CATEGORIES AND TOPOLOGY 2016 (BLOCK 1) LECTURE NOTES, PART II

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1. Introduction

In the second part of the course we will cover two important topics in homotopy theory, namely homotopy (co)limits and localizations and completions of spaces.

The main source for both of these topics still remains the "yellow monster" [BK72] by Bousfield and Kan. Other useful references for homotopy colimits include:

- The paper [DS95] by Dwyer-Spalinsky is a an excellent reference for explaining the basic philosophy, and it does model categories as well.
- The notes by Dan Dugger "A primer on homotopy colimits", which can be found on his homepage http://pages.uoregon.edu/ddugger/, are also highly recommended.

Date: November 3, 2017. Notes in progress – use at own risk.

- The book by Goerss–Jardine [GJ99] is a more modern introduction to parts of the yellow monster.
- There is also the more recently published (but quite classical in its outlook) *More concise algebraic topology* [MP12] by May and Ponto (which at 514 pages should perhaps be called "Less concise algebric topology").
- Riehl's book [Rie14] gives a nice abstract account of homotopy limits and colimits — our discussion of derived functors is essentially taken from here (though much of it is originally due to [DHKS04]).

Some references for localization and completion[1] of spaces are:

- Sullivan's MIT notes [Sul05] is a good informal introduction to localization and completion.
- Another recommended introduction is Neisendorfer's notes [Nei09].
- There's also a lot of classical material summarized in May–Ponto [MP12].
- The simplicial viewpoint on localizations is explained in Bousfield-Kan [BK72].
- General existence of localizations with respect to a homology theory was proved by Bousfield in [Bou75].
- The book of Dror-Farjoun [DF92] explains in detail how to localize with respect to a map.
- A general version of the arithmetic square is given in [DDK77] (also correcting a point in Bousfield–Kan).
- The paper [Dwy04] by Dwyer contains a more advanced survey of localizations with many examples.

2. Homotopical Categories and Abstract Localizations

We want to define the homotopy colimit to be the "derived functor" of the usual colimit functor. In this section we consider a general setting in which we can make sense of this notion.

Definition 2.1. A *relative category* is a pair $(\mathcal{C}, W)$ where $\mathcal{C}$ is a category and W is a collection of "weak equivalences" in $\mathcal{C}$; more precisely, W is a wide[2] subcategory of $\mathcal{C}$ containing all isomorphisms. A *homotopical category* is a relative category where W satisfies the *2-out-of-3 property*: if any two out of f, g, gf are in W, then so is the third. We'll often leave the class of weak equivalences implicit and just talk about a homotopical category $\mathcal{C}$.

Remark 2.2. Sometimes it's better to assume the stronger *2-out-of-6* property: if we have composable maps $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$ where gf and hg are in W, then f, g, h and hgf are also in W. (Note that this is sort of saying that g has both a left and a right "inverse".) This is assumed in the definition of homotopical category in [Rie14] or [DHKS04], but I think we'll just need 2-of-3.

The next proposition says that we can always invert a set of morphisms in a small category:

Proposition 2.3 (Gabriel-Zisman)**.** *Suppose $(\mathcal{C}, W)$ is a relative category where $\mathcal{C}$ is small. Then there exists a small category $\mathcal{C}[W^{-1}]$ and a functor $L\colon \mathcal{C} \to \mathcal{C}[W^{-1}]$ with the universal*

[1]The discussion of p-completion here also benefitted greatly from the notes on *Rational and p-adic homotopy theory* by Thomas Nikolaus.

[2]I.e. W contains all the objects of $\mathcal{C}$.

property that L is initial among functors $\mathcal{C} \to \mathcal{D}$ to a small category $\mathcal{D}$ that take the morphisms in W to isomorphisms in $\mathcal{D}$.

Sketch Proof. Define a new (small) category $\mathcal{C}[W^{-1}]$ with objects the objects of $\mathcal{C}$ and morphisms given by formal zig-zags of maps where maps in W are allowed to go the wrong way, modulo the obvious identifications given by composition, removing identities, and cancelling $a \xrightarrow{w} b \xleftarrow{w} a$ and $b \xleftarrow{w} a \xrightarrow{w} b$ where w is in W. The construction is spelled out in [GZ67, §1.1]. □

If $(\mathcal{C}, W)$ is a homotopical category we'll often write $\mathrm{Ho}(\mathcal{C})$ for $\mathcal{C}[W^{-1}]$, leaving W implicit.

Remark 2.4. We would like apply this construction to categories such as Top and $\mathrm{Top}^{\mathcal{I}}$ that are not small. However, there is a set-theoretical issue involved in doing this, as there might be a proper class of formal zig-zags. We will generally ignore this issue, but to avoid consternation let us point out that we can avoid this issue by assuming that inside our set theory there is a sub-class of *small* sets that themselves form a model of set theory; then we can use the large (i.e. not necessarily small) sets instead of worrying about classes. (The precise version of this idea is called a *Grothendieck universe*.) We use the following terminology:

- a category is *small* if its set of objects and all its Hom-sets are small,
- a category of *locally small* if its Hom-sets are all small but its set of objects is (potentially) large,
- a category is *large* if its set of objects and its Hom-sets are all (potentially) large.

We can then take Set to be the (locally small) category of small sets, Top to be the (locally small) category of topological spaces whose underlying sets are small, etc.

The construction of $\mathcal{C}[W^{-1}]$ now works just as well for large categories, so we can for example define the homotopy category Ho(Top) by starting with the category Top with objects all topological spaces and then formally inverting the weak homotopy equivalences. Similarly, we can define the (bounded) derived category of R-modules by taking all bounded chain complexes over R and formally inverting the quasi-isomorphisms (homology isomorphisms). We can also construct homotopy categories of diagrams:

Definition 2.5. Suppose $\mathcal{I}$ is a small category and $(\mathcal{C}, W)$ is a (possibly large) homotopical category. Then $\mathcal{C}^{\mathcal{I}}$ is a homotopical category if we equip it with the *natural weak equivalences*, i.e. the weak equivalences $W_{\mathcal{I}}$ are those natural transformations $\eta \colon F \to G$ such that $\eta_i \colon F(i) \to G(i)$ is in W for all $i \in \mathcal{I}$. Then we can define $\mathrm{Ho}(\mathcal{C}^{\mathcal{I}})$ as $\mathcal{C}^{\mathcal{I}}[W_{\mathcal{I}}^{-1}]$.

Remark 2.6. There is a less formal set-theoretical issue with these localizations, however: although the categories we start with (Top, $\mathrm{Top}^{\mathcal{I}}$, etc.) are all locally small, a priori the localized categories are just large categories — the Hom-sets are not necessarily small. In practice, to work with these categories it is useful to know that the localizations are also locally small. We won't deal with this issue here, however, as it won't actually affect us. That said, local smallness of the homotopy category is part of the package you get from a *model structure*, and all the categories we'll consider do have model structures, so in any case there is nothing to worry about.

Remark 2.7. We stress right away that the category $\mathrm{Ho}(\mathcal{C}^{\mathcal{I}})$ is *not* the same as the category $\mathrm{Ho}(\mathcal{C})^{\mathcal{I}}$. $\mathrm{Ho}(\mathcal{C}^{\mathcal{I}})$ is a much richer category since diagrams are required to strictly commute, whereas $\mathrm{Ho}(\mathcal{C})^{\mathcal{I}}$ is generally too weak to be of much interest (see e.g. Example 2.18 below).

Another problem with $\mathrm{Ho}(\mathcal{C})^{\mathcal{I}}$ is that $\delta : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})^{\mathcal{I}}$ in general does not have an adjoint, i.e. limits and colimits do not exist in the homotopy category (see e.g. Example 2.17 below). If we instead work with $\mathrm{Ho}(\mathcal{C}^{\mathcal{I}})$ then δ often *does* have an adjoint, and this gives one definition of the homotopy colimit: the *homotopy colimit* will be the left adjoint of $\delta\colon \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C}^{\mathcal{I}})$.

Inside a homotopical category $\mathcal{C}$ we can often find full subcategories $\mathcal{C}'$ of "good" (or "cofibrant") objects such that $\mathrm{Ho}\,\mathcal{C}'$ is equivalent to $\mathrm{Ho}\,\mathcal{C}$, but where $\mathrm{Ho}\,\mathcal{C}'$ is simpler to describe, or some functor we're interested in is better-behaved on the objects in $\mathcal{C}'$. Making this precise gives the notion of *deformation*:

Definition 2.8. Let $\mathcal{C}$ be a homotopical category. A *left deformation* of $\mathcal{C}$ is a functor $Q\colon \mathcal{C} \to \mathcal{C}$ and a natural transformation $q\colon Q \to \mathrm{id}_{\mathcal{C}}$ such that $q_c\colon Qc \to c$ is a weak equivalence for all $c \in \mathcal{C}$. We write $\mathcal{C}_Q$ for the full subcategory of $\mathcal{C}$ spanned by the essential image of Q.

Remark 2.9. Note that by the 2-of-3 property Q takes weak equivalences to weak equivalences.

Remark 2.10. Of course there is a dual notion of *right* deformation.

Lemma 2.11. *Suppose (Q, q) is a left deformation of a homotopical category $\mathcal{C}$. If $\mathcal{C}'$ is a full subcategory containing the image of Q (for instance $\mathcal{C}_Q$), then the functor $i\colon \mathrm{Ho}\,\mathcal{C}' \to \mathrm{Ho}\,\mathcal{C}$ induced by the inclusion is an equivalence of categories.*

Sketch Proof. Since the natural map $q_c\colon Qc \to c$ is a weak equivalence, and hence an isomorphism in $\mathrm{Ho}\,\mathcal{C}$, the functor i is essentially surjective. Suppose x and y are objects of $\mathcal{C}'$, then we need to prove that $\mathrm{Hom}_{\mathrm{Ho}\,\mathcal{C}'}(x, y) \to \mathrm{Hom}_{\mathrm{Ho}\,\mathcal{C}}(ix, iy)$ is a bijection. Using q we can replace any zig-zag of morphisms from x to y in $\mathcal{C}$ by an equivalent zig-zag in $\mathcal{C}_Q \subseteq \mathcal{C}'$, so this map is surjective. Similarly if two zig-zags become equivalent in $\mathcal{C}$ we can use q to show they are equivalent also in $\mathcal{C}_Q$, giving injectivity. □

For example, if $\mathcal{C}$ is Top we can take $\mathcal{C}_Q$ to be the full subcategory of CW-complexes:

Lemma 2.12. $\mathrm{Ho}(\mathrm{Top})$ *is equivalent to* hCW*, the category with objects CW-complexes and morphisms homotopy classes of maps.*

Proof. Take Q to be $|\mathrm{Sing}(-)|$, q to be the counit for the adjunction $|-| \dashv \mathrm{Sing}$, and $\mathcal{C}'$ to be CW, then the hypotheses in the previous Lemma are satisfied. Thus $\mathrm{Ho}(\mathrm{CW}) \xrightarrow{\sim} \mathrm{Ho}(\mathrm{Top})$. By Whitehead's theorem the weak homotopy equivalences between CW-complexes are the homotopy equivalences, so it remains to show that hCW is the localization of CW at the homotopy equivalences. [Exercise: Prove this, i.e. check that the homotopy category satisfies the universal property. Hint: $X \times I \to X$ is a homotopy equivalence with several homotopy inverses.] □

Similarly, if $\mathcal{C}$ is $\mathrm{Ch}_R^{\geq 0}$ we can take $\mathcal{C}'$ to be the full subcategory $\mathrm{Proj}_R^{\geq 0}$ of chain complexes of projective modules and show that $\mathrm{Ho}\,\mathrm{Ch}_R^{\geq 0} \simeq \mathrm{hProj}_R^{\geq 0}$ where the right-hand side means we take chain homotopy equivalences of maps.

We will make use of deformations for general diagram categories later. For now we will just describe how such replacements can work in some fundamental examples:

Proposition 2.13. *Let $\mathcal{I} = (0 \leftarrow 1 \to 2)$. Then $\mathrm{Ho}(\mathrm{Top}^{\mathcal{I}})$ is equivalent to the category with objects diagrams of CW-complexes and CW-complex inclusions, and morphisms homotopy classes of maps between diagrams.*

We need a technical lemma that will be useful again later. Recall that we call a map a *cofibration* (or sometimes Hurewicz or h–cofibration), if it has the homotopy extension property (cf. [Hat02, Ch. 0]).

Lemma 2.14. *Suppose given a commutative square*

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ X' & \overset{i'}{\hookrightarrow} & Y' \end{array}$$

of topological spaces, where i and i' are cofibrations and f and g are homotopy equivalences. Then every homotopy inverse of f can be extended to a homotopy inverse of g.

Proof. See [May99, §6.5]. □

Proof of Proposition 2.13. Any diagram is weakly equivalent to a diagram of this form: The map $|\operatorname{Sing}_\bullet(X)| \to X$ allows us to replace our diagram by a diagram of CW-complexes. Then turn the two maps into CW-inclusions using mapping cylinders. We conclude that for all diagrams X there exist a diagram of CW–complexes and CW-inclusions QX, and a natural weak equivalence $QX \to X$. By Lemma 2.11 this means inverting the weak equivalences in $\mathrm{Top}^{\mathcal{I}}$ is equivalent to inverting them in the subcategory of such diagrams. It remains to show that a weak equivalence $\phi\colon F \to G$ between two of these is a homotopy equivalence. By Whitehead's Theorem the maps $F(i) \to G(i)$ are all homotopy equivalences, so we need to show we can choose a natural homotopy inverse. This follows from Lemma 2.14. It then follows by the same argument as for Top that inverting homotopy equivalences is equivalent to taking homotopy classes of maps. □

As we mentioned already, the categories $\mathrm{Ho}(\mathcal{C}^{\mathcal{I}})$ and $\mathrm{Ho}(\mathcal{C})^{\mathcal{I}}$ are quite different in general. We can now give an explicit example of this:

Example 2.15. With $\mathcal{I}$ as above, we have

$$\mathrm{Hom}_{\mathrm{Ho(Top)}^{\mathcal{I}}}([D^n \leftarrow S^{n-1} \to D^n], \delta S^n) = 0$$

whereas

$$\mathrm{Hom}_{Ho(\mathrm{Top}^{\mathcal{I}})}([D^n \leftarrow S^{n-1} \to D^n], \delta S^n) = \mathbb{Z}$$

[Exercise: Try to verify this: The first claim is easy to verify. The second claim takes a bit more work, but follows later when we get our model for homotopy colimit.]

The next example is also good to keep in mind, but we will not give a full proof here:

Proposition 2.16. *For G a (finite) group, let $\mathcal{B}G$ denote the category with one object and G as morphisms. Then Top^{BG} is the category of spaces with a G-action, and $\mathrm{Ho}(Top^{BG})$ is the category where we invert G-maps that are ordinary weak equivalences. This is equivalent to the category with objects free G-CW-complexes and morphisms G-homotopy equivalence classes of maps.*

Proof Sketch. We'll not do the full proof here, but the idea is analogous to the previous proposition. As before we can functorially replace any space by a CW-complex. Now consider the space EG, the classifying space of the category with objects the elements of G and exactly one morphism between any two objects. This space is contractible and naturally carries a free G-action induced by the translation action on the category. Hence the map

$$QX := EG \times |\operatorname{Sing}_\bullet(X)| \to X$$

is a weak equivalence in Top^{BG} and replaces X by a CW-complex QX with a free G–action.

We would now like to see that we can find a G–map

$$\begin{array}{ccc} & & X \\ & \nearrow & \big\downarrow{\scriptstyle\sim} \\ QY & \xrightarrow{\sim} & Y \end{array}$$

making the diagram commute up to G–homotopy. This is easy with a bit of equivariant homotopy theory, but we will not give the details here (see e.g. [Bre67]). □

Example 2.17. Limits and colimits do not exist in the homotopy category $\mathrm{Ho}(\mathcal{C})^{\mathcal{I}}$ in general, except in very special cases like products and coproducts. Let's for instance see that $* \to K(\mathbb{Z},3) \xleftarrow{p} K(\mathbb{Z},3)$ does not have a limit in the homotopy category. The diagram has a limit if and only if the functor $F(-) = \ker(H^3(-;\mathbb{Z}) \xrightarrow{p} H^3(-;\mathbb{Z}))$ is representable in the form $[-,P]$ for some P. However, F is not exact in the middle on cofibration sequences so this is impossible.

[Exercise: Construct a concrete example. Hint: Try to realize the sequence $\mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^2$ on H^3.]

Example 2.18. If $\mathcal{I} = G$, then an object in $\mathrm{Ho}(\mathrm{Top})^{\mathcal{B}G}$ is a topological space X and a homomorphism $G \to [X,X]^{inv}$, where an element in $\mathrm{Ho}(\mathrm{Top}^{\mathcal{B}G})$ is a group action, up to homotopy. These are two different notions. An example to keep in mind is when $X = BH$. Then the genuine group actions would correspond to extensions $H \to ? \to G$, whereas the other notion would just be a map $G \to Out(H)$. There are both existence and uniqueness obstructions to this being a bijection even on objects.

[Exercise: Prove the claims in this example! There are quite a few things to check...]

3. Derived Functors via Deformations

There is an obvious natural notion of morphism between homotopical categories:

Definition 3.1. Suppose $\mathcal{C}$ and $\mathcal{D}$ are homotopical categories. We say a functor $F\colon \mathcal{C} \to \mathcal{D}$ is *homotopical* if it takes weak equivalences in $\mathcal{C}$ to weak equivalences in $\mathcal{D}$. A homotopical functor induces a unique functor $\mathrm{Ho}\,\mathcal{C} \to \mathrm{Ho}\,\mathcal{D}$ (which we also call F) that fits in a commutative square

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \big\downarrow & & \big\downarrow \\ \mathrm{Ho}\,\mathcal{C} & \xrightarrow{F} & \mathrm{Ho}\,\mathcal{D}. \end{array}$$

However, many functors we are interested in are *not* homotopical — this is why we want derived functors.

Definition 3.2. If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor, the *total left derived functor* $\mathbb{L}F$ of F is the best approximation to F from the left by a homotopical functor. More precisely, $\mathbb{L}F$ is the functor (unique up to unique isomorphism if it exists) satisfying:

(1) there is a natural transformation $\lambda\colon \mathbb{L}F \to F$,
(2) $\mathbb{L}F$ is homotopical,
(3) every natural transformation $\eta\colon G \to F$ where G is homotopy-invariant factors uniquely through λ.

Definition 3.3. A *left derived functor* of F is a homotopical functor $F' \colon \mathcal{C} \to \mathcal{D}$ together with a natural transformation $F' \to F$ such that the induced functor $L_{\mathcal{D}} \circ F \colon \mathcal{C} \to \operatorname{Ho}\mathcal{D}$ and natural transformation $L_{\mathcal{D}}F' \to L_{\mathcal{D}}F$ is a total left derived functor.

The homotopy colimit functor we'll construct will be a left derived functor of the colimit functor.

Remark 3.4. There is a natural dual notion of *right* derived functors. Given a random functor, you might wonder whether we should care about its left or right derived functor (assuming both exist, which is rare). Vaguely speaking, we typically want left derived functors of left adjoints and right derived functors of right adjoints — for one thing, as we'll see later, we then get derived adjunctions.

We can use left deformations to construct left derived functors, provided the deformation is compatible with the functor in the following sense:

Proposition 3.5 (Dwyer–Hirschhorn–Kan–Smith, cf. [Rie14, Theorem 2.2.8])**.** *Suppose given a functor $F \colon \mathcal{C} \to \mathcal{D}$. If (Q, q) is a left deformation of $\mathcal{C}$ such that*

(1) FQ is homotopical,

(2) $Fq_{Qc} \colon FQQc \to FQc$ is a weak equivalence for all c,

then (FQ, Fq) is a left derived functor of F.

Proof. We must show that given any homotopical functor $G \colon \mathcal{C} \to \operatorname{Ho}\mathcal{D}$ equipped with a natural transformation $\gamma \colon G \to L_{\mathcal{D}}F$, the natural transformation γ factors uniquely through $L_{\mathcal{D}}Fq \colon L_{\mathcal{D}}FQ \to L_{\mathcal{D}}F$. To see that such a factorization exists, consider the commutative diagram

$$\begin{array}{ccc} GQ & \xrightarrow{\gamma_Q} & L_{\mathcal{D}}FQ \\ \downarrow{\scriptstyle Gq} & & \downarrow{\scriptstyle L_{\mathcal{D}}Fq} \\ G & \xrightarrow{\gamma} & L_{\mathcal{D}}F. \end{array}$$

Here the left vertical arrow is an isomorphism, since q is a natural weak equivalence and G is homotopical. Thus γ factors as $L_{\mathcal{D}}Fq \circ \bar{\gamma}$ where $\bar{\gamma} := \gamma_Q \circ (Gq)^{-1}$.

Now suppose given any factorization of γ as $L_{\mathcal{D}}Fq \circ \gamma'$. We first show that $\gamma'_Q = \bar{\gamma}_Q$; to see this consider the diagram

$$\begin{array}{ccc} GQ & \xrightarrow{\gamma'_Q} & L_{\mathcal{D}}FQQ \\ & \searrow^{\gamma_Q} & \downarrow{\scriptstyle L_{\mathcal{D}}Fq_Q} \\ & & L_{\mathcal{D}}FQ. \end{array}$$

Here $L_{\mathcal{D}}Fq_Q$ is an isomorphism by assumption, so $\gamma'_Q = (L_{\mathcal{D}}Fq_Q)^{-1} \circ \gamma_Q$. This holds for any γ', so in particular $\gamma'_Q = \bar{\gamma}_Q$.

Next consider the commutative square

$$\begin{array}{ccc} GQ & \xrightarrow{\gamma'_Q} & L_{\mathcal{D}}FQQ \\ \downarrow{\scriptstyle Gq} & & \downarrow{\scriptstyle L_{\mathcal{D}}FQq} \\ G & \xrightarrow{\gamma'} & L_{\mathcal{D}}FQ. \end{array}$$

This implies $\gamma' = (L_{\mathcal{D}}Fq) \circ \gamma'_Q \circ (Gq)^{-1} = (L_{\mathcal{D}}Fq) \circ \bar{\gamma}_Q \circ (Gq)^{-1} = \bar{\gamma}$, as required. □

4. Recollections on Limits and Colimits

Before we get to homotopy (co)limits we begin by recalling the definition of ordinary (co)limits:

Definition 4.1. Let $\mathcal{I}$ be a small category and $\mathcal{C}$ a locally small category (i.e. its Hom's are all small sets); we write $\mathcal{C}^{\mathcal{I}}$ (or sometimes $\mathrm{Fun}(\mathcal{I},\mathcal{C})$) for the category of functors from $\mathcal{I}$ to $\mathcal{C}$. (Since $\mathcal{I}$ is small this is again locally small.) We write δ for the constant-diagram functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{I},\mathcal{C})$.

- The *limit* of a functor $F\colon \mathcal{I} \to \mathcal{C}$ is an object $\lim F$ in $\mathcal{C}$ together with a morphism $\delta(\lim F) \to F$ in $\mathcal{C}^{\mathcal{I}}$ such that the induced map
$$\mathrm{Hom}_{\mathcal{C}}(X, \lim F) \to \mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}(\delta X, \delta \lim F) \to \mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}(\delta X, F)$$
is an isomorphism for every $X \in \mathcal{C}$.
- The *colimit* of an object $F \in \mathcal{C}^{\mathcal{I}}$ is an object $\operatorname{colim} F \in \mathcal{C}$ together with a morphism $F \to \delta \operatorname{colim} F$ such that the induced map
$$\mathrm{Hom}_{\mathcal{C}}(\operatorname{colim} F, X) \to \mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}(\delta \operatorname{colim} F, \delta X) \to \mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}(F, \delta X)$$
is an isomorphism for every $X \in \mathcal{C}$.

Remark 4.2. The problem with colimits in the context of homotopy theory can already be seen from this definition, where we are making "hard" identifications. The role of homotopy colimits will be replacing these hard identifications with soft identifications.

Remark 4.3. We make some elementary remarks about limits and colimits:

- For a given functor F the (co)limit of F may or may not exist. But if it exists it is unique up to unique isomorphism.
- The limit of F exists if and only if the functor $\mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}(\delta(-), F)$ is representable. (And dually for the colimit.)
- If for a fixed $\mathcal{I}$ the limit of every functor $F \in \mathcal{C}^{\mathcal{I}}$ exists in $\mathcal{C}$, then lim by definition gives a functor *right adjoint* to δ. Similarly if all colimits exist δ is the *left adjoint* to δ.
- We say a category $\mathcal{C}$ is *(co)complete* if all (co)limits indexed by any small category $\mathcal{I}$ exist. Many categories $\mathcal{C}$ we have in mind (Set, Top, Ab,...) are both complete and cocomplete.

Proposition 4.4 (A model for colimit)**.** *Let $\mathcal{C}$ be a category with all coproducts and coequalizers. Then $\mathcal{C}$ has all colimits, and for $F \in \mathcal{C}^I$,*
$$\operatorname{colim} F \cong \operatorname{coeq}\left(\coprod_{(f\colon i \to j) \in \mathrm{Mor}(\mathcal{I})} F(i) \rightrightarrows \coprod_{k \in \mathrm{Ob}(\mathcal{I})} F(k) \right).$$
where the two morphisms in the coequalizer are given on the factor indexed by $f\colon i \to j$ by $F(i) \xrightarrow{\mathrm{id}} F(i) \to \coprod_k F(k)$ and $F(i) \xrightarrow{F(f)} F(j) \to \coprod_k F(k)$, respectively.

Remark 4.5. If $\mathcal{C}$ is Set or Top we can describe this coequalizer as the quotient $\coprod_{i \in \mathcal{I}} F(I)/\sim$ where $\sim$ is the equivalence relation generated by identifying $x \in F(i)$ with $f(x) \in F(j)$ for all morphisms $f\colon i \to j$ in $\mathcal{I}$ and all $x \in F(i)$.

Proof. Verify that it satisfies the universal property — if you map out of this, the universal property for coequalizers and coproducts gives you the standard description of limits in Set. □

Exercise 4.6. Make the above formula explicit in the categories Set, Set_*, Ab, Grp, and Ring.

Example 4.7. We give some elementary examples of limits and colimits in abelian groups:

- $\lim \begin{pmatrix} & A \\ & \downarrow f \\ 0 \longrightarrow & B \end{pmatrix} = \ker(f)$.
- $\operatorname{colim} \begin{pmatrix} & A \\ & \downarrow f \\ 0 \longrightarrow & B \end{pmatrix} = B$.
- For $\mathcal{I} = \mathcal{B}G$, a group G considered as a category with one object, and $\mathcal{C} = \text{Ab}$, $\lim_{\mathcal{B}G} M \cong M^G$, the invariants, and $\operatorname{colim}_{\mathcal{B}G} M \cong M_G$, the coinvariants.
- $\lim(\cdots \to \mathbb{Z}/p^3 \to \mathbb{Z}/p^2 \to \mathbb{Z}/p) \cong \mathbb{Z}_p$, the p-adic integers. The colimit is $\mathbb{Z}/p$.
- $\operatorname{colim}(\mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p^3 \to \cdots) = \mathbb{Z}/p^\infty$, the injective envelope of $\mathbb{Z}/p$. The limit is $\mathbb{Z}/p$.

