqslant \frac{2(\Lambda_0, \mu_{j_0})}{(\mu, \mu)} \geqslant \frac{2(\alpha, \mu_{j_0})}{(\mu, \mu)} = n_0 \left(\frac{(\mu_0, \mu_0)}{(\mu, \mu)} \right).$$

Now observe that by Corollary 3.5, $\langle \tilde{\mu}, \alpha \rangle = 1$ for any $\tilde{\mu} \in \Delta_c$ for which $(\alpha, \tilde{\mu}) > 0$. Thus, $(\mu_0, \mu_0)/(\mu, \mu) = n/n_0$. ∎

DEFINITION 4.5. For $\alpha_0 \in \Delta_n^+$ we let

$$C_{\alpha_0}^+ = \{\alpha \in \Delta_n^+ \mid \alpha \geqslant \alpha_0\} \quad \text{and} \quad C_{\alpha_0}^- = \{\alpha \in \Delta_n^+ \mid \alpha \leqslant \alpha_0\}.$$

As is suggested by the way they appear in the diagram of Δ_n^+, we think of $C_{\alpha_0}^+$ and $C_{\alpha_0}^-$ as the forward and backward cone, respectively, at α_0.

Let $\Lambda = (\Lambda_0, \lambda)$ and let $\alpha \in \Delta_n^+$. Then, by definition,

$$\langle \Lambda, \alpha \rangle = \langle \Lambda_0, \alpha \rangle + \lambda \left(\frac{(\gamma_r, \gamma_r)}{(\alpha, \alpha)} \right), \tag{4.1}$$

where $(\gamma_r, \gamma_r) = (\beta, \beta)$. Recall that Λ_0 is always assumed to be $\mathfrak{k}_1$-dominant and integral.

Lemma 4.6. *Let α_0, $\alpha \in \Delta_n^+$ and suppose that*

$$\langle \Lambda_0 + \rho, \alpha_0 \rangle + \lambda_0 \left(\frac{(\gamma_r, \gamma_r)}{(\alpha_0, \alpha_0)} \right) = 1,$$

and

$$\langle \Lambda_0 + \rho, \alpha \rangle + \lambda \left(\frac{(\gamma_r, \gamma_r)}{(\alpha, \alpha)} \right) = n > 0.$$

If $\lambda < \lambda_0$, then $\alpha \notin C_{\alpha_0}^-$.

Proof. By solving for λ and λ_0, the inequality $\lambda < \lambda_0$ is seen to be equivalent to

$$2(\Lambda_0 + \rho, \alpha) > n(\alpha, \alpha) - (\alpha_0, \alpha_0) + 2(\Lambda_0 + \rho, \alpha_0).$$

Suppose $\alpha = \alpha_0 - \mu$ with $\mu = \sum_{\mu_j \in \Sigma_c} r_j \mu_j$ and all $r_j \geqslant 0$. Unless $n = 1$ and $(\alpha_0, \alpha_0) = 2(\alpha, \alpha)$ we immediately reach an inequality contradicting the dominance of $\Lambda_0 + \rho$. In the remaining situation it follows that for any $\mu_j \in \Sigma_c$ for which $r_j \neq 0$ (and there must be at least one such), $2(\Lambda_0 + \rho, \mu_j) < (\alpha, \alpha)$ and so, since α is short, this case must also be dismissed. ∎

Remark. Under the assumptions of Lemma 4.6, α must in fact be quite a distance away from in particular the lower portions of $C_{\alpha_0}^-$. The separation increases as λ decreases.

For $\alpha \in \Delta_n^+$ let $\lambda_\alpha \in \mathbb{R}$ be determined by the equation

$$((\Lambda_0, \lambda_\alpha) + \rho)(H_\alpha) = 1. \tag{4.2}$$

Among those λ_α's for which $\Lambda_0 - \alpha$ is a highest weight for the $\mathscr{U}(\mathfrak{k}_1^{\mathbb{C}})$-module $\mathfrak{p}^- \otimes V_{\Lambda_0}$, let λ_0 denote the smallest, and let α_0 denote the corresponding element of Δ_n^+. (There are no multiplicities in $\mathfrak{p}^- \otimes V_{\Lambda_0}$.)

Corollary 4.7. *Let $\tilde{\alpha} \in \Delta_n^+$, $\tilde{\alpha} \neq \alpha_0$. If $\tilde{\alpha} \in C_{\alpha_0}^+$, $\Lambda_0 - \tilde{\alpha}$ is not a highest weight for the module $\mathfrak{p}^- \otimes V_{\Lambda_0}$.*

Proof. If $\Lambda_0 - \tilde{\alpha}$ is a highest weight for the module $\mathfrak{p}^- \otimes V_{\Lambda_0}$, $\lambda_0 < \lambda_{\tilde{\alpha}}$. By Lemma 4.6 this implies that $\alpha_0 \notin C_{\tilde{\alpha}}^-$. ∎

Let $\psi=((\Lambda_0,\lambda_0)+\rho)$, let $\alpha_1,\dots,\alpha_r\in\Delta_n^+$, and let $\chi=((\Lambda_0,\lambda)+\rho-n_1\alpha_1-\cdots-n_r\alpha_r)$ with $n_i\in\mathbb{N}$; $i=1,\dots,r$. In the sequel we shall be concerned with the situation in which the sequence $\alpha_1,\dots,\alpha_r$ satisfies condition (A) for the pair (χ,ψ). We assume for simplicity that $n_i=\langle\chi_{i-1},\alpha_i\rangle$ for $i=1,\dots,r$ (cf. Definition 3.1).

LEMMA 4.8. *Consider an α_i from above; $i=1,\dots,r$, and assume $\lambda<\lambda_0$. Then $\Lambda_0-\alpha_i$ is not a highest weight for the module $\mathfrak{p}^-\otimes V_{\Lambda_0}$. Moreover, $\alpha_i\notin C_{\alpha_0}^-$.*

Proof. α_i satisfies

$$\frac{2((\Lambda_0,\lambda)+\rho-n_1\alpha_1-\cdots-n_{i-1}\alpha_{i-1},\alpha_i)}{(\alpha_i,\alpha_i)}=n_i>0.$$

By Lemma 3.4, inner products between elements of Δ_n^+ are nonnegative and thus

$$\frac{2(\Lambda_0+\rho,\alpha_i)}{(\alpha_i,\alpha_i)}+\lambda\frac{(\gamma_r,\gamma_r)}{(\alpha_i,\alpha_i)}=\tilde{n}_i>0.$$

It follows that $\lambda_{\alpha_i}\leqslant\lambda$ and the minimality of λ_0 then gives the first part of the lemma. The second statement follows from Lemma 4.6. ∎

Let $\omega=n_1\alpha_1+\cdots+n_r\alpha_r$. The first real use of the diagram of Δ_n^+ is made in the proof of

LEMMA 4.9. *Assume that $\lambda<\lambda_0$ and that $\Lambda_0-\omega$ is a highest weight for the $\mathscr{U}(\mathfrak{k}_1^{\mathbb{C}})$-module $\mathscr{U}(\mathfrak{p}^-)\otimes V_{\Lambda_0}$. Then at least one α_i, $i=1,\dots,r$, belongs to $C_{\alpha_0}^+$.*

