Tail estimates for stochastic fixed point equations via nonlinear renewal theory

Jeffrey F. Collamore[†]

Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5,
DK-2100 Copenhagen Ø, Denmark, collamore@math.ku.dk

Anand N. Vidyashankar[‡]

Department of Statistics, George Mason University, 4400 University Drive, MS 4A7,
Fairfax, VA 22030, U.S.A., avidyash@gmu.edu

Abstract

This paper introduces a new approach, based on large deviation theory and nonlinear renewal theory, for analyzing solutions to stochastic fixed point equations of the form $V \stackrel{\mathcal{D}}{=} f(V)$, where $f(v) = A \max\{v, D\} + B$ for a random triplet $(A, B, D) \in (0, \infty) \times \mathbb{R}^2$. Our main result establishes the tail estimate $\mathbf{P}\{V > u\} \sim Cu^{-\xi}$ as $u \to \infty$, providing a new, explicit probabilistic characterization for the constant C. Our methods rely on a *dual* change of measure, which we use to analyze the path properties of the forward iterates of the stochastic fixed point equation. To analyze these forward iterates, we establish several new results in the realm of nonlinear renewal theory for these processes. As a consequence of our techniques, we develop a new characterization of the extremal index, as well as a Lundberg-type upper bound for $\mathbf{P}\{V > u\}$. Finally, we provide an extension of our main result to random Lipschitz maps of the form $V_n = f_n(V_{n-1})$, where $f_n \stackrel{\mathcal{D}}{=} f$ and $A \max\{v, D^*\} + B^* \leq f(v) \leq A \max\{v, D\} + B$.

1 Introduction

Stochastic fixed point equations (SFPE) arise in several areas of contemporary science and have been the focus of much study in applied probability, finance, analysis of algorithms, page ranking in personalized web search, risk theory, and actuarial mathematics. A general stochastic fixed point equation has the form

$$V \stackrel{\mathcal{D}}{=} f(V), \tag{1.1}$$

where f is a random function satisfying certain regularity conditions and is independent of V. When $f(v) = Av + B$, where (A, B) is independent of V and $\mathbf{E}[\log A] < 0$, this problem has a long history starting with the works of Solomon (1972), Kesten (1973), Vervaat (1979), and Letac (1986).

Tail estimates for solutions to general SFPEs have been developed by Goldie (1991) using implicit renewal theory, extending the fundamental work of Kesten (1973). Under appropriate regularity conditions, Goldie (1991) proved that

$$\mathbf{P}\{V > u\} \sim Cu^{-\xi} \quad \text{as} \quad u \to \infty \tag{1.2}$$

for certain constants C and ξ. More recently, Buraczewski et al. (2009) revisited the multi-dimensional version of (1.2) and established strong results concerning the distribution of the stationary solution.

The constant C appearing in (1.2) is defined in terms of the tails of V and $f(V)$, rendering it unsuitable for numerical purposes and for statistical inference concerning various tail parameters of scientific interest. Indeed, except in some special cases, the formulae for C presented in Goldie (1991) do not simplify to useful expressions. This issue is folklore and has been observed by several researchers including Yakin and Pollak (1998), Siegmund (2001), and Enriquez et al. (2009). Yakin and Pollak's work was motivated by likelihood ratio testing and change point problems in statistics, while the paper of Enriquez et al. was motivated by probabilistic considerations.

The primary objective of this paper is to present a general alternative probabilistic approach to deriving the above tail estimates, yielding a characterization of this constant which—beyond its theoretical interest—is also amenable to statistical inference, and to Monte Carlo estimation in particular. In the process, we draw an interesting connection between the constant C in (1.2) and the backward iterates of an SFPE described in Letac (1986). Specifically, we show that in many cases, this constant may be obtained via iterations of a corresponding perpetuity sequence, which has a comparatively simple form and is computable. As this representation is *explicit*, it also resolves questions about the positivity of the constant which have been often raised in the literature and addressed via specialised methods.

The starting point for our analysis is the quasi-linear recursion

$$f(v) = A\max\{v, D\} + B, \tag{1.3}$$

where $(A, B, D) \in (0, \infty) \times \mathbb{R}^2$, often referred to as "Letac's Model E." A key idea in our approach is the observation that the distribution of the solution to this SFPE is the stationary limit of a positively recurrent Markov chain, namely the *forward sequence* generated by this SFPE, given by

$$V_n = A_n\max\{V_{n-1}, D_n\} + B_n, \quad n = 1, 2, \ldots. \tag{1.4}$$

The tail probabilities of V can then be studied via excursions within a regeneration cycle of this Markov chain. After a large deviation change of measure argument, these excursions can be viewed as perturbations of a multiplicative random walk. While the multiplicative random walk itself can be handled via classical renewal-theoretic techniques, analysis of perturbations requires the development of new methods in nonlinear renewal theory.

In the central result of this paper, we show that $\mathbf{P}\{V > u\} \sim Cu^{-\xi}$ for a constant C which is given by

$$C = \frac{1}{\xi\lambda'(\xi)\mathbf{E}[\tau]}\mathbf{E}_\xi\left[\left(Z^{(p)} - Z^{(c)}\right)^\xi \mathbf{1}_{\{\tau=\infty\}}\right], \tag{1.5}$$

where τ is a typical regeneration time of the Markov chain $\{V_n\}$ (as described below in Lemma 2.2), λ is the moment generating function of the random variable $(\log A)$,

$$Z^{(p)} = V_0 + \sum_{n=1}^\infty \frac{B_n}{A_1 \cdots A_n},$$

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$Z^{(c)}$ is the limit of (2.16) below, often be taken to be zero, and $\mathbf{E}_\xi[\cdot]$ denotes expectation in the ξ-shifted measure (as defined below in (2.17)). In addition, we provide an error estimate which describes the convergence rate of the infinite series in (1.5) to its limiting value.

Roughly, the expectation in (1.5) quantifies the *discrepancy* between the constant appearing in Goldie's estimate (1.2) and the constant obtained in Cramér's classical ruin estimate, used to characterize the hitting probability of a positive-drift random walk into a negative barrier; see Remark 2.4 below. In practical applications, the conjugate term $Z^{(c)}$ appearing in (1.5) is often zero; thus, this expression typically represents a perpetuity sequence *killed when the transient ξ-shifted process $\{V_n\}$ regenerates within finite time*. [In Remark 2.5 below, we compare our formula to one recently obtained by Enriquez et al. (2009) for the linear recursion $f(v) = Av + B$ and independent random variables A and B.]

The next objective of this article is to describe how our probabilistic methods go beyond the scope of (1.2) in certain respects. In our first extension, we develop an analog of the classical Lundberg inequality of insurance mathematics. Namely we show that $\mathbf{P}\{V > u\} \le \bar{C}u^{-\xi}$, for all $u \ge 0$, for a certain constant $\bar{C} \in (C, \infty)$. Then, as an application of our techniques and motivated by extreme value theory, we provide a new characterization of the extremal index Θ for the forward recursive process (1.4). Namely we establish that

$$\Theta = \frac{1 - \mathbf{E}\left[e^{\xi S_{\tau^*}}\right]}{\mathbf{E}\left[\tau^*\right]}, \tag{1.6}$$

where $\tau^* = \inf\{n \ge 1 : S_n \le 0\}$ and $S_n = \sum_{i=1}^n \log A_i$. The latter expression provides a closed-form alternative to the well-known iterative solution given for the special case of the linear recursion in de Haan et al. (1989).

We conclude this paper with an extension of our main result to random maps of the form

$$V_n = f_n(V_{n-1}), \quad n = 1, 2, \ldots, \tag{1.7}$$

for $\{f_n\}$ i.i.d. copies of a random Lipschitz function f satisfying the *cancellation condition*

$$A_n \max\{v, D_n^*\} + B_n^* \le f_n(v) \le A_n \max\{v, D_n\} + B_n, \tag{1.8}$$

where $\{(A_n, B_n, D_n, B_n^*, D_n^*) : n = 1, 2, \ldots\}$ is an i.i.d. sequence taking values in $(0, \infty) \times \mathbb{R}^4$. Iterated random systems have received much recent attention in the literature (Diaconis and Freedman (1999), Stenflo (2001), Steinsaltz (2001), Carlsson (2002), Alsmeyer and Fuh (2001), Alsmeyer (2003), Athreya (2003)), and it is of interest to characterize their tail behavior; *cf.* Mirek (2011a). In fact, (1.8) describes a rather general class of recursions, and it will allow us to subsume—in a single mathematical result—all of the models considered in Goldie (1991) and to include additional processes not considered there, such as the AR(1) process with ARCH(1) errors, which we describe in Example 3.6 below. In this paper, we develop an analog of (1.5) for this general class of processes. Under our hypotheses, we obtain, once again, the asymptotic decay (1.2) with $C \in (0, \infty)$, but a simple form for the constant C is no longer feasible, and (1.5) must now be replaced with a recursive formula. It is worthwhile to observe that Mirek (2011a) also studied the above question and established related results, although without our expression for C. While our methods are probabilistic, Mirek's methods are analytic in nature; in particular, he establishes the positivity of the constant using complex analysis techniques.

The rest of the paper is organized as follows. Section 2 contains some background, notation, and a statement of the main results of the paper. Section 3 contains examples, while Section 4 describes our results in nonlinear renewal theory. Proofs of the main results are in Sections 5, 6, and 7, while proofs of the results from nonlinear renewal theory are in Section 8. Section 9 contains the proof of our main result for Lipschitz random maps and further discussion pertaining to this generalization.

2 Statement of results

2.1 Letac's principle and background from Markov chain theory

Assume that $V \stackrel{\mathcal{D}}{=} f(V)$ can be written in the general form

$$V \stackrel{\mathcal{D}}{=} F(V, Y) := F_Y(V), \tag{2.1}$$

where $F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is deterministic, measurable, and continuous in its first component.

Let v be an element of the range of F, and let $\{Y_n\}$ be an i.i.d. sequence of random variables such that $Y_n \stackrel{\mathcal{D}}{=} Y$ for all n. Then the *forward sequence* generated by the SFPE (2.1) is defined by

$$V_n(v) = F_{Y_n} \circ F_{Y_{n-1}} \circ \cdots \circ F_{Y_1}(v), \quad n = 1, 2, \ldots, \quad V_0 = v, \tag{2.2}$$

while the *backward sequence* generated by this SFPE is defined by

$$Z_n(v) = F_{Y_1} \circ F_{Y_2} \circ \cdots \circ F_{Y_n}(v), \quad n = 1, 2, \ldots, \quad Z_0 = v. \tag{2.3}$$

Note that the backward recursion need not be Markovian, but for every v and every integer n, $V_n(v)$ and $Z_n(v)$ are identically distributed. The Letac-Furstenberg principle states that, when the backward sequence converges to a random variable Z a.s. and is independent of its starting value, then the stationary distribution of $\{V_n\}$ is the same as the distribution of Z. This is described in the following lemma (Letac (1986), p. 264, or Furstenberg (1963)).

Lemma 2.1 *Let $F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be deterministic, measurable, and continuous in its first component; that is, F_Y is continuous, where Y is an $\mathbb{R}^d$-valued random variable, for some $d \in \mathbb{Z}_+$. Suppose that $\lim_{n \to \infty} Z_n(v) := Z$ exists a.s. and is independent of its initial state v. Then Z is the unique solution to the SFPE (2.1). Furthermore, $V_n(v) \Rightarrow V$ as $n \to \infty$, where V is independent of v, and the law of V is same as the law of Z.*

Starting with (2.1), we will study the tail probabilities of V using its forward recursive sequence (2.2), and noting that the distribution of V agrees with the limiting distribution of this forward sequence. We will focus especially on the SFPE

$$V \stackrel{\mathcal{D}}{=} F_Y(V), \quad \text{where} \quad F_Y(v) := A \max\{v, D\} + B, \tag{2.4}$$

for $Y = (\log A, B, D) \in (0, \infty) \times \mathbb{R}^2$, often referred to as *Letac's Model E*. This SFPE generates the forward quasi-linear recursive sequence

$$V_n = A_n \max\{V_{n-1}, D_n\} + B_n, \quad n = 1, 2, \ldots, \tag{2.5}$$

where the *driving sequence* $\{Y_n : n = 1, 2, \ldots\}$, $Y_n := (\log A_n, B_n, D_n)$, is assumed to be i.i.d. with the same distribution as $(\log A, B, D)$.

Iterating the forward recursion (2.5), we obtain with $B_0 \equiv V_0$ that

$$V_n = \max\left\{ \sum_{i=0}^{n} B_i \prod_{j=i+1}^{n} A_j, \bigvee_{k=1}^{n} \left(\sum_{i=k}^{n} B_i \prod_{j=i+1}^{n} A_j + D_k \prod_{j=k}^{n} A_j \right) \right\}. \tag{2.6}$$

Iterating the corresponding backward recursion and setting $\tilde{B}_i = B_i$ for $i = 1, \ldots, n$ and $\tilde{B}_{n+1} = V_0$, we obtain that

$$\tilde{Z}_n = \max\left\{ \sum_{i=1}^{n+1} \tilde{B}_i \prod_{j=1}^{i-1} A_j, \bigvee_{k=1}^{n} \left(\sum_{i=1}^{k} B_i \prod_{j=1}^{i-1} A_j + D_k \prod_{j=1}^{k} A_j \right) \right\}. \tag{2.7}$$

Under mild regularity conditions (to be introduced formally in the next section), the forward sequence $\{V_n\}$ in (2.6) will be Markovian, possessing a stationary distribution and a regeneration structure.

To describe this regeneration structure, let $\{V_n\}$ denote a Markov chain on a general state space $(\mathbb{S}, \mathcal{S})$, and assume that this chain is aperiodic, countably generated, and irreducible with respect to its maximal irreducibility measure φ (as defined on p. 13 of Nummelin (1984)). Let P denote the transition kernel of $\{V_n\}$. (From Section 2.2 onwards, we will take $\{V_n\}$ to be the Markov chain generated by a forward sequence (2.6), but for the remainder of this section, we will assume it to be an arbitrary chain.)

Recall that if $\{V_n\}$ is aperiodic and φ-irreducible, then it satisfies a *minorization*, namely

$$\delta \mathbf{1}_{\mathcal{C}}(x)\nu(E) \leq P^k(x, E), \quad x \in \mathbb{S}, \quad E \in \mathcal{S}, \quad k \in \mathbb{Z}_+, \tag{$\mathcal{M}$}$$

for some set $\mathcal{C}$ with $\varphi(\mathcal{C}) > 0$, some constant $\delta > 0$, and some probability measure ν on $(\mathbb{S}, \mathcal{S})$. The set $\mathcal{C}$ is called a *small set* or $\mathcal{C}$-set.

An important consequence of $(\mathcal{M})$—first established by Athreya and Ney (1978) and Nummelin (1978)—is that it yields a regeneration structure for the Markov chain. More precisely, these authors established the following result. Here and in the following, we let $\mathfrak{F}_n$ denote the σ-field generated by $(V_0, \ldots, V_n)$.

Lemma 2.2 *Assume that $(\mathcal{M})$ holds with $k = 1$. Then there exists a sequence of random times, $0 \leq K_0 < K_1 < \cdots$, such that:*
   (i) *$K_0, K_1 - K_0, K_2 - K_1, \ldots$ are finite a.s. and mutually independent;*
   (ii) *the sequence $\{K_i - K_{i-1} : i = 1, 2, \ldots\}$ is i.i.d.;*
   (iii) *the random blocks $\{V_{K_{i-1}}, \ldots, V_{K_i - 1}\}$ are independent, $i = 0, 1, \ldots$;*
   (iv) *$\mathbf{P}\{V_{K_i} \in E | \mathfrak{F}_{K_i - 1}\} = \nu(E)$, for all $E \in \mathcal{S}$.*

Let $\tau_i := K_i - K_{i-1}$ denote the ith inter-regeneration time, and let τ denote a typical regeneration time, *i.e.*, $\tau \overset{\mathcal{D}}{=} \tau_1$.

Remark 2.1 Regeneration can be related to the return times of $\{V_n\}$ to the $\mathcal{C}$-set in $(\mathcal{M})$ by introducing an augmented chain $\{(V_n, \eta_n)\}$, where $\{\eta_n\}$ is an i.i.d. sequence of Bernoulli random variables, independent of $\{V_n\}$ with $\mathbf{P}\{\eta_n = 1\} = \delta$. Then based on the proof of Lemma 2.2, we may identify $K_i - 1$ as the $(i + 1)$th return time of the chain $\{(V_n, \eta_n)\}$ to the set $\mathcal{C} \times \{1\}$. At the subsequent time, K_i, V_{K_i} has the distribution ν, where ν is given as in $(\mathcal{M})$ and is independent of the past history of the Markov chain. Equivalently, the chain $\{V_n\}$ regenerates with probability δ upon each return to the $\mathcal{C}$-set. In particular, in the special case when $\delta = 1$, in which case the chain $\{V_n\}$ is said to have an atom, the process $\{V_n\}$ regenerates upon *every* return to the $\mathcal{C}$-set.

As a consequence of the previous lemma, we obtain a representation formula relating the stationary limit distribution V of $\{V_n\}$ to the behavior of the chain over a regeneration cycle. First set

$$N_u = \sum_{n=0}^{\tau-1} \mathbf{1}_{(u,\infty)}(V_n), \tag{2.8}$$

where $V_0 \sim \nu$. By Lemma 2.2 (iv), N_u describes the number of visits of $\{V_n\}$ to the set (u, ∞) over a typical regeneration cycle. Then Theorem 2.2 yields a representation formula (Nummelin (1984), p. 75), as follows.

Lemma 2.3 *Assume that $(\mathcal{M})$ holds with $k = 1$. Then for any $u \in \mathbb{R}$,*

$$\mathbf{P}\{V > u\} = \frac{\mathbf{E}[N_u]}{\mathbf{E}[\tau]}. \tag{2.9}$$

If $(\mathcal{M})$ holds only for $k > 1$, then it is helpful to study the k-chain $\{V_{kn} : k = 0, 1, \ldots\}$ and to observe that (2.9) also holds when N_u, τ are defined relative to this k-chain. It is also helpful to note that the k-chain of the forward process generated by Letac's Model E is also a process having the form of Letac's Model E. In particular, for each $i \in \mathbb{Z}_+$, we may define

$$\mathcal{A}_i = A_{k(i-1)} \cdots A_{ki-1}, \quad \mathcal{B}_i = \sum_{j=k(i-1)}^{ki-1} B_j \left(A_{j+1} \cdots A_{ki-1} \right),$$

$$\mathcal{D}_i = \bigvee_{j=k(i-1)}^{ki-1} \left\{ D_j (A_j \cdots A_{ki-1}) - \sum_{l=k(i-1)}^{j-1} B_l (A_{l+1} \cdots A_{ki-1}) \right\}, \tag{2.10}$$

and note that $V_n = A_n \max\{V_{n-1}, D_n\} + B_n$, $n = 0, 1, \ldots$ implies

$$V_{ki} = \max\{\mathcal{A}_i V_0, \mathcal{D}_i\} + \mathcal{B}_i, \quad i = 1, 2, \ldots, \tag{2.11}$$

which has the same form as (2.4), (2.5), but with $(\mathcal{A}, \mathcal{B}, \mathcal{D}/\mathcal{A})$ in place of (A, B, D).

2.2 The main result

We now turn to the tail decay of the random variable V satisfying the SFPE

$$V \stackrel{\mathcal{D}}{=} F_Y(V), \quad \text{where} \quad F_Y(v) := A \max\{v, D\} + B. \tag{2.12}$$

Let $\{V_n\}$ and $\{Z_n\}$ denote the forward and backward recursions generated by this SFPE. Also introduce two related backward recursive sequences, which will play an essential role in the sequel.

The associated perpetuity sequence: Let $\{Y_n\}_{n \in \mathbb{Z}_+}$, $Y_n \equiv (\log A_n, B_n, D_n)$, be the i.i.d. driving sequence which generates the forward recursion $\{V_n\}$, and let $A_0 \equiv 1$ and B_0 have the distribution ν given in the minorization $(\mathcal{M})$. Assume B_0 is independent of the driving sequence $\{Y_n\}_{n \in \mathbb{Z}_+}$. Now consider the backward recursion

$$Z_n^{(p)} = F_{Y_0}^{(p)} \circ \cdots \circ F_{Y_n}^{(p)}(0), \quad n = 0, 1, \ldots,$$

where

$$F_Y^{(p)}(v) := \frac{v}{A} + \frac{B}{A}. \tag{2.13}$$

By an elementary inductive argument, it follows that

$$Z_n^{(p)} = \sum_{i=0}^{n} \frac{B_i}{A_0 \cdots A_i}, \quad n = 0, 1, \ldots. \tag{2.14}$$

The sequence $\{Z_n^{(p)}\}$ will be called the *perpetuity sequence associated to* $\{V_n\}$ and is the backward recursion generated by the SFPE (2.13). Set $Z^{(p)} = \lim_{n \to \infty} Z_n^{(p)}$ provided that this limit exists a.s.

The conjugate sequence: Let $\check{D}_0 := -B_0$, where B_0 has the distribution ν given in the minorization $(\mathcal{M})$, and let $\check{D}_i := -A_i D_i - B_i$ for $i = 1, 2, \ldots$, and consider the backward recursion

$$Z_n^{(c)} = F_{Y_0}^{(c)} \circ \cdots \circ F_{Y_n}^{(c)}(0), \quad n = 0, 1, \ldots,$$

where

$$F_Y^{(c)}(v) := \frac{1}{A} \min\{\check{D}, v\} + \frac{B}{A}. \tag{2.15}$$

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It follows by induction that

$$Z_n^{(c)} = \min \left\{ Z_n^{(p)}, 0, \bigwedge_{k=1}^{n} \sum_{i=0}^{k-1} \left(\frac{B_i}{A_0 \cdots A_i} - \frac{D_k}{A_0 \cdots A_{k-1}} \right) \right\}. \tag{2.16}$$

The sequence $\{Z_n^{(c)}\}$ will be called the *conjugate sequence to* $\{V_n\}$ and is the backward recursion generated by the SFPE (2.15). Set $Z^{(c)} = \lim_{n \to \infty} Z_n^{(c)}$ provided this limit exists a.s.

Further notation: Let A denote the multiplicative factor appearing in (2.12), and define

$$\lambda(\alpha) = \log \mathbf{E} \left[A^\alpha \right] \quad \text{and} \quad \Lambda(\alpha) = \log \lambda(\alpha), \quad \alpha \in \mathbb{R}.$$

Let μ denote the distribution of $Y := (\log A, B, D)$, and let μ_α denote the α-shifted distribution with respect to the first variable; that is,

$$\mu_\alpha(E) := \int_E e^{\alpha x} d\mu(x, y, z), \quad E \in \mathcal{B}(\mathbb{R}^3), \quad \alpha \in \mathbb{R}, \tag{2.17}$$

where, here and in the following, $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel sets of $\mathbb{R}^d$. Let $\mathbf{E}_\alpha [\cdot]$ denote expectation with respect to this α-shifted measure. For any random variable X, let $\mathfrak{L}(X)$ denote the probability law of X. Let $\text{supp}(X)$ denote the support of X. Also, given an i.i.d. sequence $\{X_i\}$, we will often write X for a "generic" element of this sequence. For any function h, let $\text{dom}(h)$ denote the domain of h and $\text{supp}(h)$ the support of h, respectively. Finally, let $\mathfrak{F}_n$ denote the σ-field generated by the forward sequence $V_0, \ldots, V_n$, where $\{V_n\}$ is obtained from the recursion (2.12).

We now state the main hypotheses of this paper.

Hypotheses:

(H_0) The random variable A has an absolutely continuous component with respect to Lebesgue measure with a nontrivial density in a neighborhood of $\mathbb{R}$.

(H_1) $\Lambda(\xi) = 0$ for some $\xi \in (0, \infty) \cap \text{dom}(\Lambda')$.

(H_2) $\mathbf{E} \left[|B|^\xi \right] < \infty$ and $\mathbf{E} \left[(A|D|)^\xi \right] < \infty$.

(H_3) $\mathbf{P} \{ A > 1, B > 0 \} > 0$ or $\mathbf{P} \{ A > 1, B \geq 0, D > 0 \} > 0$.

Next we turn to a characterization of the decay constant C in the asymptotic formula $\mathbf{P} \{ V > u \} \sim Cu^{-\xi}$ as $u \to \infty$. To describe this constant, we will compare the process $\{V_n\}$ generated by Letac's forward recursion (2.2) to that of a random walk, namely

$$S_n := \log A_1 + \cdots + \log A_n, \quad n = 1, 2, \ldots,$$

where $\{\log A_n\}$ is an i.i.d. sequence having the probability law μ_A. In Sparre-Andersen model in collective risk theory, one studies ruin for the random walk $\{S_n\}$, that is,

$$\psi^*(u) := \mathbf{P} \{ S_n < -u, \text{ for some } n \in \mathbb{Z}_+ \}.$$

It is well known that, asymptotically,

$$\psi^*(u) \sim C^* e^{-\xi u} \quad \text{as} \quad u \to \infty, \tag{2.18}$$

where $\mathbf{E} \left[A^\xi \right] = 1$. Following Iglehart (1972), the constant C^* is characterized by setting

$$\tau^* = \inf \{ n \in \mathbb{Z}_+ : S_n \leq 0 \}, \tag{2.19}$$

and then defining

$$C^* = \frac{1 - \mathbf{E}\big[e^{\xi S_{\tau^\star}}\big]}{\xi\lambda'(\xi)\mathbf{E}\left[\tau^*\right]}. \tag{2.20}$$

In collective risk theory, the constant C^* is referred to as the *Cramér-Lundberg constant*.