Let's make the problems with homotopy limits and colimits explicit:

Example 4.8. Limits and colimits have (at least) two related problems:

- In abelian groups, limits and colimits are in general *not exact functors*. E.g. the sequence $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$ is exact, but taking fixed-points under the -1 action, gives only a left-exact sequence $0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \to \mathbb{Z}/2$. This leads to homological algebra.
- In topological spaces (or simplicial sets), taking limits or colimits is in general *not homotopy invariant*. E.g. $\operatorname{colim}(* \leftarrow S^{n-1} \to D^n) \cong S^n$ but $\operatorname{colim}(* \leftarrow S^n \to *) \cong *$. This leads to homotopical algebra.

5. The Homotopy Colimit Construction

In this section we will give a definition of the homotopy colimit for functors to Top as a certain explicit construction, and look at some examples. Later we will justify this construction by proving that it is a left derived functor of the colimit functor, i.e.

(1) the functor hocolim is homotopy-invariant in the sense that it takes natural weak homotopy equivalences to weak homotopy equivalences in Top (and so induces a functor $\text{Ho}(\text{Top}^{\mathcal{I}}) \to \text{Ho}(\text{Top})$),
(2) there is a natural transformation hocolim $\to$ colim,
(3) for any homotopy-invariant functor $F\colon \text{Top}^{\mathcal{I}} \to \text{Ho}(\text{Top})$ with a natural transformation $\eta\colon F \to \operatorname{colim}$, there is a unique factorization of η as $F \to \operatorname{hocolim} \to \operatorname{colim}$.

We'll also see that $\operatorname{hocolim}\colon \text{Ho}(\text{Top}^{\mathcal{I}}) \to \text{Ho}\,\text{Top}$ is the left adjoint to the constant diagram functor $\text{Ho}\,\text{Top} \to \text{Ho}(\text{Top}^{\mathcal{I}})$.

Remark 5.1. Just because we abstractly know that hocolim is the "best" approximation to colim in the sense above doesn't mean that it is very good in concrete cases — it may be quite far from the actual colim. The homotopy colimit of a point under the action of a finite group is for example the infinite dimensional space $\text{B}G$.

Definition 5.2. Recall that if $X_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Top}$ is a simplicial space, then the *geometric realization* $|X_\bullet|$ is the coequalizer

$$\mathrm{coeq}\left(\coprod_{\phi\colon [n]\to[m]} X_n \times \Delta^m \rightrightarrows \coprod_k X_k \times \Delta^k\right)$$

where the two maps are given on the factor indexed by ϕ by $X_n \times \Delta^m \xrightarrow{\phi^*\times \mathrm{id}} X_m \times \Delta^m$ and $X_n \times \Delta^m \xrightarrow{\mathrm{id}\times\phi_*} X_n \times \Delta^n$. More explicitly, this is the quotient space obtained from $\coprod_n X_n \times \Delta^n$ by identifying $(\phi^*(x), y) \in X_n \times \Delta^n$ with $(x, \phi_* y) \in X_m \times \Delta^m$ for all $\phi\colon [n] \to [m]$, $x \in X_m$, $y \in \Delta^m$.

Remark 5.3. Compare this expression to the expression for the colimit of $X_\bullet$ that we saw earlier as

$$\mathrm{coeq}\left(\coprod_{\phi\colon [n]\to[m]} X_n \rightrightarrows \coprod_k X_k\right).$$

Thus the geometric realization is a "fattened" version of the colimit of $X_\bullet$ where we've stuck in some contractible spaces.

Definition 5.4. Let $F\colon \mathcal{I} \to \mathrm{Top}$ be a functor. We then define a simplicial space $F^\Delta_\bullet$ by $F^\Delta_n := \coprod_{i_0\to\cdots\to i_n} F(i_0)$; the map $\phi^*\colon F^\Delta_n \to F^\Delta_m$ for $\phi\colon [m] \to [n]$ in $\mathbf{\Delta}$ is given on the component indexed by $i_0 \to \cdots \to i_n$ by $F(i_0 \to \cdots \to i_{\phi(0)})\colon F(i_0) \to F(i_{\phi(0)})$ where the target is in the component indexed by $i_{\phi(0)} \to i_{\phi(1)} \to \cdots \to i_{\phi(m)}$. The *homotopy colimit* $\mathrm{hocolim}\, F$ of F is then defined to be $|F^\Delta_\bullet|$. More explicitly, we can unwind this definition to see that $\mathrm{hocolim}\, F$ is the quotient space

$$\mathrm{hocolim}\, F = \coprod_n \coprod_{i_0\to i_1\cdots\to i_n} F(i_0) \times \Delta^n / \sim$$

where $\sim$ is generated by the usual identities $(d_i x, t) \sim (x, d^i t)$ and $(s_i x, t) \sim (x, s^i t)$, with $x \in \coprod_{i_0\to i_1\cdots\to i_n} F(i_0)$ and $d_i x$ defined the "obvious way" i.e.,

$$d_0(i_0 \xrightarrow{k} \imath_1 \cdots \to i_n, x \in F(i_0)) = (i_1 \to \cdots \to i_n, k(x) \in F(i_1))$$

$$d_j(i_0 \to i_1 \cdots \to \cdots \to i_n, x \in F(i_0)) = (i_0 \to \cdots \hat{i_j} \to \cdots \to i_n, x \in F(i_0)) \text{ for } j \neq 0.$$

Remark 5.5. The colimit of F^Δ is easily seen to be the same as the colimit of F. (Check by a cofinality argument that a simplicial colimit can be replaced by a coequalizer — this gives the expression for the colimit of F as a coequalizer in Proposition 4.4.) So we form the homotopy colimit of F by taking a "fattened" version of the colimit of F^Δ instead of the ordinary colimit.

Remark 5.6. We will see later that the homotopy colimit of a simplicial diagram is often the same as its geometric realization.

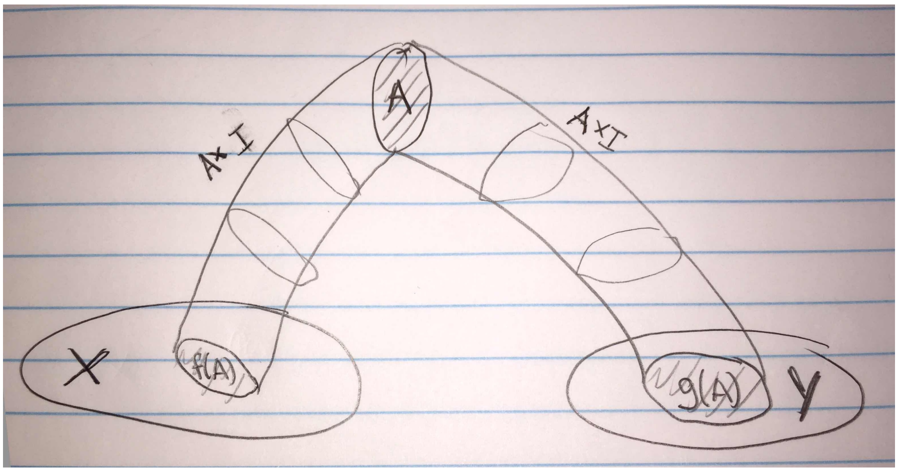
Now let us look at some examples:

Example 5.7. Suppose $F\colon \mathcal{I} \to \mathrm{Top}$ is the constant functor with value $*$. Then F^Δ is precisely the nerve $\mathrm{N}\mathcal{I}$ (viewed as a discrete simplicial space) and so $\mathrm{hocolim}_{\mathcal{I}}\, F$ is the classifying space $\mathrm{B}\mathcal{I}$. More generally if F is constant with value X then $\mathrm{hocolim}_{\mathcal{I}}\, F \cong X \times \mathrm{B}\mathcal{I}$.

Example 5.8. Consider the pushout diagram: $X \stackrel{f}{\leftarrow} A \stackrel{g}{\rightarrow} Y$. The definition reveals that the homotopy colimit is a double mapping cylinder construction. In more details:

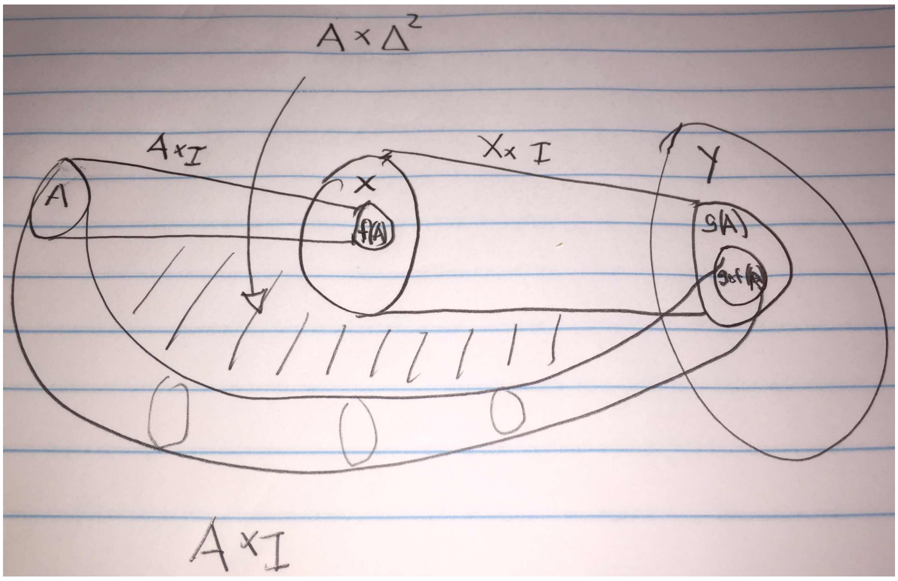
$$\text{hocolim} = ((X \coprod A \coprod Y) \coprod (A \times I \coprod A \times I))/\sim$$

where $\sim$ is as in the following picture:



To spell this out, the left $A \times I$, $A \times 0$ is glued to A and $A \times 1$ is glued to X via f, i,e, $A \times I \ni (a, 1) \sim f(a) \in X$ in the right $A \times I$, $A \times 0$ is glued to A and $A \times 1$ is glued to Y via $A \times I \ni (a, 1) \sim g(a) \in Y$

Example 5.9. The diagram $A \to X \to Y$.



Exercise 5.10. Check that the homotopy colimit of a group G acting on a space X is homotopy equivalent to the so-called Borel construction, i.e.

$$\operatorname*{hocolim}_G X \cong (X \times \mathrm{E}G)/G.$$

It is easy to see that property (1) for a derived functor holds for our definition of hocolim, i.e. there is a natural map from the homotopy colimit to the colimit:

Lemma 5.11. *There is a natural transformation* hocolim $\to$ colim*.*

Proof. If $X_\bullet$ is any simplicial space, there's a natural map $|X| \to \operatorname{colim} X$: There is an obvious map of diagrams

$$\begin{array}{ccc} \coprod_{\phi\colon [m]\to[n]} X_n \times \Delta^m & \rightrightarrows & \coprod_k X_k \times \Delta^k \\ \downarrow & & \downarrow \\ \coprod_{\phi\colon [m]\to[n]} X_n & \rightrightarrows & \coprod_k X_k, \end{array}$$

where the vertical maps are given by projecting off the Δ^n-factors. This induces a map on colimits (coequalizers) which is $|X| \to \operatorname{colim}_{\mathbf{\Delta}^{\mathrm{op}}} X$. Applying this to $F^{\Delta}_\bullet$ we get a natural map $\operatorname{hocolim}_{\mathcal{I}} F = |F^{\Delta}_\bullet| \to \operatorname{colim}_{\mathbf{\Delta}^{\mathrm{op}}} F^{\Delta} \cong \operatorname{colim}_{\mathcal{I}} F$. □

Another special case of interest is the case where the indexing category has a terminal object:

Proposition 5.12. *Let $\mathcal{I}$ be a category with a terminal object t. Then for any functor $F\colon \mathcal{I} \to \mathrm{Top}$, the natural map $\operatorname{hocolim}_{\mathcal{I}} F \to \operatorname{colim}_{\mathcal{I}} F \cong F(t)$ is a weak equivalence.*

Remark 5.13. We have already seen a special case of this situation in Example 5.9, and looking at the associated picture should give a good idea of why this is true. The formal proof is just like the proof that a nerve of a category with a terminal object is contractible, but now just for simplicial spaces keeping an extra space around[3].

Proof. We want to make a deformation retraction onto $F(t)$, i.e. a homotopy

$$\operatorname*{hocolim}_{\mathcal{I}} F \times I \to \operatorname*{hocolim}_{\mathcal{I}} F$$

contracting onto $F(t)$. Note that since geometric realizations preserve products[4] we can describe $\operatorname{hocolim}_{\mathcal{I}} F \times I$ as the realization of the simplicial space $F^{\Delta} \times \Delta^1$, which can be described as

$$(F^{\Delta} \times \Delta^1)_n := \coprod_{(i_0,0)\to\cdots(i_k,0)\to(i_k,1)\to\cdots(i_n,1)} F(i_0),$$

where we are also allowing only 0s or only 1s. (In other words the simplices correspond to elements of the nerve of the category $\mathcal{I} \times [1]$.)

We now define a map of simplicial spaces $F^{\Delta} \times \Delta^1 \to F^{\Delta}$ as follows: On the subset of sequences starting with $(i_0, 0)$ we map

$$F(i_0)_{(i_0,0)\to\cdots(i_k,0)\to(i_k,1)\to\cdots\to(i_n,1)}$$

$$F(i_0)_{i_0\to\cdots i_k\to t\to\cdots\to t}$$

by the identity, and on the sequences starting with $(i_0, 1)$ we send

$$F(i_0)_{(i_0,1)\to\cdots\to(i_n,1)}$$

[3]Note that this proof goes by defining a functor $\mathcal{I} \times [1] \to \mathcal{I}$ as the identity on $\mathcal{I} \times 0$, constant t on $\mathcal{I} \times 1$, and sending the morphism $(i, 0) \to (i, 1)$ to the unique map $i \to t$. Upon realization this gives the wanted homotopy. To see the relationship with the simplicial space case, write down what this map is on an arbirary simplex.)

[4]Technically this requires using the product of compactly generated weak Hausdorff spaces; the proof is essentially the same as for realization of simplicial sets.

to $F(t)_{t\to\cdots\to t}$ via the map induced by $i_0 \to t$.

This is easily checked to be a map of simplicial spaces, and induces the desired deformation retract on realizations. □

Remark 5.14. The above proof can be generalized to the situation called "having an extra degeneracy"; this is explained in homework.

We can use this to show how the homotopy colimit can be viewed as replacing the diagram by a homotopy equivalent "good" diagram, and then taking the ordinary colimit, generalizing the mapping cylinder and the examples in the beginning of the section.

Definition 5.15. For $F\colon \mathcal{I} \to \mathrm{Top}$, we define functor $QF\colon \mathcal{I} \to \mathrm{Top}$ by $(QF)(i) := \operatorname{hocolim}_{\mathcal{I}_{/i}} F$. The natural maps $\operatorname{hocolim}_{\mathcal{I}_{/i}} F \to \operatorname{colim}_{\mathcal{I}_{/i}} F \xleftarrow{\sim} F(i)$ give a natural transformation $QF \to F$, and by Proposition 5.12 the map $QF(i) \to F(i)$ is a weak equivalence for all i.

Proposition 5.16 (The deformation QF)**.** *For any functor F, we have a natural isomorphism* $\operatorname{colim}_{\mathcal{I}} QF = \operatorname{hocolim}_{\mathcal{I}} F$.

Sketch Proof. We unravel the definitions, using the notation $i_0 \to \cdots \to i_n \downarrow i$ for an n–fold composition in $\mathcal{I}_{/i}$:

$$\begin{aligned}
\operatorname*{colim}_{\mathcal{I}} QF &= \operatorname*{colim}_{i\in\mathcal{I}}\Big(\Big(\coprod_{i_0\to\cdots\to i_n\downarrow i} F(i_0)\times\Delta^n\Big)/\sim\Big)\\
&= \Big(\coprod_{i\in\mathcal{I}}\Big(\Big(\coprod_{i_0\to\cdots\to i_n\downarrow i} F(i_0)\times\Delta^n\Big)/\sim\Big)\Big)/\approx\\
&= \Big(\coprod_{i_0\to\cdots\to i_n} F(i_0)\times\Delta^n\Big)/\sim\\
&= \operatorname*{hocolim}_{\mathcal{I}} F
\end{aligned}$$

where $\sim$ is the equivalence relations coming from the simplicial identities and $\approx$ is the equivalence relation coming from identifying $f : j \to k$ with $id : j \to j$, for all morphisms f, which has the effect of canonically identifying the factor indexed on $i_0 \to \cdots \to i_n \downarrow i$ with the factor indexed on $i_0 \to \cdots \to i_n \downarrow i_n$. □

We'll see a slick proof of this later, when we've introduced the two-sided bar construction.

6. Homotopy Invariance of the Homotopy Colimit

Our goal in this section is to prove the homotopy invariance of the homotopy colimit. More precisely, we'll show:

Theorem 6.1. *Suppose F and F' are functors $\mathcal{I} \to \mathrm{Top}$ and $\eta\colon F \to F'$ is a natural transformation such that*

(1) for every $i \in \mathcal{I}$ the spaces $F(i)$ and $F'(i)$ are CW-complexes,
(2) for every $i \in \mathcal{I}$ the morphism $\eta_i\colon F(i) \to F'(i)$ is a weak homotopy equivalence (and hence a homotopy equivalence by (1) and Whitehead's Theorem).

Then the induced map $\operatorname{hocolim}_{\mathcal{I}} F \to \operatorname{hocolim}_{\mathcal{I}} F'$ *is a weak equivalence.*

Remark 6.2. In fact, the assumption that the functors take values in CW-complexes can be dropped, but to prove this requires more point set topology than we want to get into here. See [DI04, App. A] for a proof.

Remark 6.3. Recall that the inclusion functor $\mathrm{Ho}(\mathrm{CW}) \to \mathrm{Ho}(\mathrm{Top})$ is an equivalence of categories, where the weak equivalences in each case are the weak homotopy equivalence, which for CW are the same as the homotopy equivalences by Whitehead's theorem. We can also pass functorially between CW on and Top on a point-set level: since $|\mathrm{Sing}_\bullet(-)|$ is a functorial CW-replacement functor, given $F \in \mathrm{Top}^{\mathcal{I}}$, the functor $|\mathrm{Sing}_\bullet(F)|$ is in $\mathrm{CW}^{\mathcal{I}}$ and $|\mathrm{Sing}_\bullet(F)| \to F$ is a natural weak equivalences. In other words, $|\mathrm{Sing}_\bullet(-)|$ is a left deformation for $\mathrm{Top}^{\mathcal{I}}$. By Lemma 2.11 this means that $\mathrm{Ho}(\mathrm{CW}^{\mathcal{I}})$ and $\mathrm{Ho}(\mathrm{Top}^{\mathcal{I}})$ are isomorphic, where we invert the natural weak equivalences in both. In $\mathrm{CW}^{\mathcal{I}}$ this is equivalent to inverting maps that are *objectwise* homotopy equivalences, but note that we have *not* shown that we can choose a *natural* homotopy inverse — in fact this is *false*, this requires much stronger assumptions on the diagram than just being objectwise given by CW-complexes.

We will deduce Theorem 6.1 from a general result on homotopy invariance for the geometric realization of simplicial spaces. To state it we need some definitions:

Definition 6.4. If $X_\bullet$ is a simplicial space, its nth *latching object* L_nX is the colimit of the composite functor

$$(\mathbf{\Delta}^{\mathrm{surj}}_{[n]/})^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{X} \mathrm{Top},$$

where $\mathbf{\Delta}^{\mathrm{surj}}$ is the subcategory of $\mathbf{\Delta}$ with only surjective maps, i.e. L_nX is the colimit of X_i over all the degeneracies $[n] \twoheadrightarrow [i]$.

Remark 6.5. We can think of L_nX as the collection of *degenerate* simplices inside X_n. If X is a simplicial set, this is literally true.

Remark 6.6. Notice that $s_i : X_{n-1} \to X_n$ has a left inverse d_i, by the simplicial identities, so it is an injection and a homeomorphism onto its image $s_i(X_n)$ (since for any open $U \subseteq X_{n-1}$, $U = s_i^{-1}d_i^{-1}(U)$). Thus we can identify the colimit L_nX as the union $\bigcup_{i=0}^{n} s_i(X_{n-1})$ inside X_n.

Definition 6.7. A simplicial space $X_\bullet$ is *Reedy cofibrant* if for every n the map $L_nX \to X_n$ is a closed cofibration.

Remark 6.8. The closed condition is automatic if X_n is Hausdorff, as is seen from the expression $s_i(X_{n-1}) = \{x \in X_n : s_i d_i x = x\}$, a closed set if X is Hausdorff.[5]

The result we want is then:

Theorem 6.9. *Suppose $X_\bullet$ and $Y_\bullet$ are Reedy cofibrant simplicial spaces such that X_n and Y_n are CW-complexes for all n, and $f_\bullet \colon X_\bullet \to Y_\bullet$ is a levelwise weak equivalence. Then $|f| \colon |X| \to |Y|$ is a weak equivalence.*

To prove this we need some input from point set topology, which we summarize in a lemma:

Lemma 6.10.

(i) Suppose $X \hookrightarrow Y$ is a cofibration and $X \to X'$ is a homotopy equivalence. Then the pushout $Y \to Y \cup_X X'$ is also a homotopy equivalence.

[5]If $f, g \colon X \to Y$ are continuous and Y is Hausdorff, then $\{x : f(x) = g(x)\}$ is always a closed subset of X.

(ii) *Suppose given a commutative diagram*

$$\begin{array}{ccccc} X & \longleftarrow & A & \overset{i}{\hookrightarrow} & Y \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longleftarrow & A' & \overset{i'}{\hookrightarrow} & Y', \end{array}$$

where i and i' are cofibrations and the vertical maps are homotopy equivalences. Then the induced map on pushouts $X \cup_A Y \to X' \cup_{A'} Y'$ is a homotopy equivalence.

(iii) *Suppose given a map of sequences*

$$\begin{array}{ccccccc} X_0 & \hookrightarrow & X_1 & \hookrightarrow & X_2 & \hookrightarrow & \cdots \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\ Y_0 & \hookrightarrow & Y_1 & \hookrightarrow & Y_2 & \hookrightarrow & \cdots, \end{array}$$

where the maps $X_i \hookrightarrow X_{i+1}$ and $Y_i \hookrightarrow Y_{i+1}$ are all cofibrations, and the maps $f_i \colon X_i \to Y_i$ are all homotopy equivalences. Then the induced map on colimits $f \colon \operatorname{colim}_{n\to\infty} X_n \to \operatorname{colim}_{n\to\infty} Y_n$ is a homotopy equivalence.

(iv) *Suppose $A \hookrightarrow A'$ and $B \hookrightarrow B'$ are closed cofibrations. Then the induced map $A \times B' \amalg_{A\times B} A' \times B \to A' \times B'$ is also a closed cofibration.*

Proof. For (i) see [Hat02, Exercise 0.27]. (ii) follows formally from (i) — see [MP12, Proposition 15.4.4]. For (iii) we can use Lemma 2.14 to inductively build a sequence of compatible homotopy inverses $Y_n \to X_n$ of f_n, in the limit these then give a homotopy inverse to f. For (iv), see [May99, §6.4]. □

We also need to introduce the notion of *n-skeleton* for simplicial spaces, and to prove some basic properties of these:

Definition 6.11. For $X_\bullet$ a simplicial space, let $\operatorname{sk}_n|X|$ denote the analogue of the geometric realization where we only consider the objects $[k] \in \mathbf{\Delta}$ where $k \leq n$, i.e.

$$\operatorname{coeq}\left(\coprod_{\substack{[m]\to[k] \\ m,k\leq n}} X_k \times \Delta^m \rightrightarrows \coprod_l X_l \times \Delta^l\right).$$

Lemma 6.12. *Suppose $X_\bullet$ is a simplicial space.*

(i) *The geometric realization $|X|$ is the colimit of the natural maps*

$$\operatorname{sk}_0|X| \to \operatorname{sk}_1|X| \to \cdots.$$

(ii) *For every n there is a natural pushout square*

$$\begin{array}{ccc} L_nX \times \Delta^n \amalg_{L_nX\times\partial\Delta^n} X_n \times \partial\Delta^n & \longrightarrow & \operatorname{sk}_{n-1}|X| \\ \downarrow & & \downarrow \\ X_n \times \Delta^n & \longrightarrow & \operatorname{sk}_n|X|. \end{array}$$

(iii) *If X is Reedy cofibrant, then the maps $\operatorname{sk}_n|X| \to \operatorname{sk}_{n+1}|X|$ are all closed cofibrations.*

Proof. (i) follows formally since colimits commute and $\mathbf{\Delta}$ is the union of the subcategories with objects $\leq n$. It also makes sense geometrically since every point in $|X|$ lies in $X_n \times \Delta^n$ for some (unique) minimal n, and so $|X|$ is the union of the subspaces $\operatorname{sk}_n|X|$.

(ii) makes sense geometrically since the n-skeleton is obtained from the $(n-1)$-skeleton by gluing on $X_n \times \Delta^n$. This meets the $(n-1)$-skeleton precisely along the images of $X_{n-1} \times \Delta^n$ along a degeneracy in the first factor, and the images of $X_n \times \Delta^{n-1}$ along the inclusion of a face of Δ^n in the second factor. These are glued to the $(n-1)$-skeleton via the corresponding codegeneracy $X_{n-1} \times \Delta^n \to X_{n-1} \times \Delta^{n-1}$ in the second factor and the corresponding face map $X_n \times \Delta^{n-1} \to X_{n-1} \times \Delta^{n-1}$ in the first factor, respectively. Taking the union of these over all degeneracies and faces we get $L_nX \times \Delta^n$ and $X_n \times \partial\Delta^n$, which meet in $L_nX \times \partial\Delta^n$, giving the pullback.