Proof. Suppose not. Then, by Lemma 4.8, $\alpha_i\notin C_{\alpha_0}^+\cup C_{\alpha_0}^-$ for all $i=1,\dots,r$. Consider an arbitrary one of these, α_{i_1}. It is clear that there exists a $\gamma\in\Delta_n^+$ such that (a) $\{\alpha_{i_1},\alpha_0\}\subset C_\gamma^+$, and (b) for no $\gamma_1\in C_\gamma^+$ is $\{\alpha_{i_1},\alpha_0\}\subset C_{\gamma_1}^+$ $(\gamma_1\neq\gamma)$. It follows that there must exist two *distinct* elements, μ_{a_1} and μ_{b_1}, of Σ_c such that $\gamma+\mu_{a_1}\in\Delta_n^+$ and $\gamma+\mu_{b_1}\in\Delta_n^+$. In fact, by the way the diagrams build up, it follows that there are elements $\mu_{a_1},\dots,\mu_{a_s},\mu_{b_1},\dots,\mu_{b_t}$ of Δ_n^+ such that (i) $\alpha_0=\gamma+\mu_{a_1}+\cdots+\mu_{a_s}$, and $\alpha_{i_1}=\gamma+\mu_{b_1}+\cdots+\mu_{b_t}$, (ii) for all $j\leqslant s$ and $k\leqslant t$, $\gamma+\mu_{a_1}+\cdots+\mu_{a_j}+\mu_{b_1}+\cdots+\mu_{b_k}\in\Delta_n^+$, and (iii) $\{\mu_{a_1},\dots,\mu_{a_s}\}\cap\{\mu_{b_1},\dots,\mu_{b_t}\}=\varnothing$ (cf. Fig. 2). Let $\mu_a=\mu_{a_1}+\cdots+\mu_{a_s}$ and $\mu_b=\mu_{b_1}+\cdots+\mu_{b_t}$. By (iii) $(\mu_a,\mu_{b_k})\leqslant 0$ for $k\leqslant t$ and $(\mu_b,\mu_{a_j})\leqslant 0$ for $j\leqslant s$.

If $(\alpha_{i_1},\mu_{b_t})=0$, $\alpha_{i_1}+\mu_{b_t}+\mu_{a_1}\in\Delta_n^+$ and this is easily seen to contradict the proof of Lemma 4.1(iii). Thus, $\langle\alpha_{i_1},\mu_{b_t}\rangle\geqslant 1$, and analogously, $\langle\alpha_0,\mu_{a_s}\rangle\geqslant 1$. If we had $\langle\Lambda_0-\alpha_{i_1},\mu_{b_t}\rangle\geqslant 0$ this would imply that

$$\langle\Lambda_0,\mu_{b_t}\rangle\geqslant\langle\alpha_i,\mu_{b_t}\rangle\geqslant\max\{1,\langle\alpha_{i_1}+\mu_a,\mu_{b_t}\rangle\}.$$

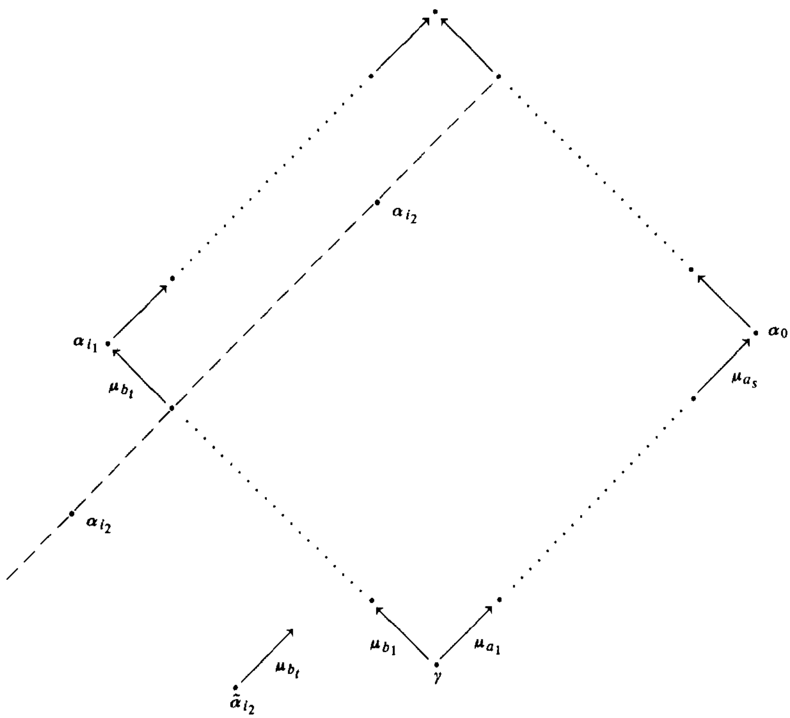


FIGURE 2

Since, by the above and by Lemma 4.4, we also have

$$\langle \Lambda_0, \mu_{a_s} \rangle \geqslant \max\{1, \langle \alpha_0 + \mu_b, \mu_{a_s} \rangle\},$$

this would further imply that $\alpha_{i_1} + \mu_a = \alpha_0 + \mu_b$ is a highest weight which, since it clearly belongs to $C^+_{\alpha_0}$, is impossible. Thus

$$\langle \Lambda_0 - \alpha_{i_1}, \mu_{b_t} \rangle < 0. \tag{4.3}$$

We now invoke the assumption that $\Lambda_0 - \omega$ is a highest weight; in particular that $\langle \Lambda_0 - \omega, \mu_{b_t} \rangle \geqslant 0$. Combined with (4.3) this implies that there exists an $\alpha_{i_2} \in \{\alpha_1, ..., \alpha_r\}$ such that $\langle \alpha_{i_2}, \mu_{b_t} \rangle < 0$. We claim that α_{i_2} has to be located on the line through $\alpha_{i_1} - \mu_{b_t}$, parallel to μ_{a_1} (cf. Fig. 2). Recall that $\alpha_{i_2} \notin C^+_{\alpha_0} \cup C^-_{\alpha_0}$. This may be proved rigorously be eliminating all other possibilities. We mention briefly that Lemma 4.1 and the fact that Dynkin diagrams can have no closed loops immediately rule out all other situations but the one indicated by an $\tilde{\alpha}_{i_2}$ in Fig. 2 (and its symmetrical counterpart). This, however; by a straightforward reduction, can be ruled out by showing that Fig. 3, in which $\gamma_1, ..., \gamma_5$ are distinct elements of Δ^+_n, can have no

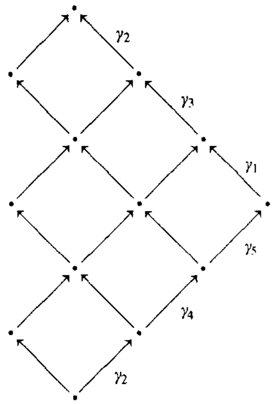


FIGURE 3

existence. But this follows easily from the nature of Dynkin diagrams, Lemma 3.4, or, equivalently and at the same time justifying our brevity; see the Appendix.

