

Categories and Topology

University of Copenhagen Lecture Notes



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Introduction

This course will be an introduction to the interplay between category theory and topology, which permeates much of modern mathematics. The prerequisite is a basic understanding of classical homotopy theory, as presented for instance in Chapters 1-5 of Hatcher's book [Hat02].

We will discuss categorical and combinatorial models for topological spaces, and conversely show how we can enhance our view of our favorite categories (be it groups, rings, modules, or something more exotic) by considering them as “topological” objects that, in a sense, can be glued together as if they were actual spaces.

We first want a better understanding of the *category* of spaces, from a homotopy point of view. Recall that we started our quest in algebraic topology with the purpose of understanding all topological spaces, and all continuous maps between them, up to a suitable notion of homotopy. This led us to consider, for any two spaces X and Y , the *space*

$$\mathrm{map}(X, Y)$$

of continuous maps between them, equipped with the compact open topology.

If we want to model topological spaces in some combinatorial way, we should require the model to be strong enough to answer the following two questions:

- (1) Given two objects (“spaces”) X and Y , are they weakly¹ homotopy equivalent?
- (2) For two objects (“spaces”) X and Y , what is the weak homotopy type of the topological space of maps between them?

We would furthermore like to be able to do this via some canonical comparison, and in particular to be able to say when two models for homotopy theory are “the same”. We have already noticed that both the category of topological spaces and the category of chain complexes are endowed with a certain notion of “homotopy”, but how are they both an instance of a general notion of “homotopy theory”? What does “being a homotopy theory” mean? Could there be a homotopy theory of homotopy theories?²

We have already strived in AlgTopII to answer these questions: we showed that every topological space is weakly homotopy equivalent to a CW complex, and that two CW complexes are homotopy equivalent if and only if they are weakly homotopy equivalent. With respect to maps, we showed at least that any map between CW complexes can be approximated by a cellular map, giving us some combinatorial control of the maps between CW complexes. And in HomAlg, we studied how we can calculate derived functors of Hom via projective or injective resolutions, and showed in particular that the specific choice of these resolutions do not matter. How are these things related?

Our first step in this story, carried out in Chapter 2, is to show that we can model topological spaces, considered up to weak equivalence, in a complete combinatorial way, using something called *simplicial sets* and denoted \mathbf{sSet} .³ The rationale for the use of simplicial sets is an expansion of the fact, known from earlier courses, that homotopy theory is controlled by CW

¹The use of the word “weak” will become clear during the course; you are now invited to read “weakly homotopy equivalent” as “equivalent”, and later “weak homotopy type” as “equivalence class”.

²The answer is yes.

³For the moment, think of \mathbf{sSet} as just a category whose objects and morphisms have a combinatorial description

complexes. Simplicial sets are similar to simplicial complexes or CW complexes, but they only remember the combinatorics of the cell attachments, and not the point-set topology.

There are a few important and technical steps in making this work out—e.g., we’ll see that just like there are preferred spaces called CW-complexes there will also be preferred simplicial sets called Kan complexes (named after Dan Kan); this parallel is mild, and in fact Kan complexes resemble, in the context of chain complexes over a ring, more the injective resolutions than the projective ones (whereas CW complexes resemble more projective resolutions). We give the rudiments of this theory in Chapter 2 (which underpins everything that follows later in the course), and point the reader to the many existing references of the full story. One important player that we introduce here is also the *nerve construction*, which associates to any category a simplicial set (and hence a topological space via the above mentioned correspondence). The strength of this construction is two-fold: on the one hand we associate a geometric object with any category, on the other hand we can try to model simplicial sets (and hence topological spaces) by categories, and writing down categories, functors, and natural transformations, is often much simpler and less ambiguous than writing down spaces, maps, and homotopies.

The next thing that will concern us, is to give a first taste of what is a general “homotopy theory”, via the theory of model categories. Remember that by a “homotopy theory” we very roughly mean somewhere where we can give answers to questions (1) and (2). We will not spend a long time on the general theory of model categories, and we will just give enough definitions to indicate how topological spaces, simplicial sets and chain complexes, as well as certain other categories constructed from them⁴, can be brought into this framework. The theory of model categories was first introduced by Daniel Quillen in the late 1960’s, and often provides a good concrete way of working with any particular, classical context in which a notion of “homotopy” is defined, for instance spaces or chain complexes. A more modern definition of homotopy theory is via the theory of infinity-categories, which is the most setting where (1) and (2) make sense: roughly speaking, an infinity-category is the datum of certain objects, called “homotopy types”, together with, for any two homotopy types, a simplicial set of “maps” between them, satisfying certain composition rules. The fact that the notion of an ∞ -category is so general will enable us to do homotopy theory in new settings (e.g., new categories constructed from our old categories) without the need to make arbitrary choices,⁵ and without the need to prove many technical lemmas that, using more classical methods, would be needed (but sometimes we are just not able to prove them, or to give a clear proof!). We will get to an example in a moment. We will not get to ∞ -categories on a technical level in this course, but we will keep this viewpoint in mind from the very beginning.⁶

Developing simplicial sets, model categories, and their basic properties will occupy the first half of this course. In the second half we will be concerned with putting these tools in use: we will study “derived/homotopy” limits and colimits, and localization techniques.

More precisely, a question that will concern us is how to describe the objects in the categories of topological spaces, chain complexes, or the like, from simpler “bits”, so that if we know the simpler bits we can glue the simpler bits together and get information about what we started with.

This will involve studying functors

$$F : \mathcal{I} \rightarrow \text{Top}$$

⁴For instance, we will consider categories of certain diagrams of spaces

⁵think of the choice of projective/injective resolutions when defining derived functors in homological algebra, with the following need to prove carefully the independence of the result, up to natural isomorphism, from the choice

⁶Just as a teaser: on a technical level, an ∞ -category will be a simplicial set satisfying an extra condition, which sounds rather innocuous; but the notion is general enough so that we can view a single space as an ∞ -category, the category of all spaces as an ∞ -category, and category of all ∞ -categories as an ∞ -category as well!

i.e., functors which to each object $i \in \mathcal{I}$ assigns a topological space, and to each morphism in \mathcal{I} assigns a continuous map of topological spaces.⁷ We want to understand how we can glue the pieces (the topological spaces obtained by evaluating the functor on objects of \mathcal{I}) together, as dictated by the functor, in such a way that the resulting output is suitably homotopy invariant. What this means exactly will need to be made precise! This will be the homotopy colimit construction

$$\operatorname{hocolim}_{\mathcal{I}} F$$

which will generalize the homotopy pushout construction that we saw in AlgTopII.⁸

Note that ideally we would want to be able to make a much more general statement. We would want to say that the entire *category* Top of topological spaces is glued together from certain pieces, smaller categories. For this we may want to consider a functor

$$F : \mathcal{I} \rightarrow \mathbf{Cat}$$

where for each $\beta \in \mathcal{I}$ we assign a category, e.g., a subcategory of the category of topological spaces, and for each morphism in \mathcal{I} we assign a functor.⁹

We are however left with a big problem: How can we “do homotopy theory” inside Cat? Given two categories X and Y , can we speak of the space $\operatorname{map}(X, Y)$? The point is that we have viewed Top, not just as a category, but as a place where we can do homotopy theory, with notions of weak equivalences, and a notion of mapping space between any two objects, as in (1) and (2), i.e., an ∞ -category; and now, voila, see that we should instead have considered F from before as functor to a suitable ∞ -category of ∞ -categories.

$$F : \mathcal{I} \rightarrow \mathbf{Cat}_{\infty}$$

Such a thing \mathbf{Cat}_{∞} exists! But we are digressing, and getting *way* ahead of ourselves....

Our concrete topics in the second half will be to learn about two things. In Chapter 5 we will develop rudiments of homotopy colimits and homotopy limits of topological spaces or simplicial sets. This will include a useful spectral sequence for calculating the homology of a homotopy colimit from its pieces, vastly generalizing the Mayer-Vietoris exact sequence in homology you learned about back in AlgTop1. There is a dual sequence for homotopy limits (such as, for example, homotopy fixed-points), which is also super useful. We will also state and prove, using the previous machinery, the so-called Quillen’s Theorems A and B, which will provide a very concrete criterion for when a functor between categories induces a homotopy equivalence between nerves, and more generally (Theorem B) identify the homotopy fiber. This is perhaps the most useful single tool in showing that maps induce homotopy equivalences. We will see some examples.

In Chapter 6 we will then move on localization and completion. These are, in essence, versions of the known constructions from algebra. The key idea for localization is that we want to invert something, classically just elements in a ring, and in general a collection of morphisms in a category, and study the resulting object. The most basic (but already super useful!) case is just inverting a prime p , and we will see how the different primes and the rational information can be glued together to recover the whole space in the so-called *Sullivan arithmetic square*. More generally we can try to invert any suitable collection of maps, that we want to treat as equivalences. We are then naturally led to the question: When does the localized category again form a homotopy theory? Here the methods from the first part come in useful.

Completion, à la Bousfield–Kan, is in good cases a special kind of localization. And it has the additional advantage that it naturally comes with a tower of fibrations whose limit is the

⁷E.g. \mathcal{I} could be a pushout diagram, a group viewed as category with one object, or the natural numbers viewed as an ordered set.

⁸There is also a dual “co-gluing” construction holim , generalizing homotopy pull-backs.

⁹We ignore for the point of this discussion any set-theoretic issues, arising from the fact that the objects of a category may not be a set, as with Top—these issues can usually be fixed by adding a few right words...

completed space. This will again allow us to describe maps into the completed space via simpler pieces, which e.g., sets up an “Adams style” spectral sequence calculating homotopy classes of maps from homological information.

If time allows, we will round off the course with a special chapter where we try to use some of the techniques mentioned so far to sketch a proof of the so-called Sullivan conjecture, which says that for a finite p -group P and for a finite P -CW complex X , fixed points and homotopy fixed-points agree at a prime p , in the precise sense that we have a homotopy equivalence

$$(X^P)_p^\wedge \rightarrow (X_p^\wedge)^{hP}$$

where $(-)_p^\wedge$ denotes Bousfield–Kan p -completion, and $(-)^{hP}$ denotes homotopy fixed-points. This is surprising, since homotopy fixed-points a priori depends on much weaker properties of the space than actual fixed-points, and this theorem, proved by Miller and Lannes in the 1980s, has led to many subsequent results.

How to use these notes

We use these notes for a eponymous course at University of Copenhagen. It is a 9 week course (usually block 1, ie. Sept–Nov), with 2 x 2 hours of lectures each week and 2 x 2 hours of exercises (including breaks). We usually roughly divide the material as follows:

Week 1: Chapter 1
 Week 2–4: Chapter 2
 Week 5: Chapter 3
 Week 6: Chapter 4
 Week 7: Chapter 5
 Week 8: Chapter 6
 Week 9: Chapter 7

About the origin of these notes

These notes are used for the course Categories and Topology, taught for many years each fall at University of Copenhagen. The first installment ran in 2009 taught by Alexander Berglund, Antonio Diaz, and Richard Hepworth (all postdocs at the time) and myself (JG). The basic outline of the course, and the first hand-written notes, trace back to this early iteration. E.g., traces of Berglund’s lectures on simplicial sets (I think again inspired by a course by Torsten Ekedahl) can still be found in Chapter 2.

I started typing these notes around 2015, in the beginning only for the second half of the course (the first being taught by Ib Madsen using a different set of notes). Rune Haugseng taught the second half of the course in 2017, where he greatly expanded the written notes on Chapters 4, 5, and 6, and those chapters are hence joint work.

Piotr Pstragowski gave some guest lectures in 2019, made a handwritten outline of the part on classification of fibrations, which was then expanded and included in the notes by Lukas Woike. Simon Gritschacher co-taught the course in 2020 and much expanded the material in Chapter 1. Lukas Woike co-taught the course in 2021 and made many additions and improvements. Andrea Bianchi co-taught the course in 2022. Shachar Carmeli and Jan Steinebrunner co-taught the course in 2023.

TAs who contributed with exercises: Maxime Ramzi, Branko Juran,

Also thanks for sending corrections and suggestions: Max Fischer, Peter Patzt, ...

Please report errors and suggestions to `jg@math.ku.dk` Thanks! :) Cookies may be on the line.

Other suggested literature

These notes have been inspired by many sources, which you are also encouraged to consult. Here is a sample:

BASIC CATEGORY THEORY

- Section 2 in Dwyer-Spalinski
- MacLane: Categories for the working mathematician
- Emily Riehl's books [Rie17] and (the more advanced) [Rie14].

SIMPLICIAL HOMOTOPY THEORY:

- Ib Madsen and Irakli Patchkoria: Notes from last year(s).
- Goerss-Jardine: Simplicial homotopy theory [GJ99] (1999)
- Gabriel and Zisman: Categories of fractions in homotopy theory. (1967)
- May: Simplicial objects in algebraic topology (1967?) [May67]
- Book in progress by Joyal–Tierney: Simplicial homotopy theory
- Introduction to infinity this-and-that: Jacob Lurie: Keredon (notes downloadable from his homepage.)

MODEL CATEGORIES:

- Dwyer-Spalinski: Homotopy theories and Model categories.
- Hovey: Model categories.
- Quillen: the original thing.

HOMOTOPY (CO)LIMITS AND LOCALIZATIONS AND COMPLETIONS OF SPACES:

In the second part of the course we will cover two important topics in homotopy theory, namely homotopy (co)limits and localizations and completions of spaces.

The main source for both of these topics still remains the “yellow monster” [BK72] by Bousfield and Kan. Other useful references for homotopy colimits include:

HOMOTOPY (CO)LIMITS:

- The paper [DS95] by Dwyer-Spalinsky is a an excellent reference for explaining the basic philosophy, and it does model categories as well.
- The notes by Dan Dugger “A primer on homotopy colimits”, which can be found on his homepage <http://pages.uoregon.edu/ddugger/>, are also highly recommended.
- The book by Goerss–Jardine [GJ99] is a more modern introduction to parts of the yellow monster.
- There is also the more recently published (but quite classical in its outlook) *More concise algebraic topology* [MP12] by May and Ponto (which at 514 pages should perhaps be called “Less concise algebraic topology”).
- Riehl’s book [Rie14] gives a nice abstract account of homotopy limits and colimits — our discussion of derived functors is essentially taken from here (though much of it is originally due to [Dwy+04]).

LOCALIZATIONS AND COMPLETIONS OF SPACES:

Some references for localization and completion

- Sullivan’s MIT notes [Sul05] is a good informal introduction to localization and completion.
- Another recommended introduction is Neisendorfer’s notes [Nei09].
- There’s also a lot of classical material summarized in May–Ponto [MP12].
- The simplicial viewpoint on localizations is explained in Bousfield-Kan [BK72].
- General existence of localizations with respect to a homology theory was proved by Bousfield in [Bou75].
- The book of Dror-Farjoun [DF92] explains in detail how to localize with respect to a map.
- A general version of the arithmetic square is given in [DDK77] (also correcting a point in Bousfield–Kan).
- The paper [Dwy04] by Dwyer contains a more advanced survey of localizations with many examples.
- The notes on *Rational and p -adic homotopy theory* by Thomas Nikolaus contain a good discussion of p -completion that we also benefited greatly from.

CHAPTER 1

Categories

The goal of this chapter is to introduce a bit of basic category theory with a view towards the later chapters. We also recommend [DS95, Sec. 2] as a readable source for some of this material.

1.1. Basic categorical definitions

We start with a definition of a category, using notation that should remind the reader of boundary maps in simplicial complexes.

DEFINITION 1.1.1. By a category \mathcal{C} we mean a class of *objects* $\text{Ob}(\mathcal{C})$ and a class of *morphisms* $\text{Mor}(\mathcal{C})$ (aka *maps* or *arrows*) together with assignments

- (1) $d_1: \text{Mor}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{C})$, the *source* map.
- (2) $d_0: \text{Mor}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{C})$, the *target* map.
- (3) $s_0: \text{Ob}(\mathcal{C}) \rightarrow \text{Mor}(\mathcal{C})$, the *identity* map.
- (4) $\circ: \text{Mor}(\mathcal{C}) \times_{\text{Ob}(\mathcal{C})} \text{Mor}(\mathcal{C}) \rightarrow \text{Mor}(\mathcal{C})$ called *composition* and written $(f, g) \mapsto g \circ f$, where we denote by $\text{Mor}(\mathcal{C}) \times_{\text{Ob}(\mathcal{C})} \text{Mor}(\mathcal{C})$ the class of pairs of morphisms $\{(f, g) \in \text{Mor}(\mathcal{C}) \times \text{Mor}(\mathcal{C}) \mid d_0 f = d_1 g\}$.

These are required to satisfy the well-known identities:

- If $d_0(f) = d_1(g)$ and $d_0(g) = d_1(h)$ then $(h \circ g) \circ f = h \circ (g \circ f)$. (Associativity.)
- $f \circ s_0(d_1(f)) = f = s_0(d_0(f)) \circ f$. (Identity)

We also write $f: x \rightarrow y$ for a morphism $f \in \text{Mor}(\mathcal{C})$ with $d_1(f) = x$ and $d_0(f) = y$, 1_x for $s_0(x)$, and $\text{Hom}_{\mathcal{C}}(x, y)$ for the collection of morphisms f with $d_1 f = x$ and $d_0 f = y$. Some authors prefer the notation $\mathcal{C}(x, y)$ for $\text{Hom}_{\mathcal{C}}(x, y)$.

We will assume that all categories \mathcal{C} satisfy the condition that $\text{Hom}_{\mathcal{C}}(x, y)$ is a set. (This condition is referred to in the literature by saying that \mathcal{C} is *locally small*.)

REMARK 1.1.2. Let us make some notational observations: In the definition of source and target in (1) and (2), it is d_1 rather than d_0 that denotes the source, as we view it as “forgetting” the target.

Also note that in the definition of composition in (4), the order gets reversed. This is due to the well-established, but in hindsight probably regrettable, convention of composing functions (of sets) from right to left.

DEFINITION 1.1.3. A category \mathcal{C} is called *small* if it has a set of objects and a set of morphisms. We will often use the letter \mathcal{I} for indexing categories (i.e. categories used to describe the shape of a diagram), and indexing categories are assumed to be small.

EXAMPLE 1.1.4. Some (very) small categories \mathcal{I} we shall consider are:

- Any set, considered as giving the objects of a category with no non-identity morphisms.
- The pushout category $(\bullet \leftarrow \bullet \rightarrow \bullet)$ and the pullback category $(\bullet \rightarrow \bullet \leftarrow \bullet)$, each having three objects and two non-identity morphisms.
- The equalizer category $\bullet \rightrightarrows \bullet$.
- The category $[n]$, consisting of the $n + 1$ numbers from 0 to n , considered as a totally ordered set, i.e. “ $[n] = (0 < 1 < \dots < n) = (\bullet \rightarrow \bullet \dots \rightarrow \bullet)$ ”

- The simplex category Δ , with objects $[0], [1], [2], \dots$. A morphism $f: [n] \rightarrow [m]$ is an order-preserving function of sets $(\{0, \dots, n\}, \leq) \rightarrow (\{0, \dots, m\}, \leq)$.
- The category $\mathcal{B}G$, with one object and a group G as morphisms. (We might even slip into the notation of denoting this category just as G .)
- The category $\mathcal{E}G$ with G as objects, and a unique morphism between any two objects.
- The natural numbers \mathbb{N} , considered as a totally ordered set $(\bullet \rightarrow \bullet \rightarrow \dots)$.
- *Poset categories*, i.e., categories with at most one morphism between any two objects. One of our favorite examples of poset categories will be the poset of a certain class of subgroups (finite, p -groups, etc.) of a fixed group G .
- The *orbit category* $\mathcal{O}_{\mathcal{F}}(G)$ for G a group, and \mathcal{F} a set of subgroups closed under conjugation. This has objects transitive G -sets with isotropy subgroup in \mathcal{F} , and morphisms G -equivariant maps. (\mathcal{F} is called a *collection* of subgroups, and if it is furthermore closed under passage to subgroups a *family* of subgroups.)
- So-called EI-categories, i.e., categories where every Endomorphism is an Isomorphism. Groups, poset categories, and orbit categories, are examples of EI-categories. EI-categories have an associated poset with the same underlying class of objects, by declaring $[x] \leq [y]$ if there exists a non-trivial map $x \rightarrow y$ — this provides a filtration of an EI-category by subcategories, and often allows proofs by induction.

EXAMPLE 1.1.5. Some larger categories we shall consider are:

- The category "Groups" of all groups and group homomorphisms.
- The category " R -mod" of all R -modules and R -module maps.
- The category "Top" of topological spaces and continuous maps.
- The category " $\text{Ch}(R)$ " of chain complexes of R -modules and chain maps of those.

One central issue is when two categories should be considered “the same”. To attack this question we first need to view our categories (including our very large ones) as objects of some larger category. The first step in this process is to define morphisms between categories. These are called functors.

DEFINITION 1.1.6. Given two categories \mathcal{C} and \mathcal{D} , a *functor* $F: \mathcal{C} \rightarrow \mathcal{D}$ assigns to each object x of \mathcal{C} an object Fx of \mathcal{D} and to each morphism $f: x \rightarrow y$ of \mathcal{C} a morphism $Ff: Fx \rightarrow Fy$ of \mathcal{D} , such that for each object x of \mathcal{C}

$$F(1_x) = 1_{Fx}$$

and for each pair of morphisms (f, g) whose composite $g \circ f$ is defined in \mathcal{C}

$$F(g \circ f) = F(g) \circ F(f).$$

EXAMPLE 1.1.7. A functor $F: [n] \rightarrow [m]$ is precisely an order preserving function $(\{0, \dots, n\}, \leq) \rightarrow (\{0, \dots, m\}, \leq)$. In other words, a morphism $f: [n] \rightarrow [m]$ in Δ is exactly a functor (where $[n]$ and $[m]$ are seen as categories).

REMARK 1.1.8. We can alternatively write the data of a small category \mathcal{C} in the following way. Setting $X_0 = \text{Ob}(\mathcal{C})$ and $X_1 = \text{Mor}(\mathcal{C})$, the maps of Definition 1.1.1 are encoded in the following diagram

$$X_0 \begin{array}{c} \xleftarrow{d_0} \\ \xleftarrow{s_0} \rightarrow \\ \xrightarrow{d_1} \end{array} X_1$$

with $d_0 s_0 = 1$, $d_1 s_0 = 1$, together with a composition operation $\circ: X_1 \times_{X_0} X_1 \rightarrow X_1$, where $X_1 \times_{X_0} X_1 = \{(f, g) \in X_1 \times X_1 \mid d_0 f = d_1 g\}$, subject to the associativity and unitality conditions as in Definition 1.1.1.

Let \mathcal{D} be another small category and write $Y_0 := \text{Ob}(\mathcal{D})$ and $Y_1 := \text{Mor}(\mathcal{D})$. Then a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ consists of maps $X_0 \rightarrow Y_0$ and $X_1 \rightarrow Y_1$ making the relevant parts of the following

diagrams commute

$$\begin{array}{ccc} X_0 & \begin{array}{c} \xleftarrow{d_0} \\ \xrightarrow{s_0} \\ \xleftarrow{d_1} \end{array} & X_1 \\ \downarrow & & \downarrow \\ Y_0 & \begin{array}{c} \xleftarrow{d_0} \\ \xrightarrow{s_0} \\ \xleftarrow{d_1} \end{array} & Y_1 \end{array} \quad \begin{array}{ccc} X_1 \times_{X_0} X_1 & \xrightarrow{\circ} & X_1 \\ \downarrow & & \downarrow \\ Y_1 \times_{Y_0} Y_1 & \xrightarrow{\circ} & Y_1 \end{array}$$

DEFINITION 1.1.9. Given two functors $F, G: \mathcal{C} \rightarrow \mathcal{D}$, a *natural transformation* $\tau: F \Rightarrow G$ is an assignment $\text{Ob}(\mathcal{C}) \rightarrow \text{Mor}(\mathcal{D})$ which assigns to each object x of \mathcal{C} a morphism $\tau_x: Fx \rightarrow Gx$ of \mathcal{D} such that for every morphism $f: x \rightarrow y$ of \mathcal{C} the following diagram in \mathcal{D} commutes

$$\begin{array}{ccc} Fx & \xrightarrow{\tau_x} & Gx \\ \downarrow Ff & & \downarrow Gf \\ Fy & \xrightarrow{\tau_y} & Gy \end{array}$$

in the sense that $Gf \circ \tau_x = \tau_y \circ Ff$ as morphisms in \mathcal{D} .

DEFINITION 1.1.10 (Functor categories). Let \mathcal{I} be a small category and \mathcal{C} any (locally small) category; we write $\mathcal{C}^{\mathcal{I}}$ (or sometimes $\text{Fun}(\mathcal{I}, \mathcal{C})$) for the category of functors from \mathcal{I} to \mathcal{C} . It has as *class* of objects the functors $F: \mathcal{I} \rightarrow \mathcal{C}$, and as morphisms from F to F' the *set* of natural transformations. ($\text{Hom}_{\mathcal{C}^{\mathcal{I}}}(F, F')$ is still a set since \mathcal{I} is assumed small and \mathcal{C} locally small.)

DEFINITION 1.1.11 (Subcategories). A *subcategory* \mathcal{D} of a category \mathcal{C} is a choice of subclasses of the classes of objects and morphisms of \mathcal{C} such that for any morphism $f: x \rightarrow y$ in \mathcal{D} , the objects x and y also lie in \mathcal{D} and such that the morphisms in \mathcal{D} are closed under composition and contain all identities of objects in \mathcal{D} . This turns a subcategory \mathcal{D} of \mathcal{C} into a category in its own right and the inclusion of \mathcal{D} into \mathcal{C} into a functor. We call a subcategory *full* if for all objects x, y in \mathcal{D} we have an equality of sets $\text{Hom}_{\mathcal{D}}(x, y) = \text{Hom}_{\mathcal{C}}(x, y)$, i.e. the inclusion functor induces a bijection on morphisms sets.

REMARK 1.1.12. Given a category \mathcal{C} , we obtain a full subcategory of \mathcal{C} by specifying any subclass of the objects of \mathcal{C} (this determines the morphism of this subcategory already because we assume it is full). We refer to this as the full subcategory spanned by these objects. For example, we will use sometimes the full subcategory $\Delta_{\leq n}$ of Δ spanned by the objects $[\ell]$ with $\ell \leq n$.

1.2. Example: The category of chain complexes

As the category of chain complexes will play an important role throughout these notes, both as a "place to do homotopy theory" and as a home for various invariants, we'll recall it's basic properties with a view towards later sections.

The reader might already have encountered these facts in a first homological algebra course, and if so can quickly skip to the next section.

1.2.1. The category $\text{Ch}(R)$. Recall that $\text{Ch}(R)$ the category of unbounded chain complexes are defined as the full subcategory of $\text{Fun}((\mathbb{Z}, \leq)^{\text{op}}, R\text{-mod})$ satisfying that any morphism between non-equal and non-consecutive objects map to zero, i.e, fancy way of saying that the objects are sequences

$$\cdots \rightarrow M_{n+1} \xrightarrow{\partial_{n+1}} M_n \xrightarrow{\partial_n} M_{n-1} \rightarrow \cdots$$

with $\partial_{n+1}\partial_n = 0$ and morphisms degree preserving maps.

We can also define a tensor product (or monoidal product) on chain complexes. For $C, D \in \text{Ch}(R)$, chain complexes over a commutative ring R , we can define the complex $C \otimes_R D$ by

$$(C \otimes_R D)_n = \bigoplus_{p+q=n} C_p \otimes_R D_q.$$

The differential is determined by the product rule $\partial(c \otimes d) = \partial c \otimes d + (-1)^{|c|} c \otimes \partial d$, where $|c|$ is the degree of c . This endows $\text{Ch}(R)$ with a symmetric monoidal structure.¹

We can furthermore define "internal" *hom complexes* i.e., a chain complex of homomorphisms

$$\text{hom}_R(C, D)_d = \prod_n \text{Hom}_R(C_n, D_{n+d})$$

equipped with a canonical differential $\partial(f_n: C_n \rightarrow D_{n+d}) = \partial_D f_n - (-1)^d f_{n-1} \partial_C$. The reader will want to verify that the signs introduced in the definition of the differential insures that $C \otimes_R D$ and $\text{hom}_R(C, D)$ are again chain complexes², and that

$$\text{Hom}_{\text{Ch}(R)}(C \otimes_R D, E) \xrightarrow{\cong} \text{Hom}_{\text{Ch}(R)}(C, \text{hom}_R(D, E)).$$

Note that this structure passes to similar structures on $\text{Ch}_{\geq 0}(R)$, the full subcategory of non-negatively graded chain complexes, so that we have inclusions of categories

$$R\text{-mod} \subseteq \text{Ch}_{\geq 0}(R) \subseteq \text{Ch}(R)$$

with matching structures.

1.2.2. The category $\mathcal{K}(R)$ of chain complexes modulo chain homotopy. As defined $\text{Ch}(R)$ will be a so-called abelian category, i.e., an abelian category, via the abelian structure on $R\text{-mod}$,

But the main interest in $\text{Ch}_{\geq 0}(R)$ and $\text{Ch}(R)$ lie in that we can construct interesting homotopical categories (∞ -categories) from them. The most useful one is the "derived category" where we formally invert morphisms which induce isomorphisms on homology, i.e., quasi-isomorphisms. Depending on how one does the inversion, one arrives either at an ∞ -category, or the homotopy category of it, which will be a so-called triangulated category. Be warned people sometimes also denote the mapping space in the ∞ -category by Hom or hom , though this is really a derived mapping space, classically denoted $\mathbb{R}\text{Hom}$.

In this subsection we will start by viewing $\text{Ch}(R)$ up to chain homotopy, i.e., where we identify two morphisms $f, g: C \rightarrow D$ if there exists a family of morphisms $h_n: C_n \rightarrow D_{n+1}$, $n \in \mathbb{Z}$ such that

$$f_n - g_n = \partial_{n+1} h_n + h_{n-1} \partial_n$$

Let us first make some observations about $\text{hom}(C, D)$: We can identify the morphisms $\text{Ch}(R)$, i.e., the chain maps, as the cycles in $\text{hom}_R(C, D)_0$:

$$Z_0(\text{hom}_R(C, D)) = \{ \{f_n\} \in \prod_n \text{Hom}_R(C_n, D_n) \mid \partial f = 0 \} = \text{Hom}_{\text{Ch}(R)}(C, D)$$

Likewise, $B_0(\text{hom}_R(C, D))$ will be those elements $\{f_n\} \in \prod_n \text{Hom}_R(C_n, D_n)$ that differ from zero by a chain homotopy.

$$B_0(\text{hom}_R(C, D)) =$$

$$\{ \{f_n\} \in \text{hom}(C, D)_0 \mid \exists \{h_n\} \in \prod_n \text{Hom}_R(C_n, D_{n+1}) \text{ s.t. } \partial_{n+1} h_n + h_{n-1} \partial_n = 0 \ \forall n \}$$

and hence

$$H_0(\text{hom}_R(C, D)) = \{ \text{chain maps } C \rightarrow D \} / \{ \text{chain homotopy} \}$$

¹The reader might ask: where did this monoidal structure come from? It is not the level-wise monoidal structure on the functor category, but rather a sort of "function convolution", as we also see eg when we multiply in the group algebra or in analysis. It can be described by first doing an "external" tensor product, obtaining a functor on $(\mathbb{Z}, \leq)^{\text{op}} \times (\mathbb{Z}, \leq)^{\text{op}}$ and then using a left Kan extension (see Section 1.10) along the addition symmetric monoidal structure $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ to obtain a symmetric monoidal structure on Ch_R . This sort of procedure is called a "Day convolution" after the Australian category theorist Brian Day.

²The notation $\text{Hom}_R(C, D)$ is also often used for the full mapping complex rather than $\text{hom}_R(C, D)$. The notation map_R is also used.

One defines the (chain) homotopy category $\mathcal{K}(R)$ of chain complexes as the category with objects chain complexes and morphism

$$\mathrm{Hom}_{\mathcal{K}(R)}(C, D) = H_0(\mathrm{hom}_R(C, D))$$

For example letting $M[n]$ (or $\Sigma^n M$) denote the module M viewed as a chain complex in degree n we have

$$\mathrm{Hom}_{\mathcal{K}(R)}(R[n], C) = H_n(C)$$

Note that objects in $\mathcal{K}(R)$ in particular will be zero if they are contractible, i.e., if the identity is chain homotopy equivalent to the zero map. By construction $\mathcal{K}(R)$ will be an additive category, i.e., the hom sets carry a natural group structure, but it will *not* in general be an abelian category.

We will later want to introduce the derived category, which is the category obtained from $\mathrm{Ch}(R)$ by formally inverting quasi-isomorphisms, i.e., morphisms which induce isomorphism on homology.

Define the interval I as the chain complex

$$\cdots 0 \rightarrow R \xrightarrow{(1, -1)} R \oplus R \rightarrow 0 \cdots$$

concentrated in degree 1 and 0, and note that $H_*(I) = R$ concentrated in degree 0.

Specifying a chain map $C \otimes I \rightarrow D$ is by adjunction the same as specifying a chain map $I \rightarrow \mathrm{hom}(C, D)$ i.e., a commutative diagram

$$\begin{array}{ccc} 0 & \longrightarrow & \mathrm{hom}_R(C, D)_2 \\ \downarrow & & \downarrow \\ R & \xrightarrow{h} & \mathrm{hom}_R(C, D)_1 \\ \downarrow (1, -1) & & \downarrow \\ R \oplus R & \xrightarrow{(f, g)} & \mathrm{hom}_R(C, D)_0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array}$$

In other words it means specifying chain maps $\{f_n: C_n \rightarrow D_n\}$, $\{g_n: C_n \rightarrow D_n\}$, together with a degree one map $\{h_n: C_n \rightarrow D_{n+1}\}$ such that $\partial_{n+1}h_n + h_{n-1}\partial_n = f_n - g_n$ for all n , which exactly the same as specifying f and g and a chain homotopy between them.

1.3. The Yoneda lemma

Let \mathcal{I} denote a small category. There are a couple of easy-to-prove, but frightfully useful things to say concerning the special functors $\mathcal{I} \rightarrow \mathrm{Set}$ of the form $\mathrm{Hom}_{\mathcal{I}}(x, -)$ and $\mathcal{I}^{\mathrm{op}} \rightarrow \mathrm{Set}$ of the form $\mathrm{Hom}_{\mathcal{I}}(-, x)$, called *(co)representable* functors (and (co)represented by x).

The most famous is the *Yoneda lemma* which has as a corollary that the *Yoneda embedding*

$$\mathcal{Y}: \mathcal{I} \rightarrow \mathrm{Set}^{\mathcal{I}^{\mathrm{op}}},$$

sending x to $\mathrm{Hom}_{\mathcal{I}}(-, x)$ is fully faithful. This allows us to study \mathcal{I} via $\mathrm{Set}^{\mathcal{I}^{\mathrm{op}}}$, which usually has much better formal properties. Conversely we shall show in Section 1.11 that $\mathrm{Set}^{\mathcal{I}^{\mathrm{op}}}$ “freely generated” by \mathcal{I} which conversely allows us to make constructions on $\mathrm{Set}^{\mathcal{I}^{\mathrm{op}}}$ by defining them on \mathcal{I} and extending “by continuity”.

LEMMA 1.3.1 (Yoneda). *For $F: \mathcal{I} \rightarrow \mathrm{Set}$ any functor we have a natural isomorphism of sets*

$$\mathrm{Nat}(\mathrm{Hom}_{\mathcal{I}}(x, -), F(-)) \xrightarrow{\cong} F(x),$$

with source the set of natural transformations, given by $\Phi \mapsto \Phi_x(\mathrm{id}_x)$.

The inverse is given by associating with any $u \in F(x)$ the natural transformation $\text{Hom}_{\mathcal{I}}(x, -) \rightarrow F(-)$ given by sending $f \in \text{Hom}_{\mathcal{I}}(x, y)$ to $F(f)(u) \in F(y)$.

Similarly for contravariant functors $F: \mathcal{I}^{\text{op}} \rightarrow \text{Set}$, we have

$$\text{Nat}(\text{Hom}_{\mathcal{I}}(-, y), F(-)) \cong F(y)$$

PROOF. The forward map is obviously surjective, since as described in the proposition any element $u \in F(x)$ determines a natural transformation $\Phi: \text{Hom}_{\mathcal{I}}(x, -) \rightarrow F(-)$ by sending $f \in \text{Hom}_{\mathcal{I}}(x, y)$ to $F(f)(u) \in F(y)$, and $\Phi_x(\text{id}_x) = F(\text{id}_x)(u) = \text{id}_{F(x)}(u) = u$.

To see injectivity, let $\Phi \in \text{Nat}(\text{Hom}_{\mathcal{I}}(x, -), F(-))$ and note that, by definition of a natural transformation, we have the following commutative diagram of sets

$$\begin{array}{ccc} \text{Hom}_{\mathcal{I}}(x, x) & \xrightarrow{\text{Hom}_{\mathcal{I}}(x, f)} & \text{Hom}_{\mathcal{I}}(x, y) \\ \downarrow \Phi_x & & \downarrow \Phi_y \\ F(x) & \xrightarrow{F(f)} & F(y). \end{array}$$

It shows that

$$\Phi_y(f) = \text{Hom}_{\mathcal{I}}(x, f)(\text{id}_x) = F(f)(\Phi_x(\text{id}_x)),$$

i.e., that the natural transformation Φ is determined by $\Phi_x(\text{id}_x) \in F(x)$. This isomorphism is obviously natural in x .

The contravariant version is just a restatement of the covariant version applied to \mathcal{I}^{op} , as $\text{Hom}_{\mathcal{I}}(x, y) = \text{Hom}_{\mathcal{I}^{\text{op}}}(y, x)$. \square

The above immediately implies:

COROLLARY 1.3.2 (Yoneda embedding).

$$\text{Nat}(\text{Hom}_{\mathcal{I}}(-, x), \text{Hom}_{\mathcal{I}}(-, y)) \cong \text{Hom}_{\mathcal{I}}(x, y)$$

In particular the functor $\mathfrak{Y}: \mathcal{I} \rightarrow \text{Set}^{\mathcal{I}^{\text{op}}}$ sending x to $\text{Hom}_{\mathcal{I}}(-, x)$ is a fully faithful embedding of \mathcal{I} (called the Yoneda embedding) \square

REMARK 1.3.3. Note our show-off use of the Japanese character \mathfrak{Y} , the first character of Yoneda; you can of course just write y , if you wish.

REMARK 1.3.4. Note that we have already encountered (co)representable functors in HomAlg , in formulas such as $(-)^G = \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}, -)$ or $_p(-) = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, -)$ and in AlgTopII , in formulas such as $\pi_n(-) = [S^n, -]_{pt}$ or $H^n(-; M) = [-, K(M, n)]$.

REMARK 1.3.5. The functor category $\mathcal{P}(\mathcal{I}) = \text{Set}^{\mathcal{I}^{\text{op}}}$ is often called Set-valued presheaves on \mathcal{I} , and the Yoneda embedding is then written $\mathcal{I} \hookrightarrow \mathcal{P}(\mathcal{I})$.

1.4. Adjoint functors

An extremely useful concept in category theory is that of adjoint functors. There are different ways of formulating the definition of an adjoint functor pair, at least one of which you may be familiar with. But let us begin by building a baby-adjunction which will illustrate the general concept.

EXAMPLE 1.4.1. Suppose that A is a set and X is a topological space. There are two obvious ways to endow A with a topology: We can give S the discrete topology (every subset is open) and denote the resulting space by A^δ . We can also equip A with the indiscrete topology (the only open subsets are \emptyset and A) and denote the resulting space by A^0 . It is clear, then, that

every map of sets $A^\delta \rightarrow X$ is continuous, and that every map of sets $X \rightarrow A^0$ is continuous too. Using the language of categories, we can phrase this more formally as follows: Define functors

$$\begin{aligned} (-)^\delta &: \text{Set} \rightarrow \text{Top} \\ (-)^0 &: \text{Set} \rightarrow \text{Top} \end{aligned}$$

and a functor

$$U: \text{Top} \rightarrow \text{Set}$$

which sends a topological space X to its underlying set UX , i.e., it “forgets the topology of X ” (therefore also called a *forgetful* functor). We observed above that for any set A and any space X there are bijections

$$\text{Hom}_{\text{Top}}(A^\delta, X) \cong \text{Hom}_{\text{Set}}(A, UX)$$

and

$$\text{Hom}_{\text{Top}}(X, A^0) \cong \text{Hom}_{\text{Set}}(UX, A),$$

and in fact these bijections are *natural* in A and X .

The observation of the preceding example is formalized by the notion of an adjoint functor pair, where it will say that $(-)^\delta$ and $(-)^0$ are respectively left and right adjoint of the forgetful functor $U: \text{Top} \rightarrow \text{Set}$.

DEFINITION 1.4.2. Given categories \mathcal{C} and \mathcal{D} an *adjunction* from \mathcal{C} to \mathcal{D} consists of a pair of functors

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{D}$$

and for each pair of objects $x \in \mathcal{C}$ and $y \in \mathcal{D}$ a bijection of sets

$$\varphi_{x,y}: \text{Hom}_{\mathcal{D}}(Fx, y) \xrightarrow{\cong} \text{Hom}_{\mathcal{C}}(x, Gy)$$

that is natural in both x and y . We call F a *left-adjoint* for G and G a *right-adjoint* for F and often write $F \dashv G$.

REMARK 1.4.3. Naturality of the bijections $\{\varphi_{x,y}\}$ can be expressed by saying that

$$\varphi: \text{Hom}_{\mathcal{D}}(F(-), -) \Rightarrow \text{Hom}_{\mathcal{C}}(-, G(-))$$

is a natural isomorphism of functors $\mathcal{C}^{\text{op}} \times \mathcal{D} \rightarrow \text{Set}$.

As a notational remark, we will usually omit the subscript on $\varphi_{x,y}$ and simply write φ for any component of the natural transformation φ .

EXAMPLE 1.4.4. In Example 1.4.1 we showed that $(-)^\delta \dashv U$ and $U \dashv (-)^0$. In particular, U is both a right-adjoint (with left-adjoint $(-)^\delta$) and a left-adjoint (with right-adjoint $(-)^0$).