(iii) follows from (ii) and the assumption of Reedy cofibrancy, since closed cofibrations are preserved under pushouts and the map $L_nX \times \Delta^n \amalg_{L_nX \times \partial\Delta^n} X_n \times \partial\Delta^n \to X_n \times \Delta^n$ is a closed cofibration by Lemma 6.10(iv). □

Remark 6.13. Let's give a more formal argument for the pushout square (ii). Let $\mathbf{\Delta}_{\leq n}$ denote the full subcategory of $\mathbf{\Delta}$ with objects $[k]$ where $k \leq n$, and write $i_n \colon \mathbf{\Delta}_{\leq n} \to \mathbf{\Delta}$ be the inclusion. If X is a simplicial space, we can define its n-skeleton sk_nX as the left Kan extension $i_{n,!}i_n^*X$ (where i_n^*X is the restriction of X to the subcategory $\mathbf{\Delta}^{\mathrm{op}}_{\leq n}$). Notice that everything in $(\mathrm{sk}_nX)_i$ is degenerate for $i > n$, so the realization $|\mathrm{sk}_nX|$ is homeomorphic to $\mathrm{sk}_n|X|$ as we defined this above. Note also that $(\mathrm{sk}_nX)_{n+1}$ is precisely $L_{n+1}X$. Since realization is a left adjoint, it preserves colimits, and as it also preserves products it's enough to show we have a pushout of simplicial spaces

$$\begin{array}{ccc} L_nX \times \Delta^n \amalg_{L_nX \times \partial\Delta^n} X_n \times \partial\Delta^n & \longrightarrow & \mathrm{sk}_{n-1}X \\ \downarrow & & \downarrow \\ X_n \times \Delta^n & \longrightarrow & \mathrm{sk}_nX, \end{array}$$

where Δ^n and $\partial\Delta^n$ here denote simplicial sets (viewed as discrete simplicial spaces) rather than spaces. These simplicial spaces are all n-skeletal (i.e. are left Kan extensions of their restrictions to $\mathbf{\Delta}^{\mathrm{op}}_{\leq n}$), so as $i_{n,!}$ preserves colimits (being a left adjoint) it suffices to prove we have a levelwise pushout when evaluated at $[k]$ for $k \leq n$. For $k < n$ we have $(\Delta^n)_k = (\partial\Delta^n)_k$ so the top left object is just $L_nX \times (\Delta^n)_k \amalg_{L_nX \times (\Delta^n)_k} X_n \times (\Delta^n)_k \cong X_n \times (\Delta^n)_k$ so both vertical arrows are isomorphisms, hence the square is a pushout. For $k = n$, we have $(\Delta^n)_n = \{*\} \amalg (\partial\Delta^n)_n$ so the top left object is $L_nX \amalg Y$ where $Y := X_n \times (\partial\Delta^n)_n$. On the other hand $X_n \times (\Delta^n)_n = X_n \amalg Y$ and so the diagram is

$$\begin{array}{ccc} L_nX \amalg Y & \longrightarrow & L_nX \\ \downarrow & & \downarrow \\ X_n \amalg Y & \longrightarrow & X_n, \end{array}$$

which is clearly a pushout.

Proof of Theorem 6.9. Using Lemma 6.10(iii) and Lemma 6.12(iii) we see that it's enough to prove that $\mathrm{sk}_n|f| \colon \mathrm{sk}_n|X| \to \mathrm{sk}_n|Y|$ is a homotopy equivalence for all n. Now applying Lemma 6.10(ii) to the natural pushout squares from Lemma 6.12(ii) (where the left vertical maps are closed cofibrations by Lemma 6.10(iv)) we see that $\mathrm{sk}_n|f|$ is a homotopy equivalence provided the maps $\mathrm{sk}_{n-1}|f|$, $f_n \times \Delta^n$, and $L_nf \times \Delta^n \amalg_{L_nf \times \partial\Delta^n} f_n \times \partial\Delta^n$ are all homotopy equivalences. For $f_n \times \Delta^n$ this holds by assumption (since X_n and Y_n are assumed to be CW-complexes), so we can complete the proof by induction if we can show that $L_nf \times \Delta^n \amalg_{L_nf \times \partial\Delta^n} f_n \times \partial\Delta^n$ is a homotopy equivalence. Using Lemma 6.10(ii) again, we see this will be true if $L_nf \colon L_nX \to L_nY$ is a homotopy equivalence. But L_nX can be naturally written as an iterated pushout of copies of X_{n-1} along inclusions of $L_{n-1}X$, so again this follows by induction and Lemma 6.10(ii). □

To deduce Theorem 6.1 it only remains to check that the simplicial space that defines the homotopy colimit is Reedy cofibrant:

Lemma 6.14. *For any functor $F\colon \mathcal{I} \to \mathrm{Top}$, the simplicial space F^{Δ} is Reedy cofibrant.*

Proof. The latching object $L_n(F^{\Delta})$ can be identified with the part of the disjoint union F_n^{Δ} corresponding to sequences $i_0 \to i_1 \to \cdots \to i_n$ where at least one of the maps is an identity. Thus F_n^{Δ} is a coproduct of $L_n(F^{\Delta})$ and the coproduct over the remaining n-simplices in $\mathrm{N}\mathcal{I}$, and the inclusion is obviously a closed cofibration. □

7. The Two-Sided Bar Construction

To prove the other properties of the homotopy colimits we're interested in, it turns out to be convenient put our construction of the homotopy colimit into a slightly more general context. We start by the non-homotopical version of this, which is the notion of *coends*.

Recall our expression for the colimit of a functor $F\colon \mathcal{I} \to \mathcal{C}$ as

$$\operatorname{colim} F \cong \operatorname{coeq}\left(\coprod_{(f\colon i \to j) \in \mathrm{Mor}(\mathcal{I})} F(i) \rightrightarrows \coprod_{k \in \mathrm{Ob}(\mathcal{I})} F(k) \right).$$

Now we introduce a variant of this construction: given a functor $\Phi\colon \mathcal{I}^{\mathrm{op}} \times \mathcal{I} \to \mathcal{C}$, its *coend* is the coequalizer

$$\operatorname{coend} \Phi := \operatorname{coeq}\left(\coprod_{(f\colon i \to j) \in \mathrm{Mor}(\mathcal{I})} \Phi(j, i) \rightrightarrows \coprod_{k \in \mathrm{Ob}(\mathcal{I})} \Phi(k, k) \right).$$

where the two morphisms are given on the component $\Phi(j,i)$ corresponding to $f\colon i \to j$ by $\Phi(f, \mathrm{id})\colon \Phi(j,i) \to \Phi(i,i)$ and $\Phi(\mathrm{id}, f)\colon \Phi(j,i) \to \Phi(j,j)$. The coend of Φ is sometimes denoted $\int^{\mathcal{I}} \Phi$; we'll generally avoid doing this though.

Remark 7.1. You might ask what this construction "means", or more precisely whether it has a universal property. In the literature this is usually discussed in terms of a rather obscure notion of "extranatural transformations". However, there is a very natural way to understand coends as ordinary colimits: If $\mathcal{I}$ is a category, the *twisted arrow category* $\mathrm{Tw}(\mathcal{I})$ of $\mathcal{I}$ is the category whose objects are the morphisms of $\mathcal{I}$ and whose morphisms from $i \to j$ to $i' \to j'$ are the commutative diagrams of the form

$$\begin{array}{ccc} i & \longrightarrow & i' \\ \downarrow & & \downarrow \\ j & \longleftarrow & j'. \end{array}$$

There is an obvious forgetful functor $\mathrm{Tw}(\mathcal{I}) \to \mathcal{I}^{\mathrm{op}} \times \mathcal{I}$, and the coend of a functor Φ can be identified with the colimit of the composite functor

$$\mathrm{Tw}(\mathcal{I}) \to \mathcal{I}^{\mathrm{op}} \times \mathcal{I} \xrightarrow{\Phi} \mathcal{C}.$$

(This is not completely obvious, it requires a cofinality argument.)

A key special case of the coend construction is the so-called *functor tensor product*: given functors $W\colon \mathcal{I}^{\mathrm{op}} \to \mathcal{C}$ and $F\colon \mathcal{I} \to \mathcal{C}$ we write $W \otimes_{\mathcal{I}} F$ for the coend of $W \times F\colon \mathcal{I}^{\mathrm{op}} \times \mathcal{I} \to \mathcal{C}$.

Examples 7.2.

(i) If $F\colon \mathcal{I} \to \mathrm{Top}$ is any functor, then $* \otimes_{\mathcal{I}} F$ is the colimit of F — this is the description of the colimit we've repeatedly used before.

(ii) If $X_\bullet$ is a simplicial space, then the geometric realization $|X|$ is nothing but the tensor $\Delta^\bullet \otimes_{\mathbf{\Delta}^{\mathrm{op}}} X_\bullet$.

(iii) We have $\mathcal{I}(-, d) \otimes_{\mathcal{I}} F \cong F(d)$: the expression for $\mathcal{I}(-, d) \otimes_{\mathcal{I}} F$ can be rewritten as

$$\mathrm{coeq}\left(\coprod_{i \to j \to d} F(i) \rightrightarrows \coprod_{k \to d} F(k) \right),$$

which is $\mathrm{colim}_{\mathcal{I}_{/d}} F$ and that's naturally isomorphic to $F(d)$.

(iv) Given functors $F\colon \mathcal{I} \to \mathrm{Top}$ and $\phi\colon \mathcal{I} \to \mathcal{J}$, the left Kan extension $\phi_! F\colon \mathcal{J} \to \mathrm{Top}$ can by a similar argument be described as

$$d \mapsto \mathcal{J}(\phi(-), d) \otimes_{\mathcal{I}} F.$$

Now we introduce a homotopical version of coends, analogous to our construction of the homotopy colimit: Given $\Phi\colon \mathcal{I}^{\mathrm{op}} \times \mathcal{I} \to \mathrm{Top}$ we define a simplicial space we'll abusively denote $\Phi^\Delta_\bullet$ by

$$\Phi^\Delta_n = \coprod_{i_0 \to \cdots \to i_n} \Phi(i_n, i_0).$$

For $\phi\colon [m] \to [n]$ in $\mathbf{\Delta}$, the structure map $\phi^*\colon \Phi^\Delta_n \to \Phi^\Delta_m$ is given on the component $\Phi(i_n, i_0)$ indexed by $i_0 \to \cdots \to i_n$ by the map $\Phi(i_n, i_0) \to \Phi(i_{\phi(n)}, i_{\phi(m)})$ with the target in the component indexed by $i_{\phi(0)} \to \cdots \to i_{\phi(m)}$. The *homotopy coend* of Φ is then the realization $|\Phi^\Delta_\bullet|$; just as the homotopy colimit is a "fattened" version of the ordinary colimit, this is a "fattened" version of the colimit of the simplicial diagram $\Phi^\Delta_\bullet$, which is the coend of Φ.

We're interested in the homotopy version of the functor tensor product; for historical reasons this has a special name:

Definition 7.3. Suppose given functors $F\colon \mathcal{I} \to \mathrm{Top}$ and $W\colon \mathcal{I}^{\mathrm{op}} \to \mathrm{Top}$. The *two-sided simplicial bar construction* $\mathbb{B}_\bullet(W, \mathcal{I}, F)$ is the simplicial space $(W \times F)^\Delta$, and the *two-sided bar construction* $B(W, \mathcal{I}, F)$ is the geometric realization $|\mathbb{B}_\bullet(W, \mathcal{I}, F)|$.

Example 7.4. $B(*, \mathcal{I}, *)$ is the classifying space $B\mathcal{I}$. More generally, $\mathbb{B}_\bullet(*, \mathcal{I}, F)$ is the simplicial space we previously denote F^Δ, so $B(*, \mathcal{I}, F)$ is $\mathrm{hocolim}\, F$.

Remark 7.5. Example 7.2(iv) suggests that we can define the *homotopy Kan extension* of F along ϕ using the two-sided bar construction as

$$d \mapsto B(\mathcal{J}(\phi(-), d), \mathcal{I}, F).$$

We'll abbreviate this to $\phi_!^{\mathbb{L}} F$.

We'll now prove some formal properties of the two-sided bar construction. First of all, from our result on the homotopy invariance of geometric realizations we immediately get:

Lemma 7.6. *Suppose $W\colon \mathcal{I}^{\mathrm{op}} \to \mathrm{Top}$ is a functor valued in CW-complexes. Then the functor $B(W, \mathcal{I}, -)\colon \mathrm{Top}^{\mathcal{I}} \to \mathrm{Top}$ preserves weak equivalences between functors $\mathcal{I} \to \mathrm{Top}$ that take values in CW-complexes (and similarly in the other variable).*

Proof. By Theorem 6.9, as $\mathbb{B}_\bullet(W, \mathcal{I}, F)$ is always Reedy cofibrant. □

Lemma 7.7. *The two-sided bar construction commutes with functor tensor products in each variable, i.e. given functors* $W\colon \mathcal{J}\times \mathcal{I}^{\mathrm{op}} \to \mathrm{Top}$*,* $F\colon \mathcal{I}\times\mathcal{K}^{\mathrm{op}}\to \mathrm{Top}$*,* $\Phi\colon \mathcal{J}^{\mathrm{op}}\to\mathrm{Top}$*, and* $\Psi\colon\mathcal{K}\to\mathrm{Top}$*, there is a canonical isomorphism*

$$\Phi\otimes_{\mathcal{J}} B(W,\mathcal{I},F)\otimes_{\mathcal{K}}\Psi \cong B(\Phi\otimes_{\mathcal{J}} W,\mathcal{I},F\otimes_{\mathcal{K}}\Psi).$$

Proof. The geometric realization commutes with colimits and products, so this is amounts to expanding everything out and commuting some colimits. □

Definition 7.8. For convenience, we'll use the standard (but perhaps slightly confusing) notation $B(\mathcal{I},\mathcal{I},F)$ for the functor $i\mapsto B(\mathcal{I}(-,i),\mathcal{I},F)$, and $B(W,\mathcal{I},\mathcal{I})$ for $i\mapsto B(W,\mathcal{I},\mathcal{I}(i,-)$. We use similar notation for functor tensor products.

Remark 7.9. With this notation we noted above that $\mathcal{I}\otimes_{\mathcal{I}} F\cong F$, and similarly $W\otimes_{\mathcal{I}}\mathcal{I}\cong W$. Thus we can think of the functors $B(\mathcal{I},\mathcal{I},F)$ and $B(W,\mathcal{I},\mathcal{I})$ as "fattened up" versions of F and W. These functors have an alternative description that will be important to us:

Lemma 7.10. *There are natural equivalences* $B(\mathcal{I},\mathcal{I},F)\cong B(*,\mathcal{I}_{/i},F)$ *and* $B(W,\mathcal{I},\mathcal{I})\cong B(W,\mathcal{I}_{i/},*)$*.*

Proof. Expand out the definitions, we see that the simplicial spaces $\mathbb{B}_\bullet(\mathcal{I}(-,i),\mathcal{I},F)$ and $\mathbb{B}_\bullet(*,\mathcal{I}_{/i},F)$ are isomorphic, and similarly in the other case. □

Using this we get a slick proof of our alternative description of the homotopy colimit, as well as a new expression:

Proposition 7.11. *For a functor* $F\colon\mathcal{I}\to\mathrm{Top}$*, there are canonical isomorphisms*

$$\operatorname*{hocolim}_{\mathcal{I}} F\cong B(\mathcal{I}_{-/})\otimes_{\mathcal{I}} F,$$

$$\operatorname*{hocolim}_{\mathcal{I}} F\cong \operatorname*{colim}_{\mathcal{I}} B(\mathcal{I},\mathcal{I},F)\cong \operatorname*{colim}_{i\in\mathcal{I}}(\operatorname*{hocolim}_{\mathcal{I}_{/i}} F).$$

Proof. We have $\operatorname{hocolim} F\cong B(*,\mathcal{I},F)\cong B(*,\mathcal{I},\mathcal{I}\otimes_{\mathcal{I}} F)\cong B(*,\mathcal{I},\mathcal{I})\otimes_{\mathcal{I}} F$. Here the space $B(*,\mathcal{I},\mathcal{I}(i,-)$ can be identified with the realization $B(\mathcal{I}_{i/})$. On the other hand, we also have $B(*,\mathcal{I},F)\cong B(*\otimes_{\mathcal{I}}\mathcal{I},\mathcal{I},F)\cong *\otimes_{\mathcal{I}} B(\mathcal{I},\mathcal{I},F)\cong \operatorname{colim} B(\mathcal{I},\mathcal{I},F)$. □

Remark 7.12. Suppose X is a simplicial space. Then we now have a description of the homotopy colimit of X as $B(\mathbf{\Delta}_{/-})^{\mathrm{op}}\otimes_{\mathbf{\Delta}^{\mathrm{op}}} X$. On the other hand, the geometric realization we can write as $\Delta^\bullet\otimes_{\mathbf{\Delta}^{\mathrm{op}}} X$. A variant of our proof of homotopy invariance for geometric realizations implies that if X is Reedy cofibrant then the functor $-\otimes_{\mathbf{\Delta}^{\mathrm{op}}} X$ preserves weak equivalences between cosimplicial CW-complexes that satisfy an analogue of the Reedy cofibrancy condition: the map from the nth "matching object", which is obtained as the colimit over all the face maps into $[n]$ (so for $\Delta^\bullet$ we get $\partial\Delta^n$) into the nth space is a cofibration. Using this we can show that if X is a Reedy cofibrant simplicial space then $\operatorname{hocolim}_{\mathbf{\Delta}^{\mathrm{op}}} X$ is weakly equivalent to $|X|$.

Here is a key fact about coends — sometimes called the "Fubini theorem":

Proposition 7.13. *For a functor* $\Phi\colon\mathcal{I}^{\mathrm{op}}\times\mathcal{J}^{\mathrm{op}}\times\mathcal{J}\to\mathcal{I}\to\mathcal{C}$ *there are natural isomorphisms*

$$\int^{\mathcal{I}}\int^{\mathcal{J}}\Phi\cong\int^{\mathcal{I}\times\mathcal{J}}\Phi\cong\int^{\mathcal{J}}\int^{\mathcal{I}}\Phi.$$

Proof. We have a commutative diagram

$$\begin{array}{ccc}
\coprod_{\substack{i \to i' \\ j \to j'}} \Phi(i', j', j, i) & \rightrightarrows & \coprod_{i, j \to j'} \Phi(i, j', j, i) \\
\downdownarrows & & \downdownarrows \\
\coprod_{i \to i', j} \Phi(i', j, j, i) & \rightrightarrows & \coprod_{i,j} \Phi(i, j, j, i)
\end{array}$$

Taking the colimit of this in three different ways gives the three expressions: First, we can do the coequalizers of the two columns, to get

$$\coprod_{i \to i'} \int^{\mathcal{J}} \Phi(i', -, -, i) \rightrightarrows \coprod_{i} \int^{\mathcal{J}} \Phi(i, -, -, i),$$

and then take the coequalizer of this to get $\int^{\mathcal{I}} \int^{\mathcal{J}} \Phi$. (The first step is a left Kan extension a along the functor from our original diagram shape to $\bullet \rightrightarrows \bullet$ that collapses the columns, and the colimit of a left Kan extension is the colimit of the original diagram.) Second, we can do the same thing but interchanging rows and columns to get $\int^{\mathcal{J}} \int^{\mathcal{I}} \Phi$. And third, we can take the coequalizer of the two diagonal maps (by a cofinality argument), which gives $\int^{\mathcal{I} \times \mathcal{J}} \Phi$. □

As a special case, we get an associativity isomorphism for the functor tensor product: given $W \colon \mathcal{I}^{\mathrm{op}} \to \mathcal{C}$, $\Phi \colon \mathcal{I} \times \mathcal{J}^{\mathrm{op}} \to \mathcal{C}$, and $F \colon \mathcal{J} \to \mathcal{C}$, we have a natural isomorphism

$$(W \otimes_{\mathcal{I}} \Phi) \otimes_{\mathcal{J}} F \cong W \otimes_{\mathcal{I}} (\Phi \otimes_{\mathcal{J}} F).$$

We now prove an analogous associtivity result for the two-sided bar construction that we'll use several times below:

Proposition 7.14. *Suppose given functors* $F \colon \mathcal{J} \to \mathrm{Top}$, $\Phi \colon \mathcal{I} \times \mathcal{J}^{\mathrm{op}} \to \mathrm{Top}$ *and* $W \colon \mathcal{I}^{\mathrm{op}} \to \mathrm{Top}$. *Then there is a natural isomorphism*

$$B(W, \mathcal{I}, B(\Phi, \mathcal{J}, F)) \cong B(B(W, \mathcal{I}, \Phi), \mathcal{J}, F).$$

Proof. Since geometric realization commutes with products and colimits, we can identify the two sides with the two double[6] geometric realizatons of the bisimplicial space where the space at index $([n], [m])$ is

$$\coprod_{\substack{i_0 \to \cdots \to i_n \\ j_0 \to \cdots \to j_n}} W(i_n) \times \Phi(i_0, j_m) \times F(j_0).$$

Another special case of the "Fubini theorem" tells us that these two double realizations are naturally isomorphic. □

As a final preliminary result, we note a generalization of Proposition 5.12:

Proposition 7.15.

(i) For any functor $F \colon \mathcal{I} \to \mathrm{Top}$, *there is a natural transformation* $B(\mathcal{I}, \mathcal{I}, F) \to F$ *that is a levelwise weak equivalence.*

(ii) For any functor $W \colon \mathcal{I}^{\mathrm{op}} \to \mathrm{Top}$, *there is a natural transformation* $B(W, \mathcal{I}, \mathcal{I}) \to W$ *that is a levelwise weak equivalence.*

[6] I.e. we realize at each level in one coordinate, and then in the other.

Proof. For (i), we can identify $B(\mathcal{I}(-,i),\mathcal{I},F)$ with $B(*,\mathcal{I}_{/i},F)$ so the map we want is just the map $\operatorname{hocolim}_{\mathcal{I}_{/i}} F \to F(i)$ we proved was a weak equivalence in Proposition 5.12. In (ii), we can similarly identify $B(W,\mathcal{I},\mathcal{I}(i,-)$ with $B(W,\mathcal{I}_{i/},*)$. A similar construction to that in the proof of Proposition 5.12 gives that there is a natural map $B(W,\mathcal{I}_{i/},*) \to W(i)$ that is a weak equivalence. $\square$

8. The Homotopy Colimit as a Derived Functor

We are now ready to prove that the homotopy colimit is indeed a derived functor. We saw above that we can write $\operatorname{hocolim} F$ as

$$B(*,\mathcal{I},F) \cong * \otimes_{\mathcal{I}} B(\mathcal{I},\mathcal{I},F) \cong \operatorname*{colim}_{\mathcal{I}} B(\mathcal{I},\mathcal{I},F).$$

For a functor F let us write $Q'F := B(\mathcal{I},\mathcal{I},F)$ and $Q'' := |\operatorname{Sing}_\bullet(F)|$; then Q' and Q'' are functors $\mathrm{Top}^{\mathcal{I}} \to \mathrm{Top}^{\mathcal{I}}$. Note that $Q''F$ is a functor weakly equivalent to F that is valued in CW-complexes, and $\operatorname{colim} Q'F$ is isomorphic to $\operatorname{hocolim} F$. We also have natural transformations $q'\colon Q' \to \mathrm{id}$ (using the natural maps $\operatorname{hocolim}_{\mathcal{I}_{/i}} F \to F(i)$) and $q''\colon Q'' \to \mathrm{id}$ (the counit for the adjunction of geometric realization and singular complex). We want to show that $Q := Q' \circ Q''$ together with the composite natural transformation $q := q'' \circ q'_{Q''}$ is a left deformation for $\mathrm{Top}^{\mathcal{I}}$ that is compatible with colim, so that $\operatorname{colim} Q$ is a derived functor by Proposition 3.5.

Lemma 8.1. *Q is a deformation of $\mathrm{Top}^{\mathcal{I}}$.*

Proof. We must show that for any $F\colon \mathcal{I} \to \mathrm{Top}$, the map $q_F\colon QF \to F$ is a natural weak equivalence. But we know that $q''_F(i)\colon |\operatorname{Sing}_\bullet F(i)| \to F(i)$ is a weak equivalence for any F and $i \in \mathcal{I}$, and $q'_F(i)\colon \operatorname{hocolim}_{\mathcal{I}_{/i}} F \to F(i)$ is a homotopy equivalence by Proposition 5.12. $\square$

Using Proposition 3.5 we need to show:

Proposition 8.2. *The functor* $\operatorname{colim} Q\colon \mathrm{Top}^{\mathcal{I}} \to \mathrm{Top}$ *preserves weak equivalences, and the map $q_{QF}\colon QQF \to QF$ is taken to a weak equivalence by* colim.

Proof. The first statment follows from Theorem 6.1, since $Q''F$ is a functor valued in CW-complexes for any F. The homotopy invariance of hocolim also implies that the natural map $\operatorname{hocolim} QF \to \operatorname{hocolim} B(\mathcal{I},\mathcal{I},|\operatorname{Sing}_\bullet(F)|)$ is a weak equivalence. It's therefore enough to show that the natural map

$$B(*,\mathcal{I},B(\mathcal{I},\mathcal{I},F)) \to B(*,\mathcal{I},F)$$

is a weak equivalence for any functor F valued in CW-complexes. But by associativity the left-hand side is isomorphic to $B(B(*,\mathcal{I},\mathcal{I}),\mathcal{I},F)$, and expanding out the definitions of the bar constructions we can identify our map with the map

$$B(B(*,\mathcal{I},\mathcal{I}),\mathcal{I},F) \to B(*,\mathcal{I},F)$$

induced by the natural map $B(*,\mathcal{I},\mathcal{I}) \to *$. Now by homotopy invariance in the first variable it follows that we get a weak equivalence if $B(*,\mathcal{I},\mathcal{I}(i,-)) \to *$ is a weak equivalence for all i, which is clear since $B(*,\mathcal{I},\mathcal{I}(i,-))$ is isomorphic to $B(*,\mathcal{I}_{/i},*) \cong B\mathcal{I}_{/i}$ — this classifying space is contractible since $\mathcal{I}_{/i}$ has a terminal object. $\square$

Remark 8.3. The same argument shows that for any $W\colon \mathcal{I}^{\mathrm{op}} \to \mathrm{Top}$, the functor

$$B(W,\mathcal{I},-) \cong W \otimes_{\mathcal{I}} B(\mathcal{I},\mathcal{I},-)$$

is a left derived functor of $W \otimes_{\mathcal{I}} -$. (Here we use that $B(W, \mathcal{I}, \mathcal{I}) \to W$ is a natural weak equivalence for any W, not just for $W = *$.)

Remark 8.4. It can also be shown using this description of the homotopy colimit as a deformation of colim that hocolim viewed as a functor $\mathrm{Ho}(\mathrm{Top}^{\mathcal{I}}) \to \mathrm{Ho}(\mathrm{Top})$ is a left adjoint to the constant diagram functor $\mathrm{Ho}(\mathrm{Top}) \to \mathrm{Ho}(\mathrm{Top}^{\mathcal{I}})$. This follows from a general result about *deformable adjunctions*: if $F \colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ is an adjunction between homotopical categories, Q is left deformation of $\mathcal{C}$ compatible with F, and R is a right deformation of $\mathcal{D}$ compatible with G, then the total derived functors $\mathbb{L}F$ and $\mathbb{R}G$ form an adjunction

$$\mathbb{L}F : \mathrm{Ho}\,\mathcal{C} \rightleftarrows \mathrm{Ho}\,\mathcal{D} : \mathbb{R}G.$$

Here $\mathbb{L}F$ is the total left derived functor of F, induced by $F \circ Q$, and $\mathbb{R}G$ is the total right derived functor of G, induced by $G \circ R$. We won't prove this here, but it's not hard — there is an abstract proof in [Rie14] and a more explicit proof in [DHKS04] (though written in Kan's usual extremely compact style).