Our aim is to develop an analogous characterization for the constants of tail decay associated with $\psi(u) := \mathbf{P}\left\{V > u\right\}$. Expectedly, this will involve the return times τ, where τ is a typical regeneration time of the Markov chain or, more precisely, the first regeneration time given that regeneration occurs at time 0.

Recall that $\{Z_n^{(p)}\}$ and $\{Z_n^{(c)}\}$ are the associated perpetuity and conjugate sequences, respectively, and that $Z^{(p)}$ and $Z^{(c)}$ are their a.s. limits (which exist under (H_0)-(H_3); see Lemma 5.5 below).

Theorem 2.1 *Assume Letac's Model E, and suppose that (H_0), (H_1), (H_2), and (H_3) are satisfied. Then*

$$\lim_{u\to\infty} u^\xi \mathbf{P}\left\{V > u\right\} = C, \tag{2.21}$$

for the finite positive constant

$$C = \frac{1}{\xi\lambda'(\xi)\mathbf{E}\left[\tau\right]}\mathbf{E}_\xi\left[\left(Z^{(p)} - Z^{(c)}\right)^\xi \mathbf{1}_{\{\tau=\infty\}}\right]. \tag{2.22}$$

Moreover, $C = \lim_{n\to\infty} C_n$, where

$$C_n = \frac{1}{\xi\lambda'(\xi)\mathbf{E}\left[\tau\right]}\mathbf{E}_\xi\left[\left(\left(Z_n^{(p)} - Z_n^{(c)}\right)^+\right)^\xi \mathbf{1}_{\{\tau>n\}}\right], \tag{2.23}$$

and $\mathfrak{R}_n := C - C_n = o(e^{-\epsilon n})$ as $n \to \infty$, for some $\epsilon > 0$.

Remark 2.2 The constant ϵ in the previous theorem can be related to the tail distribution of τ, as follows. In Lemma 5.1 below, it will be shown that the Markov chain $\{V_n\}$ is geometrically recurrent. Thus, for some $\gamma > 0$,

$$\mathbf{P}\left\{\tau > n\right\} = o(e^{-\gamma n}) \quad \text{as} \quad n \to \infty.$$

From the proof of Theorem 2.1, we identify $\epsilon = \gamma/\xi$ for $\xi \geq 1$, and $\epsilon = \gamma$ for $\xi \in (0, 1)$. The constant γ may also be identified more explicitly by a slight variant of the proof of Theorem 2.1; see Remark 6.2 below.

Remark 2.3 Hypothesis (H_0) can be replaced with the weaker condition that $\log A$ is nonarithmetic and $\{V_n\}$ is nondegenerate, although this leads to in a more complicated expression for C. Indeed, Hypothesis (H_0) is only needed to establish the minorization condition $(\mathcal{M})$ with $k = 1$. (While this assumption also appears in the section on nonlinear renewal theory, it may be replaced there with the assumption that A is non arithmetic.) However, if the recursion is nondegenerate, then under our remaining hypotheses, it follows that the forward process $\{V_n\}$ is φ-irreducible and hence Harris recurrent (*cf.* Lemma 5.1 (ii) below). Consequently $(\mathcal{M})$ always holds, although not necessarily with $k = 1$. But then we may study the k-chain in place of the 1-chain, replacing the transition kernel P with the kernel P^k everywhere in the proofs. This leads to the same representation formula as in (2.22), except that τ is now the first regeneration time of the k-chain $\{V_{kn}\}$. The other hypotheses remain the same as before, since if (H_1), (H_2), and (H_3) hold for the 1-chain, they also hold for the k-chain by an elementary argument. Nonetheless, it is important to emphasise

that the expression for C obtained in this way is not always useful, especially compared with (2.22), mainly because the regeneration time will now be relative to the k-chain. In particular, in Example 3.2 below, we will show that the conjugate sequence for the 1-chain is *zero*, greatly simplifying (2.22). However, it is not possible to show the same is true of the k-chain in this example. (For a more detailed discussion of the k-chain, see Section 9.)

Remark 2.4 The above theorem may be viewed as an extension of Cramér's famous ruin estimate, as follows. By a classical duality argument, it can be shown that the ruin probability in (2.18) is the same as the steady-state exceedance probability of the reflected random walk

$$W_n := \max\left\{W_{n-1} + \log A_n, 0\right\}, \quad n = 1, 2, \ldots \quad \text{and} \quad W_0 = 0;$$

cf. Siegmund (1985), Appendix 2. Specifically, if $W := \lim_{n\to\infty} W_n$ and ψ^* is given as in (2.18), then $\psi^*(u) = \mathbf{P}\{W > u\}$, and we now compare $\psi^*(u)$ to $\psi(u) := \mathbf{P}\{V > u\}$.

By Theorem 2.1, $(\psi(u)/\psi^*(\log u)) \sim C/C^*$ as $u \to \infty$. Moreover, recalling that $\tau^* := \inf\{n : S_n \leq 0\}$, it follows by a simple change-of-measure argument (as in the proof of Lemma 6.1 below) that

$$\mathbf{E}\left[e^{\xi S_{\tau^*}}\right] = \mathbf{P}_\xi\left\{\tau^* < \infty\right\}.$$

Hence it follows from Theorem 2.1 that

$$\frac{C}{C^*} = \frac{\mathbf{E}\left[\tau^*\right]}{\mathbf{E}\left[\tau\right]} \cdot \frac{\mathbf{E}_\xi\left[\mathcal{W}\right]}{\mathbf{P}_\xi\left\{\tau^* = \infty\right\}}, \quad \text{where} \quad \mathcal{W} := \left(Z^{(p)} - Z^{(c)}\right)\mathbf{1}_{\{\tau=\infty\}}. \tag{2.24}$$

Thus, the discrepancy between an exceedance for classical random walk and Letac's Model E is quantified via the inclusion of an additional term $\left(Z^{(p)} - Z^{(c)}\right)$ and the replacement of the regeneration time τ^* for the random walk with the regeneration time τ for Letac's Model E.

It is worth observing that, if $(B, D) = (0, A^{-1})$ in Letac's Model E, then we obtain the multiplicative random walk $V_n = \max\{A_n V_{n-1}, 1\}$, and the sample paths of $\{\log V_n\}$ are equivalent to those of $\{W_n\}$, and then we expect to obtain that $C = C^*$. Indeed, in Example 3.1 below, we will revisit this example and verify that $C = C^*$ in this special case.

Remark 2.5 In a recent work, specialized to the linear recursion $f(v) = Av + B$, Enriquez et al. (2009) provided an alternative probabilistic representation for this constant when A and B are independent. Utilizing a coupling satisfied by the doubly-infinite random walk $\{S_n : n \in \mathbb{N}\}$, where $S_n = \sum_{i=1}^n \log A_i$ for $n > 0$, $S_n = \sum_{i=-n+1}^0 \log A_i$ for $n \leq 0$ (where $A_n \stackrel{\mathcal{D}}{=} A$), these authors established another representation formula for C, namely

$$C = C^*\mathbf{E}^*\left[Z_F^\xi\right] + \mathbf{E}^*\left[Z_B^\xi\right], \tag{2.25}$$

where

$$Z_F := B_0 + \sum_{n=1}^\infty \frac{B_n}{A_1 \cdots A_n} \quad \text{and} \quad Z_B := \sum_{n=-\infty}^{-1} \frac{B_n}{A_1 \cdots A_n}.$$

Here, C^* is given as in the previous remark, and $\{A_n\}$ and $\{B_n\}$ are i.i.d. copies of A and B, respectively, and these sequences are mutually independent. Here, the expectations $\mathbf{E}^*[\cdot]$ are evaluated in the ξ-shifted measure and are conditional on the event that the random walk stays positive over its infinite-time evolution when viewed forward in time ($n = 1, 2, \ldots$), and nonnegative over its infinite-time evolution when viewed backward in time ($n = 0, -1, \ldots$).

It is difficult to directly compare (2.25) with our formula, since they are based on very different techniques. However, from an applied perspective, we note that Eq. (2.25) is still computationally complex, as it depends on paths of infinite length, both forward and backward in time, which proves

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cumbersome in applications, *e.g.*, to statistics and Monte Carlo simulation. In these contexts, our formula and method can be applied more directly (*cf.* Collamore et al. (2011)) and thus gives an appealing alternative. A further advantage of our formula is its generality, which allows one to study a wide variety of recursions beyond the linear case, as we will see in Section 3 below. Moreover, there are also other variants on our formula for C which can be deduced directly from our method of proof. We will discuss these variants in Remarks 6.2 and 6.3 below.

In practice, Theorem 2.1 only yields a simple representation for the constant C when the complicated "conjugate" term can be removed from the expression in (2.22). Indeed, in many applications of practical interest, $Z_n^{(c)} = 0$ for all n along paths for which $\{\tau = \infty\}$. Verifying this additional condition is problem-dependent, but it will always be our strategy when addressing the various applications of Letac's Model E. Thus, it is worth emphasizing that a trivial consequence of Theorem 2.1 is the following:

Corollary 2.1 *Assume the conditions of the previous theorem, and further assume that $Z_n^{(c)} = 0$, $n \in \mathbb{Z}_+$, along all paths such that $\{\tau = \infty\}$. Then (2.21) holds with*

$$C = \frac{1}{\xi \lambda'(\xi) \mathbf{E}[\tau]} \mathbf{E}_\xi \left[\left(Z^{(p)} \right)^\xi \mathbf{1}_{\{\tau = \infty\}} \right]. \tag{2.26}$$

Moreover, $C = \lim_{n \to \infty} C_n$, where

$$C = \frac{1}{\xi \lambda'(\xi) \mathbf{E}[\tau]} \mathbf{E}_\xi \left[\left((Z_n^{(p)})^\xi \vee 0 \right) \mathbf{1}_{\{\tau > n\}} \right] + o(e^{-\epsilon n}) \quad as \ \ n \to \infty, \tag{2.27}$$

for some $\epsilon > 0$.

For specific examples where the conditions of Corollary 2.1 applies, see Examples 3.1-3.4 below.

2.3 Two auxiliary results

We now supplement the previous estimate with a sharp upper bound. First define

$$\bar{Z}^{(p)} = |V_0| + \sum_{n=1}^\infty \frac{\left(|B_n| + A_n |D_n| \right)}{A_1 \cdots A_n} \mathbf{1}_{\{\tau > n\}}. \tag{2.28}$$

Proposition 2.1 *Assume Letac's Model E, and suppose that (H_0), (H_1), (H_2), and (H_3) are satisfied. Then*

$$\mathbf{P}\{V > u\} \le \bar{C}(u) u^{-\xi}, \quad \text{for all } u \ge 0, \tag{2.29}$$

where for certain positive constants $C_1(u)$ and $C_2(u)$,

$$\bar{C}(u) = \mathbf{E}_\xi \left[(\bar{Z}^{(p)})^\xi \right] \sup_{z \ge 0} \left\{ e^{-\xi z} \left(z C_1(u) + C_2(u) \right) \right\} < \infty \quad \text{for all } u. \tag{2.30}$$

Furthermore,

$$C_1(u) \to \frac{1}{m}, \quad \text{and} \quad C_2(u) \to 1 + \frac{\sigma^2}{m^2} \quad as \ \ u \to \infty, \tag{2.31}$$

where $m := \mathbf{E}_\xi [\log A]$ and $\sigma^2 := \mathrm{Var}_\xi (\log A)$.

The constants $C_1(u)$ and $C_2(u)$ will be identified explicitly in (8.44) below.

Next, we conclude our discussion of Letac's Model E with an alternative characterization of the extremal index of its forward iterates. The extremal index—which measures the tendency of a dependent process to cluster—is defined for a strictly stationary process $\{\tilde{V}_n\}$ via the equation

$$\lim_{n\to\infty} \mathbf{P}\left\{ \max_{1\leq i\leq n} \tilde{V}_n \leq u_n \right\} = e^{-\Theta t}, \tag{2.32}$$

where $\{u_n\}$ is chosen such that

$$\lim_{n\to\infty} n\left(1 - \mathbf{P}\{\tilde{V}_n > u_n\}\right) = t > 0. \tag{2.33}$$

As argued in Rootzén (1988), p. 380, when $\{V_n\}$ is a Markov sequence and the conditions of Lemma 2.2 are satisfied, this quantity may be written as

$$\Theta = \lim_{u\to\infty} \frac{\mathbf{P}\{V_n > u, \text{ some } n < \tau\}}{\mathbf{E}[N_u]}, \tag{2.34}$$

where N_u denotes the number of exceedances above level u occurring over a regeneration cycle. Consequently, using a slight modification of the methods of Theorem 2.1, we obtain the following:

Proposition 2.2 *Assume that $\{V_n\}$ satisfies the forward recursion (2.5), and suppose that (H_0), (H_1), (H_2), and (H_3) are satisfied. Then*

$$\Theta = \frac{1 - \mathbf{E}\left[e^{\xi S_{\tau^*}}\right]}{\mathbf{E}[\tau^*]}, \tag{2.35}$$

where $\tau^* = \inf\{n \geq 1 : S_n \leq 0\}$ for $S_n = \sum_{i=1}^n \log A_i$.

The above expression provides a closed-form expression for Θ, complementing the iterative solution provided for the linear recursion $f(v) = Av + B$ in de Haan et al. (1989). We note that the quantities τ^* and S_{τ^*} in this representation appear naturally in the context of risk theory; *cf.* Iglehart (1972) and (2.20) above.

2.4   General random maps

Finally, we turn to an extension of our main result to the setting of general random maps. Suppose that

$$V_n = f_n(V_{n-1}), \quad n = 1, 2, \ldots, \quad V_0 = v, \tag{2.36}$$

where $\{f_n\}$ are i.i.d. copies of a random function f, and f is approximated by Letac's Model E in a sense which we make precise below.

Before introducing this approximation, we need to impose a regularity condition which assures that the process $\{V_n\}$ in (2.36) has a unique stationary solution. Namely, we assume the following.

Lipschitz condition $(\mathfrak{L})$**:** There exists a random variable L with $\mathbf{E}[\log L] < 0$ such that

$$\sup_{v\neq w} \frac{|f(v) - f(w)|}{|v - w|} = L, \tag{$\mathcal{L}$}$$

where $\mathbf{E}\left[|\log L| + \log^+ |f(v_0)|\right] < \infty$, for some $v_0 \in \text{supp}(V)$.

It is shown in Elton (1990), Alsmeyer (2003), and Mirek (2011a,b) that under $(\mathfrak{L})$, the sequence $\{V_n\}$ has a unique stationary solution which is independent of its starting state. Moreover, in Alsmeyer (2003) it is shown that, under these assumptions, the Markov chain $\{V_n\}$ is Harris recurrent. By Theorem 2.2 of Alsmeyer (2003), the maximal Harris set either has empty interior (which we call the "degenerate" case), or the maximal Harris set is the entire space (which we call the "nondegenerate" case). In particular, for the linear recursion $f(v) = Av + B$, it is well known that the necessary and sufficient condition for nondegeneracy is that $\mathbf{P}\{B = (1 - A)c\} < 1$ for all c (cf. Goldie and Maller (2000), Theorem 2.1). We now turn to the approximation alluded to above.

Cancellation condition $(\mathfrak{C})$: There exists a nonarithmetic random variable $A \in (0, \infty)$ and a random vector $(B, D, B^*, D^*) \in \mathbb{R}^4$ such that, for all $v \in \operatorname{supp}(V)$,

$$A\max\{v, D^*\} + B^* \leq f(v) \leq A\max\{v, D\} + B, \tag{$\mathcal{C}$}$$

where Hypotheses (H_1), (H_2), and (H_3) are satisfied by (A, B, D) and by (A, B^*, D^*).

Now let $\{(A_n, B_n) : n \in \mathbb{Z}_+\}$ be an i.i.d. sequence of random variables having the same probability law as (A, B). Then our strategy is to view $\{V_n\}$ as a perturbation of the linear recursion $f(v) = Av + B$, where the remainder term

$$\mathcal{R}_n(V_{n-1}) := V_n - A_n V_{n-1}$$

plays the role of the sequence $\{B_n\}$ for the linear recursion. For the linear recursion with B supported on the nonnegative axis, we will see in Example 3.3 below that the conjugate term in Theorem 2.1 is zero and hence

$$C = \frac{1}{\xi \lambda'(\xi)\mathbf{E}[\tau]}\mathbf{E}_\xi\left[\left(V_0 + \frac{B_1}{A_1} + \frac{B_2}{A_1 A_2} + \cdots\right)^\xi \mathbf{1}_{\{\tau=\infty\}}\right].$$

In this section, we will see than an analogous result holds more generally, but with each B_n in this formula replaced with the remainder term $\mathcal{R}_n(V_{n-1})$.

Now if $\{V_n\}$ is Harris recurrent, then it is well known that $(\mathcal{M})$ holds for some $k \in \mathbb{Z}_+$. If $k = 1$, then a proper regeneration structure exists, as described in Lemma 2.2. Let $\{K_i\}$ denote the regeneration times and $\tau_i := K_i - K_{i-1}$, $i = 1, 2, \ldots$ denote the inter-regeneration times. By Remark 2.1, the regeneration times correspond to the returns of an adjoined process $\{(V_n, \eta_n)\}$ to the set $\mathcal{C} \times \{1\}$, where we assume, without loss of generality, that $\mathcal{C} \subset [-M, M]$ for some $M > 0$. Also let τ denote a typical regeneration time; more precisely, if regeneration occurs at time 0, then τ denotes the subsequent regeneration time.

If $k > 1$, then a proper regeneration structure exists for the k-chain $\{V_{kn} : n = 0, 1, \ldots\}$, and we take $\{\tau_i^{(k)}\}$ to be the inter-regeneration times of this chain.

We note that for the k-chain, the relevant remainder term changes slightly, becoming

$$\mathcal{R}_n^{(k)}(V_{kn}) := V_{kn} - \mathcal{A}_n V_{k(n-1)}, \quad n = 1, 2, \ldots,$$

where $\mathcal{A}_n := A_{k(n-1)} \cdots A_{kn-1}$ (cf. the discussion at the end of Section 2.1).

Then by a slight modification of the proof of Theorem 2.1, we obtain:

Theorem 2.2 *Assume (2.36), and suppose that $(\mathfrak{C})$ and $(\mathfrak{L})$ are satisfied and that $\{V_n\}$ is nondegenerate. Then*

$$\lim_{u \to \infty} u^\xi \mathbf{P}\{V > u\} = C \tag{2.37}$$

for the finite positive constant C. If $k = 1$, then

$$C = \frac{1}{\xi \lambda'(\xi) \mathbf{E}[\tau]} \mathbf{E}_\xi \left[\left(V_0 + \sum_{i=1}^{\infty} \frac{\mathcal{R}_i(V_{i-1})}{A_1 \cdots A_i} \right)^\xi \mathbf{1}_{\{\tau = \infty\}} \right]. \tag{2.38}$$

Moreover $C = \lim_{n \to \infty} C_n$, where

$$C_n = \frac{1}{\xi \lambda'(\xi) \mathbf{E}[\tau]} \mathbf{E}_\xi \left[\left(\left(V_0 + \sum_{i=1}^{n} \frac{\mathcal{R}_i(V_{i-1})}{A_1 \cdots A_i} \right)^+ \right)^\xi \mathbf{1}_{\{\tau > n\}} \right], \tag{2.39}$$

and $\mathfrak{R}_n := C - C_n = o(e^{-\epsilon n})$ as $n \to \infty$, for some $\epsilon > 0$. In addition, if the minorization $(\mathcal{M})$ holds for $k > 1$, then (2.37), (2.38), and (2.39) still hold, but with the following modifications. In (2.38), the stopping time τ must be replaced with $\tau^{(k)}$, where $\tau^{(k)}$ denotes first regeneration time of the k-chain $\{V_{kn} : n = 0, 1, \ldots\}$, and, moreover, the quantity $\lambda'(\xi)$ must be replaced with $k\lambda'(\xi)$. In (2.39), the quantities $\{A_n\}$, $\{\mathcal{R}_n(V_{n-1})\}$, $\lambda'(\xi)$, τ must be replaced with $\{\mathcal{A}_n\}$, $\{\mathcal{R}_n^{(k)}(V_{k(n-1)})\}$, $k\lambda'(\xi)$, $\tau^{(k)}$, respectively.

It will be seen in the proof that the quantity inside the parentheses in (2.38) is necessarily positive relative to the ξ-shifted measure; thus, the ξth moment of the quantity in (2.38) makes sense.

Remark 2.6 In the proof of Theorem 2.2, it will be seen that $(V_0 + \sum_{i=1}^{n} (\mathcal{R}_i(V_{i-1})/A_1 \cdots A_i)) = Z_n$, where $Z_n \to Z$ for a random variable Z which is strictly positive. Hence the constant C in (2.38) is well-defined and positive.

Remark 2.7 Recently, an alternative expression for the constant C was given for the linear recursion $f(v) = Av + B$ with $B \equiv 1$ in Bartkiewicz et al. (2011). In fact, this constant can be obtained as a corollary to the proof of Theorem 2.2. Indeed, as we illustrate in Remark 6.3 and Section 9 below, their formula is valid for Lipschitz random maps under the conditions of Theorem 2.2 and the assumption that $\{V_n\}$ is nonnegative. Under these conditions,

$$C = \frac{1}{\xi \lambda'(\xi)} \lim_{n \to \infty} \frac{1}{n} \mathbf{E} \left[\tilde{Z}_n^\xi \right], \tag{2.40}$$

where $\{\tilde{Z}_n\}$ is the backward process corresponding to the forward process $\{V_n\}$. See Remark 6.3 (for Letac's Model E) and the discussion following the proof of Theorem 2.2 in Section 9 (for the general case). A liability of (2.40) is that it involves an averaging and an infinite limit, while an advantage of this formula is that it does not depend on the regeneration times. (We observe, however, that determining the regeneration times is often not difficult in specific applications, and in all of the examples of the next section, one can easily verify $(\mathcal{M})$ with $k = 1$ and $\mathcal{C} = [-M, M]$ for some $M \geq 0$.) Noting that the backward sequence for the linear recursion $f(v) = Av + B$ is the perpetuity sequence $\{Z_n^{(p)}\}$, we see that there is an appealing correspondence with our original representation formula (2.22) in that special case, although (2.40) introduces a limit and scaling factor to deal with the fact that $\mathbf{E}\left[(Z_n^{(p)})^\xi\right] \uparrow \infty$ as $n \to \infty$.

3 Examples

We begin by describing a variety of examples satisfying the conditions of Theorem 2.1. First we revisit the classical ruin problem, showing that our estimate reduces to the classical Cramér-Lundberg estimate in that case.

Example 3.1 As described in the previous section, the classical ruin problem is concerned with $\psi^*(u) := \mathbf{P}\{S_n > u\}$ as $u \to \infty$, where $S_n := \sum_{i=1}^n \log A_i$. Now in Remark 2.4, we observed that by a duality argument, $\psi^*(\log u) = \psi(u) := \mathbf{P}\{V > u\}$, where V is the stationary limit of the process

$$V_n = \max\{A_n V_{n-1}, 1\}, \quad n = 1, 2, \ldots.$$

Note that $\{V_n\}$ is a special case of Letac's Model E, obtained by taking $(B, D) = (0, A^{-1})$.

By Theorem 2.1, setting $v = \log u$,

$$\psi^*(v) \sim Ce^{-\xi v} \quad \text{as} \quad v \to \infty,$$

and thus we obtain the classical Cramér-Lundberg estimate (2.18), provided that we can verify that $C = C^*$. Now by Remark 2.4, $C = C^*$ if and only if

$$\mathbf{P}_\xi\{\tau^* = \infty\} = \mathbf{E}_\xi\left[\left(Z^{(p)} - Z^{(c)}\right)\mathbf{1}_{\{\tau=\infty\}}\right]; \tag{3.1}$$

cf. (2.24). Here, τ^* is the first return time of $W_n := \log V_n$ to zero or, equivalently, the first return time of the process $\{V_n\}$ to one.

To verify that (3.1) holds, first observe that $\{V_n\}$ satisfies the minorization condition $(\mathcal{M})$ with $\mathcal{C} = \{1\}$, $\delta = 1$, and $\nu \sim \mathfrak{L}(A \vee 1)$. Thus, by Remark 2.1, $\{V_n\}$ regenerates subsequent to its return to $\{1\}$, *i.e.*, $\tau = \inf\{n : V_n = 1\} + 1$. (Thus $\tau = \tau^* + 1$.)

Now $B_i = 0$ for $i = 1, 2, \ldots$ implies $Z^{(p)} = B_0$, and, by definition, B_0 has the distribution ν appearing in the minorization $(\mathcal{M})$ satisfied by $\{V_n\}$. Thus, $Z^{(p)} = B_0 \sim \mathfrak{L}(A \vee 1)$.

