Københavns Universitet

Examination in Mathematics 3MI

Solutions to selected questions (sketched).

Exercise 2 (15 points). Let $\underline{a} = (a_n)_{n=0}^{\infty}$ be a complex sequence which belongs to $\ell_4 = \mathcal{L}_4(\mathbb{N}_0, \mathcal{P}(\mathbb{N}_0), \mu)$, where μ is the counting measure. Assume that $\|\underline{a}\|_4^4 = 15^3$. Prove that

$$\sum_{n=0}^{\infty} \frac{|a_n|}{2^{3\cdot n}} \leq 8.$$

Solution: Let $\underline{b} = (b_n)_{n=0}^{\infty}$ be the complex sequence with $b_n = \frac{1}{2^{3\cdot n}}$. Then $\underline{b} \in \ell_{4/3}$, where $4/3$ is the dual exponent to 4. A small computation shows that $\|\underline{b}\| = (\frac{16}{15})^{3/4}$, and the result now follows from the stated norm of $\underline{a}$ in conjunction with Hölder's Inequality. One must, however, also introduce the sequence $\underline{|a|} = (|a_n|)_{n=0}^{\infty}$ and observe that $\underline{|a|}$ has the same ℓ_4 norm as $\underline{a}$.

Exercise 3 (20 points). Let $(X, \mathbb{E}, \mu)$ be a measure space. Let $A, B, C \in \mathbb{E}$ satisfy

$$\mu(A \cup B) = 2, \quad \mu(A \setminus C) = \frac{3}{4}, \quad \mu(A) = \frac{3}{2}, \quad \mu(B) = \frac{1}{2}, \quad \mu(C) = 3.$$

(i) Determine $\mu(A \cap B)$.

(ii) If, furthermore, $\mu(C \setminus (A \cup B)) = 2$, prove that $\mu(B \cap C) = \frac{1}{4}$.

(iii) Let $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_N \supseteq \ldots$ be a decreasing sequence of measurable sets, and assume $\forall n : \mu(B_n \setminus B_{n+1}) = 2^{-n}$. Determine

$$\mu\left(\cup_{n=1}^{\infty}(B_n \setminus B_{n+1})\right).$$

(iv) Let $C_1 \supseteq C_2 \supseteq \cdots \supseteq C_N \supseteq \ldots$ be a decreasing sequence of measurable sets, and assume that it now are the measures of the following set-differences; $\forall n : \mu(C_n \setminus C_{n+2}) = 2^{-n}$, that are given, but where furthermore you are informed that $\mu(C_1 \setminus C_2) = \frac{1}{4}$. Determine

$$\mu\left(\cup_{n=1}^{\infty}(C_n \setminus C_{n+1})\right).$$

(v) Is the number computed in (iv) always equal to $\mu(C_1)$? - Explain!

Solutions to (iv) and (v): We have that

$$\mu(C_n \setminus C_{n+2}) = \mu(C_n \setminus C_{n+1}) + \mu(C_{n+1} \setminus C_{n+2})$$

which is the basic observation. Then,

$$\begin{aligned}\sum_{n=1}^{\infty}\mu(C_n\setminus C_{n+2}) &= \sum_{n=1}^{\infty}\frac{1}{2^n}=1\\ &= \sum_{n=1}^{\infty}\mu(C_n\setminus C_{n+1})+\sum_{n=1}^{\infty}\mu(C_{n+1}\setminus C_{n+2})\\ &= \sum_{n=1}^{\infty}\mu(C_n\setminus C_{n+1})+\sum_{n=1}^{\infty}\mu(C_n\setminus C_{n+1})-\mu(C_1\setminus C_2),\end{aligned}$$

from which the result follows.