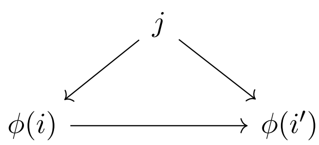
Remark 8.5. We again stress what we have said earlier: While one can choose many point-set models for homotopy colimit, the derived functor of colim is unique if it exists, so all models will agree in the homotopy category. We choose to work with "our" point-set version of hocolim, because of its relative simplicity. And the fact that it is the realization of a simplicial space in fact gives us a spectral sequence for computing hocolim which we will set up in a later section.

9. Cofinality for Homotopy Colimits

The final standard property about homotopy colimits that we want to mention is cofinality. Let's first discuss cofinality for ordinary colimits:

Definition 9.1. A functor $\phi \colon \mathcal{I} \to \mathcal{J}$ between small categories is *cofinal* if for every $j \in \mathcal{J}$ the category $\mathcal{I}_{j/} := \mathcal{I} \times_{\mathcal{J}} \mathcal{J}_{j/}$ is connected (i.e. any two objects are connected by a pair of morphisms, or, equivalently, the classifying space $B\mathcal{I}_{j/}$ is connected).

Remark 9.2. The category $\mathcal{I}_{j/}$ here has objects pairs $(i \in \mathcal{I}, j \to \phi(i))$ and morphisms $(i, j \to \phi(i)) \to (i', j \to \phi(i'))$ are given by morphisms $f \colon i \to i'$ such that the diagram



commutes.

Proposition 9.3. *If the* $\phi \colon \mathcal{I} \to \mathcal{J}$ *is cofinal then for every diagram* $F \colon \mathcal{J} \to \mathcal{C}$ *the natural map* $\mathrm{colim}_{\mathcal{I}} F\phi \to \mathrm{colim}_{\mathcal{J}} F$ *is an isomorphism (assuming either colimit exists, in which case both do).*

Proof. Recall that we have $\mathcal{J}(-, j) \otimes_{\mathcal{J}} F \cong F(j)$ for any $j \in \mathcal{J}$. Thus we can write the functor $F \circ \phi$ as $\mathcal{J}(-, \phi(-)) \otimes_{\mathcal{J}} F$, giving

$$\operatorname*{colim}_{\mathcal{I}} F\phi \cong * \otimes_{\mathcal{I}} F\phi \cong * \otimes_{\mathcal{I}} (\mathcal{J}(-, \phi(-)) \otimes_{\mathcal{J}} F).$$

Now using associativity for the functor tensor product we see that this isomorphic to $(* \otimes_{\mathcal{I}} \mathcal{J}(-, \phi(-))) \otimes_{\mathcal{J}} F$. For $j \in \mathcal{J}$, the colimit $* \otimes_{\mathcal{I}} \mathcal{J}(j, \phi(-))$ is given by the coequalizer of

$$\coprod_{i \to i'} \mathcal{J}(j, \phi(i)) \rightrightarrows \coprod_{i} \mathcal{J}(j, \phi(i)).$$

But we can identify this with

$$\mathrm{Mor}(\mathcal{I}_{j/}) \rightrightarrows \mathrm{Ob}(\mathcal{I}_{j/})$$

where the two maps send a morphism to its source and target object. The coequalizer is thus the quotient of $\mathrm{Ob}(\mathcal{I}_{j/})$ by the equivalence relation $\sim$ generated by $x \sim y$ if there is a morphism from x to y. This quotient is precisely the set of connected components of $B\mathcal{I}_{j/}$. Thus if ϕ is cofinal we get that $* \otimes_{\mathcal{I}} \mathcal{J}(-, \phi(-))$ is the constant functor at $*$, and so

$$\operatorname*{colim}_{\mathcal{I}} F\phi \cong (* \otimes_{\mathcal{I}} \mathcal{J}(-, \phi(-))) \otimes_{\mathcal{J}} F \cong * \otimes_{\mathcal{J}} F \cong \operatorname*{colim}_{\mathcal{J}} F. \qquad \square$$

Remark 9.4. In fact, this is an if and only if statement: the functors that induce isomorphisms on all colimits are precisely the cofinal functors. To see this, take F to be $\mathcal{J}(j, -)$; then as we saw above $\operatorname{colim}_{\mathcal{I}} \mathcal{J}(j, \phi(-)) \cong \pi_0 B\mathcal{I}_{j/}$ whereas $\operatorname{colim}_{\mathcal{J}} \mathcal{J}(j, -) \cong \pi_0 B\mathcal{J}_{j/} \cong *$.

This criterion can be used to prove a lot of assertions in the literature that various colimits are "obviously" the same. Here's an example we've made use of already:

Lemma 9.5. *The inclusion $\mathbf{\Delta}^{\mathrm{op}}_{\leq 1} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}$ is cofinal.*

Proof. We need to show that for every $[n] \in \mathbf{\Delta}^{\mathrm{op}}$, the category $((\mathbf{\Delta}_{\leq 1})^{\mathrm{op}})_{[n]/} \cong (\mathbf{\Delta}_{\leq 1/[n]})^{\mathrm{op}}$ is connected. We can describe the objects as (i) for $i = 0, \dots, n$ and (i, j) with $0 \leq i \leq j \leq n$ (corresponding to the maps $[0] \to [n]$ and $[1] \to [n]$ in $\mathbf{\Delta}$, respectively). To show that this is connected there are a few cases to check, let's do one of them: for two objects (i, j) and (i', j') suppose $i \leq i'$, then we have morphisms

$$(i, j) \leftarrow (i) \to (i, i') \leftarrow (i') \to (i', j'),$$

so these two objects are connected. The other cases are just as easy. $\square$

Now we want to prove an analogous result about homotopy colimits:

Definition 9.6. A functor $\phi \colon \mathcal{I} \to \mathcal{J}$ between small categories is *homotopy cofinal* if for every $j \in \mathcal{J}$ the category $\mathcal{I}_{j/} := \mathcal{I} \times_{\mathcal{J}} \mathcal{J}_{j/}$ is weakly contractible, i.e. its classifying space $B\mathcal{I}_{j/}$ is contractible.

Theorem 9.7 (Cofinality for hocolim)**.** *Suppose $\phi \colon \mathcal{I} \to \mathcal{J}$ is a homotopy cofinal functor. Then for every functor $F \colon \mathcal{J} \to \mathrm{Top}$ taking values in CW-complexes, the natural map*

$$\operatorname*{hocolim}_{\mathcal{I}} F\phi \to \operatorname*{hocolim}_{\mathcal{J}} F$$

is a weak equivalence.

Proof. Our proof will be a homotopical version of the previous proof: Recall first that we have a natural weak equivalence $B(\mathcal{J}(-, j), \mathcal{J}, F) \to F(j)$ for any $j \in \mathcal{J}$. By homotopy invariance of the two-sided bar construction we then get a weak equivalence

$$\operatorname*{hocolim}_{\mathcal{I}} F\phi \cong B(*, \mathcal{I}, F\phi) \leftarrow B(*, \mathcal{I}, B(\mathcal{J}(-, j), \mathcal{J}, F)).$$

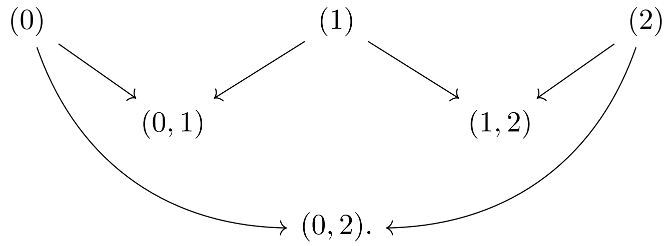
Now using associativity for the bar construction we get an isomorphism between this and the space $B(B(*, \mathcal{I}, \mathcal{J}(-, \phi(-))), \mathcal{J}, F)$. For $j \in \mathcal{J}$, the simplicial diagram $\mathbb{B}_\bullet(*, \mathcal{I}, \mathcal{J}(j, \phi(-)))$ is naturally isomorphic to $\mathrm{N}\mathcal{I}_{j/}$ (by the usual rewriting of the indices). Thus the homotopy colimit $B(*, \mathcal{I}, \mathcal{J}(j, \phi(-))$ is natural isomorphic to $B\mathcal{I}_{j/}$. Using homotopy invariance again, we see that if ϕ is homotopy cofinal, then we get a weak equivalence

$$B(B(*, \mathcal{I}, \mathcal{J}(-, \phi(-))), \mathcal{J}, F) \cong B(B\mathcal{I}_{-/}, \mathcal{J}, F) \to B(*, \mathcal{J}, F) \cong \operatorname*{hocolim}_{\mathcal{J}} F,$$

as required. □

Remark 9.8. Again this is an if and only if statement: the functors that induce weak equivalences on all homotopy colimits are precisely the homotopy cofinal functors. As before, we see this by taking F to be $\mathcal{J}(j, -)$; then we have $\operatorname{hocolim}_{\mathcal{I}} \mathcal{J}(j, \phi(-)) \simeq B\mathcal{I}_{j/}$ whereas $\operatorname{hocolim}_{\mathcal{J}} \mathcal{J}(j, -) \simeq B\mathcal{J}_{j/} \cong *$.

Remark 9.9. The inclusion $\mathbf{\Delta}^{\mathrm{op}}_{\leq 1} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}$ is *not* homotopy cofinal — the slice category $\mathbf{\Delta}_{\leq 1,/[2]}$ is connected, but not simply connected: we get a non-trivial loop from the diagram



More generally, it can be shown that for $\mathbf{\Delta}^{\mathrm{op}}_{\leq n} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}$ the slice categories are $(n-1)$-connected, but *not* n-connected, so to compute homotopy colimits we can't restrict to any finite part of $\mathbf{\Delta}^{\mathrm{op}}$.

Corollary 9.10 (Quillen's Theorem A)**.** *Suppose $F \colon \mathcal{I} \to \mathcal{J}$ is a homotopy cofinal functor. Then induced map on classifying spaces $BF \colon B\mathcal{I} \to B\mathcal{J}$ is a weak equivalence.*

Proof. Apply Theorem 9.7 to the constant functor $\mathcal{J} \to \mathrm{Top}$ with value $*$. □

10. The Grothendieck Construction as a Homotopy Colimit

As an application of the machinery we've set up, in this section we'll give a more explicit description of homotopy colimits for diagrams of classifying spaces that come from diagrams of categories. First we must introduce some notation:

Definition 10.1. Suppose $F \colon \mathcal{I} \to \mathrm{Cat}$ is a functor. The *Grothendieck construction* $\mathrm{Gr}(F)$ is the category with objects pairs $(i \in \mathcal{I}, x \in F(i))$ and a morphism $(i, x) \to (i', x')$ given by a map $f \colon i \to i'$ in $\mathcal{I}$ and a map $\phi \colon F(f)(x) \to x'$ in $F(i')$. Note that there is an obvious projection functor $\mathrm{Gr}(F) \to \mathcal{I}$ that takes an object (i, x) to i.

Remark 10.2. It is possible to characterize the functors $p \colon \mathcal{E} \to \mathcal{I}$ that arise in this way: namely, if p is a so-called "Grothendieck opfibration", then there exists an essentially unique functor $F \colon \mathcal{I} \to \mathrm{Cat}$ such that $\mathcal{E}$ is equivalent to $\mathrm{Gr}(F)$ over $\mathcal{I}$.

Proposition 10.3 (Thomason [Tho79])**.** *Suppose given a functor $F \colon \mathcal{I} \to \mathrm{Cat}$. Then the spaces* $\operatorname{hocolim}_{i \in \mathcal{I}} BF(i)$ *and* $B\mathrm{Gr}(F)$ *are weakly equivalent.*

We'll prove this as a special case of a more general result that describes homotopy colimits indexed by Grothendieck constructions:

Proposition 10.4. *Suppose given a functor $F\colon \mathcal{I} \to \mathrm{Cat}$ and a functor $\Phi\colon \mathrm{Gr}(F) \to \mathrm{Top}$. Then we have a weak equivalence*

$$\operatorname*{hocolim}_{\mathrm{Gr}(F)} \Phi \simeq \operatorname*{hocolim}_{i\in\mathcal{I}} \operatorname*{hocolim}_{F(i)} \Phi|_{F(i)}.$$

(Here we are identifying $F(i)$ with the full subcategory of $\mathrm{Gr}(F)$ with objects of the form (i, x) for $x \in F(i)$.)

For the proof, we first need an observation about homotopy left Kan extensions. Recall that for $\phi\colon \mathcal{I} \to \mathcal{J}$ and $F\colon \mathcal{I} \to \mathcal{C}$ the left Kan extension $\phi_!F$ of F along ϕ is given by

$$j \mapsto \mathcal{J}(\phi(-), j) \otimes_{\mathcal{I}} F \cong \operatorname*{colim}_{\mathcal{I}_{/j}} F,$$

where $\mathcal{I}_{/j}$ denotes the category $\mathcal{I} \times_{\mathcal{J}} \mathcal{J}_{/j}$. Similarly, for $F\colon \mathcal{I} \to \mathrm{Top}$ we can define the *homotopy left Kan extension* $\phi_!^{\mathbb{L}}F$ of F along ϕ by

$$j \mapsto B(\mathcal{J}(\phi(-), j), \mathcal{I}, F) \cong \operatorname*{hocolim}_{\mathcal{I}_{/j}} F$$

(where the isomorphism follows from the usual rewriting).

Lemma 10.5. *Given functors $\mathcal{I} \xrightarrow{\phi} \mathcal{J} \xrightarrow{\psi} \mathcal{K}$ and $F\colon \mathcal{I} \to \mathrm{Top}$, there is a natural weak equivalence*

$$\psi_!^{\mathbb{L}}\phi_!^{\mathbb{L}}F \simeq (\psi\phi)_!^{\mathbb{L}}F.$$

Proof. For $k \in \mathcal{K}$, the left-hand side is the iterated bar construction

$$B(\mathcal{K}(\psi(-), k), \mathcal{J}, B(\mathcal{J}(\phi(-), -), \mathcal{I}, F)).$$

By associativity this is naturally isomorphic to

$$B(B(\mathcal{K}(\psi(-), k), \mathcal{J}, \mathcal{J}(\phi(-), -)), \mathcal{I}, F).$$

But recall that for any $j \in \mathcal{J}$ we have for any W a natural weak equivalence $B(W, \mathcal{J}, \mathcal{J}(j, -)) \to W(j)$, so in particular we have a weak equivalence

$$B(\mathcal{K}(\psi(-), k), \mathcal{J}, \mathcal{J}(\phi(i), -)) \to \mathcal{K}(\psi\phi(i), k).$$

By homotopy invariance of the bar construction, this gives a natural weak equivalence

$$B(B(\mathcal{K}(\psi(-), k), \mathcal{J}, \mathcal{J}(\phi(-), -)), \mathcal{I}, F) \to B(\mathcal{K}(\psi\phi(-), k), \mathcal{J}, F),$$

where the right-hand side is precisely $(\psi\phi)_!^{\mathbb{L}}F$. □

Proof of Proposition 10.4. For $\mathcal{C}$ a category, let's write $\mathcal{C}_+$ for the category where we freely adjoin a terminal objects. More precisely, $\mathcal{C}_+$ has objects $\mathrm{Ob}(\mathcal{C}) \amalg \{\infty\}$ and its morphisms are defined by

$$\mathrm{Hom}_{\mathcal{C}_+}(x, y) = \begin{cases} \mathrm{Hom}_{\mathcal{C}}(x, y) & x, y \in \mathcal{C}, \\ *, & y = \infty, \\ \emptyset, & x = \infty, y \neq \infty. \end{cases}$$

For $F\colon \mathcal{I} \to \mathrm{Cat}$, we similarly write F_+ for the functor that takes $i \in \mathcal{I}$ to $F(i)_+$. There is a fully faithful inclusion $I\colon \mathrm{Gr}(F) \hookrightarrow \mathrm{Gr}(F_+)$ coming from the obvious inclusions $F(i) \hookrightarrow F(i)_+$.

By Lemma 10.5 we then have a weak equivalence

$$\operatorname*{hocolim}_{\mathrm{Gr}(F)} \Phi \simeq \operatorname*{hocolim}_{\mathrm{Gr}(F_+)} I_!^{\mathbb{L}}\Phi.$$

Next, define $J\colon \mathcal{I} \to \mathrm{Gr}(F_+)$ by $i \mapsto (i, \infty \in F(i)_+)$. The functor J is homotopy cofinal: for $(i, x \in F(i)_+)$ the category $\mathcal{I}_{(i,x)/}$ has objects $(i', (i,x) \to (i', \infty))$. We see that here $(i, (i,x) \to (i,\infty))$ is an initial object, so the category is weakly contractible.

Thus we have

$$\operatorname*{hocolim}_{\mathrm{Gr}(F)} \Phi \simeq \operatorname*{hocolim}_{\mathcal{I}} I_!^{\mathbb{L}}\Phi \circ J.$$

The functor $I_!^{\mathbb{L}}\Phi \circ J$ takes $i \in \mathcal{I}$ to $\mathrm{hocolim}_{\mathrm{Gr}(F)_{/(i,\infty)}} \Phi$.

There is a natural inclusion $F(i) \to \mathrm{Gr}(F)_{/(i,\infty)}$ given by $x \in F(i) \mapsto (i,x)$. This functor is also homotopy cofinal: Objects of $\mathrm{Gr}(F)_{/(i,\infty)}$ are given by $(i', x \in F(i'), f\colon i' \to i)$, and the category $F(i)_{(i',x,i' \to i)/}$ has objects consisting of $y \in F(i)$ together with a commutative triangle

$$\begin{array}{ccc} (i',x) & \longrightarrow & (i,y) \\ & \searrow \quad \swarrow & \\ & (i,\infty), & \end{array}$$

i.e. $(y \in F(i), F(f)(x) \to y)$ We see that $(F(f)(x)F(f)(x) \xrightarrow{\mathrm{id}} F(f)(x))$ is an initial object of this category, hence it is weakly contractible.

It follows that

$$\operatorname*{hocolim}_{\mathrm{Gr}(F)} \Phi \simeq \operatorname*{hocolim}_{i \in \mathcal{I}} \operatorname*{hocolim}_{F(i)} \Phi|_{F(i)},$$

as required. □

Proof of Proposition 10.3. Taking Φ in Proposition 10.4 to be the constant functor with value $*$, we get $\mathrm{BGr}(F) \xrightarrow{\sim} \mathrm{hocolim}_{\mathcal{I}} \mathrm{B}F(i)$. □

11. Quillen's Theorem B

In this section we indicate how our results so far lead to a proof of Quillen's Theorem B. Recall that this says:

Theorem 11.1 (Quillen's Theorem B). *Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is a functor such that for all maps $f\colon b \to b'$ in $\mathcal{B}$, the map $B(\mathcal{E}_{/b}) \to B(\mathcal{E}_{/b'})$ is a (weak) homotopy equivalence. Then the homotopy fibre of $B\mathcal{E} \to B\mathcal{B}$ at $b \in \mathcal{B}$ is $B(\mathcal{E}_{/b})$.*

We'll deduce this from our result in the last section together with the following result of Quillen [Qui73]:

Proposition 11.2. *Suppose $F\colon \mathcal{I} \to \mathrm{Top}$ is a functor such that for every morphism $\phi\colon i \to j$ in $\mathcal{I}$, the map $F(\phi)\colon F(i) \to F(j)$ is a homotopy equivalence. Then the homotopy fibre of the natural map*

$$\operatorname*{hocolim}_{\mathcal{I}} F \to \operatorname*{hocolim}_{\mathcal{I}} * \cong B\mathcal{I}$$

at i is weakly equivalent to $F(i)$.

Remark 11.3. Quillen's proof of this result is not hard, but we won't give it here as it uses some results about *quasifibrations* that we don't have time to discuss. A map of spaces is called a *quasifibration* if the natural map from its strict fibres to its homotopy fibres are weak equivalences. It is easy to see that the strict fibre of the map is $F(i)$ (for example, using that geometric realization commutes with pullbacks). Quillen shows that the map is a quasifibration using an induction over the skeleta of $B\mathcal{I}$ and some basic results on quasifibrations from [DT58] (which incidentally is probably the only important paper in homotopy theory that's in German).

Combining this with Proposition 10.3, we immediately get:

Corollary 11.4. *Suppose $F\colon \mathcal{I} \to \mathrm{Cat}$ is a functor such that for every map $f\colon i \to i'$ the map $BF(f)\colon BF(i) \to BF(i')$ is a (weak) homotopy equivalence. Then the homotopy fibre of $B\mathrm{Gr}(F) \to B\mathcal{I}$ at i is $BF(i)$.*

From this Theorem B follows easily:

Proof of Theorem B. Let $F\colon \mathcal{B} \to \mathrm{Cat}$ be the functor $b \mapsto \mathcal{E}_{/b} = \mathcal{E} \times_{\mathcal{B}} \mathcal{B}_{/b}$, and set $\overline{\mathcal{E}} := \mathrm{Gr}(F)$. Then the objects of $\overline{\mathcal{E}}$ are triples $(b \in \mathcal{B}, e \in \mathcal{E}, \phi\colon p(e) \to b)$ and a morphism $(b, e, \phi) \to (b', e', \phi')$ is given by maps $f\colon b \to b'$ and $g\colon e \to e'$ such that the square

$$\begin{array}{ccc} p(e) & \xrightarrow{p(g)} & p(e') \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} \\ b & \xrightarrow{f} & b' \end{array}$$

commutes. Let $\overline{p}$ denote the projection $\overline{\mathcal{E}} \to \mathcal{B}$ that takes (b, e, ϕ) to b; there is also a projection $q\colon \overline{\mathcal{E}} \to \mathcal{E}$ that takes this to e. Moreover, we have an inclusion $i\colon \mathcal{E} \to \overline{\mathcal{E}}$ given by $i(e) = (p(e), e, p(e) \xrightarrow{\mathrm{id}} p(e))$ and a commutative triangle

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\quad i \quad} & \overline{\mathcal{E}} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \overline{p}} & \\ & \mathcal{B}. & \end{array}$$

Clearly $qi = \mathrm{id}_{\mathcal{E}}$, and there is also a natural transformation $iq \to \mathrm{id}_{\overline{\mathcal{E}}}$ given at (b, e, ϕ) by the map $(pe, e, \mathrm{id}_{p(e)}) \to (b, e, \phi)$ determined by ϕ and id_e. But then Bq is a homotopy inverse to Bi, so $Bi\colon B\mathcal{E} \to B\overline{\mathcal{E}}$ is a homotopy equivalence. Thus the homotopy fibres of $\overline{p}$ and p are also (weakly) homotopy equivalent, and now applying Corollary 11.4 to F completes the proof. □

12. The Homology Spectral Sequence of a Simplicial Space

In this section we'll discuss a spectral sequence that computes the homology of a simplicial space, and derive a description of its E^2-page. This is originally due to Segal [Seg68]; there is a more detailed presentation of Segal's result on the first differential in the book [KT06] (but unfortunately some details are still omitted).

Let's start by briefly recalling the spectral sequence of a filtered chain complex. A *filtered chain complex* is a sequence of inclusions of chain complexes

$$C(0) \hookrightarrow C(1) \hookrightarrow C(2) \hookrightarrow \dots.$$

(One can also consider more general kinds of filtration, but this suffices for us.) Recall that from a filtered chain complex we get a spectral sequence of the form

$$E^1_{s,t} = H_{s+t}(F(s)) \Rightarrow H_{s+t}(C)$$

where $F(s)$ is the quotient $C(s)/C(s-1)$ and $C = \mathrm{colim}_{i\to\infty} C(i)$. The differentials are of the form

$$d_r\colon E^r_{s,t} \to E^r_{s-r,t+r-1}.$$

The spectral sequence converges if $H_{s+t}(F(s)) = 0$ for $t < 0$ — this implies that the spectral sequence lives in the first quadrant.

Remark 12.1. The first differential d_1 can be described as follows: Given a class $[x] \in H_{s+t}(F(s))$ let x be a lift of $[x]$ to $F(s)_{s+t}$. The map $C(s) \to F(s)$ is surjective, so we can choose $\bar{x} \in C(s)$ mapping to x. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(s-1)_{s+t} & \longrightarrow & C(s)_{s+t} & \longrightarrow & F(s)_{s+t} & \longrightarrow & 0 \\ \downarrow & & \downarrow d & & \downarrow d & & \downarrow d & & \downarrow \\ 0 & \longrightarrow & C(s-1)_{s+t-1} & \longrightarrow & C(s)_{s+t-1} & \longrightarrow & F(s)_{s+t-1} & \longrightarrow & 0, \end{array}$$

where the rows are exact. Thus, as $dx = 0$ in $F(s)_{s+t-1}$, the class $d\bar{x}$ must be the image of a (unique) $x' \in C(s-1)_{s+t-1}$ (and $dx' = 0$ since this is maps to $d^2\bar{x} = 0$ under an injective map). We define $d_1[x]$ to be the image of the cycle x' in $H_{s+t-1}F(s-1)$.

Exercise 12.2. Check that the class $d_1[x]$ in $H_{s+t-1}(F(s-1))$ is independent of the choices we made.

Remark 12.3. If $d_1[x] = 0$ then this means we can choose $y \in F(s-1)_{s+t}$ such that dy is the image of x'. Choose a $\bar{y} \in C(s-1)_{s+t}$ mapping to y, then $d\bar{y} \in C(s-1)_{s+t-1}$ maps to the image of x' in $F(s-1)_{s+t-1}$ and so $d\bar{y} - x'$ maps to 0 and thus by exactness lies in the image of $C(s-2)_{s+t-1}$. Let y' be the unique element that maps to $d\bar{y} - x'$; this is a cycle, and we define $d_2[x]$ to be the image of this in $E^2_{s,t}$. Iterating this process gives d_r for all r.

Now suppose we have a sequence of cofibrations of topological spaces

$$X(0) \hookrightarrow X(1) \hookrightarrow X(2) \hookrightarrow \cdots .$$

Let $X = \operatorname{colim}_{i\to\infty} X(i)$. Then taking (singular chains gives a filtered chain complex

$$C_*(X(0)) \hookrightarrow C_*(X(1)) \hookrightarrow C_*(X(2)) \hookrightarrow \cdots .$$

The associated spectral sequence is of the form

$$E^1_{s,t} = H_{s+t}(X(s), X(s-1)) \Rightarrow H_{s+t}(X);$$

it converges if $H_{s+t}(X(s), X(s-1))$ vanishes for $t < 0$, i.e. if the quotient $X(s)/X(s-1)$ has no homology in degrees $\leq (s-1)$.