1.4.1. Units and counits of adjunctions. There are other, equivalent ways of talking about adjunctions. For this observe that if F is a left adjoint for G , then the adjunction provides a natural transformation $\eta: 1 \Rightarrow GF$ with

$$\eta_x := \varphi_{x, Fx}(1_{Fx}) \in \text{Hom}_{\mathcal{C}}(x, GFx).$$

Naturality follows from the commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{D}}(Fx, Fx) & \xrightarrow{\cong} & \text{Hom}_{\mathcal{C}}(x, GFx) \\ \downarrow (Ff)_* & & \downarrow (GFf)_* \\ \text{Hom}_{\mathcal{D}}(Fx, Fy) & \xrightarrow{\cong} & \text{Hom}_{\mathcal{C}}(x, GFy) \\ \uparrow (Ff)^* & & \uparrow f^* \\ \text{Hom}_{\mathcal{D}}(Fy, Fy) & \xrightarrow{\cong} & \text{Hom}_{\mathcal{C}}(y, GFy) \end{array}$$

which shows (by chasing 1_{Fx} and 1_{Fy} through the diagram) that $GF(f) \circ \eta_x = \eta_y \circ f$ for every morphism $f: x \rightarrow y$. Similarly, we get a natural transformation $\varepsilon: FG \Rightarrow 1$ by setting

$$\varepsilon_y := \varphi_{Gy,y}^{-1}(1_{Gy}).$$

These natural transformations are called the *unit* and *co-unit* of the adjunction. They encode the natural bijection φ in the following fashion:

$$(1.4.1) \quad \varphi(f) = Gf \circ \eta_x, \quad \varphi^{-1}(g) = \varepsilon_y \circ Fg.$$

for $f: Fx \rightarrow y$ and $g: x \rightarrow Gy$. The first identity follows by chasing 1_{Fx} through the commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{D}}(Fx, Fx) & \xrightarrow{\varphi} & \text{Hom}_{\mathcal{C}}(x, GFx) \\ \downarrow f_* & & \downarrow (Gf)_* \\ \text{Hom}_{\mathcal{D}}(Fx, y) & \xrightarrow{\varphi} & \text{Hom}_{\mathcal{C}}(x, Gy) \end{array}$$

and the second identity follows similarly.

Unit and co-unit can be used to give the following equivalent characterization of an adjoint functor pair.

PROPOSITION 1.4.5. *An adjunction*

$$\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$$

with F left-adjoint to G determines and is determined by natural transformations $\eta: 1 \Rightarrow GF$ and $\varepsilon: FG \Rightarrow 1$ satisfying the triangle identities

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & \searrow = & \downarrow \varepsilon F \\ & & F \end{array} \quad \begin{array}{ccc} G & \xrightarrow{\eta G} & GFG \\ & \searrow = & \downarrow G\varepsilon \\ & & G. \end{array}$$

REMARK 1.4.6. Here, by definition, $F\eta$ is the natural transformation whose components are $(F\eta)_x := F(\eta_x)$ and εF is the natural transformation whose components are $(\varepsilon F)_x := \varepsilon_{Fx}$.

PROOF OF PROPOSITION 1.4.5. We have already proved almost half of the proposition. Let us check that η and ε , when they come from an adjunction $F \dashv G$, indeed satisfy the triangle identities. This follows from (1.4.1) as

$$1_{Fx} = \varphi^{-1}(\eta_x) = \varepsilon_{Fx} \circ F(\eta_x),$$

and

$$1_{Gy} = \varphi(\varepsilon_y) = G\varepsilon_y \circ \eta_{Gy}.$$

Conversely, suppose we are given the natural transformations η and ε and they satisfy the triangle identities. Then define

$$\varphi: \text{Hom}_{\mathcal{D}}(Fx, y) \rightleftharpoons \text{Hom}_{\mathcal{C}}(x, Gy) : \psi$$

by $\varphi(f) := Gf \circ \eta_x$ and $\psi(g) := \varepsilon_y \circ Fg$. These maps are natural, because η and ε are. Now for every $g: x \rightarrow Gy$ we have

$$\varphi(\psi(g)) = G(\varepsilon_y \circ Fg) \circ \eta_x = G\varepsilon_y \circ GFG \circ \eta_x = G\varepsilon_y \circ \eta_{Gy} \circ g = g.$$

In the third equality we used naturality of η , and in the last equality we used the triangle identity. Hence, $\varphi \circ \psi = \text{id}$. Similarly, one shows that $\psi \circ \varphi = \text{id}$. \square

Often the formal triangle identities of units and counits are easier to manipulate than the explicit bijections of Hom-sets. This also gives a concrete way of checking that two functors are adjoint, by specifying units and counits.

Adjoint functors have proved an amazingly useful concept. A reason is, as we shall return to in Section 1.8, that left adjoints preserve colimits and right adjoints preserve limits (and in fact, modulo set-theoretic issues, this characterizes left and right adjoints). Seeing that a functor is an adjoint is usually easier than verifying that it preserves (co)limits.

1.4.2. Uniqueness of adjoints. Another important point to mention is that adjoints are unique up to natural isomorphism. Sometimes one can prove that two functors are naturally isomorphic by showing that they are a left (or right) adjoint for the same functor.

PROPOSITION 1.4.7. *Any two left-adjoints F and F' of a functor $G: \mathcal{D} \rightarrow \mathcal{C}$ are naturally isomorphic. The analogous statement for right-adjoints holds as well.*

PROOF. One can prove this using the Yoneda lemma 1.3.1. By definition of adjunctions we have for every morphism $f: x \rightarrow y$ of \mathcal{C} a commutative diagram of natural transformations between functors $\mathcal{D} \rightarrow \text{Set}$

$$\begin{array}{ccccc} \text{Hom}_{\mathcal{D}}(Fy, -) & \xrightarrow{\cong} & \text{Hom}_{\mathcal{C}}(y, G(-)) & \xrightarrow{\cong} & \text{Hom}_{\mathcal{D}}(F'y, -) \\ \downarrow (Ff)^* & & \downarrow f^* & & \downarrow (F'f)^* \\ \text{Hom}_{\mathcal{D}}(Fx, -) & \xrightarrow{\cong} & \text{Hom}_{\mathcal{C}}(x, G(-)) & \xrightarrow{\cong} & \text{Hom}_{\mathcal{D}}(F'x, -). \end{array}$$

By Lemma 1.3.1 the composite natural transformation in the top row is induced by a unique isomorphism $\tau_y: F'y \rightarrow Fy$ and, likewise, the composition of the natural transformations in the bottom row is induced by a unique isomorphism $\tau_x: F'x \rightarrow Fx$. Moreover, there is a unique arrow $F'x \rightarrow Fy$ which induces the natural transformation obtained by either going through the top right or the bottom left corner of the diagram. Both $\tau_y \circ F'f$ as well as $Ff \circ \tau_x$ induce this natural transformation, so they must be equal: $\tau_y \circ F'f = Ff \circ \tau_x$. This shows that τ is a natural isomorphism $F' \cong F$. \square

1.4.3. Examples of adjoint functors. We will now give some examples of types of adjoint functors that occur frequently. We urge the reader to think of which of their other favorite functors are part of adjoint pairs, and can also consult the classic [Mac71] or its modern counterpart [Rie17] for inspiration.

EXAMPLE 1.4.8 (Free-forgetful adjunction). The functor $R[-]: \text{Sets} \rightarrow R\text{-modules}$ given by $X \mapsto RX$, the free R -module on X has a right adjoint given by forgetful functor from R -modules to sets. I.e., the left adjoint is given by freely generating the extra structure. The unit is given $\eta_X: X \rightarrow RX$ in Set given by inclusion and counit $\varepsilon_M: RM \rightarrow M$ given by evaluating a formal sum using the R -module structure on M . (Note that the same construction gives a functor $R[-]$ from monoids into R -algebras, which is again left adjoint to the forgetful functor from R -algebras to monoids, forgetting the R -module structure and only remembering the multiplication.)

EXAMPLE 1.4.9 (Base change). The restriction functor along ring homomorphism $\phi: R \rightarrow S$ induces a functor from S -modules into R -modules. This functor has a left adjoint is given by “extension of scalars” $S \otimes_R (-)$.

The unit is the map from R -modules to R -modules $M \rightarrow S \otimes_R M$ given by $m \mapsto 1 \otimes m$. The counit is the map from S -modules to S -modules $S \otimes_R M \rightarrow M$ given by $s \otimes m \mapsto sm$.

EXAMPLE 1.4.10 (Group of units). The functor $R[-]: \text{Groups} \rightarrow R\text{-algebras}$, assigning to G its groupring RG has a right adjoint given by the group of units $(-)^{\times}$, which can again be viewed as the composite of two right adjoints: The forgetful functor from R -algebras to monoids, followed by the groupoid core functor $(-)^{\simeq}$ to groups, which assigns the subobject of invertible elements.

1.4.3.1. *Reflective and coreflective subcategories.* Another special case is in the case of a full subcategory where the inclusion has a left or a right adjoint, in which case the subcategory enjoys special properties.

EXAMPLE 1.4.11 (Reflective subcategories). These are full subcategories where the inclusion $i: \mathcal{C} \rightarrow \mathcal{D}$ has a left adjoint L (“the reflector”).

One example is that the inclusion of groupoids into categories, which has a left adjoint where we formally adjoin inverses f^{-1} to all morphisms f .

Another example is the inclusion of abelian groups into all groups, where the left adjoint is abelianization, or torsion-free abelian groups into all abelian groups, where the left adjoint quotients out by torsion. Or rational vector spaces into abelian groups, where the left adjoint is given by rationalization $- \otimes \mathbb{Q}$.

In all of these examples we are potentially adding some generators, suitably freely, and then modding out by some relations.

We may think of the unit of the adjunction $\mathbb{1} \Rightarrow i \circ L: \mathcal{D} \rightarrow \mathcal{D}$ as a “localization”. Beware that as $i \circ L$ is the composite of a left and a right adjoint, it may not itself have adjoints.

EXAMPLE 1.4.12 (Coreflective subcategories). These are full subcategories where the inclusion $i: \mathcal{C} \rightarrow \mathcal{D}$ has a right adjoint R (“the coreflector”). The inclusion of groupoids into categories is again an example, which also has a right adjoint, namely the *groupoid* core functor $(-)^{\simeq}$ that assigns the subcategory of isomorphisms (in particular $(-)^{\simeq}$ sends a monoid to its largest subgroup). Another example is torsion-abelian groups into abelian groups. A third example is the inclusions of topological spaces having the homotopy type of a CW complex into the homotopy category of all topological spaces, where the right adjoint is CW approximation. A related example is the subcategory $k\text{Top}$ of Top given by spaces having the compactly generated topology, also called “Kelly spaces”, where a set is open iff its restriction every compact set is open. Here the right adjoint k is the functor which re-topologizes a space to make it compactly generated. We will work this out in Proposition 1.4.15 below.

We may think of the counit of the adjunction $i \circ R \Rightarrow \mathbb{1}: \mathcal{D} \rightarrow \mathcal{D}$ as a “colocalization”, and repeat the warning that this may not have adjoints from earlier.

1.4.3.2. *hom-tensor adjunctions.*

EXAMPLE 1.4.13 (Hom- \otimes -adjunctions). In this example we give some examples of adjoint functors arising as “Hom- \otimes -adjunctions”, a discussion we continue in Section 1.5.

- (1) The original Hom- \otimes -adjunction: Let R be a commutative ring, and let A, B, C be R -modules. Then there are natural bijections

$$\text{Hom}_R(A \otimes_R B, C) \cong \text{Hom}_R(A, \text{Hom}_R(B, C)),$$

so $- \otimes_R B \dashv \text{Hom}_R(B, -)$. Note that in order to make sense of the right hand side we must view $\text{Hom}_R(B, C)$ as an R -module and not just as a set. As an exercise you can convince yourself that the co-unit of the adjunction is given by evaluation

$$\begin{aligned} \text{Hom}_R(B, C) \otimes B &\rightarrow C \\ (f, b) &\mapsto f(b) \end{aligned}$$

while the unit takes the form

$$\begin{aligned} A &\rightarrow \text{Hom}_R(B, A \otimes B) \\ a &\mapsto (b \mapsto a \otimes b). \end{aligned}$$

- (2) Let $X, Y, Z \in \text{Set}$. Then there are natural bijections

$$\text{Hom}_{\text{Set}}(X \times Y, Z) \cong \text{Hom}_{\text{Set}}(X, \text{Hom}_{\text{Set}}(Y, Z)),$$

and co-unit and unit take much the same form as in the Hom- \otimes -adjunction.

- (3) It is desirable to have the previous adjunction available also in the category \mathbf{Top} . Let X, Y, Z be topological spaces. Define the *mapping space*

$$\mathrm{map}(Y, Z)$$

to be the set $\mathrm{Hom}_{\mathbf{Top}}(Y, Z)$ equipped with the *compact-open topology*: That is, a subbasis for the topology is the collection of sets

$$\{f \in \mathrm{Hom}_{\mathbf{Top}}(Y, Z) \mid f(K) \subseteq U\}_{K, U},$$

where K runs through compact subsets of Y and U runs through open subsets of Z . One can then show that there are natural bijections

$$\mathrm{Hom}_{\mathbf{Top}}(X \times Y, Z) \cong \mathrm{Hom}_{\mathbf{Top}}(X, \mathrm{map}(Y, Z))$$

provided that X, Y are Hausdorff and Y is locally compact.

These point-set topological assumptions are of course unfortunate, as we do not get an adjunction on all of \mathbf{Top} —note that even when Y and Z are locally compact, $\mathrm{map}(Y, Z)$ need not be. A way out is to restrict the category \mathbf{Top} appropriately to what some refer to as a “convenient category of topological spaces”. One such category is the category \mathbf{CGHaus} of all compactly generated Hausdorff spaces. As mentioned above it turns out that the full embedding $\mathbf{CGHaus} \hookrightarrow \mathbf{Haus}$ into all Hausdorff spaces is coreflective, with right-adjoint $k: \mathbf{Haus} \rightarrow \mathbf{CGHaus}$. The category \mathbf{CGHaus} is easily seen to have products given as $k(X \times Y)$ (see also Proposition 1.9.9 below). With these modifications, one then obtains for all $X, Y, Z \in \mathbf{CGHaus}$ a bijection

$$\mathrm{Hom}_{\mathbf{CGHaus}}(X \times_k Y, Z) \cong \mathrm{Hom}_{\mathbf{CGHaus}}(X, k \mathrm{map}(Y, Z)),$$

where $(- \times_k -) = k(- \times -)$ is the categorical product and $k(\mathrm{map}(-, -))$ an “internal hom object” in \mathbf{CGHaus} . We will elaborate on this in Proposition 1.4.15 and Section 1.5 below.

- (4) There is also a basepointed version of the previous example: Let \mathbf{CGHaus}_* be the category of compactly generated Hausdorff spaces equipped with a basepoint, and basepoint preserving maps between them. Let $X, Y, Z \in \mathbf{CGHaus}_*$.³ Then there is a natural bijection

$$\mathrm{Hom}_{\mathbf{CGHaus}_*}(X \wedge Y, Z) \cong \mathrm{Hom}_{\mathbf{CGHaus}_*}(X, k \mathrm{map}_*(Y, Z)),$$

where $\mathrm{map}_*(Y, Z) \subseteq k \mathrm{map}(Y, Z)$ is the subspace of basepoint preserving maps and

$$X \wedge Y := X \times_k Y / X \vee Y$$

is the *smash-product* of X and Y (see AlgTop II).

In particular, if $Y = S^1$, then $X \wedge S^1 = \Sigma X$ is the reduced suspension of X , and $\mathrm{map}_*(S^1, Z) = \Omega Z$ is the loop space of Z . Consequently, we obtain an adjunction $\Sigma \dashv \Omega$ on the category \mathbf{CGHaus}_* .

1.4.4. Topologies on topological spaces. Let us end this section on adjoint functors by for completeness verifying the properties of the subspace $k\mathbf{Top}$ of \mathbf{Top} claimed in Example 1.4.13(3), as it plays a role later, when discussing the so-called nerve functor—the impatient reader may charge ahead to the next section and only return to this later. Let us repeat the definition.

DEFINITION 1.4.14. Define $k\mathbf{Top}$, the category of compactly generated topological spaces, to be the full subcategory of \mathbf{Top} consisting of spaces X with the property that a subset $A \subseteq X$ is open if and only if $A \cap K$ is open in K for all compact subsets $K \subseteq X$.

³It is standard to write “ $x \in \mathcal{C}$ ” meaning that x is an object of the category \mathcal{C} .

PROPOSITION 1.4.15. *Let $k: \text{Top} \rightarrow k\text{Top}$ be the functor which re-topologizes a topological space X , by the topology declaring a subset U to be open in $k(X)$ if $U \cap K$ is open in K for all compact subsets $K \subseteq X$.*

The functor k is right adjoint to the inclusion functor $k\text{Top} \rightarrow \text{Top}$, i.e., for $X \in k\text{Top}$ and $Y \in \text{Top}$ we have

$$\text{Hom}_{\text{Top}}(X, Y) \cong \text{Hom}_{k\text{Top}}(X, k(Y)) = \text{Hom}_{\text{Top}}(X, k(Y)).$$

Any CW complex is in $k\text{Top}$, and Top and $k\text{Top}$ have the same compact subsets.

PROOF. One verifies directly that this indeed defines a topology and that the functor k is idempotent (i.e. $kkY \cong kY$ naturally in Y). One also checks that if X is compactly generated and Y is any space then $\text{Hom}_{\text{Top}}(X, Y) \cong \text{Hom}_{k\text{Top}}(X, kY)$.

Namely $\text{Hom}_{k\text{Top}}(X, kY) \subseteq \text{Hom}_{\text{Top}}(X, Y)$ since kY has more open sets than Y . Assume that $f: X \rightarrow Y$ is continuous, and let $U \subseteq Y$ be open in kY , i.e., $U \cap L$ open in L for all compact subsets $L \subseteq Y$. To see that $f^{-1}(U)$ is open in X it is enough to see that $f^{-1}(U) \cap K$ is open in K for all compact K , by the assumption $X \in k\text{Top}$. But $f^{-1}(U) \cap K = (f|_K)^{-1}(U) = (f|_K)^{-1}(U \cap f(K))$. And as $f: X \rightarrow Y$ is continuous, $f(K)$ is compact, and therefore $f(K) \cap U$ is open in $f(K)$ by assumption on U . But then $f^{-1}(U) \cap K$ is open in K as desired, as $f|_K$ is continuous.

That CW complexes are in $k\text{Top}$ follows by the definition of topology on CW complexes (as finite subcomplexes are compact, and any compact subset lies in a finite subcomplex, see Hatcher [Hat02, Prop. A.1]). It follows by the definition that they have the same compact subsets. \square

PROPOSITION 1.4.16. *The counit map $kY \rightarrow Y$ in Top is a weak equivalence for any topological space Y .*

PROOF. It follows from the adjunction that $\text{Hom}_{\text{Top}}(X, Y) = \text{Hom}_{\text{Top}}(X, kY)$ for any finite CW-complex X and arbitrary Y as any finite CW complex lies in $k\text{Top}$ by Proposition 1.4.15. \square

1.5. Example: The formalism of monoidal and closed categories

In example Example 1.4.13 we gave a bunch of examples of categories \mathcal{C} with what we called $\text{Hom} - \otimes$ -adjunctions. These were categories that had some sort of tensor product \otimes (a monoidal structure) and where that product fit in an adjoint pair with an "internal hom object" of maps in \mathcal{C} , i.e., giving an "exponential law". The general framework for these examples are *monoidal* category and *closed* categories. Monoidal categories are also called *tensor categories*, for the obvious reason. In this subsection we'll briefly explain this formalism—it can be skipped at first reading.⁴

Very briefly, a monoidal category consists of

- a category \mathcal{C}
- a functor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ (product), see also Proposition 1.7.1
- an object $\mathbf{1} \in \mathcal{C}$ (unit)

together with specified natural isomorphisms

- $x \otimes (y \otimes z) \cong (x \otimes y) \otimes z$ (associativity)
- $x \otimes \mathbf{1} \cong x \cong \mathbf{1} \otimes x$ (unitality)

for all $x, y, z \in \mathcal{C}$. These isomorphisms are required to make certain "coherence diagrams" commute, which we will not spell out. For the purposes here, it suffices to know that a monoidal category is a category with some kind of product on it that behaves more or less how we

⁴Sometimes one may also want the morphisms between objects x and y to carry extra structure, e.g., be a vector, space, module; this leads to the related but distinct notion on enriched category. A closed category will be enriched over itself, but necessarily not vice versa.

expect it. Let us write $(\mathcal{C}, \otimes, \mathbf{1})$ for a monoidal category, omitting the associativity and identity isomorphisms from the notation.

If \mathcal{C} is a category with finite products, then we can define a monoidal structure by letting the monoidal product be the categorical product and the unite the terminal object (as we the case in some of the examples in Example 1.4.13 such as \mathbf{Set} or \mathbf{Top}), a so-called *cartesian monoidal category* but of course other interesting examples such as the tensor product of R -modules do not arise this way.

A *braiding* on a monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ consists of natural isomorphisms $x \otimes y \cong y \otimes x$ for all $x, y \in \mathcal{C}$ which are again required to make certain diagrams commute. A *symmetric monoidal category* is a braided monoidal category subject to the condition that the double braiding $x \otimes y \cong y \otimes x \cong x \otimes y$ is the identity for all $x, y \in \mathcal{C}$. Examples of symmetric monoidal categories are all of the above mentioned examples

$$(R\text{-Mod}, \otimes, R), \quad (\mathbf{Set}, \times, \{*\}), \quad (\mathbf{CGHaus}, \times, \{*\}), \quad (\mathbf{CGHaus}_*, \wedge, S^0).$$

(Recall that indeed $X \wedge S^0 \cong X \cong S^0 \wedge X$).

Now one calls a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ *closed* if for each $x \in \mathcal{C}$ the functor

$$- \otimes x: \mathcal{C} \rightarrow \mathcal{C}$$

has a specified right-adjoint, which is usually denoted $(-)^x$ (also called an “exponential”). And one says that \mathcal{C} is *cartesian closed* if $\otimes = \times$ happens to be the categorical product.

Phrased in this language we can summarize Example 1.4.13 by saying that $R\text{-Mod}$ and \mathbf{CGHaus}_* are closed symmetric monoidal categories under tensor product and smash product, respectively, and \mathbf{Set} and \mathbf{CGHaus} are cartesian closed. Their “exponentials” are mapping objects like $\text{map}(Y, Z)$ or $\text{Hom}_R(B, C)$. Note that these are just the Hom-sets of the respective categories equipped with some extra structure making them objects of this very same category! For this reason they are also referred to as “internal Hom-objects”.

1.6. Equivalence of categories

We have an obvious notion of isomorphism of categories: a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is an isomorphism, if there is a functor $G: \mathcal{D} \rightarrow \mathcal{C}$ such that $FG = 1$ and $GF = 1$. But in practice, categories are rarely isomorphic. *Equivalence* of categories is a much more flexible and useful notion than isomorphism of categories.

DEFINITION 1.6.1. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is an *equivalence* if there exists a functor $G: \mathcal{D} \rightarrow \mathcal{C}$ and natural isomorphisms $GF \cong 1$ and $FG \cong 1$.

Any equivalence $F: \mathcal{C} \rightarrow \mathcal{D}$ gives rise to an adjunction (which we may then call an *adjoint equivalence*), but of course not every adjunction is an equivalence. The following proposition makes the relationship precise, and at the same time gives us a criterion to check that a given functor is an equivalence.

PROPOSITION 1.6.2. For a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ the following are equivalent:

- (1) F is an equivalence
- (2) F has a right-adjoint G such that unit $\eta: 1 \Rightarrow GF$ and co-unit $\varepsilon: FG \Rightarrow 1$ are natural isomorphisms
- (3) F has a left-adjoint G' such that unit $\eta': 1 \Rightarrow FG'$ and co-unit $\varepsilon': G'F \Rightarrow 1$ are natural isomorphisms
- (4) F is fully faithful and essentially surjective.

In the above, we say that F is *full* if for any two objects x, y of \mathcal{C} the map of sets induced by F

$$(1.6.1) \quad \text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Hom}_{\mathcal{D}}(Fx, Fy)$$

is surjective. We call F *faithful* if for any two objects x, y of \mathcal{C} the map (1.6.1) is injective. We say that F is *essentially surjective* if for every object y of \mathcal{D} there is an object x of \mathcal{C} and an isomorphism $Fx \cong y$.

PROOF OF PROPOSITION 1.6.2. We first show that (2) \Rightarrow (3). Suppose that $F \dashv G$ is an adjunction and that unit $\eta: 1 \Rightarrow GF$ and co-unit $\varepsilon: FG \Rightarrow 1$ are natural isomorphisms. We claim that G is then also a left-adjoint for F with unit $\eta' := \varepsilon^{-1}$ and co-unit $\varepsilon' := \eta^{-1}$. By Proposition 1.4.5 it is enough to check the triangle identities, that is, we must check that $F\eta^{-1} \circ \varepsilon^{-1}F = 1$ and $\eta^{-1}G \circ G\varepsilon^{-1} = 1$. But clearly, this is equivalent to $\varepsilon F \circ F\eta = 1$ and $G\varepsilon \circ \eta G = 1$, which are the triangle identities for the adjunction $F \dashv G$. This proves that G is a left-adjoint for F . In the same fashion we show that (3) \Rightarrow (2).

It is clear that (2) \Rightarrow (1). Next we show that (1) \Rightarrow (4). Let $G: \mathcal{D} \rightarrow \mathcal{C}$ denote the inverse equivalence. The natural isomorphism $FG \cong 1$ shows that for each object y of \mathcal{D} there is an isomorphism $FGy \cong y$. This proves that F is essentially surjective. To show that F is faithful and full let $\theta: GF \cong 1$ be a natural isomorphism. For every morphism $f: x \rightarrow y$ of \mathcal{C} there is a commutative diagram

$$\begin{array}{ccc} GFx & \xrightarrow[\cong]{\theta_x} & x \\ GFf \downarrow & & \downarrow f \\ GFy & \xrightarrow[\cong]{\theta_y} & y \end{array}$$

which shows that the map

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \xrightarrow{F} \mathrm{Hom}_{\mathcal{D}}(Fx, Fy) \xrightarrow{G} \mathrm{Hom}_{\mathcal{C}}(GFx, GFy)$$

is given by $f \mapsto GFf = \theta_y^{-1} \circ f \circ \theta_x$. Since this map is a bijection, F is faithful. By symmetry, G is faithful, too. To show that F is full, let $h \in \mathrm{Hom}_{\mathcal{D}}(Fx, Fy)$. Define $f := \theta_y \circ Gh \circ \theta_x^{-1}$. Then $GFf = Gh$, but since G is faithful, this implies that $Ff = h$. Hence, F is full.

To prove that (4) \Rightarrow (2) we *choose*⁵ for every object y of \mathcal{D} an object " Gy " of \mathcal{C} and an isomorphism $\varepsilon_y: FGy \cong y$. If $f: y \rightarrow y'$ is a morphism, then, since F is full and faithful, there is a unique morphism $Gf: G(y) \rightarrow G(y')$ with $FGf = \varepsilon_{y'}^{-1} \circ f \circ \varepsilon_y$. By construction, the diagram

$$\begin{array}{ccc} FG(y) & \xrightarrow{\varepsilon_y} & y \\ \downarrow FG(f) & & \downarrow f \\ FG(y') & \xrightarrow{\varepsilon_{y'}} & y' \end{array}$$

commutes. By uniqueness, $G(f \circ f') = G(f) \circ G(f')$, hence G is a functor and $\varepsilon: FG \Rightarrow 1$ is a natural isomorphism. In fact, F and ε induce bijections

$$\mathrm{Hom}_{\mathcal{C}}(x, Gy) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{D}}(Fx, FGy) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{D}}(Fx, y),$$

so G is a right-adjoint for F and ε is the counit of the adjunction. It remains to see that the unit of the adjunction $\eta: 1 \Rightarrow GF$ is also a natural isomorphism, but this follows from the triangle identities. \square

EXAMPLE 1.6.3. Part (4) of Proposition 1.6.2 provides a way to replace a category \mathcal{C} by an equivalent (possibly smaller) category $\mathrm{sk}(\mathcal{C})$ in the following way: For every isomorphism class of objects of \mathcal{C} choose precisely one representative. Then take $\mathrm{sk}(\mathcal{C})$ to be the full subcategory of \mathcal{C} on those representatives. By construction, the inclusion $\mathrm{sk}(\mathcal{C}) \subset \mathcal{C}$ satisfies (4), so it is an equivalence. We call $\mathrm{sk}(\mathcal{C})$ a *skeleton* of \mathcal{C} .

As a concrete example, take \mathcal{C} to be the category Vect_k^{fd} of finite dimensional k -vectorspaces and k -linear maps. From Linear Algebra we know that a finite dimensional vector space is

⁵In this step, we need a strong enough version of the axiom of choice: note indeed that there is a choice to be made for each object of \mathcal{D} , and these form a class that, possibly, is not a set!

isomorphic to k^n for some $n \in \mathbb{N}$. The linear maps $k^n \rightarrow k^m$ are precisely the $m \times n$ matrices over k . So the category whose objects are $0, 1, 2, \dots$ and whose Hom-sets are $\text{Hom}(n, m) = \text{Mat}_{m \times n}(k)$ is equivalent to Vect_k^{fd} . They are certainly not isomorphic, because the objects of Vect_k^{fd} form a proper class.

The following proposition give criterions when an adjunction determines a reflective or coreflective subcategory.

PROPOSITION 1.6.4. *Suppose that we are given an adjunction*

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{D}$$

Then the unit $\eta: 1 \Rightarrow GF$ is an equivalence if and only if F is fully faithful. In this case the essential image of F ⁶ is a coreflective subcategory of \mathcal{D} equivalent to \mathcal{C} .

Furthermore the coreflective subcategory is equivalent to \mathcal{D} if G is conservative⁷.

The dual statements holds about counits also hold, i.e., $\varepsilon: FG \Rightarrow 1$ is an equivalence if and only if G is fully faithful, and in that case the essential image of G is a reflective subcategory of \mathcal{C} equivalent to \mathcal{I} .

PROOF. For $x, y \in \mathcal{C}$, we have a map

$$\phi: \text{Hom}_{\mathcal{C}}(x, y) \xrightarrow{\eta} \text{Hom}_{\mathcal{C}}(x, GFy) \cong \text{Hom}_{\mathcal{D}}(Fx, Fy)$$

which is an isomorphism if and only if F is fully faithful. Hence if $\eta: 1 \Rightarrow GF$ is an equivalence, then ϕ is an isomorphism and F is fully faithful. And conversely, if F is fully faithful then $\eta: 1 \Rightarrow GF$ is an equivalence by the Yoneda lemma.

If G is furthermore conservative, then counit ε is also an equivalence by the triangle equality

$$\begin{array}{ccc} G & \xrightarrow{\eta G} & GFG \\ & \searrow = & \downarrow G\varepsilon \\ & & G. \end{array}$$

The dual statements follow by passing to opposite category. See also Exercise 1.12.48 \square

1.7. Example: The category \mathbf{Cat} of small categories

An important category for us will be the (big!!)⁸ category \mathbf{Cat} of all small categories. It has objects small categories and morphisms functors. (It even has a bit of higher structure in that we can define 2-morphisms as natural transformations, but that shall not concern us here.)

Let's start with the following easy observation:

PROPOSITION 1.7.1. *For two categories \mathcal{C} and \mathcal{D} in \mathbf{Cat} the categorical product exists. The product category $\mathcal{C} \times \mathcal{D}$ is the category whose objects are pairs (c, d) , where c is an object in \mathcal{C} and d is an object in \mathcal{D} ; a morphism $(c, d) \rightarrow (c', d')$ in $\mathcal{C} \times \mathcal{D}$ is a pair (f, g) , where $f: c \rightarrow c'$ is a morphism in \mathcal{C} and $g: d \rightarrow d'$ a morphism in \mathcal{D} . \square*

EXERCISE 1.7.2. Explicitly describe the objects and morphisms in $[n] \times [m]$.

We have the following "exponential law" for categories:

⁶The essential image of F is defined to be the full subcategory of \mathcal{D} spanned by objects isomorphic to $F(x)$ for some x in \mathcal{I}

⁷A functor G is said to be conservative, or to reflect isomorphisms, if it is true that if $G(f)$ is an isomorphism so is f .

⁸As the word "big" suggests, \mathbf{Cat} is not itself a small category, but it is still locally small, in the sense of Definition 1.1.1.

PROPOSITION 1.7.3. *For small categories \mathcal{C} and \mathcal{D} , let $\text{Fun}(\mathcal{C}, \mathcal{D})$ be the category with objects functors $\mathcal{C} \rightarrow \mathcal{D}$ and morphisms natural transformations between them. Then for small categories \mathcal{C} , \mathcal{D} and \mathcal{E} have a natural bijection*

$$\text{Hom}_{\text{Cat}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \xrightarrow{\cong} \text{Hom}_{\text{Cat}}(\mathcal{C}, \text{Fun}(\mathcal{D}, \mathcal{E}))$$

inducing an isomorphism of functors $\text{Cat}^{\text{op}} \times \text{Cat}^{\text{op}} \times \text{Cat} \rightarrow \text{Set}$.

In other words, $\text{Fun}(\mathcal{D}, -)$ is right adjoint to $- \times \mathcal{D}$ as endofunctors of Cat , i.e., $\text{Fun}(\mathcal{D}, \mathcal{E})$ is an internal hom functor in Cat , endowed with the cartesian monoidal structure $\otimes = - \times - : \text{Cat} \times \text{Cat} \rightarrow \text{Cat}$, in the language of Section 1.5.

PROOF. This is an exercise in unraveling the definitions: For a functor $F : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ we get for a fixed $c \in \mathcal{C}$ a functor $F(c, -) : \mathcal{D} \rightarrow \mathcal{E}$ and for each $f : c \rightarrow c'$ we get a natural transformation $F(f, -) : F(c, -) \rightarrow F(c', -)$. Hence we have described a functor $\tilde{F} : \mathcal{C} \rightarrow \text{Fun}(\mathcal{D}, \mathcal{E})$. These steps are obviously reversible.

It is also clear this bijection satisfies the stated naturality properties. \square

Let us look at what homotopies look like, from a categorical point of view:

PROPOSITION 1.7.4. *For small categories \mathcal{C}, \mathcal{D} , the functor category $\text{Fun}(\mathcal{C} \times [1], \mathcal{D})$ has objects that can be identified with triples (F, G, κ) where $F, G : \mathcal{C} \rightarrow \mathcal{D}$ are functors and $\kappa : F \Rightarrow G$ a natural transformation.*

PROOF. The objects $\mathcal{C} \times [1]$ are of the form $(c, 0)$ and $(c, 1)$, and the morphisms generated by $(c, 0) \xrightarrow{(f, id)} (d, 0)$, $(c, 1) \xrightarrow{(f, id)} (d, 1)$, and $(c, 0) \rightarrow (c, 1)$. Hence specifying a functor $H : \mathcal{C} \times [1] \rightarrow \mathcal{D}$, amounts to specifying functors $H(-, 0) : \mathcal{C} \times 0 \rightarrow \mathcal{D}$, $H(-, 1) : \mathcal{C} \times 1 \rightarrow \mathcal{D}$, and $H(c, 0 < 1) : H(c, 0) \rightarrow H(c, 1)$ such that

$$\begin{array}{ccc} H(c, 0) & \xrightarrow{H(c, 0 < 1)} & H(c, 1) \\ \downarrow H(f, 0) & & \downarrow H(f, 1) \\ H(d, 0) & \xrightarrow{H(d, 0 < 1)} & H(d, 1) \end{array}$$

commutes for all $f : c \rightarrow d$. Hence identifying $F(-) = H(-, 0)$, $G = H(-, 1)$ and $\kappa_c = H(c, 0 < 1)$, this is exactly the data (F, G, κ) as claimed. \square

REMARK 1.7.5. From this point of view, natural transformations look like “categorical homotopies”, at least if we regard the category $[1]$ as the counterpart of an interval. We shall return to this viewpoint. Note however now that we see an asymmetry in the above description, as natural transformations cannot in general be reversed! We need much work from higher algebra to properly deal with this issue.

1.8. Limits and colimits

DEFINITION 1.8.1 (Limits and colimits). Suppose \mathcal{I} is a small category. We write δ for the constant-diagram (aka diagonal) functor $\delta : \mathcal{C} \rightarrow \text{Fun}(\mathcal{I}, \mathcal{C})$, induced by the constant functor $\mathcal{I} \rightarrow *$.

- (1) The *limit* of a functor $F : \mathcal{I} \rightarrow \mathcal{C}$ is an object $\lim_{\mathcal{I}} F$ in \mathcal{C} together with a morphism $\delta(\lim F) \rightarrow F$ in $\mathcal{C}^{\mathcal{I}}$ such that the induced map

$$\text{Hom}_{\mathcal{C}}(X, \lim_{\mathcal{I}} F) \rightarrow \text{Hom}_{\mathcal{C}^{\mathcal{I}}}(\delta X, \delta \lim F) \rightarrow \text{Hom}_{\mathcal{C}^{\mathcal{I}}}(\delta X, F)$$

is an isomorphism for every $X \in \mathcal{C}$. We can illustrate this as follows:

$$\begin{array}{ccccc}
 X & & & & \\
 \swarrow & \searrow & & \searrow & \\
 & \lim F & \longrightarrow & Y_j & \\
 & \downarrow & & \downarrow & \\
 & Y_i & \longrightarrow & Y_k &
 \end{array}$$

- (2) The *colimit* of an object $F \in \mathcal{C}^{\mathcal{I}}$ is an object $\text{colim}_{\mathcal{I}} F \in \mathcal{C}$ together with a morphism $F \rightarrow \delta \text{colim}_{\mathcal{I}} F$ such that the induced map

$$\text{Hom}_{\mathcal{C}}(\text{colim}_{\mathcal{I}} F, X) \rightarrow \text{Hom}_{\mathcal{C}^{\mathcal{I}}}(\delta \text{colim}_{\mathcal{I}} F, \delta X) \rightarrow \text{Hom}_{\mathcal{C}^{\mathcal{I}}}(F, \delta X)$$

is an isomorphism for every $X \in \mathcal{C}$.

For an arbitrary category \mathcal{C} the limit of over \mathcal{I} may or may not exist. However, if they exist they are unique, as follows from the following definition in terms of representable functors.

PROPOSITION 1.8.2. *The limit of $F: \mathcal{I} \rightarrow \mathcal{C}$ exists if and only if the functor*

$$\text{Hom}_{\mathcal{C}^{\mathcal{I}}}(\delta(-), F): \mathcal{C}^{\text{op}} \rightarrow \text{Set}$$

is representable. If it exists, $\lim_{\mathcal{I}} F$ is the object satisfying

$$\text{Hom}_{\mathcal{C}^{\mathcal{I}}}(\delta(-), F) \cong \text{Hom}_{\mathcal{C}}(-, \lim_{\mathcal{I}} F),$$

unique up to isomorphism.

Similarly $\text{colim}_{\mathcal{I}} F$, if it exists, is the, up to isomorphism, unique object satisfying

$$\text{Hom}_{\mathcal{C}^{\mathcal{I}}}(F, \delta(-)) \cong \text{Hom}_{\mathcal{C}}(\text{colim}_{\mathcal{I}} F, -).$$

PROOF. The fact that $\lim_{\mathcal{I}} F$ and $\text{colim}_{\mathcal{I}} F$ have these properties is a consequence of the definition. Uniqueness follows from the Yoneda Lemma 1.3.1. \square

EXAMPLE 1.8.3. Here are some special cases:

- (1) The limit over the unique functor on the empty set is called a/the *terminal object*. It is by definition an object $\lim_{\emptyset} F \in \mathcal{C}$ such that $\text{Hom}_{\mathcal{C}}(X, \lim_{\emptyset} F)$ is a point for all $X \in \mathcal{C}$.
- (2) The colimit over the unique functor on the empty set is called a/the *initial object*. It is by definition an object $\text{colim}_{\emptyset} F \in \mathcal{C}$ such that $\text{Hom}_{\mathcal{C}}(\text{colim}_{\emptyset} F, X)$ is a point for all $X \in \mathcal{C}$.
- (3) A limit over a discrete category (a category with only identity morphisms) is called a (categorical) *product*. A colimit over a discrete category is called a (categorical) *coproduct*.
- (4) A colimit over \mathbb{N} is a standard *direct limit*. A limit over \mathbb{N}^{op} is an *inverse limit*.
- (5) Limits over $(\bullet \rightarrow \bullet \leftarrow \bullet)$ are called *pull-backs*. Colimits over $(\bullet \leftarrow \bullet \rightarrow \bullet) = (\bullet \rightarrow \bullet \leftarrow \bullet)^{\text{op}}$ are called *push-outs*.
- (6) Limits and colimits over $(\bullet \rightrightarrows \bullet)$ are called *equalizers* and *coequalizers* respectively (note that $(\bullet \rightrightarrows \bullet)$ is isomorphic to its own opposite).
- (7) Limits over $\mathcal{B}G$ are called *fixed-points* (or invariants, if the target category is abelian). Colimits over $\mathcal{B}G$ are called *orbits* (or coinvariants if the target category is abelian).

EXAMPLE 1.8.4 (Some examples in abelian groups).

- $\lim(0 \rightarrow B \xleftarrow{f} A) = \ker(f)$ and $\text{colim}(0 \rightarrow B \xleftarrow{f} A) = B$.
- For $\mathcal{I} = \mathcal{B}G$, a group G considered as a category with one object, and $\mathcal{C} = \text{Ab}$, $\lim_{\mathcal{B}G} M \cong M^G$, the invariants, and $\text{colim}_{\mathcal{B}G} M \cong M_G$, the coinvariants.
- $\lim(\cdots \rightarrow \mathbb{Z}/p^3 \rightarrow \mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p) \cong \mathbb{Z}_p$, the p -adic integers. The colimit is \mathbb{Z}/p .

- $\text{colim}(\mathbb{Z}/p \rightarrow \mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p^3 \rightarrow \cdots) = \mathbb{Z}/p^\infty$, the injective envelope of \mathbb{Z}/p . The limit is \mathbb{Z}/p .

The following example is also fundamental

PROPOSITION 1.8.5. *Suppose that \mathcal{I} is a small category with an initial object s . Then for any functor $F : \mathcal{I} \rightarrow \mathcal{C}$, $\lim_{\mathcal{I}} F$ exists and $F(s) \xrightarrow{\cong} \lim_{\mathcal{I}} F$ via the maps $F(s) \rightarrow F(i)$ for all $i \in \mathcal{I}$.*

Dually if \mathcal{I} is a small category with a terminal object t , then for any functor $F : \mathcal{I} \rightarrow \mathcal{C}$, $\text{colim}_{\mathcal{I}} F$ exists and $\text{colim}_{\mathcal{I}} F \xrightarrow{\cong} F(t)$ via $F(i) \rightarrow F(t)$ for all $i \in \mathcal{I}$.

PROOF. This follows from the universal property. \square

If limits exist for all functors F , they tie together to give a functor $\lim_{\mathcal{I}} : \text{Fun}(\mathcal{I}, \mathcal{C}) \rightarrow \mathcal{C}$ adjoint to the diagonal functor.

PROPOSITION 1.8.6. *If every functor $F : \mathcal{I} \rightarrow \mathcal{C}$ has a limit, then $\lim_{\mathcal{I}}$ defines a functor $\text{Fun}(\mathcal{I}, \mathcal{C}) \rightarrow \mathcal{C}$ which is right adjoint to the diagonal functor δ .*

Similarly if every functor $F : \mathcal{I} \rightarrow \mathcal{C}$ has a colimit, then $\text{colim}_{\mathcal{I}}$ defines a functor $\text{Fun}(\mathcal{I}, \mathcal{C}) \rightarrow \mathcal{C}$ which is left adjoint to the diagonal functor δ .