The proof is completed by consecutively applying the arguments that led from α_{i_1} to α_{i_2}, to each, from the previously thus obtained element of $\{\alpha_1,..., \alpha_r\}$. To wit, it follows that there are elements $\alpha_{i_3},...,$ of $\{\alpha_1,..., \alpha_r\}$ such that for each $s = 3,..., \alpha_{i_s}$ is located on the line through $\alpha_{i_1} - \mu_{b_t} - \cdots - \mu_{b_{t+2-s}}$ and parallel to the previously described line through α_{i_2}. Clearly, for $s = t + 1$, this is a contradiction. ▮

5. K-TYPES

As a $\mathscr{U}(\mathfrak{k}^{\mathbf{C}})$-module, $\mathscr{U}(\mathfrak{g}^{\mathbf{C}}) \otimes_{\mathscr{U}(\mathfrak{k}^{\mathbf{C}} \oplus \mathfrak{p}^+)} V_\Lambda$ is equal to $\mathscr{U}(\mathfrak{p}^-) \otimes V_\Lambda$. The restriction of B_Λ to each $\mathfrak{k}$-irreducible subspace is, because it is $\mathfrak{k}$-invariant, either zero, strictly positive definite, or strictly negative definite. The problem to which we address ourselves is, for Λ_0 fixed, that of determining the set of λ's for which W_Λ, $\Lambda = (\Lambda_0, \lambda)$, is infinitesimally unitary. This is the case exactly when there are no subspaces on which B_Λ is strictly negative definite.

To begin with consider an irreducible subspace of $\mathscr{U}(\mathfrak{p}^-) \otimes V_\Lambda$ and let $q \neq 0$ be the highest weight vector. Observe that the degree of q is well defined; assume that it is d. Observe moreover that since λ only makes its presence felt through the action of the center of $\mathfrak{k}$, q is, for Λ_0 fixed, a highest weight vector for all λ. Let Λ_0 and q be fixed and consider the function

$$f_q(\lambda) = B_\Lambda(q, q). \tag{5.1}$$

The following is straightforward, and in part well known.

LEMMA 5.2. *$f_q(\lambda) = (-1)^d C_q \lambda^d$ + lower order terms in λ. $C_q > 0$.*

The zeros of $f_q(\lambda)$ are the only places where the restriction of B_Λ to the irreducible subspace in question can change signature. If q is a first order polynomial and if $f_q(\lambda_q) = 0$ it is clear that W_Λ cannot be unitary for $\lambda < \lambda_q$. The smallest λ_q determined by a highest weight vector of degree 1; λ_0, was named "the last possible place of unitarity," and was explicitly determined for an arbitrary Λ_0 in [6].

PROPOSITION 5.3. *Let $\Lambda_0 - \alpha_1, ..., \Lambda_0 - \alpha_t$ be the set of highest weights in the $\mathscr{U}(\mathfrak{k}_1^{\mathbb{C}})$-module $\mathfrak{p}^- \otimes V_{\Lambda_0}$; $\alpha_1, ..., \alpha_t \in \Delta_n^+$. Let, for $i = 1, ..., t$, λ_i be determined by the equation $((\Lambda_0, \lambda_i) + \rho)(H_{\alpha_i}) = 1$. Then $\lambda_0 = \min\{\lambda_1, ..., \lambda_t\}$.*

We shall see below that W_Λ, $\Lambda = (\Lambda_0, \lambda_0)$, in fact *is* unitary.

The determination of the zeros for all f_q's as q varies in the set of highest weight vectors in $\mathscr{U}(\mathfrak{p}^-) \otimes V_\Lambda$ would of course yield a complete solution to our original problem. However, as we shall see, it is sufficient to determine N_Λ for $\lambda < \lambda_0$, and to describe this ideal it is sufficient to determine a set of generators. This set is finite, as is intuitively clear, and as follows from, e.g., the theory of the category $\mathscr{O}$ [1].

Consider now a highest weight vector q in $\mathscr{U}(\mathfrak{p}^-) \otimes V_\Lambda$ of weight $(\Lambda_0, \lambda_q) - n_1\alpha_1 - \cdots - n_r\alpha_r$; $\alpha_1, ..., \alpha_r \in \Delta_n^+$, and $n_1, ..., n_r \in \mathbb{N}$. Let $\omega_q = n_1\alpha_1 + \cdots + n_r\alpha_r$ and observe that the degree of q is $d = n_1 + \cdots + n_r$. Let I_d denote the ideal generated by the elements of degree less than d in N_Λ, $\Lambda = (\Lambda_0, \lambda_q)$. Assume that $q \in N_\Lambda$ and that $q \notin I_d$.

LEMMA 5.4. *The module $(\mathscr{U}(\mathfrak{g}^{\mathbb{C}}) \cdot q + I_d)/I_d$ is a standard cyclic module of highest weight $(\Lambda_0, \lambda_q) - \omega_q$.*

Proof. By assumption q is a highest weight vector for $\mathfrak{k}^{\mathbb{C}}$. Moreover, for any $z^+ \in \mathfrak{p}^+$, z^+q is in N_Λ and is of strictly lower degree than q. ∎

In the following proposition the assumptions on q are maintained.

PROPOSITION 5.5. *$L_{\Lambda + \rho - \omega_q} \in \mathrm{JH}(M_{\Lambda + \rho})$, $\Lambda = (\Lambda_0, \lambda_q)$.*

Proof. It follows from the various identifications of Section 2 that there exist two $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})$-invariant subspaces, A and B, of $M_{\Lambda+\rho}$ such that $B \subset A$, and such that A/B is isomorphic to the unique irreducible quotient of the standard cyclic module in Lemma 5.4. A, B, and $M_{\Lambda+\rho}/A$ belong to the category $\mathscr{O}$ and thus have finite Jordan–Hölder series. ∎

6. $SO^*(2n)$

For later reference we list here a set of unitary representations of $G_0 = SO^*(2n)$. They are obtained along the lines of [4]. Specifically, the Hermitian symmetric space corresponding to $G = SU(n, n)$ may be represented as $\mathscr{D} = \{z \in M(n, \mathbb{C}) \mid z^*z < 1\}$ and G_0 is isomorphic to a subgroup of G that leaves invariant the intersection $\mathscr{D}_0$ of $\mathscr{D}$ with the space of skew-symmetric elements of $M(n, \mathbb{C})$. One easily checks that the results of [4] may be applied, and thus the restriction to G_0 of any unitary holomorphic representation of G yields a series of unitary representations of G_0. The representations of G in question may be taken to live in Hilbert spaces of vector-valued holomorphic functions on $\mathscr{D}$ and hence, in particular, the representation of G_0 obtained from a unitary holomorphic representation of G by restricting the functions to $\mathscr{D}_0$, is unitary.

Let $e_1, \ldots, e_n$ denote the standard orthonormal basis of $\mathbb{R}^n$. Then, for G_0,

$$\Delta_c^+ = \{e_i - e_j \mid 1 \leqslant i < j \leqslant n\},$$

and

$$\Delta_n^+ = \{e_i + e_j \mid 1 \leqslant i < j \leqslant n\}.$$

$\Lambda = (\lambda_1, \ldots, \lambda_n)$ is $\mathfrak{k}_1$-integral and dominant if and only if $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ and $\lambda_i - \lambda_j \in \mathbb{Z}$. $\rho = (n-1, n-2, \ldots, 1, 0)$, and $\lambda = \lambda_1 + \lambda_2$.

