TAG Lecture 4: Algebraic Stacks

Paul Goerss

17 June 2008

Sheaves of groupoids

Let S be a scheme (usually $\mathrm{Spec}(R)$). Stacks are built from sheaves of groupoids $\mathcal{G}$ on S.

Example

Let (A, Γ) be Hopf algebroid over R. Then

$$\mathcal{G} = \{ \ \mathrm{Spec}(\Gamma) \rightrightarrows \mathrm{Spec}(A) \ \}$$

is a sheaf of groupoids in all our topologies.

Given $U \to S$ and $x \in \mathcal{G}(U)$, get a presheaf Aut_x

$$\mathrm{Aut}_x(V \to U) = \mathrm{Iso}_{\mathcal{G}(V)}(x|_V, x|_V).$$

$\mathcal{G}$ is a prestack if this is sheaf. Hopf algebroids give prestacks.

Effective descent and stacks

Let $\mathcal{G}$ be a prestack on S and let $\mathcal{NG}$ be its nerve; this is a presheaf of simplicial sets.

Definition

$\mathcal{G}$ is a **stack** *if $\mathcal{NG}$ is a fibrant presheaf of simplicial sets.*

This is equivalent to $\mathcal{G}$ satisfying the following:

Effective Descent Condition: Given

1. a cover $V_i \to U$ and $x_i \in \mathcal{G}(U_i)$;
2. isomorphisms $\phi_{ij} : x_i|_{V_i \times_U V_j} \to x_j|_{V_i \times_U V_j}$;
3. subject to the evident cocyle condition;

Then there exists $x \in \mathcal{G}(U)$ and isomorphisms $\psi_i : x_i \to x|_{V_i}$.

Example: Principal G-bundles

Hopf algebroids hardly ever give stacks. Let's fix this.
Let Λ be a Hopf algebra over a ring k and

$$G = \mathrm{Spec}(\Lambda) \to \mathrm{Spec}(k) = S$$

the associated group scheme.

Definition

*A G-scheme $P \to U$ over U is a a G-***torsor** *if it locally of the form $U \times_S G$.*

The functor from schemes to groupoids

$$U \mapsto \{ \ G\text{-torsors over } U \text{ and their isos } \}$$

is a stack. This is the **classifying** stack BG.

Example: Algebraic homotopy orbits

Let X be a G-scheme. Form a functor to groupoids

$$U \mapsto \left\{ \begin{array}{ccc} & P & \xrightarrow{G\text{-map}} & X \\ \text{torsor} & \downarrow & & \\ & U & & \end{array} \right\}$$

This is the **quotient stack** $X \times_G EG = [X/G/S]$.
If Λ is our Hopf algebra, A a comodule algebra, then $(A, \Gamma = A \otimes \Lambda)$ is a **split** Hopf algebroid and

$$\mathrm{Spec}(A) \times_G EG$$

is the associated stack to the sheaf of groupoids we get from (A, Λ).

Example: Projective space

Consider the action

$$\begin{aligned} \mathbb{A}^{n+1} \times \mathbb{G}_m &\longrightarrow \mathbb{A}^{n+1} \\ (a_0, \ldots, a_n) \times \lambda &\mapsto (a_0\lambda, \ldots, a_n\lambda) \end{aligned}$$

Define $\mathbb{P}^n \to \mathbb{A}^{n+1} \times_{\mathbb{G}_m} E\mathbb{G}_m$ by

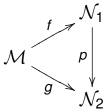
$$\{ N \to R^{n+1} \} \mapsto \left\{ \begin{array}{ccc} \mathrm{Iso}(R, N) & \longrightarrow & \mathbb{A}^{n+1} \\ \downarrow & & \\ \mathrm{Spec}(R) & & \end{array} \right\}$$

We get an isomorphism

$$\mathbb{P}^n \cong (\mathbb{A}^{n+1} - \{0\}) \times_{\mathbb{G}_m} E\mathbb{G}_m.$$

Morphisms and pullbacks

A morphism of stack $\mathcal{M} \to \mathcal{N}$ is a morphism of sheaves of groupoids. A 2-commuting diagram



is specified natural isomorphism $\phi : pf \to g$.

Given $\mathcal{M}_1 \xrightarrow{f} \mathcal{N} \xleftarrow{g} \mathcal{M}_2$ the pull-back $\mathcal{M}_1 \times_{\mathcal{N}} \mathcal{M}_2$ has objects

$$(x \in \mathcal{M}_1, y \in \mathcal{M}_2, \phi : f(x) \to g(y) \in \mathcal{N}).$$

Representable morphisms

Definition

A morphism $\mathcal{M} \to \mathcal{N}$ is **representable** *if for all morphisms $U \to \mathcal{N}$ of schemes, the pull-back*

$$U \times_{\mathcal{N}} \mathcal{M}$$

is equivalent to a scheme.