PROOF. That $\lim_{\mathcal{I}}$ defines a functor in F is a consequence of Yoneda's lemma, as we get unique maps between the representing objects. It now follows that we have an isomorphism $\text{Hom}_{\mathcal{C}}(\delta(X), F) \cong \text{Hom}_{\mathcal{C}}(X, \lim_{\mathcal{I}} F)$ functorial in both X and F which is the definition of δ being left adjoint to $\lim_{\mathcal{I}}$.

The proof for colim is dual. \square

EXERCISE 1.8.7. If $F : \mathcal{I} \rightarrow \mathcal{C}$ is a functor, then we also have a functor $F^{\text{op}} : \mathcal{I}^{\text{op}} \rightarrow \mathcal{C}^{\text{op}}$. Show that if an object $x \in \mathcal{C}$, together with a morphism $\phi : \delta x \rightarrow F$ in $\mathcal{C}^{\mathcal{I}}$, is a limit of F , then the “same” object x in \mathcal{C}^{op} , together with the morphism $\phi^{\text{op}} : F^{\text{op}} \rightarrow (\delta x)^{\text{op}}$ in $(\mathcal{C}^{\text{op}})^{\mathcal{I}^{\text{op}}}$, is a colimit of F^{op} . In few words: passing to opposite categories transforms limits into colimits. And vice-versa.

PROPOSITION 1.8.8 (Limits and colimits exist in Set). *For $\mathcal{C} = \text{Set}$ all limits and colimits exist and can be described as follows:*

Products and coproducts are given by cartesian product and disjoint union respectively. The equalizer of $X \rightrightarrows Y$ can be identified with the set $\{x \in X \mid f(x) = g(x)\}$ together with the inclusion into X . The coequalizer of $X \rightrightarrows Y$ is given by Y modulo the equivalence relation defined by $f(x) \cong g(x)$ for all $x \in X$, together with the map from Y .

More generally for a functor $F : \mathcal{I} \rightarrow \text{Set}$,

$$\lim_{\mathcal{I}} F \cong \{ \{x_i\}_{i \in \mathcal{I}} \in \prod_{i \in \mathcal{I}} F(i) \mid f(x_i) = x_j \text{ for all } f : i \rightarrow j \}$$

$$\text{colim}_{\mathcal{I}} F \cong \left(\coprod_{i \in \mathcal{I}} F(i) \right) / \sim \quad \text{where } x \sim F(f)(x) \text{ for all } f : i \rightarrow j$$

PROOF. It is clear that these sets satisfy the universal property. \square

Having described the limits and colimits in Set, we can now formulate the universal property of $\lim_{\mathcal{I}} F$ and $\text{colim}_{\mathcal{I}} F$ in general in terms of them.

PROPOSITION 1.8.9. *The limit of F exists if and only if $\lim_{i \in \mathcal{I}} \text{Hom}_{\mathcal{C}}(-, F(i)) \cong \text{Hom}_{\mathcal{C}^{\mathcal{I}}}(\delta(-), F)$ is a representable functor $\mathcal{C}^{\text{op}} \rightarrow \text{Set}$, and if so it is represented by $\lim_{\mathcal{I}} F$ i.e.,*

$$\text{Hom}_{\mathcal{C}}(X, \lim_{\mathcal{I}} F) \cong \lim_{i \in \mathcal{I}} \text{Hom}_{\mathcal{C}}(X, F(i))$$

natural in X .

Likewise the colimit of F exists if and only if $\lim_{i \in \mathcal{I}} \text{Hom}_{\mathcal{C}}(F(i), -) \cong \text{Hom}_{\mathcal{C}^{\mathcal{I}}}(F, \delta(-))$ is a representable functor $\mathcal{C} \rightarrow \text{Set}$, and if so it is represented by $\text{colim}_{\mathcal{I}} F$, i.e.,

$$\text{Hom}_{\mathcal{C}}(\text{colim}_{\mathcal{I}} F, Y) \cong \lim_{i \in \mathcal{I}} \text{Hom}_{\mathcal{C}}(F(i), Y)$$

natural in Y .

This also allows us to give a model for limits and colimits in general, in categories which have products and coequalizers.

PROPOSITION 1.8.10 (A model for colimit). *Let \mathcal{C} be a category with all coproducts and coequalizers. Then \mathcal{C} has all colimits, and for $F \in \mathcal{C}^{\mathcal{I}}$,*

$$\text{colim}_{\mathcal{I}} F \cong \text{coeq} \left(\coprod_{(f: i \rightarrow j) \in \text{Mor}(\mathcal{I})} F(i) \rightrightarrows \coprod_{k \in \text{Ob}(\mathcal{I})} F(k) \right).$$

where the two morphisms in the coequalizer are given on the factor indexed by $f: i \rightarrow j$ by $F(i) \xrightarrow{\text{id}} F(i) \rightarrow \coprod_k F(k)$ and $F(i) \xrightarrow{F(f)} F(j) \rightarrow \coprod_k F(k)$, respectively.

Dually, if \mathcal{C} has products and equalizers

$$\lim_{\mathcal{I}} F \cong \text{eq} \left(\prod_{k \in \text{Ob}(\mathcal{I})} F(k) \rightrightarrows \prod_{(f: i \rightarrow j) \in \text{Mor}(\mathcal{I})} F(j) \right).$$

where the two morphisms are given by $\prod_k F(k) \rightarrow F(j) \xrightarrow{\text{id}} F(j)$ and $\prod_k F(k) \rightarrow F(i) \xrightarrow{F(f)} F(j)$ on the factor corresponding to $f: i \rightarrow j$.

PROOF. First note that the statements are true when $\mathcal{C} = \text{Set}$, by the formula in Proposition 1.8.8. The general statement will follow from this. Let us focus on the limit statement, as the colimit statement is dual:

By our assumption we can form $Y = \text{eq} \left(\prod_{k \in \text{Ob}(\mathcal{I})} F(k) \rightrightarrows \prod_{(f: i \rightarrow j) \in \text{Mor}(\mathcal{I})} F(j) \right)$. Likewise, applying $\text{Hom}_{\mathcal{C}}(X, -)$ and using Proposition 1.8.9 "backwards" yields

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(X, Y) &\cong \text{eq} \left(\prod_{k \in \text{Ob}(\mathcal{I})} \text{Hom}_{\mathcal{C}}(X, F(k)) \rightrightarrows \prod_{(f: i \rightarrow j) \in \text{Mor}(\mathcal{I})} \text{Hom}_{\mathcal{C}}(X, F(j)) \right) \\ &\cong \lim_{i \in \mathcal{I}} \text{Hom}_{\mathcal{C}}(X, F(i)) \end{aligned}$$

where the last isomorphism is by the description of limits in sets, Proposition 1.8.8. As this is natural in X we conclude that $Y \cong \lim_{\mathcal{C}} F$ by Proposition 1.8.9. \square

We say a category \mathcal{C} is (co)complete if all (co)limits indexed by any small category \mathcal{I} exist. Many suitably big categories that you know such as Set , Top , Ab ,... are both complete and cocomplete. Other categories, such as many diagram categories, or categories arising as homotopy categories are not.

Functor categories where the target is complete or cocomplete inherit the same property, which explains the usefulness of considering functors to, say, Set :

PROPOSITION 1.8.11. *Given a functor $F: \mathcal{J} \rightarrow \mathcal{C}^{\mathcal{I}}$ such that the composite $F_i: \mathcal{J} \rightarrow \mathcal{C}^{\mathcal{I}} \xrightarrow{ev_i} \mathcal{C}$ has a limit for each $i \in \mathcal{I}$. Then F has a limit as well given by the "pointwise" functor $i \mapsto \lim_{\mathcal{J}} F(i)$.*

In particular if \mathcal{C} is (co)complete then so is $\mathcal{C}^{\mathcal{I}}$ and (co)limits are calculated "objectwise".

PROOF. Note that

$$\text{Fun}(\mathcal{J}, \mathcal{C}^{\mathcal{I}}) \cong \text{Fun}(\mathcal{J} \times \mathcal{I}, \mathcal{C}) \cong \text{Fun}(\mathcal{I}, \mathcal{C}^{\mathcal{J}})$$

by Proposition 1.7.3. Using this the claim can easily be verified from the universal property. \square

EXERCISE 1.8.12. Prove by hand that \mathbf{Cat} is complete and cocomplete, extending Proposition 1.7.1. (We will later get a slick proof of this using how \mathbf{Cat} sits inside the functor category of simplicial sets, which is again complete and cocomplete by Proposition 1.8.11.)

EXERCISE 1.8.13. Try to give five examples of where a limit does NOT exist. In particular, can you come up with examples of categories, which do have products, but not more complicated limits? [Hint: Try to work out what a limit would look like in certain derived categories or homotopy categories; if you get stuck, try to look at Exercise 1.12.16]

EXERCISE 1.8.14. Which colimits exist in $\mathbf{\Delta}$? Show that the pushout along a surjection exists, and describe the pushout. Give examples of pushouts that do not exist.

REMARK 1.8.15 (Problems with limits and colimits). The problem with colimits in the context of homotopy theory can already be seen from the definition, where we are making “hard” identifications. The role of homotopy colimits will be replacing these hard identifications with soft identifications.

More precisely, in topological spaces (or simplicial sets), taking limits or colimits is in general *not homotopy invariant*. E.g. $\operatorname{colim}(* \leftarrow S^{n-1} \rightarrow D^n) \cong S^n$ but $\operatorname{colim}(* \leftarrow S^n \rightarrow *) \cong *$. This diagram does in fact have a colimit in the homotopy category of topological spaces, namely the homotopy type of a point (verify this!). However, most of the time limits and colimits will not exist in the homotopy category of spaces. Exercise 1.12.16 provides an example of this.

A related problem occurs already in algebra: In abelian groups, limits and colimits do exist, but are in general *not exact functors*. E.g. the sequence $0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 0$ is exact, but taking fixed-points under the -1 action, gives only a left-exact sequence $0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/2$. This is why we need homological algebra.

1.9. When do functors commute with limits and colimits?

In the previous section we introduced limits and colimits of a functor $\kappa : \mathcal{I} \rightarrow \mathcal{C}$, and explained the colimit can be thought of as a “gluing” procedure, and the limit can be thought of as subjecting the values $\kappa(i)$, for i in the indexing category \mathcal{I} to certain compatibility conditions.

The next obvious question is when are limits and colimits upon applying a functor, i.e., when do $\lim_{\mathcal{I}}$ and $\operatorname{colim}_{\mathcal{I}}$ commute with $F : \mathcal{C} \rightarrow \mathcal{D}$? For a functor to preserve all colimits is essentially equivalent to F having right adjoint, and dually preserving limits is essentially equivalent to having a left adjoint. But some functors preserve some limits or colimits, but not all, and this again carries important information.

Let us first give a formal definition:

DEFINITION 1.9.1. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor, and $\kappa : \mathcal{I} \rightarrow \mathcal{C}$ a diagram such that the colimit $\operatorname{colim}_{i \in \mathcal{I}} \kappa(i)$ exists in \mathcal{C} .

We say F commutes with (or preserves) the colimit of κ if $F(\operatorname{colim}_{i \in \mathcal{I}} \kappa(i))$, together with the canonical structure maps $F(\kappa(j)) \rightarrow \operatorname{colim}_{i \in \mathcal{I}} F(\kappa(i))$ in \mathcal{D} , is a colimit of $F \circ \kappa$, i.e.,

$$\operatorname{colim}_{i \in \mathcal{I}} F(\kappa(i)) \xrightarrow{\cong} F(\operatorname{colim}_{i \in \mathcal{I}} \kappa(i))$$

We use the similar terminology when F commutes with colimits of all diagrams κ indexed over a certain class of indexing categories \mathcal{I} , e.g., “ F commutes with pushouts” or “ F commutes with all small colimits”. Dually for limits.

1.9.1. Adjoints and limits. It turns out that we have already encountered a functor that commutes with limits, namely the universal functor $\operatorname{Hom}_{\mathcal{C}}(x, -) : \mathcal{C} \rightarrow \mathbf{Set}$:

PROPOSITION 1.9.2. *The functor*

$$\operatorname{Hom}_{\mathcal{C}}(x, -) : \mathcal{C} \rightarrow \mathbf{Set}$$

commutes with limits and the contravariant functor $\text{Hom}_{\mathcal{C}}(-, y): \mathcal{C} \rightarrow \text{Set}$ sends colimits to limits (i.e. $\text{Hom}_{\mathcal{C}}(-, y): \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ commutes with limits).

PROOF. This is a restatement of Proposition 1.8.9. \square

The observation of Proposition 1.9.2 immediately implies that right adjoints preserve limits, and dually for left adjoints, which is the key to scary amount of results:

PROPOSITION 1.9.3. *Suppose that*

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{D}.$$

are adjoint functors with F left adjoint to G . Then F commutes with colimits and G commutes with limits.

PROOF. Let $\kappa: \mathcal{I} \rightarrow \mathcal{C}$ be a diagram admitting a colimit $\text{colim}_{i \in \mathcal{I}} \kappa(i)$. Then for all $y \in \mathcal{D}$ we have

$$\begin{aligned} \text{Hom}_{\mathcal{D}} \left(F(\text{colim}_{i \in \mathcal{I}} \kappa(i)), y \right) &\cong \text{Hom}_{\mathcal{C}} \left(\text{colim}_{i \in \mathcal{I}} (\kappa(i)), Gy \right) \cong \lim_{i \in \mathcal{I}} \text{Hom}_{\mathcal{C}} (\kappa(i), Gy) \\ &\cong \lim_{i \in \mathcal{I}} \text{Hom}_{\mathcal{D}} (F(\kappa(i)), y), \end{aligned}$$

using Proposition 1.9.2. This shows that $F(\text{colim}_{i \in \mathcal{I}} \kappa(i))$ has the universal property of $\text{colim}_{i \in \mathcal{I}} F(\kappa(i))$ (which hence in particular exists).

The lim statement is similar. \square

REMARK 1.9.4. Freyd's adjoint functor Theorem says that this is pretty close to being an if and only if, i.e., modulo a set theoretic condition, functors which commute with all colimits have a right adjoint (and the dual statement holds for functors commuting with limits). In fact, there are several different sufficient conditions which ensure that the converse holds, for instance the so-called solution set condition.

1.9.2. Preservation of limits over specific diagrams. Usually limits or colimits over \mathcal{I} in \mathcal{C} will not be preserved by all functors. It does happen in some very special cases though, known as *absolute (co)limits*, e.g., when one assumes that maps have enough splittings, analogous to that additive functors are exact on split exact sequences. These were first studied in [Par71].

We'll need one case in the next chapter on simplicial sets, where we take a push-out along surjections in the simplex category Δ to get a new simplex, and would like to know that it remains a pushout under the Yoneda embedding. (In fact, it turns out that if it is preserved by the Yoneda embedding, then it is preserved by all functors, as one can see using the universality of the Yoneda embedding which we treat in Section 1.11.)

PROPOSITION 1.9.5. *Suppose that we have a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow p & & \downarrow p' \\ A' & \xrightarrow{f'} & B' \end{array}$$

in a category \mathcal{C} such that there exists a section s of p and a section s' of p' which are mutually compatible in the sense that $f \circ s = s' \circ f'$. Then the square is a pushout square in \mathcal{C} .

It is furthermore an absolute pushout in the sense that if $F: \mathcal{C} \rightarrow \mathcal{D}$ is any functor, then the diagram remains a pushout in \mathcal{D} .

PROOF. The universality claim is clear, if we can prove that it is a pushout, as the assumptions are clearly preserved by any functor.

Now suppose that we are given $\phi : A' \rightarrow X$ and $\psi : B \rightarrow X$ such that $\phi \circ p = \psi \circ f$. We claim that $\xi = \psi \circ s' : B' \rightarrow X$ will be the universal morphism. The fact that it fits in the diagram follows as

$$\xi \circ f' = \psi \circ s' \circ f' = \psi \circ f \circ s = \phi \circ p \circ s = \phi.$$

Likewise it is the unique such map as p' is split surjective, i.e., if ξ and ξ' are two such maps then $\xi = \xi \circ p' \circ s' = \psi \circ s' = \xi' \circ p' \circ s' = \xi'$. \square

The following special case will be useful to later, as it

COROLLARY 1.9.6. *Let $p : [n] \rightarrow [n']$ be a surjection in Δ and $f : [n] \rightarrow [m]$ any map. Let $[m']$ be the totally ordered set constructed from $[m]$ identifying the interval $f(p^{-1}(i))$ to a single point, for all $i \in [n']$. Then we have an absolute pushout square in Δ*

$$\begin{array}{ccc} [n] & \xrightarrow{f} & [m] \\ \downarrow p & & \downarrow p' \\ [n'] & \xrightarrow{f'} & [m'] \end{array}$$

PROOF. We can construct compatible sections s and s' for p and p' by setting $s(i)$ to be the initial element in the interval $p^{-1}(i)$ and similarly for p' . Hence the result follows from Proposition 1.9.5. \square

COROLLARY 1.9.7. *With the notation above, and letting Δ^k denote the image of $[k]$ under the Yoneda embedding, we have the following pushout in $\text{Fun}(\Delta^{\text{op}}, \text{Set})$.*

$$\begin{array}{ccc} \Delta^n & \xrightarrow{f} & \Delta^m \\ \downarrow p & & \downarrow p' \\ \Delta^{n'} & \xrightarrow{f'} & \Delta^{m'} \end{array}$$

\square

In the next chapter we will introduce simplicial sets as $\text{sSet} = \text{Fun}(\Delta^{\text{op}}, \text{Set})$, and call Δ^n the simplicial n -simplex.

1.9.3. Preservation of limits over inclusions of full subcategories.

PROPOSITION 1.9.8. *Suppose $\iota : \mathcal{C}_0 \rightarrow \mathcal{C}$ is an inclusion of a full subcategory. If a diagram $F : \mathcal{I} \rightarrow \mathcal{C}_0$ has a limit or colimit in \mathcal{C} which lies in \mathcal{C}_0 , then it is also a limit or colimit in \mathcal{C}_0 .* \square

Simple examples, such as the inclusion of the abelian group \mathbb{Z} as a full subcategory of the category of all abelian groups, show that the assumption that the limit lies in \mathcal{C}_0 is not automatic.

Recall that we mentioned reflective and coreflective subcategories in Examples 1.4.11 and 1.4.12. One of the nice features of them is that, unlike arbitrary full subcategories, they are both automatically complete and cocomplete if the ambient category and limits and colimits are related in a nice way to those of the ambient category.

PROPOSITION 1.9.9. *Suppose $\iota : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ is the inclusion of a reflective subcategory, with L the left adjoint to ι , and consider a diagram $F : \mathcal{I} \rightarrow \mathcal{C}_0$.*

If ιF has a limit $\lim_{\mathcal{I}}(\iota F)$ in \mathcal{C} , then $\lim_{\mathcal{I}} F$ exists in \mathcal{C}_0 and $\iota(\lim_{\mathcal{I}} F) \cong \lim_{\mathcal{I}} \iota F$, i.e., “the limit is already in \mathcal{C}_0 ”.

If ιF has a colimit in \mathcal{C} , then F also has a colimit in \mathcal{C}_0 , given by $L(\text{colim}_{\mathcal{I}}(\iota F))$.

Dually if \mathcal{C}_0 is coreflective then colimits can be taken in \mathcal{C} , and limits are obtained by applying the right adjoint R to the limit in \mathcal{C} .

PROOF. This follows from the existence of a unit $1 \Rightarrow \iota L$ which is an isomorphism for each object of \mathcal{C}_0 , i.e. $\iota L \iota \cong \iota$:

Applying the unit to $\lim(\iota F)$ provides a map $\lim(\iota F) \rightarrow \iota L(\lim(\iota F))$ which gives a factorization

$$\delta(\lim(\iota F)) \rightarrow \delta(\iota L(\lim(\iota F))) \rightarrow \iota L \iota F \cong \iota F$$

But this says that $\iota L(\lim(\iota F))$ also has the universal property of the limit in \mathcal{C} , so $\lim(\iota F) \xrightarrow{\cong} \iota L(\lim(\iota F))$, by uniqueness of limits in \mathcal{C} , and $L(\lim(\iota F))$ is the limit in \mathcal{C}_0 by Proposition 1.9.8.

For colimits we likewise get maps $\text{colim}(\iota F) \rightarrow \iota L(\text{colim}(\iota F))$ which for any $Y \in \mathcal{C}_0$ provides universal maps

$$\begin{array}{ccccc} \iota F & \longrightarrow & \delta(\text{colim}(\iota F \iota)) & \longrightarrow & \delta \iota(Y) \\ & & \downarrow & & \downarrow \cong \\ & & \delta(\iota L(\text{colim}(\iota F))) & \longrightarrow & \delta \iota L \iota(Y) \end{array}$$

As ι is fully faithful, this shows that $L(\text{colim}(\iota F))$ satisfies the universal property of a colimit in \mathcal{C}_0 as wanted. \square

EXAMPLE 1.9.10. Here are some examples to keep in mind: The inclusion of abelian groups in all groups, where L is given by abelianization. Products agree, but the coproduct in abelian groups (=the cartesian product) is the abelianization of the coproduct in groups (=the free product). Another example is the inclusion of compact Hausdorff spaces in all Hausdorff spaces, here L is the Stone-Cech compactification, and coproducts in compact Hausdorff spaces are obtained by disjoint union, and then applying Stone-Cech compactification. We'll later see that the inclusion of the category of categories into simplicial sets is another example, giving a concrete description of colimits in the category of categories.

In the dual case of coreflective subcategories a good example to keep in mind is $k\text{Top} \hookrightarrow \text{Top}$ where products in $k\text{Top}$ are given by first taking product in Top and then retopologizing.

1.10. Kan extensions

Given a functor $\phi : \mathcal{I} \rightarrow \mathcal{J}$ we can in general consider the functor

$$\phi^* : \mathcal{C}^{\mathcal{J}} \rightarrow \mathcal{C}^{\mathcal{I}}$$

induced by precomposing with ϕ i.e., $F \mapsto F \circ \phi$. In the special case where \mathcal{J} is a point, this will be the diagonal functor, and when \mathcal{I} is a subcategory of \mathcal{J} is a subcategory it will be the restriction functor. We can ask when this functor has right and left adjoints.

In the case where \mathcal{J} is a point we've already introduced a name for the left and right adjoints of ϕ^* namely colimits and limits. And in the case of an inclusion the adjoints sometimes go under the name of coinduction and induction⁹. In general the adjoints, when they exist, are called left and right Kan extension of F along ϕ , and denoted $\phi_!$ or ϕ_* , and are extremely useful¹⁰.

As with colimits and limits we can also study these concepts for a single functor F , which leads to the following definition.

DEFINITION 1.10.1 (Kan extensions). Consider the diagram

⁹Beware that parts of representation theory confusingly switch the naming convention for what is coinduction and what is induction

¹⁰In the words of Mac Lane, "all concepts are Kan extensions" [Mac71, Ch. X]

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{F} & \mathcal{C} \\ \downarrow \phi & & \\ \mathcal{J} & & \end{array}$$

The *right Kan extension* of F along ϕ is a functor $\phi^*F = \text{Ran}_\phi F: \mathcal{J} \rightarrow \mathcal{C}$ together with a natural transformation $\phi^*F \circ \phi \Rightarrow F$, that is terminal amongst such pairs of a functor and a natural transformation, i.e., for any $G: \mathcal{J} \rightarrow \mathcal{C}$, with a natural transformation $G \circ \phi \Rightarrow F$ there exists a unique natural transformation $G \Rightarrow \phi^*F$ making the diagram of natural transformations to F commute

$$\begin{array}{ccc} G \circ \phi & \xRightarrow{\quad} & \phi^*F \circ \phi \\ & \searrow \quad \swarrow & \\ & F & \end{array}$$

Dually the *left Kan extension* of F along ϕ , denoted either $\phi_!F$ or $\text{Lan}_\phi F$ is a functor $\phi_!F = \text{Lan}_\phi F: \mathcal{J} \rightarrow \mathcal{C}$ together with a natural transformation $F \Rightarrow \phi_!F \circ \phi$, that is initial amongst such functors, i.e., for any $G: \mathcal{J} \rightarrow \mathcal{C}$, with a natural transformation $F \Rightarrow G \circ \phi$ there exists a unique natural transformation $\phi_!F \Rightarrow G$ making the diagram of natural transformations to F commute.

Let us also record the adjoint functor viewpoint from earlier in a proposition.

PROPOSITION 1.10.2. *Let \mathcal{I} and \mathcal{J} be small categories. Assume that, for a fixed ϕ , the right Kan extension exists for all F in $\mathcal{C}^\mathcal{I}$.*

Then the functor $\phi_ = \text{Ran}_\phi: \mathcal{C}^\mathcal{I} \rightarrow \mathcal{C}^\mathcal{J}$ is right adjoint to the “restriction along ϕ ” functor $\phi^* = \text{Res}_\phi: \mathcal{C}^\mathcal{J} \rightarrow \mathcal{C}^\mathcal{I}$ given by precomposing with ϕ , i.e.,*

$$\text{Hom}_{\mathcal{C}^\mathcal{I}}(\phi^*G, F) \cong \text{Hom}_{\mathcal{C}^\mathcal{J}}(G, \phi_*F)$$

*naturally in G and F , sending $G \Rightarrow \phi_*F$ to $\phi^*G \Rightarrow \phi^*\phi_*F \Rightarrow F$.*

Dually, if the left Kan extension exists for all F in $\mathcal{C}^\mathcal{I}$, then $\phi_! = \text{Lan}_\phi$ will be left adjoint to ϕ^ . \square*

The proof amounts to writing down the definitions, and is left as an exercise.

1.10.1. A pointwise formula for Kan extensions and over and under categories.

In the important special case where \mathcal{C} is furthermore complete and cocomplete, the Kan extensions exist as well, and their calculation reduces to that of limits and colimits. In order to introduce this we need a couple of natural

Given a functor between two categories there a couple of natural categories constructed from $\phi: \mathcal{I} \rightarrow \mathcal{J}$ that one can examine, reminiscent of fibers and cofibers in homotopy theory.

DEFINITION 1.10.3 (Over and under categories). Given a functor $\phi: \mathcal{I} \rightarrow \mathcal{J}$ we can form, for each object $j \in \mathcal{J}$, the *overcategory* ϕ/j , as the category with objects pairs $(i, \phi(i) \rightarrow j) \in \text{Ob}(\mathcal{I}) \times \text{Mor}(\mathcal{J})$ and morphisms $(i, \phi(i) \rightarrow j) \rightarrow (i', \phi(i') \rightarrow j)$ given by morphisms $g: i \rightarrow i'$ such that the following triangle commutes in \mathcal{J}

$$\begin{array}{ccc} \phi(i) & \xrightarrow{\phi(g)} & \phi(i') \\ & \searrow \quad \swarrow & \\ & j & \end{array}$$

We also refer to this as the *slice category*, and also denoted $\mathcal{I}/_j$, suppressing ϕ from the notation.

Similarly, we define the *undercategory* (or *coslice category*) j/ϕ (or $\mathcal{I}_{j/}$) as the category with objects pairs $(i, j \rightarrow \phi(i)) \in \text{Ob}(\mathcal{I}) \times \text{Mor}(\mathcal{J})$ and morphisms given by morphisms $g: i \rightarrow i'$ such that the dual diagram commutes.¹¹

A special case is when ϕ is the identity functor, where we get over and under categories in a more classical sense, which we then write \mathcal{I}/i and i/\mathcal{I} , for $i \in \mathcal{I}$.

EXERCISE 1.10.4. Prove that the undercategory under ϕ can be described as a pull-back in Cat of an undercategory in \mathcal{J} :

$$j/\phi = \mathcal{I}_{j/} \cong \mathcal{I} \times_{\mathcal{J}} \mathcal{J}_{j/}$$

And similarly for overcategories.

EXAMPLE 1.10.5. If $\phi: \mathcal{I} \rightarrow \mathcal{J}$ is the inclusion of a full subcategory and j lies in the essential image of ϕ , then j/ϕ has an initial object given an isomorphism $j \rightarrow \phi(i)$ for some $i \in \mathcal{I}$. Dually for overcategories.

EXAMPLE 1.10.6. Suppose ϕ is an inclusion of groups $H < G$ viewed as categories with one element. Then j/ϕ will have objects the elements of G and a unique morphism from g to g' if there exists h such that $gh = g'$, a category equivalent to the discrete category G/H .

We can now give the pointwise formula for Kan extensions.

PROPOSITION 1.10.7. Let $\phi: \mathcal{I} \rightarrow \mathcal{J}$ be a functor between small categories, and assume that \mathcal{C} is complete. Then $\phi_*: \mathcal{C}^{\mathcal{I}} \rightarrow \mathcal{C}^{\mathcal{J}}$ exists, and is calculated “pointwise” via the formula:

$$\phi_* F(j) = \lim_{(j \rightarrow \phi(i)) \in \mathcal{I}_{j/}} F(i)$$

Similarly, if \mathcal{C} is cocomplete then $\phi_!: \mathcal{C}^{\mathcal{I}} \rightarrow \mathcal{C}^{\mathcal{J}}$ exists and is given by

$$\phi_! F(j) = \text{colim}_{(\phi(i) \rightarrow j) \in \mathcal{I}_{/j}} F(i).$$

PROOF. Exercise. □

While the above formula is useful in many contexts, and generalize all the usual induction constructions from algebra (see the exercises), there are also many cases where we want to study Kan extensions but the assumptions do *not* apply, e.g., where \mathcal{I} and \mathcal{J} are large, and \mathcal{C} is a homotopy category, hence generally without limits and colimits. Indeed this is the setup we have when studying derived functors, where \mathcal{J} will be the homotopy category of \mathcal{I} , and \mathcal{C} some other homotopy category, and we want to study the “best approximation” of F by a homotopy invariant functor—we return to this formally in Chapter 4.

1.11. The density theorem and the universal property of the Yoneda embedding

In the latter parts of the course, many of our categories of interest will occur as functor categories to sets. With the notion of limits and colimits under our belt, we can state and prove the density theorem, which says that any functor in a functor category to sets can be written as a colimit of representable functors. This reduces checking statements for all functors to checking it on representables, and seeing that class of functors for which the statement is true is closed under colimits.

In fact we will see that the Yoneda embedding $\mathcal{I} \rightarrow \text{Fun}(\mathcal{I}^{\text{op}}, \text{Set})$ can be characterized as freely adding colimits to \mathcal{I} . This will motivate our definition of simplicial sets in the next

¹¹Someone should come up with a good notation for slice and coslice categories: $\phi \downarrow j$ is potentially confusing as in practice j may itself be an object in a functor category, and may e.g. already be denoted F : then one would write $\phi \downarrow F$, losing the apparent asymmetry; $\mathcal{I}_{/j}$ can also create confusion, as it drops ϕ from the notation, and $\phi_{/j}$ is also confusing as it is not apparent that the notation refers to a category rather than say to a functor (the same issue occurs for $\phi \downarrow j$); $\phi_{/j}$ is anyway not widely used. Maybe $\mathcal{I}_{\phi/j}$ for the overcategory and $\mathcal{I}_{\phi \setminus j}$ or $\mathcal{I}_{j/\phi}$ for the undercategory?

chapter, which, in light of this, can be viewed as the category obtained by freely allowing all gluings of simplices along face and degeneracy maps.

DEFINITION 1.11.1 (Category of elements). For a given functor $F \in \text{Set}^{\mathcal{I}}$ consider the category *category of elements* in F , denoted $\text{el}(F)$ or $\int^{\mathcal{I}} F$, with objects the pairs $(i, u_i \in F(i))$, and morphisms from $(i, u_i \in F(i))$ to $(i', u_{i'} \in F(i'))$ given by those $(i \rightarrow i') \in \mathcal{I}$ such that $F(\tau)(u_i) = u_{i'}$.

We will encounter a generalization of this category later in the so-called Grothendieck construction, where Set is replaced by Cat .

We can describe $\text{el}(F)$ as $(F_{\setminus *})^{\text{op}}$, the opposite undercategory of $* \in \text{Set}$ with respect to the functor $F: \mathcal{I} \rightarrow \text{Set}$. Via the Yoneda Lemma 1.3.1 it can also be described as follows: The objects of $\text{el}(F)$ identify with a representable functor $\text{Hom}(i, -)$ together with a natural transformation to F . More precisely, let $\kappa: \mathcal{I}^{\text{op}} \rightarrow \text{Set}^{\mathcal{I}}$ be given by $i \mapsto \text{Hom}_{\mathcal{I}}(i, -)$ then the category $\text{el}(F)$ identifies with the opposite overcategory $((\mathcal{I}^{\text{op}})_{/\kappa})^{\text{op}}$. The objects are natural transformations $\phi: \text{Hom}(i, -) \rightarrow F(-)$ and the morphisms are morphisms $(\tau: i \rightarrow i') \in \mathcal{I}$ such that the following triangle commutes in $\text{Set}^{\mathcal{I}}$

$$\begin{array}{ccc} \text{Hom}(i, -) & & \\ \uparrow \tau^* & \searrow \phi & \\ & & F \\ & \nearrow \phi' & \\ \text{Hom}(i', -) & & \end{array}$$

EXAMPLE 1.11.2. If $F = \text{Hom}_{\mathcal{I}}(j, -)$ then $\text{el}(F)$ consists of pairs $(i, f_i: j \rightarrow i)$. In particular in this case $\text{el}(F)$ has an initial object namely (j, id_j) .

In general, by definition, $\text{el}(F)$ parametrizes approximations of F from the left by representable functors, and the next result, the density theorem, says that the colimit over these recovers the functor F , much like any module can be written as a cokernel of free modules. It will allow us to reduce certain verifications for simplicial sets (like proving that geometric realization preserves products) to a small family of cases, e.g. checking only for an n -simplex for all n .

PROPOSITION 1.11.3 (Density theorem). *Let \mathcal{I} be a small category and $F: \mathcal{I} \rightarrow \text{Set}$ a functor. Then there is a natural isomorphism in $\text{Set}^{\mathcal{I}}$*

$$\text{colim}_{(i, u_i) \in \text{el}(F)} \text{Hom}_{\mathcal{I}}(i, -) \xrightarrow{\cong} F$$

where $\text{el}(F)$ is the category of elements, built from pairs $(i, u_i \in F(i))$. The isomorphism is induced by the natural transformations $\text{Hom}_{\mathcal{I}}(i, -) \rightarrow F$, for each object $(i, u_i \in F(i))$ in $\text{el}(F)$, sending the element $(\phi: i \rightarrow j) \in \text{Hom}_{\mathcal{I}}(i, j)$ to $F(\phi)(u_i) = u_j \in F(j)$.

PROOF. By the Yoneda embedding it is enough to see that we have a natural isomorphism after applying $\text{Hom}_{\text{Set}^{\mathcal{I}}}(-, G)$, for all $G \in \text{Set}^{\mathcal{I}}$. Indeed we have

$$\begin{aligned} \text{Hom}_{\text{Set}^{\mathcal{I}}} \left(\text{colim}_{(i, u_i) \in \text{el}(F)} \text{Hom}_{\mathcal{I}}(i, -), G(-) \right) &\cong \lim_{(i, u_i) \in \text{el}(F)} \text{Hom}_{\text{Set}^{\mathcal{I}}}(\text{Hom}_{\mathcal{I}}(i, -), G(-)) \\ &\cong \lim_{(i, u_i) \in \text{el}(F)} G(i) \cong \text{Hom}_{\text{Set}^{\mathcal{I}}}(F, G) \end{aligned}$$

where the last isomorphism amounts to unravelling the definitions: Giving a natural transformation between F and G amounts to, for each i and each element $u_i \in F(i)$, specifying an element $\phi_i(u_i) \in G(i)$, such that $\phi_j(F(f)(u_i)) = G(f)(\phi_i(u_i))$ for all $f: i \rightarrow j$. But this is the same as specifying an element in $\lim_{(i, u_i) \in \text{el}(F)} G(i) \subseteq \prod_i \text{Hom}(F(i), G(i))$. \square

REMARK 1.11.4. The proposition is often applied to an index category “ \mathcal{I} ” of the form \mathcal{I}^{op} , i.e. in the context of presheaves. We leave it to the reader to explicitly state this dual version.

The density theorem can be used to prove the following universal property of the Yoneda embedding, saying that a functor has “a unique colimit preserving (aka cocontinuous) extension to presheaves on \mathcal{I} ”:

PROPOSITION 1.11.5 ($\text{Fun}(\mathcal{I}^{\text{op}}, \text{Set})$ as the free cocompletion of \mathcal{I}). **[check that the op’s are correct]** Let \mathcal{I} be a small category and $F: \mathcal{I} \rightarrow \mathcal{D}$ a functor. Consider the Yoneda embedding $\mathbf{y}: \mathcal{I} \rightarrow \text{Fun}(\mathcal{I}^{\text{op}}, \text{Set})$.

If \mathcal{D} has all colimits, then there exists a unique (up to natural isomorphism) colimit preserving functor

$$\bar{F}: \text{Fun}(\mathcal{I}^{\text{op}}, \text{Set}) \rightarrow \mathcal{D},$$

called the Yoneda extension, making the diagram

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{\mathbf{y}} & \text{Fun}(\mathcal{I}^{\text{op}}, \text{Set}) \\ \downarrow F & \searrow \bar{F} & \\ \mathcal{D} & \xrightarrow{\quad} & \end{array}$$

commute, i.e., $\bar{F}(\text{Hom}(-, i)) = F(i)$.

Indeed, restriction induces an equivalence of categories

$$\text{Fun}^{\text{cocont}}(\text{Fun}(\mathcal{I}^{\text{op}}, \text{Set}), \mathcal{D}) \xrightarrow{\cong} \text{Fun}(\mathcal{I}, \mathcal{D})$$

Furthermore $\bar{F}: \text{Fun}(\mathcal{I}^{\text{op}}, \text{sSet}) \rightarrow \mathcal{C}$ has a right adjoint $G: \mathcal{C} \rightarrow \text{Fun}(\mathcal{I}^{\text{op}}, \text{sSet})$ which takes $c \in \mathcal{C}$ to the functor $\mathcal{I}^{\text{op}} \rightarrow \text{Set}$ given by

$$G(c)(i) = \text{Hom}_{\mathcal{C}}(F(i), c).$$

PROOF. By Proposition 1.11.3, any functor $\Phi \in \text{Fun}(\mathcal{I}^{\text{op}}, \text{Set})$ can be written as a colimit of representable functors.

$$\text{colim}_{(i, u_i) \in \text{el}(\Phi)} \text{Hom}_{\mathcal{I}}(-, i) \xrightarrow{\cong} \Phi$$

and by definition $\bar{F}(\text{Hom}_{\mathcal{I}}(-, i)) = F(i)$ Hence we must define

$$\bar{F}(\Phi) = \text{colim}_{(i, u_i) \in \text{el}(\Phi)} F(i)$$

if \bar{F} is to be colimit preserving. This gives a well-defined canonical extension of F , which at least preserves colimits of a certain form. If a colimit preserving extension exists it will furthermore be the unique such extension.

We will not check now that \bar{F} in fact preserves all colimits, but just instead prove that \bar{F} has a right adjoint G , which in particular means that it preserves all colimits by Proposition 1.9.3.

Let us indicate how the unit and counit of the adjunction are constructed, and leave you to fill the details for why these are natural transformations that gives an adjunction. Using the formula for \bar{F} , we see that

$$\bar{F}G(c) = \text{colim}_{(i, \alpha_i) \in \text{el}(G(c))} F(i) = \text{colim}_{i \in \mathcal{I}, \alpha_i: F(i) \rightarrow c} F(i).$$

This has a canonical map to c , namely map which on the copy of $F(i)$ indexed by (i, α_i) associates $\alpha_i: F(i) \rightarrow c$ itself. This gives to counit.

For the unit, we need a natural map $X \rightarrow G\bar{F}(X)$ for every functor $X: \mathcal{I}^{\text{op}} \rightarrow \mathcal{C}$. Now, we have

$$G\bar{F}(X)(j) \simeq G(\text{colim}_{(i, u_i) \in \text{el}(X)} F(i))(j) = \text{Hom}_{\mathcal{C}}(F(j), \text{colim}_{(i, u_i) \in \text{el}(X)} F(i)).$$

That is, the value of $G\bar{F}(X)$ on $j \in \mathcal{I}$ is collection of maps from $F(j)$ to the colimit of the objects $F(i)$, indexed by the diagram of pairs $i \in \mathcal{I}$, $u_i \in X(i)$. Thus, a map $X \rightarrow G\bar{F}(X)$ amounts

to a rule that to every $j \in \mathcal{I}$ and every $v_j \in X(j)$, gives a map $F(j) \rightarrow \operatorname{colim}_{(i,u_i) \in \operatorname{el}(X)} F(i)$. But we have a canonical such map which to j and v_j associates the structure map $F(j) \rightarrow \operatorname{colim}_{(i,u_i) \in \operatorname{el}(X)} F(i)$ determined by (j, v_j) , as a colimit over a diagram by definition receives compatible maps from all the values of the functor. This gives the unit transformation of the adjunction. \square

[Wishlist to be added in preparation for infinity category chapter: Grothendieck construction, Localization of small categories]

1.12. Exercises

Homological algebra background exercises:

EXERCISE 1.12.1. Consider a functor $F : \mathcal{I} \rightarrow \operatorname{Vect}_k$, for some field k , where $\mathcal{I} = (\bullet \rightarrow \bullet \leftarrow \bullet)$ is the pull-back category. What are the projective objects in this category? Calculate the (right) derived functors of the (left exact) functor $\lim : \operatorname{Vect}_k^{\mathcal{I}} \rightarrow \operatorname{Vect}$.

EXERCISE 1.12.2. Do Exercise 1.12.1 with $\mathcal{I} = \mathcal{BG}$ instead.

EXERCISE 1.12.3. Let $\mathcal{I} = \mathcal{BC}_n$, the cyclic group of order n , and let M be a $\mathbb{Z}\mathcal{C}_n$ -module with trivial action. Calculate $\operatorname{colim}_{\mathcal{I}}^* M$ and $\lim_{\mathcal{I}}^* M$ as abelian groups. Now let $M = \mathbb{F}_p$ and do the same calculation as rings.

[Hint: This being doable probably requires knowledge of group cohomology. Look up in a book on group cohomology if you get stuck. At the very least, you should know the answer.]

EXERCISE 1.12.4 (Group cohomology). Let $\mathcal{I} = \mathcal{B}(\mathbb{Z}/p)^r$, calculate $\lim_{\mathcal{I}}^* \mathbb{F}_p$ as a ring.

[Hint: Did you ever calculate the \mathbb{F}_2 -cohomology of \mathbf{RP}^∞ ? If not, now is a good time. Look up in a book on group cohomology if you get stuck. Again, at the very least, you should know the answer.]