Lemma 12.4. *In this spectral sequence the first differential d_1 is given by*

$$H_{s+t}(X(s), X(s-1)) \xrightarrow{\partial} H_{s+t-1}(X(s-1)) \to H_{s+t-1}(X(s-1), X(s-2)).$$

Proof. Immediate from the definition of d_1 above and the definition of the boundary map ∂. □

We now want to apply this to the geometric realization $|X|$ of a simplicial space $X_\bullet$. Suppose X is Reedy cofibrant, so the inclusion $L_nX \hookrightarrow X_n$ is a closed cofibration. Then, as we saw before, in the pushout square

$$\begin{array}{ccc} L_nX \times \Delta^n \amalg_{L_nX\times\partial\Delta^n} X_n \times \partial\Delta^n & \longrightarrow & \operatorname{sk}_{n-1}|X| \\ \downarrow & & \downarrow \\ X_n \times \Delta^n & \longrightarrow & \operatorname{sk}_n|X|. \end{array}$$

from Lemma 6.12(ii) the left vertical map is also a cofibration, hence so is the right vertical map. Thus the skeletal filtration

$$\mathrm{sk}_0|X| \hookrightarrow \mathrm{sk}_1|X| \hookrightarrow \cdots \hookrightarrow |X|$$

is given by cofibrations. We then have a spectral sequence of the form

$$E^1_{s,t} = H_{s+t}(\mathrm{sk}_s|X|, \mathrm{sk}_{s-1}|X|) \Rightarrow H_{s+t}(|X|).$$

Our pushout square means we have isomorphisms

$$\begin{aligned} H_{s+t}(\mathrm{sk}_s|X|, \mathrm{sk}_{s-1}|X|) &\cong H_{s+t}(X_s \times \Delta^s, L_sX \times \Delta^s \amalg_{L_sX\times\partial\Delta^s} X_s \times \partial\Delta^s) \\ &\cong H_{s+t}((X_s, L_sX) \wedge (\Delta^s, \partial\Delta^s)). \end{aligned}$$

Here we use the notation $(A, B) \wedge (A', B') = (A \times A', A \times B' \amalg_{B\times B'} B \times A')$ — note that $(A \times A')/(A \times B' \amalg_{B\times B'} B \times A') \cong A/B \wedge A'/B'$. In our case we thus have

$$H_{s+t}(\mathrm{sk}_s|X|, \mathrm{sk}_{s-1}|X|) \cong \tilde{H}_{s+t}(\Sigma^s(X_s/L_sX)) \cong H_t(X_s, L_sX).$$

In particular this clearly vanishes for $t < 0$, so the spectral sequence converges. In summary, we have:

Proposition 12.5. *Suppose $X_\bullet$ is a Reedy cofibrant simplicial space. Then there is a convergent spectral sequence of the form*

$$E^1_{s,t} = H_t(X_s, L_sX) \Rightarrow H_{s+t}(|X|).$$

Our next goal is to identify the first differential in this spectral sequence: we'll show that it is given by the alternating sum of the face maps from $X_\bullet$.

As a warm-up to proving this, let's consider the case where X is a simplicial set. Then writing X^{nd}_s for the set of non-degenerate s-simplices, we have a pushout square

$$\begin{array}{ccc} X^{\mathrm{nd}}_s \times \partial\Delta^s & \longrightarrow & \mathrm{sk}_{s-1}|X| \\ \downarrow & & \downarrow \\ X^{\mathrm{nd}}_s \times \Delta^s & \longrightarrow & \mathrm{sk}_s|X|. \end{array}$$

Consider $X^{\mathrm{nd}}_s \times \Delta^s$ equipped with the skeletal filtration, i.e. $X^{\mathrm{nd}}_s \times \mathrm{sk}_i\Delta^s$. With this filtration the map $X^{\mathrm{nd}}_s \times \Delta^s \to \mathrm{sk}_s|X|$ is a map of filtered spaces. This gives a commutative diagram

$$\begin{array}{ccccc} H_i(X^{\mathrm{nd}}_s \times \Delta^s, X^{\mathrm{nd}}_s \times \partial\Delta^s) & \xrightarrow{\partial} & H_{i-1}(X^{\mathrm{nd}}_s \times \partial\Delta^s) & \longrightarrow & H_{i-1}(X^{\mathrm{nd}}_s \times \partial\Delta^s, X^{\mathrm{nd}}_s \times \mathrm{sk}_{s-2}\Delta^s) \\ \downarrow\cong & & \downarrow & & \downarrow \\ H_i(\mathrm{sk}_s|X|, \mathrm{sk}_{s-1}|X|) & \xrightarrow{\partial} & H_{i-1}(\mathrm{sk}_{s-1}|X|) & \longrightarrow & H_{i-1}(\mathrm{sk}_{s-1}|X|, \mathrm{sk}_{s-2}|X|). \end{array}$$

Here the bottom row gives d_1. On the other hand, we have

$$(X^{\mathrm{nd}}_s \times \partial\Delta^s)/(X^{\mathrm{nd}}_s \times \mathrm{sk}_{s-2}\Delta^s) \cong \bigvee_{j=0}^{s} (X^{\mathrm{nd}}_s \times \Delta^{s-1})/(X^{\mathrm{nd}}_s \times \partial\Delta^{s-1}).$$

On the ith copy the map to $H_{i-1}(\mathrm{sk}_{s-1}|X|, \mathrm{sk}_{s-2}|X|)$ comes from the map

$$X_s \times \Delta^{s-1} \xrightarrow{d_i\times\mathrm{id}} X_{s-1} \times \Delta^{s-1} \to \mathrm{sk}_{s-1}|X|.$$

Taking the orientation of $\partial\Delta^s$ into account, we get $\sum(-1)^i d_i$, as we expected.

Now of course we have

$$H_t(X_s, L_sX) \cong \begin{cases} \mathbb{Z}X_s^{\mathrm{nd}} & t = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the E^1-page of the spectral sequence is 0 except for $t = 0$ where we have a chain complex

$$\cdots \to \mathbb{Z}X_s^{\mathrm{nd}} \to \mathbb{Z}X_{s-1}^{\mathrm{nd}} \to \cdots \to \mathbb{Z}X_0^{\mathrm{nd}} = \mathbb{Z}X_0.$$

Since the spectral sequence converges the homology of this is isomorphic to the homology $H_*(|X|)$ of the realization. We can interpret this using the chain complex associated to the simplicial set X; this uses a standard result on simplicial abelian groups:

Lemma 12.6. *Suppose $A_\bullet$ is a simplicial abelian group, and write $C(A)$ for the associated chain complex with $C(A)_n = A_n$ and $d = \sum_i (-1)^i d_i$. Let $D_k(A)$ be the subgroup of degenerate elements. Then the differential on $C(A)$ restricts to a differential on $D(A)$. Moreover, the chain complex $D(A)$ is contractible, so the projection $C(A) \to \bar{C}(A) := C(A)/D(A)$ is a quasi-isomorphism.*

The chain complex we have is $\bar{C}(\mathbb{Z}X)$, so this is quasi-isomorphic to the usual chain complex $C(\mathbb{Z}X)$. Thus, if we want we can add back in the degenerate simplices and as the spectral sequence converges we've proved

$$H_*(|X|) \cong H_*(C(\mathbb{Z}X)).$$

Before we turn to the case of general simplicial spaces, we state a lemma:

Lemma 12.7. *For a Reedy cofibrant simplicial space $X_\bullet$, let $F_{-k}X_s$ denote the union of $s_{i_1} \cdots s_{i_k} X_{s-k}$ in X_s, and let $F_{-k}H_*X_s$ be the subgroup generated by the images of H_*X_{s-k} under the degeneracies. Then*

(1) $H_*(F_{-k}X_s) \to H_*(F_{1-k}X_s)$ is injective.

(2) The image of $H_*(F_{-k}X_s)$ in H_*X_s is precisely $F_{-k}H_*X_s$. In particular, $H_*(L_sX) \cong D(H_*X_*)_s$.

This is supposed to follow from the simplicial identities, but we won't prove it. In any case it is obvious for the case we're interested in, the bar construction, because the degenerate part splits off. In fact, in this case we can use the same proof as for simplicial sets.

Proposition 12.8. *Let $X_\bullet$ be a Reedy cofibrant simplicial space. Then the tth row in the E^1-page of the spectral sequence is $\bar{C}(H_t(X_*))$. Thus we have*

$$E^2_{s,t} \cong H_s(H_tX_*).$$

Proof. Filter X_s as before, and filter $X_s \times \Delta^s$ by taking

$$F_k := F_k(X_s \times \Delta^s) = \bigcup_{i+j=k} F_iX_s \times \mathrm{sk}_j\Delta^s.$$

Thus $F_s = X_s \times \Delta^s$ and $F_{s-1} = L_sX \times \Delta^s \amalg_{L_sX \times \partial\Delta^s} X_s \times \partial\Delta^s$. With this filtration the map $X_s \times \Delta^s \to \mathrm{sk}_s|X|$ is a map of filtered spaces. This means we get a commutative diagram

$$\begin{array}{ccccc} H_i(F_s, F_{s-1}) & \xrightarrow{\partial} & H_{i-1}(F_{s-1}) & \longrightarrow & H_{i-1}(F_{s-1}, F_{s-2}) \\ \downarrow{\cong} & & \downarrow & & \downarrow \\ H_i(\mathrm{sk}_s|X|, \mathrm{sk}_{s-1}|X|) & \xrightarrow{\partial} & H_{i-1}(\mathrm{sk}_{s-1}|X|) & \longrightarrow & H_{i-1}(\mathrm{sk}_{s-1}|X|, \mathrm{sk}_{s-2}|X|). \end{array}$$

Here the space F_{s-1}/F_{s-2} splits as

$$((\Delta^s/\partial\Delta^s) \wedge (L_sX/F_{-2}X_s)) \vee ((\partial\Delta^s/\mathrm{sk}_{s-2}\Delta^s) \wedge (X_s/L_sX)).$$

On the first summand, we can identify the differential using the diagram (also from a map of filtered spaces)

$$\begin{array}{ccccc} H_t(X_s, L_sX) & \xrightarrow{\partial} & H_{t-1}(L_sX) & \longrightarrow & H_{t-1}(L_sX, F_{-2}X_s) \\ \downarrow{\cong} & & \downarrow & & \downarrow \\ H_{s+t}(F_s, F_{s-1}) & \xrightarrow{\partial} & H_{s+t-1}(F_{s-1}) & \longrightarrow & H_{s+t-1}(F_{s-1}, F_{s-2}). \end{array}$$

But here the boundary map $H_t(X_s, L_sX) \to H_{t-1}(L_sX)$ is zero, since $H_*(L_s) \to H_*(X_s)$ is injective. Thus the only contribution to d_1 comes from

$$H_{s+t-1}((\partial\Delta^s/\mathrm{sk}_{s-2}\Delta^s) \wedge (X_s/L_sX)) \cong \bigoplus_{i=0}^{s} H_t(X_s, L_sX) \to H_{t-1}(X_{s-1}, L_{s-1}X),$$

and this is given by the alternating sum of face maps, as before. □

13. The Homology Spectral Sequence of a Homotopy Colimit

We now want to give a more conceptual (and potentially morecomputable) description of the E^2-page of the homology spectral sequence for the simplicial space $\mathbb{B}_\bullet(*, \mathcal{I}, F)$ that defines the homotopy colimit of F. This will be in terms of the derived functors of the colimit functor for abelian groups. We will now introduce these, and then derive the required expression for relating them to the E^2-page.

First of all, observe that for any small category $\mathcal{I}$, the functor category $\mathrm{Ab}^{\mathcal{I}}$ is an abelian category. To define derived functors, we first need to know this category has enough projective:

Definition 13.1. Let $\mathcal{I}$ be a small category. For $i \in \mathcal{I}$, write L_i for the functor $\mathcal{I} \to \mathrm{Ab}$ given by $i \mapsto \mathbb{Z}\mathcal{I}(i, -)$.

Example 13.2. If $\mathcal{I} = 1 \leftarrow 0 \rightarrow 2$, then $L_0 = (\mathbb{Z} \leftarrow \mathbb{Z} \rightarrow \mathbb{Z})$, $L_1 = (\mathbb{Z} \leftarrow 0 \rightarrow 0)$, and $L_2 = (0 \leftarrow 0 \rightarrow \mathbb{Z})$.

Lemma 13.3. *The object $L_i \in \mathrm{Ab}^{\mathcal{I}}$ is projective.*

Proof. Since L_i is free, for $F \in \mathrm{Ab}^{\mathcal{I}}$ we have

$$\mathrm{Hom}_{\mathrm{Ab}^{\mathcal{I}}}(L_i, F) \cong \mathrm{Hom}_{\mathrm{Set}}(\mathcal{I}(i, -), F) \cong F(i)$$

by the Yoneda Lemma. Thus given a surjective map $G \twoheadrightarrow F$ and a map $L_i \to F$, this map corresponds to $x \in F(i)$ and a lift $L_i \to G$ corresponds to an element of $G(i)$ that maps to x. Since $G(i) \to F(i)$ was by assumption surjective, this exists. □

Exercise 13.4. This exercise is somewhat paranthetical, but some may enjoy it: Show that the argument in this lemma can be seen directly as an instance of the Yoneda lemma in enriched category theory, where the enrichment is over abelian groups (and find out that these words mean). [Hint: Introduce the linear category $\mathbb{Z}\mathcal{I}$ with objects $\mathcal{I}$ and morphisms the $\mathbb{Z}$–linear span of the morphisms in $\mathcal{I}$ ie, $\mathrm{Hom}_{\mathbb{Z}\mathcal{I}}(i, j) = \mathbb{Z}\,\mathrm{Hom}_{\mathcal{I}}(i, j)$, and observe that $L_i(j) = \mathrm{Hom}_{\mathbb{Z}\mathcal{I}}(i, j)$.]

Lemma 13.5. *The category $\mathrm{Ab}^{\mathcal{I}}$ has enough projectives.*

Proof. Given $F \in \mathrm{Ab}^{\mathcal{I}}$ we can take $P = \bigoplus_{i\in\mathcal{I}} \bigoplus_{x\in F(i)} L_i$; this is projective and has a surjective map $P \to F$ given an the component corresponding to $x \in F(i)$ by the map $L_i \to F$ corresponding to x. □

The functor $\mathrm{colim}\colon \mathrm{Ab}^{\mathcal{I}} \to \mathrm{Ab}$ is clearly right exact (since it commutes with all colimits) and by Lemma 13.5 the category $Ab^{\mathcal{I}}$ has enough projectives. We can therefore make the following definition:

Definition 13.6. Define colim_s as the sth left derived functor of colim. In other words, for a functor $F \in \mathrm{Ab}^{\mathcal{I}}$, the abelian group $\mathrm{colim}_i \mathbb{X} \in \mathrm{Ab}$ is $H_i(\mathrm{colim}\, P_\bullet)$ where $P_\bullet$ is a projective resolution of $F \in \mathrm{Ab}^{\mathcal{I}}$.

Example 13.7. If $\mathcal{I} = \mathcal{B}G$, meaning a group G viewed as a category with one object $*$, then the functor L_* is just the group algebra $\mathbb{Z}G$ viewed as a free module of rank one, and $\mathrm{colim}_s M$ identifies with $H_s(G; M)$.

Example 13.8. [This will probably be done in the exercise session.] Suppose $\mathcal{I}$ is the category $\bullet \leftarrow \bullet \to \bullet$, and consider $M = A \xleftarrow{f} C \xrightarrow{g} B$. Then

$$\mathrm{colim}_s M \cong \begin{cases} \mathrm{colim}\, M \cong \mathrm{coker}(C \xrightarrow{(f,-g)} A \oplus B) & s = 0, \\ \ker(C \xrightarrow{(f,-g)} A \oplus B) & s = 1, \\ 0 & s > 1. \end{cases}$$

To see this, consider the exact sequence

$$0 \to (A \leftarrow 0 \to B) \to (A \leftarrow C \to B) \to (0 \leftarrow C \to 0) \to 0$$

By the long exact sequence of derived functors, the claim will follow from the following claims:

$$\mathrm{colim}_s((A \leftarrow 0 \to B)) = \begin{cases} A \oplus B & s = 0 \\ 0 & s > 0 \end{cases}$$

$$\mathrm{colim}_s((0 \leftarrow C \to 0)) = \begin{cases} 0 & s \neq 1 \\ C & s = 1 \end{cases}$$

For the first claim, the resolution will just be a resolution in Ab at the spots of A and B, so the claim follows easily.

For the second, let $P \xrightarrow{p} C$ be the projective cover, and let $K = \ker(p)$.

Let $C_0 = (P \leftarrow P \to P)$ which maps onto $(0 \leftarrow C \to 0)$ with kernel $(P \leftarrow K \to P)$ Set

$$C_1 = ((K \leftarrow K \to K) \oplus (0 \leftarrow 0 \to P) \oplus (P \leftarrow 0 \to 0))$$

which maps onto $(P \leftarrow K \to P)$ with kernel $C_2 := (K \leftarrow 0 \to 0) \oplus (0 \leftarrow 0 \to K)$

We have hence produced a projective resolution of length 2.

Taking colim gives

$$K \oplus K \to (K \oplus K)/\Delta(K) \oplus P \oplus P \to (P \oplus P)/\Delta(P)$$

which has homology

$$0 \to C \to 0$$

[Alternative proof of the second part: First observe that the claim holds for diagrams $(0 \leftarrow P \to 0)$ and use the long the long exact sequence for

$$(0 \leftarrow K \to 0) \to (0 \leftarrow P \to 0) \to (0 \leftarrow C \to 0)$$

to prove it in general.]

Remark 13.9. Notice that colim only has a first derived functor in the category of pushouts of abelian groups, despite the fact that the projective dimension of the category of pushouts is 2. An explanation of this fact is given in Proposition 13.11, and has to do with considering $\operatorname{colim}_{\mathcal{I}} F = \mathbb{Z} \otimes_{\mathcal{I}} F$ as a bi-functor.

Definition 13.10. For $F\colon \mathcal{I} \to \mathrm{Ab}$, let F^{Δ} denote the simplicial abelian group where

$$F_n^{\Delta} := \bigoplus_{i_0 \to \cdots \to i_n} F(i_0),$$

with the by now familiar structure maps. We write $E(F) := C(F^{\Delta})$ for the associated chain complex.

Proposition 13.11. *For* $F : \mathcal{I} \to \mathrm{Ab}$, *we have a natural isomorphism*

$$\operatorname{colim}_n F \cong H_n E(F).$$

Proof. For $W\colon \mathcal{I}^{\mathrm{op}} \to \mathrm{Ab}$ and $F\colon \mathcal{I} \to \mathrm{Ab}$, in this proof we'll write $W \otimes_{\mathcal{I}} F$ for the coend of $W \otimes F\colon \mathcal{I}^{\mathrm{op}} \times \mathcal{I} \to \mathrm{Ab}$. Then

$$\operatorname*{colim}_{\mathcal{I}} F \cong \mathbb{Z} \otimes_{\mathcal{I}} F,$$

since $\mathbb{Z}$ is the unit for the tensor product.

Moreover, $- \otimes_{\mathcal{I}} -$ is right exact in both variables (since it preserves colimits in both), and by the same double complex argument as for the derived functors of $\otimes$ we can use a projective resolution in either variable to compute derived functors.

We claim that the functor $i \mapsto E(L_i)$ is a projective resolution of $\mathbb{Z}$ in $\mathrm{Ab}^{\mathcal{I}^{\mathrm{op}}}$. Note that $i \mapsto L_i(x) = \mathbb{Z}\mathcal{I}(i,x) = \mathbb{Z}\mathcal{I}^{\mathrm{op}}(x,i)$ is projective in $\mathrm{Ab}^{\mathcal{I}^{\mathrm{op}}}$, so $E(L_{(-)})$ is a chain complex of projectives since each term is a direct sum of such functors.

Now by the usual rewriting we have $E(L_i)_n = \bigoplus_{i \to i_0 \to \cdots \to i_n} \mathbb{Z} \cong \mathbb{Z}(\mathrm{N}\mathcal{I}_{i/})_n$. Thus $E(L_i) \cong C(\mathbb{Z}\mathrm{N}\mathcal{I}_{i/}$ and so

$$H_* E(L_i) \cong H_* B\mathcal{I}_{i/} \cong \begin{cases} \mathbb{Z}, & * = 0 \\ 0, & * \neq 0, \end{cases}$$

since $B\mathcal{I}_{i/}$ is contractible.

It follows that $\operatorname{colim}_s F$ is computed by the chain complex $E(L_{(-)}) \otimes_{\mathcal{I}} F$. Now we need to identify this with $E(F)$. We have

$$\begin{aligned}
E(L_{(-)})_n \otimes_{\mathcal{I}} F &\cong \operatorname{coeq}\left(\bigoplus_{i \to i'} \bigoplus_{i_0 \to \cdots \to i_n} \mathbb{Z}\mathcal{I}(i', i_0) \otimes F(i) \rightrightarrows \bigoplus_{i} \bigoplus_{i_0 \to \cdots \to i_n} \mathbb{Z}\mathcal{I}(i, i_0) \otimes F(i) \right) \\
&\cong \operatorname{coeq}\left(\bigoplus_{i \to i' \to i_0 \to \cdots \to i_n} F(i) \rightrightarrows \bigoplus_{i \to i_0 \to \cdots \to i_n} F(i) \right) \\
&\cong \bigoplus_{i_0 \to \cdots \to i_n} \operatorname{coeq}\left(\bigoplus_{i \to i' \to i_0} F(i) \rightrightarrows \bigoplus_{i \to i_0} F(i) \right) \\
&\cong \bigoplus_{i_0 \to \cdots \to i_n} \operatorname*{colim}_{\mathcal{I}_{/i_0}} F \\
&\cong \bigoplus_{i_0 \to \cdots \to i_n} F(i_0) \\
&= E(F)_n.
\end{aligned}$$

Thus $\operatorname{colim}_s F \cong H_s(E(L_{(-)}) \otimes_{\mathcal{I}} F) \cong H_s E(F)$, as required. ☐

Theorem 13.12. *Given $F \colon \mathcal{I} \to \mathrm{Top}$ there is a convergent spectral sequence of the form*

$$E^2_{s,t} = \operatorname{colim}_s H_t F \Rightarrow H_{s+t}(\operatorname{hocolim} F).$$

Proof. We consider the homology spectral sequence of the simplicial space $\mathbb{B}_\bullet(*, \mathcal{I}, F)$. By Proposition 12.5 this converges, and by Proposition 12.8 it has E^2-term given by

$$E^2_{s,t} \cong H_s(H_t(\mathbb{B}_*(*, \mathcal{I}, F))).$$

But here $H_t(\mathbb{B}_n(*, \mathcal{I}, F)) \cong \bigoplus_{i_0 \to \cdots \to i_n} H_t F(i_0) \cong E(H_t F)_n$, and the differentials clearly agree too. So

$$E^2_{s,t} \cong H_s E(H_t F) \cong \operatorname{colim}_s H_t F.$$ ☐

Exercise 13.13. By Example 13.8, the E_2-term of the spectral sequence for a homotopy pushout $(X \leftarrow A \to Y)$ degenerates to two lines:

$$\operatorname{colim}_0(H_i(-)) = \operatorname{coker}(H_i(C) \to H_i(X) \oplus H_i(Y))$$

$$\operatorname{colim}_1(H_i(-)) = \ker(H_i(C) \to H_i(X) \oplus H_i(Y))$$

Check that the corresponding long exact sequence identifies with the Meyer-Vietoris sequence.

14. Localizations and Reflective Subcategories

Definition 14.1. Let W be a collection of morphisms (not necessarily a set) in a category $\mathcal{C}$. We say an object $c \in \mathcal{C}$ is *W-local* if for every morphism $\phi \colon x \to y$ in W, the map

$$\operatorname{Hom}_{\mathcal{C}}(y, c) \xrightarrow{\phi^*} \operatorname{Hom}_{\mathcal{C}}(x, c)$$

is an isomorphism.

Definition 14.2. For $x \in \mathcal{C}$, a *W-localization* of x is a morphism $\lambda_x \colon x \to Lx$ where Lx is a W-local object of $\mathcal{C}$ and the map λ_x lies in W.

Remark 14.3. If x, y are W-local and $\phi\colon x \to y$ is in W, then ϕ is an isomorphism: composition with ϕ gives isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(y, x) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(x, x), \quad \mathrm{Hom}_{\mathcal{C}}(y, y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(x, y),$$

from which it follows that ϕ has both a left and a right inverse. Thus if x is W-local a W-localization $x \to Lx$ is necessarily an isomorphism.

Lemma 14.4. *Suppose $x \to Lx$ is a W-localization.*

(i) A map $f\colon x \to y$ with y W-local factors uniquely as $x \xrightarrow{\lambda_x} Lx \xrightarrow{f'} y$.
(ii) For any map $g\colon x \to z$ in W there exists a unique map $l\colon z \to Lx$ such that $\lambda_x = l \circ g$.

Proof. Since y is W-local and λ_x is in W, we have an isomorphism $\mathrm{Hom}_{\mathcal{C}}(Lx, y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(x, y)$. Then f has a unique preimage f', which gives (i).

For (ii), observe that since Lx is W-local and f is in W, we have an isomorphism $\mathrm{Hom}_{\mathcal{C}}(z, Lx) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(x, Lx)$. Then λ_x has a unique preimage l. □

Remark 14.5. We can reformulate this as: a W-localization $\lambda_x\colon x \to Lx$ is the initial map from x to a W-local object, and the terminal map in W out of x. In particular, we see that the W-localization is unique (up to unique isomorphism) if it exists.

Proposition 14.6. *Suppose every object of $\mathcal{C}$ has a W-localization. Let $\mathcal{C}_W$ denote the full subcategory of $\mathcal{C}$ spanned by the W-local objects. Then the inclusion $i\colon \mathcal{C}_W \hookrightarrow \mathcal{C}$ has a left adjoint $L\colon \mathcal{C} \to \mathcal{C}_W$ (where the unit map $x \to Lx$ is a W-localization of x) and L exhibits $\mathcal{C}_W$ as the localization $\mathcal{C}[W^{-1}]$.*

Proof. If $x \to Lx$ is a W-localization of x, then the induced isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(x, iy) \cong \mathrm{Hom}_{\mathcal{C}}(Lx, y) \cong \mathrm{Hom}_{\mathcal{C}_W}(Lx, y)$$

is natural in y, so the functor $\mathrm{Hom}_{\mathcal{C}}(x, i(-))\colon \mathcal{C}_W \to \mathrm{Set}$ is representable (by Lx) for every x. By the Yoneda Lemma it follows that i has a left adjoint with $x \to Lx$ as the unit map at x. (More explicitly, we can define the functor L on morphisms by taking $L(f)$ for $f\colon x \to y$ to be the unique map $Lx \to Ly$ that factors $\lambda_y \circ f\colon x \to Ly$ through Lx.)

Now suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a functor that takes the morphisms in W to isomorphisms. Let $F' := F \circ i\colon \mathcal{C}_W \to \mathcal{D}$. Then $F \cong F'L$: the unit transformation $\lambda\colon \mathrm{id} \to iL$ is given by maps in W, so since F takes these maps to isomorphisms, $F\lambda\colon F \to FiL = F'L$ is a natural isomorphism. Thus F factors through L.

To see that this factorization is unique, suppose we have a natural isomorphism $F \cong GL$ for some $G\colon \mathcal{C}_W \to \mathcal{D}$. Then we get $F' = Fi \cong GLi = G$ since $Li = \mathrm{id}$. This shows that $\mathcal{C}_W$ satisfies the universal property of $\mathcal{C}[W^{-1}]$. □

Definition 14.7. A full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ such that the inclusion has a left adjoint is called a *reflective* subcategory.

