Mathematics 3MI **Hans Plesner Jakobsen**
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Weekly notice #8

The lectures in week 12: This was more or less a guided tour through sections §4.3–4.6. Only a cursory knowledge of Sections §4.5–4.6 will be expected at the exam (you should be familiar with the material but not necessarily with the proofs).

The lectures in week 13: First it will be shown that the Riemann and Lebesgue integrals coincide when they are both defined (Problem W7.6). Then we will prove the uniqueness of the Lebesgue measure on $\mathbb{R}^k$ (§5.1) and continue with §5.2 and portions of §5.3.

Homework - to be handed in to the TA in week 14: 3MI-exam January 2001: Problem 3.

The problem sessions in week 14: A) First do exercise 4.38 from the book. Then prove "Sætning 4.24" in the case $J = \mathbb{N}$. State "Sætning 4.26" and "Sætning 4.28" for this special case. Discuss when and how the content of Exercise 4.38 may be generalized to doubly indexed sequences $\{a_{j,k}\}$ of complex numbers. [Hint: "Bemærkning 4.27".]

B) Do the following 3MI-exam problems: January '01: Problem 2; June '01: Problem 3; January '02: Problems 3, 7. Finally, do W8.1.

[An English version of the above exams can be found below. For a Danish version, use e.g. the link "Old 3MI-exams (in Danish)" from the course home page.]

Problem W8.1. Consider the function $f : \mathbb{R} \to \mathbb{R}_+$ given by

$$f(x) = \frac{1}{1+x^4}\ .$$

Let μ be the measure on $\mathbb{B}(\mathbb{R}$ given by

$$\mu(E) = \int_E f\ dm,$$

where m is the Lebesgue measure. Compute
$$\int_0^\infty x\ d\mu(x).$$

3MI-exam, January '01 - Problem 2 (10 points): Determine the limit
$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\cos\left(\frac{x^2+7n}{n}\right)e^{-|x|}\ dx.$$
You must argue carefully for each step in the computation.

3MI-exam, January '01 - Problem 3 (20 points): Let $f:\mathbb{R}\to\mathbb{R}$ be given by
$$f(x)=\begin{cases}1, & x\in[-1,0[,\\ -2 & x\in[0,4],\\ 0, & x\in]-\infty,-1[\,\cup\,]4,\infty[.\end{cases}$$
Let m be the usual Lebesgue measure on $\mathbb{R}$, let ε_0 be the Dirac measure in 0 on $\mathbb{R}$, and let μ denote the counting measure on $\mathbb{R}$.

(i) Argue that f is a Borel function.
(ii) Determine whether $f\in\mathcal{L}(m)$, determine whether $f\in\mathcal{L}(\varepsilon_0)$, and determine whether $f\in\mathcal{L}(\mu)$.
(iii) Determine those of the integrals below, that are defined (see question (ii))
$$\int_{\mathbb{R}} f\ dm,\quad \int_{\mathbb{R}} f\ d\varepsilon_0,\quad \int_{\mathbb{R}} f\ d\mu.$$

3MI-exam, June '01 - Problem 3 (20 points): For every Borel set $E\subseteq\mathbb{R}$ we define
$$\mu(E)=\int_E\frac{dm(x)}{1+x^2},$$
where m as usual denotes the Lebesgue measure on $\mathbb{R}$.

(i) Justify (e.g. by reference to a relevant place in the notes) that μ is a measure on $(\mathbb{R},\mathbb{B}(\mathbb{R}))$ and that
$$\int|x|^a\mu(x)=\int\frac{|x|^a dm(x)}{1+x^2}\qquad(\star)$$
for every $a\in\mathbb{R}$. (As usual, $0^a=\infty$ for $a<0$, and $|x|^0=1$ for all x in $\mathbb{R}$.)
(ii) Show that the value of $(\star)$ is finite when $-1<a<1$.

3MI-exam, January '02 - Problem 3 (15 points): Set

$$F(t) = \int_0^\pi \sin(x + t\cos(x))dx, \quad t \in \mathbb{R}.$$

Justify that F is continuous and differentiable, and that

$$F'(t) = \int_0^\pi \cos(x + t\cos(x))\cos(x)dx, \quad t \in \mathbb{R}.$$

3MI-exam, January '02 - Problem 7 (25 points): Let $(X, \mathbb{E}, \mu)$ be a measure space, let $f : X \to [0, \infty[$ be an $\mathbb{E}$-measurable function, and set

$$E_t = \{x \in X \mid f(x) \geq t\}, \quad t \in [0, \infty[.$$

a) Justify that $E_t \in \mathbb{E}$ for $t \in [0, \infty[$.

Consider the function

$$\varphi(t) = \mu(E_t), t \in [0, \infty[.$$

b) Show that φ is decreasing, that is,

$$\forall s, t \in [0, \infty[: s < t \Rightarrow \varphi(s) \geq \varphi(t).$$

c) Show that

$$\lim_{n\to\infty} \varphi(1/n) = \mu(X_0),$$

where $X_0 = \{x \in X \mid f(x) > 0\}$.