EXERCISE 1.12.5 (Group cohomology). Let $\mathcal{I} = \mathcal{BD}_8 = \mathcal{B}(\mathbb{Z}/4 \rtimes \mathbb{Z}/2)$, calculate $\lim_{\mathcal{I}}^* \mathbb{F}_p$ as a ring.

[Hint: You'll probably need to know the Lyndon-Hochschild-Serre spectral sequence. Try applying it to the central extension $0 \rightarrow \mathbb{Z}/2 \rightarrow D_8 \rightarrow (\mathbb{Z}/2)^2 \rightarrow 0$, and look at the extensions arising by pulling back to the 3 subgroups of order 2 in $\mathbb{Z}/2 \times \mathbb{Z}/2$. Look up in a book on group cohomology if you get stuck.]

Category exercises:

Recall the examples in Example 1.1.4.

EXERCISE 1.12.6 (Structure of morphisms in Δ). Consider the following distinguished morphisms in the simplicial category Δ :

- (1) For all $n \geq 0$ and $0 \leq i \leq n+1$, the i^{th} face map (at n), denoted $d^i : [n] \rightarrow [n+1]$, is the unique order preserving injection whose image does not contain i .
- (2) For all $n \geq 1$ and $0 \leq i \leq n-1$, the i^{th} degeneracy map (at n), denoted $s^i : [n] \rightarrow [n-1]$, is the unique order preserving surjection that sends i and $i+1$ to i .

(Here we are slightly abusing the notation by ignoring n in the notations d^i and s^i . This is more convenient in computations.)

Show that any morphism $\varphi : [n] \rightarrow [m] \in \operatorname{Mor}(\Delta)$ factors uniquely as follows:

$$\varphi = d^{i_1} d^{i_2} \dots d^{i_l} s^{j_1} s^{j_2} \dots s^{j_t},$$

where $0 \leq i_l < \dots < i_1 \leq m$ and $0 \leq j_1 < \dots < j_t < n$.

EXERCISE 1.12.7 (Structure of the orbit category $\mathcal{O}_{\mathcal{F}}(G)$).

- (1) Find the automorphisms of an object G/H in $\mathcal{O}_{\mathcal{F}}(G)$
- (2) Make a diagram of the orbit category $\mathcal{O}_{\mathcal{F}}(\mathfrak{S}_4)$, where \mathcal{F} is the collection of 2-subgroups the symmetric group on 4 letters.

EXERCISE 1.12.8 (Coverings and $\mathcal{O}(G)$). Strengthen the classification of covering spaces as given in say Hatcher, to prove an equivalence of categories between the category of connected coverings of X (with the usual hypotheses) and the orbit category of the fundamental group.

EXERCISE 1.12.9. Let $G = \mathbb{Z}/2$ and k a field of characteristic 2. What are the simple and projective objects in $\text{Vect}^{\mathcal{O}(G)^{\text{op}}}$? Do the same exercise with $G = \mathfrak{S}_3$, and the category $\mathcal{O}_{\mathcal{F}}(G)$, where \mathcal{F} is the collection of 2-subgroups of G .

Adjoint exercises:

EXERCISE 1.12.10. Show that the inclusion functor $\text{Set} \rightarrow \text{Cat}$ has a left adjoint. Describe it explicitly. (We call this functor π_0 , the set of components of the category; if $\pi_0(\mathcal{C})$ is a point, we call \mathcal{C} *connected*.)

EXERCISE 1.12.11. Show that the inclusion functor $\text{Grpd} \rightarrow \text{Cat}$, from groupoids to categories has a left adjoint, and describe it explicitly. (We call this functor π , the fundamental groupoid of \mathcal{C} .)

Limits exercises:

EXERCISE 1.12.12. Make the formula in Proposition 1.8.10 explicit in the categories Set , Set_* , Ab , Grp , and Ring .

EXERCISE 1.12.13. Does the category of compact Hausdorff spaces have arbitrary colimits? What are they? Is this a special case of a more general result? [To formulate this you may need a definition: A full subcategory is called *reflective* if the inclusion has a left adjoint, and *coreflective* if it has a right adjoint.

EXERCISE 1.12.14 (Coequalizers as initial objects). Let \mathcal{C} be a category.

The first part of the exercise gives a low-tech approach to initial objects (compare Example 1.8.3):

- (1) An object $c_0 \in \mathcal{C}$ is called an *initial* (or sometimes *universal*) object of \mathcal{C} if for any object $c \in \mathcal{C}$, there exists a unique morphism $c_0 \rightarrow c$ in \mathcal{C} . Show that if c_0 and c_1 are initial objects of the category \mathcal{C} , then there is a unique isomorphism between them (i.e., initial objects are unique up to unique isomorphism).
- (2) Give an example of a category which does not have an initial object.
- (3) What are the initial objects of the categories Set , Ab , Ring , Top ?

Next we want to view coequalizers as initial objects in a certain category:

- (4) Let $f: A \rightarrow B$ be a homomorphism of abelian groups. Consider the following category denoted by \mathcal{C}_f :

Objects: Pairs (C, g) , where C is an abelian group, $g: B \rightarrow C$ is a homomorphism and $gf = 0$;

Morphisms: A morphism from (C, g) to (C', g') is a homomorphism $h: C \rightarrow C'$ such that the following diagram commutes

$$\begin{array}{ccc} B & \xrightarrow{g} & C \\ \downarrow g' & \swarrow h & \\ C' & & \end{array}$$

What is the initial object of \mathcal{C}_f ?

- (5) Using the ideas from the previous paragraph show that coequalizers are initial objects of certain categories and hence unique up to unique isomorphism.

EXERCISE 1.12.15 (Tensor products as initial objects). Let R be an associative ring, M a right R -module and N a left R -module. Describe the *tensor product* $M \otimes_R N$ as an initial object of some category.

Let R be a commutative ring and let A , B and C be R -modules. Show that there is a natural isomorphism of R -modules

$$\mathrm{Hom}_R(A \otimes_R B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C)).$$

EXERCISE 1.12.16 (No colimits in the homotopy category.). Let \mathcal{H} denote the category of topological spaces and homotopy classes of maps between them (all are pointed). We will show that colimits do not always exist in \mathcal{H} .

Consider the following diagram in \mathcal{H} , where $2: S^n \rightarrow S^n$ is any map of degree 2:

$$\begin{array}{ccc} S^n & \xrightarrow{2} & S^n \\ \downarrow & & \\ * & & \end{array}$$

Suppose that this diagram has a colimit, and denote the colimit by T .

- (1) Show that $\mathrm{Mor}_{\mathcal{H}}(T, X) \cong \{a \in \pi_n(X) \mid 2a = 0\}$ for any pointed space X .
- (2) Describe a fibration sequence $A \rightarrow B \rightarrow C$ whose long exact homotopy sequence

$$\cdots \pi_{n+1}(C) \rightarrow \pi_n(A) \rightarrow \pi_n(B) \rightarrow \pi_n(C) \rightarrow \pi_{n-1}(A) \rightarrow \cdots$$

is equal to $\cdots 0 \rightarrow \mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2 \rightarrow 0 \rightarrow \cdots$.

- (3) It follows that $\mathrm{map}(T, A) \rightarrow \mathrm{map}(T, B) \rightarrow \mathrm{map}(T, C)$ is a fibration sequence. What are the homotopy groups of the three spaces?
- (4) Derive a contradiction.

EXERCISE 1.12.17 (Cofinality). Consider a functor $F: \mathcal{J} \rightarrow \mathcal{C}$, with \mathcal{C} cocomplete. Suppose that $\phi: \mathcal{I} \rightarrow \mathcal{J}$ is an inclusion of small categories which satisfies that $\mathcal{I}_{j/}$ is connected (in the sense of Exercise 1.12.10) for every $j \in \mathcal{J}$.

Prove that the natural morphism

$$\mathrm{colim}_{\mathcal{I}}(F \circ \phi) \xrightarrow{\sim} \mathrm{colim}_{\mathcal{J}} F$$

is an isomorphism in \mathcal{C} .

[Functors ϕ with this property are called *cofinal*; we will later see the homotopy invariant strengthening of this property. There is also dual notion call *final* (find this by passing to op !)]

[Note that when taking $\mathcal{J} = \mathbb{N}$, this notion agrees with the notion of a “cofinal sequence”, as you learned in your first-year calculus class....]

EXERCISE 1.12.18 (Cofinality of a poset of subgroups...). This exercise works out a non-trivial example of a cofinal subcategory, in continuation of Exercise 1.12.17. Let \mathcal{C} be a subposet of the poset of non-trivial p -subgroups $\mathcal{S}_p(G)$ in a finite group G , and assume that \mathcal{C} is closed under passage to p -supergroups.

Prove that the inclusion $\mathcal{C} \hookrightarrow \mathcal{S}_p(G)$ is cofinal iff \mathcal{C} contains all non-trivial p -subgroups P such that $\mathcal{S}_p(N_G(P)/P)$ is disconnected or empty.

[Note that $\mathcal{S}_p(N_G(P)/P)$ is empty iff P is a Sylow p -subgroup in G .]

[Bonus: Can you find a group theoretic criterion for when $\mathcal{S}_p(G)$ is disconnected?]

Kan extension exercises:

EXERCISE 1.12.19. Consider an inclusion $H < G$ of groups, viewed as a functor $\mathcal{B}H \rightarrow \mathcal{B}G$. If \mathcal{C} is the category of abelian groups, write down a formula for the left and right Kan extension of functors $\mathcal{B}H \rightarrow \mathcal{C}$.

[Bonus question: Do you know another, old-school, name for these? ;)]

PS0.

EXERCISE 1.12.20 (Posets as categories). Let (P, \leq) be a preordered set. Define a category \underline{P} whose objects are the elements of P , and whose morphisms are defined by $\text{hom}(x, y) = \{(x, y)\}$ if $x \leq y$, \emptyset otherwise.

- (a) What is composition in this category ?
- (b) Show that a category is equivalent to \underline{P} for some preordered set P if and only if it is isomorphic to one. Describe a necessary and sufficient condition for a category to be a preorder.
- (c) Do you know another name for a skeletal preordered set ? Describe two distinct constructions of an equivalent skeletal category for a poset.
- (d) Show that a functor $\underline{P} \rightarrow \underline{Q}$ is the same thing as nondecreasing map $P \rightarrow Q$.
- (e) When does \underline{P} have limits, colimits ? (if you are not familiar with co/limits yet, feel free to skip this question)
- (f) Describe in more elementary terms what an adjunction between \underline{P} and \underline{Q} is.

We will probably abuse notation and write P for both the preordered set and the category.

EXERCISE 1.12.21 (Sets as categories). Let X, Y be two sets, viewed as *discrete* categories, that is, their objects are their elements and the only morphisms are identity morphisms.

- (a) What is a functor $X \rightarrow Y$?
- (b) What is an adjunction between X and Y ?

EXERCISE 1.12.22 (Adjunctions in posets). (a) Let L/K be a field extension, and let $G_{L/K}$ be the group of automorphisms of L leaving K fixed (pointwise). Show that the following is an adjunction between the poset of subextensions and the opposite of the poset of subgroups of G : $E \mapsto G_{L/E}$, $H \mapsto L^H$ (remember that L^H is the set of fixed points of L under H)

(b) Consider \mathbb{R} and \mathbb{Z} as posets, and let $i : \mathbb{Z} \rightarrow \mathbb{R}$ denote the inclusion. Show that i has both a left and a right adjoint, and describe them.

EXERCISE 1.12.23 (Fully faithfulness). Recall that a functor $F : C \rightarrow D$ is full (resp. faithful, resp. fully faithful) if the induced map $\text{hom}_C(x, y) \rightarrow \text{hom}_D(F(x), F(y))$ is surjective (resp. injective, resp. bijective) for all x, y .

- (a) Give a lot of examples of functors that are (resp. are not) full (resp. faithful, fully faithful).
- (b) Suppose $F : C \rightarrow D$ is a functor. Show how to define a functor $F_* : \text{Fun}(I, C) \rightarrow \text{Fun}(I, D)$ by postcomposition with F , for any I . If F is full (resp. faithful, resp. fully faithful), is F_* full (resp. faithful, resp. fully faithful) ?
- (c) Suppose F is fully faithful. Show that F is *conservative*: if $F(f)$ is an isomorphism, then f is an isomorphism. In fact F creates isomorphisms : if $F(x) \cong F(y)$, then $x \cong y$. Show that this can fail if one only assumes that F is full or faithful. Give examples of conservative functors that are not fully faithful; and of conservative functors that do not create isomorphisms.
- (d) When is a functor between discrete categories full ? faithful ?

EXERCISE 1.12.24 (Groupoids). A category is called a groupoid if every morphism is an isomorphism.

For G a group, let \mathbf{BG} denote the groupoid with only one object whose endomorphism group is G .

- (a) Let \mathcal{G} be a groupoid where any two objects are isomorphic. Show that \mathcal{G} is either empty or equivalent to \mathbf{BG} for some group G .
- (b) Give an example of a groupoid that is not empty or equivalent to \mathbf{BG} for any G .
- (c) Let G, H be two groups. Show that a functor $\mathbf{BG} \rightarrow \mathbf{BH}$ is the same thing as a group morphism $G \rightarrow H$. What is a natural transformation between two such functors in group-theoretic terms ?
- (d) Let \mathcal{G} be a groupoid. Show that \mathcal{G} is equivalent (in fact, isomorphic) to \mathcal{G}^{op} .

EXERCISE 1.12.25 (Core groupoids). Given a category C , let C^\simeq denote the category that has the same objects, and whose arrows are the isomorphisms of C .

- (a) Show that this is a category, and in fact a groupoid.
- (b) Let $C = \mathbf{Fin}$ denote the category of finite sets. Give a description of C^\simeq up to equivalence in terms of certain \mathbf{BG} 's.
- (c) Do (b) for a general category C .

EXERCISE 1.12.26 (Action groupoids). (a) When is a groupoid equivalent to the terminal groupoid $*$? We call such groupoids *contractible* by analogy with topology.

(b) Let X be a set with an action of the group G . Define a category $X//G$ whose objects are elements of x , and $\text{hom}(x, y) = \{g \in G \mid gx = y\}$. Show that $X//G$ is a groupoid and that $*//G \cong \mathbf{BG}$.

(c) Show that one can make $X \mapsto X//G$ a functor from left G -sets to groupoids. Show that we can actually make it into a functor to the comma-category $\mathbf{Grpd}/\mathbf{BG}$.

(d) Define $\mathbf{EG} := G//G$, where G acts on itself by translation. Show that \mathbf{EG} is contractible and comes with a functor $\mathbf{EG} \rightarrow \mathbf{BG}$. This functor is a groupoid-analogue of a (universal) covering map. Determine its deck transformation group.

EXERCISE 1.12.27 (An example). Let G be a group and consider a subgroup $H \leq G$. What are the automorphisms of G/H as a G -set?

EXERCISE 1.12.28 (Another example). Let X be a fixed set, and consider the functor $X \times - : Y \mapsto X \times Y$ (can you describe it on morphisms?).

Prove that there is only one natural transformation $X \times - \Rightarrow \text{id}_{\mathbf{Set}}$.

EXERCISE 1.12.29 (Actions). (a) Let G be a group. Prove that the category of G -sets is equivalent to the category of functors $\text{Fun}(\mathbf{BG}, \mathbf{Set})$ (see Exercise 1.12.24 for the definition of \mathbf{BG}). More generally, given a category C , can you give a more “concrete” description of the category $\text{Fun}(\mathbf{BG}, C)$?

(b) Give an example of a “transformation” between functors that is not natural.

EXERCISE 1.12.30 (Linear duals). Let \mathbf{Vect} denote the category of vector spaces over a fixed field K .

- (a) Show that the linear dual defines a functor $(-)^* : \mathbf{Vect} \rightarrow \mathbf{Vect}^{op}$.
- (b) Explain how to define a bidual functor $(-)^{**} : \mathbf{Vect} \rightarrow \mathbf{Vect}$.
- (c) Show that there exists a natural transformation $\text{id}_{\mathbf{Vect}} \rightarrow (-)^{**}$ which is an isomorphism when restricted to finite dimensional vector spaces. Is there a natural transformation $(-)^{**} \rightarrow \text{id}_{\mathbf{Vect}}$?
- (d) Let $\mathbf{Vect}_{fd}^\simeq$ denote the category of finite dimensional vector spaces and linear isomorphisms. Explain how to define a functor $(-)^* : \mathbf{Vect}_{fd}^\simeq \rightarrow \mathbf{Vect}_{fd}^\simeq$ (not op). Is there a natural isomorphism between this functor and the identity?

EXERCISE 1.12.31 (The Eckmann-Hilton argument). Suppose M is a monoid, and assume that there is a morphism of monoids $\mu : M \times M \rightarrow M$ which equips the underlying set of M with another monoid structure, call it M' .

- (a) Show that the neutral elements of M, M' agree.
- (b) Show that μ is the multiplication of M , and that M is commutative.
- (c) Show that, in fact, you didn't need to assume that the multiplication of M (or μ) was associative.
- (d) (Bonus) Give a pictorial proof of (a),(b).

EXERCISE 1.12.32. Let $U : \mathbf{Top} \rightarrow \mathbf{Set}$ denote the forgetful functor from topological spaces to sets. It has both a left and a right adjoint: describe them, and prove that they are indeed adjoints.

PS1.

EXERCISE 1.12.33 (Initial objects and colimits).

Let \mathcal{C} be a category. An object $c_0 \in \mathcal{C}$ is called an *initial* (or sometimes *universal*) object of \mathcal{C} if for any object $c \in \mathcal{C}$, there exists a unique morphism $c_0 \rightarrow c$ in \mathcal{C} .

- (1) Show that if c_0 and c_1 are initial objects of the category \mathcal{C} , then there is a unique isomorphism between them (i.e. initial objects are unique up to unique isomorphism).
- (2) Give an example of a category which does not have an initial object.
- (3) What are the initial objects of the categories **Set**, **Ab**, **Ring**, **Top**? What about a poset viewed as a category? Does **BG** have an initial object? Does **EG**? More generally, when does a groupoid have an initial object? (see sheet 0, exercise 5. for definitions of **BG**, **EG**)
- (4) Let $f: A \rightarrow B$ be a homomorphism of abelian groups. Consider the following category denoted by \mathcal{C}_f :

Objects: Pairs (C, g) , where C is an abelian group, $g: B \rightarrow C$ is a homomorphism and $gf = 0$;

Morphisms: A morphism from (C, g) to (C', g') is a homomorphism $h: C \rightarrow C'$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{g} & C \\ g' \downarrow & \swarrow h & \\ C' & & \end{array}$$

commutes.

Does \mathcal{C}_f have an initial object? If yes, describe it.

- (5) Using the ideas from the previous paragraph show that coequalizers are initial objects of certain categories and hence unique up to unique isomorphism.
- (6) Do the “dual” version of this exercise (the corresponding notion is that of a *terminal* object). Do you need to do separate, “different but similar proofs” for these statements?
- (7) (Bonus) Provide a similar statement for general colimits (and dually, limits).

EXERCISE 1.12.34 (Tensor products). (1) Let R be an associative ring, M a right R -module and N a left R -module. Describe the *tensor product* $M \otimes_R N$ as an initial object of some category.

- (2) Let R be a commutative ring and let A , B , and C be R -modules. Show that there is a natural isomorphism of R -modules

$$\mathrm{Hom}_R(A \otimes_R B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C)).$$

EXERCISE 1.12.35 (π_0). (1) Show that the inclusion functor $\mathrm{Set} \rightarrow \mathrm{Cat}$ that views every set as a discrete category has a left adjoint. Describe it explicitly. (We call this functor π_0 , the set of components of the category; if $\pi_0(\mathcal{C})$ is a point, we call \mathcal{C} *weakly connected* or sometimes just connected)

- (2) Contrast the situation with the “discrete” functor $\mathrm{Set} \rightarrow \mathrm{Top}$: does it have a left adjoint? Does it preserve limits?
- (3) Does the answer change if one restricts to **LocCon**, the category of locally connected topological spaces? To **LocPathCon**, the category of locally path-connected spaces?

EXERCISE 1.12.36 (Co/limits). (1) Assume that all diagrams in \mathcal{C}^I have a limit and a colimit. Verify that the functors $\lim: \mathcal{C}^I \rightarrow \mathcal{C}$ and $\mathrm{colim}: \mathcal{C}^I \rightarrow \mathcal{C}$ are indeed respectively right and left adjoint to constant diagram functor $\delta: \mathcal{C} \rightarrow \mathcal{C}^I$.

- (2) Initial objects are colimits. Of what diagrams? What about terminal objects?
- (3) Let \mathcal{C} be a category and X an object of \mathcal{C} . A retract of X is an object Y together with maps $i: Y \rightarrow X, r: X \rightarrow Y$ such that $r \circ i = \mathrm{id}_Y$. Consider a limit diagram X in $\mathcal{C}^{I^\triangleleft}$, where

I^\triangleleft is the same category as I , except we added an initial object). Show that any retract of X is a limit diagram.

EXERCISE 1.12.37 (Adjoint and universal properties). Prove that left adjoint functors commute with colimits. Dually, prove that right adjoint functors commute with limits.

EXERCISE 1.12.38 (Some examples). Compute the following limits and colimits in **Ab**:

- (1) $\lim \left(\begin{array}{ccc} & & A \\ & & \downarrow f \\ 0 & \longrightarrow & B \end{array} \right)$
- (2) $\operatorname{colim} \left(\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \\ 0 & & \end{array} \right)$
- (3) $\lim (\cdots \rightarrow \mathbb{Z}/p^3 \rightarrow \mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p)$
- (4) $\operatorname{colim} (\mathbb{Z}/p \rightarrow \mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p^3 \rightarrow \cdots)$

EXERCISE 1.12.39 (Co/limits over **BG**). Let G be a group, and X a set with an action of G .

- (1) Recall from sheet 0 how this corresponds to a functor $\mathbf{BG} \rightarrow \mathbf{Set}$.
- (2) Compute its limit and colimit.

EXERCISE 1.12.40 (The Yoneda lemma). Let \mathcal{C} be a category and consider the functor

$$\mathcal{Y} : \mathcal{C}^{op} \rightarrow \mathbf{Set}^{\mathcal{C}}$$

that sends $x \in \mathcal{C}$ to $\mathcal{Y}(x) = h_x := \operatorname{Hom}_{\mathcal{C}}(x, -)$ and a morphism to the natural transformation given by precomposition.

- (1) Given a functor $F : \mathcal{C} \rightarrow \mathbf{Set}$. Find a bijection between $\operatorname{Nat}(\mathcal{Y}(x), F)$ and $F(x)$.
- (2) Conclude that \mathcal{Y} is fully faithful (this is the Yoneda lemma, and \mathcal{Y} is called the *Yoneda embedding*).
- (3) How does this imply the phrasing of the Yoneda Lemma in the lecture ? Recall that it was “Let $x, y \in \mathcal{C}$. Every natural transformation

$$\operatorname{Hom}_{\mathcal{C}}(x, -) \Longrightarrow \operatorname{Hom}_{\mathcal{C}}(y, -)$$

is given by the precomposition with a unique $f \in \operatorname{Hom}_{\mathcal{C}}(y, x)$ ”.

- (4) Show that

$$f^* : \operatorname{Hom}_{\mathcal{C}}(x, -) \Longrightarrow \operatorname{Hom}_{\mathcal{C}}(y, -)$$

is a natural isomorphism if and only if $f \in \operatorname{Hom}_{\mathcal{C}}(y, x)$ is an isomorphism.

- (5) State a dual version of this Yoneda lemma.
- (6) Use the Yoneda lemma in the form “ \mathcal{Y} is fully faithful” and exercise 4. from sheet 0 to prove that given a functor G , any two left adjoints of G are isomorphic.

EXERCISE 1.12.41 (An equivalent formulation of adjunctions). (a) Prove proposition 1.2.5. from the lecture notes : An adjunction between $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ is determined and determines two natural transformations $\eta : \operatorname{id}_{\mathcal{C}} \rightarrow GF$ (the unit) and $\epsilon : FG \rightarrow \operatorname{id}_{\mathcal{D}}$ (the co-unit) satisfying the triangle identities: $(\epsilon F) \circ (F\eta) = \operatorname{id}_F$ and $(G\epsilon) \circ (\eta G) = \operatorname{id}_G$

(draw the commutative diagrams corresponding to these identities)

(b) As an application, show that if F is left adjoint to G , then precomposition by F is *right* adjoint to precomposition by G as functors $\operatorname{Fun}(\mathcal{D}, \mathcal{E}) \rightarrow \operatorname{Fun}(\mathcal{C}, \mathcal{E})$ and $\operatorname{Fun}(\mathcal{C}, \mathcal{E}) \rightarrow \operatorname{Fun}(\mathcal{D}, \mathcal{E})$. (Bonus : try to prove this with the definition of adjunction from the lecture notes)

EXERCISE 1.12.42 (An example of Kan extension). [1+2+3+1+2] Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be categories and $F: \mathbf{A} \rightarrow \mathbf{B}$ and $X: \mathbf{A} \rightarrow \mathbf{C}$ functors. Recall that a *left Kan extension* of X along F is a functor $L: \mathbf{B} \rightarrow \mathbf{C}$ together with a natural transformation $\eta: X \Rightarrow LF$ such that for every functor $M: \mathbf{B} \rightarrow \mathbf{C}$ and natural transformation $\theta: X \Rightarrow MF$ there is a unique natural transformation $\sigma: L \rightarrow M$ such that $\theta = \sigma_F \circ \eta$.

- (1) Let $X: \mathcal{I} \rightarrow \mathbf{C}$ be a diagram and consider $F: \mathcal{I} \rightarrow *$ the functor sending every object to the one object and every morphism to the one (identity) morphism of $*$. Observe that the left Kan extension of X along F is

$$\operatorname{colim}_{\mathcal{I}} X.$$

- (2) Let H be a subgroup of G and A a (left) $\mathbb{Z}H$ -module. Prove that the induction $\mathbb{Z}G \otimes_{\mathbb{Z}H} A$ is the left Kan extension of A (considered as a functor $\mathcal{B}H \rightarrow \mathbf{Ab}$) along the inclusion $\mathcal{B}H \rightarrow \mathcal{B}G$.
- (3) Let $M: \mathcal{C} \rightarrow \mathbf{Ab}$ and $N: \mathcal{C}^{op} \rightarrow \mathbf{Ab}$ be functors. Define

$$M \otimes_{\mathcal{C}} N := \operatorname{coeq} \left(\bigoplus_{f \in \operatorname{Hom}_{\mathcal{C}}(c_1, c_2)} M_{c_1} \otimes N_{c_2} \rightrightarrows \bigoplus_{c \in \mathcal{C}} M_c \otimes N_c \right).$$

Let $F: \mathcal{C} \rightarrow \mathcal{I}$ be a functor and $M: \mathcal{C} \rightarrow \mathbf{Ab}$. Prove that

$$M \otimes_{\mathcal{C}} \mathbb{Z} \operatorname{Hom}_{\mathcal{I}}(F(-), -)$$

is a left Kan extension of M along F .

- (4) Show that $- \otimes_{\mathcal{C}} N: \mathbf{Ab}^{\mathcal{C}} \rightarrow \mathbf{Ab}$ is the left adjoint to $\operatorname{Hom}_{\mathbf{Ab}}(N, -): \mathbf{Ab} \rightarrow \mathbf{Ab}^{\mathcal{C}}$.
- (5) Let $\underline{\mathbb{Z}}: \mathcal{C}^{op} \rightarrow \mathbf{Ab}$ be the functor that sends every object to \mathbb{Z} and every morphism to the identity on \mathbb{Z} . Show that

$$M \otimes_{\mathcal{C}} \underline{\mathbb{Z}} \cong \operatorname{colim}_{\mathcal{C}} M$$

if $M: \mathcal{C} \rightarrow \mathbf{Ab}$ is a functor.

You can also think about specializing these last three questions to $\mathcal{C} = \mathcal{B}G$ for some group G (and $\mathcal{I} = \mathcal{B}Q$ for some other group Q) and describe the constructions in representation-theoretic terms.

EXERCISE 1.12.43 (A criterion for equivalences). Prove part of proposition 1.2.12. from the lecture notes, namely: $F: \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence of categories if and only if it is fully faithful (see sheet 0, exercise 4.) and essentially surjective (for any $y \in \mathcal{D}$, there exists x with $F(x) \cong y$).

EXERCISE 1.12.44 (Co/limits in functor categories). Let $\mathcal{C}, \mathcal{D}, \mathcal{I}$ be categories. Suppose \mathcal{C} has \mathcal{I} -shaped colimits. Prove that $\operatorname{Fun}(\mathcal{D}, \mathcal{C})$ also has \mathcal{I} -shaped colimits, and prove that they are computed pointwise (formulate this last part precisely).

EXERCISE 1.12.45 (“All” co/limits). (a) Prove Proposition 1.4.8. from the notes: suppose \mathcal{C} has all coproducts and coequalizers, then \mathcal{C} has all colimits. Moreover, give an expression of a general colimit in terms of coproducts and coequalizers.

(b) State the dual version.

(c) State and prove a similar version for “finite colimits” instead of colimits.

EXERCISE 1.12.46 (Preserving co/limits). Let \mathcal{I} be a category and $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor. We say that F *preserves* \mathcal{I} -shaped limits if for any limit $\delta L \rightarrow X$ of a diagram $X: \mathcal{I} \rightarrow \mathcal{C}$, $\delta F(L) \rightarrow F \circ X$ is a limit of the composite diagram $F \circ X: \mathcal{I} \rightarrow \mathcal{D}$.

(a) Compare this with the notion of “commutes with limits” from the lecture notes.

(b) Using exercise 12., show that if \mathcal{C} is complete and F preserves products and equalizers, it preserves all limits. State an analogous result for finite limits.

(c) Let F, G be two functors $\mathcal{C} \rightarrow \mathcal{D}$, and assume F is a retract of G in $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$. Show that if G preserves \mathcal{I} -shaped limits, so does F .

(d) Suppose C is cocomplete, and let $\theta : F \Rightarrow G$ be a natural transformation of colimit preserving functors $C \rightarrow D$. Show that if θ_x is an isomorphism for all $x \in S$, for some $S \subset C$, then θ_y is an equivalence for all $y \in \langle S \rangle$, where the latter is the smallest subcategory of C containing S and closed under colimits.

EXERCISE 1.12.47 (The homotopy category does not have enough colimits). Let \mathcal{H}_* denote the homotopy category of pointed spaces and pointed maps. Let $n \geq 1$ and let $2 : S^n \rightarrow S^n$ denote any degree 2 pointed map.

Suppose the following diagram has a colimit T :

$$\begin{array}{ccc} S^n & \xrightarrow{2} & S^n \\ \downarrow & & \\ \text{pt} & & \end{array}$$

Observe that $\text{hom}_{\mathcal{H}_*}(T, X) \cong \{a \in \pi_n(X) \mid 2a = 0\}$ naturally in $X \in \mathcal{H}_*$.

Using the fiber sequence $K(\mathbb{Z}/2, n) \rightarrow K(\mathbb{Z}/4, n) \rightarrow K(\mathbb{Z}/2, n)$, conclude that T does not exist.

EXERCISE 1.12.48 (Reflective subcategories). Let C be a category. A *reflective subcategory* of C is a full¹² subcategory $C_0 \subset C$ such that the inclusion functor $i : C_0 \rightarrow C$ admits a left adjoint.

The dual notion, where i admits a right adjoint, is that of a *coreflective subcategory*.

(a) Show that the following are reflective subcategories :

$$\mathbf{Ab} \subset \mathbf{Grp}, \mathbf{CRing} \subset \mathbf{Ring}, \mathbf{Q} - \mathbf{Vect} \subset \mathbf{Ab}$$

Show that the last one is also coreflective. If you know what sheaves are, show that sheaves inside of presheaves is a reflective subcategory.

(b) Show that if \mathbf{A}, \mathbf{B} are abelian categories, the full subcategory of 0-preserving functors is reflective and coreflective inside the category of functors: $\text{Fun}^*(\mathbf{A}, \mathbf{B}) \subset \text{Fun}(\mathbf{A}, \mathbf{B})$ (if you know what this means, it suffices to assume \mathbf{A}, \mathbf{B} additive).

(c) Let $C_0 \subset C$ be a reflective subcategory, let i denote the inclusion and L its left adjoint. Show that the co-unit $\epsilon : Li \rightarrow \text{id}_{C_0}$ (see exercise 1.12.41) is an isomorphism. Show that for $x \in C$, the unit $\eta_x : x \rightarrow iL(x)$ is an isomorphism if and only if x is isomorphic to some object in C_0 .

(d) Conversely, let $F \dashv G$ denote an adjunction between C and D . Show that if the co-unit $\epsilon : FG \rightarrow \text{id}_D$ is an isomorphism, then G is fully faithful, and its *essential image* (the full subcategory of C on objects x isomorphic to some $G(y), y \in D$) is a reflective subcategory of C , equivalent to D via G .

Show that this essential image is equivalently the full subcategory on those objects x such that the unit η_x is an isomorphism.

EXERCISE 1.12.49 (Co/limits in reflective subcategories). (a) Using the notation from question (c) in the previous exercise, call a morphism f an L -equivalence if $L(f)$ is an isomorphism; and an object $x \in C$ L -local if for any L -equivalence $f : y \rightarrow z$, precomposition by f induces an isomorphism $\text{hom}_C(z, x) \rightarrow \text{hom}_C(y, x)$. Show that x is L -local if and only if it is isomorphic to some object of C_0 .

(b) Using (a), show that $C_0 \subset C$ is closed under any limits that exist in C , i.e. if $X : I \rightarrow C_0$ has a limit in C , then that limit is isomorphic to some object of C_0 . In particular, if C has I -shaped limits, then C_0 does so too, and they are preserved by the inclusion - why was the latter clear ?

(c) Show that if C has I -shaped colimits, then C_0 does too. Are they preserved by i ? If so, prove it, if not, give a counterexample.

¹²Some authors don't insist on it being full; here, we do

EXERCISE 1.12.50 (Co/ends). Let C, D be categories, $x \in D$ and $G : C^{op} \times C \rightarrow D$ be a functor. A *wedge* from x to F is a family of morphisms $e_c : x \rightarrow F(c, c)$ such that for every morphism $f : c \rightarrow c'$ in C , the following diagram commutes:

$$\begin{array}{ccc} x & \xrightarrow{e_{c'}} & F(c', c') \\ \downarrow e_c & & \downarrow F(f, c') \\ F(c, c) & \xrightarrow{F(c, f)} & F(c, c') \end{array}$$

An *end* of F is a terminal wedge, i.e. a wedge $(x, (e_c)_{c \in C})$ such that for any wedge $(y, (h_c)_{c \in C})$ there is a unique morphism $\alpha : y \rightarrow x$ such that for all c , $e_c \circ \alpha = h_c$.

(a) Define the dual notion of a cowedge and a coend.

(b) Let $Tw(C)$ (read : “twisted arrow category of C ”) denote the following category: its objects are arrows in C , and a morphism from $x \xrightarrow{f} y$ to $x' \xrightarrow{f'} y'$ is a commutative diagram of the following form (pay attention to the orientation of the arrows!):

$$\begin{array}{ccc} x & \xleftarrow{p} & x' \\ \downarrow f & & \downarrow f' \\ y & \xrightarrow{q} & y' \end{array}$$

(Can you describe the composition of two such morphisms ?)

Show that $(x \xrightarrow{f} y) \mapsto (x, y)$ is a functor $Tw(C) \rightarrow C^{op} \times C$.

(c) Letting π denote the functor from (b), prove that the data of a wedge from x to F is exactly the same thing as a natural transformation from the constant functor x on $Tw(C)$ to $F \circ \pi$. State the dual version for cowedges.

(d) Deduce that an end of F exists if and only if $F \circ \pi$ admits a limit, and that they agree. State the dual version for coends and colimits.

The following notation is often used for co/ends: $\int_{c \in C} F(c, c)$ is the end, and $\int^{c \in C} F(c, c)$ is the coend (if they exist).

EXERCISE 1.12.51 (An important end). Let $F, G : C \rightarrow D$ be two functors, and let $\text{hom}_D(F(-), G(-))$ be the functor on $C^{op} \times C$ defined by $\text{hom}_D(F(-), G(-))(c, c') = \text{hom}_D(F(c), G(c'))$ (can you describe it on arrows ?).

Show that it has an end, given by the set of natural transformations from F to G .

With the notation from above: $\int_{c \in C} \text{hom}_D(F(c), G(c)) \cong \text{Nat}(F, G)$.

EXERCISE 1.12.52 (A coend computation). We let \mathbf{Vect} denote the category of vector spaces over some field K (which will remain fixed), and \mathbf{Vect}_{fd} the full subcategory of finite dimensional vector spaces.

(a) Let C be a small category, and $F : C^{op} \times C \rightarrow D$ be a functor to a cocomplete category. Show that it has a coend, and furthermore, show that $\text{hom}_D(\int^{c \in C} F(c, c), X) \cong \int_{c \in C} \text{hom}_D(F(c, c), X)$, naturally in X (you have to make sense of the latter).

(b) Let $F : \mathbf{Vect}_{fd}^{op} \times \mathbf{Vect}_{fd} \rightarrow \mathbf{Vect}$ be the functor defined by $(V, W) \mapsto V^* \otimes_K W$. Show that it is naturally isomorphic to $\mathcal{L}(V, W)$.

(c) Show that for any vector space E , the functor $(V, W) \mapsto \text{hom}_{\mathbf{Vect}}(F(V, W), E)$ is naturally isomorphic to $(V, W) \mapsto \text{hom}_{\mathbf{Vect}}(W, \mathcal{L}(V^*, E))$.

(d) Prove that the set of natural transformations $\text{id}_{\mathbf{Vect}_{fd}} \rightarrow E \otimes_K -$ is isomorphic to E , naturally in E .

(e) Using the previous questions, as well as the previous exercise and the Yoneda lemma, deduce that $\int^{V \in \mathbf{Vect}_{fd}} V^* \otimes_K V \cong K$. Can you unravel all this work to describe more explicitly the component $\text{End}_K(V) \cong V^* \otimes V \rightarrow K$ of this cowedge ?

EXERCISE 1.12.53 (Co/ends and Kan extensions). Let $f : I \rightarrow J$ be a functor of small categories, and C a cocomplete category. Let $L : I \rightarrow C$ be a functor.

Show that the left Kan extension of L along f , $f_!L$ can be described by $f_!L(j) \cong \int^{i \in I} \text{hom}(f(i), j) \cdot L(i)$, where for a set X and an object $c \in C$, $X \cdot c$ is a coproduct of an X -indexed diagram of c 's ($\coprod_{x \in X} c$).

State a dual formula for right Kan extensions in terms of ends and products.

EXERCISE 1.12.54 (The universal property of presheaves, 1+2+1+1.5+1). Let C be a small category, we let $\mathbf{Psh}(C) := \text{Fun}(C^{\text{op}}, \mathbf{Set})$ denote the category of presheaves on C .

Recall from exercise 8 that $\mathcal{Y} : C \rightarrow \mathbf{Psh}(C)$ is a fully faithful embedding (if you have not covered exercise 8, you can take this for granted). Proposition 1.5.5. in the lecture notes (the *Density theorem*) shows that any presheaf is “canonically” a colimit of elements in the essential image of \mathcal{Y} . We want to show that these colimits are *free*.

(a) Let D be a category, and $L : \mathbf{Psh}(C) \rightarrow D$ a functor. Let $R : D \rightarrow \mathbf{Psh}(C)$ be defined by $d \mapsto \text{hom}_D(L \circ \mathcal{Y}(-), d)$. Show that

$$\text{hom}_{\mathbf{Psh}(C)}(\mathcal{Y}(c), R(d)) \cong \text{hom}_D(L \circ \mathcal{Y}(c), d)$$

naturally in c, d .

(b) Define a natural morphism $\text{hom}_D(L(F), d) \rightarrow \text{hom}_{\mathbf{Psh}(C)}(F, R(d))$ which specializes to the above isomorphism when $F = \mathcal{Y}(c)$.

(c) Deduce that if L preserves colimits, then $L \dashv R$. In fact, deduce that it suffices to show that L preserves the specific colimits from proposition 1.5.5. to show that it is a left adjoint to R , and therefore that it preserves all colimits.

(d) We now assume that D is cocomplete. Let $\text{Fun}^L(\mathbf{Psh}(C), D)$ denote the full subcategory of colimit-preserving (equivalently by (c), left adjoint) functors. Using Kan extensions and (c), show that the restriction functor along \mathcal{Y} is essentially surjective

$$\text{Fun}^L(\mathbf{Psh}(C), D) \rightarrow \text{Fun}(C, D)$$

(e) Show that this restriction functor is faithful, and that it is full. Conclude that $\text{Fun}^L(\mathbf{Psh}(C), D) \cong \text{Fun}(C, D)$: this is the universal property of presheaves.

EXERCISE 1.12.55 (Cofinality and colimits, 0.5+0.5+0.5+1.5+0.5). Recall that given a functor $f : \mathcal{I} \rightarrow \mathcal{J}$ and an object $j \in \mathcal{J}$, we define $\mathcal{I}_{j/}$ by the pullback $\mathcal{I} \times_{\mathcal{J}} \mathcal{J}_{j/}$.

(a) Give a more explicit description of $\mathcal{I}_{j/}$.

(b) Say f is *cofinal* if for all $j \in \mathcal{J}$, $\mathcal{I}_{j/}$ is weakly connected (see exercise 1.12.35). Give an order-theoretic description of “ f is cofinal”, when \mathcal{I}, \mathcal{J} are two preorders, and \mathcal{I} is *directed*, i.e. for any $i_0, i_1 \in \mathcal{I}$ there exists $k \geq i_0, i_1$.

(c) Give an example of a cofinal functor between preorders; and one example between categories that aren't preorders.

(d) Prove that if f is cofinal, and $X : \mathcal{J} \rightarrow C$ is a diagram, then X has a colimit if and only if $X \circ f$ does; and that if they both have one, then they are the same (you should also make that statement precise).

(e) Formulate dual statements : define the notion of a *final* functor, and state how “limits are preserved” under precomposition by final functors.

EXERCISE 1.12.56 (Grothendieck construction). Let C be a small category. Let $F : C^{\text{op}} \rightarrow \mathbf{Set}$ be a functor. We define the category $\text{Un } F$ as follows: Objects are pairs (c, x) where $c \in C$ and $x \in F(c)$. The set of morphisms from (c, x) to (c', x') is given by morphisms $f : c \rightarrow c'$ in C such that $F(f)(x') = x$. The category $\text{Un } F$ comes with an obvious functor $\text{Un } F \rightarrow C$. This construction is called the *Grothendieck construction* or *unstraightening* of F .