Example 14.8. Let $S \subseteq \mathbb{Z}$ be a set of primes, and take W to be the collection of maps $f\colon A \to B$ in Ab such that $S^{-1}f\colon S^{-1}A \to S^{-1}B$ is an isomorphism. Then W-localizations in Ab exist and are given by the natural maps $A \to S^{-1}A$. (This is just a reformulation fo the usual universal property of $S^{-1}A$.)

Remark 14.9. If $\mathcal{C}$ is a very well-behaved category, namely a so-called *presentable* (or sometimes "locally presentable") category, then W-localizations always exist for classes of morphisms W that are in a suitable sense generated by a *set* of morphisms.

15. R-Localization of Spaces

We're interested in W-localizations in Ho(Top) where W is the class W_R of maps $f\colon X \to Y$ such that $f_*\colon H_*(X;R) \to H_*(Y;R)$ is an isomorphism for some fixed ring R. We'll call the maps in W_R *R-equivalences*, and we'll call W_R-local spaces *R-local* and talk about *R-localization* of spaces. The category Ho(Top) is far from being presentable (one requirement is the existence of colimits), so the existence of R-localizations is not formal. Nevertheless, we have the following result of Bousfield:

Theorem 15.1 (Bousfield [Bou75])**.** *Any space $X \in$ Ho(Top) admits an R-localization.*

We won't prove this here. Instead, we'll give explicit constructions of R-localizations for nice (more precisely, simply connected) spaces in two important cases:

- $R = \mathbb{Z}_{(p)} := \mathbb{Z}[q^{-1} : q \neq p \text{ prime}]$; $\mathbb{Z}_{(p)}$-localization is called *p-localization*, and we write $X_{(p)}$ for the p-localization of X. More generally we'll consider $R \subseteq \mathbb{Q}$; this gives for example the *rationalization* $X_{\mathbb{Q}}$ of a space, i.e. its $\mathbb{Q}$-localization.
- $R = \mathbb{F}_p$; $\mathbb{F}_p$-localization is often called *p-completion*, and we write $X_p^\wedge$ for the p-completion of X.

The general existence result gives an inexplicit description of the localization that is not useful for computations, so the explicit construction is in any case important. Note that with more work everything we do can be extended from simply connected spaces to *nilpotent* spaces.

Remark 15.2.

- With more care one can construct R-localization as a functor on Top rather than Ho(Top), which is (very) often convenient since it is much better to have diagrams that commute strictly than only up to homotopy, as we've seen in the previous sections.
- The R-local objects can be shown to be fibrant objects in a model category, and this is often a convenient way to construct the localization.

Remark 15.3. The name "p-completion" will make more sense when we understand $\mathbb{F}_p$-local objects. Note that there is a map from the p-localization to the p-completion (as well as to the rationalization) induced by the maps $\mathbb{Z}_{(p)} \to \mathbb{F}_p$ and $\mathbb{Z}_{(p)} \to \mathbb{Q}$. The p-localization is easier to define than p-completion, but the p–completion is in general more computable and therefore more useful. This is related to the fact that we have more methods for calculating maps into p-complete spaces — they are "more algebraic" in the sense that p-completions can be expressed as limits of $K(\mathbb{F}_p, n)$'s for different n's (in fact in several different ways). This ultimately gives us ways of understanding the maps into p-complete spaces in terms of mod p cohomology.

We will also examine how a space X can be recovered from these localizations. This is called Sullivan's arithmetic square or, as Sullivan writes, a Hasse principle for spaces. There is both a Sullivan square for p-localization and one for p-completion.

We'll now consider some examples of R-local spaces:

Lemma 15.4. *Suppose R is a principal ideal domain. Then the Eilenberg-MacLane space $K(M,n)$ is R-local for any R-module M.*

Proof. We have $[X, K(M,n)] \cong H^n(X;M)$. Since R is a principal ideal domain we have a universal coefficient sequence

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(X;R), M) \to H^n(X;M) \to \mathrm{Hom}(H_n(X;R), M) \to 0.$$

Using the 5-Lemma we see from this that any map that gives an isomorphism on R-homology gives an isomorphism on M-cohomology, hence $K(M,n)$ is R-local. □

Lemma 15.5. *The spaces $K(\mathbb{Z}/p^k, n)$ and $K(\mathbb{Z}_p\hat{}, n)$ are $\mathbb{F}_p$-local.*

Proof. To see this for $K(\mathbb{Z}/p^k, n)$ we induct on k (note that the case $k = 1$ follows from Lemma 15.4) and use the long exact sequence in cohomology induced by the short exact sequence of groups

$$0 \to \mathbb{Z}/p^{k-1} \to \mathbb{Z}/p^k \to \mathbb{Z}/p \to 0$$

plus the 5-Lemma. For $\mathbb{Z}_p\hat{}$ we use the Milnor $\lim^1$-sequence, which is

$$0 \to \lim\nolimits^1_k H^{n-1}(X; \mathbb{Z}/p^k) \to H^n(X; \mathbb{Z}_p\hat{}) \to \lim_k H^n(X; \mathbb{Z}/p^k) \to 0,$$

to see we get an isomorphism on $\mathbb{Z}_p\hat{}$-cohomology. □

16. p-Localization of Simply Connected Spaces

Suppose that $R \subseteq \mathbb{Q}$ — this implies that $R = S^{-1}\mathbb{Z}$ for some set S of primes. Homology with R-coefficients for such R is easy to describe: we have $H_*(-; R) = H_*(-) \otimes R$ by the Universal Coefficient Theorem, since $- \otimes R$ is exact for torsion-free abelian groups. Furthermore, we have an alternative description of R-equivalences between simply connected spaces:

Lemma 16.1. *Suppose $R \subseteq \mathbb{Q}$. Let $f : X \to Y$ be a map between simply connected spaces. Then f is an R-equivalence if and only if it induces an isomorphism $\pi_*(X) \otimes R \to \pi_*(Y) \otimes R$.*

Proof. This follows from the Whitehead theorem modulo Serre Classes (cf. [Hat]), taking as Serre class the class of abelian groups A such that $A \otimes R = 0$, i.e. the class of torsion groups whose torsion is of order □

Our first main result gives a characterization of R-local spaces:

Theorem 16.2. *Suppose X is simply connected and $R \subseteq \mathbb{Q}$. Then X is R-local if and only if $\pi_n(X)$ is R-local for all n.*

Let us first prove the easy direction:

Lemma 16.3. *Suppose $R \subseteq \mathbb{Q}$ and X is a simply connected R-local space. Then $\pi_n(X)$ is R-local for all n.*

Proof. The degree-p map $S^n \xrightarrow{p} S^n$ is in W_R for all primes p that are invertible in R: We have $H_*(S^n; R) \cong H_*(S^n) \otimes R$, and on $H_*(S^n)$ this map is given by multiplication by p. Thus if X is R-local, this map induces an isomorphism on $[S^n, X] = \pi_n(X)$, i.e. $\pi_n(X)$ is R-local. □

For the less trivial direction, we will use an induction going up the Postnikov tower, so we first briefly review this:

Proposition 16.4 (Postnikov towers)**.** *Suppose that X is a simply connected CW-complex. Then there exists a tower of principal fibrations*

$$\cdots \to P_n X \to P_{n-1} X \to P_1 X$$

and a map from X into the tower, satisfying the following:

- $X \to \lim_n P_n X$ *is a homotopy equivalence,*
- $\pi_i(X) \to \pi_i(P_n X)$ *is an isomorphism for $i \leq n$ and $\pi_i(P_n X) = 0$ for $i > n$.*

See for example [Hat02, Theorem 4.69] for a proof. Here a fibration $F \to E \to B$ is called *principal* if we have a commutative diagram

$$\begin{array}{ccccccc} F & \longrightarrow & E & \longrightarrow & B & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \Omega X & \longrightarrow & E' & \longrightarrow & B' & \longrightarrow & X, \end{array}$$

where $B' \to X$ is a fibration, the vertical maps are weak equivalences, and the bottom row is the start of a Puppe sequence. For the principal fibration $P_n X \to P_{n-1}X$ it follows from the long exact sequence that the fibre is a $K(\pi_n X, n)$, so the "classifying space" X is $K(\pi_n X, n+1)$. Thus we have a homotopy pullback square

$$\begin{array}{ccc} P_n X & \longrightarrow & P_{n-1}X \\ \downarrow & & \downarrow{\scriptstyle k_{n-1}} \\ * & \longrightarrow & K(\pi_n X, n+1). \end{array}$$

The maps $k_{n-1}\colon P_{n-1}X \to K(\pi_n X, n+1)$ correspond to cohomology classes $k_{n-1} \in H^{n+1}(P_{n-1}X; \pi_n X)$ called the *k-invariants* of X.

Remark 16.5. For a general space the Postnikov tower still exists as a tower of fibrations, but in general they do not have to be *principal*. In fact, it can be shown that the fibrations in the Postnikov tower are principal if and only if the space X is *simple*, which means that $\pi_1 X$ is abelian and acts trivially on $\pi_* X$. (The homotopy fibre of a map of simply connected spaces is always simple; this example will come up later.) More generally, Postnikov towers are also well-behaved for *nilpotent* spaces, meaning the fundamental group is nilpotent (in the sense of group theory) and acts nilpotently on the higher homotopy groups, but this is a bit more complicated.

For inductive arguments using the Postnikov tower we will also need a variant of the 5-Lemma that works for Puppe sequences:

Lemma 16.6. *Let*

$$\begin{array}{ccccccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & X_4 & \longrightarrow & X_5 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_4} & & \downarrow{\scriptstyle \alpha_5} \\ Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & Y_4 & \longrightarrow & Y_5 \end{array}$$

be a diagram of pointed sets, with exact rows, where furthermore X_1,X_2,Y_1, and Y_2 are groups and the maps between them are group homomorphisms. Suppose furthermore that X_2 acts on X_3 and Y_2 acts on Y_3, and that α_2 and α_3 are compatible with these actions in the obvious way; exactness at at X_3 and Y_3 means exactness in terms of these groups actions. Then if α_1,α_2, α_4, and α_5 are isomorphisms, so is α_3.

Proof. The proof is just the standard proof of the 5-lemma. □

Using this we have:

Lemma 16.7 (Fibre Lemma)**.** *Suppose $E \to B$ is a fibration where E and B are both simply connected and W-local, where W is a class of maps such that if $f\colon X \to Y$ is in W, so is $\Sigma f\colon \Sigma X \to \Sigma Y$. (This holds for W_R for any ring R.) Then the fibre F is also W-local.*

Proof. Given a map $f\colon X \to Y$ in W, we have a commutative diagram

$$\begin{array}{ccccccccc} [Y,\Omega E] & \longrightarrow & [Y,\Omega B] & \longrightarrow & [Y,F] & \longrightarrow & [Y,E] & \longrightarrow & [Y,B] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [X,\Omega E] & \longrightarrow & [X,\Omega B] & \longrightarrow & [X,F] & \longrightarrow & [X,E] & \longrightarrow & [X,B]. \end{array}$$

This satisfies the hypotheses of Lemma 16.6 (for the two leftmost vertical maps we use the adjunction between Σ and Ω and the assumption that Σf is in W). Thus $[Y,F] \to [X,F]$ is an isomorphism for all $f\colon X \to Y$ in W, i.e. F is W-local. (Note that there is no difference between pointed and unpointed homotopy classes since all targets are simple spaces.) □

Lemma 16.8 (Tower Lemma)**.** *Suppose* $\cdots \to Z_2 \to Z_1 \to Z_0$ *is a tower of fibrations between pointed simply connected* W*-local spaces, where* W *is a class of maps such that if* $f\colon X \to Y$ *is in* W*, so is* $\Sigma f\colon \Sigma X \to \Sigma Y$*. Then* $\lim_{n\to\infty} Z_n$ *is also* W*-local.*

Proof. For any space X we have a natural $\lim^1$ exact sequence:

$$0 \to \lim^1_n [X, \Omega Z_n] \to [X, \lim_n Z_n] \to \lim_n [X, Z_n] \to 0.$$

Therefore we can again use Lemma 16.6 to see that any map $X \to Y$ in W_R induces an isomorphism on the middle terms $[Y, \lim_n Z_n] \to [X, \lim_n Z_n]$. □

We can now complete the proof of the Theorem:

Proof of Theorem 16.2. It remains to prove that if a simply connected space X has R-local homotopy groups, then X is R-local. We will first show by induction that the spaces P_nX in the Postnikov tower of X are R-local. Since X is simply connected, the first of these is P_2X which is $K(\pi_2X, 2)$; this is R-local by Lemma 15.4. Assuming $P_{n-1}X$ is R-local, we have a homotopy pullback square

$$\begin{array}{ccc} P_nX & \longrightarrow & P_{n-1}X \\ \downarrow & & \downarrow{\scriptstyle k_{n-1}} \\ * & \longrightarrow & K(\pi_nX, n+1). \end{array}$$

Here $K(\pi_nX, n+1)$ is again R-local by Lemma 15.4, so P_nX is R-local by Lemma 16.7. Then $X \simeq \lim_n P_nX$ is R-local by Lemma 16.8. □

Example 16.9. Let $J = \{p_1, p_2, \ldots\}$ be the set of all primes but p. Then

$$S^n_{(p)} = \operatorname{hocolim}(S^n \xrightarrow{p_1} S^n \xrightarrow{p_1p_2} S^n \xrightarrow{p_1p_2p_3} \cdots)$$

is the p-localization of of S^n: We can describe this homotopy colimit by replacing the maps by cofibrations and then taking the ordinary colimits. Since π_* commutes with sequential colimits along cofibrations (this just follows from a compactness argument), we get

$$\pi_*(S^n_{(p)}) \cong \operatorname*{colim}_k \pi_*(S^n) \cong \pi_*(S^n)_{(p)}.$$

Thus $S^n_{(p)}$ is p-local by Theorem 16.2, and the map $S^n \to S^n_{(p)}$ is a $\mathbb{Z}_{(p)}$-equivalence by Lemma 16.1.

Theorem 16.10. *Suppose* X *is simply connected and* $R \subseteq \mathbb{Q}$*. Then the* R*-localization* $X \to X_R$ *exists and is characterized as the unique (up to homotopy) map* $X \to Y$ *that induces an isomorphism* $\pi_*(X) \otimes R \xrightarrow{\sim} \pi_*(Y)$*.*

We will again prove this by an induction using the Postnikov tower. The base case is the following:

Lemma 16.11. *Suppose M is an abelian group and $R \subseteq \mathbb{Q}$. Then the map $K(M,n) \to K(M \otimes R, n)$ exhibits $K(M \otimes R, n)$ as the R-localization of $K(M,n)$.*

Proof. The space $K(M \otimes R, n)$ is R-local by Theorem 16.2, and the map $K(M,n) \to K(M \otimes R, n)$ is an R-equivalence by Lemma 16.1. □

Since the universal property of R-localizations only produces homotopy commutative squares, for the induction we will need an observation about homotopy fibres:

Definition 16.12. Recall that the homotopy fiber of a based map $f : (X,x) \to (Y,y)$ is defined as the space $F_f = X \times_Y PY$, where PY is the path space consisting of paths $p \colon I \to Y$ such that $p(0) = y$ with the pullback taken along f and the map $PY \to Y$ given by evaluation at 1. Note that the projection map $F_f \to X$ is a fibration, since $PY \to Y$ is, and that F_f can be viewed as the actual fiber of the fibration $X \times_Y Y^I \to Y$, given by evaluating p at 0, where the map $X \to X \times_Y Y^I$ induced by the constant-path map $Y \to Y^I$ is a homotopy equivalence.

(See also the discussion after Proposition 4.64 in [Hat02].)

Lemma 16.13. *Suppose that we have a homotopy-commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ Z & \xrightarrow{g} & W \end{array}$$

Then there exists a map $h : F_f \to F_g$ such that we have a diagram

$$\begin{array}{ccccccc} \Omega Y & \longrightarrow & F_f & \longrightarrow & X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \Omega\beta} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ \Omega W & \longrightarrow & F_g & \longrightarrow & Z & \xrightarrow{g} & W \end{array}$$

where the first and third squares commute up to homotopy and the second square commutes strictly*.*

For a proof see for example [MP12, Lemma 1.2.3.].

Proof of Theorem 16.10. Note that it suffices to construct a map of spaces $X \to X'$ such that $\pi_*(X) \otimes R \to \pi_*(X')$ is an isomorphism — then X' is R-local by Theorem 16.2, and the map $X \to X'$ is an R-equivalence by Lemma 16.1. We construct such a map by going up the Postnikov tower. Since X is simply connected, the base case is $P_2X = K(\pi_2 X, 2)$ where $P_2X \to (P_2X)_R$ is given by $K(\pi_2X, 2) \to K(\pi_2 X \otimes R, 2)$ by Lemma 16.11. Now suppose we have an R-localization $P_{n-1}X \to (P_{n-1}X)_R$. The composite $k_{n-1} \colon P_{n-1}X \to K(\pi_n X, n+1) \to K(\pi_n X \otimes R, n+1)$ is a map from $P_{n-1}X$ to an R-local space, so by the universal property of R-localization in Ho(Top) there is a unique commutative square

$$\begin{array}{ccc} P_{n-1}X & \longrightarrow & (P_{n-1}X)_R \\ \downarrow{\scriptstyle k_{n-1}} & & \downarrow{\scriptstyle k_{n-1,R}} \\ K(\pi_n X, n+1) & \longrightarrow & K(\pi_n X \otimes R, n+1). \end{array}$$

By Lemma 16.13 there is then a diagram

$$\begin{array}{ccccccc} K(\pi_n X, n) & \longrightarrow & F_{k_{n-1}} & \longrightarrow & P_{n-1} & \xrightarrow{k_{n-1}} & K(\pi_n X, n+1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K(\pi_n X \otimes R, n) & \longrightarrow & F_{k_{n-1,R}} & \longrightarrow & (P_{n-1}X)_R & \xrightarrow{k_{n-1,R}} & K(\pi_n X \otimes R, n+1), \end{array}$$

where the middle square commutes strictly. The homotopy fibre $F_{k_{n-1}}$ is homotopy equivalent to $P_n X$, and from the fibration sequence we see that $(P_n X)_R := F_{k_{n-1,R}}$ is an R-localization of $P_n X$. Now we define $X_R := \lim_{n\to\infty}(P_n X)_R$. This is R-local by Lemma 16.8, and $\pi_i X_R \cong \pi_i((P_n X)_R) \cong \pi_i X \otimes R$ (for $n \geq i$; since the homotopy groups of $(P_n X)_R$ stabilize we have no $\lim^1$-term). □

17. Sullivan's Arithmetic Square for p-Localization

In this section we construct Sullivan's arithmetic square for p-localization, which is easy.

Theorem 17.1 (Sullivan's arithmetic square for p-localization)**.** *Suppose X is simply connected. Then the canonical square*

$$\begin{array}{ccc} X & \longrightarrow & \prod_p X_{(p)} \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & \left(\prod_p X_{(p)}\right)_{\mathbb{Q}} \end{array}$$

is a homotopy pullback.

Remark 17.2. The bottom horizontal map in the diagram is the $\mathbb{Q}$-localization of the top map, so the diagram commutes — strictly if we have a functorial model for $\mathbb{Q}$-localization, otherwise just in the homotopy category.

For the proof we need an algebraic lemma:

Lemma 17.3. *Let M be a abelian group. Then the commutative square*

$$\begin{array}{ccc} M & \longrightarrow & \prod_p M \otimes \mathbb{Z}_{(p)} \\ \downarrow & & \downarrow \\ M \otimes \mathbb{Q} & \longrightarrow & \left(\prod_p M \otimes \mathbb{Z}_{(p)}\right) \otimes \mathbb{Q} \end{array}$$

gives isomorphisms on the kernels and cokernels of the rows.

Proof. We first observe that it suffices to prove that this holds after localizing at all primes q: A map of abelian groups $f\colon A \to B$ is an isomorphism if and only if the maps $f \otimes \mathbb{Z}_{(q)}$ are isomorphisms for all primes q, and since $-\otimes \mathbb{Z}_{(q)}$ is exact this implies we get an isomorphism on (co)kernels in the square if and only if we get such isomorphisms after tensoring the square with $\mathbb{Z}_{(q)}$ for all q.

We can rewrite the square as

$$\begin{array}{ccc} M & \longrightarrow & (M \otimes \mathbb{Z}_{(q)}) \oplus \prod_{p\neq q} M \otimes \mathbb{Z}_{(p)} \\ \downarrow & & \downarrow \\ M \otimes \mathbb{Q} & \longrightarrow & (M \otimes \mathbb{Q}) \oplus \left(\prod_{p\neq q} M \otimes \mathbb{Z}_{(p)}\right) \otimes \mathbb{Q} \end{array}$$

Tensoring this with $\mathbb{Z}_{(q)}$ now gives

$$\begin{array}{ccc} M \otimes \mathbb{Z}_{(q)} & \longrightarrow & (M \otimes \mathbb{Z}_{(q)}) \oplus \left(\prod_{p\neq q} M \otimes \mathbb{Z}_{(p)}\right) \otimes \mathbb{Q} \\ \downarrow & & \downarrow \\ M \otimes \mathbb{Q} & \longrightarrow & (M \otimes \mathbb{Q}) \oplus \left(\prod_{p\neq q} M \otimes \mathbb{Z}_{(p)}\right) \otimes \mathbb{Q} \end{array}$$

(here we use that $\left(\prod_{p\neq q} M \otimes \mathbb{Z}_{(p)}\right) \otimes \mathbb{Z}_{(q)} \cong \left(\prod_{p\neq q} M \otimes \mathbb{Z}_{(p)}\right) \otimes \mathbb{Q}$ as q is already inverted in this product); this square is clearly both a pushout and a pullback, and so it gives isomorphisms on the kernels and cokernels of the rows. □

We also make use of the following criteria for homotopy pullbacks:

Lemma 17.4. *A commutative square of spaces*

$$\begin{array}{ccc} P & \longrightarrow & Q \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{\phi} & B \end{array}$$

is a homotopy pullback square if and only if the induced map on homotopy fibres $F_f \to F_g$ is a weak equivalence.

Proof. We can replace the maps by fibrations. Then this is equivalent to the map $P \to \phi^*Q$ of fibrations over A being a weak equivalence if and only if the map on fibres is a weak equivalence, which is immediate from the long exact sequences on homotopy groups and the 5-Lemma. □

Proof of Theorem 17.1. We assume our square commutes strictly — otherwise we can replace it by a weakly equivalent square that does. Let F and F' denote the homotopy fibres of $\alpha\colon X \to \prod_p X_{(p)}$ and $\beta\colon X_{\mathbb{Q}} \to \left(\prod_p X_{(p)}\right)_{\mathbb{Q}}$, respectively. Then we want to show that the induced map $F \to F'$ is a weak equivalence. We consider the map of long exact sequences

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \pi_n F & \longrightarrow & \pi_n X & \xrightarrow{\pi_n \alpha} & \pi_n\left(\prod_p X_{(p)}\right) & \longrightarrow & \pi_{n-1} F & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \pi_n F' & \longrightarrow & \pi_n X_{\mathbb{Q}} & \xrightarrow{\pi_n \beta} & \pi_n\left(\prod_p X_{(p)}\right)_{\mathbb{Q}} & \longrightarrow & \pi_{n-1} F' & \longrightarrow & \cdots \end{array}$$

By Theorem 16.10 we can identify the middle square with

$$\begin{array}{ccc} \pi_n X & \longrightarrow & \prod_p \pi_n X \otimes \mathbb{Z}_{(p)} \\ \downarrow & & \downarrow \\ \pi_n X \otimes \mathbb{Q} & \longrightarrow & \left(\prod_p \pi_n X \otimes \mathbb{Z}_{(p)}\right) \otimes \mathbb{Q}, \end{array}$$

and by Lemma 17.3 we have isomorphisms on the kernels and cokernels of the rows, i.e. the induced maps $\ker \pi_n\alpha \to \ker \pi_n\beta$ and $\operatorname{coker} \pi_n\alpha \to \operatorname{coker} \pi_n\beta$ are isomorphisms.

For each n we have a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{coker} \pi_{n+1}\alpha & \longrightarrow & \pi_n F & \longrightarrow & \ker \pi_n\alpha & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \operatorname{coker} \pi_{n+1}\beta & \longrightarrow & \pi_n F' & \longrightarrow & \ker \pi_n\beta & \longrightarrow & 0. \end{array}$$

The 5-Lemma now implies that the map $\pi_n F \to \pi_n F'$ is an isomorphism, which completes the proof. □

18. p-Completion of Simply Connected Spaces I (The Easy Case)

We now turn to p-completion of spaces, i.e. localization with respect to $\mathbb{F}_p$. In this section we focus on the case of spaces with finitely generated homotopy groups, which explains where the term "p-completion" comes from. We'll return to the general case below in §21 after constructing Sullivan's arithmetic square in the next section. Our goal in this section is to prove

Theorem 18.1.

(i) If X is a simply connected space such that $\pi_n X$ is a finitely generated $\mathbb{Z}_p^\wedge$-module for all n, then X is p-complete.

(ii) If X is a simply connected space with finitely generated homotopy groups, then the p-completion $X \to X_p^\wedge$ of X exists and is characterized by the map $\pi_* X \otimes \mathbb{Z}_p^\wedge \to \pi_*(X_p^\wedge)$ being an isomorphism.

(iii) If X and Y are simply connected spaces with finitely generated homotopy groups, then a map $f\colon X \to Y$ is an $\mathbb{F}_p$-equivalence if and only if $\pi_* f \otimes \mathbb{Z}_p^\wedge$ is an isomorphism.

Warning 18.2. The finiteness hypotheses are essential here — for more general homotopy groups simply taking their p-completion does not give the $\mathbb{F}_p$-localization.

We can already prove (i):

Proof of Theorem 18.1(i). We saw in Lemma 15.5 that the Eilenberg-MacLane spaces $K(\mathbb{Z}/p^k, n)$ and $K(\mathbb{Z}_p^\wedge, n)$ are p-complete. A finitely generated $\mathbb{Z}_p^\wedge$-module M is a finite sum of these modules, so it follows that $K(M, n)$ is also p-complete.

We now consider the Postnikov tower of X. The space $P_2 X = K(\pi_2 X, 2)$ is p-complete since $\pi_2 X$ is a finitely generated $\mathbb{Z}_p^\wedge$-module. And if $P_{n-1}X$ is p-complete then we see that $P_n X$ is p-complete by applying Lemma 16.7 to the homotopy pullback square

$$\begin{array}{ccc} P_n X & \longrightarrow & P_{n-1} X \\ \downarrow & & \downarrow \\ * & \longrightarrow & K(\pi_n X, n+1), \end{array}$$

where $K(\pi_n X, n+1)$ and $P_{n-1}X$ are both p-complete. Since X is weakly equivalent to $\lim_n P_n X$ it follows that X is p-complete using Lemma 16.8. □

Before we prove part (ii) of the theorem we prove a criterion for a map to be an $\mathbb{F}_p$-equivalence in terms of the homotopy groups of the homotopy fibre:

Theorem 18.3. *Let $f\colon X \to Y$ be a map between simply connected spaces, and let F be the homotopy fibre of f. Then the following are equivalent:*

(1) f is an $\mathbb{F}_p$-equivalence,
(2) $\tilde{H}_*(F;\mathbb{F}_p) = 0$,
(3) $\tilde{H}_*(F;\mathbb{Z})$ is a $\mathbb{Z}[\frac{1}{p}]$-module,
(4) $\pi_*(F)$ is a $\mathbb{Z}[\frac{1}{p}]$-module.