A representable morphism of stacks $\mathcal{N} \to \mathcal{M}$ is smooth or étale or quasi-compact or $\cdots$ if

$$U \times_{\mathcal{N}} \mathcal{M} \longrightarrow U$$

has this property for all $U \to \mathcal{N}$.

Algebraic Stacks

Definition

A stack $\mathcal{M}$ is **algebraic** *if*

1. *all morphisms from schemes $U \to \mathcal{M}$ are algebraic; and,*
2. *there is a smooth surjective map $q : X \to \mathcal{M}$.*

$\mathcal{M}$ is *Deligne-Mumford* if P can be chosen to be étale.

$X \times_G EG$ is algebraic with presentation

$$X \longrightarrow X \times_G EG$$

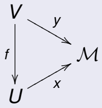
if G is smooth. Deligne-Mumford if G is étale.

Quasi-coherent sheaves

Definition

A **quasi-coherent sheaf** *$\mathcal{F}$ on an algebraic stack $\mathcal{M}$:*

1. *for each smooth $x : U \to \mathcal{M}$, a quasi-coherent sheaf $\mathcal{F}(x)$;*
2. *for 2-commuting diagrams*



coherent isomorphisms $\mathcal{F}(\phi) : \mathcal{F}(y) \to f^*\mathcal{F}(x)$.

Descent: If $X \to \mathcal{M}$ is a presentation then

$$\{ \text{ QC-sheaves on } \mathcal{M} \ \} \simeq \{ \text{ Cartesian sheaves on } X_\bullet \ \}$$

Example: Quasi-coherent sheaves and comodules

Suppose $\mathcal{M} = X \times_G EG$ where

- $G = \mathrm{Spec}(\Lambda)$ with Λ smooth over the base ring;
- $X = \mathrm{Spec}(A)$ where A is comodule algebra.

Then $X = \mathrm{Spec}(A) \to \mathcal{M}$ is a presentation and

$$\mathrm{Spec}(A) \times_{\mathcal{M}} \mathrm{Spec}(A) \cong \mathrm{Spec}(A \otimes \Lambda) = \mathrm{Spec}(\Gamma).$$

We have

$$\{ \text{ Cartesian sheaves on } X_\bullet \} \simeq \{ (A, \Gamma)\text{-comodules} \}.$$

Derived Deligne-Mumford stacks

Theorem (Lurie)

Let $\mathcal{M}$ be a stack and $\mathcal{O}$ a sheaf of ring spectra on $\mathcal{M}$. Then $(\mathcal{M}, \mathcal{O})$ is a derived Deligne-Mumford stack if

1. *$(\mathcal{M}, \pi_0\mathcal{O})$ is a Deligne-Mumford stack; and*
2. *$\pi_i\mathcal{O}$ is a quasi-coherent sheaf on $(\mathcal{M}, \pi_0\mathcal{O})$ for all i.*

Again there is a technical condition on $\mathcal{O}$ which I am suppressing.

Exercise

Let $\mathbb{G}_m$ be the multiplicative group and $B\mathbb{G}_m$ its classifying stack: this assigns to each commutative ring the groupoid of $\mathbb{G}_m$-torsors over A. Show that $B\mathbb{G}_m$ classifies locally free modules of rank 1; that is, the groupoid of $\mathbb{G}_m$-torsors is equivalent to the groupoid of locally free modules of rank 1.

The proof is essentially the same as that of equivalence between line bundles over a space X and the principle $\mathrm{Gl}_1(\mathbb{R})$-bundles over X. Here are two points to consider:

1. If N is locally free of rank 1, then $\mathrm{Iso}_A(A, N)$ is a $\mathbb{G}_m$-torsor;

2. If P is a $\mathbb{G}_m$ torsor, choose a faithfully flat map $f : A \to B$ so that we can choose an isomorphism $\phi : f^*P \cong \mathbb{G}_m$. If $d_i : B \to B \otimes B$ are the two inclusions then ϕ determines an isomorphism $d_1^*\mathbb{G}_m \to d_0^*\mathbb{G}_m$ – which must be given by a $\mu \in (B \otimes_A)^\times$. Then (B, μ) is the descent data determining a locally free module of rank 1 over A.