By letting τ_2 be the trivial representation in Proposition 2.3 of [5], and taking $j = k$, it follows from the above that the representations

$$(0, \ldots, 0, -m_1, \ldots, -m_j) - (j, j, \ldots, j) \tag{6.1}$$

of G_0 are unitary.

Consider a Λ_0 for which $(\Lambda_0, e_1 - e_2) = 0$; i.e., $\lambda_1 = \lambda_2$, and let i be determined by $\lambda_1 = \lambda_2 = \cdots = \lambda_i \neq \lambda_{i+1}$. It is clear that the λ_0 of Proposition 5.3 is attained at $\alpha_0 = (e_{i-1} + e_i)$, and it follows that

$$\begin{aligned}\lambda_0 = \lambda_1 + \lambda_2 &= \lambda_{i-1} + \lambda_i \\ &= 1 - (n - (i-1)) - (n - i) \\ &= -2n + 2i\end{aligned}$$

and thus, $\lambda_1 = \lambda_2 = \cdots = \lambda_i = -(n-i)$. A comparison with (6.1) then gives the following:

Proposition 6.1. *Let $(\Lambda_0, e_1 - e_2) = 0$. Then W_Λ, $\Lambda = (\Lambda_0, \lambda_0)$, is unitarizable.*

We observe that the case $(\Lambda_0, e_1 - e_2) \neq 0$ cannot be treated by this method.

7. Unitarity at the Last Possible Place

Let Λ_0 be $\mathfrak{k}_1$-integral and dominant. Suppose that for a given λ', W_Λ, $\Lambda = (\Lambda, \lambda')$, is *not* unitarizable. Observe that $\mathfrak{k}$-irreducible subspaces of different highest weights are perpendicular, independently of λ. It follows that the Hermitian form, restricted to the space spanned by the K-types of some highest weight, is not positive semidefinite. On the other hand, for λ sufficiently negative, W_Λ *is* unitarizable. Moreover, the form is a smooth function of λ (polynomial) and changes in the signature can only happen at points at which it is degenerate. It follows easily that for some $\lambda_q < \lambda'$ and some ω_q, we must be in the situation of Proposition 5.5. It thus follows that there exists a sequence $\alpha_1, \ldots, \alpha_r \in \Delta^+$ which satisfies condition (A) for the pair $((\Lambda_0, \lambda_q) - \omega_q + \rho, (\Lambda_0, \lambda_q) + \rho)$. According to Proposition 3.6 we may assume that $\alpha_i \in \Delta_n^+$, $i = 1, \ldots, r$.

Let V_{α_0} denote the $\mathfrak{k}$-invariant subspace of $\mathfrak{p}^- \otimes V_{\Lambda_0}$ of highest weight $\Lambda_0 - \alpha_0$.

Lemma 7.1. *If q is a highest weight vector in $\mathscr{U}(\mathfrak{p}^-) \otimes V_{\Lambda_0}$ of weight $\Lambda_0 - \omega_q = \Lambda_0 - n_1\alpha_1 - \cdots - n_r\alpha_r$, and if for all $i = 1, \ldots, r$, $\alpha_i \in C_{\alpha_0}^+$, then $q \in \mathscr{U}(\mathfrak{p}^-) \cdot V_{\alpha_0}$. Moreover, q does not belong to the $\mathscr{U}(\mathfrak{p}^-)$-ideal generated by any other $\mathfrak{k}$-invariant subspace of $\mathfrak{p}^- \otimes V_{\Lambda_0}$.*

Proof. Since q evidently must belong to one of the $\mathscr{U}(\mathfrak{p}^-)$-ideals generated by the $\mathfrak{k}$-irreducible subspaces of $\mathfrak{p}^- \otimes V_{\Lambda_0}$, it is sufficient to prove that it does not belong to any one different from $\mathscr{U}(\mathfrak{p}^-) \cdot V_{\alpha_0}$. Let $V_{\tilde{\alpha}}$ denote a $\mathfrak{k}$-irreducible subspace of $\mathfrak{p}^- \otimes V_{\Lambda_0}$ of weight $\Lambda_0 - \tilde{\alpha}$ with $\tilde{\alpha} \neq \alpha_0$. To begin with, observe that $\tilde{\alpha} \in C_{\alpha_0}^-$. Using Corollary 4.7 and the fact that $\Lambda_0 - \tilde{\alpha}$ is a highest weight this follows by arguments similar to those that led to (4.3) in the proof of Lemma 4.9. Let Σ_c' denote the smallest subset of Σ_c for which any element of $C_{\alpha_0}^+$ can be written as a linear combination of α_0 and the elements of the subset (coefficient 0 allowed). Let $\mu \in \Sigma_c$ and assume that $\alpha_0 - \mu \in \Delta_n^+$. Our next observation is that $\mu \notin \Sigma_c'$. To wit, if it did, by passing to the "boundary" of $C_{\alpha_0}^+$, one would, through Lemma 4.4, obtain an $\alpha' \in C_{\alpha_0}^+$, $\alpha' \neq \alpha_0$, for which $\Lambda_0 - \alpha'$ is a highest weight of the module $\mathfrak{p}^- \otimes V_{\Lambda_0}$ and this is impossible by Corollary 4.7. Now assume that $q \in \mathscr{U}(\mathfrak{p}^-) \cdot V_{\tilde{\alpha}}$. By the same arguments that gave that a highest weight of $\mathfrak{p}^- \otimes V_{\Lambda_0}$ is of the form $\Lambda_0 - \alpha$ for some $\alpha \in \Delta_n^+$ it follows that there are elements $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_s$ of Δ_n^+ such that $\omega_q = \tilde{\alpha} + \tilde{\alpha}_1 + \cdots + \tilde{\alpha}_s$. Observe that $s = n_1 + \cdots + n_r - 1$. Now, $\tilde{\alpha} \in C_{\alpha_0}^-$ and $\tilde{\alpha} \neq \alpha_0$, and hence there exists a $\mu \in \Sigma_c$ such that $\alpha_0 - \mu \in \Delta_n^+$ and such that, moreover, the coefficient to μ in $\tilde{\alpha}$ is strictly less than the coefficient to μ in α_0. Thus one of the $\tilde{\alpha}_i$'s must have a μ-coefficient which is strictly larger than that of α_0. However, by the observation that $\mu \notin \Sigma_c'$ and by the way the diagrams are built up, this is easily seen to be impossible. ∎