Next, we claim that the conjugate sequence is zero on $\{\tau = \infty\}$, and hence the conditions of Corollary 2.1 are satisfied. To see that this is the case, observe that in (2.16), after setting $A_0' = B_0 \sim \mathfrak{L}(A \vee 1)$ and recalling that $A_0 \equiv 1$, we have

$$\sum_{i=0}^{k-1} \frac{B_i}{A_0 \cdots A_i} - \frac{D_k}{A_0 \cdots A_{k-1}} = A_0'\left(1 - \frac{1}{A_0' A_1 \cdots A_{k-1}}\right). \tag{3.2}$$

Observe that on $\{\tau = \infty\}$, the multiplicative process $A_0' A_1 \cdots A_{k-1} > 1$ for all k. Thus the term on the right-hand side of (3.2) is positive. Since (2.16) involves the minimum of these terms with zero, we deduce that the conjugate term must be zero on $\{\tau = \infty\}$.

Consequently,

$$\mathbf{E}_\xi\left[\left(Z^{(p)} - Z^{(c)}\right)\mathbf{1}_{\{\tau=\infty\}}\right] = \mathbf{E}_\xi\left[(A_0')^\xi \mathbf{1}_{\{\tau=\infty\}}\right]. \tag{3.3}$$

Next, by a change of measure, applied now to the initial term A_0' (which, up to now, has *not* been shifted in the expectation on the right-hand side), we further obtain that

$$\mathbf{E}_\xi\left[(A_0')^\xi \mathbf{1}_{\{\tau=\infty\}}\right] = \mathbf{P}_\xi'\{\tau = \infty\}, \tag{3.4}$$

where $\mathbf{P}_\xi'\{\cdot\}$ denotes that the probability is evaluated in the ξ-shifted measure with respect to the *entire* sequence $A_0', A_1, A_2, \ldots$ (and not just with respect to $A_1, A_2, \ldots$). Finally, recall that $\tau = \infty \iff \tau^* = \infty$. Indeed, τ^* is the first return time of the random walk $\sum_{i=1}^n \log A_i$ to $(-\infty, 0]$ (starting at time one), while $\tau - 1$ is the first return time of the multiplicative random walk $\{V_n\}$, starting at time zero, to $\{1\}$. Thus $\mathbf{P}_\xi'\{\tau = \infty\} = \mathbf{P}_\xi\{\tau^* = \infty\}$. Substituting this equation and (3.3) into (3.4), we conclude that (3.1) holds and thus $C = C^*$.

Example 3.2 Consider an extension of the classical ruin problem, where the insurance company invests its surplus capital and earns stochastic interest on its investments. More precisely, let L_i

denote the net *losses* of the insurance business incurred during the ith discrete time interval. Now in classical risk theory, the surplus process is a Lévy process, and typically it is assumed that the insurance company earns premiums at a faster rate than the expected losses due to the claims; thus, the sequence $\{L_i\}$ is i.i.d. with a *negative* mean. Next, assume that the company earns a stochastic interest R_i on its surplus capital at time $i-1$. Then the total capital of the company at time i is given by

$$Y_i = R_i Y_{i-1} - L_i, \quad i = 1, 2, \ldots, \quad Y_0 = u,$$

and we further assume that the process $\{R_i\}$ is i.i.d. (but that L_i and R_i may be dependent for any given i).

By an elementary computation, one can show that $\{Y_n < 0, \text{ for some } n \in \mathbb{Z}_+\}$ if and only if $\{\mathcal{L}_n > u, \text{ for some } n \in \mathbb{Z}_+\}$, where $\{\mathcal{L}_n\}$ is the *cumulative discounted loss process*

$$\mathcal{L}_n := A_1 L_1 + (A_1 A_2) L_2 + \cdots + (A_1 \cdots A_n) L_n, \quad n = 1, 2, \ldots; \quad A_i := \frac{1}{R_i}. \tag{3.5}$$

Thus

$$\mathbf{P}\left\{\text{ruin}\right\} = \mathbf{P}\left\{\mathcal{L} > u\right\}, \quad \text{where} \quad \mathcal{L} := \left(\sup_{n \in \mathbb{Z}_+} \mathcal{L}_n\right) \vee 0. \tag{3.6}$$

To see that the random variable $\mathcal{L}$ satisfies an SFPE of the form of Letac's Model E, we begin by observing that, in (3.5), $\mathcal{L}_{n-1} \overset{\mathcal{D}}{=} A_2 L_2 + \cdots + (A_2 \cdots A_n) L_n$. Hence

$$\mathcal{L}_n \overset{\mathcal{D}}{=} B + A \mathcal{L}_{n-1}, \quad \text{where} \quad A \overset{\mathcal{D}}{=} A_1 \quad \text{and} \quad B \overset{\mathcal{D}}{=} A_1 L_1, \tag{3.7}$$

and (A, B) is independent of $\mathcal{L}_{n-1}$ on the right-hand side. Recalling that $\mathcal{L} := \left(\sup_{n \in \mathbb{Z}_+} \mathcal{L}_n\right) \vee 0$, we then conclude that $\mathcal{L}$ satisfies the SFPE

$$\mathcal{L} \overset{\mathcal{D}}{=} (A\mathcal{L} + B)^+. \tag{3.8}$$

(An alternative SFPE is obtained by setting $\tilde{\mathcal{L}} = \sup_{n \in \mathbb{Z}_+} \mathcal{L}_n$ and repeating the above argument. Then $\mathbf{P}\left\{\text{ruin}\right\} = \mathbf{P}\{\tilde{\mathcal{L}} > u\}$, where $\tilde{\mathcal{L}}$ satisfies the SFPE $\tilde{\mathcal{L}} \overset{\mathcal{D}}{=} A \max\{\tilde{\mathcal{L}}, 0\} + B$. However, for our current discussion, the SFPE (3.8) is slightly simpler to analyze.)

Observe that the forward process $\{V_n\}$ generated by the SFPE (3.8) agrees with the linear process $\tilde{V}_n := A_n \tilde{V}_{n-1} + B_n$, except that $\{V_n\}$ is reflected upon its return to the origin. Focusing on this return to the origin, one can verify that $\{V_n\}$ satisfies the minorization $(\mathcal{M})$ with $\delta = 1$, $\mathcal{C} = 0$, and $\nu = \mathfrak{L}(B^+)$. By Remark 2.1, we conclude that regeneration occurs with probability one subsequent to the return of $\{V_n\}$ to the origin; that is, $\tau = \inf\{n : \tilde{V}_n \leq 0\} + 1$.

Next observe that the conjugate sequence is zero on $\{\tau = \infty\}$. Indeed, setting $D_i = -B_i/A_i$ in (2.16), we obtain

$$Z^{(c)} = \min\left\{0, \bigwedge_{n=1}^{\infty} \sum_{i=0}^{n} \frac{B_i}{A_0 \cdots A_i}\right\}. \tag{3.9}$$

But the terms on the right-hand side all have the general form $\sum_{i=0}^{n} B_i/(A_0 \cdots A_i)$, which upon multiplication by $(A_0 \cdots A_n)$ yields

$$\sum_{i=0}^{n} B_i(A_{i+1} \cdots A_n), \quad n = 0, 1, \ldots. \tag{3.10}$$

Now for $Z^{(c)}$ to be nonzero, we would need the quantity in (3.10) to be negative for some n. But (3.10) is just the nth iteration of the SFPE $\tilde{V} \overset{\mathcal{D}}{=} A\tilde{V} + B$ with forward recursive sequence $\{\tilde{V}_n\}$, which, as we have already observed, agrees with the process $\{V_n\}$ prior to the first entrance of

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either process to $(-\infty, 0]$. Since $\tau := \inf\{n : V_n = 0\}$, it follows that these two processes agree and are positive-valued along sample paths where $\{\tau = \infty\}$. Thus, the quantity in (3.10) is positive on $\{\tau = \infty\}$, and hence so is $\sum_{i=0}^n B_i/(A_0 \cdots A_i)$ for all n. Thus, by (3.9), we conclude that $Z^{(c)} = 0$.

Hence by Corollary 2.1, $\mathbf{P}\{\mathcal{L} > u\} \sim Cu^{-\xi}$, where

$$C = \frac{1}{\xi \lambda'(\xi) \mathbf{E}[\tau]} \mathbf{E}_\xi \left[\left(B_0 + \frac{B_1}{A_1} + \frac{B_2}{A_1 A_2} + \cdots \right)^\xi \mathbf{1}_{\{\tau = \infty\}} \right]; \tag{3.11}$$

here, $B_0 \sim B^+$ and τ is identified as the time subsequent to the return of $\{V_n\}$ to $\{0\}$.

Note that if $\xi \geq 1$, then Minkowskii's inequality followed by a change of measure argument yields the upper bound

$$\mathbf{E}_\xi \left[\left(\sum_{i=0}^\infty \frac{|B_i|}{A_0 \cdots A_i} \right)^\xi \mathbf{1}_{\{\tau = \infty\}} \right]^{1/\xi} \leq \mathbf{E}\left[|B|^\xi \right]^{1/\xi} \left(1 + \sum_{i=1}^\infty \mathbf{P}\{\tau > i-1\}^{1/\xi} \right). \tag{3.12}$$

Thus the constant C may be explicity related to the return times of $\{V_n\}$ to the origin. Since $\{V_n\}$ is geometrically ergodic (by Lemma 5.1 below), these regeneration times have exponential moments, and so this last expression is also finite. (The case $\xi \in (0,1)$ can be handled similarly, since then the inequality $|x+y|^\xi \leq |x|^\xi + |y|^\xi$ may be used in place of Minkowskii's inequality.)

From (3.11), we obtain a qualitative description of how the insurance surplus process influences ruin of the company, and we note that this information is *not* obtained directly from the decay constant ξ (since this constant depends only on A and hence only on the investment process). For some specific examples of investment and insurance processes, see Collamore (2009), Section 3.

Example 3.3 In the well-known ARCH(1) and GARCH(1,1) financial time series models, one typically studies the forward process $\{V_n\}$ generated by the linear recursion $f(v) = Av + B$, where A and B are strictly positive random variables. Here it is of interest to determine $\mathbf{P}\{V > u\}$, where $V := \lim_{n \to \infty} V_n$ (*cf.* Embrechts et al. (1997), Mikosch (2003), or Section 3 of Collamore (2009)).

For these processes, a minorization is obtained by first observing that $V(v) := Av + B$ has a density, which we call h_v, and $h_v(x)$ is monotonically decreasing for large x. Hence

$$\hat{\nu}(E) := \int_E \left(\inf_{v \in [0,M]} h_v(x) \right) dx$$

is not identically zero, and then $P(x, E) \geq \mathbf{1}_{[0,M]}(x)\hat{\nu}(E)$. Normalizing $\hat{\nu}$ so that it is a probability measure, we then obtain $(\mathcal{M})$ with $k = 1$ and δ the normalizing constant. Thus, according to Remark 2.1, we see that regeneration occurs w.p. δ following return to the set $\mathcal{C} = [0, M]$.

Since $(A, B) \in (0, \infty)^2$, the process $V_n = A_n V_{n-1} + B_n$ is *always* positive; thus, the quantity in (3.10) is positive. Now setting $D_i \equiv 0$ in (2.16), we again obtain (3.9), and, as before, we conclude that this minimum is equal to zero. Thus, the conjugate term is zero on the *entire* probability space. Hence, once again, the conditions of Corollary 2.1 are satisfied. Consequently, $\mathbf{P}\{V > u\} \sim Cu^{-\xi}$, where C assumes the same form as in (3.11), but where $B_0 \sim \hat{\nu}$ and, by Corollary (2.1), τ denotes the first time subsequent to the return of $\{(V_n, \delta_n)\}$ to $[0, M] \times \{1\}$, where $\{\delta_n\}$ is an i.i.d. sequence of Bernoulli(δ) random variables, independent of $\{V_n\}$.

In this example, we have seen that for positive-valued random variables A and B, the linear recursion $f(v) = Av + B$ is a rather special case of Letac's Model E. However, it is worth observing that for random variables B taking values in $(-\infty, \infty)$, this reduction would not be possible (and, in particular, we would not be able to deduce the positivity of the constant C from our result).

Example 3.4 In a life insurance context, one often studies the future discounted losses of the company, that is, the perpetuity sequence (3.5), where $\{A_n\}$ denotes a stochastic discount factor (as before) and $\{L_n\}$ denotes the future obligations of the company. Then it is of interest to study the steady-state tail behavior of $\mathcal{L}_\infty := \lim_{n\to\infty} \mathcal{L}_n$, and a simple argument yields that $\mathcal{L}_\infty \overset{\mathcal{D}}{=} A\mathcal{L}_\infty + B$. Thus we are back in the previous example, and the constants of decay assume the same form as in that case.

Example 3.5 As a slight extension of an example studied in Goldie (1991), consider the polynomial SFPE

$$V \overset{\mathcal{D}}{=} f(V) := \tilde{A}_k V + \tilde{A}_{k-1} V^{(k-1)/k} + \cdots + \tilde{A}_1 V^{1/k} + \tilde{A}_0, \tag{3.13}$$

where the driving sequence $Y := (\tilde{A}_0, \ldots, \tilde{A}_k) \in (0, \infty)^d$, and where we assume that (H_1) is satisfied with $\tilde{A}_k$ in place of A, and that $\mathbf{E}\big[\tilde{A}_i^\xi\big] < \infty$ for all i.

Note that (3.13) can be written in a more suggestive form as

$$V \overset{\mathcal{D}}{=} AV + \mathcal{R}(V), \text{ where } A := \tilde{A}_k \text{ and } \mathcal{R}(V) := \sum_{i=0}^{j-1} \tilde{A}_i V^{i/j}. \tag{3.14}$$

Thus, when the random coefficients $(\tilde{A}_0, \ldots, \tilde{A}_{k-1})$ are fixed, $v^{-1}\mathfrak{B}(v) \downarrow 0$ as $v \to \infty$, and hence for large V, the random function $\mathcal{R}(V)$ may be viewed as a remainder term, whereas the asymptotic behavior of the process is dominated by the linear term, AV. Thus, from a qualitative perspective, we are in the basic setting of Theorem 2.2.

In this example, it is easy to see that the minorization condition $(\mathcal{M})$ holds with $k = 1$. Moreover, by repeating the argument on p. 712 of Mirek (2011a), we obtain that

$$\frac{|f(v) - f(q)|}{|v - q|} \leq \sqrt{A} \tag{3.15}$$

and

$$\left| f(v) - \sqrt{A}|v| \right| \leq \frac{1}{\sqrt{A}} \sum_{i=1}^{j-1} \tilde{A}_i + \frac{\tilde{A}_0}{\sqrt{Q}}, \quad Q := \tilde{A}_0 - \frac{1}{4A}\left(\sum_{i=1}^{j-1} \tilde{A}_i \right)^2. \tag{3.16}$$

Hence the Lipschitz and cancellation conditions hold, provided that $Q > 0$ and

$$\mathbf{E}\left[\left(\frac{1}{\sqrt{A}} \sum_{i=1}^{j-1} \tilde{A}_i + \frac{\tilde{A}_0}{\sqrt{Q}} \right)^\xi \right] < \infty. \tag{3.17}$$

Admittedly, the latter condition is not optimal. In Section 9 below, we revisit this example and discuss an alternative approach, leading to more natural moment assumptions for this problem.

Example 3.6 As another illustration, consider the AR(1) model with ARCH(1) errors studied in Borkovec (2000). In this case, the model takes the form

$$V_n = \left(\gamma V_{n-1} + \sqrt{\left(\beta + \lambda V_{n-1}^2\right)} A_n \right), \tag{3.18}$$

where $\{A_n : n = 1, 2, \ldots\}$ is a collection of i.i.d. symmetric random variables with continuous density with respect to the Lebesgue measure. We also assume that $\mathbf{E}\left[\log \left| \gamma + \sqrt{\lambda}|A| \right| \right] < 0$; this

condition guarantees the existence and uniqueness of the stationary distribution for the process $\{V_n\}$. It is easy to see that $\{V_n\}$ satisfies the minorization $(\mathcal{M})$ with $k = 1$. Consider

$$Q_n := \left| \left(\gamma Q_{n-1} + \sqrt{(\beta + \lambda V_{n-1}^2)} A_n \right) \right|, \quad n = 1, 2, \ldots. \tag{3.19}$$

Then it is shown in Borkovec (2000) that $\{Q_n\}_{n \in \mathbb{Z}_+} \overset{\mathcal{D}}{=} \{V_n\}_{n \in \mathbb{Z}_+}$. Hence, the tail behavior of $|V|$ can be inferred from the tail of $Q := \lim_{n \to \infty} Q_n$. To verify that the cancellation condition holds, set

$$f(v) = \gamma |v| + \sqrt{(\beta + \lambda v^2)} A. \tag{3.20}$$

Notice that f is Lipschitz, and

$$\left| f(v) - (\gamma + \sqrt{\lambda} A)|v| \right| \leq \sqrt{\beta} A \tag{3.21}$$

(*cf.* Mirek (2011a), p. 713), implying the cancellation condition. Thus the tail of $|V|$ is governed by the constant ξ satisfying $\mathbf{E}\left[(\gamma + \sqrt{\lambda} A)^\xi \right] = 1$. The constant C in this case can be determined from (2.38) and is evidently positive.

4   Some results from nonlinear renewal theory

The main tools needed in the proofs of the main theorems will involve ideas from nonlinear renewal theory, which we now present in detail.

The crux of the proof of Theorem 2.1 will be to utilize the representation formula of Lemma 2.2, namely,

$$\mathbf{P}\{V > u\} = \frac{\mathbf{E}[N_u]}{\mathbf{E}[\tau]}.$$

Thus we will need to estimate $\mathbf{E}[N_u]$, the number of exceedances above level u which occur *over a regeneration cycle*. To study this quantity, we will employ the following *dual* change of measure:

$$\mathfrak{L}\big(\log A_n, B_n, D_n \big) = \begin{cases} \mu_\xi & \text{for } n = 1, \ldots, T_u, \\ \mu & \text{for } n > T_u, \end{cases} \tag{4.1}$$

where $T_u = \inf\{n : V_n > u\}$ and μ_ξ is defined as in (2.17). Roughly speaking, this dual measure shifts the distribution of $\log A_n$ on a path terminating at time T_u, and reverts to the original measure thereafter. Let $\mathbf{E}_{\mathfrak{D}}[\cdot]$ denote expectation with respect to the dual measure described in (4.1).

We now focus on two quantities:

(i) the overjump distribution at the first time V_n exceeds the level u, calculated in the ξ-shifted measure; and

(ii) the expected number of exceedances above level u which then occur until regeneration, calculated in the original measure.

Note that in the ξ-shifted measure, $\mathbf{E}_\xi[\log A] > 0$ and hence $V_n \uparrow \infty$ w.p.1 as $n \to \infty$ (*cf.* Lemma 5.2 below). Consequently,

$$V_n = A_n \max\{D_n, V_{n-1}\} + B_n$$

implies

$$V_n \approx A_n V_{n-1} \quad \text{for large } n.$$

In other words, in an asymptotic sense, the process $\{V_n\}$ will ultimately resemble a perturbation of multiplicative random walk. Consequently, the problem described in (i) may be viewed as a variant of a standard problem in nonlinear renewal theory (*cf.* Woodroofe (1982)).

Lemma 4.1 *Assume Letac's Model E, and suppose that (H_0), (H_1), (H_2), and (H_3) are satisfied. Then*

$$\lim_{u \to \infty} \mathbf{P}_\xi \left\{ \frac{V_{T_u}}{u} > y \,\middle|\, T_u < \tau \right\} = \mathbf{P}_\xi \{ \hat{V} > y \} \tag{4.2}$$

for some random variable $\hat{V}$. The distribution of this random variable $\hat{V}$ is independent of the initial distribution of V_0 and is described as follows. If A^l is a typical ladder height of the process $S_n = \sum_{i=1}^n \log A_i$ in the ξ-shifted measure, then

$$\mathbf{P}_\xi \{ \log \hat{V} > y \} = \frac{1}{\mathbf{E}_\xi [A^l]} \int_y^\infty \mathbf{P}_\xi \{ A^l > z \} dz, \quad \text{for all } y \geq 0. \tag{4.3}$$

While Lemma 4.1 follows easily from known results in nonlinear renewal theory, this is not the case for the problem stated in (ii) above. Here we would like to determine $\mathbf{E}_\mathfrak{D} [N_u | \mathfrak{F}_{T_u}]$, *i.e.*, the expected number of exceedances, in the dual measure, which occur over a typical regeneration cycle prior to the regeneration time τ. We provide a precise estimate for this quantity next. In the following result, note that we *start* the process at a level vu, where $v > 1$, and so the dual measure actually agrees with the original measure in this case.

Theorem 4.1 *Assume Letac's Model E, and suppose that (H_0), (H_1), (H_2), and (H_3) are satisfied. Then for any $v > 1$,*

$$\lim_{u \to \infty} \mathbf{E} \left[N_u \,\middle|\, \frac{V_0}{u} = v \right] = U(\log v), \tag{4.4}$$

where $U(z) := \sum_{n \in \mathbb{N}} \mu_A^{\star n}(-\infty, z)$ and μ_A is the marginal distribution of $-\log A$.

Roughly speaking, the function U may be interpreted as the renewal function of the random walk $-S_n = -\sum_{i=1}^n \log A_i$. More precisely, if the distribution of $\log A$ is continuous, so that the open interval $(-\infty, z)$ could be replaced with the closed interval $(-\infty, z]$, then $U(z)$ agrees with the standard definition of the renewal function of $\{-S_n\}$.

We now supplement the estimate in the previous theorem with an upper bound.

Proposition 4.1 *Assume that the conditions of the previous theorem. Then there exist finite positive constants $C_1(u)$ and $C_2(u)$ such that*

$$\mathbf{E}_\mathfrak{D} \left[N_u \,\big|\, \mathfrak{F}_{T_u \wedge (\tau-1)} \right] \leq \left(C_1(u) \log \left(\frac{V_{T_u}}{u} \right) + C_2(u) \right) \mathbf{1}_{\{T_u < \tau\}}, \quad \text{for all } u, \tag{4.5}$$

where $C_1(u)$ and $C_2(u)$ are the positive finite constants that converge as $u \to \infty$ to m^{-1} and $1 + (\sigma^2/m^2)$, respectively, where $m := \mathbf{E}_\xi [\log A]$ and $\sigma^2 := \mathrm{Var}_\xi (\log A)$.

We emphasize that this last proposition will prove crucial for obtaining the upper bound described in Proposition 2.1.

Finally, using the previous results it is now possible to state an extension of Lemma 4.1, which will be particularly useful in the sequel. Let

$$\mathcal{Q}_u = \mathbf{E}_\mathfrak{D} \left[N_u \,\big|\, \mathfrak{F}_{T_u \wedge (\tau-1)} \right] \left(\frac{V_{T_u}}{u} \right)^{-\xi} \mathbf{1}_{\{T_u < \tau\}}.$$

Theorem 4.2 *Assume the conditions of the previous theorem. Then conditional on $\{T_u < \tau\}$,*

$$\mathcal{Q}_u \Rightarrow U(\log \hat{V})\hat{V}^{-\xi} \quad \text{as} \quad u \to \infty, \tag{4.6}$$

where $\hat{V}$ is given as in (4.2) and (4.3). That is,

$$\lim_{u \to \infty} \mathbf{P}_{\mathfrak{D}} \left\{ \mathcal{Q}_u \le y \mid T_u < \tau \right\} = \mathbf{P}_{\mathfrak{D}} \left\{ U(\log \hat{V})\hat{V}^{-\xi} \le y \right\}, \quad \text{for all } y \ge 0.$$

Remark 4.1 For the previous results, Hypothesis (H_0) is only needed to assure that the minorization condition holds with $k = 1$. If $k > 1$, then the same results hold for the k-chain, and then (H_0) may be replaced with the weaker condition that $\log A$ is nonarithmetic.

5 Proofs of the main results: preliminary lemmas

We begin with a series of preparatory lemmas, which will be needed in the proofs of the main theorems given below in Section 6. The first two lemmas describe path properties of the Markov chain $\{V_n\}$, where $V_n = A_n \max\{V_{n-1}, D_n\} + B_n$, $n = 1, 2, \ldots$; that is, the forward iterates of Letac's Model E described in (2.5).

First recall that a Markov chain satisfies a *drift condition* if

$$\int_{\mathbb{S}} h(y)P(x, dy) \le \rho h(x) + \beta \mathbf{1}_{\mathcal{C}}(x), \quad \text{for some } \rho \in (0, 1), \tag{$\mathcal{D}$}$$

where h is a function taking values in $[1, \infty)$, β is a positive constant, and $\mathcal{C}$ is a Borel subset of $\mathbb{R}$.

In the next theorem, we will work with the assumption that the process $\{V_n\}$ is nondegenerate, namely that we do not have $\mathbf{P}\{F_Y(v) = v\} = 1$. This assumption is subsumed by (H_3), but we state it here as a separate hypothesis.

Lemma 5.1 *Assume Letac's Model E, and let $\{V_n\}$ denote the forward recursive sequence corresponding to this SFPE. Suppose that (H_1) and (H_2) are satisfied and $\{V_n\}$ is nondegenerate. Then:*

(i) $\{V_n\}$ satisfies the drift condition $(\mathcal{D})$ with $\mathcal{C} = [-M, M]$ for some constant $M \ge 0$.

(ii) $\{V_n\}$ is φ-irreducible, where φ is the stationary distribution of $\{V_n\}$.

(iii) Suppose (H_0) is satisfied. Then $\{V_n\}$ satisfies the minorization condition $(\mathcal{M})$ with $k = 1$ both with respect to the measure μ and the measure μ_ξ. Furthermore, for any $M > 0$, the set $[-M, M]$ is petite.