Another good idea is to use either the first or the second equality below:

$$\cup_{n=1}^{\infty}(C_n\setminus C_{n+1})=\cup_{n\in\mathbb{N},n\ \mathrm{odd}}^{\infty}(C_n\setminus C_{n+2})=(C_1\setminus C_2)\cup\left(\cup_{n\in\mathbb{N},n\ \mathrm{even}}^{\infty}(C_n\setminus C_{n+2})\right).$$

Since

$$C_1=\cup_{n=1}^{\infty}(C_n\setminus C_{n+1})\cup(\cap_{n=1}^{\infty}C_n)\quad(\text{disjoint union}),$$

and we do not assume $\mu(\cap_{n=1}^{\infty}C_n)=0$, we cannot in general find the measure of C_1 in this way.

Exercise 4 (15 points). Let $\forall x\in\mathbb{R}: f(x)=e^{\sin x}$. Determine

$$lim_{n\to\infty}\int_{-1}^{1}f(\frac{x}{n})\,dx.$$

Exercise 5 (25 points). Let X,Y be non-empty sets and let $\phi: X\mapsto Y$ be a function from X to Y.

(i) Let $\mathbb{D}_X$ be a Dynkin system (also called a σ-class) in X. Prove that

$$\widetilde{\mathbb{D}}_Y=\{F\in\mathcal{P}(Y)\mid\phi^{-1}(F)\in\mathbb{D}_X\}$$

is a Dynkin system in Y.

(ii) Let $\mathbb{E}_X$ and $\mathbb{E}_Y$ be σ-algebras in X and Y, respectively, and set

$$\widetilde{\mathbb{E}}_Y=\{F\in\mathbb{E}_Y\mid\phi^{-1}(F)\in\mathbb{E}_X\}.$$

Prove that $\widetilde{\mathbb{E}}_Y$ is a σ-algebra in Y.

(iii) Let $\widehat{\mathbb{E}}_X\subseteq\mathcal{P}(X)$ be defined by

$$\widehat{\mathbb{E}}_X=\{\phi^{-1}(F)\mid F\in\mathbb{E}_Y\}.$$

Prove that $\widehat{\mathbb{E}}_X$ is a σ-algebra in X.

(iv) Let $\mathbb{K}_Y$ be a generating system for $\mathbb{E}_Y$. Prove that

$$\widehat{\mathbb{K}}_X=\{\phi^{-1}(K)\mid K\in\mathbb{K}_Y\}$$

is a generating system for $\widehat{\mathbb{E}}_X$.

Solution to (iv): It is clear that $\widehat{\mathbb{K}}_X \subseteq \widehat{\mathbb{E}}_X$, hence it will generate a σ-algebra $\sigma(\widehat{\mathbb{K}}_X) = \check{\mathbb{E}}_X \subseteq \widehat{\mathbb{E}}_X$. If $\check{\mathbb{E}}_X \neq \widehat{\mathbb{E}}_X$ then

$$\mathbb{K}_Y \subseteq \widetilde{\check{\mathbb{E}}_Y} = \{F \in \mathbb{E}_Y \mid \phi^{-1}(F) \in \check{\mathbb{E}}_X\} \neq \mathbb{E}_Y,$$

and by (ii), this is a contradiction.

Exercise 6 (15 points). a) Let $g \in \mathcal{L}_2(\mathbb{R}, \mathbb{B}, dx)$, where dx denotes the Lebesgue measure on $\mathbb{R}$. Prove that the function

$$F(t) = \int_{\mathbb{R}} \frac{1}{(x^2 + t^2 + 1)^{1/2}}\, g(x)\, dx$$

belongs to $C^1(\mathbb{R})$ and argue (or prove) that it actually belongs to $C^\infty(\mathbb{R})$.

b) Let $\psi \in \mathcal{S}$ be a Schwartz function on $\mathbb{R}$. Prove that

$$\int \left(\int \hat{\psi}(t) e^{-(x-t)^2/2}\, dt \right) dx = (\sqrt{2\pi})^3 \psi(0),$$

where $\hat{\psi}$ denotes the Fourier transform of ψ. You must carefully justify all steps in the computation.