We now begin to examine $\Lambda = (\Lambda_0, \lambda_0)$. Even though in general there will be some λ_q's less than λ_0 at which $L_{\Lambda+\rho-\omega_q} \in \mathrm{JH}(M_{\Lambda+\rho})$ it seems to be generally true that any $\mathfrak{k}$-type whose highest weight occurs in $\mathscr{U}(\mathfrak{p}^-) \otimes V_{\Lambda_0-\omega_q}$ belongs to $\mathscr{U}(\mathfrak{p}^-)\, V_{\alpha_0}$, and thus vanishes at λ_0. Here we shall only investigate the situation for some groups and representations. First we wish to establish that if $\alpha_1,\dots,\alpha_r \in \Delta_n^+$ satisfies condition (A) for the pair $((\Lambda_0,\lambda_q)-\omega_q+\rho, (\Lambda_0,\lambda_q)+\rho)$, where $\Lambda_0-\omega_q$ is the weight of a highest weight vector q in the $\mathscr{U}(\mathfrak{k}_1^{\mathbb{C}})$-module $\mathscr{U}(\mathfrak{p}^-) \otimes V_{\Lambda_0}$ and $\lambda_q < \lambda_0$, then, for all $i=1,\dots,r$, $\alpha_i \in C_{\alpha_0}^+$. For the general case, the combinatorics has so far proved to be too complicated, but we can establish this for sufficiently many cases. To begin with, let $n_1,\dots,n_r$ be the integers of Definition 3.1. In particular,

$$n_i = \langle (\Lambda_0,\lambda_q)+\rho - n_1\alpha_1 - \cdots - n_{i-1}\alpha_{i-1}, \alpha_i\rangle. \tag{7.1}$$

Because inner products between positive noncompact roots are nonnegative, and because $\lambda_q < \lambda_0$, it follows that

$$\langle (\Lambda_0,\lambda_0)+\rho, \alpha_i\rangle > \langle (\Lambda_0,\lambda_q)+\rho, \alpha_i\rangle > 0. \tag{7.2}$$

(Recall that $\langle (\Lambda_0,\lambda_0)+\rho, \alpha_0\rangle = 1$.) Equation (7.1) implies that the α_i's are distinct. Suppose, namely, that $\alpha_i = \alpha_j$, with $j > i$. Then

$$\begin{aligned} n_j &= \langle (\Lambda_0,\lambda_q)+\rho - n_1\alpha_1 - \cdots - n_i\alpha_i \\ &\quad - (n_{i+1}\alpha_{i+1} + \cdots + n_{j-1}\alpha_{j-1}), \alpha_i\rangle \\ &= -n_i - \langle n_{i+1}\alpha_{i+1} + \cdots + n_{j-1}\alpha_{j-1}, \alpha_i\rangle < 0. \end{aligned}$$

Naturally, Eq. (7.1) contains much more information, but for our limited purposes, what has been stated is sufficient.

Consider now the diagram corresponding to so $(2n-1,2)$. (See the Appendix.) Let Λ_0 be $\mathfrak{k}_1$-integral and dominant, and let i denote the smallest integer such that $\langle \Lambda_0, \mu_i\rangle \neq 0$. Assume that $i<n$. In this case it is easy to see that $\alpha_0 = \gamma_r - \mu_2 - \cdots - \mu_{i-1}$ (if $i=2$, $\alpha_0=\gamma_r$) and that any root $\alpha \in \Delta_n^+$ that satisfies (7.2) must be in $C_{\alpha_0}^+$. However, it is also clear that for any such α one can find a μ_j, $j<i$, such that $\langle \alpha, \mu_j\rangle > 0$ and such that α is the only element of $C_{\alpha_0}^+$ with a nonzero inner product with μ_j. But this means that it is impossible to construct a highest weight of the form $\Lambda_0 - n_1\alpha_1 - \cdots - n_r\alpha_r$ with elements α_j, $j=1,\dots,r$, of $C_{\alpha_0}^+$. The conclusion is the same when $i=n$. It suffices to observe that $\langle \Lambda_0, \mu_n\rangle = 1$ is the only case in which $\alpha_0 = \gamma_r - \mu_2 - \cdots - \mu_n$, and that this α_0 satisfies $\langle \alpha_0, \mu_n\rangle = 0$ (in fact, $\langle \alpha_0, \Sigma_c\rangle = 0$). Analogous arguments yield the same conclusion for so $(2n-2,2)$ and all cases for e_6 and e_7 but the ones described below:

e_6. Following the notation of the Appendix, consider a Λ_0 for which

$\langle \Lambda_0, \mu_2 \rangle = \cdots = \langle \Lambda_0, \mu_5 \rangle = 0$, and $\langle \Lambda_0, \mu_6 \rangle = n > 0$. Clearly, $\alpha_0 = \beta + \mu_3 + \mu_4 + \mu_5 + \mu_6$. If $n = 1$ the root $\alpha' = \beta + \mu_3 + \mu_4 + \mu_5 + \mu_2 + \mu_4 + \mu_3$ satisfies (7.2) but is not in $C^+_{\alpha_0}$. However, it is impossible to have α' belong to the set $S = \{\alpha_1, ..., \alpha_r\}$ because of the μ_3 pointing towards it. Suppose, namely, that $\alpha' \in S$. Since $\Lambda_0 - n_1\alpha_1 - \cdots - n_r\alpha_r$ is a highest weight and $\langle \alpha', \mu_3 \rangle > 0$, either $\alpha' - \mu_3 + \mu_6$ or $\alpha' - \mu_3 + \mu_6 + \mu_5$ (or both) must belong to S. If, say, $\alpha' - \mu_3 + \mu_6 \in S$, because of the μ_4 pointing towards this root, we must have $\alpha' + \mu_6 + \mu_5 \in S$, and the coefficient of that root must be at least equal to that of $\alpha' - \mu_3 + \mu_6$. But there is a μ_3 pointing towards that root as well, and the net effect is that $\langle \omega_q, \mu_3 \rangle > 0$. By considering μ_5, the case of $\alpha' - \mu_3 + \mu_6 + \mu_5$ is excluded as well.

e_7. In the case $\langle \Lambda_0, \mu_1 \rangle = \cdots = \langle \Lambda_0, \mu_5 \rangle = 0$ and $\langle \Lambda_0, \mu_6 \rangle = n > 0$, $\alpha_0 = \beta + \mu_6 + \mu_5 + \mu_4 + \mu_3 + \mu_2 + \mu_4 + \mu_5 + \mu_6$. When $n = 1$ the roots $\alpha_0 - \mu_6 + \mu_1 + \mu_3 + \mu_4$ and $\alpha_0 - \mu_6 + \mu_1 + \mu_3 + \mu_4 + \mu_2$ must be excluded, and for $n = 2$ the latter must be dismissed whereas the former no longer satsifies (7.2). This is done by arguments analogous to those of e_6. We stress that in the mentioned cases for e_6 and e_7 there *are* solutions to (7.1) with $\lambda_q < \lambda_0$.

Finally we consider $so^*(2n)$ and the case left open in Section 6: $\langle \Lambda_0, e_1 - e_2 \rangle \neq 0$. Let j be the biggest integer such that $\lambda_2 = \cdots = \lambda_j$. Evidently then, $\alpha_0 = e_1 + e_j$. Observe that $C^+_{\alpha_0}$ is "one-dimensional," We claim that, in this case, there can be no solutions to (7.1) with $\lambda_q < \lambda$. Indeed, if there were, according to Lemma 4.9 there would be an α among the $\alpha_1, ..., \alpha_r$ which belongs to $C^+_{\alpha_0}$, and hence also a smallest such, α'. However, by considering the simple compact root μ pointing towards α' it follows by an exhaustion argument along the lines of the proof of Lemma 4.9 that this is not possible. Together with Section 6, this extablishes the unitarity at the last possible place for $so^*(2n)$.