(iv) $\{V_n\}$ is geometrically ergodic. Moreover $\mathbf{E}\left[e^{\epsilon \tau}\right] < \infty$, where τ is the inter-regeneration time of the process $\{V_n\}$ under any minorization $(\mathcal{M})$ for which the $\mathcal{C}$-set is bounded.

Proof (i) Using the defining equation $V_n = A_n \max\{D_n, V_{n-1}\} + B_n$, we obtain the inequality

$$|V_n| \le A_n|V_{n-1}| + (A_n|D_n| + |B_n|). \tag{5.1}$$

Now hypothesis (H_1) implies that $\mathbf{E}[A^\alpha] < 1$ for all $\alpha \in (0, \xi)$. Fix $\alpha \in (\xi \wedge 1)$ and $V_0 = v$. Then from the deterministic inequality $|x + y|^\alpha \le |x|^\alpha + |y|^\alpha$, $\alpha \in (0, 1)$, we obtain

$$\mathbf{E}\left[|V_1|^\alpha\right] \le \mathbf{E}[A^\alpha] v^\alpha + \mathbf{E}\left[|B_1|^\alpha\right] + \mathbf{E}\left[(A_1|D_1|)^\alpha\right]. \tag{5.2}$$

Thus $(\mathcal{D})$ holds with $h(x) = |x|^\alpha + 1$, $\rho = (\mathbf{E}[A^\alpha] + 1)/2$, and $\mathcal{C} = [-M, M]$, where M is chosen sufficiently large such that

$$\frac{1}{2}\left(1 - \mathbf{E}[A^\alpha]\right) M^\alpha \ge \mathbf{E}\left[|B_1|^\alpha\right] + \mathbf{E}\left[(A_1|D_1|)^\alpha\right].$$

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(ii) To verify that $\{V_n\}$ is φ-irreducible, it is sufficient to show that $\varphi(A) > 0$ implies that $\mathcal{U}(x, A) > 0$ for all x, where

$$\mathcal{U}(x, A) := \sum_{n=1}^{\infty} P^n(x, A)$$

(*cf.* Meyn and Tweedie (1993), Proposition 4.2.1). Now to identify φ, note that under our hypotheses, the conditions of Proposition 6.1 of Goldie (1991) are satisfied. Hence the backward recursion (2.7) converges to a random variable $\tilde{Z}$ which is independent of v. By Lemma 2.1, it follows that $\{V_n\}$ is stationary and the law of the limiting random variable V is the same as the law of $\tilde{Z}$. Now choose $\varphi \equiv \mathfrak{L}(\tilde{Z})$. Since $\{V_n\}$ is assumed to be nondegenerate, it follows by Theorem 1.3.1 and (1.16) of Meyn and Tweedie (1993) that

$$\sup_{A \in \mathcal{B}(\mathbb{R})} |P^n(x, A) - \varphi(A)| \to 0 \quad \text{as } n \to \infty, \tag{5.3}$$

for all $x \in \mathbb{R}$ (where we have used (i) to verify their condition (1.14) with $h(x) = |x|^{\alpha} + 1$). Hence

$$\mathcal{U}(x, A) \geq \lim_{n \to \infty} P^n(x, A) = \varphi(A) > 0 \tag{5.4}$$

for any φ-positive set A.

(iii) We note that the main issue is to verify that the minorization holds with $k = 1$ (rather than for general k, which is well known to follow immediately from (ii)).

We will show that for any $v \in \mathbb{R}$, there exists an ϵ-neighborhood $\mathfrak{B}_\epsilon(v)$ such that

$$P(w, E) \geq \delta \mathbf{1}_{\mathfrak{B}_\epsilon(v)}(w)\nu(E), \quad \text{for all } w \in \mathbb{R} \text{ and } E \in \mathcal{B}(\mathbb{R}), \tag{5.5}$$

for some positive constant δ and some probability measure ν, both of which will typically depend on v. Since *some* such interval $\mathfrak{B}_\epsilon(v)$ will necessarily be φ-positive with $\varphi \equiv \mathfrak{L}(V)$ as in (ii), this will imply the existence of a minorization with $k = 1$.

Set $v^* = \inf \{v : \mathbf{P}\{D \leq v\} = 1\} \in (-\infty, \infty]$. We will consider three different cases, namely $v < v^*$, $v > v^*$, and $v = v^*$.

If $v < v^*$, then $\mathbf{P}\{D > v\} > 0$. Since $\mathbf{P}\{D > w\}$ is nonincreasing as a function of w, $\mathbf{P}\{D > v + \epsilon\} > 0$ for some $\epsilon > 0$. For this choice of v and ϵ, let $\mathfrak{B}_\epsilon(v)$ be the required ϵ-neighborhood in (5.5). Now if $V_0 \in \mathfrak{B}_\epsilon(v)$ and $D_1 > v + \epsilon$, then $\max\{D_1, V_0\} = D_1$ and hence $V_1 = A_1 D_1 + B_1$. Thus, for any initial state $V_0 \in \mathfrak{B}_\epsilon(v)$,

$$V_1 = (A_1 D_1 + B_1)\mathbf{1}_{\{D_1 > v+\epsilon\}} + V_1 \mathbf{1}_{\{D_1 \leq v+\epsilon\}}. \tag{5.6}$$

Note that the first term on the right-hand side is independent of V_0. Taking ν to be the probability law of $(A_1 D_1 + B_1)$ conditional on $\{D_1 > v + \epsilon\}$ and $\delta = \mathbf{P}\{D > v + \epsilon\}$, we obtain (5.5).

Next suppose $v > v^*$. Then for some $\epsilon > 0$, $\mathbf{P}\{D > v - \epsilon\} = 0$, and for this choice of v and ϵ, let $\mathfrak{B}_\epsilon(v)$ be the ϵ-neighborhood in (5.5). Then $\mathbf{P}\{V_1 = A_1 V_0 + B_1 | V_0 \in \mathfrak{B}_\epsilon(v)\} = 1$. Hence, to obtain a minorization for V_1 in this case, it is sufficient to derive a minorization for $V_1(v) := A_1 v + B_1$ a.s. To this end, begin by observing that since A_1 has an absolutely continuous component with a nontrivial density in a neighborhood of $\mathbb{R}$, so does the pair $(A_1, A_1 v + B_1)$ in $\mathbb{R}^2$. Let $\gamma^{(v)}$ denote the probability law of $(A_1, A_1 v + B_1)$. Then by the Lebesgue decomposition theorem, $\gamma^{(v)}$ can be decomposed into its absolutely continuous and singular components relative to Lebesgue measure, and the continuous component satisfies

$$\gamma_c^{(v)}(E) = \int_E \frac{d\gamma^{(v)}}{dl}(z)dl(z), \quad \text{for all } E \in \mathcal{B}(\mathbb{R}^2),$$

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where l denotes Lebesgue measure on $\mathbb{R}^2$. Since there is a nontrivial density, by (H_0), there exists a rectangle where $\frac{d\gamma^{(v)}}{dl}$ is bounded away from zero; that is,

$$\frac{d\gamma^{(v)}}{dl}(x,y) \geq \delta > 0, \quad \text{for all } (x,y) \in [a,b] \times [c,d]. \tag{5.7}$$

Now suppose $w \in \mathfrak{B}_\epsilon(v)$ and let $\gamma^{(w)}$ denote the probability law of $V_1(w) := A_1 w + B_1$. Then $V_1(w) - V_1(v) = A_1(w - v)$, and hence (5.7) yields

$$\frac{d\gamma^{(w)}}{dl}(x,y) \geq \delta, \quad \text{for all} \quad (x,y) \in [a,b] \times [c + w^*, d - w^*],$$

where $w^* = b|w - v|$. Notice that the constant b was obtained by observing that the first component describes the distribution of A, where $\text{supp}(A) \subset (0, \infty)$, implying that $0 < a < b < \infty$. Since $w^* \downarrow 0$ as $w \to v$, it follows that for sufficiently small ϵ, there exists a subinterval $[c', d']$ of $[c, d]$ such that

$$\frac{d\gamma^{(w)}}{dl}(x,y) \geq \delta, \quad \text{for all } (x,y) \in [a,b] \times [c',d'] \text{ and all } w \in \mathfrak{B}_\epsilon(v). \tag{5.8}$$

Hence the minorization (5.5) holds with

$$\nu(E) = \int_{\mathbb{R} \times E} \left(\inf_{w \in \mathfrak{B}_\epsilon(v)} \frac{d\gamma^{(w)}}{dl}(x,y) \right) dl(x,y), \quad \text{for all } E \in \mathcal{B}(\mathbb{R}),$$

and by (5.7), this measure is not identically equal to zero.

It remains to consider the case where $v = v^*$. First observe that, similar to (5.6), $\mathbf{P}\{V_1 = A_1 V_0 + B_1 \,|\, V_0 \in \mathfrak{B}_\epsilon(v^*), \ D_1 \leq v^* - \epsilon\} = 1$. Hence, we clearly have

$$V_1 = (A_1 V_0 + B_1)\mathbf{1}_{\{D_1 \leq v^* - \epsilon\}} + V_1 \mathbf{1}_{\{D_1 > v^* - \epsilon\}}. \tag{5.9}$$

Now define $\gamma^{(\epsilon, v^*)}$ on any Borel set $E \subset \mathbb{R}^2$ to be

$$\gamma^{(\epsilon, v^*)}(E) = \mathbf{P}\left\{(A_1, A_1 v^* + B_1) \in E, D_1 \leq v^* - \epsilon\right\} \nearrow \gamma^{(v^*)}(E) \quad \text{as} \quad \epsilon \to 0,$$

where the last step follows from the definition of $\gamma^{(v^*)}$, since $\mathbf{P}\{D_1 \leq v^* - \epsilon\} \uparrow 1$ as $\epsilon \to 0$. Hence, since $\gamma^{(v^*)}$ has an absolutely continuous component, there exists an $\epsilon > 0$ such that $\gamma^{(\epsilon, v^*)}$ also has an absolutely continuous component. Now apply the previous argument with $\gamma^{(v)}$ replaced with $\gamma^{(\epsilon, v^*)}$ for this choice of ϵ to obtain the corresponding minorization for this case. Thus we have obtained (5.5) for all the three cases. Note that the above computations hold regardless of whether we are in the original measure μ or in the ξ-shifted measure μ_ξ.

Finally, to show that $[-M, M]$ is a petite set, note that $[-M, M] \subset \bigcup_{v \in [-M,M]} \mathfrak{B}_{\epsilon(v)}(v)$, where $\mathfrak{B}_{\epsilon(v)}(v)$ is a small set and hence is petite. Thus there exists a finite subcover of petite sets, and then by Proposition 5.5.5 of Meyn and Tweedie (1993), $[-M, M]$ is petite.

(iv) Since $[-M, M]$ is petite for any $M > 0$, it follows from (i) and Meyn and Tweedie (1993), Theorem 15.0.1, that the process $\{V_n\}$ is geometrically ergodic. It remains to show that—regardless of $(\delta, \mathcal{C}, \nu)$ in the minorization $(\mathcal{M})$—we have $\mathbf{E}[e^{\epsilon\tau}] < \infty$, where τ is the inter-regeneration time under $(\mathcal{M})$. Now since $\mathcal{C}$ is bounded, it follows by Theorem 15.2.6 of Meyn and Tweedie (1993) that $\mathcal{C}$ is h-geometrically regular with $h(x) = |x|^\alpha + 1$. Consequently, letting K denote the first return time of $\{V_n\}$ to $\mathcal{C}$, we have

$$\Gamma(t) := \sup_{v \in C} \mathbf{E}\left[t^K | V_0 = v\right] < \infty \tag{5.10}$$

for some $t > 1$.

Consider the split-chain (see Nummelin (1984), Section 4.4 or Remark 2.1 above). Originating from the $\mathcal{C}$-set, this chain has the 1-step transition measure $\nu(dy)$ w.p. δ and the 1-step transition kernel $\bar{P}(x, y) := (P(x, dy) - \nu(dy))/(1 - \delta)$ w.p. $(1 - \delta)$. Thus, conditional on either one of these transition kernels, it follows from (5.10) that $\Gamma_\nu(t) < \infty$ and $\Gamma_{\bar{P}}(t) < \infty$, where Γ_ν denotes conditioning on $V_1 \sim \nu$, and $\Gamma_{\bar{P}}(t) := \sup_{v \in C} \mathbf{E}\left[t^K | V_0 = v, V_1 \sim \bar{P}\right]$. Hence by Remark 2.1,

$$\mathbf{E}\left[t^\tau\right] \leq \Gamma_\nu(t)\left(\delta + \sum_{n=1}^{\infty} \delta(1 - \delta)^n \left(\Gamma_{\bar{P}}(t)\right)^n\right), \tag{5.11}$$

where δ is the constant appearing in $(\mathcal{M})$. Finally observe by a dominated convergence argument that $\Gamma_{\bar{P}}(t) \downarrow 1$ as $t \downarrow 1$, and thus for sufficiently small $t > 1$, $\Gamma_{\bar{P}}(t) < (1 - \delta)^{-1}$. For this choice of t, the series on the right-hand side of (5.11) is convergent, as required. $\qquad \square$

Next we turn to a critical result which establishes the transience of the process $\{V_n\}$ in its ξ-shifted measure.

Lemma 5.2 *Assume Letac's Model E, and let $\{V_n\}$ denote the forward recursive sequence corresponding to this SFPE. Assume that (H_1), (H_2) and (H_3) are satisfied and $\{V_n\}$ is nondegenerate. Then under the measure μ_ξ,*

$$V_n \nearrow +\infty \quad \text{w.p. } 1 \quad \text{as} \quad n \to \infty. \tag{5.12}$$

Thus, in particular, the Markov chain $\{V_n\}$ is transient.

Proof By (H_3), the set $[M, \infty)$ is attainable with positive probability for all M. Consequently by Meyn and Tweedie (1993), Theorem 8.3.6, it is sufficient to show that for some constant M,

$$\mathbf{P}_\xi\left\{V_n \leq M, \text{ for some } n \in \mathbb{Z}_+ | V_0 \geq 2M\right\} < 1. \tag{5.13}$$

To establish (5.13), note that by iterating the inequality $V_n \geq A_n V_{n-1} - |B_n|$, we obtain

$$\frac{V_n}{A_1 \cdots A_n} \geq V_0 - W_n, \quad \text{where} \quad W_n := \sum_{i=1}^{n} \frac{|B_i|}{A_1 \cdots A_i}.$$

Now since $\mathbf{E}_\xi[\log A] > 0$, it follows by Lemma 5.1 that $\{W_n\}$ converges a.s. to a proper random variable and, furthermore,

$$\mathbf{P}_\xi\left\{A_1 \cdots A_n \geq 1, \text{ for all } n\right\} > 0.$$

Hence (5.13) holds.

We conclude that $[-M, M]$ is uniformly transient, and thus $|V_n| \uparrow \infty$ as $n \to \infty$. Then $V_n \geq A_n D_n + B_n$ for all n, implying that $V_n \uparrow \infty$ as $n \to \infty$. $\qquad \square$

Prior to stating the next lemma, recall that $T_u := \inf\{n : V_n > u\}$, and that $\mathbf{E}_{\mathfrak{D}}[\cdot]$ denotes expectation with respect to the dual measure over a typical regeneration cycle.

Lemma 5.3 *Assume Letac's Model E, let $\{V_n\}$ denote the forward recursive sequence corresponding to this SFPE, and assume that (H_1) and (H_2) are satisfied. Let $g : \mathbb{R}^\infty \to [0, \infty]$ be a deterministic function, and let g_n denote its projection onto the first $n + 1$ coordinates; that is, $g_n(x_0, \ldots, x_n) = g(x_0, \ldots, x_n, 0, 0, \ldots)$. Then*

$$\mathbf{E}\left[g_{\tau-1}(V_0, \ldots, V_{\tau-1})\right] = \mathbf{E}_{\mathfrak{D}}\left[g_{\tau-1}(V_0, \ldots, V_{\tau-1})e^{-\xi S_{T_u}}\mathbf{1}_{\{T_u < \tau\}}\right]$$
$$+ \mathbf{E}_{\mathfrak{D}}\left[g_{\tau-1}(V_0, \ldots, V_{\tau-1})e^{-\xi S_\tau}\mathbf{1}_{\{T_u \geq \tau\}}\right]. \tag{5.14}$$

Proof If

$$\mathfrak{L}\left(\log A_i, B_i, D_i\right) = \begin{cases} \mu_\xi & \text{for } i = 1, \ldots, n, \\ \mu & \text{for } i > n, \end{cases}$$

then it can be shown using induction that

$$\mathbf{E}\left[g_n(V_0, \ldots, V_n)\right] = \mathbf{E}_\xi\left[g_n(V_0, \ldots, V_n)e^{-\xi S_n}\right]. \tag{5.15}$$

Then (5.14) follows by conditioning on $\{T_u = m, \tau = n\}$ and summing over all possible values of m and n. $\qquad\square$

Next we establish a critical result which links forward iteration of Letac's Model E to its corresponding backward iterates.

Lemma 5.4 *Assume Letac's Model E, and let $\{V_n\}$ denote the forward recursive sequence corresponding to this SFPE. Let $\{Z_n^{(p)}\}$ and $\{Z_n^{(c)}\}$ denote the associated perpetuity and conjugate sequences, respectively, assumed to have the initial values described prior to (2.13) and (2.15), and set $A_0 = 1$. Then for any $n \in \mathbb{N}$,*

$$\left(Z_n^{(p)} - Z_n^{(c)}\right)\mathbf{1}_{\{Z_n > 0\}} = Z_n \mathbf{1}_{\{Z_n > 0\}}, \quad \text{where} \quad Z_n := \frac{V_n}{A_0 \cdots A_n}. \tag{5.16}$$

Proof It follows by an inductive argument that for all $n \in \mathbb{N}$,

$$V_n = \max\left\{\sum_{i=0}^n B_i \prod_{j=i+1}^n A_j, \bigvee_{k=1}^n \left(\sum_{i=k}^n B_i \prod_{j=i+1}^n A_j + D_k \prod_{j=k}^n A_j\right)\right\}, \tag{5.17}$$

where $B_0 := V_0$. Hence

$$Z_n := \frac{V_n}{A_0 \cdots A_n} = \sum_{i=0}^n \frac{B_i}{A_0 \cdots A_i} - \min\left\{0, \mathfrak{M}_n\right\}, \tag{5.18}$$

where

$$\mathfrak{M}_n := \bigwedge_{k=1}^n \left\{\sum_{i=0}^{k-1} \frac{B_i}{A_0 \cdots A_i} - \frac{D_k}{A_0 \cdots A_{k-1}}\right\}. \tag{5.19}$$

Now

$$Z_n^{(p)} = F_{Y_0}^{(p)} \circ \cdots \circ F_{Y_n}^{(p)}(0) = \sum_{i=0}^n \frac{B_i}{A_0 \cdots A_i}. \tag{5.20}$$

To obtain a similar expression for $Z_n^{(c)}$, observe again by induction that

$$Z_n^{(c)} = \min\left\{\sum_{i=0}^n \frac{B_i}{A_0 \cdots A_i}, \bigwedge_{k=0}^n \left(\sum_{i=0}^k \frac{B_i}{A_0 \cdots A_i} + \frac{D_k^*}{A_0 \cdots A_k}\right)\right\}. \tag{5.21}$$

Then substituting $D_0^* = -B_0$ and $D_i^* = -A_i D_i - B_i$, $i = 1, 2, \ldots$, yields that, with $\mathfrak{M}_n$ given as in (5.19),

$$Z_n^{(c)} = \min\left\{Z_n^{(p)}, 0, \mathfrak{M}_n\right\}. \tag{5.22}$$

Comparing these expressions with (5.18), we conclude that

$$Z_n = Z_n^{(p)} - \min\{0, \mathfrak{M}_n\}. \tag{5.23}$$

Finally observe that if $Z_n \geq 0$, then it follows from the previous equation that $0 \leq \max\left\{Z_n^{(p)}, Z_n^{(p)} - \mathfrak{M}_n\right\}$ and hence

$$Z_n \mathbf{1}_{\{Z_n \geq 0\}} = \max\left\{0, Z_n^{(p)}, Z_n^{(p)} - \mathfrak{M}_n\right\} = Z_n^{(p)} - Z_n^{(c)}. \qquad \square$$

Finally, we study the convergence of the sequence $\{Z_n\}$ introduced in the previous lemma, and relate it to the random variable $\bar{Z}^{(p)}$ defined prior to the statement of Proposition 2.1.

Lemma 5.5 *Assume Letac's Model E, and suppose that (H_1), (H_2), and (H_3) are satisfied and $\{V_n\}$ is nondegenerate. Let $\{Z_n\}$ be defined as in* (5.16). *Then in μ_ξ-measure, $\{Z_n\}$ has the following regularity properties*:

(i) $Z_n \to Z$ *a.s. as $n \to \infty$, where Z is a proper random variable supported on $(0, \infty)$. Consequently, $Z^{(p)}$ and $Z^{(c)}$ are well defined and $Z^{(p)} > Z^{(c)}$ a.s.*

(ii) $\mathbf{E}_\xi\left[\left(\bar{Z}^{(p)}\right)^\xi\right] < \infty$. *Moreover, for all n and u,*

$$\left|Z_n \mathbf{1}_{\{n < \tau\}}\right| \leq \bar{Z}^{(p)} \qquad \text{and} \qquad \left|Z_{T_u} \mathbf{1}_{\{T_u < \tau\}}\right| \leq \bar{Z}^{(p)}. \tag{5.24}$$

Proof We begin by establishing (ii). Since

$$|V_n| \leq A_n |V_{n-1}| + \left(A_n |D_n| + |B_n|\right), \tag{5.25}$$

the process $\{|V_n|\}$ is bounded from above by $\{R_n\}$, where $R_0 = |V_0|$ and, for each $n \in \mathbb{Z}_+$,

$$R_n = \tilde{B}_n + A_n R_{n-1}, \quad \text{where} \quad \tilde{B}_n = A_n |D_n| + |B_n|.$$

Iterating the previous equation yields

$$R_n = \sum_{i=0}^{n} \tilde{B}_i \prod_{i=j+1}^{n} A_i, \quad n = 0, 1, \ldots, \tag{5.26}$$

where $\tilde{B}_0 \equiv |V_0|$. Since $|V_n| \leq R_n$ for all n, we may now apply (5.16) to obtain that $\{|Z_n|\}$ is bounded from above by the perpetuity sequence

$$\tilde{W}_n := (A_1 \cdots A_n)^{-1} R_n = \sum_{i=0}^{n} \frac{\tilde{B}_i}{A_1 \cdots A_i}, \quad n = 1, 2, \ldots, \tag{5.27}$$

and hence

$$|Z_n| \mathbf{1}_{\{n < \tau\}} \leq \tilde{W}_n \mathbf{1}_{\{n < \tau\}} \leq \sum_{i=0}^{\infty} \frac{\tilde{B}_i}{A_0 \cdots A_i} \mathbf{1}_{\{\tau > i\}} := \bar{Z}^{(p)}, \tag{5.28}$$

and similarly for $Z_{T_u} \mathbf{1}_{\{T_u < \tau\}}$.

It remains to show that $\mathbf{E}_\xi\left[\left(\bar{Z}^{(p)}\right)^\xi\right] < \infty$. To this end, observe that if $\xi \geq 1$, then by Minkowskii's inequality followed by a change of measure argument,

$$\mathbf{E}_\xi\left[\left(\bar{Z}^{(p)}\right)^\xi\right]^{1/\xi} \leq \mathbf{E}\left[|V_0|^\xi\right] + \mathbf{E}\left[\tilde{B}^\xi\right]^{1/\xi} \sum_{n=1}^{\infty} \mathbf{P}\{\tau > n - 1\}^{1/\xi} < \infty, \tag{5.29}$$

where finiteness is obtained from (H_2) and Lemma 5.1 (iv). For the first term on the right-hand side, we have used $B_0 \equiv |V_0|$, where $V_0 \sim \nu$. Note that the first expectation on the right-hand

side of (5.29) is also finite since, in the split-chain construction (as described in Nummelin (1984), Section 4.4, or or Remark 2.1),

$$\mathbf{E}\left[\left.|V_0|^\xi\right| V_{-1} = v\right] = \delta\bar{\mathbf{E}}\left[\left.|V_0|^\xi\right| V_0 \sim \nu\right] + (1-\delta)\bar{\mathbf{E}}\left[\left.|V_0|^\xi\right| V_{-1} = v\right],$$

where $\bar{\mathbf{E}}\left[\cdot\right]$ denotes expectation with respect to the kernel $\bar{P}(x, dy) := (P(x, dy) - \nu(dy))/(1-\delta)$. As the left-hand side of the last equation is finite under (H_1) and (H_2), so is the first term on the right-hand side, as required.

On the other hand, if $\xi < 1$, then an analogous result is obtained using the deterministic inequality $|x + y|^\alpha \leq |x|^\alpha + |y|^\alpha$, $0 < \xi < 1$, in place of Minkowskii's inequality, and this completes the proof of (ii).