(a) Let $G : D \rightarrow C$ be a functor. A morphism $e : y \rightarrow x$ in D is called *G-cartesian* if for every object z in D , every morphism $g : z \rightarrow x$ in D and $\bar{f} : p(z) \rightarrow p(y)$ in C such that $p(e)\bar{f} = p(g)$,

there is a unique morphism $f: z \rightarrow y$ in \mathcal{D} such that $p(f) = \bar{f}$ and $ef = g$. We can visualize this as follows:

$$\begin{array}{ccc} p(z) & \xrightarrow{p(g)} & p(x) \\ \bar{f} \downarrow & \nearrow p(e) & \\ p(y) & & \end{array} \qquad \begin{array}{ccc} z & \xrightarrow{g} & x \\ f \downarrow & \nearrow e & \\ y & & \end{array}$$

Now consider the functor $\text{Un } F \rightarrow \mathcal{C}$. Show that for every $f: c \rightarrow c'$ in \mathcal{C} and every (c', x) , there is a unique lift of f to a cartesian morphism in $\text{Un } F$ with target (c', x) . Moreover, note that for $c \in \mathcal{C}$, the fiber of this functor over c is a set.

We say that the functor $\text{Un } F \rightarrow \mathcal{C}$ is a Grothendieck fibration, fibered in Set .

(b) Show that this construction enhances to a functor

$$\text{Un}: \text{Fun}(\mathcal{C}^{\text{op}}, \text{Set}) \rightarrow \text{Cat}/\mathcal{C}$$

such that for every morphism f in $\text{Fun}(\mathcal{C}^{\text{op}}, \text{Set})$, the induced morphism $\text{Un}(f)$ in Cat/\mathcal{C} preserves cocartesian morphisms. It hence factors through the (non-full) subcategory $\text{Fib}_{\text{Set}}(\mathcal{C}) \subset \text{Cat}/\mathcal{C}$ of Grothendieck fibrations, fibered in Set , and functors over \mathcal{C} preserving cartesian morphisms.

(c) Show that the composition of two cartesian morphism is again a cartesian morphism.

(d) Let \mathcal{C} be a small category and let $F: \mathcal{E} \rightarrow \mathcal{C}$ be a Grothendieck fibration, fibered in Set . Show that there is a functor

$$\text{Str}(F): \mathcal{C}^{\text{op}} \rightarrow \text{Set}$$

which sends $c \in \mathcal{C}$ to $F^{-1}(c)$ and a morphism $\varphi: c' \rightarrow c$ to the map

$$F^{-1}(c) \rightarrow F^{-1}(c')$$

sending $e \in F^{-1}(c)$ to the source $e' \in F^{-1}(c')$ of the cartesian lift $\bar{\varphi}: e' \rightarrow e$ of $\varphi: c' \rightarrow c$.

This functor is called the *straightening* of F .

(e) Show that the above construction enhances to a functor

$$\text{Str}: \text{Fib}_{\text{Set}}(\mathcal{C}) \rightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \text{Set})$$

which is an inverse equivalence to Un .

There is also a version of this theorem for functors $\mathcal{C}^{\text{op}} \rightarrow \text{Grpd}$ which is used in the theory of stacks. You can read about this in the stacksproject.

EXERCISE 1.12.57 (Posets as categories). Let (P, \leq) be a preordered set. Define a category \underline{P} whose objects are the elements of P , and whose morphisms are defined by $\text{hom}(x, y) = \{(x, y)\}$ if $x \leq y$, \emptyset otherwise.

(a) What is composition in this category? Show that a functor $\underline{P} \rightarrow \underline{Q}$ is the same thing as nondecreasing map $P \rightarrow Q$.

(b) Give a necessary and sufficient condition for a category to be a preorder.

(c) Let I be a category and let $F: I \rightarrow \underline{P}$ be a functor. Give a necessary and sufficient condition for F to admit a (co)limit.

(d) Give a necessary and sufficient condition for a functor $\underline{P} \rightarrow \underline{Q}$ to be a left adjoint.

(e) Let L/K be a field extension, and let $G_{L/K}$ be the group of automorphisms of L leaving K fixed (pointwise). Show that the following is an adjunction between the poset of subextensions and the opposite of the poset of subgroups of G : $E \mapsto G_{L/E}$, $H \mapsto L^H$ (remember that L^H is the set of fixed points of L under H)

(f) Consider \mathbb{R} and \mathbb{Z} as posets, and let $i: \mathbb{Z} \rightarrow \mathbb{R}$ denote the inclusion. Show that i has both a left and a right adjoint, and describe them.

EXERCISE 1.12.58. Let $L: \mathcal{C} \rightarrow \mathcal{D}$ be a functor with a fully faithful right adjoint $R: \mathcal{D} \rightarrow \mathcal{C}$ (such a functor is called a Bousfield localization). Show that

(a) A morphism $f: x \mapsto y$ is sent to an isomorphism by L if and only if it is an *R-local equivalence*, i.e. if the induced map

$$f^*: \operatorname{Hom}_{\mathcal{C}}(y, z) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(x, z)$$

is a bijection for every z in the essential image of $R: \mathcal{D} \rightarrow \mathcal{C}$.

(b) The unit of the adjunction $x \mapsto RLx$ is an *R-local equivalence* for every $x \in \mathcal{C}$.

(c) Let \mathcal{E} be a category. Show that

$$L^*: \operatorname{Fun}(\mathcal{D}, \mathcal{E}) \longrightarrow \operatorname{Fun}(\mathcal{C}, \mathcal{E})$$

is fully faithful and the essential image consists of the functors which send *R-local equivalences* to isomorphisms. (We say that L exhibits \mathcal{D} as the localization of \mathcal{C} at the *R-local equivalences*).

CHAPTER 2

The homotopy theory of simplicial sets

We will start by introducing simplicial sets. Our goal is to show that we can do homotopy theory in the category of simplicial sets, just like in the category of spaces. The category of simplicial sets has a number of technical advantages over topological spaces, both with respect to formal properties, and in terms of avoiding any continuity considerations.

They also have a direct link to the category of (small) categories via the so-called *nerve* construction. We can hence both view a simplicial set as combinatorial version of a simplicial complex, *and* view it as sort of a generalized category. It is the combination of those two viewpoints that makes them so useful.

Idea: A simplicial set can be viewed as a sequence of sets X_0, X_1, X_2, \dots , where the elements in X_n are called *simplices of dimension n* (or *degree n*), or just *n -simplices* for short. These will be tied together via various combinatorial relations, like in an ordered simplicial complex. We have face maps $d_i : X_n \rightarrow X_{n-1}$, associated to the process of restricting to the i^{th} face

$$(2.0.1) \quad "d_1 \left(\begin{array}{c} 1 \\ \bullet \\ \nearrow \quad \searrow \\ 0 \quad \bullet \quad 2 \end{array} \right) = \bullet \longrightarrow \bullet \text{ "}$$

To get the right categorical properties, we also have to allow for certain simplices to be “degenerate”, i.e. obtained from lower-dimensional simplices via degeneracy maps $s_i : X_n \rightarrow X_{n+1}$: these simplices often play no substantial role, like identity maps in a category, trivial subgroups in a group etc., and can sometimes be regarded as a sort of “noise”. The remaining “non-degenerate” n -simplices correspond to the actual n -cells in the associated CW complex, the *geometric realization* introduced in Section 2.4.

In this combinatorial picture a simplicial set can be viewed as a bunch of sets and maps as follows

$$X_0 \begin{array}{c} \xleftarrow{d_0} \\ \xleftarrow{s_0} \rightarrow \\ \xleftarrow{d_1} \end{array} X_1 \begin{array}{c} \xleftarrow{d_0} \\ \xleftarrow{s_1} \rightarrow \\ \xleftarrow{s_0} \rightarrow \\ \xleftarrow{d_2} \end{array} X_2 \cdots$$

where the maps satisfy certain relations, known as simplicial identities.

2.1. The simplex category Δ and simplicial sets

We now embark on the slick formal definition of simplicial sets (which is due to Dan Kan). First we need the following category:

DEFINITION 2.1.1. The simplex category Δ is defined to be the category with objects the non-empty finite totally ordered sets

$$[n] = (0 < 1 < \cdots < n).$$

for $n = 0, 1, \dots$, and morphisms the order preserving maps between them. In other words Δ is the full subcategory of Cat with objects the categories $[0], [1], [2], \dots$

REMARK 2.1.2. The category Δ can be described via generators and relations. Namely we have a map $d^i : [n-1] \rightarrow [n]$ which “skips i ”, i.e., is the identity for $j < i$ and sends j to $j+1$ for $j \geq i$.

Similarly we have $s^i : [n+1] \rightarrow [n]$ which “repeats i ”, i.e., is the identity on $j \leq i$ and sends j to $j-1$ for $j > i$.

These satisfies some obvious relations:

$$(2.1.1) \quad \begin{aligned} d^j d^i &= d^i d^{j-1}, i < j & s^j s^i &= s^i s^{j+1}, i \leq j \\ s^j d^i &= d^i s^{j-1}, i < j & s^j d^i &= d^{i-1} s^j, i > j+1 & s^i d^i &= s^i d^{i+1} = 1 \end{aligned}$$

PROPOSITION 2.1.3. *The category Δ can be described as being the category with objects $[0], [1], \dots$, and morphisms generated by the d^i 's and s^i 's, subject to the above relations.*

In particular, a functor $F : \Delta \rightarrow \mathcal{C}$ to any category \mathcal{C} is uniquely determined by a choice of objects $F([0]), F([1]), \dots$ in \mathcal{C} and a choice of maps $F(d^i)$ and $F(s^i)$ in \mathcal{C} between them, satisfying the above relations.

The proof is left as an exercise.

DEFINITION 2.1.4. The category of *simplicial sets* \mathbf{sSet} is defined to be the functor category $\text{Fun}(\Delta^{\text{op}}, \text{Set})$ with objects functors and morphisms natural transformations. For X a simplicial set we write $X_n = X([n])$ for the set of n -simplices. We sometimes also write the whole simplicial set as X_\bullet if it is important to stress the indexing.

It is for intuition often fruitful to think of a simplicial set as a geometric object X in its own right, resembling a “space” or a CW-complex, rather than a combinatorial object consisting of a sequence of sets and relations.

PROPOSITION 2.1.5. *The category \mathbf{sSet} is complete and cocomplete and limits and colimits are calculated “levelwise”, i.e. by taking the corresponding limits and colimits in Set in each degree $n \geq 0$.*

PROOF. This follows by the definition of \mathbf{sSet} as a functor category, by Proposition 1.8.11. \square

The universal property of the Yoneda embedding, Proposition 1.11.5, helps justify why the functor category definition as object in $\text{Fun}(\Delta^{\text{op}}, \text{Set})$ is “correct” if we want to build geometric objects glued together out of simplices Δ^n . This viewpoint is furthermore powerful in proving properties, as it allows us to access all the standard methods of category theory.

Let us however also try to spell out the structure of X in a more combinatorial fashion: For a simplicial set we hence have operations $d_i = X(d^i) : X_n \rightarrow X_{n-1}$, (aka *face maps*), and $s_i = X(s^i) : X_n \rightarrow X_{n+1}$ (aka *degeneracy maps*).

As the d_i and s_i generate all morphisms in Δ , specifying a *simplicial map* $f : X_\bullet \rightarrow Y_\bullet$ (i.e. a natural transformation of functors $\Delta^{\text{op}} \rightarrow \text{Set}$) amounts to specifying a sequence of maps of sets $\{f_n : X_n \rightarrow Y_n\}$ such that

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ \downarrow d_i & & \downarrow d_i \\ X_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-1} \end{array} \quad \text{and} \quad \begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ s_i \uparrow & & s_i \uparrow \\ X_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-1} \end{array}$$

commute for all i and n . The simplicial relations also allow us to describe the objects of \mathbf{sSet} combinatorially:

PROPOSITION 2.1.6 (Combinatorial description of \mathbf{sSet}). *A simplicial set X_\bullet can be described as a collection of sets $X_0, X_1, \dots, X_n, \dots$, and maps d_i 's and s_i 's, satisfying the relations*

induced by (2.1.1), which read as follows:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \text{ if } i < j \\ s_i s_j &= s_{j+1} s_i \text{ if } i \leq j \\ d_i s_j &= s_{j-1} d_i \text{ if } i < j \\ d_j s_j &= 1 = d_{j+1} s_j \\ d_i s_j &= s_j d_{i-1} \text{ if } i > j + 1 \end{aligned}$$

□

It may be memotechnically useful for the reader to say out in words what these relations mean. E.g., the relation “ $d_i d_j = d_{j-1} d_i$ if $i < j$ ” says that if we first take the j^{th} face and then take the i^{th} face, then this is the same as first taking the i^{th} face and then taking the $(j-1)^{\text{th}}$ face when j is larger than i (as the indexing of what “ j^{th} face” means has changed). The other relations are similar.

Categorically we can think of the elements in X_0 as objects and the X_1 as arrows, with source and target given by the boundary maps d_1 and d_0 , and $s_0 : X_0 \rightarrow X_1$ giving the identity morphism.

Geometrically we can think of the map d_i as the map giving the i^{th} face of a simplex, as already drawn in Figure (2.0.1). However, not all simplices really contribute geometrically, as the ones which are in the image of some s_i are really just repeated copies of things which come from lower dimensions. The simplices in X_n which in the image of $s_i : X_{n-1} \rightarrow X_n$ for some i are called *degenerate* and if not called *non-degenerate*. We will return to them shortly, in Section 2.1.1.

DEFINITION 2.1.7. Define the *simplicial n -simplex* as the simplicial set

$$\Delta^n = \text{Hom}_{\text{Cat}}(-, [n]) = \text{Hom}_{\Delta}(-, [n])$$

REMARK 2.1.8. Note that in our categorical language from the last chapter the Δ^n 's are exactly the representable functors in $\text{sSet} = \text{Set}^{\Delta^{\text{op}}}$.

We can already now calculate our first simplicial maps:

LEMMA 2.1.9.

$$\text{Hom}_{\text{sSet}}(\Delta^n, X) \xrightarrow{\cong} X_n$$

PROOF. This follows by the Yoneda lemma, since by definition $\Delta^n = \text{Hom}_{\Delta}(-, [n])$. □

The above lemma says that we can think of an n -simplex in X as either an element x in the set X_n or as a simplicial map $f : \Delta^n \rightarrow X$. Applying the face map d_i to $x \in X_n$ corresponds restricting f to the i^{th} face by precomposing with the map $d^i : \Delta^{n-1} \rightarrow \Delta^n$. In the second notation we are treating x as an n -cell in X , via a map $\Delta^n \rightarrow X$. In this way we have a geometric picture in mind, and may pretend that “ X ” is still a space (rather than a simplicial set) and “ $\Delta^n \rightarrow X$ ” is a map of spaces (rather than a map of simplicial sets). It is the first step in our long dictionary between simplicial sets and topological spaces. The dictionary will be so effective that researchers can indeed write entire papers containing results about “spaces”, and only in the end really decide if they meant by the word “space” a topological space or a simplicial set (or perhaps another equivalent ∞ -category?).

REMARK 2.1.10. More generally we define simplicial objects in any category \mathcal{C} as the functor category $\mathcal{C}^{\Delta^{\text{op}}}$. Cosimplicial objects in \mathcal{C} are similarly defined to be functors $\Delta \rightarrow \mathcal{C}$.

REMARK 2.1.11. Note that $\text{Hom}_{\Delta}(-, -)$ defines a functor $\Delta^{\text{op}} \times \Delta \rightarrow \text{Set}$. We can hence view the functor $(\Delta^\bullet)_\bullet$ which to $([n], [m])$ assigns the set $(\Delta^m)_n = \text{Hom}([n], [m])$ of n -simplices in the simplicial set Δ^m as either a cosimplicial simplicial set, or a simplicial cosimplicial set. This dual variance is the key to many formulas.

Before we move on to the nerve functor, let us note that the simplices in X can also naturally be viewed as a category, namely by considering $\text{el}(X)$ as the category of elements $\text{el}(X)$ of X from Definition 1.11.1. In this case is sometimes called *category of simplices*, the *simplex category* or *subdivision category* of X , and occasionally also denoted $\text{sd}(X)$ or dX .

The objects are $x_n \in X_n$ and a morphism $x_n \rightarrow x_m$ consists of a morphism $(\theta : [m] \rightarrow [n])$ in Δ such that $\theta^*(x_n) = x_m$, where $\theta^* : X([n]) \rightarrow X([m])$ is the induced map.

The abstract Theorem 1.11.3 now specializes to describe X as colimit of its simplices.

PROPOSITION 2.1.12.

$$\text{colim}_{\sigma_n \in \text{el } X} \Delta^n \xrightarrow{\cong} X$$

PROOF. Follows by applying Theorem 1.11.3 to $X \in \text{sSet} = \text{Set}^{\Delta^{\text{op}}}$, as $\Delta^n = \text{Hom}_{\Delta}(-, [n])$ by definition.

[recheck co and contravariance in yoneda embedding]

□

2.1.1. Non-degenerate simplices. As already mentioned, most element in X_n don't play a role, as they are repeats of things occurring earlier, via the degeneracy operator $s_i : X_n \rightarrow X_{n+1}$, which is injective with left inverse d_i . It is the non-degenerate simplices, i.e., the ones which are not in the image of a degeneracy map which turn out to play the role of cells in a CW decomposition, as we shall see later. We've already a hint that degeneration maps don't play a role in Corollary 1.9.7, which says that pushouts of simplices along degeneration maps just produce another simplex.

DEFINITION 2.1.13. An n -simplex $x \in X_n$ is called *degenerate* if it is of the form $\theta^*(y)$ for some $\theta : [n] \rightarrow [m]$, $m < n$, and $y \in X_m$. Otherwise it is called *non-degenerate*.

LEMMA 2.1.14 (Eilenberg-Zilber Lemma). *Each n -simplex x can be written as $\theta^*(y)$ for a unique pair (y, θ) where $\theta : [n] \twoheadrightarrow [m]$, and $y \in X_m$ is non-degenerate.*

The element y can be described as follows: Let $i_1 < \dots < i_r$ be the set of indices $0 \leq i \leq n$, such that $x \in s_i X_{n-1}$. Then

$$y = d_{i_1} \cdots d_{i_r} x.$$

PROOF. Given two such factorizations (y, θ) and (y', θ') it factors through the pushout by Corollary 1.9.7

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\theta} & \Delta^m \\ \downarrow \theta' & & \downarrow \\ \Delta^{m'} & \twoheadrightarrow & \Delta^{m''} \end{array} \quad \begin{array}{c} \nearrow y \\ \searrow y' \\ \text{---} \end{array} \quad \begin{array}{c} \\ \\ X \end{array}$$

where $\Delta^{m''}$ is the simplex corresponding to the totally ordered set $[m'']$ constructed from $[m]$ identifying the interval $\theta(\theta'^{-1}(i))$ to a single point, for all $i \in [m']$. However, as m and m' are assumed minimal, this shows that $m = m'' = m'$. However a surjective map $[m] \rightarrow [m]$ is the identity, so the diagram shows that $\theta = \theta'$ and $y = y'$.

The description of y also follows from the above.

□

REMARK 2.1.15. One can try to axiomatize what properties of Δ makes such a unique factorization possible (apart from saying that one should be able to use Corollary 1.9.6). This is studied in [BM11], under the punny name EZ-categories.

2.1.2. Boundary, horns, and skeleta.

DEFINITION 2.1.16 (Boundary). Define the boundary $\partial\Delta^n$ of the n -simplex Δ^n as the functor $\Delta^{\text{op}} \rightarrow \text{Set}$ given by

$$\partial\Delta^n([m]) = \{\varphi : [m] \rightarrow [n] \mid \varphi \text{ is not surjective}\}$$

In other words $\partial\Delta^n$ is the subsimplicial set¹ of Δ^n generated by simplices of dimension strictly less than n .

Note that the identity $(1_{[n]} : [n] \rightarrow [n]) \in \text{Hom}_{\Delta}([n], [n])$ is an n -simplex in Δ^n but not in $\partial\Delta^n$, so that $\partial\Delta^n$ is a proper subsimplicial set of Δ^n .



Furthermore we define:

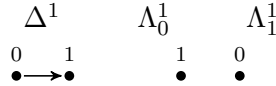
DEFINITION 2.1.17 (Horns). Define the i^{th} horn Λ_i^n as

$$\Lambda_i^n([m]) = \{\varphi : [m] \rightarrow [n] \mid \text{image}(\varphi) \notin \{[n], [n] \setminus \{i\}\}\}.$$

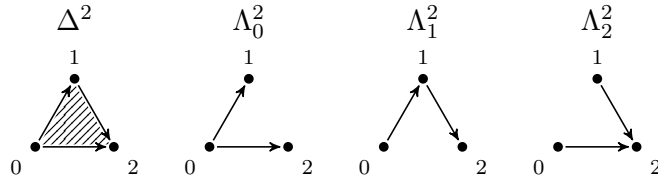
In other words Λ_i^n is the subobject of Δ^n obtained by removing the unique non-degenerate n -simplex ι , as well as the $(n-1)$ -simplex $d_i(\iota)$: it is thus the subobject generated by the $(n-1)$ -simplices $d_j(\iota)$ for $j \neq i$.

Let us draw some pictures for small n :

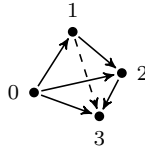
$n = 1 :$



$n = 2 :$

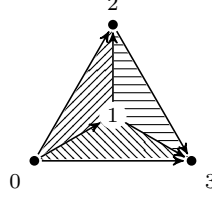


For $n = 3$, the simplicial set Δ^3 is the filled 3-simplex:



¹A subsimplicial set Y_{\bullet} of X_{\bullet} is just a selection of subsets $Y_n \subseteq X_n$ on which all face and degeneracy maps restrict to give a new simplicial set. We also refer to a subsimplicial set as a subobject in the category of simplicial sets.

and we can e.g. draw the 3-horn Λ_1^3 in the plane as follows:



DEFINITION 2.1.18 (n^{th} skeleton). Given a simplicial set X we can consider its n^{th} skeleton $\text{sk}_n X$, the subobject generated by simplices of dimension at most n .

E.g., $\partial\Delta^n = \text{sk}_{n-1}\Delta^n$. Note also that

$$X = \text{colim}_n \text{sk}_n X,$$

where the colimit is indexed by the category \mathbb{N} (it is a *direct limit* in the classical sense, and in this case it can be visualised as an infinite, increasing union).

$$\text{sk}_1 \left(\begin{array}{c} \bullet \\ \nearrow \quad \searrow \\ \bullet \quad \bullet \end{array} \right) = \begin{array}{c} \bullet \\ \nearrow \quad \searrow \\ \bullet \quad \bullet \end{array}, \quad \text{sk}_0 \left(\begin{array}{c} \bullet \\ \nearrow \quad \searrow \\ \bullet \quad \bullet \end{array} \right) = \begin{array}{c} \bullet \\ \bullet \quad \bullet \end{array}.$$

PROPOSITION 2.1.19. We have that $\text{sk}_n(-) = i_! i^*(-)$, where $i : \Delta_{\leq n} \rightarrow \Delta$ is the inclusion functor and $i^* : \text{Fun}(\Delta^{\text{op}}, \text{Set}) \rightarrow \text{Fun}(\Delta_{\leq n}^{\text{op}}, \text{Set})$ and $i_!$ denotes restriction and left Kan extension respectively.

In particular $\text{sk}_n(-)$ commutes with colimits.

PROOF. The identification $\text{sk}_n(-) = i_! i^*(-)$ is left as an exercise (or maybe better, take this description of the definition, and then deduce the description in Example 2.1.18 [do this switch?]).

As i^* also has a right adjoint (the right Kan extension, by Proposition 1.10.7) and $i_!$ is a left adjoint by definition, we have that $\text{sk}_n(-)$ is a composite of left adjoints and hence preserves colimits by Proposition 1.9.3. \square

EXERCISE 2.1.20. Define the coskeleton cosk_n dually to sk_n . Describe it completely.

2.2. The nerve functor N

Let Cat denote the category² of small categories, with objects small categories, and morphisms functors, studied in Section 1.7. We now want to introduce the nerve functor

$$N : \text{Cat} \rightarrow \text{sSet}$$

and its left adjoint h , the so-called homotopy category functor (named so for reasons that will become clear later).

By definition, it gives us a way to associate a simplicial set (which we think of as a geometric object) to any category. Furthermore it gives us an easy way of constructing interesting examples of simplicial sets. And simplicial sets that come from categories are easier to manipulate, essentially as it is in general easier to write up functors and natural transformations than verifying simplicial identities.

DEFINITION 2.2.1. The nerve functor

$$N : \text{Cat} \rightarrow \text{sSet}$$

²It is even a 2-category, with 2-morphisms natural transformations, but we will ignore this for now.

associates to a small category \mathcal{C} the *nerve* $NC : \Delta^{\text{op}} \rightarrow \text{Set}$, defined as the functor $\text{Hom}_{\text{Cat}}(-, \mathcal{C}) : \Delta^{\text{op}} \rightarrow \text{Set}$ sending

$$[n] \mapsto (NC)_n = \text{Hom}_{\text{Cat}}([n], \mathcal{C})$$

REMARK 2.2.2. We have already seen an instance of the nerve functor, in our definition of Δ^n : we can indeed write

$$\Delta^n = N[n].$$

REMARK 2.2.3. Let us spell out what the above definition means concretely: a functor $[n] \rightarrow \mathcal{C}$ can be represented as a sequence $(x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n)$ of objects and morphisms in \mathcal{C} . For $i \neq 0, n$ we have

$$d_i(x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n) = x_0 \xrightarrow{f_1} \cdots \rightarrow x_{i-1} \xrightarrow{f_{i+1} \circ \cdots \circ f_i} x_{i+1} \rightarrow \cdots \rightarrow x_n$$

whereas $d_0(x_0 \rightarrow \cdots \rightarrow x_n) = (x_1 \rightarrow \cdots \rightarrow x_n)$ and $d_n(x_0 \rightarrow \cdots \rightarrow x_n) = (x_0 \rightarrow \cdots \rightarrow x_{n-1})$.

Similarly

$$s_i(x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n) = (x_0 \rightarrow \cdots \rightarrow x_{i-1} \xrightarrow{f_i} x_i \xrightarrow{\text{id}} x_i \xrightarrow{f_{i+1}} x_{i+1} \rightarrow \cdots \rightarrow x_n).$$

For example the faces of the 2-simplex $\sigma = (x_0 \xrightarrow{f} x_1 \xrightarrow{g} x_2)$ look as expected:

$$\begin{array}{ccc} & x_1 & \\ f \nearrow & & \searrow g \\ x_0 & \xrightarrow{g \circ f} & x_2 \end{array}$$

we have $d_0\sigma = g$ (really, $(x_1 \xrightarrow{g} x_2)$), $d_1\sigma = g \circ f$, and $d_2\sigma = f$.

Let us also record the non-degenerate simplices of the nerve NC :

LEMMA 2.2.4. *Consider NC for some small category \mathcal{C} . The non-degenerate 0-simplices are the set of objects of \mathcal{C} , the non-degenerate 1-simplices are the set of non-identity morphisms, and the non-degenerate n -simplices are the n -fold compositions of non-identity morphisms. \square*

PROPOSITION 2.2.5. *The nerve functor $N : \text{Cat} \rightarrow \text{sSet}$ is fully faithful, i.e., it induces a bijection on morphism sets*

$$\text{Hom}_{\text{Cat}}(\mathcal{C}, \mathcal{D}) \xrightarrow{\cong} \text{Hom}_{\text{sSet}}(NC, ND).$$

PROOF. It is clear that $\text{Hom}_{\text{Cat}}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Hom}_{\text{sSet}}(NC, ND)$ is injective, as a functor is specified by what it does on objects and morphisms, and this information is captured by the behavior of the corresponding simplicial map between nerves in degrees 0 and 1.

We now verify surjectivity: Suppose $\phi : NC \rightarrow ND$ is a morphism of simplicial sets. We want to show that $\phi = NF$ for a functor F . We first verify that ϕ restricted to $(NC)_{\leq 1}$ indeed defines a functor F . By the definition of a category and a functor (Definitions 1.1.1 and 1.1.6), for this we need to check that it respects composition: Given morphisms $f_1 : x_0 \rightarrow x_1$ and $f_2 : x_1 \rightarrow x_2$ in $\text{Mor}(\mathcal{C}) = (NC)_1$ with $d_0(f_1) = x_1 = d_1(f_2)$, we have a unique 2-simplex $\sigma = (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2)$ in NC . Write $\phi(\sigma) = (y_0 \xrightarrow{g_1} y_1 \xrightarrow{g_2} y_2) \in (ND)_2$. Then $F(f_1) = F(d_2\sigma) = d_2\phi(\sigma) = g_1$. Likewise $F(f_2) = F(d_0\sigma) = d_0\phi(\sigma) = g_2$ and $F(f_2 \circ f_1) = F(d_1\sigma) = d_1\phi(\sigma) = g_2 \circ g_1 = F(f_2) \circ F(f_1)$. So, when restricted to $(NC)_{\leq 1}$, ϕ indeed comes from a (unique) functor $F : \mathcal{C} \rightarrow \mathcal{D}$. But then $\phi = NF$, as the map $NC_n \rightarrow NC_1 \times_{NC_0} NC_1 \cdots \times_{NC_0} NC_1$ is injective (in fact bijective). \square

We now want to see that the inclusion $N : \text{Cat} \rightarrow \text{sSet}$ as a full subcategory furthermore has a left adjoint h , i.e., that it is a reflective subcategory.

given by considering the free category on the set of objects X_0 with morphisms generated by the 1-simplices X_1 under formal composition \star , subject to the following relation: whenever

$f, g \in X_1$ and there exist a 2-simplex $\sigma \in X_2$ with $d_2\sigma = f$ and $d_0\sigma = g$, then we identify the formal composition $g \star f$ with the 1-simplex $d_1\sigma$.³ Written down more formally:

PROPOSITION 2.2.6. *The nerve functor $N : \text{Cat} \rightarrow \text{sSet}$ has a left adjoint*

$$h : \text{sSet} \rightarrow \text{Cat}.$$

and in particular N preserves limits.

The functor h associates to X the category with objects X_0 , and morphism set \tilde{X}_1 defined by letting the morphisms from $x \rightarrow y$ be freely generated strings $f_n \star \cdots \star f_1$ with $f_i \in X_1$ satisfying that $d_1f_1 = x$, $d_0f_n = y$, and $d_0f_i = d_1f_{i+1}$ for all i , subject to the equivalence relation \sim generated by imposing that $d_1\sigma \sim d_0\sigma \star d_2\sigma$ for all 2-simplices σ :

$$\begin{array}{ccc} & x_1 & \\ d_2\sigma \nearrow & & \searrow d_0\sigma \\ x_0 & \xrightarrow{d_1\sigma} & x_2. \end{array}$$

Hence, as a set, we have

$$\tilde{X}_1 = \left(\coprod_{n \geq 1} X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1 \right) / \sim,$$

where the n^{th} set in the disjoint union is a fiber product of n copies of X_1 . The identity morphisms in this category are given by the \sim -classes 1_x of $s_0x \in X_1$, for $x \in X_0$.

In particular h only depends on the 2-skeleton of X , i.e., $h(X) = h(\text{sk}_2(X))$.

PROOF. It is clear that the above definition defines a category hX with identity maps $1_x = s_0x$ and composition $\circ = \star$. To show that 1_y is a left identity for all $y \in X_0$, fix a 1-simplex $f : x \rightarrow y$ and note that the 2-simplex s_1f ensures commutativity of the following triangle in hX :

$$\begin{array}{ccc} & y & \\ f \nearrow & & \searrow s_0y \\ x & \xrightarrow{f} & y, \end{array}$$

so $f = 1_y \circ f$; and similarly for right identity, using s_0f . It is also clear that \circ is associative, since $(h \circ g) \circ f$ by definition is the class of the string of morphisms $h \star g \star f$, and so is $h \circ (g \circ f)$. As the assignment is obviously functorial, we conclude that we have a well-defined functor $h : \text{sSet} \rightarrow \text{Cat}$.

We want to check that this is indeed left adjoint, i.e., that

$$\text{Hom}_{\text{sSet}}(X, NC) \xrightarrow{\cong} \text{Hom}(hX, C)$$

Each simplex $\Delta^n \rightarrow NC$ is determined by its restriction to $\text{sk}_1\Delta^n \rightarrow NC$, and a map $\text{sk}_1\Delta^n \rightarrow NC$ extends to Δ^n iff it extends to $\text{sk}_2\Delta^n \rightarrow NC$ i.e.,

$$\text{Hom}_{\text{sSet}}(\Delta^n, NC) \xrightarrow{\cong} \text{Hom}_{\text{sSet}}(\text{sk}_2\Delta^n, NC) \hookrightarrow \text{Hom}_{\text{sSet}}(\text{sk}_1\Delta^n, NC).$$

As any simplicial set is a colimit of its simplices by density Proposition 2.1.12, and sk_i preserves colimits (Proposition 2.1.19), we more generally have

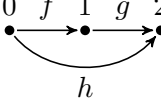
$$\text{Hom}_{\text{sSet}}(X, NC) \xrightarrow{\cong} \text{Hom}_{\text{sSet}}(\text{sk}_2X, NC) \hookrightarrow \text{Hom}_{\text{sSet}}(\text{sk}_1X, NC).$$

Furthermore the condition that simplicial map $F_{\leq 1} : \text{sk}_1X \rightarrow NC$ extends to $F_{\leq 2} : \text{sk}_2X \rightarrow NC$ is that for each 2-simplex σ in X_2 we have $F(d_1\sigma) = F(d_0\sigma) \circ F(d_2\sigma)$. But this is exactly the condition imposed by setting $d_0\sigma \circ d_2\sigma \sim d_1\sigma$, and requiring that F be a functor. Hence

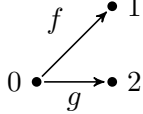
³This means that if for given f, g there are several such σ 's, we identify all morphisms $d_1\sigma$ with each other.

maps of simplicial sets $X \rightarrow N\mathcal{C}$ are in one-to-one correspondence with functors $hX \rightarrow \mathcal{C}$ as wanted. \square

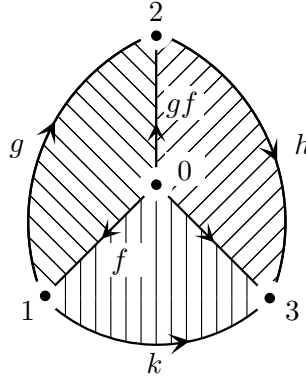
EXAMPLE 2.2.7. Let us describe h on $\partial\Delta^n$ and Λ_i^n both to get some feeling for the construction and as it will be useful for us later.

- (1) $h(\Delta^n) = [n]$.
 (2) $h(\partial\Delta^n) = [n]$ for $n \geq 3$.
 (3) $h(\partial\Delta^2) =$  and no relations among morphisms. (“ $= \partial\Delta^2$ viewed as a category”).

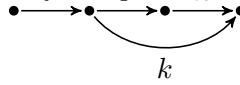
- (4) $h(\Lambda_i^n) = [n]$ for $n \geq 4$.
 (5) $h(\Lambda_i^n) = [n]$ for $0 < i < n$.

- (6) $h(\Lambda_0^2) =$  (“ $= \Lambda_0^2$ viewed as a category”), and likewise $h(\Lambda_2^2)$ is “ Λ_2^2 viewed as a category”.

- (7) $h\Lambda_0^3$ is described by the picture



where, as the “back simplex” is missing, we are not imposing that the composite hg equals k . In other words it has generators $0 \xrightarrow{f} 1 \xrightarrow{g} 2 \xrightarrow{h} 3$ and one relation $hgf = kf$: the



(filled) triangles in the diagram imply that both morphisms hgf and kf are identified in $h\Lambda_0^3$ with the morphism given by the 1-simplex of Λ_0^3 having the 0-simplices 0 and 3 as endpoints. There is a dual description of $h\Lambda_3^3$.

As products both in \mathbf{sSet} and \mathbf{Cat} are described via a cartesian product, one directly checks the following: **[expand]**

PROPOSITION 2.2.8. *The functor $h : \mathbf{sSet} \rightarrow \mathbf{Cat}$ preserves finite products.* \square

REMARK 2.2.9 (An alternative approach to N and h). Note that the existence of the left adjoint h to N shows that \mathbf{Cat} is a reflective subcategory of \mathbf{sSet} , and in particular shows that \mathbf{Cat} is cocomplete by Proposition 1.9.9, identifying the colimit as h applied to the colimit in \mathbf{sSet} .

One can also run this machine in reverse: If one already has established that \mathbf{Cat} is cocomplete by some argument, one can construct N and h using the universal property of the Yoneda embedding: As Δ is a full subcategory of \mathbf{Cat} , and \mathbf{Cat} is cocomplete, the universal property

of the Yoneda embedding (Proposition 1.11.5) implies that we have the inclusion extends to a unique colimit preserving functor

$$h: \mathbf{sSet} = \mathbf{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set}) \rightarrow \mathbf{Cat}.$$

Moreover, by Proposition 1.11.5 the functor h has a right adjoint $N: \mathbf{Cat} \rightarrow \mathbf{sSet}$, given by

$$N\mathcal{C}_n = \mathbf{Hom}_{\mathbf{Cat}}([n], \mathcal{C}).$$

2.3. Cell decompositions

In this section we shall see how the non-degenerate simplices can be used as building blocks for all of X , by gluing simplices along attaching maps, in a way similar to the cells in a CW-decomposition. Later, when we introduce the geometric realization functor we shall make this correspondence precise.

PROPOSITION 2.3.1 (Cell attachments). *Any simplicial set X is isomorphic to the colimit of its skeleta along the canonical map*

$$\operatorname{colim}_{n \geq 0} \operatorname{sk}_n X \xrightarrow{\cong} X,$$

and for $n \geq 0$, $\operatorname{sk}_n X$ is obtained from $\operatorname{sk}_{n-1} X$ via a pushout.⁴

$$\begin{array}{ccc} \coprod_{\sigma \in X_n^{\mathrm{nd}}} \partial \Delta^n & \longrightarrow & \operatorname{sk}_{n-1} X \\ \downarrow & & \downarrow \\ \coprod_{\sigma \in X_n^{\mathrm{nd}}} \Delta^n & \longrightarrow & \operatorname{sk}_n X. \end{array}$$

Here $X_n^{\mathrm{nd}} \subset X_n$ denotes the subset of non-degenerate n -simplices in X .

In particular any simplicial set X can be constructed via (potentially infinitely many) simplex attachments, i.e., pushouts of the form

$$\begin{array}{ccc} \partial \Delta^n & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & Z. \end{array}$$

PROOF. To see that $\operatorname{colim}_{n \geq 0} \operatorname{sk}_n X \xrightarrow{\cong} X$, we make a Yoneda argument: for any simplicial set Y we have

$$\begin{aligned} \mathbf{Hom}_{\mathbf{sSet}} \left(\operatorname{colim}_n \operatorname{sk}_n X, Y \right) &\cong \lim_n \mathbf{Hom}_{\mathbf{sSet}} (\operatorname{sk}_n X, Y) \cong \\ &\lim_n \mathbf{Hom}_{\mathbf{Set}^{\Delta^{\leq n}}} (\operatorname{res}_{\Delta^{\leq n}} X, \operatorname{res}_{\Delta^{\leq n}} Y) \cong \mathbf{Hom}_{\mathbf{sSet}} (X, Y). \end{aligned}$$

For the first pushout diagram we just need to verify that in all degrees k , this is a push-out in \mathbf{Set} . This is clear in degree $k < n$ as the vertical maps are bijections. It is also a pushout for $k = n$, as $(\operatorname{sk}_n X)_n \setminus (\operatorname{sk}_{n-1} X)_n$ consists exactly of the non-degenerate n -simplices, which equals $\coprod_{\sigma \in X_n^{\mathrm{nd}}} (\Delta^n)_n \setminus (\partial \Delta^n)_n$.

It is also a pushout in degrees $k > n$ as each simplex in $(\operatorname{sk}_n X)_k \setminus (\operatorname{sk}_{n-1} X)_k$ or in $(\Delta^n)_k \setminus (\partial \Delta^n)_k$ is degenerate, and by Lemma 2.1.14 such simplices are uniquely obtained from non-degenerate ones in degree $\leq n$.

The statement about attachments is just a restatement of the previous ones. \square

⁴By convention $\operatorname{sk}_{-1} X$ is the empty simplicial set, as well as $\partial \Delta^0$.

2.3.1. Products. In preparation for the next section on geometric realization, let us also look at non-degenerate simplices in some products.

EXAMPLE 2.3.2. Let us work out the non-degenerate simplices in $\Delta^1 \times \Delta^1$. We have already seen that $\Delta^1 \times \Delta^1 = N([1] \times [1])$, as N commutes with products. We can easily describe the category $[1] \times [1]$: There are 4 objects, 5 non-identity morphisms, and 2 possible compositions, not involving identities. Hence we see that the corresponding simplicial set looks as follows

$$\begin{array}{ccc} (0, 1) & \longrightarrow & (1, 1) \\ \uparrow & \nearrow & \uparrow \\ (0, 0) & \longrightarrow & (1, 0) \end{array}$$

with 4 non-degenerate 0-simplices 5 non-degenerate 1-simplices, and 2 non-degenerate 2-simplices. Note in particular that the non-degenerate 2-simplices correspond to $(0 \leq 0 \leq 1) \times (0 \leq 1 \leq 1)$ and $(0 \leq 1 \leq 1) \times (0 \leq 0 \leq 1)$. In particular, both of them are the product of a pair of degenerate 2-simplices in Δ^1 and Δ^1 , but the resulting product 2-simplex is non-degenerate because the two factors are degenerate “for different reasons” (are in the image of different degeneracy maps).⁵

EXAMPLE 2.3.3 (Non-degenerate simplices of $\Delta^n \times \Delta^m$). Note that an object in $[n] \times [m]$ is a pair (i, j) with $0 \leq i \leq n$ and $0 \leq j \leq m$, which we can picture as picking a vertex in a grid of size $(n+1) \times (m+1)$. A non-identity morphism in $[n] \times [m]$ amounts to picking two distinct pairs (i, j) and (i', j') with $i \leq i'$ and $j \leq j'$. More generally a composite of k non-identity morphisms in $[n] \times [m]$, i.e., a non-degenerate k -simplex in $N([n] \times [m]) = \Delta^n \times \Delta^m$ is then a choice of $(k+1)$ distinct vertices in this grid in increasing order, from bottom-left to top right.

In particular the non-degenerate $(n+m)$ -simplices in $\Delta^n \times \Delta^m = N([n] \times [m])$ correspond to paths from $(0, 0)$ to (n, m) where we at each step we increase one of the two coordinates by one. These are in one-to-one correspondence with (n, m) -shuffles, and thus there are $\binom{n+m}{n}$ of them.

For example the $n+1$ non-degenerate $(n+1)$ -simplices in $\Delta^n \times \Delta^1$, correspond to the sequences

$$\begin{aligned} (0, 0) &< (0, 1) < (1, 1) < (2, 1) < \cdots < (n, 1) \\ (0, 0) &< (1, 0) < (1, 1) < (2, 1) < \cdots < (n, 1) \\ &\vdots \\ (0, 0) &< (1, 0) < \cdots < (n, 0) < (n, 1) \end{aligned}$$

which relate to the “prisms” which occur in first topology courses when proving homotopy invariance of singular homology. (See also [GJ99, Prop I.4.2].)