This will take some work to prove. We start with a 2-of-3 criterion for $\mathbb{F}_p$-equivalences:

Proposition 18.4. *Suppose given a commutative square*

$$\begin{array}{ccc} X & \xrightarrow{\psi} & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow{\phi} & W \end{array}$$

where Z and W are simply connected, and let $\eta\colon F_f \to F_g$ be the induced map on homotopy fibres. If R is a principal ideal domain and two of the three morphisms ϕ, ψ, and η are R-equivalences, then so is the third.

We will not prove this here. The case where ϕ and η are R-equivalences is an immediate consequence of naturality for the Serre spectral sequences. The other two cases are also proved using the Serre spectral sequences by more complicated arguments. See [Hat, Proposition 1.12] for an argument in one of these cases.

As a special case, we get:

Corollary 18.5. *Let*

$$\begin{array}{ccc} X & \xrightarrow{\psi} & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow{\phi} & W \end{array}$$

be a homotopy pullback square of spaces with Z and W simply connected. If R is a principal ideal domain, then ϕ is an R-equivalence if and only if ψ is an R-equivalence.

Proof. Since the square is a homotopy pullback, the induced map of homotopy fibres $F_f \to F_g$ is a weak equivalence by Lemma 17.4, so this follows from Proposition 18.4. □

This gives a first approximation to the Theorem:

Lemma 18.6. *Let $f\colon X \to Y$ be a map between simply connected spaces, and let F be the homotopy fibre of f. Then the following are equivalent:*

(1) f is an $\mathbb{F}_p$-equivalence,
(2) $\tilde{H}_*(F;\mathbb{F}_p) = 0$,
(3) $\tilde{H}_*(F;\mathbb{Z})$ is a $\mathbb{Z}[\frac{1}{p}]$-module.

Proof. The equivalence of (1) and (2) is a special case of Corollary 18.5. The equivalence of (2) and (3) follows from the Bockstein long exact sequence (i.e. the long exact sequence of homology groups induced by the short exact sequence of abelian groups $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p \to 0$):

$$\cdots \to H_i(F;\mathbb{Z}) \xrightarrow{p} H_i(F;\mathbb{Z}) \to H_i(F;\mathbb{F}_p) \to H_{i-1}(F;\mathbb{Z}) \to \cdots .$$

Here we see that $\tilde{H}_i(F;\mathbb{F}_p) = 0$ for all i, then multiplication by p is an isomorphism on $\tilde{H}_i(F;\mathbb{Z})$ for all i, and vice versa. □

The fibre of a map of simply connected spaces is not necessarily simply connected, but it is a *simple* space, meaning its fundamental group is abelian and acts trivially on the higher homotopy groups. Lemma 18.6 therefore reduces the proof of Theorem 18.3 to showing:

Proposition 18.7. *Suppose X is a simple space. Then $\tilde{H}_*(X;\mathbb{F}_p) = 0$ if and only if $\pi_n X$ is a $\mathbb{Z}[\frac{1}{p}]$-module for all n.*

We first consider the case of Eilenberg-MacLane spaces:

Lemma 18.8. *An abelian group A is a $\mathbb{Z}[\frac{1}{p}]$-module if and only if $\tilde{H}_*(K(A,n);\mathbb{F}_p) = 0$ (for any $n \geq 1$).*

Proof. First suppose $\tilde{H}_*(K(A,n);\mathbb{F}_p) = 0$. Then as we saw above, the Bockstein long exact sequence implies that $\tilde{H}_i(K(A,n);\mathbb{Z})$ is a $\mathbb{Z}[\frac{1}{p}]$-module for all i. But since A is abelian we have $A \cong H_n(K(A,n);\mathbb{Z})$, so A is a $\mathbb{Z}[\frac{1}{p}]$-module.

Now suppose A is a $\mathbb{Z}[\frac{1}{p}]$-module. We first consider the case where $n = 1$. Since $\tilde{H}_*(-;\mathbb{F}_p)$ preserves filtered colimits it suffices to consider A a finitely generated $\mathbb{Z}[\frac{1}{p}]$-module. Then A is a direct sum of copies of $\mathbb{Z}/q^k$ where k is some prime $\neq p$ and $\mathbb{Z}[\frac{1}{p}]$. We won't show that $\tilde{H}_*(K(\mathbb{Z}/q^k,1);\mathbb{F}_p) = 0$; this can for example be done using group cohomology. [Exercise: Check this for $K(\mathbb{Z}/2,1) = \mathbb{RP}^\infty$ using cellular homology.] In the case of $\mathbb{Z}[\frac{1}{p}]$ we have $\mathbb{Z}[\frac{1}{p}] = \operatorname{colim}(\mathbb{Z} \xrightarrow{p} \mathbb{Z} \cdots)$ so $K(\mathbb{Z}[\frac{1}{p}],1) \cong \operatorname{colim}(K(\mathbb{Z},1) \xrightarrow{p} K(\mathbb{Z},1) \cdots)$ and so as homology commutes with filtered colimits we have $\tilde{H}_*(K(\mathbb{Z}[\frac{1}{p}],1),\mathbb{Z}) \cong \tilde{H}_*(S^1;\mathbb{Z})[\frac{1}{p}]$, which is a $\mathbb{Z}[\frac{1}{p}]$-module, so $\tilde{H}_*(K(\mathbb{Z}[\frac{1}{p}],1),\mathbb{F}_p) = 0$.

We can now inductively extend this to $n > 1$: Applying Corollary 18.5 to the homotopy pullback square

$$\begin{array}{ccc} K(A,n-1) & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & K(A,n). \end{array}$$

gives that $\tilde{H}_*(K(A,n);\mathbb{F}_p) = 0$ if and only if $\tilde{H}_*(K(A,n-1);\mathbb{F}_p) = 0$. □

Proof of Proposition 18.7. We first consider the case where X is a simply connected space. By applying Corollary 18.5 to the homotopy pullback square

$$\begin{array}{ccc} K(\pi_n X,n) & \longrightarrow & P_n X \\ \downarrow & & \downarrow \\ * & \longrightarrow & P_{n-1} X \end{array}$$

we see that if $\tilde{H}_*(P_{n-1}X;\mathbb{F}_p)=0$ then $\tilde{H}_*(K(\pi_n X,n);\mathbb{F}_p)\xrightarrow{\sim}\tilde{H}_*(P_nX;\mathbb{F}_p)$. In particular, by Lemma 18.8 we have that if $\tilde{H}_*(P_{n-1}X;\mathbb{F}_p)=0$ then $\tilde{H}_*(P_nX;\mathbb{F}_p)=0$ if and only if π_nX is a $\mathbb{Z}[\frac{1}{p}]$-module.

The map $X\to P_nX$ is $(n+1)$-connected (meaning the homotopy fibre F_n is n-connected), and since P_nX is simply connected the Serre spectral sequence for this map has E^2-term

$$E^2_{s,t}=H_s(P_nX;\mathbb{F}_p)\otimes_{\mathbb{F}_p}H_t(F_n;\mathbb{F}_p)\Rightarrow H_{s+t}(X;\mathbb{F}_p).$$

As $H_t(F_n;\mathbb{F}_p)=0$ for $t\leq n$ there is no room for differentials in the range $s+t\leq n$, and thus $H_i(X;\mathbb{F}_p)\xrightarrow{\sim}H_i(P_nX;\mathbb{F}_p)$ for $i\leq n$. Moreover, there are no differentials out of $H_{n+1}(P_nX;\mathbb{F}_p)$ so the map $H_{n+1}(X;\mathbb{F}_p)\to H_{n+1}(P_nX;\mathbb{F}_p)$ is surjective.

We can therefore conclude that if π_iX is a $\mathbb{Z}[\frac{1}{p}]$-module for all $i\leq n$ then $\tilde{H}_*(P_nX;\mathbb{F}_p)=0$ and so $\tilde{H}_i(X;\mathbb{F}_p)=0$ for $i\leq n$. In particular, if all the homotopy groups of X are $\mathbb{Z}[\frac{1}{p}]$-modules then $\tilde{H}_*(X;\mathbb{F}_p)=0$.

Now suppose $\tilde{H}_*(X;\mathbb{F}_p)=0$. Then for every n we have $\tilde{H}_i(P_nX;\mathbb{F}_p)=0$ for $i\leq n+1$. Assume we know that π_iX is a $\mathbb{Z}[\frac{1}{p}]$-module for all $i<n$. Then we get that $\tilde{H}_i(K(\pi_nX,n);\mathbb{F}_p)=0$ for $i\leq n+1$. But then from the Bockstein long exact sequence we see that multiplication by p on $H_n(K(\pi_nX,n);\mathbb{F}_p)\cong\pi_nX$ is an isomorphism, i.e. π_nX is a $\mathbb{Z}[\frac{1}{p}]$-module. By inducting on n this completes the proof for X simply connected.

If X is not simply connected, we consider the Serre spectral sequence for the map $X\to B\pi_1X$ whose homotopy fibre is the universal cover $\tilde{X}$. Since X is simple there is no local system in the E^2-term, so this spectral sequence is of the form

$$E^2_{s,t}=H_s(\tilde{X};\mathbb{F}_p)\otimes_{\mathbb{F}_p}H_t(B\pi_1X;\mathbb{F}_p)\Rightarrow H_{s+t}(X;\mathbb{F}_p).$$

If π_nX is a $\mathbb{Z}[\frac{1}{p}]$-module for all n, then since $\tilde{X}$ is simply connected we have $\tilde{H}_*(\tilde{X};\mathbb{F}_p)=0$, and $\tilde{H}_*(B\pi_1X;\mathbb{F}_p)=0$ by Lemma 18.8. Thus $E^2_{s,t}=0$ except when $s=t=0$ and thus $\tilde{H}(X;\mathbb{F}_p)=0$.

On the other hand, if $\tilde{H}(X;\mathbb{F}_p)=0$, then we know $\tilde{H}(X;\mathbb{Z})$ is a $\mathbb{Z}[\frac{1}{p}]$-module, hence $\pi_1X\cong H_1(X;\mathbb{Z})$ is a $\mathbb{Z}[\frac{1}{p}]$-module, and so $\tilde{H}(B\pi_1X;\mathbb{F}_p)=0$ by Lemma 18.8. Thus $E^2_{s,t}=0$ except when $t=0$, so the spectral sequence collapses and we have $\tilde{H}_*(\tilde{X};\mathbb{F}_p)\cong\tilde{H}_*(X;\mathbb{F}_p)=0$. Then since $\tilde{X}$ is simply connected we have that $\pi_n\tilde{X}=\pi_nX$ is a $\mathbb{Z}[\frac{1}{p}]$-module for all $n\geq 2$. □

Warning 18.9. The "proof" of this that I gave in the lecture is wrong (or at least incomplete — I did not show that if $\tilde{H}_*(X;\mathbb{F}_p)=0$ then π_*X are $\mathbb{Z}[\frac{1}{p}]$-modules).

This completes the proof of Theorem 18.3, and we are ready to prove the base case of (ii) in Theorem 18.1:

Lemma 18.10. *Suppose A is a finitely generated abelian group. Then $K(A,n)\to K(A\hat{_p},n)$ is a p-completion.*

Proof. We know from Theorem 18.1(i) that $K(A\hat{_p},n)$ is p-complete, so it remains to show that the map is a $\mathbb{F}_p$-equivalence. Since A is finitely generated, it suffices to check separately the cases where A is $\mathbb{Z}/p^k$, $\mathbb{Z}/q^k$ (q a prime $\neq p$), and $\mathbb{Z}$. In these cases $A\hat{_p}$ is $\mathbb{Z}/p^k$, 0, and $\mathbb{Z}\hat{_p}$, respectively. For $\mathbb{Z}/p^k$ the space $K(\mathbb{Z}/p^k,n)$ is already p-complete and there is nothing to prove. The group $\mathbb{Z}/q^k$ is a $\mathbb{Z}[\frac{1}{p}]$-module, so by Theorem 18.3 it follows that $K(\mathbb{Z}/q^k,n)$ is $\mathbb{F}_p$-equivalent to a point, as required.

The remaining case $A = \mathbb{Z}$ is the more interesting one. Here the map $\phi\colon \mathbb{Z} \to \mathbb{Z}\hat{_p}$ is clearly injective, so we can identify the homotopy fibre of our map with $K(\operatorname{coker}\phi, n-1)$. It thus suffices to show that $\operatorname{coker}\phi$ is a $\mathbb{Z}[\frac{1}{p}]$-module, i.e. multiplication by p is an isomorphism. An element x of $\mathbb{Z}\hat{_p}$ can be written as $\sum_{i=0}^{\infty} a_i p^i$, where $0 \leq a_i < p$. The image of $\mathbb{Z}$ is precisely those sums with only finitely many non-zero a_i. Modulo $\mathbb{Z}$ we then see that x equals $p\left(\sum_{i=1}^{\infty} a_i p^{i-1}\right)$, so multiplication by p is surjective. On the other hand, if $p\left(\sum_{i=0}^{\infty} a_i p^i\right)$ is 0 modulo $\mathbb{Z}$ then $a_i = 0$ except for finitely many i, i.e. $\sum a_i p^i$ is is in the image of $\mathbb{Z}$. Hence multiplication by p is injective. This completes the proof. □

Proof of Theorem 18.1(ii). We once again work our way up the Postnikov tower. First we can take $P_2X\hat{_p} := K(\pi_2 X \otimes \mathbb{Z}\hat{_p}, 2)$, then $P_2X \to P_2X\hat{_p}$ is a p-completion by Lemma 18.10. Next if we have a p-completion $\phi\colon P_{n-1}X \to P_{n-1}X\hat{_p}$, we get from the universal property a homotopy-commutative square

$$\begin{array}{ccc} P_{n-1}X & \xrightarrow{\phi} & P_{n-1}X\hat{_p} \\ \downarrow{\scriptstyle k_{n-1}} & & \downarrow{\scriptstyle (k_{n-1})\hat{_p}} \\ K(\pi_n X, n+1) & \xrightarrow{\psi} & K(\pi_n X \otimes \mathbb{Z}\hat{_p}, n+1). \end{array}$$

On homotopy fibres we get a map $P_nX \to P_nX\hat{_p} := F_{(k_{n-1})\hat{_p}}$, where the space $P_nX\hat{_p}$ is p-complete by Lemma 16.7. Moreover, since ϕ and ψ are $\mathbb{F}_p$-equivalences it follows from Proposition 18.4 that this map is an $\mathbb{F}_p$-equivalence, so it is a p-completion.

Now define $X\hat{_p} := \lim_n P_nX\hat{_p}$. This is p-complete by Lemma 16.8. It remains to show that $X \to X\hat{_p}$ is an $\mathbb{F}_p$-equivalence. To see this, let F_n be the homotopy fibre of the map $P_nX \to P_nX\hat{_p}$. Then the homotopy fibre of $X \to X\hat{_p}$ is weakly equivalent to $\lim_n F_n$. Since the homotopy groups of P_nX and $P_nX\hat{_p}$ stabilize, we see from the long exact sequence that so do those of F_n. Thus there is no $\lim^1$-term and we have $\pi_i(\lim_n F_n) \cong \lim_n \pi_i F_n \cong \pi_i F_k$ for k sufficiently large. Since the homotopy groups of the spaces F_n are $\mathbb{Z}[\frac{1}{p}]$-modules, it follows that the same is true for those of $\lim_n F_n$, hence the map $X \to X\hat{_p}$ is an $\mathbb{F}_p$-equivalence by Theorem 18.3. □

Proof of Theorem 18.1(iii). Applying the 2-of-3 property for $\mathbb{F}_p$-equivalences to the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X\hat{_p} & \xrightarrow{f\hat{_p}} & Y\hat{_p} \end{array}$$

we see that f is an $\mathbb{F}_p$-equivalence if and only if $f\hat{_p}$ is one. But the spaces $X\hat{_p}$ and $Y\hat{_p}$ are p-complete, so $f\hat{_p}$ is an $\mathbb{F}_p$-equivalence if and only if it is an isomorphism in Ho(Top), i.e. if it gives an isomorphism $\pi_*X\hat{_p} \to \pi_*Y\hat{_p}$. Using Theorem 18.1(iii) we see from this that f is an $\mathbb{F}_p$-equivalence if and only if $\pi_*(f) \otimes \mathbb{Z}\hat{_p}$ is an isomorphism. □

19. Sullivan's Arithmetic Square for p-Completion

We now consider Sullivan's arithmetic square for p-completion:

Theorem 19.1. *Suppose X is simply connected. Then we have a homotopy pullback square*

$$\begin{array}{ccc} X & \longrightarrow & \prod_p X_{\hat{p}} \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & \left(\prod_p X_{\hat{p}}\right)_{\mathbb{Q}} . \end{array}$$

We'll prove this in 3 steps:

(1) Let Y be the homotopy pullback in the square, then we'll show that the induced map $\phi\colon X \to Y$ is a $\mathbb{Z}$-equivalence (isomorphism in $\mathbb{Z}$-homology).
(2) Next, we'll see that Y is a $\mathbb{Z}$-local space. Combined with (1), this shows that Y is the $\mathbb{Z}$-localization of X.
(3) Finally, we'll observe that X is itself $\mathbb{Z}$-local, so that (2) implies that $X \xrightarrow{\sim} Y$.

For the proof of the first step we need the following result. With finite generation assumptions this follows from the results of the previous section, and the general case will follow from the results on p-completion for general simply connected spaces we'll prove later.

Proposition 19.2. *If X is a simply connected p-complete space, then π_*X are $\mathbb{Z}[\frac{1}{q}]$-modules for any prime $q \neq p$.*

To see that ϕ is a $\mathbb{Z}$-equivalence we'll use the following criterion:

Lemma 19.3. *A map of spaces $f\colon X \to Y$ induces an isomorphism on $H_*(-;\mathbb{Z})$ if and only if it induces an isomorphism on $H_*(-;\mathbb{Q})$ and on $H_*(-;\mathbb{F}_p)$ for all primes p.*

Proof. The forward direction is obvious. For the backward direction, we can assume that f is an inclusion, and the result follows if we can show that $H_*(Y,X) = 0$ under the stated assumptions. The universal coefficient sequence gives us an exact sequence

$$0 \to H_n(Y,X) \otimes A \to H_n(Y,X;A) \to \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(Y,X),A) \to 0.$$

If $H_n(Y,X;\mathbb{F}_p) = 0$, then it follows that $H := H_n(Y,X)$ satisfies $H \otimes \mathbb{F}_p = 0$ and $\mathrm{Tor}_1^{\mathbb{Z}}(H,\mathbb{F}_p) = 0$. This implies that H is uniquely p-divisible for all p. If this holds for all p, then H is uniquely p-divisible for all primes p, i.e. it is a $\mathbb{Q}$-vector space. But then $H \cong H \otimes \mathbb{Q}$, and if $H_n(Y,X;\mathbb{Q}) = 0$ the exact sequence implies that $H \otimes \mathbb{Q} = 0$. □

Proposition 19.4. *The map $\phi\colon X \to Y$ is a $\mathbb{Z}$-equivalence.*

Proof. By Lemma 19.3 it is enough to prove that ϕ gives isomorphisms on $H_*(-;\mathbb{F}_p)$ for all primes p and on $H_*(-;\mathbb{Q})$.

Consider the following diagram, where the isomorphisms are by the universal property of $\mathbb{Q}$-localization:

$$\begin{array}{ccccc} H_*(X;\mathbb{Q}) & \longrightarrow & H_*(Y;\mathbb{Q}) & \longrightarrow & H_*(\prod_p X_{\hat{p}};\mathbb{Q}) \\ & \searrow^{\cong} & \downarrow & & \downarrow \cong \\ & & H_*(X_{\mathbb{Q}};\mathbb{Q}) & \longrightarrow & H_*(\left(\prod_p X_{\hat{p}}\right)_{\mathbb{Q}};\mathbb{Q}), \end{array}$$

Corollary 18.5 implies that the left vertical map is also an isomorphism, which shows the $X \to Y$ is an isomorphism on $H_*(-;\mathbb{Q})$.

Now consider the following diagram:

$$\begin{array}{ccccc} & & \overset{\cong}{\frown} & & \\ H_*(X;\mathbb{F}_q) & \longrightarrow & H_*(Y;\mathbb{F}_q) & \longrightarrow & H_*(\prod_p X\hat{_p};\mathbb{F}_q) \\ & & \downarrow & & \downarrow \\ & & H_*(X_{\mathbb{Q}};\mathbb{F}_q) & \xrightarrow{\cong} & H_*(\big(\prod_p X\hat{_p}\big)_{\mathbb{Q}};\mathbb{F}_q), \end{array}$$

Here the top map is an isomorphism since $\prod_p X\hat{_p} = X\hat{_q} \times \prod_{p\neq q} X\hat{_p}$ where the homotopy groups of the second space are $\mathbb{Z}[\frac{1}{q}]$-modules by Proposition 19.2, and so its reduced $\mathbb{F}_q$-homology is trivial by Theorem 18.3. By the Künneth theorem this means

$$H_*(\prod_p X\hat{_p};\mathbb{F}_q) \cong H_*(X\hat{_q};\mathbb{F}_q) \otimes_{\mathbb{F}_q} H_*(\prod_{p\neq q} X\hat{_p};\mathbb{F}_q) \cong\cong H_*(X\hat{_q};\mathbb{F}_q),$$

and so the map $H_*(X;\mathbb{F}_q) \to H_*(\prod_p X\hat{_p};\mathbb{F}_q)$ is an isomorphism. The bottom horizontal map is also an isomorphism, by the same argument. (Indeed, since the homotopy groups of the spaces at the bottom are $\mathbb{Q}$-vector spaces, the $\mathbb{F}_q$-homology in the bottom row is trivial.)

Now using Corollary 18.5 again the top horizontal map is also an isomorphism, which shows that $X \to Y$ gives an isomorphism on $H_*(-;\mathbb{F}_q)$ for all q. Thus $X \to Y$ gives an isomorphism on $H_*(-;\mathbb{Z})$, as required. □

For step (2), we need the following result:

Proposition 19.5. *If*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

is a homotopy pullback square and Y, Z, and W are R-local for some ring R, then X is R-local.

To see this we need some observations about R-equivalences and weak equivalences:

Lemma 19.6. *If $f\colon X \to Y$ is an R-equivalence, then so is $f \times T\colon X \times T \to Y \times T$ for any space T.*

Proof. We have a Künneth spectral sequence

$$E^2_{p,q} = \bigoplus_{i+j=p} \mathrm{Tor}^q_R(H_i(X;R), H_j(T;R)) \Rightarrow H_{i+j}(X \times T;R).$$

This is natural, and the map induced by f gives an isomorphism on the E^2-terms. Therefore it gives an isomorphism $H_*(X \times T;R) \xrightarrow{\sim} H_*(Y \times T;R)$, as required. □

Lemma 19.7. *A space Z is R-local if and only if for every R-equivalence $f\colon X \to Y$ the induced map*

$$\mathrm{Map}(Y,Z) \to \mathrm{Map}(X,Z)$$

is a weak equivalence.

To show this, we will use the following fact:

Fact 19.8. *A map of spaces $f\colon X \to Y$ is a weak equivalence if and only if the induced map on homotopy classes $[T, X] \to [T, Y]$ is an isomorphism for all CW-complexes T. (Equivalently, f induces an isomorphism in* $\mathrm{Ho}(\mathrm{Top})$*.)*

This is a special case of a standard fact about model categories, for example.

Warning 19.9. These are unpointed homotopy classes. To show that $X \to Y$ is a weak equivalence it is *not* enough to know that $[S^n, X] \to [S^n, Y]$ is an isomorphism for all spheres.

Proof of Lemma 19.7. If $\mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$ is a weak equivalence for all R-equivalences f, then on π_0 we get an isomorphism $[Y, Z] \to [X, Z]$ for every R-equivalence, i.e. Z is R-local.

Conversely, suppose Z is R-local and $f\colon X \to Y$ is an R-equivalence. The map $\mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$ is a weak equivalence if and only if for all spaces T the induced map

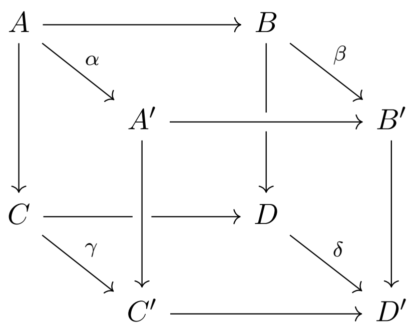
$$[T, \mathrm{Map}(Y, Z)] \to [T, \mathrm{Map}(X, Z)]$$

is an isomorphism. But this is isomorphic to the map

$$[T \times Y, Z] \to [T \times X, Z]$$

induced by $T \times f$, which is an R-equivalence by Lemma 19.6, and so this map is an isomorphism since Z is R-local. □

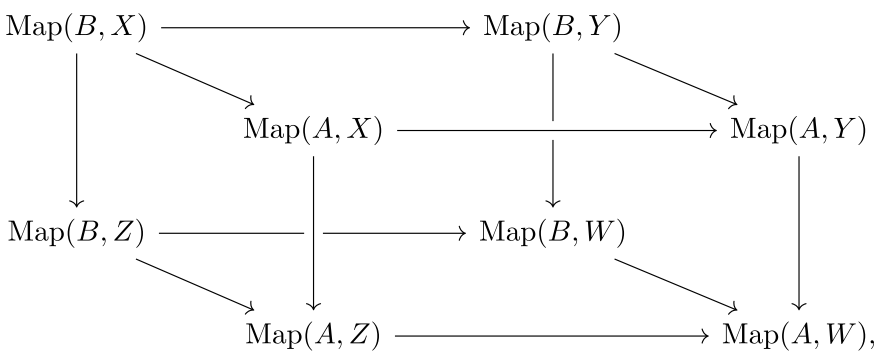
Lemma 19.10. *Given a commutative cube*



where the front and back faces are homotopy pullbacks, if the maps β, γ, and δ are weak equivalences, so is α.