To establish (i), set $Z_- = \liminf_{n \to \infty} Z_n$ and observe that if $\mathbf{P}_\xi\{Z_- \leq 0\} = p > 0$, then by Fatou's lemma

$$p \leq \liminf_{n \to \infty} \mathbf{E}_\xi\left[\mathbf{1}_{\{Z_n \leq 0\}}\right] = \liminf_{n \to \infty} \mathbf{P}_\xi\left\{V_n \leq 0\right\}, \tag{5.30}$$

where, in the last step, we have used Lemma 5.4 to replace Z_n by V_n. But by Lemma 5.2, $V_n \uparrow +\infty$ w.p.1 as $n \to \infty$, and thus $p = 0$. We conclude $Z_- \in (0, \infty)$.

Next observe by definition of $\{V_n\}$ that if $V_{n-1} \geq 0$,

$$A_n V_{n-1} - |B_n| \leq V_n \leq A_n V_{n-1} + (A_n|D_n| + |B_n|). \tag{5.31}$$

Let $\delta(n)$ denote the indicator function on the event $\{V_l \geq 0, \text{ for all } l \geq n\}$. Then by Lemma 5.2, $\lim_{n \to \infty} \delta(n) = 1$ a.s. By iterating the left- and right-hand sides of (5.31), we obtain that for any $m > n$,

$$\left|V_m - (A_{n+1} \cdots A_m) V_n\right| \delta(n) \leq \sum_{i=n+1}^{m} \tilde{B}_i \prod_{i=j+1}^{n} A_i, \tag{5.32}$$

where $\tilde{B}_i := A_i|D_i| + |B_i|$. Hence for any $m > n$,

$$\left|Z_m - Z_n\right| \delta(n) \leq \sum_{i=n+1}^{\infty} \frac{\tilde{B}_i}{A_1 \cdots A_i}. \tag{5.33}$$

Now recall that $\mathbf{E}_\xi\left[\log A\right] = \Lambda'(\xi) > 0$. Hence the process $\tilde{S}_n := -\left\{\sum_{i=1}^{n} \log A_i\right\} + \log \tilde{B}_n$ has a negative drift, and by (H_1) we have $\mathbf{E}_\xi\left[(A^{-1})^\alpha\right] < \infty$ for some $\alpha > 0$. Hence by Cramér's large deviation theorem (Dembo and Zeitouni (1998), Section 2.2), it follows that

$$\mathbf{P}_\xi\left\{\frac{\tilde{B}_n}{A_1 \cdots A_n} > a^n\right\} \leq e^{-tn}, \tag{5.34}$$

where $a := \exp\left\{-\mathbf{E}_\xi\left[\log A\right] + \epsilon\right\} \in (0, 1)$, and $t \in (0, \infty)$ and $\epsilon \in (0, \mathbf{E}_\xi\left[\log A\right])$. Then by the Borel-Cantelli lemma,

$$\mathbf{P}_\xi\left\{\frac{\tilde{B}_n}{A_1 \cdots A_n} > a^n \text{ i.o.}\right\} = 0. \tag{5.35}$$

Since $\lim_{n \to \infty} \delta(n) = 1$ a.s., it follows as a consequence of (5.33) and (5.35) that Z_n converges a.s. to a proper random variable on $(0, \infty)$, which we denote by Z.

Since the above argument also holds for the recursion $f(v) = Av + B$, it follows that $Z_n^{(p)}$ converges a.s. to a random variable taking values in $(0, \infty)$. Then by Lemma 5.4, the limit in the definition of $Z^{(c)}$ must also exist and we must have that $Z^{(p)} - Z^{(c)} = Z > 0$ a.s. $\qquad \square$

6   Proof of Theorem 2.1

The key to the proof is the following result, which shows that $u^\xi \mathbf{E}\left[N_u\right]$ behaves asymptotically as a product of two terms, the first being influenced mainly by the "short-time" behavior of the process, the second term being determined by the "large-time" behavior. We will later identify this second term as a classical limit related to the random walk $S_n = \sum_{i=1}^n \log A_i$.

Proposition 6.1 *Assume Letac's Model E, and suppose that* (H_0), (H_1), (H_2), *and* (H_3) *are satisfied. Then*

$$\lim_{u\to\infty} u^\xi \mathbf{E}\left[N_u\right] = \mathbf{E}_\xi\left[Z^\xi \mathbf{1}_{\{\tau=\infty\}}\right] \lim_{u\to\infty} \mathbf{E}_\mathfrak{D}\left[N_u \left(\frac{V_{T_u}}{u}\right)^{-\xi} \middle| T_u < \tau\right]. \tag{6.1}$$

Remark 6.1 A similar asymptotic independence is utilized in Enriquez et al. (2009). However, their precise results and techniques differ considerably from ours.

Proof Since $N_u = g_{\tau-1}(V_0, \ldots, V_{\tau-1})$, where $g(v_0, v_1, \ldots) = \sum_{n=0}^\infty \mathbf{1}_{\{v_n > u\}}$ and g_τ denotes the projection onto the first $\tau + 1$ coordinates, it follows by Lemma 5.3 that

$$\mathbf{E}\left[N_u\right] = \mathbf{E}_\mathfrak{D}\left[N_u e^{-\xi S_{T_u}}\right].$$

Moreover by the definition of Z_n in (5.21),

$$e^{-\xi S_{T_u}} \mathbf{1}_{\{T_u < \tau\}} = \left((A_1 \cdots A_{T_u})^{-\xi} V_{T_u}^\xi \mathbf{1}_{\{T_u < \tau\}}\right) V_{T_u}^{-\xi} = \left(Z_{T_u}^\xi \mathbf{1}_{\{T_u < \tau\}}\right) V_{T_u}^{-\xi}.$$

Combining the last two equations, we obtain

$$\mathbf{E}\left[N_u\right] = \mathbf{E}_\mathfrak{D}\left[\left(Z_{T_u}^\xi \mathbf{1}_{\{T_u < \tau\}}\right) N_u V_{T_u}^{-\xi}\right]. \tag{6.2}$$

Consequently, by conditioning on $\mathfrak{F}_{T_u \wedge (\tau-1)}$ and rearranging the terms,

$$u^\xi \mathbf{E}\left[N_u\right] = \mathbf{E}_\mathfrak{D}\left[\left(Z_{T_u}^\xi \mathbf{1}_{\{T_u < \tau\}}\right) \mathcal{Q}_u\right], \tag{6.3}$$

where

$$\mathcal{Q}_u := \mathbf{E}_\mathfrak{D}\left[N_u \middle| \mathfrak{F}_{T_u \wedge (\tau-1)}\right] \left(\frac{V_{T_u}}{u}\right)^{-\xi} \mathbf{1}_{\{T_u < \tau\}}. \tag{6.4}$$

Note

$$\mathbf{E}_\mathfrak{D}\left[\left(Z_{T_u}^\xi \mathbf{1}_{\{T_u < \tau\}}\right) \mathcal{Q}_u\right] = \mathbf{E}_\mathfrak{D}\left[Z_n^\xi \mathcal{Q}_u \mathbf{1}_{\{n \le T_u < \tau\}}\right]$$
$$+ \mathbf{E}_\mathfrak{D}\left[\left(Z_{T_u}^\xi \mathbf{1}_{\{T_u < \tau\}} - Z_n^\xi \mathbf{1}_{\{n \le T_u < \tau\}}\right) \mathcal{Q}_u\right]. \tag{6.5}$$

Now take the limit first as $u \to \infty$ and then as $n \to \infty$.

We begin by analyzing the first term on the right-hand side. By Proposition 4.1 and the definition of $\mathcal{Q}_u$ in (6.4), $\{\mathcal{Q}_u\}$ is uniformly bounded in u. Also, by Lemma 5.5 (ii), $\mathbf{E}_\xi\left[|Z_n|^\xi \mathbf{1}_{\{n \le T_u < \tau\}}\right] < \infty$, for all n. Consequently, for any given n,

$$\mathbf{E}_\mathfrak{D}\left[|Z_n^\xi \mathcal{Q}_u| \mathbf{1}_{\{n \le T_u < \tau\}}\right] \le K \mathbf{E}_\xi\left[|Z_n|^\xi \mathbf{1}_{\{n \le T_u < \tau\}}\right] \le K' \tag{6.6}$$

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for finite constants K and K'. In the middle term, we have replaced $\mathbf{E}_{\mathfrak{D}}[\cdot]$ with $\mathbf{E}_{\xi}[\cdot]$, since these will be the same on the set $\{n \leq T_u < \tau\}$. Hence an application of the dominated convergence theorem yields

$$
\lim_{u \to \infty} \mathbf{E}_{\mathfrak{D}}\left[Z_n^{\xi} \mathcal{Q}_u \mathbf{1}_{\{n \leq T_u < \tau\}}\right]
$$
$$
= \mathbf{E}_{\mathfrak{D}}\left[Z_n^{\xi} \lim_{u \to \infty} \mathbf{1}_{\{n \leq T_u < \tau\}} \mathbf{E}_{\mathfrak{D}}\left[\mathcal{Q}_u \,|\, \mathfrak{F}_n, T_u < \tau\right]\right]
$$
$$
= \mathbf{E}_{\mathfrak{D}}\left[Z_n^{\xi} \lim_{u \to \infty} \mathbf{1}_{\{n \leq T_u < \tau\}}\right] \mathbf{E}_{\mathfrak{D}}[\mathcal{Q}], \tag{6.7}
$$

where the last step of (6.7) follows by Theorem 4.2 (and, in particular, the independence of the limiting distribution $\mathcal{Q}$ from the initial distribution of $\{V_k\}$, which is here taken to be the distribution of V_n for some fixed n). In the second expectation on the right-hand side, the limit and expectation may be exchanged due to the weak convergence of $\mathcal{Q}_u$ to $\mathcal{Q}$ and the uniform boundedness of $\{\mathcal{Q}_u\}$, by Proposition 4.1 and Theorem 4.2.

To identify the first expectation on the right-hand side of (6.7), observe once again that $Z_n \mathbf{1}_{\{n \leq T_u < \tau\}}$ is the same in the ξ-shifted measure as it is in the dual measure, since the dual measure agrees with the ξ-shifted measure up until time T_u. Hence

$$
\mathbf{E}_{\mathfrak{D}}\left[Z_n^{\xi} \lim_{u \to \infty} \mathbf{1}_{\{n \leq T_u < \tau\}}\right] = \mathbf{E}_{\xi}\left[Z_n^{\xi} \lim_{u \to \infty} \mathbf{1}_{\{n \leq T_u < \tau\}}\right] = \mathbf{E}_{\xi}\left[Z_n^{\xi} \mathbf{1}_{\{\tau = \infty\}}\right],
$$

since an elementary argument yields that $T_u \uparrow \infty$ a.s. as $u \to \infty$.

Substituting into (6.7) and now taking the limit as $n \to \infty$ yields

$$
\lim_{n \to \infty} \lim_{u \to \infty} \mathbf{E}_{\mathfrak{D}}\left[Z_n^{\xi} \mathcal{Q}_u \mathbf{1}_{\{n \leq T_u < \tau\}}\right] = \lim_{n \to \infty} \mathbf{E}_{\xi}\left[Z_n^{\xi} \mathbf{1}_{\{\tau = \infty\}}\right] \mathbf{E}_{\mathfrak{D}}[\mathcal{Q}]
$$
$$
= \mathbf{E}_{\xi}\left[Z^{\xi} \mathbf{1}_{\{\tau = \infty\}}\right] \mathbf{E}_{\mathfrak{D}}[\mathcal{Q}], \tag{6.8}
$$

where the last step follows from the dominated convergence theorem and Lemma 5.5 (i) and (ii). Finally, observe by Theorem 4.2 that

$$
\mathbf{E}_{\mathfrak{D}}[\mathcal{Q}] = \lim_{u \to \infty} \mathbf{E}_{\mathfrak{D}}[\mathcal{Q}_u] \left(\mathbf{P}\{T_u < \tau\}\right)^{-1}
$$
$$
= \lim_{u \to \infty} \mathbf{E}_{\mathfrak{D}}\left[\mathbf{E}_{\mathfrak{D}}\left[N_u \,|\, \mathfrak{F}_{T_u \wedge (\tau-1)}\right] \left(\frac{V_{T_u}}{u}\right)^{-\xi} \mathbf{1}_{\{T_u < \tau\}}\right] \left(\mathbf{P}\{T_u < \tau\}\right)^{-1}
$$
$$
= \lim_{u \to \infty} \mathbf{E}_{\mathfrak{D}}\left[N_u \left(\frac{V_{T_u}}{u}\right)^{-\xi} \,\Big|\, T_u < \tau\right].
$$

Substituting the previous equation into (6.7) yields

$$
\lim_{n \to \infty} \lim_{u \to \infty} \mathbf{E}_{\mathfrak{D}}\left[Z_n^{\xi} \mathcal{Q}_u \mathbf{1}_{\{n \leq T_u \wedge (\tau-1)\}}\right]
$$
$$
= \mathbf{E}_{\xi}\left[Z \mathbf{1}_{\{\tau = \infty\}}\right] \lim_{u \to \infty} \mathbf{E}_{\mathfrak{D}}\left[N_u \left(\frac{V_{T_u}}{u}\right)^{-\xi} \,\Big|\, T_u < \tau\right]. \tag{6.9}
$$

Returning to (6.3) and (6.5), we see that the proof of the proposition will now be complete, provided that we can show that

$$
\lim_{n \to \infty} \lim_{u \to \infty} \mathbf{E}_{\mathfrak{D}}\left[\left(Z_{T_u}^{\xi} \mathbf{1}_{\{T_u < \tau\}} - Z_n^{\xi} \mathbf{1}_{\{n \leq T_u < \tau\}}\right) \mathcal{Q}_u\right] = 0. \tag{6.10}
$$

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But recall once again that $\{\mathcal{Q}_u\}$ is uniformly bounded. Thus, to establish (6.10), it is sufficient to show

$$\lim_{n\to\infty}\lim_{u\to\infty}\mathbf{E}_\xi\left[\left(Z_{T_u}^\xi\mathbf{1}_{\{T_u<\tau\}}-Z_n^\xi\mathbf{1}_{\{n\le T_u<\tau\}}\right)^+\right]=0 \qquad (6.11)$$

and

$$\lim_{n\to\infty}\lim_{u\to\infty}\mathbf{E}_\xi\left[\left(Z_{T_u}^\xi\mathbf{1}_{\{T_u<\tau\}}-Z_n^\xi\mathbf{1}_{\{n\le T_u<\tau\}}\right)^-\right]=0. \qquad (6.12)$$

Note that in these last expectations, we have again replaced $\mathbf{E}_\mathfrak{D}\left[\cdot\right]$ with $\mathbf{E}_\xi\left[\cdot\right]$, since these expectations involve random variables on $\{T_u<\tau\}$, and on that set these expectations are the same.

To establish (6.11), first apply Lemma 5.5 (ii). Namely observe that the integrand on the left-hand side is dominated by $2\big(\bar{Z}^{(p)}\big)^\xi$, which is integrable. Hence, applying the dominated convergence theorem twice, first with respect to the limit in u and then with respect to the limit in n, we obtain

$$\mathbf{E}_\xi\left[\left(\lim_{u\to\infty}Z_{T_u}^\xi\mathbf{1}_{\{T_u<\tau\}}-\lim_{n\to\infty}Z_n^\xi\mathbf{1}_{\{\tau=\infty\}}\right)^+\right]=0, \qquad (6.13)$$

where, in the last equality, we have used that $T_u\uparrow\infty$ as $u\to\infty$. This establishes (6.11). The proof of (6.12) is analogous. $\qquad\square$

Next we identify second term on the right-hand side of (6.1) by relating it to the classical Cramér-Lundberg constant. First recall that τ^* is a typical return time of the random walk $S_n=\sum_{i=1}^n\log A_i$ to the origin, while τ is a typical regeneration for the process $\{V_n\}$.

Lemma 6.1 *Assume* (2.12), *and suppose that* (H_1), (H_2), *and* (H_3) *are satisfied. Then*

$$\frac{1-\mathbf{E}\big[e^{\xi S_{\tau^*}}\big]}{\mathbf{E}\left[\tau^*\right]}\lim_{u\to\infty}\mathbf{E}_\mathfrak{D}\left[N_u\left(\frac{V_{T_u}}{u}\right)^{-\xi}\Big|T_u<\tau\right]=C^*, \qquad (6.14)$$

where C^* *is the Cramér-Lundberg constant defined in* (2.20).

Proof Define
$$W_n=\left(\log A_n+W_{n-1}\right)^+,\ \text{for}\ n=0,1,\dots.$$

Then $\{W_n\}$ is a random walk reflected at the origin, and since $\mathbf{E}\left[\log A\right]<0$, it is well known that this process is a recurrent Markov chain which converges to a random variable W whose distribution is the stationary distribution. Also, by Iglehart (1972), Lemma 1,

$$\lim_{u\to\infty}u^{-\xi}\mathbf{P}\left\{W>\log u\right\}=C^*, \qquad (6.15)$$

where $W:=\lim_{n\to\infty}W_n$.

Set $\tau^*=\inf\left\{n\in\mathbb{Z}_+:W_n=0\right\}$. Since $\{W_n\}$ has an atom at the origin, τ^*+1 is a regeneration time of the Markov chain $\{W_n\}$. Hence by the representation formula in Lemma 2.2,

$$\mathbf{P}\left\{W>\log u\right\}=\frac{\mathbf{E}\big[N_u^*\big]}{\mathbf{E}[\tau^*]}, \qquad (6.16)$$

where $N_u^*:=\sum_{n=1}^{\tau^*}\mathbf{1}_{(\log u,\infty)}(W_n)$.

Let μ_A denote the marginal distribution of $\log A$, and set

$$\mu_{A,\xi}(E)=\int_E e^{\xi x}d\mu_A(x),\quad E\in\mathcal{B}(\mathbb{R}).$$

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Let $T_u^* = \inf\{n \in \mathbb{Z}_+ : W_n > \log u\}$, and define the dual measure

$$\mathfrak{L}(\log A_n) = \begin{cases} \mu_{A,\xi} & \text{for } n = 1, \ldots, T_u^*, \\ \mu_A & \text{for } n > T_u^*. \end{cases} \qquad (6.17)$$

With a slight abuse of notation, let $\mathbf{E}_\mathfrak{D}[\cdot]$ denote expectation with respect to this dual measure. Then by a change of measure, analogous to Lemma 5.3, we obtain

$$\begin{aligned} \mathbf{E}[N_u^*] &= u^{-\xi} \mathbf{E}_\mathfrak{D}\left[N_u^* e^{-\xi(W_{T_u^*} - \log u)}\right] \\ &= u^{-\xi} \mathbf{E}_\mathfrak{D}\left[\mathbf{E}_\mathfrak{D}\left[N_u^* \,\big|\, \mathfrak{F}_{T_u^* \wedge \tau^*}\right] e^{-\xi(W_{T_u^*} - \log u)} \mathbf{1}_{\{T_u^* \le \tau^*\}}\right], \end{aligned} \qquad (6.18)$$

which is equal to

$$u^{-\xi} \mathbf{E}_\mathfrak{D}\left[\mathbf{E}_\mathfrak{D}\left[N_u^* \,\big|\, \mathfrak{F}_{T_u^* \wedge \tau^*}\right] e^{-\xi(W_{T_u^*} - \log u)} \,\Big|\, T_u^* < \tau^* + 1\right] \mathbf{P}_\mathfrak{D}\{T_u^* < \tau^* + 1\}.$$

To complete the proof, notice that the multiplicate random walk process $\{\exp(W_n)\}$ also satisfies the conditions of Theorem 4.2, and the random variable inside the last expectation converges to the same weak limit as the corresponding quantity for the process $\{V_n\}$. [Note that we write "$T_u^* < \tau^*+1$" while we write "$T_u < \tau$," since 0 and τ are regeneration times for the original process, while $\{W_n\}$ regenerates subsequent to returning to the origin, so 1 and τ^*+1 are regeneration times for this process.] Thus by Theorem 4.2, $(J_1(u)/J_2(u)) \to 1$ as $u \to \infty$, where

$$J_1(u) := \mathbf{E}_\mathfrak{D}\left[\mathbf{E}_\mathfrak{D}\left[N_u \,|\, \mathfrak{F}_{T_u}\right] \left(\frac{V_{T_u}}{u}\right)^{-\xi} \,\Big|\, T_u < \tau\right],$$

$$J_2(u) := \mathbf{E}_\mathfrak{D}\left[\mathbf{E}_\mathfrak{D}\left[N_u^* \,|\, \mathfrak{F}_{T_u^* \wedge \tau^*}\right] e^{-\xi(W_{T_u^*} - \log u)} \,\Big|\, T_u^* \le \tau^*\right].$$

Consequently we conclude

$$\begin{aligned} \lim_{u \to \infty} \mathbf{E}_\mathfrak{D}\left[N_u \left(\frac{V_{T_u}}{u}\right)^{-\xi} \,\Big|\, T_u < \tau\right] &= \lim_{u \to \infty} \frac{u^\xi \mathbf{E}[N_u^*]}{\mathbf{P}_\mathfrak{D}\{T_u^* \le \tau^*\}} \\ &= \lim_{u \to \infty} \frac{C^* \mathbf{E}[\tau^*]}{\mathbf{P}_\mathfrak{D}\{T_u^* \le \tau^*\}} \quad \text{by (6.15) and (6.16).} \end{aligned} \qquad (6.19)$$

Finally observe that $\lim_{u \to \infty} \mathbf{P}_\mathfrak{D}\{T_u^* \le \tau^*\} = \mathbf{P}_\mathfrak{D}\{T_u^* = \infty\}$, and by a change of measure argument,

$$\begin{aligned} \mathbf{P}_\mathfrak{D}\{\tau^* = \infty\} &= 1 - \mathbf{P}_\xi\{\tau^* < \infty\} = 1 - \sum_{n=1}^\infty \mathbf{E}_\xi\left[\mathbf{1}_{\{\tau^*=n\}}\right] \\ &= 1 - \sum_{n=1}^\infty \mathbf{E}\left[e^{\xi S_n}\mathbf{1}_{\{\tau^*=n\}}\right] = 1 - \mathbf{E}\left[e^{\xi S_{\tau^*}}\right]. \end{aligned} \qquad (6.20)$$

The required result then follows from (6.19) and (6.20). $\qquad\square$

Proof of Theorem 2.1 By Lemma 2.2, $\mathbf{P}\{V > u\} = \mathbf{E}[N_u]/\mathbf{E}[\tau]$, and hence by Proposition 6.1,

$$\begin{aligned} \lim_{u \to \infty} u^\xi \mathbf{P}\{V > u\} &= \mathbf{E}_\xi\left[Z^\xi \mathbf{1}_{\{\tau=\infty\}}\right] \\ &\qquad \cdot (\mathbf{E}[\tau])^{-1} \lim_{u \to \infty} \mathbf{E}_\mathfrak{D}\left[N_u \left(\frac{V_{T_u}}{u}\right)^{-\xi} \,\Big|\, T_u < \tau\right]. \end{aligned} \qquad (6.21)$$

Since $Z > 0$ w.p. 1 in the ξ-shifted measure, by Lemma 5.5 (i), we may identify the random variable Z in the previous display as $Z = Z^{(p)} - Z^{(c)}$ a.s., by Lemma 5.4. Then by Lemma 6.1, the last limit on the right-hand side of (6.21) may be identified as

$$\frac{C^* \mathbf{E}\left[\tau^*\right]}{1 - \mathbf{E}\left[e^{\xi S_{\tau^*}}\right]} = \frac{1}{\xi \lambda'(\xi)} \qquad \text{by } (2.20). \tag{6.22}$$

Thus we have shown that (2.22) holds. Also, the positivity of the constant C is obtained from Lemma 5.5 (i), which yields $Z > 0$ w.p. 1, while the finiteness of this constant is obtained from Lemma 5.5 (ii), which yields $Z\mathbf{1}_{\{\tau=\infty\}} \le \bar{Z}^{(p)}$, where $\mathbf{E}_\xi\left[(\bar{Z}^{(p)})^\xi\right] < \infty$.

