Mathematics 3MI **Hans Plesner Jakobsen**
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Weekly notice #7

The lectures in week 11: We defined the space $\mathcal{L}$ of real integrable functions, proved Lebesgue's Dominated Convergence Theorem and applied this to a study of continuity and differentiation "under the integral sign".

The lectures in week 12: We will discuss some of of the remaining topics from §4. In §4.4 we will bluntly *define* the integral of a subset $V \in \mathbb{E}$ of a measure space $(X, \mathbb{E}, \mu)$ as

$$\int_V f d\mu = \int_X f \cdot 1_V d\mu,$$

provided $f \in \mathcal{L}(X, \mathbb{E}, \mu)$ or $f \in \mathcal{M}^+(X, \mathbb{E})$. Notice, however, that the right hand side makes sence as long as just $f \cdot 1_V \in \mathcal{L}(X, \mathbb{E}, \mu)$ or $f \cdot 1_V \in \mathcal{M}^+(X, \mathbb{E})$.

The problem sessions in week 13: <u>4.26</u>, W7.1, W7.2, W7.3*, W7.4, <u>W7.6</u>

Problem W7.1. Show that (c.f. p. 4.15)

$$\mid \int f d\mu \mid \leq \int \mid f \mid d\mu$$

for a μ-integrable function $f : X \to \mathbb{C}$.

Problem W7.2. Compute

$$\int_{\mathbb{R}} (\sum_{n=1}^{\infty} \frac{1}{n} 1_{]0,1/n]}) dm \quad \text{and} \quad \int_{\mathbb{R}} (\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} 1_{]n,n+1]}) dm.$$

Problem W7.3. Let $(X, \mathbb{E}, \mu)$ be a measure space.

(i) Suppose that $f, g \in \mathcal{L}_{\mathbb{C}}(X, \mathbb{E}, \mu)$. Prove that $f = g$ μ-a.e. if and only if

$$\forall E \in \mathbb{E} : \int_E f d\mu = \int_E g d\mu$$

[Hint: Consider the real and the imaginary part of $f - g$ and the subsets on which these functions are positive, resp. negative.]

(ii) Let $f \in \mathcal{L}_{\mathbb{C}}(X, \mathbb{E}, \mu)$. Prove that

$$\int_E f d\mu = 0$$

for all $E \in \mathbb{E}$ with $\mu(E) = 0$.

The results (i) and (ii) also hold for functions in $\mathcal{M}^+(X, \mathbb{E})$ – but this claim is not a part of the problem.

Problem W7.4. Let μ and ν be measures on a measurable space $(X, \mathbb{E})$. Show that

$$\int_X f d(\mu + \nu) = \int_X f d\mu + \int_x f d\nu \tag{1}$$

for all $f \in \mathcal{M}^+$. [Hint: "Hovedsætning 4.2"]. Then prove that $\mathcal{L}(\mu+\nu) = \mathcal{L}(\mu) \cap \mathcal{L}(\nu)$ and that (1) holds for all $f \in \mathcal{L}(\mu + \nu)$.

Problem W7.5. Consider the Dirac measure ε_a on $(X, \mathcal{P}(X))$ for a point $a \in X$. Prove that

$$\int_X f d\varepsilon_a = f(a) \tag{2}$$

for all $f \in \mathcal{M}^+(X)$. After this, prove that $\mathcal{L}(\varepsilon_a)$ consists of all complex functions on X and that (2) holds for all $f \in \mathcal{L}(\varepsilon_a)$.

Problem W7.6. The purpose of this exercise is to show that the Riemann and Lebesgue integrals coincide when they both are defined. For a function $f : [a, b] \to \mathbb{C}$ we let $\int_a^b f(x)dx$ denote its Riemann integral (if it exists) and we let $\int_{[a,b]} f \, dm$ denote its Lebesgue integral (if it exists). We assume that a and b are finite.

(i) Explain why a Riemann integrable function $f : [a, b] \to \mathbb{C}$ belongs to $\mathcal{L}_{\mathbb{C}}([a, b], \mathbb{B}, m)$ if and only it is a Borel function[1].[Hint: Riemann integrable functions are bounded.]

[1]There *are* Riemann integrable functions which are not Borel functions. However, it can be shown that a Riemann integrable functions is measurable with respect to a completion of the Borel algebra - the so-called Lebesgue measurable sets.

(ii) Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Show that for all $\epsilon > 0$ there exist simple measurable functions $s_1, s_2 : [a, b] \to \mathbb{R}$ such that $s_1 \leq f \leq s_2$ and such that

$$\mid \int_{[a,b]} s_1 d\mu - \int_a^b f(x)\ dx \mid < \epsilon \text{ and } \mid \int_{[a,b]} s_2 d\mu - \int_a^b f(x)\ dx \mid < \epsilon.$$

[Hint: A lower sum $\sum_{i=1}^n l_i(x_i - x_{i-1})$ for f, i.e. where for all x in $]x_{i-1}, x_i] : l_i \leq f(x)$, may be viewed as the Lebesgue integral of the simple function $\sum_{i=1}^n l_i \cdot 1_{]x_i - 1, x_i]}$.]

(iii) Show that

$$\int_{[a,b]} f d\mu = \int_a^b f(x)\ dx$$

if $f[a, b] \to \mathbb{C}$ is a Riemann integrable Borel function. [Hint: First reduce to the case where f is real. Then use (i) and (ii).]