Let us now return to the case left open for e_6. Observe that $\lambda_0 = -4$. At $\lambda = -4 - i$, $L_{\Lambda + \rho - (4-i)(\alpha_0 + \gamma_r)} \in \mathrm{JH}(M_{\Lambda + \rho})$, $i = 1, 2, 3$, but $\Lambda_0 - s(\alpha_0 + \gamma_r)$ is a highest weight if and only if $\langle \Lambda_0, \mu_6 \rangle \geqslant s$. Let V_q denote a subspace of $\mathscr{U}(\mathfrak{p}^-) \otimes V_{\Lambda_0}$ of highest weight $\Lambda_0 - s(\alpha_0 + \gamma_r)$, $s = 1, 2, 3, ...$. According to Lemma 7.1, $V_q \subseteq \mathscr{U}(\mathfrak{p}^-) \cdot V_{\alpha_0}$. More generally, any irreducible subspace of $\mathscr{U}(\mathfrak{p}^-) \otimes V_{\Lambda_0}$ whose highest weight occurs as a such in $\mathscr{U}(\mathfrak{p}^-)\, V_q$ actually belongs to $\mathscr{U}(\mathfrak{p}^-)\, V_{\alpha_0}$. This is intuitively clear since the coordinate functions of V_{α_0} corresponding to the highest weight vector v_0 in V_{Λ_0} include the terms $z_{-\alpha_i}$, $\alpha_i \in C^+_{\alpha_0}$. Any highest weight vector $\tilde{q}$ in $\mathscr{U}(\mathfrak{p}^-)\, V_q$ must, because of μ_6, have a coordinate function $\tilde{q}_0$ relative to v_0 of the form $\tilde{q}_0 = \sum_{i=0}^{N} z_{-\alpha_i} p_i$, where $\alpha_i \in C^+_{\alpha_0}$ and $p_i \in \mathscr{U}(\mathfrak{p}^-)$. This observation can easily be made rigorous since it is enough to consider the case in which none of the p_i's contain elements corresponding to $-C^+_{\alpha_0}$. ($\mathscr{U}(\mathfrak{p}^-) \otimes V_{\Lambda_0}$ is built up by consecutively tensoring representations with $\mathfrak{p}^-$). In this case the coordinate function is readily seen to coincide with the coordinate function of a highest weight vector in $\mathscr{U}(\mathfrak{p}^-) \cdot V_{\alpha_0}$. The remaining case for e_7 may be treated analogously but we observe that the representations in question ($\lambda_0 = -8$) exactly are those that occur in the decomposition of the tensor product of the most singular, unitary, nontrivial representation with $\Lambda_0 = 0$, with itself. The λ-parameter of that representation is -4.

The only remaining cases now are $sp(n, \mathbb{R})$ and $su(p, q)$. The unitarity at the last possible place for those groups has been established in [5, 6]. We can then state

PROPOSITION 7.2. *W_Λ, with $\Lambda = (\Lambda_0, \lambda_0)$, is unitarizable.*

8. THE UNITARY HIGHEST WEIGHT MODULES

With the unitarity at the last possible place thus established the description of the general situation follows by forming tensor products, along the lines of [4], of W_Λ; $\Lambda = (\Lambda_0, \lambda_0)$, with the most singular, nontrivial, unitary module W_{λ_s} corresponding to $\Lambda_0 = 0$ (and $\lambda = \lambda_s$). The value of λ_s has been determined in [8, 9]. W_{λ_s} contains, of course, all first order polynomials, but the $\mathscr{U}(\mathfrak{p}^-)$-ideal generated by the $\mathfrak{k}_1$-irreducible subspace of the second order polynomials of highest weight $-\beta - \gamma'$ is missing. Here, γ' denotes the smallest element of Δ_n^+ perpendicular to β. The restriction of $W_\Lambda \otimes W_{\lambda_s}$ to the diagonal (cf. [4]) is the unitarizable module $W_{\Lambda'}$, $\Lambda' = (\Lambda_0, \lambda_0 + \lambda_s)$. If a second order polynomial is missing from W_Λ, it is clear that it must be in the ideal $\mathscr{U}(\mathfrak{p}^-) \cdot V_{\alpha_0}$ and in no other such. Moreover, in this case it follows (cf. Lemma 5.2) that the modules W_Λ, $\Lambda = (\Lambda_0, \lambda)$ with $\lambda_0 + \lambda_s < \lambda < \lambda_0$, are not unitarizable. From these observations it is straightforward to decide in the explicit cases at hand exactly in which cases second order polynomials are missing from $W_{\Lambda'}$, and by repeatedly forming tensor products with W_{λ_s} the complete description of the set of λ's below λ_0 at which certain polynomials are missing from the corresponding module W_Λ, follows. Moreover, if a third order polynomial q is missing at $\lambda_0 + 2\lambda_s$ (this can only happen for $\mathfrak{g} = su(p, q)$, $sp(n, \mathbb{R})$, or $so^*(2n)$), it is in the ideal generated by the second order polynomial that is missing at $\lambda_0 + \lambda_s$. Hence q is missing at $\lambda_0 + 2\lambda_s$, $\lambda_0 + \lambda_s$, and λ_0. Further, the Hermitian form restricted to q is positive for $\lambda \to -\infty$, hence it is negative between $\lambda_0 + 2\lambda_s$ and $\lambda_0 + \lambda_s$. Continuing along these lines we see that if a jth order polynomial is missing at $\lambda_0 + (j-1)\lambda_s$, the Hermitian form cannot be positive semi-definite in the open interval from $\lambda_0 + (j-1)\lambda_s$ to $\lambda_0 + (j-2)\lambda_s$.

Observe that the description of the generators of the missing K-types fits nicely into the diagrammatic approach presented here. As an example with $\mathfrak{g} = su(p, q)$, the missing fifth order polynomial four steps below a λ_0 is indicated in Fig. 4. The 1's indicate that the roots occur exactly once in ω_q.

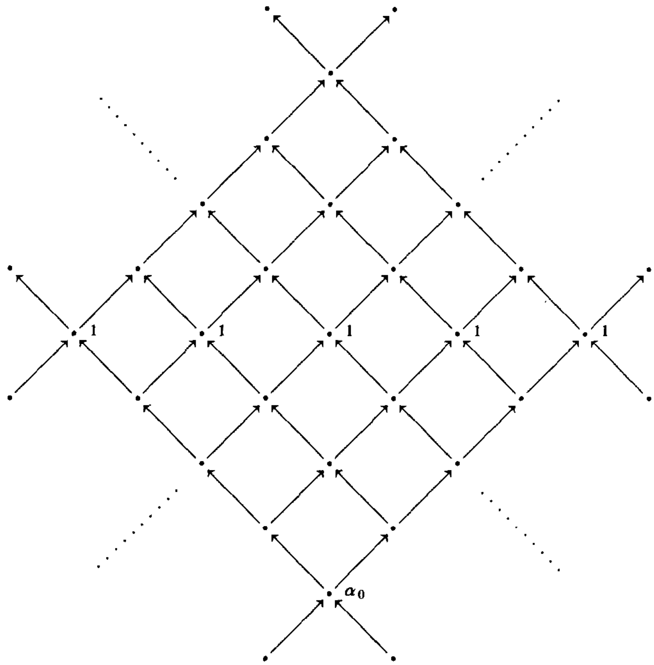


FIGURE 4

Naturally, Λ_0 as well as (p, q) must satisfy certain conditions for this configuration to be possible. (The horizontal string of 1's must hit both walls of $C^+_{\alpha_0}$, cf. below).