To complete the proof, it suffices to show that

$$\mathbf{E}_\xi\left[\left(Z_n^+\right)^\xi \mathbf{1}_{\{\tau>n\}}\right] - \mathbf{E}_\xi\left[\left(Z_{n-1}^+\right)^\xi \mathbf{1}_{\{\tau>n-1\}}\right] \le Ke^{-\epsilon n}, \quad n \in \mathbb{Z}_+, \tag{6.23}$$

for finite positive constants K and γ. Now the left-hand side of (6.23) is equal to

$$\mathbf{E}_\xi\left[\left((Z_n^+)^\xi - (Z_{n-1}^+)^\xi\right)\mathbf{1}_{\{\tau>n\}}\right] - \mathbf{E}_\xi\left[(Z_{n-1}^+)^\xi \mathbf{1}_{\{\tau=n\}}\right]. \tag{6.24}$$

For the second term, observe that if regeneration occurs at time n, then (according to the split-chain described in Nummelin (1984), Section 4.4, or Remark 2.1 above), we must have $V_{n-1} \in \mathcal{C} = [-M, M]$. Moreover, by Lemma 5.1, $\{V_n\}$ is geometrically ergodic, and thus $\mathbf{P}\{\tau > n\} \le Je^{-\gamma n}$ for certain positive constants J and γ. Hence, since $Z_n = V_n/(A_1 \cdots A_n)$, it follows by Lemma 5.3 that

$$\mathbf{E}_\xi\left[(Z_{n-1}^+)^\xi \mathbf{1}_{\{\tau=n\}}\right] = \mathbf{E}\left[(V_{n-1}^+)^\xi \mathbf{1}_{\{\tau=n\}}\right] \le M\mathbf{P}\{\tau=n\} \le MJe^{-\gamma(n-1)}. \tag{6.25}$$

Next consider the first term in (6.24). It follows immediately by definition in (2.12) that $-|B_n| \le V_n - A_n V_{n-1} \le A_n |D_n| + |B_n|$. Hence, similar to (5.33),

$$\left|Z_n^+ - Z_{n-1}^+\right| \mathbf{1}_{\{\tau>n\}} \le \frac{\tilde{B}_n}{A_1 \cdots A_n} \mathbf{1}_{\{\tau>n\}}, \quad \text{where} \quad \tilde{B}_n = A_n |D_n| + |B_n|. \tag{6.26}$$

If $\xi \ge 1$, then from the previous equation and an application of Minkowskii's inequality, we obtain

$$\mathbf{E}_\xi\left[(Z_n^+)^\xi \mathbf{1}_{\{\tau>n\}}\right] \le \mathbf{E}_\xi\left[(Z_{n-1}^+)^\xi \mathbf{1}_{\{\tau>n\}}\right]\left(1 + \frac{\mathbf{E}_\xi\left[\tilde{B}_n^\xi \mathbf{1}_{\{\tau>n\}}/(A_1 \cdots A_n)^\xi\right]}{\mathbf{E}_\xi\left[(Z_{n-1}^+)^\xi \mathbf{1}_{\{\tau>n\}}\right]}\right)^\xi. \tag{6.27}$$

Moreover by Lemma 5.3,

$$\mathbf{E}_\xi\left[\frac{\tilde{B}_n^\xi}{(A_1 \cdots A_n)^\xi}\mathbf{1}_{\{\tau>n\}}\right] = \mathbf{E}\left[\tilde{B}_n^\xi \mathbf{1}_{\{\tau>n\}}\right] \le Ke^{-\gamma(n-1)}, \tag{6.28}$$

where the last step was obtained by the independence of $\tilde{B}_n$ from $\{\tau>n-1\}$ and by the geometric ergodicity of $\{V_n\}$. Substituting (6.28) into (6.27) yields, after a Taylor expansion,

$$\mathbf{E}_\xi\left[(Z_n^+)^\xi \mathbf{1}_{\{\tau>n\}}\right] \le \mathbf{E}_\xi\left[(Z_{n-1}^+)^\xi \mathbf{1}_{\{\tau>n\}}\right] + o(e^{-\epsilon n}), \tag{6.29}$$

where $\epsilon = \gamma/\xi$. Repeating this argument (but using the lower bound rather than the upper bound for the difference $(Z_n^+ - Z_{n-1}^+)$ in (6.26)) yields the same bound, but with Z_n^+ and Z_{n-1}^+ interchanged in (6.29). Using this observation together with (6.29) and (6.25), we conclude that (6.23) holds for $\xi \ge 1$. On the other hand, if $\xi < 1$, then we may use the deterministic inequality $|x+y|^\xi \le |x|^\xi + |y|^\xi$ in place of Minkowskii's inequality to obtain the same result, obtaining in this case that $\epsilon = \gamma$. The proof of the theorem is now complete. $\qquad \square$

Remark 6.2 The proof of Theorem 2.1 could be slightly modified to obtain other representations for the constant C. For example, applying Theorem 2.2 of Collamore et al. (2011) (and its proof), we obtain that

$$\mathbf{P}\{V > u\} = \pi(\mathcal{C})\mathbf{E}[N_u], \tag{6.30}$$

where $\pi_{\mathcal{C}}(E) := \pi(E)/\pi(\mathcal{C})$ for any Borel subset $E \subset \mathcal{C}$ (and π denotes the stationary measure of $\{V_n\}$), and where N_u denotes the number of exceedances above level u over a cycles eminating from and returning to $\mathcal{C} = [-M, M]$, where $M \geq 0$ and $\varphi(\mathcal{C}) > 0$. (In other words, the regeneration time in our original definition of N_u needs to be replaced with the first return time to $\mathcal{C}$.) Using (6.30) in place of (2.9), we obtain essentially the same representation formula for C, *viz.* (2.21), except that the initial measure of V_0 is taken to be $\pi_{\mathcal{C}}$ (rather than the measure ν appearing in the minorization $(\mathcal{M})$), and the stopping time τ is interpreted as the *first* return time to $\mathcal{C}$. In this way, we arrive at a canonical representation for the constant C, which does not depend on the minorization $(\mathcal{M})$.

Now in the proof of Theorem 2.1, the remainder term satisfies $\mathfrak{R}_n := C - C_n = o(e^{-\epsilon n})$, where $\epsilon = \gamma/\xi$ if $\xi \geq 1$ and $\epsilon = \gamma$ if $\xi < 1$. Here, γ is obtained from the geometric ergodicity of $\{V_n\}$; in particular, $\mathbf{P}\{\tau > n\} \leq Ke^{-\gamma n}$ for some positive constant K. But if (6.30) is used in place of (2.9), then γ derives from the *return times* to $\mathcal{C}$ rather than the regeneration times, and these return times can be quantified more easily. In particular, the drift condition $(\mathcal{D})$ derived in Lemma 5.1 (i) yields that

$$\mathbf{E}[h(V_n)\mathbf{1}_{\{V_0 \notin C\}}] \leq \rho h(V_{n-1}),$$

where $h(v) = |v|^\alpha + 1$ for some $\alpha \leq 1$ chosen such that $\lambda(\alpha) \in (0, 1)$. Hence by an inductive argument,

$$\mathbf{E}\left[h(V_n)\prod_{i=1}^{n-1}\mathbf{1}_{\{V_i \notin C\}}\right] \leq \rho^n h(V_1)\mathbf{1}_{\{V_1 \notin C\}}. \tag{6.31}$$

Since the function $h(v)$ is increasing in $|v|$, it follows that $\mathbf{E}[h(V_\tau)] \leq h(V_1)$ for any $V_1 \notin C$. Consequently, (6.31) implies that $\mathbf{P}\{\tau = n\} \leq \rho^n$.

Remark 6.3 When the process $\{V_n\}$ is nonnegative, an alternative respresentation for the constant C may be obtained by noting that, using the stationarity of $\{V_n\}$,

$$\mathbf{P}\{V > u\} = \lim_{n \to \infty}\frac{1}{n}\mathbf{E}\left[\mathfrak{N}_u^{(n)}\right], \quad \text{where} \quad \mathfrak{N}_u^{(n)} := \sum_{i=1}^{n}\mathbf{1}_{(u,\infty)}(V_i). \tag{6.32}$$

Next, let $\kappa^{(n)}$ denote the first regeneration time *after* time n. Then the proof of Theorem 2.1 can be repeated with $\kappa^{(n)}$ in place τ. The only significant change occurs in proof of Proposition 6.1, which plays a critical role in the proof of Theorem 2.1. Specifically, in place of (6.5) in that proof, we obtain

$$n^{-1}\mathbf{E}_{\mathfrak{D}}\left[\left(Z_{T_u}^\xi \mathbf{1}_{\{T_u < \kappa^{(n)}\}}\right)\mathcal{Q}_u\right] = n^{-1}\mathbf{E}_{\mathfrak{D}}\left[Z_n^\xi \mathcal{Q}_u\mathbf{1}_{\{T_u < \kappa^{(n)}\}}\right]$$
$$+ n^{-1}\mathbf{E}_{\mathfrak{D}}\left[\left(Z_{T_u}^\xi - Z_n^\xi\right)\mathcal{Q}_u\mathbf{1}_{\{T_u < \kappa^{(n)}\}}\right], \tag{6.33}$$

where $\mathcal{Q}_u$ is defined as in (6.4). As in the proof of Proposition 6.1, we let $u \to \infty$ and then $n \to \infty$.

To handle the first term on the right-hand side of (6.33), first note by a slight modification of the results in Section 4 that, as $u \to \infty$,

$$\mathbf{E}_{\mathfrak{D}}[\mathcal{Q}_u] \to \mathbf{E}_{\mathfrak{D}}[\mathcal{Q}], \quad \mathbf{E}_{\mathfrak{D}}\left[\mathcal{Q}_u\mathbf{1}_{\{T_u \geq \kappa^{(n)}\}}\big|\mathfrak{F}_{\kappa^{(n)}}\right]\mathbf{E}_{\mathfrak{D}}[\mathcal{Q}]\mathbf{1}_{\{T_u \geq \kappa^{(n)}\}}.$$

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Hence, by mimicking the proof of Proposition 6.1, we obtain

$$\lim_{u \to \infty} \mathbf{E}_{\mathfrak{D}} \left[Z_n^{\xi} \mathcal{Q}_u \mathbf{1}_{\{T_u < \kappa^{(n)}\}} \right] = \mathbf{E}_{\xi} \left[Z_n^{\xi} \mathbf{1}_{\{\kappa^{(n)} = \infty\}} \right] \mathbf{E}_{\mathfrak{D}} \left[\mathcal{Q} \right]. \tag{6.34}$$

Next, consider a modification of the ξ-shifted measure, where the first n innovations are taken in the *original* measure, but all the remaining innovations are taken according to the ξ-shifted measure. We let $\mathbf{E}_{\xi}^{(n)}[\cdot]$ denote expectation with respect to this modified ξ-shifted measure. Since $Z_n := V_n/(A_1 \cdots A_n)$, we then obtain by the change of measure formula in Lemma 5.3 that, for any positive constant $\gamma < 1/\xi$,

$$n^{-1}\mathbf{E}_{\xi} \left[Z_n^{\xi} \mathbf{1}_{\{\kappa^{(n)}=\infty\}} \right] = n^{-1}\mathbf{E}_{\xi}^{(n)} \left[V_n^{\xi} \mathbf{1}_{\{V_n < n^{\gamma}\}} \mathbf{1}_{\{\kappa^{(n)}=\infty\}} \right]$$
$$+ n^{-1}\mathbf{E}_{\xi}^{(n)} \left[V_n^{\xi} \mathbf{1}_{\{V_n \geq n^{\gamma}\}} \mathbf{1}_{\{\kappa^{(n)}=\infty\}} \right]. \tag{6.35}$$

Recall that for regeneration to occur, the process must first return to the set $\mathcal{C} = [-M, M]$ (as described in Remark 2.1). Hence, since $V_n \geq A_n V_{n-1} - |B_n|$ for all n, it follows by an application of the proof of Lemma 5.2 that

$$\sup_{s \geq n^{\gamma}} \mathbf{P}_{\xi}^{(n)} \left\{ \kappa^{(n)} = \infty \,\big|\, V_n = s \right\} \nearrow 1 \quad \text{as} \quad n \to \infty.$$

Since the first term on the right-hand side of (6.35) is zero, we conclude that

$$\lim_{n \to \infty} \frac{1}{n} \mathbf{E}_{\xi} \left[Z_n^{\xi} \mathbf{1}_{\{\kappa^{(n)}=\infty\}} \right] = \lim_{n \to \infty} \frac{1}{n} \mathbf{E}_{\xi} \left[Z_n^{\xi} \right]. \tag{6.36}$$

Next consider the second term on the right-hand side of (6.33). First take that limit as $u \to \infty$. By Lemma 5.5 (ii), we have $\left| Z_{T_u} \mathbf{1}_{\{T_u < \kappa^{(n)}\}} \right| \leq \mathcal{Z}^{(p)}$, where $\mathcal{Z}^{(p)}$ is defined the same way as $\bar{Z}^{(p)}$, but with τ replaced with $\kappa^{(n)}$. Then by repeating the argument in (5.29), we obtain that $\mathbf{E} \left[(\mathcal{Z}^{(p)})^{\xi} \right] \leq L(n) < \infty$. (The finiteness of $L(n)$ follows from the geometric ergodicity of $\{V_n\}$ and (6.43) below.) Hence, since $\{Q_u\}$ is uniformly bounded (as noted in the proof of Proposition 6.1), it follows by the dominated convergence theorem that the second term in (6.33) tends to

$$\frac{1}{n} \mathbf{E}_{\xi} \left[\left(Z^{\xi} - Z_n^{\xi} \right) \mathcal{Q} \mathbf{1}_{\{\kappa^{(n)}=\infty\}} \right] \quad \text{as} \quad u \to \infty. \tag{6.37}$$

We claim that as $n \to \infty$, this quantity tends to zero. Since $\mathcal{Q}$ is bounded from above, it suffices to show that

$$\limsup_{n \to \infty} \frac{1}{n} \mathbf{E}_{\xi} \left[|Z^{\xi} - Z_n^{\xi}| \mathbf{1}_{\{\kappa^{(n)}=\infty\}} \right] = 0. \tag{6.38}$$

To this end, we employ the elementary inequalities

$$\left| |x+y|^{\xi} - |x|^{\xi} \right| \leq \begin{cases} |y|^{\xi} & , \quad \xi \in (0, 1], \\ \xi |y| \left(|x| + |y| \right)^{\xi-1} & , \quad \xi > 1. \end{cases} \tag{6.39}$$

Applying (6.39) with $x = Z_n$, $y = Z - Z_n$, we see that (6.38) will hold for $\xi \in (0, 1]$ provided that

$$\limsup_{n \to \infty} \frac{1}{n} \mathbf{E}_{\xi} \left[|Z - Z_n|^{\xi} \mathbf{1}_{\{\kappa^{(n)}=\infty\}} \right] = 0. \tag{6.40}$$

When $\xi > 1$, we require that (6.40) holds and, in addition,

$$\limsup_{n \to \infty} \frac{1}{n} \mathbf{E}_{\xi} \left[|Z - Z_n| Z_n^{\xi-1} \mathbf{1}_{\{\kappa^{(n)}=\infty\}} \right] = 0. \tag{6.41}$$

Note by (5.33) that

$$|Z - Z_n| \leq \sum_{i=n+1}^{\infty} \frac{\tilde{B}_i}{A_1 \cdots A_n}, \quad \tilde{B}_i := A_i |D_i| + |B_i|.$$

Then, by repeating the argument in (3.12) or (5.29), we obtain

$$\mathbf{E}_\xi \left[|Z - Z_n|^\xi \mathbf{1}_{\{\kappa^{(n)} = \infty\}} \right]^{1/\xi} \leq \mathbf{E}\big[\tilde{B}^\xi\big]^{1/\xi} \sum_{i=n+1}^{\infty} \mathbf{P}\big\{\kappa^{(n)} > i - 1\big\}^{1/\xi} < \infty. \tag{6.42}$$

To see that the quantity on the right-hand side is finite, apply a renewal argument along the lines of Iscoe et al. (1985), Lemma 6.2. Recall that $\{K_i\}$ denotes the regeneration times of $\{V_n\}$ and let $\mathcal{N}_n = \sup\{i : K_i \leq n\}$. Then set $\gamma_n = K_{\mathcal{N}_{n+1}-1} - K_{\mathcal{N}_n}$ and notice that $\gamma_n \leq \kappa^{(n)} - n$. Now assume that regeneration occurs at time zero and let τ denote the subsequent regeneration time. Then

$$u_n := \mathbf{E}\left[e^{\epsilon \gamma_n} \right] = \mathbf{E}\left[e^{\epsilon \tau}; \tau > n \right] + \sum_{k=0}^{n} \mathbf{P}\left\{ \tau = k \right\} \mathbf{E}\left[e^{\epsilon \gamma_{n-k}} \right] := a_n + \sum_{k=0}^{n} p_k u_{n-k}.$$

Then the argument on p. 396 of Iscoe et al. (1985) can be applied to obtain that

$$\lim_{n \to \infty} \mathbf{E}\left[e^{\epsilon \gamma_n} \right] = \frac{1}{\mathbf{E}[\tau]} \mathbf{E}\big[\tau e^{\epsilon \tau} \big]. \tag{6.43}$$

Hence $\kappa^{(n)} - n$ has exponential moments, as can now be seen from the geometric ergodicity of $\{V_n\}$ established in Lemma 5.1 (iv). Consequently, the right-hand side of (6.42) must be finite. Thus we conclude that (6.40) holds.

Next, to establish (6.41) when $\xi > 1$, first apply Hölder's inequality to obtain that

$$\frac{1}{n} \mathbf{E}_\xi \left[|Z - Z_n| Z_n^{\xi-1} \mathbf{1}_{\{\kappa^{(n)} = \infty\}} \right] \leq \left(\frac{1}{n} \mathbf{E}_\xi \left[|Z - Z_n|^\xi \mathbf{1}_{\{\kappa^{(n)} = \infty\}} \right] \right)^{1/\xi} \left(\frac{1}{n} \mathbf{E}_\xi \big[Z_n^\xi \big] \right)^{\xi-1/\xi}.$$

As we have just shown, the first term on the right-hand side tends to zero. Moreover, by a repetition of (5.29), the second term is finite, since $\mathbf{E}_\xi \big[Z_n^\xi \big] \leq \mathbf{E}\left[V_0 \right] + n \mathbf{E}\big[\tilde{B} \big]$. This establishes (6.41).

In this way, we have obtained an alternative expression for the constant C in Theorem 2.1 for nonnegative processes $\{V_n\}$ governed by Letac's Model E, namely,

$$C = \frac{1}{\xi \lambda'(\xi)} \lim_{n \to \infty} \frac{1}{n} \mathbf{E}_\xi \big[Z_n^\xi \big]. \tag{6.44}$$

Moreover,

$$\mathbf{E}_\xi \big[Z_n^\xi \big] := \mathbf{E}_\xi \big[V_n^\xi / (A_1 \cdots A_n)^\xi \big] = \mathbf{E}\big[V_n^\xi \big],$$

and—as long as the initial state is the same—the backward and forward sequences are equal in distribution for every n; that is, $V_0 = \tilde{Z}_0 \implies V_n = \tilde{Z}_n$ for all n, where $\{\tilde{Z}_n\}$ denotes the corresponding backward sequence. Hence

$$C = \frac{1}{\xi \lambda'(\xi)} \lim_{n \to \infty} \frac{1}{n} \mathbf{E}\big[\tilde{Z}_n^\xi \big]. \tag{6.45}$$

Thus, we have arrived at an alternative expression for the constant C, which was established for the special case $f(v) = Av + B$ and $B \equiv 1$ in Bartkiewicz et al. (2011). However, we note that our original expression in Theorem 2.1 is more useful in our applications, since it is *exact* and does not involve limiting moments of $\{\tilde{Z}_n\}$. A further, very important, advantage of our original representation is that in applications such as Example 3.2, we are thereby able to eliminate the complicated conjugate term, which often turns out to be *zero* when taken over paths which fail to regenerate in the ξ-shifted measure. Nonetheless, for general Lipschitz maps, the formula we have just obtained appears to provide an appealing alternative.[*]

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[*]We thank Prof. Ewa Damek for pointing out the paper of Bartkiewicz et al. (2011).

7 Proofs of Propositions 2.1 and 2.2

Proof of Proposition 2.1 A repetition of (6.3) and (6.4) yields

$$u^\xi \mathbf{E}\left[N_u\right] = \mathbf{E}\left[\left(Z^\xi_{T_u}\mathbf{1}_{\{T_u<\tau\}}\right)\mathbf{E}\left[N_u\,|\mathfrak{F}_{T_u\wedge\tau-1}\right]\left(\frac{V_{T_u}}{u}\right)^{-\xi}\right]$$

$$\leq \mathbf{E}\left[(\bar{Z}^{(p)})^\xi\left(C_1(u)\log\left(\frac{V_{T_u}}{u}\right)+C_2(u)\right)\left(\frac{V_{T_u}}{u}\right)^{-\xi}\right], \tag{7.1}$$

where the last step follows from Proposition 4.1 and Lemma 5.5 (ii). Moreover, since $V_{T_u}\geq u$, we obviously have

$$\left(C_1(u)\log\left(\frac{V_{T_u}}{u}\right)+C_2(u)\right)\left(\frac{V_{T_u}}{u}\right)^{-\xi}\leq \sup_{z\geq 0}\left\{e^{-\xi z}\left(zC_1(u)+C_2(u)\right)\right\}, \tag{7.2}$$

which is bounded. Substituting this deterministic bound into the right-hand side of (7.1) establishes the theorem. The limiting values of $C_1(u)$ and $C_2(u)$ are obtained by a further application of Proposition 4.1. $\qquad\square$

Proof of Proposition 2.2 By (2.34),

$$\Theta = \lim_{u\to\infty}\frac{\mathbf{P}\{V_n>u,\ \text{for some }n<\tau\}}{\mathbf{E}\left[N_u\right]} = \lim_{u\to\infty}\frac{\mathbf{E}\left[\mathbf{1}_{\{T_u<\tau\}}\right]}{\mathbf{E}\left[N_u\right]}. \tag{7.3}$$

Moreover, by a repetition of the proof of Proposition 6.1,

$$\lim_{u\to\infty}u^\xi\mathbf{E}\left[\mathbf{1}_{\{T_u<\tau\}}\right] = \mathbf{E}_\xi\left[Z^\xi\mathbf{1}_{\{\tau=\infty\}}\right]\lim_{u\to\infty}\mathbf{E}_\xi\left[\mathbf{1}_{\{T_u<\tau\}}\left(\frac{V_{T_u}}{u}\right)^{-\xi}\Big|T_u<\tau\right]. \tag{7.4}$$

(In the second term on the right-hand side, we have used the fact that the dual measure and the ξ-shifted measures are the same prior to time T_u.)

To identify the last term on the right-hand side, let $S_n=\sum_{i=1}^n\log A_i$, let $T_u^*=\inf\{n:S_n>u\}$, and let $\tau^*=\inf\{n:S_n\leq 0\}$. Then by applying Lemma 4.1 to the process $\{V_n\}$ and to the multiplicative random walk $\{\exp(S_n)\}$, we see that V_{T_u}/u and $\exp S_{T_u^*}/u$ converge to the same weak limit. Hence

$$\lim_{u\to\infty}\mathbf{E}_\xi\left[\left(\frac{V_{T_u}}{u}\right)^{-\xi}\Big|T_u<\tau\right] = \lim_{u\to\infty}\mathbf{E}_{\mathfrak{D}}\left[e^{-\xi(S_{T_u^*}-\log u)}\Big|T_u^*<\tau^*+1\right] \tag{7.5}$$

(where by definition, we recall that $\{V_n\}$ regenerates at times 0 and τ, while the process $\{S_n\}$, but reflected at the origin, regenerates subsequent to its returns to the origin, *i.e.,* at times 1 and τ^*+1).

Next, we observe that the quantity on the right-hand side is just the Cramér-Lundberg constant C^*. To see that this is the case, first note that if $T_u^*\not\leq\tau^*$, then the process $\{S_n\}$ returns to $(-\infty,0]$ and, starting from this new level $x\in(-\infty,0]$ until its return to $(-\infty,x]$, the overjump distribution will have the same limiting distribution (as $u\to\infty$) as it had when starting its first cycle from the origin. Moreover, since we are in the ξ-shifted measure, the number of such returns will be finite, with each return occuring with a probability bounded above by some $p\in(0,1)$. From these considerations, we conclude that on the right-hand side of (7.5), the conditioning on the event $\{T_u^*<\tau^*+1\}$ may be dropped. Consequently, the right-hand side of this equation may be identified as

$$\lim_{u\to\infty}\mathbf{E}_\xi\left[e^{-\xi(S_{T_u^*}-\log u)}\right] = \lim_{u\to\infty}u^\xi\mathbf{P}\{T_u^*<\infty\} = C^*,$$

where the second equality follows from a change of measure argument. Thus by (7.4),

$$\lim_{u \to \infty} u^\xi \mathbf{E}\left[\mathbf{1}_{\{T_u < \tau\}}\right] = C^* \mathbf{E}_\xi\left[Z^\xi \mathbf{1}_{\{\tau = \infty\}}\right]. \tag{7.6}$$

But by Proposition 6.1 and Lemma 6.1, we also have that

$$\lim_{u \to \infty} u^\xi \mathbf{E}\left[N_u\right] = \mathbf{E}_\xi\left[Z^\xi \mathbf{1}_{\{\tau = \infty\}}\right] \frac{C^* \mathbf{E}\left[\tau^*\right]}{1 - \mathbf{E}\left[e^{\xi S_{\tau^*}}\right]}. \tag{7.7}$$

Substituting (7.6) and (7.7) into (7.3), we conclude (2.35). □

8 Proofs of the results from nonlinear renewal theory

First we turn to the proof of Lemma 4.1, which extends a classical result from nonlinear renewal theory (given in Woodroofe (1982), Theorem 4.2) to the setting of our problem.

Proof of Lemma 4.1 By definition of Z_n in (5.16),

$$V_n = (A_1 \cdots A_n)\left(Z_n \mathbf{1}_{\{Z_n > 0\}} + Z_n \mathbf{1}_{\{Z_n \le 0\}}\right). \tag{8.1}$$

Now by Lemma 5.2, $V_n \uparrow \infty$ w.p.1 under the measure μ_ξ. Thus $T_u < \infty$ a.s., and at this exceedance time we obviously have $\mathbf{1}_{\{Z_n \le 0\}} = 0$. Thus, letting $Y_n = Z_n$ on $\{Z_n > 0\}$ and $Y_n = 1$ otherwise, we obtain that on $\{Z_n > 0\}$,

$$V_{T_u} = V'_{T_u} \quad \text{for} \quad V'_n := (A_1 \cdots A_n) Y_n, \quad n = 1, 2, \ldots, \tag{8.2}$$

Note that Y_n is everywhere *positive*.