2.4. The functors geometric realization $|\cdot|$ and singular functor Sing_\bullet

We are now ready to construct the comparison functors between sSet and Top . Luckily with the Yoneda extension theorem from Proposition 1.11.5 this turns out to be formal from what it does on the n -simplex. First a definition.

DEFINITION 2.4.1. Denote the *topological* n -simplex as

$$\Delta_{\text{top}}^n = \left\{ \lambda_0 e_0 + \lambda_1 e_1 + \cdots + \lambda_n e_n \in \mathbb{R}^{n+1} \mid \lambda_i \geq 0, \sum_{i=0}^n \lambda_i = 1 \right\}.$$

⁵This example shows the utility of degenerate simplices and degeneracies in the theory of simplicial sets: it makes the product of simplicial sets agree with the product of the topological spaces associated along geometric realisation.

DEFINITION 2.4.2. Note that we have a functor $\Delta \rightarrow \text{Top}$ on objects given by $[n] \mapsto \Delta_{\text{top}}^n$, and on morphisms given by associating to $\theta : [n] \rightarrow [m]$ the continuous map $\theta_* : \Delta_{\text{top}}^n \rightarrow \Delta_{\text{top}}^m$ given by

$$\theta_* \left(\sum_{i=0}^n \lambda_i e_i \right) = \sum_{i=0}^n \lambda_i e_{\theta(i)}$$

In other words the above functor defines a cosimplicial object in Top (aka a cosimplicial space). This cosimplicial space is usually denoted $\Delta_{\text{top}}^\bullet$.

EXERCISE 2.4.3. Draw this in low dimensions, and check that it is indeed a functor.

Definition 2.4.2 gives us a way to associate the topological n -simplex Δ_{top}^n to $[n]$, which we can also identify with Δ^n via the Yoneda embedding. Yoneda extension now produces a pair of adjoint functors, which will allow us to relate simplicial sets to topological spaces.

Proposition 1.11.5 now immediately implies the following:

PROPOSITION 2.4.4. *The Yoneda extension of the functor $\Delta \rightarrow \text{Top}$, $[n] \mapsto \Delta_{\text{top}}^n$, defines a pair of adjoint functors*

$$\text{sSet} \xrightleftharpoons[\text{Sing}_\bullet]{|\cdot|} \text{Top},$$

The left adjoint $|\cdot| : \text{sSet} \rightarrow \text{Top}$ is called geometric realization and given by

$$|X| = \text{colim}_{\sigma_n \in \text{el } X} \Delta_{\text{top}}^n.$$

The right adjoint $\text{Sing}_\bullet(-) : \text{Top} \rightarrow \text{sSet}$ is called the singular functor and given by

$$\text{Sing}_\bullet(Y) = ([n] \mapsto \text{Hom}_{\text{Top}}(\Delta_{\text{top}}^n, Y))$$

i.e., $\text{Sing}_\bullet(Y) : \Delta^{\text{op}} \rightarrow \text{Set}$ is the composition of $\Delta^{\text{op}} \rightarrow \text{Top}^{\text{op}}$ from Definition 2.4.2 with $\text{Hom}_{\text{Top}}(-, Y) : \text{Top}^{\text{op}} \rightarrow \text{Set}$. \square

Note that by definition $|\Delta^n| = \Delta_{\text{top}}^n$, so can retire the notation Δ_{top}^n and just use $|\Delta^n|$ instead.

REMARK 2.4.5. Let us spell out how " $|\Delta^\bullet|$ " is interpreted as a functor $\text{el}(X) \rightarrow \text{Top}$. On objects it sends $\sigma_n \in X_n$ to $|\Delta^n|$, and on morphisms it goes as follows: A morphism $\sigma_n \rightarrow \tau_m$ in $\text{el}(X)$ arises as a map $\theta : [n] \rightarrow [m]$ in Δ such that $\sigma \in X_n$, $\tau \in X_m$ and $\theta^*(\tau) = \sigma$, and we send this morphism to the map $\theta_* : |\Delta^n| \rightarrow |\Delta^m|$ from Definition 2.4.2.

REMARK 2.4.6. Note that for a space Y we have $C_n^{\text{sing}}(Y; \mathbb{Z}) = \mathbb{Z} \text{Sing}_n(Y)$, and the differential in the singular chain complex is given by $d = \sum_i (-1)^i d_i$; this shows how $\text{Sing}_\bullet(Y)$ is a non-linear version of singular chain complex defining singular homology. We will see that $\text{Sing}_\bullet(Y)$ describes the entire weak homotopy type of Y , and show how to obtain invariants such as homotopy groups of Y directly from $\text{Sing}_\bullet(Y)$.

REMARK 2.4.7. For $X \in \text{sSet}$, the geometric realization $|X|$ can also be described more concretely as follows:

$$|X| = \left(\prod_{n \geq 0} X_n \times |\Delta^n| \right) / \sim$$

where \sim is the equivalence relation generated by the identifications $(\theta^*(x), \lambda) \sim (x, \theta_*(\lambda))$, for varying $\theta : [n] \rightarrow [m]$, $\lambda \in |\Delta^n|$ and $x \in X_m$.

Using the notion of a categorical notion of a coend, which we will introduce in Section 5.3.1, this can more succinctly be written as

$$|X| = X_\bullet \otimes_\Delta |\Delta^\bullet|.$$

REMARK 2.4.8. In the coend language of Remark 2.4.7 the adjunction from Proposition 2.4.4 becomes a “Hom-tensor” adjunction

$$\text{Hom}_{\text{Top}}(X_\bullet \otimes_\Delta |\Delta^\bullet|, Y) \simeq \text{Hom}_{\text{sSet}}(X_\bullet, \text{Hom}_{\text{Top}}(|\Delta^\bullet|, Y))$$

REMARK 2.4.9. It is worth checking the isomorphism

$$\text{Hom}_{\text{Top}}(|X_\bullet|, Y) \xrightarrow[\simeq]{\phi} \text{Hom}_{\text{sSet}}(X_\bullet, \text{Sing}_\bullet(Y))$$

implied by Proposition 2.4.4 is the “obvious” one: The map ϕ sends $f : |X_\bullet| \rightarrow Y$, to the map $\phi(f)_n : X_n \rightarrow \text{Sing}_n(X)$, which assigns to $x \in X_n$ the map that sends $\lambda \in |\Delta^n|$ to $(\phi(f)_n(x))(\lambda) = f((x, \lambda)) \in Y$. The inverse ψ takes $g : X_\bullet \rightarrow \text{Sing}_\bullet(Y)$ to $\psi(g) : |X_\bullet| \rightarrow Y$ given by $\psi(g)(x, \lambda) = g(x)(\lambda) \in Y$.

In other words, for each n we use the adjunction in spaces

$$\text{Hom}_{\text{Top}}(X_n \times |\Delta^n|, Y) \cong \text{Hom}_{\text{Top}}(X_n, \text{map}(|\Delta^n|, Y)),$$

and note that a collection of such maps on the right glue together to form a map $f : |X_\bullet| \rightarrow Y$ if and only if $\{\phi(f)_n : X_n \rightarrow \text{Hom}_{\text{Top}}(|\Delta^n|, Y)\}_n$ defines a morphism of simplicial sets.

PROPOSITION 2.4.10. *The geometric realisation $|X|$ of a simplicial set X has a natural structure of CW complex, whose n -cells correspond to the non-degenerate n -simplices in X .*

PROOF. As $|\cdot|$ is a left adjoint it commutes with colimits. In particular

$$|X| \cong \text{colim}_n |\text{sk}_n X|$$

Furthermore Proposition 2.3.1, combined again with the fact that $|\cdot|$ commutes with colimits, implies that we have a pushout square:

$$\begin{array}{ccc} \coprod_{\sigma \in X_n^{\text{nd}}} |\partial \Delta^n| & \longrightarrow & |\text{sk}_{n-1} X| \\ \downarrow & & \downarrow \\ \coprod_{\sigma \in X_n^{\text{nd}}} |\Delta^n| & \longrightarrow & |\text{sk}_n X|. \end{array}$$

This puts a CW structure on $|X|$. □

EXAMPLE 2.4.11. Consider $\mathcal{B}\mathbb{Z}/2$, i.e., the category with one object and two morphisms 1 and g , where $g^2 = 1$. For all n there is exactly one non-degenerate n -simplex which we can write (g, g, \dots, g) , n times. The boundary maps d_0 and d_n send the non-degenerate n -simplex to the non-degenerate $(n-1)$ -simplex, whereas d_i for $0 < i < n$ maps it to a degenerate simplex.

In the geometric realization, the 2-skeleton is obtained by gluing a 2-simplex to S^1 along the map that wraps two of the edges (the first and the third) the same way around S^1 , and sends the other edge (the second) constantly to the point representing the unique 0-simplex. A similar but more involved description holds for the attaching maps of the higher simplices. One checks that $\mathcal{B}\mathbb{Z}/2$ is homotopy equivalent to \mathbf{RP}^∞ : see also Exercise 2.12.1.

REMARK 2.4.12. The composite functor $B(-) = |N(-)| : \text{Cat} \xrightarrow{N} \text{sSet} \xrightarrow{|\cdot|} \text{Top}$ is also called the *classifying space* functor, in particular when evaluated on categories of the form $\mathcal{B}G$. (The name stems from the fact that, when G is a group, the space $B(\mathcal{B}G)$, also denoted BG , classifies principal G -bundles; more about this later!). Sometimes this notation becomes a bit heavy, and the classifying space functor is therefore abbreviated in the literature as just $|\cdot| : \text{Cat} \rightarrow \text{Top}$, by an abuse of notation. In yet other parts of the literature the word “space” is taken to mean a simplicial set, and it is then natural to use the notation $|\cdot|$ for the functor $\text{Cat} \xrightarrow{N} \text{sSet}$, here denoted N . For the even more advanced, distinguishing at all between a category and its image in sSet under the fully faithful embedding $N : \text{Cat} \hookrightarrow \text{sSet}$ is rather cumbersome, and one can simply get rid of both N and $|\cdot|$ altogether, by only considering simplicial sets (some of which are (nerves of) categories). The confluence of categories and

topology has reached such a pinnacle that one does not even notationally distinguish between the two!!

2.5. Geometric realization preserves products

It is also true that geometric realization preserves products, suitably interpreted, though this is not formal.⁶ We will now discuss this. The key point is that it holds for products of simplices:

LEMMA 2.5.1.

$$|\Delta^n \times \Delta^m| \xrightarrow{\cong} |\Delta^n| \times |\Delta^m|$$

IDEA OF PROOF. As the map is continuous by construction, and the source is compact and the target is Hausdorff, it is enough to check that the map is a bijection.

This essentially follows from our description of the non-degenerate simplices in $N([n] \times [m])$ in Example 2.3.3. This provides a triangulation of $|\Delta^n \times \Delta^m|$ with top cells corresponding to the $\binom{n+m}{n}$ (n, m) -shuffles. This is identical to the standard triangulation of $|\Delta^n| \times |\Delta^m|$.

[Picture: We imagine an n times m grid and a path from $(0, 0)$ to (n, m) takes $m + n$ steps going up and right. We now have to decide where to place the n steps in which we go right, by selecting n moments in a sequence of $n + m$ (e.g., the sublist $1, 2, 3, 4, \dots, n$ would mean that we first go all the way right, and then all the way up.)

(Note again that when $m = 1$ this is just the “prism” division of $|\Delta^n| \times |\Delta^1|$ as a union of $(n + 1)$ many $(n + 1)$ -simplices, which occurs when proving homotopy invariance of homology.)

The details are left as an exercise. See also Exercise 2.12.13 and [GZ67, p. III.3.4]. \square

THEOREM 2.5.2 (Realization preserves products, v.1). *Let X and Y be simplicial sets, and assume that Y is finite, in the sense that there are only finitely many non-degenerate simplices in Y , considering all dimensions of simplices at the same time.⁷ Then*

$$|X \times Y| \xrightarrow{\cong} |X| \times |Y|$$

PROOF. First note that the statement holds for X and Y of the form Δ^n and Δ^m , by Lemma 2.5.1. Also note that $X \cong \operatorname{colim}_{\sigma_n \in \operatorname{el}(X)} \Delta^n$ by the density theorem, that $|-|$ commutes with colimits being a left adjoint, and that by definition $|X| \cong \operatorname{colim}_{\sigma_n \in \operatorname{el}(X)} |\Delta^n|$. We now assume that Y is a *finite* simplicial set: as a consequence of Proposition 2.3.1, $|Y|$ is a finite CW complex, and in particular it is a compact space. We can then calculate:

$$\begin{aligned} |X \times Y| &\cong \left| \operatorname{colim}_{\sigma_n \in \operatorname{el}(X)} \operatorname{colim}_{\tau_m \in \operatorname{el}(Y)} \Delta^n \times \Delta^m \right| \\ &\cong \operatorname{colim}_{\sigma_n \in \operatorname{el}(X)} \operatorname{colim}_{\tau_m \in \operatorname{el}(Y)} |\Delta^n \times \Delta^m| \\ &\cong \operatorname{colim}_{\sigma_n \in \operatorname{el}(X)} \operatorname{colim}_{\tau_m \in \operatorname{el}(Y)} (|\Delta^n| \times |\Delta^m|) \\ &\cong \operatorname{colim}_{\sigma_n \in \operatorname{el}(X)} (|\Delta^n| \times |Y|) \\ &\cong \left(\operatorname{colim}_{\sigma_n \in \operatorname{el}(X)} |\Delta^n| \right) \times |Y| \\ &\cong |X| \times |Y|. \end{aligned}$$

In the fourth and fifth line we use that $|\Delta^n|$ and $|Y|$ are compact, as otherwise there is only a set-theoretic bijection and the two topologies do not necessarily agree. [**also use colimits**

⁶By not “formal” here we mean that there is no simple proof such as “products are limits, and geometric realisation is a right adjoint, hence it preserves products”, since geometric realisation is a *left* adjoint but not a right adjoint. An ad hoc argument has to be found!

⁷This condition is *not the same* as asking Y_n to be finite for each n (we would say then that Y is *locally finite*).

commute with products in \mathbf{ss} , as true in sets, which is again because $(-) \times Y$ is left adjoint. ref to earlier? Example 1.4.13(2)] \square

Theorem 2.5.2 can be weakened to Y only having finitely many cells in each dimension, but is not true without it. This is due to the fact, which the reader probably has encountered in earlier courses on elementary homotopy theory, say [Hat02], that the product in \mathbf{Top} of CW complexes will not carry the standard topology on CW complexes, which has the property that a set is open if and only if its restriction to every compact set is open, i.e., the topology. However using Proposition 1.4.15 we get.

THEOREM 2.5.3 (Realization preserves products, v.2). *Let $X, Y \in \mathbf{sSet}$, then $|X \times Y| \xrightarrow{\cong} |X| \times_k |Y|$, if the product on the left is taken in the category of compactly generated spaces $k\mathbf{Top}$.*

PROOF. The argument in Theorem 2.5.2 shows that the map is continuous bijection, but where the left-hand side is topologized by the topology where a set is open if its restriction to every compact is open, which is the same as the topology on $|X| \times_k |Y|$, as explained in Proposition 1.4.15. [write more formally?] \square

2.6. Mapping spaces and simplicial homotopy

Recall that $\Delta^n = N([n])$, and d^i is the map $[n-1] \rightarrow [n]$ that skips i , so that e.g., $N(d^1) : N([0]) \rightarrow N([1])$ geometrically realises to the embedding of a point $\{0\} = |\Delta^0|$ in the interval $I = [0, 1] \cong |\Delta^1|$ at 0. There is hence an obvious candidate definition for simplicial homotopy:

DEFINITION 2.6.1. Let X and Y be simplicial sets. We say that $f : X \rightarrow Y$ is homotopic to $g : X \rightarrow Y$ if there exists a map $H : X \times \Delta^1 \rightarrow Y$ such that

$$X \cong X \times \Delta^0 \xrightarrow{1 \times d^1} X \times \Delta^1 \xrightarrow{H} Y$$

equals f and

$$X \cong X \times \Delta^0 \xrightarrow{1 \times d^0} X \times \Delta^1 \xrightarrow{H} Y$$

equals g .

For general simplicial sets X this definition however has the alarming feature that simplicial homotopy is *not* an equivalence relation in general, due to the “direction” of the homotopy! (I.e., it may not be symmetric.)

EXAMPLE 2.6.2. By Proposition 2.2.5, $\mathrm{Hom}(\Delta^1, \Delta^1) = \mathrm{Hom}([1], [1])$ consists of 3 maps: the identity map, the constant map 0, and the constant map 1. In particular, while the map $Nd^1 : \Delta^0 \rightarrow \Delta^1$ is homotopic to Nd^0 via the identity map of Δ^1 , Nd^0 is *not* homotopic to Nd^1 , as there is no map Δ^1 to Δ^1 flipping the endpoints.

We will describe in the next sections how to rectify this problem by restricting our attention to “nice” simplicial sets, called Kan complexes (or ∞ -groupoids), where such issues cannot occur. However this notion of “niceness” means having sufficiently many simplices to solve natural extension problems. The payback is that “nice” simplicial sets, in this sense, have to be rather large in general, just like an injective resolution of a module or chain complex in homological algebra, is in general much larger than the thing we started out with.

Before moving on to this, let us introduce the internal hom object in simplicial sets:

DEFINITION 2.6.3. For simplicial sets $X, Y \in \mathbf{sSet}$ define a new simplicial set $\mathrm{map}(X, Y)_\bullet$, the *simplicial mapping space*, by setting, for $n \geq 0$, $\mathrm{map}(X, Y)_n = \mathrm{Hom}_{\mathbf{sSet}}(X \times \Delta^n, Y)$, with the simplicial structure induced from the cosimplicial structure on Δ^\bullet , i.e.,

$$\mathrm{map}(X, Y)_\bullet = \mathrm{Hom}_{\mathbf{sSet}}(X \times \Delta^\bullet, Y).$$

REMARK 2.6.4. Note that we have in particular

$$\mathrm{map}(X, Y)_0 = \{ \text{simplicial maps } X \rightarrow Y \};$$

$$\mathrm{map}(X, Y)_1 = \{ \text{simplicial homotopies } X \times \Delta^1 \rightarrow Y \}.$$

Simplicially we are hence encoding the mapping space $\mathrm{map}(X, Y)_\bullet$ a bit differently than topologically. We let the points be *all* maps, and then, instead of introducing a topology, we introduce higher order simplices in order to say which maps are homotopic to which, which homotopies are homotopic to which, etc.

Let us check that this makes $\mathrm{map}(Y, -)$ into a right adjoint to $(-) \times Y$:

PROPOSITION 2.6.5. *We have an adjunction isomorphism*

$$\mathrm{Hom}_{\mathrm{sSet}}(X \times Y, Z) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{sSet}}(X, \mathrm{map}(Y, Z))$$

natural in X , Y , and Z .

PROOF. Naturality in X , Y and Z is evident. Both the right-hand side and the left-hand side send colimits in the first variable $X \in \mathrm{sSet}$ to limits in the category Set ; hence it is enough to check the bijection between hom-sets with $X = \Delta^n$.

But in this case the lefthand side reads $\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \times Y, Z)$ and the right-hand side reads $\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, \mathrm{map}(Y, Z)) = \mathrm{map}(Y, Z)_n$, so this is by definition. \square

REMARK 2.6.6. The reader may wonder if a functor category always inherits the property of being complete and cartesian closed from the target. This turns out to be the case, as is explained here

<https://mathoverflow.net/questions/104152/exponentials-in-functor-categories>

The following proposition says that the nerve functor takes the internal hom in Cat to the internal hom in sSet .

PROPOSITION 2.6.7. *Let \mathcal{C}, \mathcal{D} be small categories. Then the evaluation functor $\mathcal{C} \times \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{D}$ induces via adjunction an isomorphism of simplicial sets*

$$N(\mathrm{Fun}(\mathcal{C}, \mathcal{D})) \cong \mathrm{map}(N\mathcal{C}, N\mathcal{D}).$$

PROOF. We see directly that

$$N(\mathrm{Fun}(\mathcal{C}, \mathcal{D} \mathrm{map}(N\mathcal{C}, N\mathcal{D})))_n = \mathrm{Hom}_{\mathrm{sSet}}(N\mathcal{C} \times \Delta^n, N\mathcal{D}) = \mathrm{Hom}_{\mathrm{Cat}}(\mathcal{C} \times [n], \mathcal{D})$$

which is a string of n natural transformations, which is the same as $N(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))_n$.

For a more "no-hands" arguments one can also argue as follows:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{sSet}}(-, N\mathrm{Fun}(\mathcal{C}, \mathcal{D})) &\cong \mathrm{Hom}_{\mathrm{Cat}}(h(-) \times \mathcal{C}, \mathcal{D}) \cong \mathrm{Hom}_{\mathrm{Cat}}(h(- \times N\mathcal{C}), \mathcal{D}) \\ &\cong \mathrm{Hom}_{\mathrm{sSet}}(- \times N\mathcal{C}, N\mathcal{D}) \cong \mathrm{Hom}_{\mathrm{sSet}}(-, \mathrm{map}(N\mathcal{C}, N\mathcal{D})) \end{aligned}$$

where he have used that h commutes with finite products (Proposition 2.2.8). \square

2.7. Horn filling conditions and Kan complexes

In this section we will introduce the different horn filling conditions. Let us first introduce them, before we try to discuss what they "mean".

DEFINITION 2.7.1 (Kan complex, ∞ -category, etc.).

- (1) X is said to satisfy the *horn filling condition* if for any $n \geq 1$ and $0 \leq k \leq n$, and for any $\Lambda_k^n \rightarrow X$, there exists a map $\Delta^n \rightarrow X$ making the following diagram commute:⁸

$$(2.7.1) \quad \begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \downarrow & \nearrow \exists & \\ \Delta^n & & \end{array}$$

- (2) X is said to satisfy the *inner horn filling condition* if for any $n \geq 2$ and $0 < k < n$, and for any $\Lambda_k^n \rightarrow X$, (i.e., only for “inner horns”) there exists a map $\Delta^n \rightarrow X$ making the diagram (2.7.1) commute.

Simplicial sets that satisfy the horn filling condition are called *Kan complexes* (or ∞ -groupoids). If they just satisfy the inner horn filling condition are called *weak Kan complexes*, *quasi-categories*, or ∞ -categories.

We say that a simplicial set X satisfies the *unique horn filling condition* if the filler in Definition 2.7.1(1) is unique when $n \geq 2$. We define the *unique inner horn filling condition* similarly.

For $n = 2$ the unique inner horn filling condition says that given 1-simplices $f : x \rightarrow y$ and $g : y \rightarrow z$ there exists a unique 2-simplex filling in the diagram

$$\begin{array}{ccc} & y & \\ f \nearrow & & \searrow g \\ x & & z \end{array}$$

If X is the nerve of a category, then it is clear that such a unique 2-simplex exists namely $(x \xrightarrow{f} y \xrightarrow{g} z)$. And it is not hard to see that nerves of categories satisfy the unique inner horn filling condition for higher simplices as well as we shall see in Proposition 2.7.3 (the point being that the n -simplex $(x_0 \xrightarrow{f_1} x_1 \rightarrow \cdots \xrightarrow{f_n} x_n)$ is uniquely determined by its edges f_1, \dots, f_n). The inner horn filling condition hence says that we can compose morphisms, but gives up on uniqueness—however the fact that we fill in *all* inner horns nevertheless turns out to ensure that we get a well defined composition up to “a contractible space of choices”. It also turns out to be enough structure to define a whole mapping space from x to y , mimicking the topological space of maps in classical topology.

If we think of what the horn filling conditions would say *after* taking geometric realizations, they would be ways of saying that the map $|X| \rightarrow *$ is a Serre fibration, letting n vary and for each n picking just one k arbitrarily. Of course, this property is satisfied by any topological space Y replacing $|X|$. The two extreme values $k = 0, n$ would not play a special role. And the lift would also be far from unique. The Kan property is a formulation of this property at the level of simplicial sets, and Kan complexes turn out to model topological spaces — this is a fundamental theorem, see Section 2.10. For now, let us just record that the above argument, combined with adjointness, implies the following proposition.

PROPOSITION 2.7.2. $\text{Sing}_\bullet(Y)$ is a Kan complex for all $Y \in \text{Top}$.

PROOF. Filling in the dotted arrow in the commutative diagram

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & \text{Sing}_\bullet(Y) \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

⁸Note that for $n = 1$ the horn filling condition is always satisfied.

is by adjunction equivalent to filling in the dotted arrow in the following diagram in spaces

$$\begin{array}{ccc} |\Lambda_k^n| & \longrightarrow & Y \\ \downarrow & \nearrow \text{dotted} & \\ |\Delta^n| & & \end{array}$$

which is possible as $|\Lambda_k^n|$ is a (deformation) retract of $|\Delta^n|$. \square

2.7.1. Horn filling conditions for nerves. In this subsection we will see how to identify \mathbf{Cat} inside \mathbf{sSet} as the subcategory of simplicial sets with the unique inner horn filling condition, and also see that $N\mathcal{C}$ will satisfy the general horn filling condition if and only if \mathcal{C} is a groupoid, i.e., a category in which every morphism is invertible.

PROPOSITION 2.7.3. *[Unique inner horn filling condition] A simplicial set X is in the essential image of $N : \mathbf{Cat} \rightarrow \mathbf{sSet}$ iff X satisfies the unique inner horn filling condition. Hence \mathbf{Cat} identifies with the full subcategory of \mathbf{sSet} with objects simplicial sets satisfying the unique inner horn filling condition.*

PROOF. Let us first check that any simplicial set of the form $N\mathcal{C}$ has the unique inner horn filling condition:

Or, said in a more fancy-pants way: By adjunction, the lifting problem

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & N\mathcal{C} \\ \downarrow & \nearrow \text{dotted} & \\ \Delta^n & & \end{array}$$

for $n \geq 2$ and $0 < i < n$ is equivalent to the lifting problem

$$\begin{array}{ccc} [n] = h\Lambda_i^n & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow \text{dotted} & \\ [n] = h\Delta^n & & \end{array}$$

which of course has a unique solution, where we have used $[n] = h\Lambda_i^n = h\Delta^n$ by Example 2.2.7.

Conversely, suppose that X is a simplicial set with the unique inner horn filling condition. We need to define a category \mathcal{C} such that $N\mathcal{C} \cong X$. For this we can take $\mathcal{C} = hX$.

Let us just for concreteness spell out what \mathcal{C} is in this case. Consider the diagram

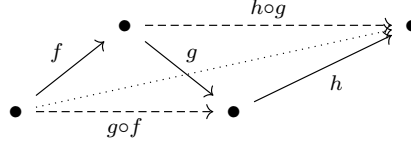
$$\begin{array}{ccc} \Lambda_1^2 & \xrightarrow{(f,g)} & X \\ \downarrow & \nearrow \text{dotted} & \\ \Delta^2 & & \end{array}$$

and define the composition $g \circ f$ as d_1 of the filler

$$\begin{array}{ccc} & y & \\ f \nearrow & & \searrow g \\ x & \xrightarrow{\text{dotted } g \circ f} & z \end{array}$$

which by our assumption exists and is unique. This assignment indeed defines a category \mathcal{C} , with X_1 as morphisms. $s_0x \in X_1$ provides identity maps 1_x for all $x \in X_0$ and associativity

follows from the uniqueness of dotted filler in the following diagram:



That $X \cong N\mathcal{C}$ again follows by the uniqueness of the filler map. \square

PROPOSITION 2.7.4 (Horn filling condition for nerves). *The nerve $N\mathcal{C}$ satisfies the horn filling condition if and only if \mathcal{C} is a groupoid.*

PROOF. Note by adjointness that a lift in the diagram

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & N\mathcal{C} \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

is equivalent to a lift in

$$\begin{array}{ccc} h\Lambda_k^n & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \\ h\Delta^n & & \end{array}$$

Furthermore recall by Example 2.2.7 that $h(\Lambda_i^n) = [n]$ for almost all $n \geq 2$ and $0 \leq i \leq n$, making the lifting problem trivial, except the four outer horns $h(\Lambda_0^2)$, $h(\Lambda_2^2)$, $h(\Lambda_0^3)$, $h(\Lambda_3^3)$.

By Example 2.2.7, the lifting problem with $h(\Lambda_0^2)$ is solvable if for any two morphisms $f : x \rightarrow y$ and $g : x \rightarrow z$ in \mathcal{C} there exists $h : y \rightarrow z$ with $h \circ f = g$

$$\begin{array}{ccc} & \bullet & \\ f \uparrow & \nearrow \exists h & \\ \bullet & \xrightarrow{g} & \bullet \end{array}$$

This holds if and only if every morphism $f : x \rightarrow y$ in \mathcal{C} has a left inverse (this is clearly sufficient, and setting $g = 1_x$ for some object x shows it is also necessary).

Similarly the lifting problem with $h(\Lambda_2^2)$ is solvable if and only if every morphism $f : x \rightarrow y$ in \mathcal{C} has a right inverse. Hence an extension exist for all diagrams of type $h(\Lambda_0^2)$ and $h(\Lambda_2^2)$ if and only if every morphism in \mathcal{C} is invertible, and in this case the extension is unique. This already forces \mathcal{C} to be a groupoid.

To verify that extensions always exist and are unique for $n \geq 2$ in the groupoid case, let us check what the extension condition says for $h(\Lambda_0^3)$, a category we already described in Example 2.2.7 (you checked the details of that example, right?): In this case, the requirement on \mathcal{C} is that if we are given morphisms $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$ and $y \xrightarrow{k} w$ with $h \circ g \circ f = k \circ f$, then $h \circ g = k$. In other words the condition is equivalent to that \mathcal{C} satisfies right cancellation, and this is always true in a groupoid. The extension is obviously unique as $\begin{array}{ccccccc} & 0 & & 1 & & 2 & & 3 \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \end{array}$ is a subcategory of $h(\Lambda_0^3)$. Similarly the lifting problem for $h(\Lambda_3^3)$ is equivalent to left cancellation, and the extension is unique for the same reason as before. \square

REMARK 2.7.5. The above remark explains why a Kan complex, i.e., “space”, is sometimes referred to as an ∞ -groupoid: A groupoid is a category such that also the outer horns admit fillers. Analogously an ∞ -groupoid is an ∞ -category such that the outer horns admit fillers.

2.7.2. The fundamental groupoid: the functor h on Kan complexes.

PROPOSITION 2.7.6. *Suppose that X is a Kan complex, then hX (a priori just a category) is a groupoid, called the fundamental groupoid of X , and also denoted $\pi(X)$.*

Furthermore we have the following alternative description: For X is a Kan complex, define a category $\pi(X)$ with objects X_0 and morphisms given by

$$\text{Hom}_{\pi(X)}(x, y) \cong \{f \in X_1 \mid d_1 f = x, d_0 f = y\} / \sim^h$$

where \sim^h is an equivalence relation defined by $f \sim^h g$ iff there exists $\sigma \in X_2$ with $\partial(\sigma) := (d_0 \sigma, d_1 \sigma, d_2 \sigma) = (s_0 y, g, f)$. Composition is given by $g \circ f = d_1 \sigma$, where $\sigma \in X_2$ is any simplex satisfying $d_2 \sigma = f$ and $d_0 \sigma = g$ (such σ exist by the Kan condition):

$$\begin{array}{ccc} & y & \\ f \nearrow & \sigma & \searrow g \\ x & \xrightarrow{d_1 \sigma} & z. \end{array}$$

Then the identity map $X_{\leq 1} \rightarrow X_{\leq 1}$ induces an isomorphism $\pi(X) \xrightarrow{\cong} h(X)$.

PROOF. Let $f \in X_1$. Then f is invertible in hX , since by the Kan condition on Λ_0^2 there exists $\sigma \in X_2$, so that $d_0 \sigma \circ f = 1$, so f has a left inverse in hX , and similarly for a right inverse.

Now let us check the isomorphism $\pi(X) \xrightarrow{\cong} h(X)$. As we can always find σ as in the diagram above, any morphism in hX can be represented by a single element in X_1 , i.e., we have a surjective map $X_{\leq 1} \rightarrow h(X)$.

We need to check that \sim^h agrees with the \sim from the description of h . Let us start by verifying that \sim^h is itself an equivalence relation. Obviously $f \sim^h f$ via $s_1 f$. Also $f \sim^h g$ implies $g \sim^h f$ as we can find a 3-simplex τ filling in the following Λ_3^3

$$\begin{array}{ccc} & y & \\ f \nearrow & \downarrow 1 & \nwarrow 1 \\ & y & \\ \sigma \nearrow & \nwarrow s_1 g & \\ x & \xrightarrow{g} & y \end{array}$$

and $g \sim^h f$ is witnessed by $d_3 \tau$.

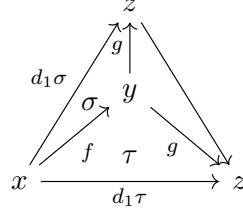
Likewise it is transitive: if $f \sim^h g$ and $g \sim^h h$, we can fill in the following Λ_2^3 with a 3-simplex τ and use $d_2 \tau$ to witness $f \sim^h h$:

$$\begin{array}{ccc} & y & \\ f \nearrow & \downarrow 1 & \nwarrow 1 \\ & y & \\ \sigma \nearrow & \nwarrow \sigma' & \\ x & \xrightarrow{h} & y. \end{array}$$

We now verify that \sim agrees with \sim^h . If $f \sim^h g$, then there exists a 2-simplex σ in the following diagram.

$$\begin{array}{ccc} & y & \\ f \nearrow & \sigma & \nwarrow 1 \\ x & \xrightarrow{g} & y \end{array}$$

so $f \sim 1_y \circ f \sim g$. On the other hand if σ and τ are two 2-simplices such that $d_2\sigma = d_2\tau = f$ and $d_0\sigma = d_0\tau = g$, then we can fill in the following Λ_1^3



with a 3-simplex, whose d_1 -face shows that $d_1\sigma \stackrel{h}{\sim} d_1\tau$. And since $d_1\sigma \sim d_1\tau$ generates \sim , we conclude that \sim and $\stackrel{h}{\sim}$ have to be the same equivalence relation. \square

COROLLARY 2.7.7. *Suppose X is a Kan complex, then $\pi(X)$ is a category with set of components $\pi_0(\pi(X))$ given by X_0 modulo the equivalence relation that $x \sim y$ if there exists $(f : x \rightarrow y) \in X_1$. Furthermore $\text{Aut}_{\pi(X)}(x) = \{(f : x \rightarrow x) \in X_1\} / \sim$ where $f \sim g$ iff there exists $\sigma \in X_2$ with $\partial(\sigma) = (s_0y, g, f)$.* \square

COROLLARY 2.7.8. *Suppose X is a Kan complex, then $\pi(X) \xrightarrow{\sim} \pi(|X|)$ is an equivalence of categories, where the last notation means the topological fundamental groupoid.*

PROOF. The functor is given on objects by sending a 0-simplex to the corresponding point in the geometric realisation, and on morphisms by sending the class of a 1-simplex to the class of the corresponding path in $|X|$. It is essentially surjective, as any point in $|X|$ is equivalent to a point coming from X_0 by the description of the CW structure.

Likewise surjective on morphisms: Any path from x to y in $|X|$ can be approximated by a simplicial path, i.e. a composition of edges and inverses of edges; using the Kan property we can always replace an edge run in the wrong direction by two edges run in the correct direction. So any path from x to y in $|X|$ can be approximated by a concatenation of paths corresponding to 1-simplices in X . Using the inner horn filling property of X and the fact that a 2-simplex σ in X can be used to give a homotopy (relative to its endpoints) between the paths $|d_0\sigma| \star |d_2\sigma|$ and $|d_1\sigma|$, we can further approximate our path in $|X|$ by one coming from a single 1-simplex in X .

To show faithfulness, if two 1-simplices $f, g \in X_1$ are such that $\partial(f) = \partial(g) = (y, x)$ and moreover $|f| \sim |g|$ rel. x, y , then we can write a homotopy $H : [0, 1] \times [0, 1] \rightarrow |X|$ between $|f|$ and $|g|$, i.e. a continuous map which is already simplicial on $\partial([0, 1] \times [0, 1])$, in that it sends the edges of the square to $|f|$, 1_y , g and 1_x respectively. We can replace H by a simplicial approximation that agrees with H on $\partial([0, 1] \times [0, 1])$, by suitably subdividing $[0, 1] \times [0, 1]$ (but without the need to subdivide $\partial[0, 1] \times [0, 1]$). We can then use the 2-simplices of the new H to witness the equality $f \sim g$ in $\text{h}X$: here it is crucial that whenever a 2-simplex is oriented “wrongly”, we can use the Kan condition to change H so that it is oriented correctly. \square

2.8. Kan complexes and simplicial homotopy

The following proposition should be compared with the fact from homological algebra that if R is a commutative ring, and M, N are R -modules with M projective (or flat) and N injective, then the R -module $\text{Hom}_R(M, N)$ is again injective.

PROPOSITION 2.8.1. *If Y is a Kan complex then so is $\text{map}(X, Y)$.*

PROOF. We need to solve the following lifting problem:

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & \text{map}(X, Y) \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

which by the adjunction of $(-) \times X$ and $\text{map}(X, -)$ is equivalent to

$$\begin{array}{ccc} \Lambda_k^n \times X & \longrightarrow & Y \\ \downarrow & \nearrow & \\ \Delta^n \times X & & \end{array}$$

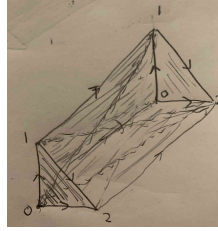
Now recall that by Proposition 2.3.1, X can be constructed by iterative simplex attachments.

Suppose that X is obtained from X' by attaching an m -simplex, and we have constructed the map on $\Delta^n \times X'$. Then we have a diagram of simplicial sets

$$\begin{array}{ccc} (\Lambda_k^n \times X) \cup (\Delta^n \times X') & \longrightarrow & Y \\ \downarrow & \nearrow & \\ \Delta^n \times X & & \end{array}$$

However, extending along this map is equivalent to extending along the inclusion $(\Lambda_k^n \times \Delta^m \cup \Delta^n \times \partial\Delta^m) \hookrightarrow \Delta^n \times \Delta^m$. (In $\Lambda_k^n \times \Delta^m \cup \Delta^n \times \partial\Delta^m$ the interior of $\Delta^n \times \Delta^m$ and one $\Delta^{n-1} \times \Delta^m$ faces is missing.) This we can do, by extending over the $\binom{n+m}{n}$ many $(m+n)$ -simplices, one at a time, using the horn filling condition for X . This should be geometrically fairly obvious, but we will not provide the combinatorial details of this here.

Just consider for example the following picture for $n = 2$, $k = 0$ and $m = 1$:



Here the outwards facing side and interior is removed, and we want to fill in the three 3-simplices $(0, 0) < (0, 1) < (1, 1) < (2, 1)$, $(0, 0) < (1, 0) < (1, 1) < (2, 1)$ and $(0, 0) < (1, 0) < (2, 0) < (2, 1)$ (remember the shuffle picture from Example 2.3.3). The first 3-simplex is filled in with the hornfilling condition, and similarly we can fill the 2-simplex $(0, 0) < (1, 0) < (2, 1)$; we can then fill in the remaining 3-simplices using the horn filling condition.

The argument in general is obtained by passing to a colimit to attach all m -simplices, then moving on to the $m + 1$ -simplices etc. See also [GJ99, Prop I.4.2]. \square

REMARK 2.8.2 (∞ -categories). It turns out that weak Kan complexes, i.e., simplicial sets X with the weak inner horn filling condition (also sometimes called quasi-categories), model “ ∞ -categories”. The elements in X_0 are the objects (viewed as “homotopy types”). And for two points $x, y \in X_0$ we can associate a space of maps between them “ $\text{Hom}_X(x, y)$ ”, given as the fiber over (x, y) of the map $\text{map}(\Delta^1, X) \rightarrow X \times X$ given by evaluation at endpoints. It turns out to be a Kan complex (see e.g., [Lur09, Prop. 2.4.1.8]).

We will combine the results of the previous subsections by applying h to $\text{map}(X, Y)$ to see that simplicial homotopy defines an equivalence relation when mapping into Kan complexes. This fact could of course also have been established directly (Exercise 2.8.8), but we do it this way to illustrate our “higher” viewpoint, where properties of morphisms can be reinterpreted as properties of an object, the mapping *space*.

COROLLARY 2.8.3. *Suppose X is an arbitrary simplicial set, and Y a Kan complex. Then $\pi(\text{map}(X, Y)) \cong h\text{map}(X, Y)$ is a groupoid with objects maps $f : X \rightarrow Y$, and*

$$\text{Hom}_{\pi(\text{map}(X, Y))}(f, g) = \{H : X \times I \rightarrow Y \mid H(-, 0) = f, H(-, 1) = g\} / \sim,$$

where $H \sim H'$ if there exists a map $G : X \times \Delta^2 \rightarrow Y$ which restricts to (s_0g, H', H) on the boundary pieces $(X \times d_0\Delta^2, X \times d_1\Delta^2, X \times d_2\Delta^2)$. Elements of $\text{Hom}_{\pi(\text{map}(X, Y))}(f, g)$ are thus “homotopy classes of homotopies between f and g ”. In particular homotopy between maps into a Kan complex is an equivalence relation, corresponding to connected components of the groupoid $\pi(\text{map}(X, Y))$.

PROOF. By Proposition 2.8.1 $\text{map}(X, Y)$ is a Kan complex, and the description can now be obtained by writing out Proposition 2.7.6 in that case. \square

DEFINITION 2.8.4. Suppose Y is a Kan complex, then we denote by $[X, Y]$ the (free) homotopy classes of maps $X \rightarrow Y$ (being homotopic is an equivalence relation on the set of maps $X \rightarrow Y$ by Corollary 2.8.3).

If X and Y are pointed simplicial sets, with Y still Kan, then we denote by $[X, Y]_*$ the set of pointed homotopy classes of pointed maps, i.e., the set of equivalence classes of pointed maps $X \rightarrow Y$ under the equivalence relation $f \sim g$ if there is a homotopy $H : X \times \Delta^1 \rightarrow Y$ restricting to f on $X \times d_1\Delta^1$, to Y on $X \times d_0\Delta^1$ and to the constant map to $*$ on $* \times \Delta^1$: here $*$ denotes the subsimplicial set of X , respectively of Y , generated by the 0-simplex giving the basepoint (along degeneracies).

EXERCISE 2.8.5. Check that pointed homotopy indeed defines an equivalence relation when the target simplicial set Y is Kan.