When $\mathfrak{g} = sp(n, \mathbb{R})$ or $so^*(2n)$ the natural scene for the presentation of the missing K-types is the union of Δ_n^+ with its mirror image around the line joining β and γ_r in the Δ_n^+ of $sp(n, \mathbb{R})$ (cf. the remark following the proof of Lemma 4.2). In these terms the pictures are identical to those for $\mathfrak{g} = su(n, n)$ with the exception that for $\mathfrak{g} = so^*(2n)$ one must exclude those pictures that contain points on the above mentioned line.

Finally for e_6 and e_7 the missing second order polynomials are easily localized. Observe that (α_0, Σ_c') is a basis for a root system (Σ_c' as in the proof of Lemma 7.1); for e_6 it corresponds to $so(8, 2)$ and for e_7 to $so(10, 2)$. For e_6, the corresponding λ is -7, whereas for e_7 it is -12.

To complete the description of the set of unitarizable highest weight modules we observe that by tensoring with W_{λ_s} a finite number of times we reach a point λ_c beyond which there will be unitarity for all $\lambda < \lambda_c$. (λ_c might—though unitary—be called the first possible place for nonunitarity). This is so because of the structure of the diagrams combined with the fact that as λ decreases, the number of points in the diagram that can be used in

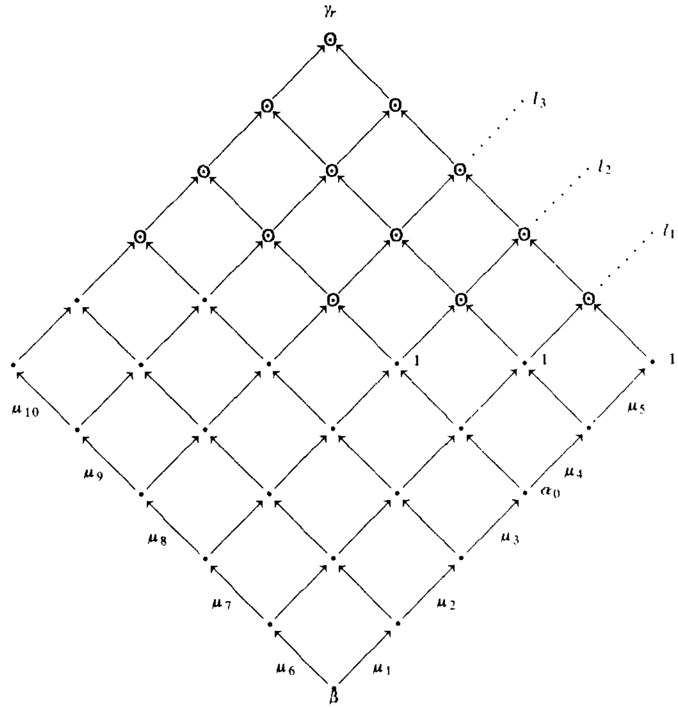


FIGURE 5

a sequence satisfying condition (A), decreases. Recall that by Proposition 5.5, there must be such a sequence at the first possible place of nonunitarity. Thus it may be seen that beyond the mentioned points in the examples of e_6 and e_7 there is unitarity. In all other cases for e_6 and e_7, as well as all cases for $so(n, 2)$, $\lambda_c = \lambda_0$ $(\Lambda_0 \neq 0)$. The cases of $su(p, q)$, $sp(n, \mathbb{R})$, and $so^*(2n)$ are straightforward. We conclude with an example for $\mathfrak{g} = su(6, 6)$ that contains all the relevant features of this remark.

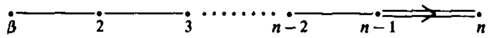
EXAMPLE. $\mathfrak{g} = su(6, 6)$. Assume $\langle \Lambda_0, \mu_i \rangle = 0$ for $i = 1, 2, 4, \ldots, 10.$ and $\langle \Lambda_0, \mu_3 \rangle = 1$. Then $\alpha_0 = \beta + \mu_1 + \mu_2 + \mu_3$, and $\lambda_0 = -3$. At $\lambda = -4$ a second order polynomial is missing, and at $\lambda = -5$ a third order polynomial is missing. The latter is indicated by the string of 1's. For $\lambda < -5$ one can at most use the roots indicated by circles to form a sequence satisfying condition (A). However, the result must be $\mathfrak{k}_1$-dominant and so, because of μ_6, there can be no points in such a hypothetical sequence on the line l_1. But then, because of μ_7, there can be no points on l_2, etc. We conclude that there is unitarity for $\lambda < -5$.

Appendix

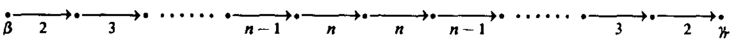
The diagrams of Δ_n^+

$\mathfrak{g} = so(2n-1, 2)$.

B_n:

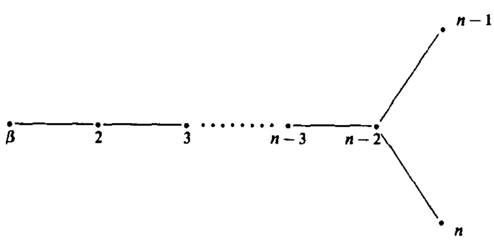


Δ_n^+:

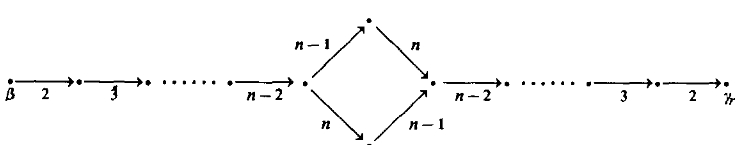


$\mathfrak{g} = so(2n-2, 2)$.

D_n:

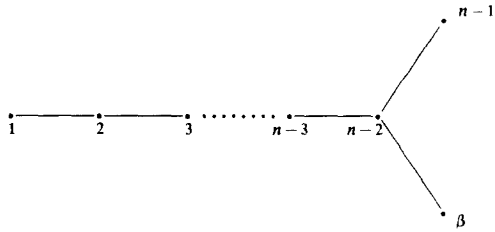


Δ_n^+:

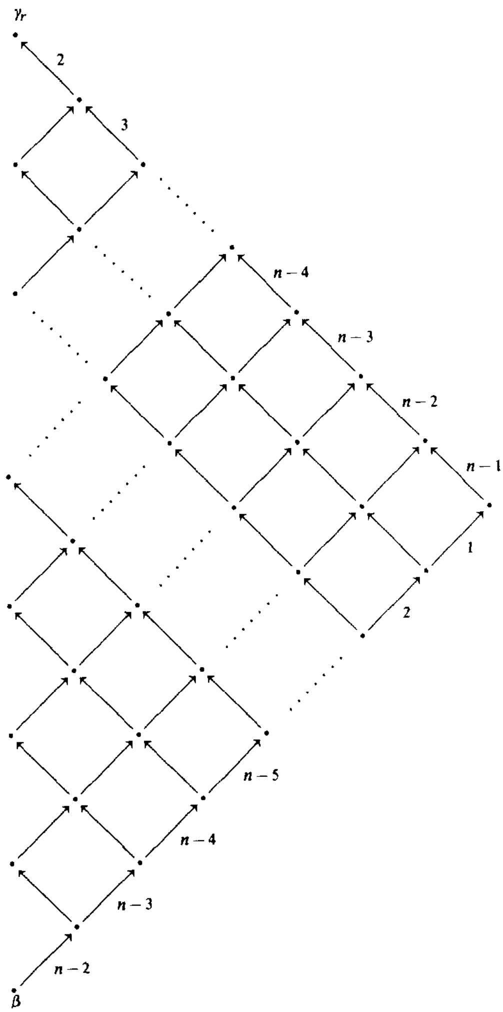


$\mathfrak{g} = so^*(2n)$.