We begin by showing that

$$\frac{V'_{T_u}}{u} \Rightarrow \hat{V} \quad \text{as} \quad u \to \infty, \tag{8.3}$$

where $\hat{V}$ has the distribution described in (4.3). To this end, note by (8.2) that

$$\log V'_n = S_n + \delta_n, \tag{8.4}$$

where $S_n := \sum_{i=1}^n \log A_i$ and $\delta_n := \log Y_n$. Hence $\{\log V'_n\}$ may be viewed as a perturbed random walk where $\{S_n\}$ has a positive drift under μ_ξ-measure, and the sequence $\{\delta_n\}$ has the property that $\{(\log A_i, \delta_i) : i = 1, \ldots, n\}$ is independent of $\log A_j$ for all $j > n$. This puts us in the setting of classical nonlinear renewal theory.

Next observe by Lemma 5.5 and the fact that $\mathbf{1}_{\{Z_n \le 0\}} \to 0$ a.s. that $\delta_n := \log Y_n$ converges a.s. to a proper random variable, and hence $\delta_n/n \to 0$ a.s. as $n \to \infty$. Thus $\{\delta_n\}$ is slowly changing. By Woodroofe (1982), Theorem 4.2, it follows that (8.3) holds and hence, by (8.2), this equation also holds with V_{T_u} in place of V'_{T_u}.

To show that the result holds conditional on $\{T_u < \tau\}$, let $V_0 \sim \nu$, where ν is given in Lemma 2.2 (iv); let $K_0 = 0$; and let $K_1, K_2, \ldots$ denote the successive regeneration times. For each i, let R_i denote the distribution of (V_{T_u}/u) conditional on the event that $T_u \in [K_i, K_{i+1})$. By independence of the regeneration cycles, $\{R_i\}$ is an i.i.d. sequence of random variables. Consequently, the conditional distribution of V_{T_u} given that $T_u \in [K_i, K_{i+1})$ is the same for all i. Thus, the conditional distribution of V_{T_u}/u given $\{T_u < \tau\}$ must agree with the unconditional distribution of V_{T_u}/u, completing the proof. □

Next we turn to the proofs of Proposition 4.1 and Theorems 4.1 and 4.2. An important part of the proofs will be the study the process $\{V_n\}$ as it returns from the set (u, ∞). To do so, we

will introduce a new barrier at a lower level u^t, where $t \in (0,1)$, and divide the trajectory into two parts, namely that occuring prior to the event $V_n \in [-u^t, u^t]$, and that occuring after this time. Intuitively, the process $\{V_n\}$ will closely resemble a multiplicative random walk away from the set $[-u^t, u^t]$, and so a critical aspect of the proofs will be to characterize the behavior of $\{V_n\}$ after it returns to $[-u^t, u^t]$, but prior to regeneration. Indeed, it is precisely in this region that $\{V_n\}$ fails to resemble the random walk process. In the next proposition, we show that the behavior of the process in this critical region may be neglected in an appropriate sense, and we also provide a reasonably sharp estimate for the number of visits above the level u which arise after returning to $[-u^t, u^t]$.

For any $v \geq 0$, first define
$$K(v) = \inf\{n : |V_n| \leq v\}.$$

Proposition 8.1 *Assume Letac's Model E, suppose that $\{V_n\}$ is nondegenerate and (H_1) and (H_2) are satisfied, and assume that $V_0 > u$. Then for any $t \in (0,1)$, there exist finite positive constants α, M, and $\rho \in (0,1)$ such that*

$$\mathbf{E}\left[\sum_{n=K(u^t)}^{\tau-1} \mathbf{1}_{\{V_n > u\}}\right] \leq \Delta(u). \tag{8.5}$$

The constant $\Delta(u)$ is characterized as follows. Set $\bar{V}_1 := A_1 \max\{D_1, V_0\} + |B_1|$. Then

$$\Delta(u) := \frac{u^{-\alpha(1-t)}}{1-\rho}\left\{1 + u^{-\alpha t} \sup_{w \in [-M,M]} \mathbf{E}_w[\tau]\, \mathbf{E}_M\left[\bar{V}_1^\alpha\right]\right\} < \infty \tag{8.6}$$

(where $\mathbf{E}_w[\cdot]$ denotes expectation conditional on $V_0 = v$).

Note that we have dropped the dependence on the dual measure in this proposition, since we assume that $V_0 > u$, and hence the entire trajectory will take place in the original measure.

Proof of Proposition 8.1 Recall by Lemma 5.1 (ii) that $\{V_n\}$ satisfies a drift condition; namely, for some $\alpha > 0$,
$$\mathbf{E}\left[|V_n|^\alpha\, \big|\, V_{n-1} = v\right] \leq \rho|v|^\alpha + \beta \mathbf{1}_{\mathcal{C}_M}(v), \tag{8.7}$$

where $\rho \in (0,1)$, $\mathcal{C}_M := [-M, M]$, and M and β are constants.

We will divide the proof into three steps. In the first step, we study the number of exceedances above level u which occur in an excursion beginning at time $K(u^t)$ (i.e., when the process $\{|V_n|\}$ first falls below the level u^t for $t < 1$) and ending at time $K(M)$ (i.e., when this process first enters the set $\mathcal{C}_M := [-M, M]$). In the next step, we consider the number of exceedances above level u which occur between time $K(M)$ and the actual regeneration time. Combining these two estimates in Step 3, we obtain the desired upper bound.

Step 1: For any $t \in (0,1)$,

$$\mathbf{E}\left[\sum_{n=K(u^t)}^{K(M)} \mathbf{1}_{\{V_n > u\}}\right] \leq \frac{u^{-\alpha(1-t)}}{1-\rho}. \tag{8.8}$$

Proof: If $V_0 = v$, where $|v| > M$, then by iterating (8.7) we obtain
$$\mathbf{E}\left[|V_n|^\alpha \mathbf{1}_{\{K(M)>n\}}\big|V_0 = v\right] \leq \rho^n |v|^\alpha, \quad n = 0, 1, \ldots, \tag{8.9}$$

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and hence

$$\mathbf{E}\left[\mathbf{1}_{\{V_n>u\}}\mathbf{1}_{\{K(M)>n\}}\Big|V_0=v\right]\leq u^{-\alpha}\rho^n|v|^\alpha. \tag{8.10}$$

Consequently,

$$\mathbf{E}\left[\sum_{n=0}^{K(M)-1}\mathbf{1}_{\{V_n>u\}}\Big|V_0=v\right]\leq u^{-\alpha}|v|^\alpha\left(1+\rho+\rho^2+\cdots\right). \tag{8.11}$$

Then by the previous equation and the strong Markov property,

$$\mathbf{E}\left[\sum_{n=K(u^t)}^{K(M)-1}\mathbf{1}_{\{V_n>u\}}\Big|V_{K(u^t)}\right]\leq u^{-\alpha}|V_{K(u^t)}|^\alpha\left(1+\rho+\rho^2+\cdots\right). \tag{8.12}$$

Now by definition, we have $|V_{K(u^t)}|\leq u^t$. Thus we have established (8.8).

Step 2: We have

$$\mathbf{E}\left[\sum_{n=K(M)}^{\tau-1}\mathbf{1}_{\{V_n>u\}}\right]\leq\frac{u^{-\alpha}}{1-\rho}\left\{\sup_{w\in[-M,M]}\mathbf{E}_w\left[\tau\right]\mathbf{E}_M\left[\bar{V}_1^\alpha\right]\right\}. \tag{8.13}$$

Proof: It suffices to show that for any $v\in[-M,M]$,

$$\mathbf{E}\left[N_u|V_0=v\right]\leq\frac{u^{-\alpha}}{1-\rho}\left\{\sup_{w\in[-M,M]}\mathbf{E}_w\left[\tau\right]\mathbf{E}_M\left[\bar{V}_1^\alpha\right]\right\}. \tag{8.14}$$

To this end, introduce the augmented chain (as described in Remark 2.1); namely consider the process $\{(V_n,\eta_n)\}$, where $\{\eta_n\}$ is an i.i.d. sequence of Bernoulli random variables with $\mathbf{P}\{\eta_n=1\}=\delta$ and $\mathbf{P}\{\eta_n=0\}=1-\delta$. Then we may identify τ as the return time of this augmented chain to the set $\mathbb{C}:=\mathcal{C}\times\{1\}$, where $\mathcal{C}$ and δ are obtained from the minorization condition $(\mathcal{M})$ (which holds by Lemma 5.1 (iii)). We assume without loss of generality that $\mathcal{C}\subset\mathcal{C}_M$, and define $\mathbb{M}=\mathcal{C}_M\times\{0,1\}$. (We may possibly have that $\mathcal{C}$ is strictly contained in $\mathcal{C}_M$.)

With a slight abuse of notation, let P denote the transition kernel of the augmented chain $\{(V_n,\eta_n)\}$. Introduce the taboo transition kernel

$$_GP(x,E):=\int_E\mathbf{1}_{G^c}(y)P(x,dy),$$

where G and E are Borel subsets of $\mathbb{R}\times\{0,1\}$. Then by the last-exit decomposition (*cf.* Meyn and Tweedie (1993), Section 8.2),

$$_\mathbb{C}P^n(\mathbf{v},E)=_\mathbb{M}P^n(\mathbf{v},E)+\sum_{k=1}^{n-1}\int_{\mathbb{M}-\mathbb{C}}{}_\mathbb{C}P^k(\mathbf{v},d\mathbf{w})_\mathbb{M}P^{n-k}(\mathbf{w},E). \tag{8.15}$$

Now set $E=(u,\infty)\times\{0,1\}$ and fix $\mathbf{v}=(v,q)$, where $v<u$ and $q\in\{0,1\}$. Then sum (8.15) over all $n\in\mathbb{Z}_+$. On the left-hand side of (8.15), the term $_\mathbb{C}P^n\big(\mathbf{v},(u,\infty)\times\{0,1\}\big)$ describes the probability that regeneration does *not* occur during the first n time increments and that $V_n\in(u,\infty)$. Thus

$$\sum_{n=1}^\infty{}_\mathbb{C}P^n\big(\mathbf{v},(u,\infty)\times\{0,1\}\big)$$

$$=\sum_{n=1}^\infty\mathbf{E}\left[\mathbf{1}_{\{V_n>u\}}\mathbf{1}_{\{\tau>n\}}\big|V_0=v\right]=\mathbf{E}\left[N_u|V_0=v\right]. \tag{8.16}$$

Next consider the second term on the right-hand side of (8.15). Applying Tonelli's theorem to interchange the order of summation and integration, then interchanging the order of summation, we obtain

$$
\sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \int_{\mathbb{M}-\mathbb{C}} {}_{\mathbb{C}}P^k(\mathbf{v}, d\mathbf{w}) {}_{\mathbb{M}}P^{n-k}\big(\mathbf{w}, (u, \infty) \times \{0, 1\}\big)
$$

$$
= \int_{\mathbb{M}-\mathbb{C}} \sum_{k=1}^{\infty} {}_{\mathbb{C}}P^k(\mathbf{v}, d\mathbf{w}) \left(\sum_{n=k+1}^{\infty} {}_{\mathbb{M}}P^{n-k}\big(\mathbf{w}, (u, \infty) \times \{0, 1\}\big) \right)
$$

$$
= \int_{\mathbf{w} \in (\mathbb{M}-\mathbb{C})} \int_{\mathbf{z} \in \mathbb{M}^c} \sum_{k=1}^{\infty} {}_{\mathbb{C}}P^k(\mathbf{v}, d\mathbf{w}) {}_{\mathbb{M}}P(\mathbf{w}, d\mathbf{z})
$$

$$
\cdot \left(\sum_{n=0}^{\infty} {}_{\mathbb{M}}P^n\big(\mathbf{z}, (u, \infty) \times \{0, 1\}\big) \right). \tag{8.17}
$$

For the last term in parentheses, note that ${}_{\mathbb{M}}P^n\big(\mathbf{z}, (u, \infty) \times \{0, 1\}\big)$ describes the probability that $\{V_n\}$ avoids the set $[-M, M]$ during the first n time increments and that $V_n \in (u, \infty)$. Hence setting $\mathbf{z} = (z, r)$ yields, by (8.11),

$$
\sum_{n=0}^{\infty} {}_{\mathbb{M}}P^n\big(\mathbf{z}, (u, \infty) \times \{0, 1\}\big) = \mathbf{E}\left[\sum_{n=0}^{\infty} \mathbf{1}_{\{V_n > u\}} \mathbf{1}_{\{K(M) > n\}} \Big| V_0 = z \right]
$$

$$
= \mathbf{E}\left[\sum_{n=0}^{K(M)-1} \mathbf{1}_{\{V_n > u\}} \Big| V_0 = z \right] \leq \frac{u^{-\alpha} |z|^\alpha}{1 - \rho}. \tag{8.18}
$$

Substituting this inequality into the previous equation, we see that the right-hand side of (8.17) is bounded above by

$$
\frac{u^{-\alpha}}{1 - \rho} \sum_{k=1}^{\infty} \int_{\mathbb{M}-\mathbb{C}} {}_{\mathbb{C}}P^k(\mathbf{v}, d\mathbf{w}) \left(\int_{\mathbb{M}^c} {}_{\mathbb{M}}P(\mathbf{w}, d\mathbf{z}) |z|^\alpha \right). \tag{8.19}
$$

Note that with $\mathbf{w} = (w, s)$,

$$
\int_{\mathbb{M}^c} {}_{\mathbb{M}}P(\mathbf{w}, d\mathbf{z}) |z|^\alpha \leq \mathbf{E}\left[\bar{V}_1^\alpha \big| V_0 = M \right],
$$

where $\bar{V}_1 := A_1 \max\{D_1, V_0\} + |B_1|$. Moreover, for the remaining integral in (8.19), we have by definition that

$$
\int_{\mathbb{M}-\mathbb{C}} {}_{\mathbb{C}}P^k(\mathbf{v}, d\mathbf{w}) = \mathbf{P}\left\{ \tau > k, V_k \in [-M, M] \big| V_0 = v \right\}.
$$

Substituting the last two estimates into (8.19), we obtain that (8.19) is bounded above by

$$
\frac{u^{-\alpha}}{1 - \rho} \sum_{k=1}^{\infty} \mathbf{P}\left\{ \tau > k \big| V_0 = v \right\} \mathbf{E}\left[\bar{V}_1^\alpha \big| V_0 = M \right]. \tag{8.20}
$$

Now repeat the same argument, but applied to the first term on the right of (8.15). Essentially, this can be viewed as one of the terms in the previous sum, namely the term $k = 0$. In fact, the previous argument may be repeated without change to obtain

$$
{}_{\mathbb{M}}P^n(\mathbf{v}, E) \leq \mathbf{P}\left\{ \tau > 0 \big| V_0 = v \right\} \mathbf{E}\left[\bar{V}_1^\alpha \big| V_0 = M \right]. \tag{8.21}
$$

Substituting these last two equations into the right-hand side of (8.15) and substituting (8.16) into the left-hand side of (8.15) yields

$$\mathbf{E}\left[N_u | V_0 = v\right] = \frac{u^{-\alpha}}{1 - \rho} \sum_{k=0}^{\infty} \mathbf{P}\left\{\tau > k \, | V_0 = v\right\} \mathbf{E}\left[\bar{V}_1^{\alpha} \, | V_0 = M\right]$$

$$\leq \frac{u^{-\alpha}}{1 - \rho} \left\{ \sup_{w \in [-M, M]} \mathbf{E}_w\left[\tau\right] \mathbf{E}_M\left[\bar{V}_1^{\alpha}\right] \right\}, \tag{8.22}$$

which is (8.14).

Finally observe that the quantity on the right-hand side of (8.22) is finite. To this end, it is sufficient to verify that the set $\mathbb{M}$ is regular. For this purpose, observe that since $\{\eta_n\}$ is an i.i.d. sequence, it follows by a slight modification of Lemma 5.1 (iii) that $\mathbb{M}$ is petite. Moreover, letting κ denote the first return time of $\{V_n\}$ to $\mathcal{C}_M$, then by Lemma 5.1 (i) and Theorem 15.0.1 of Meyn and Tweedie (1993), we have that $\sup_{v \in \mathcal{C}_M} \mathbf{E}_v\left[t^{\kappa}\right] < \infty$ for some $t > 1$. Then by definition, it follows that this last equation also holds for the augmented chain with $\mathbb{M} := \mathcal{C}_M \times \{0, 1\}$ in place of $\mathcal{C}_M$. Consequently, the conditions of Meyn and Tweedie (1993), Theorem 11.3.14 (i), are fulfilled and hence $\mathbb{M}$ is regular.

Step 3: Finally observe that by summing the expectations studied in Steps 1 and 2, we immediately obtain (8.5). $\square$

Proof of Theorem 4.1 By Proposition 8.1, it is sufficient to show that for some $t \in (0, 1)$,

$$\mathbf{E}\left[\sum_{n=0}^{K(u^t)-1} \mathbf{1}_{\{V_n > u\}} \, \middle| \, \frac{V_0}{u} = v\right] = U(\log v), \tag{8.23}$$

where $U(z) := \sum_{n \in \mathbb{N}} \mu_A^{*n}(-\infty, z)$ and μ_A is the marginal distribution of $-\log A$. As explained earlier in Section 4, under a minor continuity condition, U may be viewed as the renewal function of $-S_n = -\sum_{i=1}^{n} \log A_i$, while the expectation on the left-hand side may be viewed as a *truncated* renewal function of the nonlinear process $\{V_n\}$.

In the remainder of the proof, we will suppress the conditioning in (8.23).

To prove (8.23), we first establish an upper bound and then a corresponding lower bound. For the upper bound, begin by observing (*cf.* (5.25)) that

$$\frac{|V_n|}{|V_{n-1}|} \leq A_n + \frac{(A_n |D_n| + |B_n|)}{|V_{n-1}|}. \tag{8.24}$$

Note by definition that $|V_n| > u^t$ for all $n \leq K(u^t)$. Hence

$$\log\left(\frac{|V_n|}{|V_{n-1}|}\right) < \log\left(A_n + u^{-t}\left(A_n |D_n| + |B_n|\right)\right), \quad \text{all } n < K(u^t). \tag{8.25}$$

Now introduce the random walk

$$S_n^{(u)} := \sum_{i=1}^{n} X_i^{(u)}, \quad \text{where } X_i^{(u)} := \log\left(A_i + u^{-t}\left(A_i |D_i| + |B_i|\right)\right), \tag{8.26}$$

and $S_0^{(u)} = 0$. From the previous two equations, we obtain $(\log |V_n| - \log |V_0|) \leq S_n^{(u)}$. Since $(V_0/u) = v$, it follows that

$$\log |V_n| - \log u \leq S_n^{(u)} + \log v. \tag{8.27}$$

Now suppose that $|V_n| > u$. Then the left-hand side of (8.27) is positive, so $S_n^{(u)} > -\log v$. Consequently,

$$\mathbf{E}\left[\sum_{n=0}^{K(u^t)-1} \mathbf{1}_{\{|V_n|>u\}}\right] \leq \mathbf{E}\left[\sum_{n=0}^{\infty} \mathbf{1}_{\{S_n^{(u)}>-\log v\}}\right] := U^{(u)}(\log v). \tag{8.28}$$

To establish an upper bound, it remains to show that

$$\limsup_{u\to\infty} U^{(u)}(\log v) \leq U(\log v). \tag{8.29}$$

To this end, observe by (H_2) and (8.26) that $\mathbf{E}\left[\exp\left\{\epsilon X_i^{(u)}\right\}\right] < \infty$ for some $\epsilon > 0$. Moreover, $\mathbf{E}\left[\log A\right] < 0 \implies \mathbf{E}\left[X_i^{(u)}\right] < 0$ for sufficiently large u. Since $U^{(u)}$ can itself be viewed as a renewal function for the random walk $\{S_n^{(u)}\}$ (describing the number of visits of this negative-drift process to the interval $(-\log v, \infty)$), it follows that $U^{(u)}(\log v) < \infty$. Since $X_i^{(u)}$ decreases monotonically to $\log A_i$ as $u \to \infty$, it follows by a dominated convergence argument that

$$\lim_{u\to\infty} U^{(u)}(\log v) = \mathbf{E}\left[\sum_{n=0}^{\infty} \mathbf{1}_{\{S_n>-\log v\}}\right] := U(\log v), \tag{8.30}$$

which establishes the required upper bound.

Turning now to the lower bound, fix $\epsilon > 0$ and choose N sufficiently large such that

$$\left|\mathbf{E}\left[\sum_{n=0}^{N} \mathbf{1}_{\{S_n>-\log v\}}\right] - U(\log v)\right| < \epsilon. \tag{8.31}$$

Then

$$\mathbf{E}\left[\sum_{n=0}^{K(u^t)-1} \mathbf{1}_{\{V_n>u\}}\right] \geq \mathbf{E}\left[\sum_{n=0}^{N} \mathbf{1}_{\{V_n>u\}}\right] - N\mathbf{P}\left\{K(u^t) \leq N\right\}. \tag{8.32}$$

To bound the first term on the right-hand side, note that

$$V_n \geq A_n V_{n-1} - |B_n|, \tag{8.33}$$

and iterating this equation yields

$$V_n \geq (A_1 \cdots A_n)V_0 - \sum_{i=1}^{n-1} \prod_{j=i+1}^{n} A_j |B_i|, \quad \text{for all } n. \tag{8.34}$$

Thus, setting $(V_0/u) = v$ and letting $u \to \infty$ while *fixing* $n \in \mathbb{Z}_+$, we obtain that for any given n,

$$\liminf_{u\to\infty} \frac{V_n}{u} \geq (A_1 \cdots A_n)v := \exp\{S_n + \log v\} \quad \text{a.s.} \tag{8.35}$$

Hence, if $S_n > -\log v$, then we will necessarily have $\liminf_{u\to\infty}(V_n/u) > 1$. Consequently,

$$\liminf_{u\to\infty} \mathbf{E}\left[\sum_{i=1}^{N} \mathbf{1}_{\{V_n>u\}}\right] \geq \mathbf{E}\left[\sum_{n=1}^{N} \mathbf{1}_{\{S_n>-\log v\}}\right] \geq U(\log v) - \epsilon, \tag{8.36}$$

where the last step was obtained by (8.31). Finally, observe by the definition of $K(u^t)$ together with (8.34) that, for any $s \in (t, 1)$,

$$\mathbf{P}\left\{K(u^t) \leq N\right\} \leq \mathbf{P}\left\{(A_1 \cdots A_N)V_0 \leq u^s\right\} + \mathbf{P}\left\{\sum_{i=1}^{N-1} \prod_{j=i+1}^{N} A_j |B_i| > u^s - u^t\right\}. \tag{8.37}$$

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In the first term on the right-hand side, $(V_0/u) = v$, and so the probability in question reduces to $\mathbf{P}\left\{(A_1 \cdots A_N)\, v \le u^{s-1}\right\} = \mathbf{P}\left\{S_N + \log v \le (s-1)\log u\right\}$, and the latter probability tends to zero as $u \to \infty$, as N is fixed and $(s-1) < 0$. Similar reasoning shows that the second term on the right of (8.37) also tends to zero as $u \to \infty$. Thus we conclude $\lim_{u \to \infty} \mathbf{P}\left\{K(u^t) \le N\right\} = 0$ for any fixed N. Substituting this last equation and (8.36) into (8.32) yields the desired lower bound.

$\square$

Proof of Proposition 4.1 First assume that $(V_0/u) = v$ for some $v > 1$. Then it follows by (8.28) and Proposition 8.1 that

$$\mathbf{E}\left[N_u \,\bigg|\, \frac{V_0}{u} = v\right] = \mathbf{E}\left[\sum_{n=0}^{\infty} \mathbf{1}_{\{V_n > u\}} \,\bigg|\, \frac{V_0}{u} = v\right] \le U^{(u)}(\log v) + \Delta(u), \tag{8.38}$$

where, if $S_n^{(u)}$ be defined as in (8.26), then

$$U^{(u)}(\log v) := \mathbf{E}\left[\sum_{n=0}^{\infty} \mathbf{1}_{\{S_n^{(u)} > -\log v\}}\right] = \mathbf{E}\left[\sum_{n=0}^{\infty} \mathbf{1}_{\{-S_n^{(u)} < \log v\}}\right]. \tag{8.39}$$

Notice that $U^{(u)}(\cdot)$ is the renewal function of $\{-S_n^{(u)}\}$. Consequently, by Lorden's inequality (Asmussen (2003), Proposition V.6.2), it follows that

$$U^{(u)}(\log v) \le \frac{\log v}{m_u} + \left(1 + \frac{\sigma_u^2}{m_u^2}\right), \tag{8.40}$$

where

$$m_u := \mathbf{E}\left[X^{(u)}\right] \quad \text{and} \quad \sigma_u^2 := \operatorname{Var}\left(X^{(u)}\right) \tag{8.41}$$

and $X^{(u)}$ is defined as in (8.26).