COROLLARY 2.8.6. The adjunction between $|-|$ and Sing_\bullet passes to homotopy categories, i.e. for any simplicial set X and any topological space Y we have a natural bijection

$$[|X|, Y] \cong [X, \text{Sing}_\bullet Y];$$

and if X and Y are pointed we also have a natural bijection

$$[|X|, Y]_* \cong [X, \text{Sing}_\bullet Y]_*.$$

PROOF. By Corollary 2.8.3, simplicial homotopy is an equivalence relation when mapping into Kan complexes, and by Proposition 2.7.2 $\text{Sing}_\bullet(Y)$ is Kan. By the $|-|$ - $\text{Sing}_\bullet(-)$ adjunction, maps $X \times \Delta^1 \rightarrow \text{Sing}_\bullet(Y)$ are in bijection with maps $|X \times \Delta^1| \rightarrow Y$. Since $|X \times \Delta^1| \cong |X| \times |\Delta^1|$, by Theorem 2.5.2, the two equivalence relations on maps agree. We leave it as an exercise to verify that this adjunction restricts to the pointed category. \square

EXERCISE 2.8.7. Check that the equivalence relation $\overset{h}{\sim}$ on the set of 1-simplices of a Kan complex can also be described as $f \overset{h}{\sim} g$ if there exists 2-simplex σ filling in the diagram

$$\begin{array}{ccc} & x & \\ 1_x = s_0(x) \nearrow & \sigma & \searrow f \\ x & \xrightarrow{g} & y. \end{array}$$

EXERCISE 2.8.8. Establish directly (without referring to the fundamental groupoids of mapping spaces) that simplicial homotopy is an equivalence relation when mapping into Kan complexes.

EXERCISE 2.8.9. Verify that $|-|$ and $\text{Sing}_\bullet(-)$ provide an adjunction between pointed simplicial sets and pointed topological spaces, and verify that this adjunction passes to homotopy classes. (That is, fill in the step left to the reader in Proposition 2.8.6.)

2.9. Simplicial homotopy groups and fibrations

In this section we will define simplicial homotopy groups of Kan complexes and establish their basic properties. As homotopy groups require basepoints, we shall mainly work in the category of pointed simplicial sets, which we denote by sSet_* .

2.9.1. Simplicial homotopy groups.

DEFINITION 2.9.1. Define the simplicial n -sphere as $S^n = \Delta^n / \partial\Delta^n$ for $n > 0$, and set $S^0 = \partial\Delta^1$.

REMARK 2.9.2. Note that $\Delta^n / \partial\Delta^n$ has one non-degenerate 0-simplex and one non-degenerate n -simplex. In particular there are exactly two maps $\Delta^n / \partial\Delta^n \rightarrow \Delta^n / \partial\Delta^n$, the trivial map and the identity. This shows the importance of replacing complexes with Kan complexes to get enough maps: the naive simplicial definition of $\pi_n(S^n)$ as $\{f: \Delta^n / \partial\Delta^n \rightarrow \Delta^n / \partial\Delta^n\} / \sim$ would lead to a set with (at most) two elements, instead of “ \mathbb{Z} ” many as we expect from the topological, classical notion.

REMARK 2.9.3. By the Yoneda lemma

$$\mathrm{Hom}_{\mathbf{Set}_*}(S^n, X) \xrightarrow{\cong} \{x \in X_n \mid \partial x := (d_0x, d_1x, \dots, d_nx) = (*, *, \dots, *)\},$$

where by abuse of notation we write $*$ for the $(n-1)$ -simplex $s_0^{n-1}(*)$ (a practice we will continue to simplify notation!).

DEFINITION 2.9.4 (Simplicial homotopy groups). Let X be a Kan complex with basepoint $*$. Define

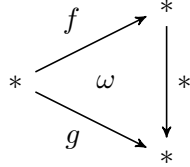
$$\pi_n(X, *) = [S^n, X]_* \quad \text{for } n \geq 0.$$

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Let us start by verifying that this gives the right thing for $n = 0, 1$. The following lemma gives an alternative way of viewing π_0 and π_1 in terms of the fundamental groupoid.

LEMMA 2.9.5. *Let X be a Kan complex. Then*

- $\pi_0(X) \cong \{\text{isomorphism classes of objects in } \pi(X)\} = X_0 / \sim$, where $x \sim y$ iff there exists $(f: x \rightarrow y) \in X_1$ (in particular this is an equivalence relation).
- $\pi_1(X, *) \cong \mathrm{Hom}_{\pi(X)}(*) = \{(f: * \rightarrow *) \in X_1\} / \sim$, where $f \sim g$ iff there exists $\omega \in X_2$ with $\partial\omega = (*, g, f)$ (in particular this is an equivalence relation).



PROOF. We start with $\pi_0(X)$. The proof can be summarized by the following picture:

$$\begin{array}{ccc} * & \xrightarrow{\quad * \quad} & * \\ & H & \\ x & \xrightarrow[\quad f \quad]{} & y \end{array}$$

In slightly more detail: By definition S^0 is isomorphic to the coproduct $d_1\Delta^1 \sqcup d_0\Delta^1$ of two 0-simplices, where $*$ = $d_1\Delta^1$ is the basepoint. A map $\tilde{x}: S^0 \rightarrow X$ is hence equivalent to a choice of $x \in X_0$ where to send $d_0\Delta^1$, i.e. an object in $\pi(X)$. A pointed homotopy $H: S^0 \times \Delta^1 \rightarrow X$ is a map

$$(* \sqcup d_0\Delta^1) \times \Delta^1 \cong * \times \Delta^1 \sqcup d_0\Delta^1 \times \Delta^1 \rightarrow X$$

restricting to the constant map to $*$ in X on $* \times \Delta^1$: such a map is uniquely determined by its behavior on $d_0\Delta^1 \times \Delta^1 \cong \Delta^1$, corresponding to an $f \in X_1$. In other words: Saying that H witnesses that \tilde{x} and \tilde{y} are equivalent in $\pi_0(X)$, with \tilde{x} and \tilde{y} corresponding to x and y in X_0 ,

⁹One could also set $\pi_n(X, *) = [(I^n, \partial I^n), (X, *)]$, and indeed this may be the right way of doing it initially, as checking things like associativity etc. will be more straightforward (e.g. as things reduce to a π_0 statement via the mapping space), just like for homotopy groups of topological spaces

respectively, means that $d_1 f = x$ and $d_0 f = y$ and that $f: x \rightarrow y$ witnesses that x and y are equivalent in $\pi(X)$.

We now turn to $\pi_1(X, *)$. Note that giving a pointed map $S^1 = \Delta^1 / \partial \Delta^1 \rightarrow X$ means specifying a simplex $(f: * \rightarrow *) \in X_1$. Furthermore, a pointed simplicial homotopy $H: S^1 \times \Delta^1 \rightarrow X$ from f to g is uniquely determined by a 1-simplex $h \in X_1$ and two 2-simplices $\sigma, \tau \in X_2$ satisfying $\partial \sigma = (*, h, f)$ and $\partial \tau = (g, h, *)$, as pictured below:

$$\begin{array}{ccc} * & \xrightarrow{f} & * \\ * & \searrow \sigma & \downarrow * \\ * & \xrightarrow{g} & * \end{array} \quad \begin{array}{c} \tau \\ h \end{array}$$

Now given $\omega \in X_2$ with $\partial \omega = (*, g, f)$ we can construct a simplicial homotopy H as follows

$$\begin{array}{ccc} * & \xrightarrow{f} & * \\ * & \searrow \omega & \downarrow * \\ * & \xrightarrow{g} & * \end{array} \quad \begin{array}{c} s_0 g \\ \tau \end{array}$$

Conversely, given h, σ and τ , we can fill in the following horn Λ_3^3 :

$$\begin{array}{ccc} & * & \\ f \nearrow & \downarrow & \nwarrow g \\ * & \xrightarrow{h} & * \\ \tau \nearrow & \downarrow & \nwarrow \end{array}$$

which shows $f \sim g$, using Exercise 2.8.7. □

LEMMA 2.9.6. *$f, g \in \pi_n(X, *)$ are homotopic if and only if there exists $\omega: \Delta^{n+1} \rightarrow X$ such that $\omega \circ d^n = g$ and $\omega \circ d^{n+1} = f$, and all other faces of ω are constantly $*$ (i.e., $\omega \in X_{n+1}$ such that $\partial \omega = (*, \dots, *, g, f)$).*

PROOF. It is clear by definition that elements in $\pi_n(X, *)$ are represented by n -simplices whose boundary faces are constantly $*$.

The statement about the equivalence relations is not just the definition: A simplicial homotopy is a map $\Delta^n \times \Delta^1 \rightarrow X$ sending $\partial \Delta^n \times \Delta^1$ to the basepoint $*$ in X . The datum of such a homotopy is equivalent to the datum of $n+1$ many $(n+1)$ -simplices $\sigma_0, \dots, \sigma_n \in X_{n+1}$ such that one face of σ_0 is f , one face of σ_n is g , and the other faces are partially required to be equal in pairs, and partially required to be constantly $*$.

We have a simplicial collapse map $\Delta^n \times \Delta^1 \rightarrow \Delta^{n+1}$, which shows that an ω as above induces a simplicial homotopy.

Conversely, if we have a simplicial homotopy, then we see that f is equivalent to g in the sense above by going through the $n+1$ levels of the prism. More precisely, there are n -simplices $f = f_{n+1}, f_n, \dots, f_1, f_0 = g$ such that $\partial \sigma_i = (*, \dots, *, f_i, f_{i+1}, *, \dots, *)$, where f_i occurs as the i^{th} face.

It is a general principle that if σ is a $(n+1)$ -simplex with $\partial \sigma = (*, \dots, *, a, b, *, \dots, *)$, for some n -simplices a, b occurring as i^{th} and $(i+1)^{\text{st}}$ faces, with $i \leq n$, then we can form a horn with faces $(*, \dots, *, \sigma, s_i a, ?, s_{i+2} b, *, \dots, *)$, whose $(i+2)^{\text{nd}}$ face, denoted $? = \tilde{\sigma}$, satisfies $\partial \tilde{\sigma} = (*, \dots, *, a, b, *, \dots, *)$, where now a, b occur as $(i+1)^{\text{st}}$ and $(i+2)^{\text{nd}}$ faces. To justify this principle, use also Lemma 2.9.9 in the following. Using this principle many times we find simplices $\tilde{\sigma}_i$ with $\partial \tilde{\sigma}_i = (*, \dots, *, f_i, f_{i+1})$.

It is another general principle that if a, b, c are n -simplices with $\partial a = \partial b = \partial c = (*, \dots, *)$, then as soon as we have two of the following three types of $(n+1)$ -simplices:

- a simplex α with $\partial\alpha = (*, \dots, *, a, b)$;
- a simplex β with $\partial\alpha = (*, \dots, *, b, c)$;
- a simplex γ with $\partial\alpha = (*, \dots, *, a, c)$,

then we can construct the third by filling a horn whose faces are $(*, \dots, *, \alpha, \gamma, \beta)$ (where one of α, γ, β replaced by a $?$), and filling this horn gives an example of the missing type of the three. Using this principle many times we can construct a simplex ω with $\partial\omega = (*, \dots, *, g, f)$. Exercise: check that this indeed works (and that I have not made mistakes with the indices). **[todo: Add details. Draw this for $\Delta^1 \times \Delta^1$.]** \square

PROPOSITION 2.9.7. *For a pointed topological space $(Y, *)$, let $* \in \text{Sing}_\bullet(Y)_0 = \text{Hom}_{\text{Top}}(|\Delta^0|, Y)$ be given by the composite $|\Delta^0| \rightarrow * \hookrightarrow Y$. Then*

$$\pi_n(\text{Sing}_\bullet(Y), *) = [\Delta^n / \partial\Delta^n, \text{Sing}_\bullet(Y)]_* \cong [|\Delta^n / \partial\Delta^n|, Y]_* = \pi_n(Y, *),$$

PROOF. This follows from the adjunction of Corollary 2.8.6, as $|\Delta^n / \partial\Delta^n| \cong S^n$. \square

PROPOSITION 2.9.8. *Let \mathcal{G} be a groupoid, $* \in \mathcal{G}$. Then*

$$\pi_n(N\mathcal{G}, *) = 0 \text{ for } n \geq 2$$

and in particular if $\mathcal{G} = BG$ for a group G , then $N\mathcal{G}$ is a $K(G, 1)$.

PROOF. Note that $N\mathcal{G}$ is Kan by Proposition 2.7.4, so the homotopy groups are defined. Furthermore if $x \in (N\mathcal{G})_n$, $n \geq 2$ and $\partial x = (s_0^{n-1}(*), \dots, s_0^{n-1}(*))$, then $x = s_0^n(*)$, because x is determined by its 1-skeleton, as this is true for the nerve of any category. \square

It may be useful at this point to restate the horn filling condition in concrete simplicial terms:

LEMMA 2.9.9. *Specifying a simplicial map $f : \Lambda_k^n \rightarrow X$ is equivalent to specifying elements $x_0, \dots, \hat{x}_k, \dots, x_n \in X_{n-1}$ such that $d_i x_j = d_{j-1} x_i$ for $0 \leq i < j \leq n$, with $i, j \neq k$. Extending f to Δ^n is equivalent to finding $y \in X_n$ such that $d_i y = x_i$ for $i \neq k$.*

PROOF. Recall that the simplicial identities say also that $d_i d_j = d_{j-1} d_i$ if $i < j$; these identities correspond to how the $(n-1)$ -faces of an n -simplex identify along the $(n-2)$ -faces, cf. Proposition 2.1.6. Correspondingly, specifying a map on Λ_k^n exactly amounts to specifying x_i for $i \neq k$, subject to these constraints for $i, j \neq k$. And extending a map to all of Δ^n means finding $x \in X_n$ such that $d_i x = x_i$ for all $i \neq k$. As a general principle, if X is finite then a simplicial map $X \rightarrow Y$ can be defined by declaring its behavior only on those simplices of X that are not in the image of any face or degeneracy map (we might call these the “free simplices” of X), under all constraints ensuring that the above choices actually glue to a map out of X (It is a very good exercise to check all the details of this). \square

PROPOSITION 2.9.10 (Group structure). *Let $n \geq 1$. The following operation \star makes $\pi_n(X, *)$ into a group, which is abelian if $n \geq 2$ (in this case we write $+$ for \star). Given $x, y \in X_n$ with $\partial x = \partial y = *$, let ω be any lift in*

$$\begin{array}{ccc} \Lambda_{n+1}^{n+1} & \xrightarrow{(*, \dots, *, x, -, y)} & X \\ \downarrow & \nearrow \exists \omega & \\ \Delta_{n+1}^{n+1} & & \end{array},$$

i.e. let $\omega \in X_{n+1}$ be such that $\partial\omega = (, \dots, *, x, d_n\omega, y)$, and set*

$$[x] \star [y] = [d_n\omega].$$

PROOF. $\partial(d_n\omega)$ is determined by the behavior of the map $\Lambda_n^{n+1} \rightarrow X$ on a part of the $(n-1)$ -skeleton of Λ_n^{n+1} , and in particular $\partial(d_n\omega) = (*, \dots, *)$, so $d_n\omega \in X_n$ represents a homotopy class. Associativity of \star and existence of inverses is similar to verifying that $\pi(X)$ is a groupoid. The proof that $\pi_n(X, *)$ is abelian when $n \geq 2$ uses an Eckmann-Hilton argument [elaborate]. \square

OBSERVATION 2.9.11. We saw in Lemma 2.9.5 that $\pi_1(X, *) \cong \text{Hom}_{\pi(X)}(*, *)$. We can now add that the group operation on $\pi_1(X, *) \cong \text{Hom}_{\pi(X)}(*, *)$ coincides with the composition on $\text{Hom}_{\pi(X)}(*, *)$:

$$(x, -, y) \text{ gives } \omega \in X_2 \text{ with } \begin{array}{c} \bullet \\ \swarrow y \quad \searrow x \\ \omega \\ \downarrow d_1\omega \\ \bullet \end{array} \text{ and } [x] \star [y] = [d_1\omega].$$

In the same way ω witnesses that $[x] \circ [y] = [d_1\omega]$ in $\pi(X)$. [reformulate?]

2.9.2. Kan fibrations. Extending the notion of a Kan complex, we can define a Kan fibration:

DEFINITION 2.9.12 (Kan fibration). A simplicial map $p: X \rightarrow Y$ is a *Kan fibration* if it has the following lifting property ($\forall n > 0, 0 \leq k \leq n$)

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{f} & X \\ \downarrow & \nearrow \exists \lambda & \downarrow p \\ \Delta^n & \xrightarrow{g} & Y \end{array}$$

i.e. given f, g making the square commute, there exists a λ making the two triangles commute.

REMARK 2.9.13. Note that a simplicial set X is Kan if and only if the map $X \rightarrow *$ is a Kan fibration.

PROPOSITION 2.9.14. Let $f: Z \rightarrow W$ be a map in Top . Then $\text{Sing}_\bullet(f)$ is a Kan fibration if and only if f is a Serre fibration.

PROOF. By adjunction, for a continuous map of spaces $f: Z \rightarrow W$, the lifting problem in sSet

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & \text{Sing}_\bullet(Z) \\ \downarrow & \nearrow & \downarrow \text{Sing}_\bullet(f) \\ \Delta^n & \longrightarrow & \text{Sing}_\bullet(W) \end{array}$$

is equivalent to the lifting problem in Top

$$\begin{array}{ccc} |\Lambda_k^n| & \longrightarrow & Z \\ \downarrow & \nearrow & \downarrow f \\ |\Delta^n| & \longrightarrow & W \end{array}$$

\square

The following is a vast generalization of Proposition 2.8.1:

THEOREM 2.9.15 (The notorious SM7). *If $p : X \rightarrow Y$ is a (Kan) fibration of simplicial sets and $i : A \hookrightarrow B$ is an inclusion of simplicial sets, then*

$$\mathrm{map}(B, X) \xrightarrow{(i^*, p_*)} \mathrm{map}(A, X) \times_{\mathrm{map}(A, Y)} \mathrm{map}(B, Y)$$

is a (Kan) fibration. In particular, $\mathrm{map}(B, X) \rightarrow \mathrm{map}(A, X)$ is a fibration for X Kan, and $\mathrm{map}(A, X) \rightarrow \mathrm{map}(A, Y)$ is a fibration if p is a fibration.

The result can be strengthened: Below in Definition 2.10.4, we define the notion of a weak equivalence of simplicial sets. With this notion, the map (i^*, p_*) is a weak equivalence if i and p are.

ABOUT THE PROOF OF THEOREM 2.9.15. Note that it is a generalization of Proposition 2.8.1. We will not give the proof here: it can be found in [GJ99, Prop. I.5.2], following an approach of Gabriel–Zisman [GZ67].

It involves the same ingredients as the baby case Proposition 2.8.1, which we also did not prove in details—the key input is constructing a lifting cell-by-cell, which again involves keeping track of the “prism” division of products of simplices into simplices. A good way of keeping track of this is via the theory of “anodyne extensions”. (“Anodyne” might sound like a scary word, but it actually means roughly “inoffensive”; they are inclusions such as $\Lambda_i^n \rightarrow \Delta^n$, $(\Lambda_i^n \times \Delta^1 \cup \Delta^n \times \partial\Delta^1) \rightarrow \Delta^n \times \Delta^1$ etc., obtained by iterately attaching a new simplex along a “contractible” part of its boundary (in a precise sense), and you prove that you can lift over anodyne extensions.) \square

The (non-formal) fact that geometric realization commutes with products in fact has the following generalization:

THEOREM 2.9.16 (Quillen). *If $p : X \rightarrow Y$ is a Kan fibration, then $|p| : |X| \rightarrow |Y|$ is a Serre fibration.*

ABOUT THE PROOF. The result is certainly not a formality, and is a bit surprising since $|-|$ is a left adjoint rather than a right adjoint, in the same way that it was surprising that $|-|$ commuted with products. We will skip the proof, which requires some work. The original reference is [Qui68]. The statement and proof is also given in [GJ99, Thm. I.10.10]. \square

REMARK 2.9.17. Note that the converse of Theorem 2.9.16 is false: The map $|X| \rightarrow *$ is always a Serre fibration, even if X is not Kan.

PROPOSITION 2.9.18. *The composition of Kan fibrations is a Kan fibration, and the pullback of a Kan fibration is a Kan fibration.*

PROOF. This follows directly from the lifting property, and the universal property of the pullback. \square

DEFINITION 2.9.19. For a Kan complex X , define the path space and loop space via the pullback squares

$$\begin{array}{ccc} \Omega X & \longrightarrow & \mathrm{map}(\Delta^1, X) \\ \downarrow & & \downarrow \mathrm{ev}_0 \times \mathrm{ev}_1 \\ * & \longrightarrow & X \times X \end{array} \quad \text{and} \quad \begin{array}{ccc} PX & \longrightarrow & \mathrm{map}(\Delta^1, X) \\ \downarrow & & \downarrow \mathrm{ev}_0 \times \mathrm{ev}_1 \\ X & \xrightarrow{* \times \mathrm{id}_X} & X \times X. \end{array}$$

Note in particular:

PROPOSITION 2.9.20. *PX and ΩX are Kan complexes, and $PX \rightarrow X$ is a Kan fibration.*

PROOF. Theorem 2.9.15 guarantees that the right-hand vertical maps in the two squares in Definition 2.9.19 are Kan fibrations, and then so are the left-hand ones by the pull-back property, Proposition 2.9.18. \square

2.9.3. The long exact sequence in homotopy groups.

THEOREM 2.9.21. *Let $p: X \rightarrow Y$ be a Kan fibration between Kan complexes and let $F = p^{-1}(*) \xrightarrow{i} X$ be the fiber of a vertex $* \in Y_0$. Assume a lift $* \in F_0 \subset X_0$ of $* \in Y_0$ is given. Then F is a Kan complex and there is a long exact sequence of homotopy groups*

$$\begin{aligned} \cdots \longrightarrow \pi_n(F, *) &\xrightarrow{i_*} \pi_n(X, *) \xrightarrow{p_*} \pi_n(Y, *) \xrightarrow{\partial} \pi_{n-1}(F, *) \longrightarrow \cdots \\ \cdots \longrightarrow \pi_1(Y, *) &\xrightarrow{\partial} \pi_0(F) \xrightarrow{i_*} \pi_0(X) \xrightarrow{p_*} \pi_0(Y). \end{aligned}$$

Moreover, there is an action of the group $\pi_1(Y, *)$ on the set $\pi_0(F)$ such that $i_*[x] = i_*[y] \in \pi_0(X)$ if and only if $[x], [y] \in \pi_0(F)$ are in the same orbit.

The entire statement is natural in the Kan fibration: a commutative diagram of simplicial sets

$$\begin{array}{ccccc} F & \longrightarrow & X & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ F' & \longrightarrow & X' & \longrightarrow & Y' \end{array}$$

with p and p' being Kan fibrations (and compatible choices of basepoints in Y and in Y' as well as in $F := p^{-1}(*) \subset X$ and $F' := (p')^{-1}(*) \subset X'$) gives rise to a commutative diagram of sets (for $n \geq 0$)/of groups (for $n \geq 1$)

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial} & \pi_n(F, *) & \longrightarrow & \pi_n(X, *) & \xrightarrow{p_*} & \pi_n(Y, *) \xrightarrow{\partial} \pi_{n-1}(F, *) \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \xrightarrow{\partial} & \pi_n(F', *) & \longrightarrow & \pi_n(X', *) & \xrightarrow{p'_*} & \pi_n(Y', *) \xrightarrow{\partial} \pi_{n-1}(F', *) \longrightarrow \cdots \end{array}$$

PROOF. The boundary map $\partial: \pi_n(Y, *) \rightarrow \pi_{n-1}(F, *)$ is defined as follows: Let $y \in Y_n$ with $\partial y = *$ be given. Let $\lambda \in X_n$ be a solution of the lifting problem

$$\begin{array}{ccc} \Lambda_0^n & \xrightarrow{(-, *, \dots, *)} & X \\ \downarrow & \nearrow \exists \lambda & \downarrow p \\ \Delta^n & \xrightarrow{y} & Y, \end{array}$$

i.e. $\partial \lambda = (d_0 \lambda, *, \dots, *)$, $p(\lambda) = y$. So $p(d_0 \lambda) = d_0 p(\lambda) = d_0 y = *$ and therefore $d_0 \lambda \in F$. Moreover, $\partial(d_0 \lambda) = *$. So $d_0 \lambda$ represents a class in $\pi_{n-1}(F, *)$. Define

$$\partial[y] = [d_0 \lambda].$$

One checks that ∂ is well-defined and a homomorphism for $n > 1$ (exercise!). For the exactness, we check it at

$$\pi_n(X, *) \xrightarrow{p_*} \pi_n(Y, *) \xrightarrow{\partial} \pi_{n-1}(F, *).$$

For $x \in X_n$, $\partial x = *$, we have

$$\partial p_*[x] = \partial[p(x)] = [d_0 x] = [*] = *,$$

since we can choose $\lambda = x$. Conversely, suppose that $0 = \partial[y] = [d_0 \lambda]$. Then by Lemma 2.9.6, there exists $\omega \in F_n$ such that $\partial \omega = (*, \dots, *, d_0 \lambda)$. The lifting problem

$$\begin{array}{ccc}
\Lambda_n^{n+1} & \xrightarrow{(\omega, *, \dots, *, -, \lambda)} & X \\
\downarrow & \nearrow \exists \Theta & \\
\Delta^{n+1} & &
\end{array}$$

gives $\Theta \in X_{n+1}$ with $\partial\Theta = (\omega, *, \dots, *, \eta, \lambda)$ for some $\eta \in X_n$ with $\partial\eta = *$. Since $p\omega = *$ (indeed $\omega \in F$), it follows $\partial p\Theta = p\partial\Theta = (*, \dots, *, p\eta, y)$. So $p\Theta$ exhibits $[y] = [p\eta] = p_*[\eta]$. The rest of the proof is left as exercise. \square

2.10. The equivalence between sSet and Top

THEOREM 2.10.1. *For any Kan complex X , the canonical map*

$$X \xrightarrow{\eta_X} \text{Sing}_\bullet |X|$$

*induces an isomorphism $\pi_n(X, *) \xrightarrow{\cong} \pi_n(\text{Sing}_\bullet |X|, *) = \pi_n(|X|, *)$ in homotopy groups.*

PROOF. The proof goes by induction on n , using the path-loop fibration $\Omega X \rightarrow PX \rightarrow X$, where PX and ΩX are defined through the pullback squares

$$\begin{array}{ccc}
PX & \longrightarrow & \text{map}(\Delta^1, X) \\
\downarrow & & \downarrow \text{ev}_0 \\
* & \longrightarrow & X
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\Omega X & \longrightarrow & PX \\
\downarrow & & \downarrow \text{ev}_1 \\
* & \longrightarrow & X.
\end{array}$$

We do the induction step, using two facts:

- (1) $PX \rightarrow X$ is indeed a fibration (Proposition 2.9.20).
- (2) PX has trivial homotopy groups.

By Theorem 2.9.16, $|\Omega X| \rightarrow |PX| \rightarrow |X|$ is a (Serre) fibration in Top and hence, $\text{Sing}_\bullet(|\Omega X|) \rightarrow \text{Sing}_\bullet(|PX|) \rightarrow \text{Sing}_\bullet(|X|)$ is a (Kan) fibration in sSet. The commutative diagram of Kan fibrations

$$\begin{array}{ccccc}
\Omega X & \longrightarrow & PX & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
\text{Sing}_\bullet(|\Omega X|) & \longrightarrow & \text{Sing}_\bullet(|PX|) & \longrightarrow & \text{Sing}_\bullet(|X|)
\end{array}$$

induces a commutative diagram

$$\begin{array}{ccccccc}
\pi_n(PX) & \longrightarrow & \pi_n(X) & \xrightarrow{\partial} & \pi_{n-1}(\Omega X) & \longrightarrow & \pi_{n-1}(PX) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_n(\text{Sing}_\bullet |PX|) & \longrightarrow & \pi_n(\text{Sing}_\bullet |X|) & \xrightarrow{\partial} & \pi_{n-1}(\text{Sing}_\bullet(|\Omega X|)) & \longrightarrow & \pi_{n-1}(\text{Sing}_\bullet |PX|).
\end{array}$$

By the second fact (the fact the PX has trivial homotopy groups), it follows that also $\text{Sing}_\bullet |PX|$ is contractible. So the boundary maps ∂ in the diagram are isomorphisms and since by induction hypothesis, $\pi_{n-1}(\Omega X) \rightarrow \pi_{n-1}(\text{Sing}_\bullet(|\Omega X|))$ is an isomorphism, we conclude the induction step. \square

COROLLARY 2.10.2. *For any topological space Y , the canonical map*

$$|\mathrm{Sing}_\bullet(Y)| \xrightarrow{\varepsilon_Y} Y$$

is a weak homotopy equivalence.

PROOF. Applying π_n to the commutative diagram

$$\begin{array}{ccc} \mathrm{Sing}_\bullet(Y) & \xrightarrow{\eta_{\mathrm{Sing}_\bullet(Y)}} & \mathrm{Sing}_\bullet |\mathrm{Sing}_\bullet(Y)| \\ & \searrow 1 & \downarrow \mathrm{Sing}_\bullet(\varepsilon_U) \\ & & \mathrm{Sing}_\bullet(Y) \end{array}$$

yields a commutative diagram

$$\begin{array}{ccc} \pi_n(\mathrm{Sing}_\bullet(Y)) & \xrightarrow{\cong} & \pi_n(\mathrm{Sing}_\bullet |\mathrm{Sing}_\bullet(Y)|) = \pi_n(|\mathrm{Sing}_\bullet(Y)|) \\ & \searrow 1 & \downarrow \\ & & \pi_n(\mathrm{Sing}_\bullet(Y)) = \pi_n(Y) \end{array},$$

where the horizontal morphism is an isomorphism Theorem 2.10.1. So also the map $\pi_n(|\mathrm{Sing}_\bullet(Y)|) \rightarrow \pi_n(Y)$ induced by the counit of the adjunction is an isomorphism. \square

[would still like to see that weak homotopy equivalence of simplicial sets agrees with “strong” homotopy equivalence defined via the path object on Kan complexes]

REMARK 2.10.3. $|\mathrm{Sing}_\bullet(U)| \xrightarrow{\varepsilon_U} U$ is a functorial CW approximation of U , i.e. the functor $|\mathrm{Sing}_\bullet(-)|: \mathrm{Top} \rightarrow \mathrm{Top}$ produces for every topological space a CW approximation.

Recall that for any adjunction

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{D},$$

the functors F, G are equivalences of categories if and only if $X \xrightarrow{\eta_X} GF(X)$ and $FG(Y) \xrightarrow{\varepsilon_Y} Y$ are isomorphisms for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$.

In the case of the adjunction

$$\mathrm{sSet} \begin{array}{c} \xrightarrow{|\cdot|} \\ \xleftarrow{\mathrm{Sing}_\bullet} \end{array} \mathrm{Top},$$

the morphisms $X \xrightarrow{\eta_X} \mathrm{Sing}_\bullet |X|$ and $|\mathrm{Sing}_\bullet(Y)| \xrightarrow{\varepsilon_Y} Y$ are not necessarily isomorphisms, but they are weak homotopy equivalences, and therefore they induce an equivalence of homotopy categories

$$\mathrm{Ho}(\mathrm{sSet}) \begin{array}{c} \xrightarrow{|\cdot|} \\ \xleftarrow{\mathrm{Sing}_\bullet} \end{array} \mathrm{Ho}(\mathrm{Top}),$$

where $\mathrm{Ho}(\mathcal{C})$ is \mathcal{C} with weak equivalences formally inverted (this will be discussed in more detail in Section 4.1).

DEFINITION 2.10.4. A simplicial map between any simplicial sets $f: X \rightarrow Y$ is a *weak equivalence* if $|f|: |X| \rightarrow |Y|$ is a weak homotopy equivalence.

We conclude that for any simplicial set X , the morphism $X \xrightarrow{\eta_X} \text{Sing}_\bullet |X|$ is a weak equivalence by the commutative diagram

$$\begin{array}{ccc} |X| & \xrightarrow{|\eta_X|} & |\text{Sing}_\bullet |X|| \\ & \searrow 1 & \downarrow \varepsilon_{|X|} \\ & & |X| \end{array}$$

and Corollary 2.10.2. Similarly we have the following:

PROPOSITION 2.10.5. *For X and Y Kan, $f: X \rightarrow Y$ is a weak equivalence if and only if $f_*: \pi_0(X) \rightarrow \pi_0(Y)$ and $f_*: \pi_n(X, *) \rightarrow \pi_n(Y, f(*))$ are isomorphisms for all $n \geq 1$ and all $* \in X_0$.*

REMARK 2.10.6 (A small digression on model categories). The above comparison between simplicial sets and topological spaces can be formulated within the framework of model categories. A *model category* is a category that comes equipped with three classes of morphisms, namely *cofibrations*, *fibrations* and *weak equivalences* subject to a list of axioms, see Chapter 2 of [GJ99] or also [Hov99]. One should think of cofibrations (fibrations) as a nice class of injections (surjections) although this is nothing more than an intuition. A weak equivalence should be thought of as morphism that preserves all essential information, but is typically not an isomorphism (not an *invertible* morphism). A cofibration (fibration) that is also a weak equivalence is called a *trivial cofibration* (*fibration*) (instead of the word ‘trivial’, the word ‘acyclic’ is also in use). Bicompleteness is part of the definition of a model category, in particular there is always an initial object \emptyset and a terminal object \star . An object X is called *cofibrant* if the unique map $\emptyset \rightarrow X$ is a cofibration; it is called *fibrant* if the unique map $X \rightarrow \star$ is a fibration. In any model category, the classes of weak equivalences and fibrations determine the class of cofibrations; dually, the class of fibrations can be recovered from knowledge of the classes of weak equivalences and cofibrations.

One can show that simplicial sets form a model category in which the cofibrations are monomorphisms (concretely, simplicial maps that are injective in each degree), fibrations are Kan fibrations and weak equivalences are exactly the ones from Definition 2.10.4. As a consequence, all objects in \mathbf{sSet} with this model structure are cofibrant, but only the Kan complexes are fibrant.

The category of topological spaces with its usual weak equivalences and Serre fibrations as fibrations is a model category. The cofibrations (that are determined by this choice of weak equivalences and fibrations) are the cofibrations encountered in \mathbf{AlgTop}_2 , and defined in terms of the homotopy extension property.

A good notion for the comparison of model categories \mathcal{C} and \mathcal{D} is a *Quillen adjunction*, i.e. an adjoint pair $L \dashv R$

$$\begin{array}{ccc} & L & \\ \mathcal{C} & \xrightleftharpoons[R]{L} & \mathcal{D} \end{array}$$

such that L preserves cofibrations and trivial cofibrations (equivalently, we can ask that R preserves fibrations and trivial fibrations). A Quillen adjunction $L \dashv R$ is called *Quillen equivalence* if for any cofibrant object X in \mathcal{C} and any fibrant object Y in \mathcal{D} a morphism $X \rightarrow RY$ is a weak equivalence if and only if the adjoint morphism $LX \rightarrow Y$ is a weak equivalence (an equivalent definition asks the so-called *derived unit* and *derived counit* to be weak equivalences).

From the results above, one concludes that the adjunction

$$\begin{array}{ccc} & | - | & \\ \text{sSet} & \xrightleftharpoons{\quad} & \text{Top} \\ & \text{Sing}_\bullet & \end{array}$$

is a Quillen equivalence for the model structures on sSet and Top that we have just described. This adjunction induces an equivalence of homotopy categories

$$\begin{array}{ccc} & | - | & \\ \text{Ho}(\text{sSet}) & \xrightleftharpoons{\quad} & \text{Ho}(\text{Top}) \\ & \text{Sing}_\bullet & \end{array}$$

between the homotopy categories of sSet and Top that are obtained by localization at weak equivalences. As a word of warning: It is not true that any Quillen adjunction will descend to the homotopy categories. This is true for $| - | \dashv \text{Sing}_\bullet$ because both functors preserve weak equivalences. If this is not the case, we need to *derive*, using the techniques developed in Section 4.2.

2.11. The classification of fibrations

In this section, we discuss the simplicial approach to the classification of fibrations or fiber bundles. In other lectures, you might have encountered the problem of classifying all complex (or real) vector bundles of rank n over a given space X . If X satisfies some basic requirements, more precisely paracompactness in order to allow partition of unity arguments, then homotopy theory offers the following solution to this problem.

For instance, if we want to classify complex vector bundles of rank n , we can consider the group of linear automorphisms of the fiber, here \mathbb{C}^n . This gives us the Lie group $\text{GL}(n, \mathbb{C})$ which is homotopy equivalent to the Lie group $\text{U}(n)$ of \mathbb{C} -linear *isometries* of \mathbb{C}^n , with respect to the standard hermitian metric; the inclusion $\text{U}(n) \xrightarrow{\sim} \text{GL}(n, \mathbb{C})$ is a homotopy equivalence, and in fact classifying \mathbb{C}^n -vector bundles over X is as difficult as classifying *hermitian* vector bundles, i.e. vector bundles all of whose fibres are endowed with a hermitian metric (positive definite, symmetric and sesquilinear form): this follows from the observation that the *space* of hermitian structures that one can put on a vector bundle has a natural “convex” structure, and is hence contractible.

One may then observe that instead of classifying hermitian complex vector bundles of rank n over X , we may equivalently classify principal $\text{U}(n)$ -bundles over X . From the topological group $\text{U}(n)$, as for any other topological group, we can construct a certain space $\text{BU}(n)$, the *classifying space* of $\text{U}(n)$. One can then prove that the isomorphism classes of principal $\text{U}(n)$ -bundles over X (and hence the isomorphism classes of complex vector bundles of rank n over X) are in bijection with the set $[X, \text{BU}(n)]$ of homotopy classes of maps from X to $\text{BU}(n)$ (a very detailed treatment is given in Section 14 of [Die08]).

If for example we are interested in complex line bundles over X , we need to compute $[X, \text{BU}(1)]$; and we have seen in AlgTop2 that $\text{BU}(1)$ is actually a $K(\mathbb{Z}, 2)$ -space. Therefore, line bundles over X up to isomorphism are in bijection with $H^2(X; \mathbb{Z})$.

We will formulate a simplicial version of the solution to the problem of classifying fibrations. It is a little different in flavor (there will be no technical issues coming from point-set topology). Moreover, a lot of the concepts developed along the way will be of independent interest. For example, the notion of a minimal fibration introduced below is key to the proof given for Theorem 2.9.16 in [GJ99].

Our goal is to classify, up to homotopy equivalence, the Kan fibrations over any simplicial set X whose fiber is homotopy equivalent to a given Kan complex F . To this end, we need a more rigid version of a Kan fibration, a so-called *minimal fibration*. It will have the property

that if two n -simplices in the total space have equal boundary and equal projections onto the base space, and are homotopic relative boundary by a homotopy that projects to the constant homotopy, they are already equal. The precise definition is as follows:

DEFINITION 2.11.1. A Kan fibration $p : X \rightarrow Y$ is called a *minimal fibration* if for every commutative diagram

$$\begin{array}{ccccc} \partial\Delta^n \times \Delta^1 & \longrightarrow & \Delta^n \times \Delta^1 & \longrightarrow & \Delta^n \\ \downarrow & & \downarrow h & & \downarrow \\ \partial\Delta^n & \longrightarrow & X & \xrightarrow{p} & Y \end{array}$$

the two compositions $h \circ d^0, h \circ d^1 : \Delta^n \rightarrow \Delta^n \times \Delta^1 \rightarrow X$ are equal.

REMARK 2.11.2. The property of being a minimal fibration is stable under pullbacks.

PROPOSITION 2.11.3. *For any Kan fibration $p : X \rightarrow Y$, there is a strong fiber-wise deformation retract $q : Z \rightarrow Y$ which is a minimal fibration, i.e. there is a simplicial subset $Z \subset X$ such that $q := p|_Z : Z \rightarrow Y$ is a minimal fibration and a fiber-wise homotopy $H : X \times \Delta^1 \rightarrow X$, i.e. the following properties are satisfied:*

- $p \circ H$ is equal to the composition of the projection $X \times \Delta^1 \rightarrow X$ with p ;
- $H(-, 0) = \text{id}_X$;
- H restricts to the projection $Z \times \Delta^1 \rightarrow Z$ on $Z \times \Delta^1$;
- $H(-, 1)$ has image in $Z \subset X$.

PROOF. This is Proposition 10.3 in [GJ99]. The idea is to construct Z skeleton by skeleton. As the 0-skeleton of Z , one takes a system of representatives for the fiber-wise homotopy classes of vertices of X . Then one continues in a similar way to the higher skeleta. \square

REMARK 2.11.4. A Kan complex X is called minimal if $X \rightarrow *$ is a minimal fibration. The above result tells us that we can replace any Kan complex by a homotopy equivalent minimal Kan (sub)complex. Moreover, using stability of being minimal under pullback, we obtain that if $p : X \rightarrow Y$ is a minimal fibration, then the fibres of p over vertices of Y are minimal Kan complexes.

LEMMA 2.11.5. *Let $p : X \rightarrow Y$ and $q : X' \rightarrow Y$ be minimal fibrations and let $f : X \rightarrow X'$ be a map of fibrations over Y , i.e. $q \circ f = p$. If f is fiber-wise a homotopy equivalence, then f is an isomorphism.*

PROOF. This is Lemma 10.4 in [GJ99]. The proof can be reduced to the case $Y = *$, i.e. to the case of a single fiber. Then the statement follows in a relatively straightforward way from the minimality. \square

Proposition 2.11.3 and Lemma 2.11.5 tell us that, instead of investigating the homotopy classes of Kan fibrations over X with fiber homotopy equivalent to F , we can equivalently consider isomorphism classes of minimal fibrations over X whose fiber is isomorphic to F^{\min} (a minimal replacement of F).

LEMMA 2.11.6. *Let $p : X \rightarrow Y$ be a Kan fibration and let $f_0, f_1 : A \rightarrow Y$ be maps such that there exists a homotopy $A \times \Delta^1 \rightarrow Y$ from f_0 to f_1 . Consider the pullback*

$$\begin{array}{ccc} P_i & \longrightarrow & X \\ \downarrow & & \downarrow p \\ A & \xrightarrow{f_i} & Y \end{array}$$

for $i = 0, 1$. Then there is a fiber-wise homotopy equivalence $P_0 \rightarrow P_1$, i.e. a homotopy equivalence $P_0 \rightarrow P_1$ over A . If p is minimal, there is even an isomorphism $P_0 \rightarrow P_1$ over A .

PROOF. The first part is Lemma 10.6 in [GJ99]. The addendum for minimal fibrations follows from Lemma 2.11.5. \square

Next we will establish the local triviality of minimal fibrations.

DEFINITION 2.11.7. A map $p : E \rightarrow B$ of simplicial sets is called a *fiber bundle* with fiber F if it is surjective and if for any $b : \Delta^n \rightarrow B$ the pullback $E \times_B \Delta^n \rightarrow \Delta^n$ of p along b is isomorphic over Δ^n to the projection $F \times \Delta^n \rightarrow \Delta^n$. If F is additionally a Kan complex, we call the fiber bundle a *Kan fiber bundle*.