Proof. This is a special case of homotopy invariance for homotopy limits. □

Proof of Proposition 19.5. If $f\colon A \to B$ is an R-equivalence, by Lemma 19.7 we need to prove that $\mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$ is a weak equivalence. This follows by applying Lemma 19.10 to the cube



where the front and back faces are homotopy pullbacks since $\mathrm{Map}(A, -)$ preserves homotopy limits. □

We need one more observation before we can prove the theorem:

Lemma 19.11. *If a space X is R-local for some ring R, then X is $\mathbb{Z}$-local.*

Proof. It suffices to show that if $f\colon Y \to Z$ is a $\mathbb{Z}$-equivalence then it is an R-equivalence for any ring R. From the universal coefficient theorem we have a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_n(Y) \otimes R & \longrightarrow & H_n(Y; R) & \longrightarrow & \mathrm{Tor}^1(H_{n-1}(Y), R) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_n(Z) \otimes R & \longrightarrow & H_n(Z; R) & \longrightarrow & \mathrm{Tor}^1(H_{n-1}(Z), R) & \longrightarrow & 0, \end{array}$$

so the 5-Lemma implies that f is an R-equivalence if it is a $\mathbb{Z}$-equivalence. □

Proof of Theorem 19.1. The spaces $X_{\mathbb{Q}}$, $X_p^{\wedge}$ and $(\prod_p X_p^{\wedge})_{\mathbb{Q}}$ are $\mathbb{Z}$-local by Lemma 19.11, and the space $\prod_p X_p^{\wedge}$ is also $\mathbb{Z}$-local since an arbitrary product of R-local spaces is always R-local. Thus the homotopy pullback Y is $\mathbb{Z}$-local by Proposition 19.5. Since $X \to Y$ is also a $\mathbb{Z}$-equivalence by Proposition 19.4, this means that Y is the $\mathbb{Z}$-localization of X. But we know from Theorem 16.2 that a simply connected space is R-local for $R \subseteq \mathbb{Q}$ if and only if its homotopy groups are R-modules — taking $R = \mathbb{Z}$ this says that every simply connected space is $\mathbb{Z}$-local. Thus the $\mathbb{Z}$-localization of X is just X, and so $X \to Y$ is a weak equivalence, as required. □

Remark 19.12. Theorem 19.1 says that we can recover a (simply connected) space X from its rationalization $X_{\mathbb{Q}}$, its p-completions $X_p^{\wedge}$ at all primes, and the map $X_{\mathbb{Q}} \to (\prod_p X_p^{\wedge})_{\mathbb{Q}}$. In fact, we can say a bit more: If we have p-complete spaces $Y(p)$ for all primes p and a rational space Q, all simply connected, together with a map $Q \to (\prod_p Y(p))_{\mathbb{Q}}$, let P be the homotopy pullback in

$$\begin{array}{ccc} P & \longrightarrow & \prod_p Y(p) \\ \downarrow & & \downarrow \\ Q & \longrightarrow & (\prod_p Y(p))_{\mathbb{Q}}. \end{array}$$

Then the same proof shows: Q is the rationalization of P and $Y(p)$ is the p-completion of P. Thus we can build a space with arbitrary rationalization and p-completions, provided we have the bottom map — this is the only "interaction" between the rational and p-complete "parts" of the space.

20. An Algebraic Interlude: Derived p-Completion

We saw above that p-completion of spaces is closely related to p-completion of abelian groups — but only for finitely generated abelian groups. The reason is that the naïve definition of the p-completion of an abelian group A as $A_p^{\wedge} := \lim_k A/p^k$ must be replaced by a "derived" p-completion in chain complexes of abelian groups. This involves using derived versions of both parts of the construction, i.e. quotienting by p^k and taking the limit:

- Instead of just taking the quotient A/p^k, i.e. the cokernel of the map $A \xrightarrow{p^k} A$ we take the "derived cokernel", meaning the *mapping cone* of this map. This is the chain complex
$$\cdots \to 0 \to A \xrightarrow{p^k} A \to 0 \to \cdots$$
with the non-zero groups in degrees 0 and 1. We'll denote this $A//p^k$. Note that $H_0(A//p^k) = A/p^k$ and $H_1(A//p^k)$ is the kernel of multiplication by p^k, i.e. the group of p^k-torsion elements in A.
- Now we want to take the limit of the chain complexes $A//p^k$ over k — but since lim is not an exact functor we must take the *derived* limit. This means we first replace the diagram $k \mapsto A//p^k$ by an injective object in the (abelian) category of diagrams, and then take the limit of that. We'll denote this derived limit of a sequence of chain complexes $C(k)$ by $\mathbb{R}\mathrm{lim}_k\, C(k)$. (This is well-defined up to quasi-isomorphism.) Note that since we're working with chain complexes rather than cochain complexes the $\lim^1$-term will appear in degree -1. I.e. if $\cdots \to M_1 \to M_0$ is a sequence of abelian groups we have
$$H_i(\mathbb{R}\mathrm{lim}_k M_k) = \begin{cases} \lim_k M_k, & i = 0, \\ \lim^1_k M_k, & i = 1, \\ 0, & \text{otherwise.} \end{cases}$$
More generally, for a sequence $C(k)$ of chain complexes we get short exact sequences
$$0 \to \lim^1_k H_{i+1}C(k) \to H_i\mathbb{R}\mathrm{lim}_k C(k) \to \lim_k H_i C(k) \to 0.$$
(This is a degenerate special case of Grothendieck's hyperhomology spectral sequence, for example.)

Definition 20.1. The *derived p-completion* of an abelian group A is $\mathbb{R}\mathrm{lim}_k\, A//p^k$. We'll denote this $\mathbb{D}A^\wedge_p$.

Remark 20.2. From the short exact sequence for $\mathbb{R}\mathrm{lim}_k$ above, we see that
$$H_1\mathbb{D}A^\wedge_p \cong \lim_k \mathrm{Tor}(\mathbb{Z}/p^k, A)$$
and there is a short exact sequence
$$0 \to \lim^1 \mathrm{Tor}(\mathbb{Z}/p^k, A) \to H_0\mathbb{D}A^\wedge_p \to A^\wedge_p \to 0.$$
Thus if the p^k-torsion in A vanishes for k sufficiently large, then $H_1\mathbb{D}A^\wedge_p = 0$ and $H_0\mathbb{D}A^\wedge_p \cong A^\wedge_p$ so $\mathbb{D}A^\wedge_p$ is (up to quasi-isomorphism) just $A^\wedge_p$. This is, for example, always true if A is finitely generated.

Definition 20.3. An abelian group A is *derived p-complete* (or Ext-*p-complete*) if the natural map $A \to \mathbb{D}A^\wedge_p$ is a quasi-isomorphism, i.e. if $A \xrightarrow{\sim} H_0\mathbb{D}A^\wedge_p$ and $H_1\mathbb{D}A^\wedge_p = 0$.

Definition 20.4. Let $\mathbb{Z}/p^\infty$ denote the abelian group obtained as the colimit of the sequence
$$\mathbb{Z}/p \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^3 \xrightarrow{p} \cdots.$$

Remark 20.5. Our description of $H_1\mathbb{D}A^\wedge_p$ above can be interpreted as $\lim_k \mathrm{Hom}(\mathbb{Z}/p^k, A) \cong \mathrm{Hom}(\mathrm{colim}\, \mathbb{Z}/p^k, A) \cong \mathrm{Hom}(\mathbb{Z}/p^\infty, A)$. We will now see that $H_0\mathbb{D}A^\wedge_p$ can similarly be interpreted as $\mathrm{Ext}(\mathbb{Z}/p^\infty, A)$. In fact, we will see that the chain complex $\mathbb{D}A^\wedge_p$ can be described as a shift of a derived Hom in the following sense:

Definition 20.6. If C and D are (bounded-below) chain complexes, we write $\mathbb{R}\mathrm{Hom}(C, D)$ for the chain complex obtained as either $\mathrm{Hom}(C', D)$ where C' is a projective replacement of C, or as $\mathrm{Hom}(C, D')$ where D' is an injective replacement of D.

Lemma 20.7. *The chain complexes* $\mathbb{D}A^{\wedge}_p$ *and* $\mathbb{R}\mathrm{Hom}(\mathbb{Z}/p^\infty, A)[1]$ *are quasi-isomorphic. In particular we have*

$$H_0\mathbb{D}A^{\wedge}_p \cong \mathrm{Ext}(\mathbb{Z}/p^\infty, A),$$
$$H_1\mathbb{D}A^{\wedge}_p \cong \mathrm{Hom}(\mathbb{Z}/p^\infty, A).$$

Sketch Proof. We will use the fact that $\mathbb{R}\mathrm{Hom}$ takes derived colimits in the first variable to derived limits. The colimit defining $\mathbb{Z}/p^\infty$ is derived, so $\mathbb{R}\mathrm{Hom}(\mathbb{Z}/p^\infty, A) \simeq \mathbb{R}\lim_k \mathbb{R}\mathrm{Hom}(\mathbb{Z}/p^k, A)$. Now the chain complex $\mathbb{Z}//p^k$ is a projective resolution of $\mathbb{Z}/p^k$, so the chain complex $\mathbb{R}\mathrm{Hom}(\mathbb{Z}/p^k, A)$ is quasi-isomorphic to $\mathrm{Hom}(\mathbb{Z}//p^k, A)$, which can be identified with $A//p^k[-1]$. Thus $\mathbb{R}\mathrm{Hom}(\mathbb{Z}/p^\infty, A) \simeq \mathbb{R}\lim_k A//p^k[-1] \simeq \mathbb{D}A^{\wedge}_p[-1]$. $\square$

Lemma 20.8. *There is a short exact sequence of abelian groups*

$$0 \to \mathbb{Z} \to \mathbb{Z}[\tfrac{1}{p}] \to \mathbb{Z}/p^\infty \to 0.$$

Proof. We have a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{p^k} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/p^k & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{p^{k+1}} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/p^{k+1} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

Since $\mathbb{Z}[\frac{1}{p}] = \mathrm{colim}_k(\mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \cdots)$, taking colimits in this diagram gives the desired short exact sequences (as taking the colimit of a sequence is an exact functor). $\square$

As a consequence, if A' is an injective resolution of an abelian group A, we get a short exact sequence of chain complexes

$$0 \to \mathrm{Hom}(\mathbb{Z}/p^\infty, A') \to \mathrm{Hom}(\mathbb{Z}[\tfrac{1}{p}], A') \to A' \to 0,$$

or

$$0 \to \mathbb{R}\mathrm{Hom}(\mathbb{Z}/p^\infty, A) \to \mathbb{R}\mathrm{Hom}(\mathbb{Z}[\tfrac{1}{p}], A) \to \mathbb{R}\mathrm{Hom}(\mathbb{Z}, A) \to 0.$$

The associated long exact sequence in homology is

$$0 \to \mathrm{Hom}(\mathbb{Z}/p^\infty, A) \to \mathrm{Hom}(\mathbb{Z}[\tfrac{1}{p}], A) \to A \to \mathrm{Ext}(\mathbb{Z}/p^\infty, A) \to \mathrm{Ext}(\mathbb{Z}[\tfrac{1}{p}], A) \to 0.$$

From this we immediately see:

Lemma 20.9. *An abelian group A is derived p-complete if and only if* $\mathrm{Hom}(\mathbb{Z}[\frac{1}{p}], A) = 0$ *and* $\mathrm{Ext}(\mathbb{Z}[\frac{1}{p}], A) = 0$*, i.e. if and only if* $\mathbb{R}\mathrm{Hom}(\mathbb{Z}[\frac{1}{p}], A) \simeq 0$.

We'll also need the following characterization of derived p-complete groups:

Proposition 20.10. *If* $A \to \mathrm{Ext}(\mathbb{Z}/p^\infty, A)$ *is an isomorphism, then* $\mathrm{Hom}(\mathbb{Z}/p^\infty, A) = 0$. *In other words,* A *is derived* p*-complete if and only if* $A \xrightarrow{\sim} \mathrm{Ext}(\mathbb{Z}/p^\infty, A)$.

Proof. From the long exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Z}/p^\infty, A) \to \mathrm{Hom}(\mathbb{Z}[\tfrac{1}{p}], A) \to A \to \mathrm{Ext}(\mathbb{Z}/p^\infty, A) \to \mathrm{Ext}(\mathbb{Z}[\tfrac{1}{p}], A) \to 0$$

we see that if $A \to \mathrm{Ext}(\mathbb{Z}/p^\infty, A)$ is an isomorphism, then the map $\mathrm{Hom}(\mathbb{Z}[\frac{1}{p}], A) \to A$ is zero. A homomorphism $\mathbb{Z}[\frac{1}{p}] \to A$ is determined by a sequence $(a_0, a_1, \ldots)$ such that $a_i = pa_{i+1}$. If the map to A is zero, then this means that for any such sequence we must have $a_0 = 0$. But $(a_k, a_{k+1}, \ldots)$ is a sequence in A of the same form, so we must have $a_k = 0$ for all k. Thus $\mathrm{Hom}(\mathbb{Z}[\frac{1}{p}], A) = 0$, and then from the exact sequence we get that $\mathrm{Hom}(\mathbb{Z}/p^\infty, A) = 0$ as well. □

Remark 20.11. The derived p-complete abelian groups are precisely the ones that are W-local where W is the class of maps $A \to B$ such that $\mathrm{Ext}(\mathbb{Z}/p^\infty, A) \xrightarrow{\sim} \mathrm{Ext}(\mathbb{Z}/p^\infty, B)$ is an isomorphism. The localization of an abelian group A with respect to W is $LA = \mathrm{Ext}(\mathbb{Z}/p^\infty, A)$, with $A \to LA$ induced by the boundary map from the sequence $0 \to \mathbb{Z} \to \mathbb{Z}[\frac{1}{p}] \to \mathbb{Z}/p^\infty \to 0$.

21. p-Completion of Simply Connected Spaces II (The General Case)

We now want to consider p-completion for general simply connected spaces. The theorem we want to prove is:

Theorem 21.1.

(i) A simply connected space X *is* p*-complete if and only if* $\pi_n X$ *is derived* p*-complete for all* n*.*

(ii) If X *is a simply connected space then its* p*-completion* $X \to X^\wedge_p$ *exists, and for every* n *there is a short exact sequence*

$$0 \to \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_n X) \to \pi_n X^\wedge_p \to \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1} X) \to 0.$$

The key step is understanding the p-completion of Eilenberg-MacLane spaces in terms of the algebraic derived p-completion we defined in the previous section. This requires introducing some notation:

Definition 21.2. Suppose C is a non-negatively graded chain complex of abelian groups, with finitely many non-zero homology groups. We can define an Eilenberg-MacLane space from C by

$$K(C, n) := \prod_i K(H_i C, n + i).$$

Remark 21.3. Using a more functorial version of this definition, it can be shown that this construction is compatible with derived limits. Moreover, if we have a short exact sequence of chain complexes

$$0 \to C \to C' \to C'' \to 0$$

then $K(C, n)$ is the homotopy fibre of the induced map $K(C', n) \to K(C'', n)$. We won't prove this, but it should be plausible since the long exact sequence in homology from the short exact sequence looks the same as the long exact sequence on homotopy groups.

Definition 21.4. Suppose A is an abelian group. We define $K(A,n)\hat{_p}$ to be $K(\mathbb{D}A\hat{_p},n)$. We can also first define $K(A,n)/p^k$ to be $K(A//p^k,n)$, or equivalently the homotopy fibre of $K(A,n+1) \xrightarrow{p^k} K(A,n+1)$. Then we can define $K(A,n)\hat{_p}$ as the homotopy limit of the maps $K(A,n)/p^{k+1} \to K(A,n)/p^k$ (where the homotopy limit is given by replacing these maps by fibrations and then taking the usual limit).

Remark 21.5. From our description of $\mathbb{D}A\hat{_p}$ above, we see that

$$\pi_* K(A,n)\hat{_p} \cong \begin{cases} 0, & * \neq n, n+1, \\ \mathrm{Ext}(\mathbb{Z}/p^\infty, A), & * = n, \\ \mathrm{Hom}(\mathbb{Z}/p^\infty, A), & * = n+1. \end{cases}$$

Proposition 21.6. *If A is an abelian group, then $K(A,n) \to K(A,n)\hat{_p}$ is a p-completion.*

Before we give the proof we need to make a simple observation:

Lemma 21.7. *Suppose M is a $\mathbb{Z}/p^k$-module. Then $K(M,n)$ is p-complete.*

Proof. We prove this by induction on k. For $k=1$ the universal coefficient sequence gives (as there is no Ext term over the field $\mathbb{F}_p$) $H^*(X,M) \cong \mathrm{Hom}_{\mathbb{F}_p}(H_*(X,\mathbb{F}_p),M)$, so an $\mathbb{F}_p$-equivalence induces an isomorphism on $H^n(-,M) \cong [-,K(M,n)]$.

Now suppose M is a $\mathbb{Z}/p^k$-module. Then there is a short exact sequence

$$0 \to pM \to M \to M/pM \to 0,$$

where pM and M/pM are $\mathbb{Z}/p^{k-1}$-modules. This induces a long exact sequence

$$\cdots \to H^n(X,pM) \to H^n(X,M) \to H^n(X,M/pM) \to H^{n+1}(X,pM) \to \cdots$$

Using the 5-Lemma this gives inductively that an $\mathbb{F}_p$-equivalence gives an isomorphism on $H^n(-,M) \cong [-,K(M,n)]$. □

Proof of Proposition 21.6. To see that $K(A,n)\hat{_p}$ is p-complete it suffices by Lemma 16.8 to see that $K(A,n)/p^k$ is p-complete for each k. But this space is weakly equivalent to $K(\mathrm{Tor}(\mathbb{Z}/p^k,A),n+1) \times K(A/p^k,n)$. Here $\mathrm{Tor}(\mathbb{Z}/p^k,A)$ and A/p^k are both $\mathbb{Z}/p^k$-modules, hence this space is p-complete by Lemma 21.7.

We saw in the previous section that we have a short exact sequence of chain complexes

$$0 \to \mathbb{R}\mathrm{Hom}(\mathbb{Z}/p^\infty, A) \to \mathbb{R}\mathrm{Hom}(\mathbb{Z}[\tfrac{1}{p}], A) \to \mathbb{R}\mathrm{Hom}(\mathbb{Z}, A) \to 0,$$

which as $\mathbb{D}A\hat{_p} \simeq \mathbb{R}\mathrm{Hom}(\mathbb{Z}/p^\infty, A)[1]$ gives a fibre sequence

$$K(A,n)\hat{_p} \to K(\mathbb{R}\mathrm{Hom}(\mathbb{Z}[\tfrac{1}{p}],A),n+1) \to K(A,n+1).$$

Continuing this (as a Puppe sequence) we get a fibre sequence

$$K(\mathbb{R}\mathrm{Hom}(\mathbb{Z}[\tfrac{1}{p}],A),n) \to K(A,n) \to K(A,n)\hat{_p}.$$

Thus the homotopy groups of the fibre are $\mathrm{Hom}(\mathbb{Z}[\frac{1}{p}],A)$ in degree n and $\mathrm{Ext}(\mathbb{Z}[\frac{1}{p}],A)$ in degree $n+1$. These are in particular both $\mathbb{Z}[\frac{1}{p}]$-modules, and so the map $K(A,n) \to K(A,n)\hat{_p}$ is an $\mathbb{F}_p$-equivalence by Theorem 18.3. □

Corollary 21.8. *An Eilenberg-MacLane space $K(A,n)$ is p-complete if and only if A is derived p-complete.*

Proof. Since $K(A,n)\hat{_p}$ is the p-completion of $K(A,n)$, the space $K(A,n)$ is p-complete if and only if the natural map $K(A,n) \to K(A,n)\hat{_p}$ is a weak equivalence. From the computation of $\pi_* K(A,n)\hat{_p}$ we see this is equivalent to $\mathrm{Hom}(\mathbb{Z}/p^\infty, A) = 0$ and $A \xrightarrow{\sim} \mathrm{Ext}(\mathbb{Z}/p^\infty, A)$. □

Now a Postnikov tower argument gives one direction of (i) in Theorem 21.1:

Proposition 21.9. *If X is a simply connected space such that the groups $\pi_* X$ are derived p-complete, then X is p-complete.*

Proof. Exactly as the proof of Theorem 18.1(i). □

Using this we can prove part (ii) of the theorem:

Proof of Theorem 21.1(ii). We construct the p-completion exactly as in the Proof of Theorem 18.1(ii): We take $P_2 X\hat{_p} := K(\pi_2 X, 2)\hat{_p}$, then $P_2 X \to P_2 X\hat{_p}$ is a p-completion by Proposition 21.6. Then if we have a p-completion $P_{n-1}X \to P_{n-1}X\hat{_p}$, we get from the universal property a homotopy-commutative square

$$\begin{array}{ccc} P_{n-1}X & \xrightarrow{\phi} & P_{n-1}X\hat{_p} \\ \downarrow{\scriptstyle k_{n-1}} & & \downarrow{\scriptstyle (k_{n-1})\hat{_p}} \\ K(\pi_n X, n+1) & \xrightarrow{\psi} & K(\pi_n X, n+1)\hat{_p}, \end{array}$$

which gives on homotopy fibres a map $P_n X \to P_n X\hat{_p} := F_{(k_{n-1})\hat{_p}}$, where the space $P_n X\hat{_p}$ is p-complete by Lemma 16.7. Finally, we take $X\hat{_p} := \lim_n P_n X\hat{_p}$, which is p-complete by Lemma 16.8. The maps $P_n X \to P_n X\hat{_p}$ and $X \to X\hat{_p}$ are $\mathbb{F}_p$-equivalences by the same arguments as in the finitely generated case.

It remains to show that we have the stated description of $\pi_n X\hat{_p}$. From the long exact sequence from the fibration $P_n X\hat{_p} \to P_{n-1}X\hat{_p} \to K(\pi_n X, n+1)\hat{_p}$ we see that there are isomorphisms

$$\pi_i P_n X\hat{_p} \cong \pi_i P_{n-1} X\hat{_p}, \quad i < n,$$
$$\mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_n X) \cong \pi_{n+1} P_n X\hat{_p},$$

and (using this isomorphism for $\pi_n P_{n-1} X\hat{_p}$) there is a short exact sequence

$$0 \to \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_n X) \to \pi_n P_n X\hat{_p} \to \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1} X) \to 0.$$

The homotopy group $\pi_i P_n X\hat{_p}$ thus stabilizes for $n \geq i$ so there is no $\lim^1$ and we get $\pi_n X\hat{_p} \cong \pi_n P_n X\hat{_p}$, and so we have the desired description of this group. □

Remark 21.10. In fact, it can be shown that the short exact sequence

$$0 \to \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_n X) \to \pi_n X\hat{_p} \to \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1} X) \to 0$$

always splits.

Finally, we end by proving the other direction in Theorem 21.1(i):

Proof of Theorem 21.1(i). It remains to show that if X is simply connected and p-completed, then the abelian groups $\pi_* X$ are derived p-complete. By the description of $X\hat{_p}$ in Theorem 21.1(ii) we see if X is p-complete then there is a short exact sequence

$$0 \to \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_n X) \to \pi_n X \to \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1} X) \to 0.$$

For $n = 2$ this says $\mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_2 X) \xrightarrow{\sim} \pi_2 X$. It follows that $\pi_2 X$ is derived p-complete, for example by Remark 20.11, or by using that the short exact sequence splits and then applying Proposition 20.10. Now if $\pi_{n-1} X$ is derived p-complete, then $\mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1} X) = 0$ so the short exact sequence for $\pi_n X$ gives $\pi_n X \cong \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_n X)$. Thus we see by induction that all the homotopy groups of X are derived p-complete, as required. □

References

[BK72] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, Vol. 304, Springer-Verlag, Berlin-New York, 1972.

[Bou75] A. K. Bousfield, *The localization of spaces with respect to homology*, Topology 14 (1975), pp. 133–150.

[Bre67] G. E. Bredon, *Equivariant cohomology theories*, Lecture Notes in Mathematics, No. 34, Berlin: Springer-Verlag, 1967.

[DDK77] E. Dror, W. G. Dwyer, and D. M. Kan, *An arithmetic square for virtually nilpotent spaces*, Illinois J. Math. 21.2 (1977), pp. 242–254.

[DF92] E. Dror Farjoun, *Homotopy localization and v_1-periodic spaces*, in: *Algebraic topology (San Feliu de Guíxols, 1990)*, vol. 1509, Lecture Notes in Math. Berlin: Springer, 1992, pp. 104–113.

[DHKS04] William G. Dwyer, Philip S. Hirschhorn, Daniel M. Kan, and Jeffrey H. Smith, *Homotopy limit functors on model categories and homotopical categories*, vol. 113, Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2004.

[DI04] D. Dugger and D. C. Isaksen, *Topological hypercovers and $\mathbb{A}^1$-realizations*, Math. Z. 246.4 (2004), pp. 667–689.

[DS95] W. G. Dwyer and J. Spaliński, *Homotopy theories and model categories*, in: *Handbook of algebraic topology*, North-Holland, 1995, pp. 73–126.

[DT58] Albrecht Dold and René Thom, *Quasifaserungen und unendliche symmetrische Produkte*, Ann. of Math. (2) 67 (1958), pp. 239–281.

[Dwy04] W. G. Dwyer, *Localizations*, in: *Axiomatic, enriched and motivic homotopy theory*, vol. 131, NATO Sci. Ser. II Math. Phys. Chem. Dordrecht: Kluwer Acad. Publ., 2004, pp. 3–28.

[GJ99] P. G. Goerss and J. F. Jardine, *Simplicial homotopy theory*, vol. 174, Progress in Mathematics, Basel: Birkhäuser Verlag, 1999.

[GZ67] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Springer-Verlag New York, Inc., New York, 1967.

[Hat] A. Hatcher, *Spectral Sequences in Algebraic Topology*.

[Hat02] A. Hatcher, *Algebraic topology*, Cambridge Univ. Press, 2002.

[KT06] Akira Kono and Dai Tamaki, *Generalized cohomology*, vol. 230, Translations of Mathematical Monographs, Translated from the 2002 Japanese edition by Tamaki, Iwanami Series in Modern Mathematics, American Mathematical Society, Providence, RI, 2006.

[May99] J. P. May, *A concise course in algebraic topology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1999.

[MP12] J. P. May and K. Ponto, *More concise algebraic topology*, Chicago Lectures in Mathematics, Chicago, IL: University of Chicago Press, 2012.

[Nei09] J. A. Neisendorfer, *A quick trip through localization*, in: *Alpine perspectives on algebraic topology*, vol. 504, Contemp. Math. Providence, RI: Amer. Math. Soc., 2009, pp. 203–236.

[Qui73] D. Quillen, *Higher algebraic K-theory. I*, in: *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, Springer, Berlin, 1973, 85–147. Lecture Notes in Math., Vol. 341.

[Rie14] Emily Riehl, *Categorical homotopy theory*, vol. 24, New Mathematical Monographs, Cambridge University Press, Cambridge, 2014.

[Seg68] Graeme Segal, *Classifying spaces and spectral sequences*, Inst. Hautes Études Sci. Publ. Math. 34 (1968), pp. 105–112.

[Sul05] D. Sullivan, *Geometric topology: localization, periodicity and Galois symmetry*, vol. 8, K-Monographs in Math, The 1970 MIT notes, Springer, 2005.

[Tho79] R. W. Thomason, *Homotopy colimits in the category of small categories*, Math. Proc. Cambridge Philos. Soc. 85.1 (1979), pp. 91–109.

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