Observe that the moments in (8.41) are actually finite. In particular, $|x|^\epsilon$ dominates $(\log x)^2$ for any $\epsilon > 0$. Hence, applying (H_2) for a sufficiently small choice of $\epsilon > 0$, we obtain

$$\mathbf{E}\left[\left(X^{(u)}\right)^2\right] \le \text{const.} \cdot \mathbf{E}\left[e^{\epsilon X^{(u)}}\right] < \infty, \quad \text{for all } u. \tag{8.42}$$

Consequently, both the constants m_u and σ_u^2 are finite. Substituting (8.40) into (8.38) yields

$$\mathbf{E}\left[N_u \,\bigg|\, \frac{V_0}{u} = v\right] \le C_1(u)\log v + C_2(u), \tag{8.43}$$

where, by (8.6) and (8.40),

$$C_1(u) := \frac{1}{m_u} \quad \text{and} \quad C_2(u) := \left(1 + \frac{\sigma_u^2}{m_u^2}\right) + \Delta(u). \tag{8.44}$$

Next recall that our primary objective is to study $\mathbf{E}_{\mathfrak{D}}\left[N_u \,\big|\, \mathfrak{F}_{T_u \wedge (\tau-1)}\right]$. But by the strong Markov property,

$$\mathbf{E}_{\mathfrak{D}}\left[N_u \,\big|\, \mathfrak{F}_{T_u \wedge (\tau-1)}\right] = \mathbf{E}_{\mathfrak{D}}\left[N_u \big| V_{T_u}\right] \mathbf{1}_{\{T_u < \tau\}}.$$

Observe that in the dual measure, the process $\{V_n\}$ reverts back to its original measure after time T_u, and if $T_u \not< \tau$ then N_u is zero. Thus, we may apply (8.43) to the right-hand side of the previous equation to obtain that

$$\mathbf{E}_{\mathfrak{D}}\left[N_u \,\big|\, \mathfrak{F}_{T_u \wedge (\tau-1)}\right] \le \left(C_1(u)\log\left(\frac{V_{T_u}}{u}\right) + C_2(u)\right) \mathbf{1}_{\{T_u < \tau\}}, \tag{8.45}$$

which is the required upper bound.

Finally observe that as $u \to \infty$, $X^{(u)}$ decreases monotonically to $\log A$ a.s. and $\Delta(u) \downarrow 0$, and hence as $u \to \infty$,

$$C_1(u) \to C_1 := \frac{1}{m} \quad \text{and} \quad C_2(u) \to C_2 := 1 + \frac{\sigma^2}{m^2}, \tag{8.46}$$

where $m := \mathbf{E}[\log A]$ and $\sigma^2 := \operatorname{Var}(\log A)$. $\qquad \square$

Proof of Theorem 4.2 By the strong Markov property,

$$\mathbf{E}\left[N_u \mid \frac{V_{T_u}}{u} = v\right] = \mathbf{E}\left[N_u \mid \frac{V_0}{u} = v\right] := H_u(v),$$

and by Lemma 4.1,

$$\lim_{u \to \infty} H_u(v) = H(v) := U(\log v). \tag{8.47}$$

Let ν_u denote the probability law of V_{T_u}/u in the dual measure. Then our objective is to show that for any $z > 1$,

$$\lim_{u \to \infty} \int_1^z H_u(v) v^{-\xi} d\nu_u(v) = \int_1^z H(v) v^{-\xi} d\hat{\mu}(v). \tag{8.48}$$

[Here we have taken the lower end point of the integral at one, since $V_{T_u}/u > 1$ for every u.]

Next observe that

$$\int_1^z H_u(v) v^{-\xi} d\nu_u(v) = R_1(u) + R_2(u), \tag{8.49}$$

where

$$R_1(u) = \int_1^z H_u(v) v^{-\xi} d\left(\nu_u(v) - \hat{\mu}(v)\right), \quad R_2(u) = \int_1^z H_u(v) v^{-\xi} d\hat{\mu}(v).$$

Now by (8.43), (8.46), (8.47), and the dominated convergence theorem, it follows that as $u \to \infty$,

$$R_2(u) \to R_2 := \int_1^z H(v) v^{-\xi} d\hat{\mu}(v). \tag{8.50}$$

Finally, $R_1(u) \to 0$ as $u \to \infty$ using (8.43), (8.46), and the weak convergence of ν_u to $\hat{\mu}$. $\qquad \square$

9 Proof of Theorem 2.2

The proof of Theorem 2.2 is a modification of the proof of Theorem 2.1. However, we first need to verify that the preparatory lemmas are satisfied in the general setting of Section 2.4.

Lemmas 5.1-5.5 for general Lipschitz maps. A complication arises with Lemma 5.1 (iii), which asserts the minorization ($\mathcal{M}$) with $k = 1$. However, in general, this strong minorization condition is not necessary and can be avoided by considering the k-chain $\{V_{kn} : n = 0, 1, \ldots\}$. (As noted in Remark 2.3, if we consider the k-chain, then Lemma 5.1 (iii) is not necessary and, as a consequence, we do not need to assume (H_0) in the case of Letac's Model E. While these remarks were stated in the setting of Theorem 2.1, the situation is the same here.)

Thus, we consider two cases. In the first case, we suppose that ($\mathcal{M}$) holds with $k = 1$ and $\mathcal{C} \subset [-M, M]$ for some $M \geq 0$. Then by the cancellation condition ($\mathfrak{C}$),

$$A \max\{v, D^*\} + B^* \leq V_n(v) \leq A \max\{v, D\} + B, \tag{9.1}$$

for triplets (A, B, D) and (A, B^*, D^*) satisfying (H_1), (H_2), and (H_3), and by considering the upper bound in (9.1), we immediately obtain Lemma 5.1 (i) and (iv). Also, we obtain Lemma 5.1 (ii)

directly from our previous proof, since assumption ($\mathfrak{L}$) in Section 2.4 imply the existence of a unique stationary measure for $\{V_n\}$, independent of the initial state. Hence, for this case, we see that all of the conclusions in Lemma 5.1 hold.

If ($\mathcal{M}$) is satisfied with $k > 1$, then we work with the k-chain $\{V_{kn} : n = 0, 1, \ldots\}$ in place of the 1-chain $\{V_n : n = 0, 1, \ldots\}$. Since $\{V_{kn}\}$ converges to the same stationary distribution as $\{V_n\}$, the representation formula in Lemma 2.3 still holds; that is,

$$\mathbf{P}\{V > u\} = \frac{\mathbf{E}\big[N_u^{(k)}\big]}{\mathbf{E}\big[\tau^{(k)}\big]}, \tag{9.2}$$

where $\tau^{(k)}$ denotes the first regeneration time *of the process* $\{V_{kn}\}$, and

$$N_u^{(k)} := \sum_{n=0}^{\tau^{(k)}-1} \mathbf{1}_{(u,\infty)}(V_{kn}),$$

Using the compositional formulas (2.10) and (2.11), we obtain in place of (9.1) that

$$\mathcal{A}\max\{v, \mathcal{D}^*\} + \mathcal{B}^* \leq f^{(k)}(v) \leq \mathcal{A}\max\{v, \mathcal{D}\} + \mathcal{B}, \tag{9.3}$$

where $f^{(k)}$ denotes the k-fold composition $f_k \circ \cdots \circ f_1$ for i.i.d. copies $\{f_i\}$ of the random map f, and where $(\mathcal{A}, \mathcal{B}, \mathcal{D})$ is given according to (2.10) and $(\mathcal{B}^*, \mathcal{D}^*)$ is given according to (2.10), but with (B_i^*, D_i^*) in place of (B_i, D_i). Moreover, $(\mathcal{A}, \mathcal{B}, \mathcal{D})$ and $(\mathcal{A}, \mathcal{B}^*, \mathcal{D}^*)$ are easily seen to satisfy (H_1), (H_2), and (H_3), provided that (A, B, D) and (A, B^*, D^*) satisfy these conditions. By (9.3) and the discussion in the previous paragraph, we conclude that Lemmas 5.1 (i), (ii), and (iv) hold for the k-chain. In part (iv), it is worth observing that the set $[-M, M]$ is petite. In particular, under the Lipschitz condition ($\mathfrak{L}$), $\{V_n\}$ is weak Feller (*cf.* Alsmeyer (2003), Remark C). Under our assumption of nondegeneracy of $\{V_n\}$, it then follows by Proposition 6.2.8 of Meyn and Tweedie (1993) that $[-M, M]$ is petite.

Now turning to the remaining lemmas of Section 5, we note that the transience asserted in Lemma 5.2 is immediately obtained from the proof given in Section 5 and the lower bound in (9.3). Similarly, Lemma 5.3 follows by the previous proof given in Section 5, and Lemma 5.5 is obtained by the proof in Section 5 and the upper and lower bounds in (9.3) (*cf., e.g.* (5.31)). In Lemma 5.5 for the k-chain, note that (9.3) implies

$$|V_{kn}| \leq \mathcal{A}_n|V_{k(n-1)}| + \tilde{\mathcal{B}}_n, \quad \text{where} \quad \tilde{B}_n := \mathcal{A}_n\big(|\mathcal{D}_n| + |\mathcal{D}_n^*|\big) + |\mathcal{B}_n| + |\mathcal{B}_n^*|.$$

Thus, as an analog to (2.28), the perpetuity sequence utilized for the upper bound now becomes

$$\bar{Z}^{(p)} := |V_0| + \sum_{n=1}^{\infty} \frac{\tilde{B}_n}{\mathcal{A}_1 \cdots \mathcal{A}_n} \mathbf{1}_{\{\tau^{(k)}>n\}}. \tag{9.4}$$

(The required upper bound is seen to hold by repeating the proof of Lemma 5.5 (ii) following (5.25).) Finally, Lemma 5.4 is not needed for our discussion of general Lipschitz maps.

Nonlinear renewal theory for general Lipschitz maps. The proofs of these results require almost no modifications, once it is understood that we obtain these results with respect to the k-chain when $k > 1$ and that the bounds (9.1) and (9.3) hold. In particular, Lemma 4.1 is proved in the same way as before and essentially follows from the transience $\{V_n\}$ (resp. $\{V_{kn}\}$) obtained by Lemma 5.2. Also, Proposition 8.1 is derived for Lipschitz maps by repeating our previous proof in Section 8, since these calculations mainly rely on the drift condition (8.7), established in Lemma 5.1 (i). The finiteness on the right-hand side of (8.6) is obtained by first noting that $[-M, M]$ is petite

(by the above discussion following (9.3)). Then by Meyn and Tweedie (1993), Theorem 11.3.11, the expected return time to $\mathcal{C}$, starting from x, is bounded above by $V(x) +$ const., where V is given as in Lemma 5.1 (iv). Turning to Theorem 4.1, we see that the main ingredient of the proof are the bounds (8.24) and (8.33), which, when adapted for the k-chain, follow from the inequalities

$$\mathcal{A}_n V_{k(n-1)} - |\mathcal{B}_n^*| \leq V_{kn} \leq \mathcal{A}_n V_{k(n-1)} + (\mathcal{A}_n |\mathcal{D}_n| + |\mathcal{B}_n|), \quad n = 1, 2, \ldots;$$

cf. (9.1) and (9.3). The proof of Proposition 4.1 also follows from the previous proof, once it is understood that these results are derived with respect to the k-chain (and thus N_u, m_u, σ_u, etc. are all calculated with respect to the k-chain and with respect to the *lower*-bounding Letac model, *i.e.*, with respect to the process generated by the lower bound in (9.3)). Finally, the proof of Theorem 4.2 is purely analytical in nature and is derived from the previous results of the section and does not require any modifications.

Proof of Theorem 2.2 The proof can be obtained from the proof of Theorem 2.1, with a few modifications involving the replacement of the 1-chain with the k-chain and the use of the upper and lower bounds in (9.3). Observe that the conclusions of Proposition 6.1 still hold under the conditions of Theorem 2.2. Indeed, if $(\mathcal{M})$ holds with $k = 1$, then we obtain (6.1) with $Z := \lim_{n \to \infty} V_n/(A_0 \cdots A_n)$. If $(\mathcal{M})$ holds with $k > 1$, then we also obtain (6.1), except that N_u is now replaced with $N_u^{(k)}$, the number of returns of the k-chain $\{V_{kn} : n = 0, 1, \ldots\}$ to (u, ∞) prior to regeneration; τ is now replaced with $\tau^{(k)}$, the first regeneration time of the k-chain; and T_u is now replaced with $T_u^{(k)} := \inf\{n : V_{kn} > u\}$. We emphasize here that the dual measure is computed with respect to the k-chain.

By Proposition 6.1 (for the k-chain),

$$\lim_{u \to \infty} u^\xi \mathbf{P}\{V > u\} = \mathbf{E}_\xi \left[Z^\xi \mathbf{1}_{\{\tau^{(k)} = \infty\}} \right]$$

$$\cdot \left(\mathbf{E}[\tau^{(k)}] \right)^{-1} \lim_{u \to \infty} \mathbf{E}_{\mathfrak{D}} \left[N_u \left(\frac{V_{T_u^{(k)}}}{u} \right)^{-\xi} \Bigg| T_u^{(k)} < \tau^{(k)} \right]. \qquad (9.5)$$

In the first term on the right-hand side of (9.5), note that

$$Z_n := \frac{V_n}{A_1 \cdots A_n} := \frac{A_n V_{n-1} + \mathcal{R}_n(V_{n-1})}{A_1 \cdots A_n} = Z_{n-1} + \frac{\mathcal{R}_n(V_{n-1})}{A_1 \cdots A_n} = V_0 + \sum_{i=1}^n \frac{\mathcal{R}_i(V_{i-1})}{A_1 \cdots A_i}, \qquad (9.6)$$

where the last step follows by induction. Hence

$$\mathbf{E}_\xi \left[Z^\xi \mathbf{1}_{\{\tau^{(k)} = \infty\}} \right] = \mathbf{E}_\xi \left[\left(V_0 + \sum_{i=1}^\infty \frac{\mathcal{R}_i(V_{i-1})}{A_1 \cdots A_i} \right)^\xi \mathbf{1}_{\{\tau^{(k)} = \infty\}} \right], \qquad (9.7)$$

and since $Z > 0$ w.p. 1 in the ξ-shifted measure (by Lemma 5.5 (i)), the signed quantity appearing on the right-hand side of (9.6) is positive. (We emphasise that this is the same expression for the k-chain and the 1-chain, except that the stopping time $\tau^{(k)}$ is used for the k-chain, while τ is used for the 1-chain. In particular, $\{Z_{kn}\}$ and $\{Z_n\}$ converge to the same a.s. limit, so the quantity inside the exponent in (9.7) does not change.)

Next we turn to the second term on the right-hand side of (9.5). Applying Lemma 6.1 (but to the multiplicative random walk $\{\mathcal{A}_i\}$ rather than $\{A_i\}$ when $k > 1$, where $\mathcal{A}_i := A_{k(i-1)\cdots ki-1}$), we obtain that

$$\frac{1 - \mathbf{E}[e^{\xi S_{\tau^*}}]}{\mathbf{E}[\tau^*]} \lim_{u \to \infty} \mathbf{E}_{\mathfrak{D}} \left[N_u \left(\frac{V_{T_u^{(k)}}}{u} \right)^{-\xi} \Bigg| T_u^{(k)} < \tau^{(k)} \right] = C^*, \qquad (9.8)$$

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where we now write $S_n = \log \mathcal{A}_1 + \cdots + \log \mathcal{A}_n$ and $\tau^* = \inf\{n : S_n < 0\}$; *cf.* (6.14). Note that the parameter ξ remains unchanged in these expressions (compared with the case $k = 1$), since $\mathbf{E}\left[\mathcal{A}^\xi\right] = \mathbf{E}\left[(A_1 \cdots A_k)^\xi\right] = 1$, by independence of $\{A_i\}$. Furthermore, by (2.20),

$$\frac{C^* \mathbf{E}\left[\tau^*\right]}{1 - \mathbf{E}\left[e^{\xi S_{\tau^*}}\right]} = \frac{1}{\xi \lambda_k'(\xi)}, \tag{9.9}$$

where λ_k is the moment generating function of $\log \mathcal{A}$. Now an elementary calculation yields $\lambda_k'(\xi) = k\lambda'(\xi)$. Hence, substituting (9.7) and (9.9) into (9.5), we obtain that $\mathbf{P}\{V > u\} \sim Cu^{-\xi}$ as $u \to \infty$, where

$$C = \frac{1}{\xi \lambda'(\xi) \cdot k\mathbf{E}\left[\tau^{(k)}\right]} \mathbf{E}_\xi \left[\left(V_0 + \sum_{i=1}^\infty \frac{\mathcal{R}_i(V_{i-1})}{A_1 \cdots A_i}\right)^\xi \mathbf{1}_{\{\tau^{(k)} = \infty\}} \right], \tag{9.10}$$

which is (2.38) adapted to the case of general $k \in \mathbb{Z}_+$.

To establish (2.39) (again for general $k \in \mathbb{Z}_+$), we first need to observe that for $k > 1$, there is a natural analog to (9.6) which is established by the same computation, namely,

$$Z_{kn} = V_0 + \sum_{i=1}^n \frac{\mathcal{R}_i^{(k)}(V_{k(i-1)})}{\mathcal{A}_1 \cdots \mathcal{A}_i}. \tag{9.11}$$

Once this has been observed, we obtain from the argument given in the proof of Theorem 2.1 that (2.39) holds.

It remains to observe that the constant C obtained in (9.10) is both finite and positive. To this end, note $Z > 0$ w.p. 1 in the ξ-shifted measure. Thus both of the terms on the right-hand side of (9.5) are positive, and hence $C > 0$. The finiteness of C is then obtained from Lemma 5.5 (ii) for Lipschitz maps. □

In Remark 6.3, we observed that an alternative expression for the constant can be obtained by applying (6.32). For the k-chain, this equation takes the form

$$\mathbf{P}\{V > u\} = \lim_{n \to \infty} \frac{1}{n} \mathbf{E}\left[\mathfrak{N}_u^{(n)}\right], \quad \text{where} \quad \mathfrak{N}_u^{(n)} := \sum_{i=1}^{\kappa^{(n)}} \mathbf{1}_{(u,\infty)}(V_{ki}) \tag{9.12}$$

and $\kappa^{(n)}$ now denotes the first regeneration time after time n *for the k-chain* $\{V_{ki} : i = 0, 1, \ldots\}$. We may then modify Proposition 6.1 exactly as described in Remark 6.3. Note that we do not use the special structure of Letac's Model E in this argument, working there with the general quantity $Z_n := V_n/(A_1 \cdots A_n)$. Also, recall that $\lambda'(\xi)$ must be replaced with $k\lambda'(\xi)$ when analyzing the k-chain (as noted in the proof of Theorem 2.2). We then obtain an analog to (6.45), namely

$$C = \frac{1}{\xi \cdot k\lambda'(\xi)} \lim_{n \to \infty} \frac{1}{n} \mathbf{E}\left[\tilde{Z}_{kn}^\xi\right] = \frac{1}{\xi \lambda'(\xi)} \lim_{n \to \infty} \frac{1}{n} \mathbf{E}\left[\tilde{Z}_n^\xi\right], \tag{9.13}$$

where $\{\tilde{Z}_n\}$ is the backward process corresponding to $\{V_n\}$, and, as in Remark 6.3, it is assumed that $\{V_n\}$ is nonnegative. In this way, we deduce an alternative expression for C, which seems of some theoretical interest.

Example 9.1 (Example 3.5 revisited). Finally, we return to the polynomial recursion studied earlier in Example 3.5. There we noted that the hypotheses of Theorem 2.2 can be verified by applying the arguments of Mirek (2011a). However, this leads to the condition (3.17) which appears to introduce a rather unnatural set of moment assumptions that we now would like to circumvent.

Specifically, we now show how the proof of Theorem 2.2 may be mimicked to obtain the same conclusion as in this example, but under a preferable set of moment assumptions. Essentially what needs to be shown—in this and any example—is the convergence of the forward process $\{V_n\}$ to a stationary distribution, and that this stationary distribution possesses appropriate moment conditions.

To *directly* verify the existence of a stationary measure, utilizing a technique of Loynes (Goldie (1991), p. 161), first observe that for any positive constant c, the remainder term in (3.14) satisfies

$$\mathcal{R}(v) \leq \left(\sum_{i=1}^{j-1} \tilde{A}_i \right) c^{j-1/j} \max\{1, v/c\} + \tilde{A}_0.$$

Thus, in (3.13), $V \overset{\mathcal{D}}{=} f(V)$ where

$$f(v) \leq \left(\tilde{A}_j c + c^{j-1/j} \sum_{i=1}^{j-1} \tilde{A}_i \right) \max\{1, v/c\} + \tilde{A}_0 := \mathfrak{A} \max\{c, v\} + \mathfrak{B}, \qquad (9.14)$$

for $\mathfrak{B} = \tilde{A}_0$ and $\mathfrak{A} = \left(A + c^{-1/j} \sum_{i=1}^{j-1} \tilde{A}_i \right)$, where $A \equiv \tilde{A}_j$. The upper bound in (9.14) is Letac's Model E, which itself converges to a stationary distribution under the conditions of Goldie (1991), namely under the assumption that $\xi > 0$ and

$$\mathbf{E}\big[\tilde{A}_i^{\xi'}\big] < \infty, \quad i = 0, \ldots, j-1, \ \text{ for some } \ \xi' > \xi. \qquad (9.15)$$

Under these assumptions, the Letac model given on the right-hand side of (9.14) converges to a stationary distribution. Moreover, observe that if B_0 is an independent copy of $\tilde{A}_0$ and

$$\mathbf{E}\left[\log\left(A + \left(\sum_{i=1}^{j-1} \tilde{A}_i \right) / 2B_0 \right) \right] < 0, \qquad (9.16)$$

then it follows by an argument on p. 162 of Goldie (1991) that the backward iterates of our polynomial SFPE converge and are independent of the initial value, and hence by Lemma 2.1, the forward recursive sequence also converges a.s. to the same random variable, which we denote by V. Thus we have established—by direct argument under a more natural set of conditions—the convergence of $\{V_n\}$. It is also worth observing as a consequence that $\mathcal{R}_n(V_n)$ then converges a.s. to

$$\mathcal{R}(V) := \sum_{i=0}^{j-1} \tilde{A}_i V^{i/j}, \qquad (9.17)$$

where $\{\tilde{A}_i\}$ is independent of V on the right-hand side.

At this point, the proof of Theorem 2.2 can be repeated where, once again, we treat the remainder term $\mathcal{R}_n(V_{n-1})$ like the random variable B_n appearing in the linear recursion $V_n = A_n V_{n-1} + B_n$. Noting that the minorization $(\mathcal{M})$ holds with $k = 1$, we then obtain that $\mathbf{P}\{V > u\} \sim Cu^{-\xi}$, where C is given as in (2.38), which can be viewed, from a practical perspective, as an recursive formula. The alternative expression (2.40) could also be used here, but that expression is also quite complex in the present setting, since the backward sequence of our polynomial recursion does not reduce to any simple form, and, in addition, (2.40) also introduces a limit as $n \to \infty$.

Finally, to see that this constant C is finite, first note that $\mathcal{R}_n(V_n)$ has a finite $(\xi')^{\text{th}}$ moment for some $\xi' > \xi$. Indeed, by (9.14), $\{V_n\}$ is dominated from above by

$$\bar{V}_n := \mathfrak{A}_n \bar{V}_{n-1} + (\mathfrak{A}_n c + \mathfrak{B}_n), \quad n = 1, 2, \ldots, \quad \bar{V}_0 = V_0,$$

which upon iteration yields $\bar{V}_n = V_0(\mathfrak{A}_1 \cdots \mathfrak{A}_n) \sum_{i=1}^{n}(\mathfrak{A}_i c + \mathfrak{B}_i)(\mathfrak{A}_{i+1} \cdots \mathfrak{A}_n)$. But given $\alpha < \xi$, we can choose a constant c such that $\mathbf{E}[\mathfrak{A}^\alpha] < 1$. Then a simple computation (using either Minkowskii's inequality in the case $\xi' \geq 1$ or otherwise the inequality $|x + y|^\xi \leq |x|^\xi + |y|^\xi$, $\xi \in (0, 1]$) yields that $\mathbf{E}[\bar{V}_n^\alpha]$ is uniformly bounded in n for any $\alpha < \xi$. Using the independence of V from the remaining terms on the right-hand side of (9.17), it now follows that

$$\lim_{n \to \infty} \mathbf{E}\left[(\mathcal{R}_n(V_{n-1}))^{\xi'}\right] = \sum_{i=1}^{j-1} \mathbf{E}[\tilde{A}_i^{\xi'}]\mathbf{E}[V^{\xi'(i/j)}] < \infty, \tag{9.18}$$

provided that ξ' has been chosen sufficiently small such that $\xi'(j-1)/j < \xi$. Then by repeating the computation in (3.12) or (5.29), we see that the expectation in (2.38) is bounded above by

$$\left(\mathbf{E}[V_0^\xi] + \sum_{n=1}^{\infty} \mathbf{E}\left[(\mathcal{R}_n(V_{n-1}))^{\xi'}\right]^{1/\xi'} \mathbf{P}\{\tau > n\}^{1/\xi''}\right)^\xi, \tag{9.19}$$

where $(1/\xi') + (1/\xi'') = 1/\xi$, and the latter expression is now seen to be finite by the geometric ergodicity of $\{V_n\}$ in its original measure. Hence $C < \infty$.

Thus, by adopting the proof of Theorem 2.2, we have obtained the constant representation in Example 3.5, but under the more natural conditions (9.15) and (9.16) (as compared with (3.17)). In conclusion, we see that the proof of Theorem 2.2 actually describes, in effect, an algorithm which may be adapted *individually* to obtain sharp asymptotics in a variety of problems beyond Letac's Model E, and the correct conditions in each of these problems may vary slightly from those given in the statement of Theorem 2.2.

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