Mathematics 3MI — **Hans Plesner Jakobsen**
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Weekly notice #4

The lectures in week 8: §§2.1 - 2.2 were covered.

Make sure that you understand all the details having to do with inverse images (as in the proof of "Sætning 2.2"). If you are in doubt, ask the instructur or I. You may also wish to consult the notes (in Danish) on set theory and logic (to be found under "Supplerende noter" in the course home page). Notice also the notes (again in Danish) about "limes superior" and related topics. These may come in handy in §0 and §2.2.

The lectures in week 9: We will finish §2 and then move on to §4.1 where we will integrate simple, and then positive, functions.

Homework - to be handed in to the teaching assistant in week 10: The winter exam in 3MI, january 1999: Exercise 2 parts (i), (v), and (vi). (See page 2 of this notice).

The problem sessions in week 10: Problem W4.1, and exercises 4.3, 4.4, and 4.6 from the book. For those needing even more challenge: Problem W4.2.

Problem W4.1. It will be shown later in this course that if a function $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then it is also Lebsegue integrable and the two integrals agree. You may use this fact in the following!

The aim of this exercise is to prove that there are (bounded, positive) Lebesgue integrable functions $f : [a, b] \to \mathbb{R}$ which are not Riemann integrable and such that **furthermore** one cannot find any function g which is Riemann integrable and such that $f = g$ a.e.

(i) Prove that if U is open and dense in $[0, 1]$ and if N is a null-set w.r.t. the Lebesgue measure m on $\mathbb{R}$, then $U \setminus N$ is also dense in $[0, 1]$.

(ii) Suppose that E is a dense Borel subset of $[0, 1]$ and that $m(E) < 1$. Show that if $g : [a, b] \to \mathbb{R}$ is a function such that $1_E \leq g$ and $1_E = g \; a.e.$, then g is *not* Riemann integrable. [Hint: First prove that any upper sum for g is ≥ 1. Then use the fact mentioned in the beginning.]

(iii) Prove what we are aiming at! [Hint: Try $f = 1_U$ for an appropriate open set U; see for instance Problem 2.3 on weekly notice $\#2$.]

Problem W4.2. (This is a continuation of Problem W3.3) Suppose that $\mathcal{A}$ satisfies (i), (ii), and (iii) in Proposition W3 on weekly nocice #3. Show that:

(i) All simple functions in $\mathcal{M}^+(X, \mathbb{B}(X))$ belong to $\mathcal{A}$. [Hint: W3.3 (iii)]

(ii) All Borel functions $f : X \to \mathbb{R}^+$ belong to $\mathcal{A}$. [Hint: Use an appropriate Proposition from §4.1]

(iii) $\mathcal{B}(X, \mathbb{C}) \subseteq \mathcal{A}$.

3MI–exam, January 1999 - Problem 2 (25 points out of 100): This problem consists of 6 short exercises which are independent of each other.

(i) Determine
$$\int_{\mathbb{R}} \sum_{n=1}^{\infty} \frac{1}{n^3} 1_{(0,n]} dm,$$
and give a careful justification for your answer.

(ii) Consider the measurable space $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}))$, and let for each $a \in \mathbb{Z}$ ε_a denote the Dirac measure (centered) in a. Let $f : \mathbb{Z} \to \mathbb{R}$ denote the function $f(x) = \cos(x^2)$. Compute
$$\int_{\mathbb{Z}} f \; d(\varepsilon_{-1} + \varepsilon_0 + \varepsilon_1).$$

(iii) Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of functions which are integrable w.r.t. the Lebesgue measure. Suppose furthermore that $f_n \to 0$ pointwise. Does it follow that
$$\lim_{n\to\infty} \int_0^1 f_n(x) dm(x) = 0 \;?$$
Either prove this, or give a counter example.

(iv) Determine
$$\lim_{n\to\infty} \int_0^{\infty} \sin(x)(\sqrt{x^2 + (1/n)} - x)e^{-x} dx,$$
and give a careful justification for your answer.

(v) Let $f, g : \mathbb{R} \mapsto \mathbb{R}$ be *Borel* functions satisfying $f = g$ m-a.e. Does it follow that $f(x) = g(x)$ for all $x \in \mathbb{R}$? Prove this, or give a counter example.

(vi) Let $f, g : \mathbb{R} \mapsto \mathbb{R}$ be *continuous* functions satisfying $f = g$ m–a.e. Does it follow that $f(x) = g(x)$ for all $x \in \mathbb{R}$? Prove this, or give a counter example.