Mathematics 3MI **Hans Plesner Jakobsen**
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Weekly notice #11

The lectures in week 15: We discussed the Cantor set Z. After this, §6.1 was completed.

The lectures in week 16: Product measures. Theorems of Fubini and Tonelli.

Homework - to be handed in to the TA in week 18: 3MI-Exam January '00; Exercises 2 and 4 (see below)

The problem sessions in week 18: W11.1, W11.2, W11.3, 6.14, 6.15, 6.17 (you may here assume that X and Y are countable), 6.18, 6.19, 6.20.

Problem W11.1: Let $(X, \mathbb{E}, \mu)$ be a σ-finite measure space and let $f : X \times X \to \mathbb{C}$ be a measurable function that satisfies $f(x, y) = -f(y, x)$. Show that if $f \in \mathcal{L}(\mu \otimes \mu)$ then

$$\int_X \left(\int_X f(x, y) \, d\mu(x) \right) d\mu(y) = 0.$$

Give an example of a function f which satisfies $f(x, y) = -f(y, x)$ and for which the integral above is defined and $\neq 0$. [Hint: Consider a suitable rational function $q(x, y)$in two variables in $[0, 1) \times [0, 1]$, i.e., $q(x, y) = p_1(x, y)/p_2(x, y)$ with p_1, p_2 polynomials.]

Problem W11.2: Let I_p, I_q be standard intervals in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively. Let f be a real Borel function on $\mathbb{R}^k$, where $k = p + q$. Show that if

$$\int_{I_q} \int_{I_p} |f(x, y)| \, dx \, dy < \infty$$

then both double integrals

$$\int_{I_q} \int_{I_p} f(x, y) \, dx \, dy \quad \text{and} \quad \int_{I_p} \int_{I_q} f(x, y) \, dy \, dx$$

exist and are equal.

Problem W11.3: (Addendum to exercise 6.14) Let $(X, \mathbb{E}, \mu)$ be a σ-finite measure space and let $f \in \mathcal{M}^+(X, \mathbb{E})$ (with finite values). Show that

$$\int_X f \, d\mu = \int_0^\infty \mu(f^{-1}(]t, \infty[)) \, dt.$$

[Hint: Look at the product space $(X \times \mathbb{R}, \mathbb{E} \otimes \mathbb{B}, \mu \otimes m)$ and at the sets

$$G(f) = \{(x,t) \mid 0 \le t < f(x)\} \subseteq X \times \mathbb{R}. \tag{1}$$

Don't forget to skow that $G(f) \in \mathbb{E} \otimes \mathbb{B}$. This may be done by first looking at the case where f is simple and then by using that $G(f) = \cup_{n=1}^\infty G(s_n)$ if $s_n \nearrow f$.]

Remark: The function $t \mapsto \mu(f^{-1}(]t, \infty[))$ is decreasing and hence, actually, the generalized Riemann integral exists. One can thus *define* the Lebesgue integral w.r.t. an arbitrary (σ-finite) measure μ by (1) above.

January 2000 Exercise 2: Let E be the set of points $(x, y) \in \mathbb{R}^2$ for which either x or y is rational. Show that E is a Borel subset of $\mathbb{R}^2$ and determine $m_2(E)$, where m_2 denotes the Lebesgue measure on $\mathbb{R}^2$.

January 2000 Exercise 4: Set

$$A = \{(x,y) \in \mathbb{R}^2 \mid 0 < x \le y \le \pi/2\} \subseteq \mathbb{R}^2,$$

and let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \cos(y)/y & \text{if } (x,y) \in A \\ 0 & \text{if } (x,y) \in \mathbb{R}^2 \setminus A \end{cases}$$

Carefully explain why

$$\int_{\mathbb{R}^2} \mathsf{f(x,y)\ dm_2(x,y)} = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathsf{f(x,y)\ dm(x)} \right) \mathsf{dm(y)} = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathsf{f(x,y)\ dm(y)} \right) \mathsf{dm(x)},$$

and then compute the integral.