# TAG Lectures 9 and 10: $p$-divisible groups and Lurie's realization result

Paul Goerss

20 June 2008



## $p$-divisible groups

Pick a prime $p$ and work over $\mathrm{Spf}(\mathbb{Z}_p)$; that is, $p$ is implicitly nilpotent in all rings. This has the implication that we will be working with $p$-complete spectra.

**Definition**

*Let $R$ be a ring and $G$ a sheaf of abelian groups on $R$-algebras. Then $G$ is a $p$-**divisible group** of **height** $n$ if*

1. $p^k : G \to G$ *is surjective for all* $k$;
2. $G(p^k) = \mathrm{Ker}(p^k : G \to G)$ *is a finite and flat group scheme over $R$ of rank* $p^{kn}$;
3. $\operatorname{colim} G(p^k) \cong G$.

This definition is valid when $R$ is an $E_\infty$-ring spectrum.

# Examples of $p$-divisible groups

Formal Example: A formal group over a field or complete local ring is $p$-divisible.

**Warning:** A formal group over an arbitrary ring may not be $p$-divisible as the height may vary "fiber-by-fiber".

Étale Example: $\mathbb{Z}/p^\infty = \operatorname{colim} \mathbb{Z}/p^k$ with

$$\mathbb{Z}/p^k = \mathrm{Spec}(\mathrm{map}(\mathbb{Z}/p^n, R)).$$

Fundamental Example: if $C$ is a (smooth) elliptic curve then

$$C(p^\infty) \overset{\text{def}}{=} C(p^n)$$

is $p$-divisible of height 2.



# A short exact sequence

Let $G$ be $p$-divisible and $G_{\text{for}}$ be the completion at $e$. Then $G/G_{\text{for}}$ is étale ; we get a natural short exact sequence

$$0 \to G_{\text{for}} \to G \to G_{\text{et}} \to 0$$

split over fields, but not in general.

**Assumption:** We will always have $G_{\text{for}}$ of dimension 1.

**Classification:** Over a field $\mathbb{F} = \bar{\mathbb{F}}$ a $p$-divisible group of height $n$ is isomorphic to one of

$$\Gamma_k \times (\mathbb{Z}/p^\infty)^{n-k}$$

where $\Gamma_k$ is the unique formal group of height $k$. Also

$$\mathrm{Aut}(G) \cong \mathrm{Aut}(\Gamma_k) \times \mathrm{Gl}_{n-k}(\mathbb{Z}_p).$$

# Ordinary vs supersingular elliptic curves

Over $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) = p$, an elliptic curve $C$ is **ordinary** if $C_{\mathrm{for}}(p^\infty)$ has height 1. If it has height 2, $C$ is **supersingular**.

**Theorem**

*Over an algebraically closed field, there are only finitely many isomorphism classes of supersingular curves and they are all smooth.*

If $p > 3$, there is a modular form of $A$ of weight $p - 1$ so that $C$ is supersingular if and only if $A(C) = 0$.



# $p$-divisible groups in stable homotopy theory

Let $E$ be a $K(n)$-local periodic homology theory with associated formal group

$$\mathrm{Spf}(E^0\mathbb{C}\mathrm{P}^\infty)) = \mathrm{Spf}(\pi_0 F(\mathbb{C}\mathrm{P}^\infty, E)).$$

We have

$$F(\mathbb{C}\mathrm{P}^\infty, C) \cong \lim F(BC_{p^n}, E).$$

Then

$$G = \mathrm{colim}\, \mathrm{Spec}(\pi_0 L_{K(n-1)} F(BC_{p^n}, E))$$

is a $p$-divisible group with formal part

$$G_{\mathrm{for}} = \mathrm{Spf}(\pi_0 F(\mathbb{C}\mathrm{P}^\infty, L_{K(n-1)} E)).$$

# Moduli stacks

Define $\mathcal{M}_p(n)$ to be the moduli stack of $p$-divisible groups

1. of height $n$ and
2. with $\dim G_{\mathrm{for}} = 1$.

There is a morphism

$$\begin{aligned} \mathcal{M}_p(n) &\longrightarrow \mathcal{M}_{\mathbf{fg}} \\ G &\mapsto G_{\mathrm{for}} \end{aligned}$$

**Remark**

1. The stack $\mathcal{M}_p(n)$ is not algebraic, just as $\mathcal{M}_{\mathbf{fg}}$ is not. Both are “pro-algebraic”.
2. Indeed, since we are working over $\mathbb{Z}_p$ we have to take some care about what we mean by an algebraic stack at all.