D_n:



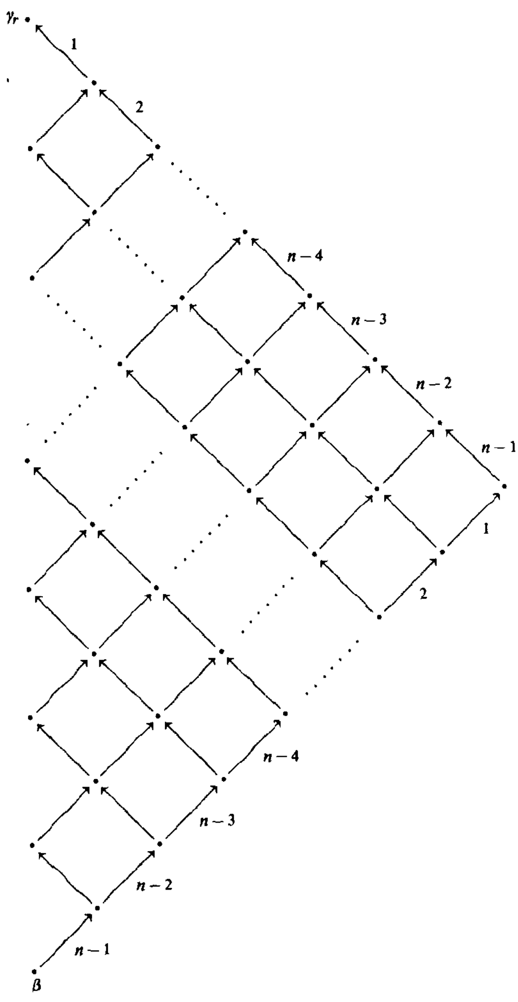
Δ_n^+:



$\mathfrak{g} = sp(n, \mathbb{R})$.

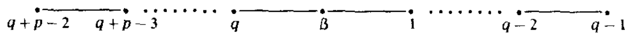
C_n:

Δ_n^+:

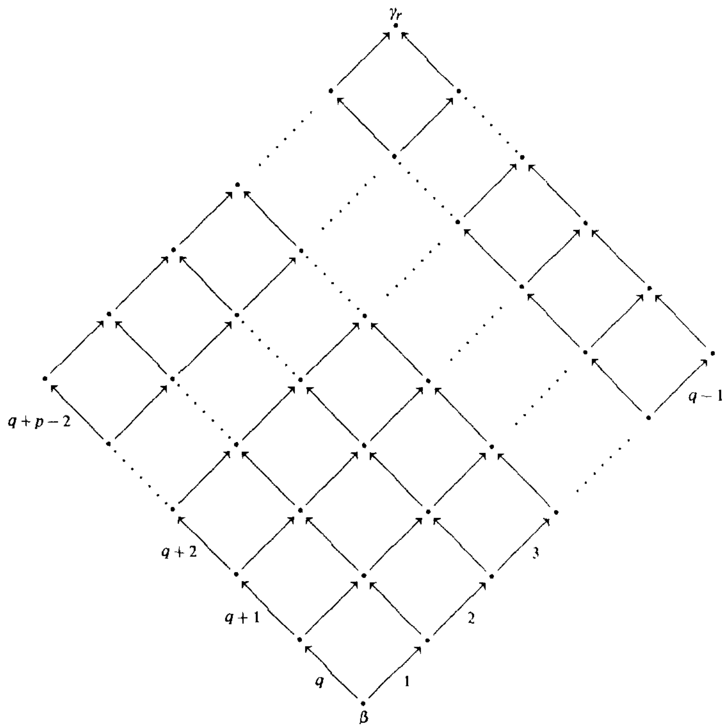


$\mathfrak{g} = su(p, q)$.

A_{q+p-1}:

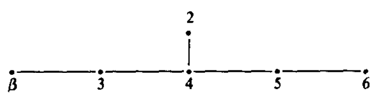


Δ_n^+:

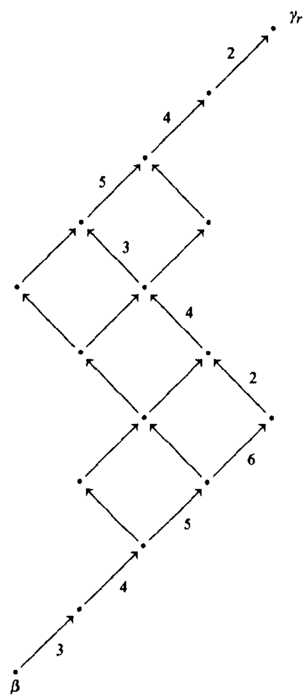


$\mathfrak{g} = e_6$.

e_6:

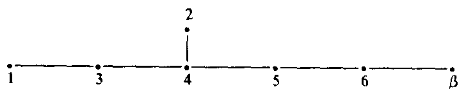


Δ_n^+:

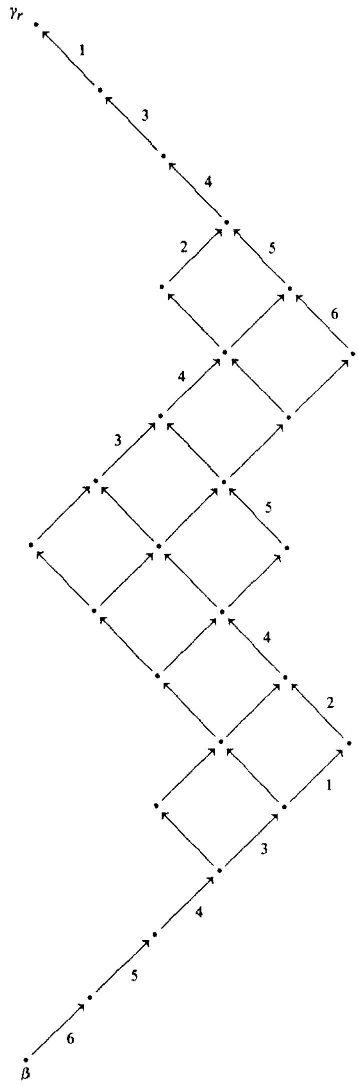


$\mathfrak{g} = e_7$.

e_7:



Δ_n^+:



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