# Some geometry

Let $\mathcal{V}(k) \subseteq \mathcal{M}_p(n)$ be the open substack of $p$-divisible groups with formal part of height $k$. We have a diagram

$$\begin{array}{ccccccc} \mathcal{V}(k-1) & \longrightarrow & \mathcal{V}(k) & \longrightarrow & \mathcal{M}_p(n) & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{U}(k-1) & \longrightarrow & \mathcal{U}(k) & \longrightarrow & \mathcal{U}(n) & \longrightarrow & \mathcal{M}_{\mathbf{fg}} \end{array}$$

1. the squares are pull backs;
2. $\mathcal{V}(k) - \mathcal{V}(k-1)$ and $\mathcal{U}(k) - \mathcal{U}(k-1)$ each have one geometric point;
3. in fact, these differences are respectively

$$B\,\mathrm{Aut}(\Gamma_k) \times B\mathrm{Gl}_{n-k}(\mathbb{Z}_p) \text{ and } B\,\mathrm{Aut}(\Gamma_k).$$

# Lurie's Theorem

**Theorem (Lurie)**

*Let $\mathcal{M}$ be a Deligne-Mumford stack of abelian group schemes. Suppose $G \mapsto G(p^\infty)$ gives a representable and formally étale morphism*

$$\mathcal{M} \longrightarrow \mathcal{M}_p(n).$$

*Then the realization problem for the composition*

$$\mathcal{M} \longrightarrow \mathcal{M}_p(n) \longrightarrow \mathcal{M}_{\mathbf{fg}}$$

*has a canonical solution. In particular, $\mathcal{M}$ is the underlying algebraic stack of derived stack.*

**Remark:** This is an application of a more general representability result, also due to Lurie.



# Serre-Tate and elliptic curves

Let $\mathcal{M}_{e\ell\ell}$ be the moduli stack of elliptic curves. Then

$$\mathcal{M}_{e\ell\ell} \longrightarrow \mathcal{M}_p(2) \qquad C \mapsto C(p^\infty)$$

is formally étale by the Serre-Tate theorem.

Let $C_0$ be an $\mathcal{M}$-object over a field $\mathbb{F}$, with $\mathrm{char}(\mathbb{F}) = p$. Let $q : A \to \mathbb{F}$ be a ring homomorphism with nilpotent kernel. A **deformation** of $C_0$ to $R$ is an $\mathcal{M}$-object over $A$ and an isomorphism $C_0 \to q^* C$. Deformations form a category $\mathbf{Def}_{\mathcal{M}}(\mathbb{F}, C_0)$.

**Theorem (Serre-Tate)**

*We have an equivalence:*

$$\mathbf{Def}_{e\ell\ell}(\mathbb{F}, C_0) \to \mathbf{Def}_{\mathcal{M}_p(2)}(\mathbb{F}, C_0(p^\infty))$$

## Topological modular forms

If $C$ is a singular elliptic curve, then $C_{\mathrm{sm}} \cong \mathbb{G}_m$ or

$$C_{\mathrm{sm}}(p^\infty) = \text{multiplicative formal group}$$

which has height 1, not 2. Thus

$$\mathcal{M}_{e\ell\ell} \longrightarrow \mathcal{M}_p(2)$$

doesn't extend over $\bar{\mathcal{M}}_{e\ell\ell}$; that is, the approach just outlined constructs **tmf**$[\Delta^{-1}]$ rather than **tmf**.

To complete the construction we could

1. handle the singular locus separately: "Tate $K$-theory is $E_\infty$"; and
2. glue the two pieces together.



## Higher heights

There are very few families of group schemes smooth of dimension 1. Thus we look for stackifiable families of abelian group schemes $A$ of higher dimension so that

- There is a natural splitting $A(p^\infty) \cong A_0 \times A_1$ where $A_0$ is a $p$-divisible group with formal part of dimension 1; and
- Serre-Tate holds for such $A$: $\mathbf{Def}_{A/\mathbb{F}} \simeq \mathbf{Def}_{A_0/\mathbb{F}}$.

This requires that $A$ support a great deal of structure; very roughly:

(E) $\mathrm{End}(A)$ should have idempotents; there is a ring homomorphism $B \to \mathrm{End}(A)$ from a certain central simple algebra;

(P) Deformations of $A(p^\infty)$ must depend only on deformations of $A_0$; there is a duality on $A$ – a **polarization.**

# Shimura varieties

Such abelian schemes have played a very important role in number theory.

**Theorem (Behrens-Lawson)**

*For each $n > 0$ there is a moduli stack $\mathrm{Sh}_n$ (a* **Shimura variety***) classifying appropriate abelian schemes equipped with a formally étale morphism*

$$\mathrm{Sh}_n \longrightarrow \mathcal{M}_p(n).$$

*In particular, the realization problem for the surjective morphism*

$$\mathrm{Sh}_n \to \mathcal{U}(n) \subseteq \mathcal{M}_{\mathbf{fg}}$$

*has a canonical solution.*

The homotopy global sections of the resulting sheaf of $E_\infty$-ring spectra is called **taf**: topological automorphic forms.