PROPOSITION 2.11.8. *Any Kan fiber bundle is a Kan fibration, and any minimal fibration with connected base is a Kan fiber bundle.*

PROOF. This is Lemma 11.9 and 11.10 in [May67]. \square

In the above recollection of the classification of vector bundles (in the topological setting), it was a key step to pass from bundles with a certain fiber to principal bundles for the automorphism group of the fiber. We will pursue essentially the same strategy in the simplicial setting.

DEFINITION 2.11.9. Let G be a simplicial group, i.e. a functor $\Delta^{\text{op}} \rightarrow \text{Grp}$: in particular, we have a sequence of groups $G_n = G([n])$ for $n \geq 0$. A *principal G -bundle over B* is a map of simplicial sets $p : E \rightarrow B$ equipped with a *free* right action of G on E (i.e. compatible, free right actions $E_n \times G_n \rightarrow E_n$ of each group G_n on the corresponding set E_n) such that $p : E \rightarrow B$ is equivariant with respect to the trivial G -action on B , and such that the induced map $E/G \rightarrow B$ is an isomorphism, where E/G is the simplicial set with E_n/G_n as set of n -simplices, and with induced face and degeneracy maps.

REMARK 2.11.10. A pullback of a principal fibration is again a principal fibration.

REMARK 2.11.11. A morphism of principal G -bundles over a fixed base is a simplicial map over that base that respects the G -action. One can prove that such a morphism is always an isomorphism, i.e. principal G -bundles over a fixed base form a groupoid.

PROPOSITION 2.11.12. *Any principal G -bundle $E \rightarrow B$ is a Kan fiber bundle with fiber G . In particular, $E \rightarrow B$ is a Kan fibration.*

PROOF. The map $E \rightarrow B$ is surjective, and its fiber, namely G , is a Kan complex because every simplicial group is a Kan complex (a lemma you might want to prove). In order to prove that for $b : \Delta^n \rightarrow B$ the pullback $E \times_B \Delta^n \rightarrow \Delta^n$ is trivial, we can assume $B = \Delta^n$ and $b = \text{id}_{\Delta^n}$ by Remark 2.11.10. By surjectivity we can find a lift $\tilde{b} : \Delta^n \rightarrow E$ of b . Now one can verify that $G \times \Delta^n \xrightarrow{G \times \tilde{b}} G \times E \xrightarrow{\text{action}} E$ is an isomorphism. \square

DEFINITION 2.11.13. For a Kan complex F , we define the *automorphism group* $\text{Aut}(F)$ of F as the subsimplicial set of $\text{map}(F, F)$ of those simplices $\sigma : F \times \Delta^n \rightarrow F$ such that

$$\sigma' : F \times \Delta^n \rightarrow F \times \Delta^n, \quad (f, x) \mapsto (\sigma(f, x), x)$$

is an isomorphism (over Δ^n). $\text{Aut}(F)$ is a simplicial group whose multiplication is in particular as follows: given $\sigma, \tau \in \text{Aut}(F)_n$, i.e. maps $F \times \Delta^n \rightarrow F$, we set $\sigma \circ \tau : (f, x) \mapsto \sigma(\tau(f, x), x)$.

The simplicial set $\text{Aut}(F)$ is a simplicial group (and, as any simplicial group, a Kan complex). We have a left $\text{Aut}(F)$ -action on F by

$$\text{Aut}_n(F) \times F_n \rightarrow F_n, \quad (\sigma, (f : \Delta^n \rightarrow F)) \mapsto (\Delta^n \xrightarrow{f \times \text{id}} F \times \Delta^n \xrightarrow{\sigma} F).$$

Given simplicial sets X and Y with right and left G -action, respectively, for any simplicial group G , we have a left action on $X \times Y$ given by $g.(x, y) = (x.g^{-1}, g.y)$ for $g \in G_n$, $x \in X_n$ and $y \in Y_n$. We denote the orbits by $X \times_G Y := (X \times Y)/G$. This will be relevant for us in

the following situation: Using the left $\text{Aut}(F)$ -action on F , we can form $E \times_{\text{Aut}(G)} F$ for any $\text{Aut}(F)$ -principal bundle $E \rightarrow B$; then the map $E \times_{\text{Aut}(G)} F \rightarrow E \times_{\text{Aut}(G)} * \cong B$ is an F -fibre bundle. In fact we have the following:

THEOREM 2.11.14. *For any simplicial set B and any Kan complex F , the assignment sending a principal $\text{Aut}(F)$ -bundle $E \rightarrow B$ to the fiber bundle $E \times_{\text{Aut}(F)} F \rightarrow B$ establishes an equivalence of groupoids*

$$\left\{ \begin{array}{c} \text{groupoid of} \\ \text{principal } \text{Aut}(F)\text{-bundles} \\ \text{over } B \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} \text{groupoid of} \\ \text{fiber bundles over } B \text{ with fiber } F \\ \text{and their isomorphisms} \end{array} \right\}.$$

PROOF. This is (in a slightly different language) Proposition 20.7 in [May67]. The proof is technical, but again the idea is to reduce to $B = \Delta^n$. Then all principal bundles and fiber bundles are trivial, hence the above equivalence reduces to a statement about the automorphism groups of the trivial $\text{Aut}(F)$ -principal bundle and the trivial bundle with fiber F over Δ^n : Indeed, the group of automorphisms for the trivial principal $\text{Aut}(F)$ -bundle over Δ^n is $\text{Aut}_n(F)$; in fact, more generally the automorphism group of the trivial principal G -bundle over Δ^n can be identified with the group of maps $\Delta^n \rightarrow G$ and hence with G_n . On the other hand, the group of automorphisms of the trivial bundle $F \times \Delta^n$ is really $\text{Aut}_n(F)$ by definition. This implies the assertion in this special case. \square

For a given simplicial group G we will need in the sequel a contractible Kan complex EG with free right G -action. This makes the quotient map $EG \rightarrow BG := EG/G$ into a principal G -bundle, and the hypotheses on EG imply that BG is also a Kan complex. We call $EG \rightarrow BG$ the *universal principal G -bundle*. One can build a concrete model for EG with n -simplices given by the set $EG_n = G_n \times \cdots \times G_n$, but we will not rely too much on this model. The contractibility of this model of EG can be seen by exhibiting extra degeneracies in the sense of [GJ99], page 200 (there will be an exercise on that).

The properties that EG has by definition characterize it uniquely up to equivariant homotopy. This follows from the following statement:

LEMMA 2.11.15. *For any simplicial group G and any simplicial set E with free G -action, there is a map $E \rightarrow EG$ unique up to equivariant homotopy.*

We will not give a proof here, but just mention that if one establishes a simplicial model structure on simplicial sets with (right) G -action, this is a direct consequence of (the analogue of) the (SM7) axiom. More precisely, one can see that the trivial Kan fibration $EG \rightarrow *$ induces a trivial Kan fibration $\text{map}_G(E, EG) \rightarrow \text{map}_G(E, *) = *$ between the mapping spaces of G -equivariant maps (this needs that E has a free G -action, i.e. it is cofibrant as a G -simplicial set). This tells us that the Kan complex $\text{map}_G(E, EG)$ is contractible and gives us Lemma 2.11.15.

THEOREM 2.11.16. *For any simplicial set X and any simplicial group G , the assignment sending a homotopy class of maps $f : X \rightarrow BG$ to the pullback f^*EG of the universal G -bundle $EG \rightarrow BG$ along f establishes a bijection of sets*

$$[X, BG] \xrightarrow{\cong} \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{principal } G\text{-bundles over } X \end{array} \right\}.$$

PROOF. The well-definedness of the map of set is Lemma 3.4 in [GJ99] (the needed statement is similar to Lemma 2.11.6). We sketch the construction of the inverse map of sets: Let a principal G -bundle $P \rightarrow X$ be given. Since the G -action on P is free, there is by Lemma 2.11.15 a G -equivariant map $P \rightarrow EG$ that is unique up to G -equivariant homotopy. Since this map is G -equivariant, it descends to a map $X = P/G \rightarrow BG = EG/G$: this gives us the inverse assignment from isomorphism classes of principal G -bundles over X to $[X, BG]$. \square

In summary, we have achieved the following result:

THEOREM 2.11.17. *For any simplicial set X and any Kan complex F with minimal replacement F^{\min} , there is a bijection*

$$[X, B \operatorname{Aut}(F^{\min})] \xrightarrow{\cong} \left\{ \begin{array}{l} \text{homotopy equivalence classes of} \\ \text{Kan fibrations over } X \text{ with fiber} \\ \text{homotopy equivalent to } F \end{array} \right\}.$$

2.12. Exercises

Basic definitions (simplicial set, nerve, (non-)degenerate simplices

EXERCISE 2.12.1. In continuation of Example 2.4.11, verify that $|N\mathbb{Z}/2| \simeq \mathbf{RP}^\infty$.

EXERCISE 2.12.2. Show that $|N\mathbb{Z}| \simeq S^1$.

Simplicial homotopy

EXERCISE 2.12.3. Show that for a group G and an element $g \in G$, the simplicial map $NG \rightarrow NG$ given by conjugation by g , given explicitly on n -simplices by

$$(* \xrightarrow{g_1} * \xrightarrow{g_2} \dots \xrightarrow{g_n} *) \mapsto (* \xrightarrow{gg_1g^{-1}} * \xrightarrow{gg_2g^{-1}} \dots \xrightarrow{gg_ng^{-1}} *)$$

is homotopic to the identity.

EXERCISE 2.12.4. For G, H groups calculate the Kan complex $\operatorname{map}(NG, NH)$. What is the set of components? What is the homotopy type of each component?

PS2.

EXERCISE 2.12.5 (Generators and relations for Δ). Recall the following distinguished morphisms in the simplicial category Δ :

- For all $n \geq 0$ and $0 \leq i \leq n+1$, the i th face map (at n), denoted $d^i : [n] \rightarrow [n+1]$, is the unique order preserving injection whose image does not contain i .
- For all $n \geq 1$ and $0 \leq i \leq n-1$, the i th degeneracy map (at n), denoted $s^i : [n] \rightarrow [n+1]$, is the unique order preserving surjection that sends i and $i+1$ to i .

(1) Show that any morphism $\varphi : [n] \rightarrow [m]$ in Δ factors uniquely as follows:

$$\varphi = d^{i_1} d^{i_2} \dots d^{i_t} s^{j_1} s^{j_2} \dots s^{j_t},$$

where $0 \leq i_1 < \dots < i_t \leq m$ and $0 \leq j_1 < \dots < j_t < n$.

(2) Show that to give a functor $X : \Delta^{op} \rightarrow \mathbf{Set}$ is equivalent to giving a sequence of sets $\{X[n]\}_{n \geq 0}$ together with maps $d_i : X[n+1] \rightarrow X[n]$, $0 \leq i \leq n+1$, and $s_j : X[n] \rightarrow X[n+1]$, $0 \leq j \leq n$, satisfying the so called *simplicial identities*:

$$d_i d_j = d_{j-1} d_i, \quad i < j,$$

$$d_i s_j = \begin{cases} s_{j-1} d_i, & i < j, \\ 1, & i = j, j+1, \\ s_j d_{i-1}, & i > j+1, \end{cases}$$

$$s_i s_j = s_{j+1} s_i, \quad i \leq j.$$

EXERCISE 2.12.6 (Geometric realization and products). Here, we take for granted the result from Homework Problem 1, that is, the canonical map $|\Delta^n \times \Delta^m| \rightarrow |\Delta^n| \times |\Delta^m|$ is a homeomorphism.

Suppose \mathbf{T} is a *convenient category of topological spaces*, that is¹⁰, a full subcategory of the category of spaces which is cocomplete and has finite products, and such that for any

¹⁰For our purposes. There may be technical definitions in the literature that differ from the one we give here.

$X \in \mathbf{T}, X \times - : \mathbf{T} \rightarrow \mathbf{T}$ preserves colimits. Suppose further that it contains compact spaces and that the forgetful functor to **Top** preserves products of compact spaces. Then, using the previous exercise, show that for all $X, Y \in \mathbf{sSet}$, the canonical map $|X \times Y| \rightarrow |X| \times |Y|$ is an isomorphism in **T**.

(Warning: we do not assume that in general the product in **T** agrees with the one in **Top**, nor in fact that colimits do).

EXERCISE 2.12.7 (Combinatorics of products). Give a description of the nondegenerate $(n+m)$ -simplices of $\Delta^n \times \Delta^m$. Are there nondegenerate simplices in higher dimension?

EXERCISE 2.12.8 (The geometric realization). Show that for a simplicial set X , $|X|$ can be described as a coend, $\int^{[n] \in \Delta^{op}} X_n \times |\Delta^n|$.

Deduce that it is left adjoint to $\text{Sing} : \mathbf{Top} \rightarrow \mathbf{sSet}$.

EXERCISE 2.12.9 (A classical invariant). Let X be a simplicial set, viewed as a functor $\Delta^{op} \rightarrow \mathbf{Set}$. Describe its colimit in geometric terms.

EXERCISE 2.12.10 (Simplicial spheres). Show that $\Delta^n / \partial \Delta^n$ and $\partial \Delta^{n+1}$ have homeomorphic geometric realizations. Are they isomorphic simplicial sets?

EXERCISE 2.12.11 (Horns and boundaries). (1) Describe $(\partial \Delta^n)_k$ as a subset of $(\Delta^n)_k = \text{hom}([k], [n])$.

(2) Do the same for $(\Lambda_i^n)_k$.

EXERCISE 2.12.12 (CW-structures). Recall the n -skeleton $\mathbf{sk}_n X$ from exercise 2.12.14.

(1) Show that for a simplicial set X , $|X| \cong \text{colim}_n |\mathbf{sk}_n X|$.

(2) Let NX_n denote the subset of X_n consisting of nondegenerate simplices¹¹. Show that there is a pushout in **sSet** as follows:

$$\begin{array}{ccc} \coprod_{\sigma \in NX_{n+1}} \partial \Delta^{n+1} & \longrightarrow & \mathbf{sk}_n X \\ \downarrow & & \downarrow \\ \coprod_{\sigma \in NX_{n+1}} \Delta^{n+1} & \longrightarrow & \mathbf{sk}_{n+1} X \end{array}$$

(you should say what the horizontal morphisms are)

(3) Explain how to construct a CW-structure on $|X|$.

EXERCISE 2.12.13. [Product of simplices]

Let us show in detail that $|\Delta^n \times \Delta^m| \cong |\Delta^n| \times |\Delta^m|$. You may freely use the description from exercise 2.12.7.

(1) Explain how to get a continuous map $|\Delta^n \times \Delta^m| \rightarrow |\Delta^n| \times |\Delta^m|$ using functoriality of $|-|$.

(2) Using general topology, show that it suffices to show that it is a bijection on underlying sets (you will want to notice that $|\Delta^n \times \Delta^m|$ is a finite colimit of ordinary simplices).

(3) Let $(x, y) \in |\Delta^n| \times |\Delta^m|$, and write $x = (x_0, \dots, x_n), y = (y_0, \dots, y_m)$.

Set $u^p = \sum_{i=0}^p x_i, v^q = \sum_{j=0}^q y_j$. Arrange the u^p 's and v^q 's in nondecreasing order $w^0 \leq \dots \leq w^{n+m}$ and let $\mu_i = w^i - w^{i-1}$ (with $w^{-1} = 0$). Note that $(\mu_0, \dots, \mu_{n+m}) \in |\Delta^{n+m}|$.

Explain why there are morphisms $f : [n+m] \rightarrow [n]$ and $g : [n+m] \rightarrow [m]$ such that $f_*(\mu_0, \dots, \mu_{n+m}) = (x_0, \dots, x_n)$ and $g_*(\mu_0, \dots, \mu_{n+m}) = (y_0, \dots, y_m)$. Deduce that our map is surjective.

(4) Show that our map is injective.

EXERCISE 2.12.14 (Skeletons). [Suggested grading: 1+3+2+3+1+3*] Let $\Delta_{\leq n}^{op}$ denote the full subcategory of Δ^{op} on the objects $[m], m \leq n$.

¹¹Can you see why $[n] \mapsto NX_n$ is *not* a simplicial set?

Then restriction along the inclusion $i_n : \Delta_{\leq n}^{op} \rightarrow \Delta^{op}$ induces a functor $i_n^* : \mathbf{sSet} \rightarrow \mathbf{sSet}_{\leq n}$ between the categories of presheaves, which has a left adjoint given by left Kan extension. We call it sk_n .

Let cosk_n denote the *right* adjoint to i_n^* and $\mathbf{cosk}_n = \mathrm{cosk}_n i_n^*$.

(1) Show that the unit map $\mathrm{id} \rightarrow i_n^* \mathrm{sk}_n$ and the co-unit map $i_n^* \mathrm{cosk}_n \rightarrow \mathrm{id}$ are isomorphism. (Can you state a more general result for when the co/unit of a “left/right Kan extension” adjunction is an isomorphism?)

(2) Let X be a simplicial set. Give a necessary and sufficient condition for the co-unit $\mathrm{sk}_n i_n^* X \rightarrow X$ to be an isomorphism. We call these simplicial sets n -skeletal, or n -dimensional. We let sk_n denote the composite $\mathrm{sk}_n i_n^*$. Do the analogue of for cosk . Be careful that the necessary and sufficient condition is slightly different.

(3) Show that $\mathrm{sk}_n Y$ is n -skeletal for any $Y \in \mathbf{sSet}_{\leq n}$ and that $\mathrm{cosk}_n Y$ is n -coskeletal for any $Y \in \mathbf{sSet}_{\leq n}$. We call $\mathbf{sk}_n X$ the n -skeleton of X and $\mathbf{cosk}_n Y$ the n -coskeleton.

(4) Show that a simplicial set X is n -coskeletal if and only if for every $m > n$ and every map $f : \partial \Delta^m \rightarrow X$, there exists a unique extension $\Delta^m \rightarrow X$.

(5) Deduce that the nerve of a category C is 2-coskeletal. Is it 1-coskeletal?

(Bonus*) Construct a diagram, natural in the simplicial set X , of the form

$$\mathbf{sk}_0 X \rightarrow \mathbf{sk}_1 X \rightarrow \cdots \rightarrow \mathbf{sk}_n X \rightarrow \cdots$$

Prove that its colimit is X .

EXERCISE 2.12.15 (The Segal condition). [Suggested grading: 1 + 3 + 3 + 3]

(1) Let M be a monoid. Let BM denote the category with exactly one object, and its endomorphism monoid is M (the composition matches M 's multiplication), and let $\mathrm{Seg}(M)$ denote its nerve. Describe $\mathrm{Seg}(M)_n$ in terms of M .

(2) Observe that Seg defines a functor $\mathbf{Mon} \rightarrow \mathbf{sSet}$. Show that it is fully faithful.

(3) We want to describe the essential image of the functor Seg . First observe that $\mathrm{Seg}(M)_0$ is always a singleton. Now, consider the following maps $\rho_k^n : [1] \rightarrow [n]$, given by the inclusion $\{0 < 1\} \cong \{(k-1) < k\} \hookrightarrow [n]$.

- (a) Show that for any monoid M , and for all n , the ρ_k^n induce an isomorphism $\mathrm{Seg}(M)_n \rightarrow \mathrm{Seg}(M)_1 \times \cdots \times \mathrm{Seg}(M)_1 = \prod_{i=1}^n \mathrm{Seg}(M)_1$.
- (b) Describe the multiplication of M in terms of the inverse morphism $\mathrm{Seg}(M)_2 \rightarrow \mathrm{Seg}(M)_1 \times \mathrm{Seg}(M)_1$.
- (c) Let X be a simplicial set, and suppose that X_0 is a singleton and that for all n , the map ρ_k^n induce an isomorphism $X_n \rightarrow \prod_{i=1}^n X_1$ as above. Explain how to define a monoid structure on X_1 , and show that $\mathrm{Seg}(X_1) \cong X$.
- (d) Describe the essential image of Seg ¹².

(4) A functor $M : \Delta^{op} \rightarrow \mathbf{Top}$ is called an E_1 -space¹³ if M_0 is contractible and the Segal maps ρ_k^n from above induce a *weak equivalence* $M_n \rightarrow \prod_{i=1}^n M_1$. We call this the *Segal condition*.

Let (X, x) be a nice pointed topological space, and let $(\Theta X)_n$ be the subspace of $\mathrm{map}(|\Delta^n|, X)$ consisting of the maps that send all vertices of $|\Delta^n|$ to x . Explain why this is a simplicial topological space, and prove the Segal condition for $n = 2$: $(\Theta X)_2 \rightarrow (\Theta X)_1 \times (\Theta X)_1$ is a weak equivalence.¹⁴

PS3.

¹²It turns out that we can perform this construction in any category with finite products instead of \mathbf{Set} , and the conclusion is the same. This is not required for the Homework problem, but you can think about it.

¹³Historically, the term “ E_1 -space” has a more complicated meaning. We owe this simplification to Segal.

¹⁴You can prove for yourself that the Segal condition holds for all n . This encodes the structure of a “monoid up to homotopy” on $(\Theta X)_1 = \Omega X$ - the multiplication itself is defined only up to homotopy, depending on a choice of a homotopy inverse for the map described above.

EXERCISE 2.12.16 (Homotopies and natural transformations). (a) Explain why the nerve functor from categories to simplicial sets preserves products.

(b) Deduce that for every natural transformation $\eta : F \Rightarrow G$ between functors $C \rightarrow D$ induces a simplicial homotopy between $N(F)$ and $N(G)$. In fact, show that the set of natural transformations is in bijection with the set of simplicial homotopies.

(c) Use (b) to find examples of simplicial homotopies that are not “reversible”.

(d) Show that an adjunction between categories induces an equivalence between their geometric realizations.

EXERCISE 2.12.17 (Examples). (a) Find an example of a category whose geometric realization is contractible but has no initial or terminal object.

(b) Show that a category with binary products has a contractible realization. Can you give at least two proofs?

EXERCISE 2.12.18 (A justification for drawings). (a) Describe $\partial\Delta^n$ as a coequalizer of coproducts of $n-1$ -simplices. Deduce a combinatorial description of maps $\partial\Delta^n \rightarrow X$.

(b) Do the same for Λ_k^n .

(c) Why does this “justify” the use of drawings for these simplicial sets?

EXERCISE 2.12.19 (More examples). (a) Let C be the category with two objects a, b , and two arrows from a to b (no other non identity arrow). Compute $|C|$.

(b) Find a preordered set whose geometric realization is S^1 . Can you find one for S^2 ?

EXERCISE 2.12.20 (Homotopies in Kan complexes). Let X be a Kan complex and $x, y \in X_0$, $f, g \in X_1$ edges from x to y .

Show that the existence of a 2-simplex filling any of the following maps $\partial\Delta^2 \rightarrow X$ implies the existence of 2-simplices filling all the other ones.

$$\begin{array}{cccc}
 \begin{array}{ccc} x & \xrightarrow{f} & y \\ \text{id}_x \uparrow & \nearrow g & \\ x & & \end{array} &
 \begin{array}{ccc} x & \xrightarrow{g} & y \\ \text{id}_x \uparrow & \nearrow f & \\ x & & \end{array} &
 \begin{array}{ccc} y & \xrightarrow{\text{id}_y} & y \\ f \uparrow & \nearrow g & \\ x & & \end{array} &
 \begin{array}{ccc} y & \xrightarrow{\text{id}_y} & y \\ g \uparrow & \nearrow f & \\ x & & \end{array}
 \end{array}$$

EXERCISE 2.12.21 (Simplicial homotopy groups). Let X be a simplicial set which is not a Kan complex. Explain everything that goes wrong if one tries to define $\pi_n(X, x)$ as a quotient of $\text{hom}((S^n, *), (X, x))$. In particular, find examples where no such quotient can be isomorphic to $\pi_n(|X|, x)$.

EXERCISE 2.12.22 (Functor categories). (a) Prove that the left adjoint to the nerve functor, h , preserves finite products.

(b) Construct a natural morphism $\text{hom}(NC, ND) \rightarrow N(\text{Fun}(C, D))$, where hom here is the internal hom of simplicial sets. Using (a), prove that it’s an isomorphism.

EXERCISE 2.12.23 (Another homotopy cofinality). Recall (from Homework problem 2) that a functor $f : I \rightarrow J$ is *homotopy cofinal* if for all $j \in J$, $|I_{j/}|$ is contractible.

Draw inspiration from Homework Problem 2, (b) to prove that the inclusion $(\Delta^{inj})^{\text{op}} \rightarrow \Delta^{\text{op}}$ is homotopy cofinal, where Δ^{inj} is the subcategory of Δ with the same objects but only injective maps.

EXERCISE 2.12.24 (Homotopy cofinality). [1.5+0.5+1+1] A functor $f : I \rightarrow J$ is called homotopy cofinal if for all $j \in J$, $|I_{j/}|$ is contractible (rather than connected).

(a) Show that the diagonal functor $\Delta_{\leq 1}^{\text{op}} \rightarrow \Delta_{\leq 1}^{\text{op}} \times \Delta_{\leq 1}^{\text{op}}$ is cofinal. Is it homotopy cofinal?¹⁵ Same questions for the inclusion $\Delta_{\leq 1}^{\text{op}} \rightarrow \Delta^{\text{op}}$.

(b) We wish to show that $\Delta^{\text{op}} \rightarrow \Delta^{\text{op}} \times \Delta^{\text{op}}$ is homotopy cofinal. So fix $([m], [p]) \in \Delta^{\text{op}} \times \Delta^{\text{op}}$, and consider the category $C_{m,p} = \Delta_{([m],[p])}^{\text{op}}$.

¹⁵In fact, one can show that the same holds when replacing 1 with n , for any $n < \infty$.

- (1) Show that there is a functor $S : C_{m,p} \rightarrow C_{m,p}$ sending the object $([k] \rightarrow [m], [k] \rightarrow [p])$ to $([k+1] \rightarrow [m], [k+1] \rightarrow [p])$ which adds a 0 at the beginning.
- (2) Explain how to construct natural transformations $\text{id} \rightarrow S$ and from a constant functor at $([0] \rightarrow [m], [0] \rightarrow [p])$ to S .
- (3) Conclude.

EXERCISE 2.12.25 (Localization). [Suggested grading: 2+2+4+2] (a) For C a small category, describe a natural morphism $\eta_C : C \rightarrow \Pi_1(|C|)$, where Π_1 denotes the fundamental groupoid. You do not have to prove naturality in C . Prove that it is in fact a morphism to $\Pi_1(|C|, C)$, the full subgroupoid of $\Pi_1(|C|)$ whose objects are the vertices of $|C|$, i.e. the objects of C .

(b) Prove that if $C = \mathbf{BG}$, the morphism from (a) is an equivalence - this amounts to proving that the natural morphism $G \rightarrow \pi_1(|\mathbf{BG}|, *)$ is an isomorphism. Deduce that the morphism from (a) is an equivalence whenever C is a groupoid. Prove that it is an isomorphism of groupoids if we restrict to $\Pi_1(|C|, C)$.

For category C , we call $L(C) := \Pi_1(|C|, C)$ for simplicity.

(c) There are two natural morphisms $L(C) \rightarrow L(L(C))$, namely $\eta_{L(C)}$ and $L(\eta_C)$. Prove that they are equal (literally equal). Deduce from this and from (b) that L is left adjoint to the inclusion $\mathbf{Gpd} \rightarrow \mathbf{Cat}$, with unit given by η .

(d) Show that restriction along η_C induces an equivalence of categories $\text{Fun}(L(C), D) \rightarrow \text{Fun}(C, D)^{\text{inv}}$, where $\text{Fun}(C, D)^{\text{inv}}$ is the full subcategory of $\text{Fun}(C, D)$ on functors that map every morphism to an isomorphism. ¹⁶

EXERCISE 2.12.26 (The space of compositions). [Suggested grading: 2+8] Let K be a Kan complex and let $f, g \in K_1$ such that the target of f agrees with the source of g . In this case, we obtain a map $(f, g) : \Delta_1^2 \rightarrow K$. Let $\text{comp}(f, g)$ denote the pullback

$$\begin{array}{ccc} \text{comp}(f, g) & \longrightarrow & \text{map}(\Delta^2, K) \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow{(f, g)} & \text{map}(\Delta_1^2, K) \end{array}$$

where the right map is induced by the inner horn inclusion.

- (a) Show that this simplicial set is non-empty and give an example where it is non-trivial (i.e. it has more than one vertex).
- (b) Show that $\pi_0(\text{comp}(f, g)) \cong *$.

You will actually only need the existence of inner horn fillers up to dimension 3. Assuming that inner horn fillers exist in all dimensions, one can actually prove that $\text{comp}(f, g)$ is contractible. This shows that composition in K is *unique up to contractible choice*.

2.12.1. PS4.

EXERCISE 2.12.27 (finite-dimensional Kan complexes). Show that a finite-dimensional Kan complex is 0-dimensional.

EXERCISE 2.12.28 (Multiplication on π_n). (a) Recall the definition of multiplication on $\pi_n(X, x)$, where X is a Kan complex.

- (b) Show that it is well-defined, i.e. that any two choices of lifts are homotopic.
- (c) Show that it does indeed provide π_n with a group structure : it is unital, associative and has inverses.
- (d) Show that it is abelian when $n \geq 2$.

¹⁶Given a category C and a class of arrows W , a functor $\eta : C \rightarrow E$ with the universal property from above with W instead of “every morphism” is called a localization of C at W , denoted $C[W^{-1}]$. It is unique up to equivalence - the above shows that $C[C^{-1}]$ can be constructed as $\Pi_1(|C|)$.

EXERCISE 2.12.29 (Simplicial groups). Our goal is to show that a simplicial group is automatically a Kan complex. So let G be a simplicial group.

(1) Show the lifting property for horns of Δ^2 . (Hint: a horn with edges x, y can be expressed as a product of horns with edges x and a degeneracy, respectively a degeneracy and y - up to some minor changes).

(2) Show that the lifting property holds for arbitrary horns. (Hint: if you have a horn consisting of $(x_0, \dots, \hat{x}_k, \dots, x_n)$ you may want to find a simplex filling (x_0, \dots, x_m) by induction on m)

EXERCISE 2.12.30 (Classifying spaces). Let BG denote the nerve of the category $\mathbf{B}G$.

(1) Give a concrete description of $\text{map}(BG, BH)$. What is its π_0 ? Its π_1 at a given map $BG \rightarrow BH$?

(2) Let X, Y be Kan complexes, and $x \in X_0$. Show that evaluation at x induces a Kan fibration $\text{map}(X, Y) \rightarrow Y$. Do you know a name for its fiber at $y \in Y$?

(3) Take $X = BG, Y = BH$, what is this fiber? Specialize to $X = B\mathbb{Z}$, and observe that there are two fiber sequences of the form $G \rightarrow ? \rightarrow BG$, but in one of them $?$ is contractible, and not in the other one. (Bonus) What is the difference between those two?

EXERCISE 2.12.31 (Path spaces). Let X be a Kan complex, $x \in X_0$. Recall that the path space PX is the fiber of $ev_0 : \text{map}(\Delta^1, X) \rightarrow X$ at x . Show that PX is contractible.

EXERCISE 2.12.32 (Homotopy equivalence of Kan complexes). In this problem, we let map denote the internal hom of simplicial sets. We take for granted the fact that if Y is a Kan complex and $X \rightarrow X'$ is an arbitrary injection of simplicial sets, $\text{map}(X', Y) \rightarrow \text{map}(X, Y)$ is a Kan fibration (see exercise 1).

A weak equivalence of Kan complexes is a morphism that induces an isomorphism on π_0 and on all simplicial homotopy groups at all basepoints.

(1) Consider a commutative diagram of Kan complexes as follows:

$$\begin{array}{ccccc} & & Z & & \\ & & \downarrow & \searrow \sim & \\ X & \longrightarrow & Y & & Z' \\ & \searrow \sim & \downarrow & \searrow \sim & \downarrow \\ & & X' & \longrightarrow & Y' \end{array}$$

where all the diagonal maps are weak equivalences (as indicated with a \sim). Suppose further that $Z \rightarrow Y$ and $Z' \rightarrow Y'$ are Kan fibrations. Using the long exact sequence in homotopy groups associated with a fibration, show that the induced morphism on pullbacks is a weak equivalence: $X \times_Y Z \rightarrow X' \times_{Y'} Z'$.

(2) Let $X \rightarrow Y$ be a weak equivalence of Kan complexes. Using (1), and working by induction on skeleta, show that for any finite dimensional simplicial set Z , $\text{map}(Z, X) \rightarrow \text{map}(Z, Y)$ is a weak equivalence.

(3) State (but don't prove!) a property similar to (1) for inverse limits along \mathbb{N} of Kan complexes along Kan fibrations.

(4) Accepting the property from (3), show that $\text{map}(Z, X) \rightarrow \text{map}(Z, Y)$ is a weak equivalence of Kan complexes for any Z .

(5) Deduce that $X \rightarrow Y$ is a simplicial homotopy equivalence: there is a map $Y \rightarrow X$ whose composite with our original weak equivalence is simplicially homotopic to the identity in either direction.

EXERCISE 2.12.33 (Principal bundles). Let G be a simplicial group and E a simplicial set. A G -action on E is the data of a map $G \times E \rightarrow E$ such that the two "obvious" morphisms $G \times G \times E \rightarrow E$ agree (one of them multiplies in G and then acts on E , and the second one acts on E twice), and such that the following composite is the identity of E : $E \rightarrow G \times E \rightarrow E$, where the first morphism picks out the neutral element of G .

(1) Define the quotient E/G to be the coequalizer of the action morphism $G \times E \rightarrow E$ and the projection onto the second factor. Show that it can also be described as $[n] \mapsto E_n/G_n$.

A *principal G -bundle over a simplicial set F* is defined to be a simplicial set E with a G -action, together with an identification $E/G \cong F$, and such that the action of G_n on E_n is free for all n .

(2) Let E be a G -bundle over F . Suppose that it has a section, that is, a morphism $s : F \rightarrow E$ such that the composite $F \rightarrow E \rightarrow E/G \cong F$ is the identity. Prove that $E \cong G \times F$ as G -simplicial sets, and that the composite $E \rightarrow E/G \cong F$ gets identified with the projection on the second coordinate $G \times F \rightarrow F$. Deduce that a principal G -bundle over Δ^n is always trivial, i.e. of the form $G \times \Delta^n \rightarrow \Delta^n$ ¹⁷.

(3) Show that if E is a principal G -bundle over F , then the projection $E \rightarrow E/G \cong F$ is a Kan fibration.

HOMEWORK PROBLEM 1 (Mapping spaces). $[1+1+2+2+4+2^*+2^*]$ Our goal is to show that if $A \rightarrow B$ is an injection of simplicial sets, and X is a Kan complex, then $\text{map}(B, X) \rightarrow \text{map}(A, X)$ is a Kan fibration, where map is the internal hom of simplicial sets. In fact, more generally, we will see that if $X \rightarrow Y$ is a Kan fibration, and $A \rightarrow B$ is an injection, then $\text{map}(B, X) \rightarrow \text{map}(B, Y) \times_{\text{map}(A, Y)} \text{map}(A, X)$ is a Kan fibration.

(1) Show that the first goal is indeed a special case of the general one, and that “for any Kan complex X , $\text{map}(A, X)$ is a Kan complex” is a special case of the first one.

(2) Given a commutative diagram consisting of solid arrows

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{\quad} & \text{map}(B, X) \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ \Delta^n & \xrightarrow{\quad} & \text{map}(B, Y) \times_{\text{map}(A, Y)} \text{map}(A, X) \end{array}$$

show that the existence of a dotted arrow making both triangles commute is equivalent to the existence of a dotted lift in a certain diagram of the form:

$$\begin{array}{ccc} \Lambda_k^n \times B \cup \Lambda_k^n \times A & \xrightarrow{\quad} & X \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ \Delta^n \times B & \xrightarrow{\quad} & Y \end{array}$$

(3) Show that the class of arrows $A \rightarrow B$ for which such a lift exists is closed under pushouts (against arbitrary maps), under composition, and under infinite compositions, in the sense that if $A_0 \rightarrow A_1, A_1 \rightarrow A_2, \dots, A_n \rightarrow A_{n+1}, \dots$ belong to this class, then the arrow $A_0 \rightarrow \text{colim}_n A_n =: A_\infty$ is also in this class.

(4) Deduce from (3) that it suffices to show that $\partial\Delta^m \rightarrow \Delta^m$ is in this class, for all m to prove that every injection $A \rightarrow B$ is in there.

(5) (This is the hard combinatorics part) Prove that $\partial\Delta^m \rightarrow \Delta^m$ is in this class.

(6) (Bonus) Show that if $0 < k < n$, then you only need $X \rightarrow Y$ to have lifts against inner horns.

(7) (Bonus') Do the same exercise, but with $\partial\Delta^n \rightarrow \Delta^n$ instead of $\Lambda_k^n \rightarrow \Delta^n$, and observe that if $X \rightarrow Y$ is a fibration and a weak equivalence (an *acyclic fibration*), then so is our map $\text{map}(B, X) \rightarrow \text{map}(B, Y) \times_{\text{map}(A, Y)} \text{map}(A, X)$ ¹⁸.

HOMEWORK PROBLEM 2 (The extra degeneracy trick). $[3+4+3]$ We define two categories that extend Δ : $\Delta \subset \Delta_+ \subset \Delta_\infty$. Δ_+ has just an extra object, namely \emptyset , but it still has ordered

¹⁷It follows that a principal G -bundle is always *locally trivial* - this is a condition we usually have to additionally require in topological spaces, but is automatic in simplicial sets

¹⁸With a bit more work, we could prove that it also suffices for $A \rightarrow B$ to be an injection and a weak equivalence.

maps as arrows, while Δ_∞ has extra maps, namely we allow partial order-preserving functions defined only on a beginning section, e.g. $[2] \rightarrow [1]$, $0 \mapsto 0$, $1 \mapsto 2$, undefined on 2 (so defined on $[n]$ up to some k). We write $[-1] := \emptyset$

(1) Given a simplicial set X , explain what the data of an extension of X to Δ_+ amounts to. What about to Δ_∞ ?

We call a presheaf over Δ_+ an *augmented simplicial set*.

(2) Show that given an augmented simplicial set X with $X_{-1} = *$, the data of an extension to $\Delta_\infty^{\text{op}}$ amounts to the data of a section $X_{-1} \rightarrow X_{|\Delta^{\text{op}}}$ of the canonical map $X_{|\Delta^{\text{op}}} \rightarrow X_{-1}$ together with a homotopy between $\text{id}_{X_{|\Delta^{\text{op}}}}$ and the composite $X_{|\Delta^{\text{op}}} \rightarrow X_{-1} \rightarrow X_{|\Delta^{\text{op}}}$.

(3) Let $X : \Delta_\infty^{\text{op}} \rightarrow C$ be a functor to an arbitrary category. Show that the cocone $X_{|\Delta^{\text{op}}} \rightarrow X_{-1}$ is a colimit diagram.¹⁹

¹⁹This is actually true in for homotopy colimits too, which is extremely useful.

CHAPTER 3

Chain complexes, simplicial abelian groups, and the Dold–Kan correspondence

In a basic course on homological algebra, like `HomAlg`, notions like chain homotopy, and other homotopical-sounding notions were introduced, some of which were recalled in Section 1.2. Our goal in this chapter is to relate those concepts to the simplicial theory introduced in Chapter 2, via the category of simplicial abelian groups i.e., functors $\Delta^{\text{op}} \rightarrow \text{Ab}$.

In particular we'll introduce the Dold–Kan correspondence, which is a slightly funky equivalence between the category of simplicial abelian groups and the category of non-negatively graded chain complexes $\text{Ch}_{\geq 0}(\mathbb{Z})$ of abelian groups, which also descends to the level of homotopy theories. The statement in fact holds verbatim for any abelian category (instead of abelian groups) and suggests a way of ‘doing homological algebra’ in a non-abelian category by studying simplicial objects in that category.

Additional references for this section is [Wei94], Sections 8.3 and 8.4, [GJ99], III.2, and the online resource [Kerodon](#).

3.1. Simplicial groups

DEFINITION 3.1.1. A simplicial object in a category \mathcal{C} is a functor $\Delta^{\text{op}} \rightarrow \mathcal{C}$. In particular a simplicial group G is a functor $\Delta^{\text{op}} \rightarrow \text{Gr}$.

LEMMA 3.1.2. *The underlying simplicial set of any simplicial group is a Kan complex.*

PROOF. [not hard. eg in weibel] □

Recall that we introduced $\pi_n(X, x)$ of a simplicial set as $\pi_n(X, x) = [(\Delta^n, \partial\Delta^n), (X, x)]$ and make the same definition for simplicial groups, applying it to the underlying simplicial set. But for simplicial groups it turns out that there is a much more economical model.

PROPOSITION 3.1.3. *For a simplicial group G set*

$$N_n(G) = \{x \in G_n \mid d_i x = 1 \text{ for } i < n\}$$

and consider the chain complex of (non-abelian) groups

$$1 \leftarrow N_0(G) \xleftarrow{d_1} N_1(G) \xleftarrow{d_2} N_2(G) \cdots$$

where $N_n(G) = \{x \in G_n \mid d_i x = 1 \text{ for } i < n\}$

Then

$$\pi_i(G, 1) \xrightarrow{\cong} Z_n(G)/B_n(G)$$

cycles modulo boundaries, via $(f : (S^n, 1) \rightarrow (G, 1)) \mapsto f(i_n) \in G_n$, where i_n is the unique non-degenerate simplex in $(S^n)_n$.

Furthermore $x_ : \pi_i(G, 1) \xrightarrow{\cong} \pi_i(G, x)$, for any $x \in G_0$, i.e., $\pi_i(G, x)$ is independent of the basepoint, up to canonical isomorphism.*

PROOF. First note that $N_*(G)$ is indeed a chain complex. If $x \in N_n(G)$ then $d_{n-1}d_n x = d_{n-1}d_{n-1}x = 1$. Likewise, if $x \in B_n(G)$ and $g \in Z_n(G)$, then $B_n(G)$ is normal in $Z_n(G)$ as

$$d_n(gd_{n+1}(y)g^{-1}) = d_n(g)d_nd_{n+1}(y)d_n(g)^{-1} = d_n(g)d_n(g)^{-1} = 1$$

Also note that by Lemma 3.1.2 G is Kan, so the definition of $\pi_i(G, 1)$ via homotopy classes of simplicial maps make sense.

Also note that we have an isomorphism

$$(3.1.1) \quad \text{Hom}_{\text{Set}}((\Delta^n, \partial\Delta^n), (G, 1)) \xrightarrow{\cong} Z_n(G)$$

since, unravelling the definitions, a simplicial map $f : (\Delta^n, \partial\Delta^n) \rightarrow (G, 1)$ identifies via $f \mapsto f(i_n)$ with an element $x \in G_n$ such that $d_i(x) = 1$ for all i . Furthermore, this is a group homomorphism.

By Lemma 2.9.6, f is null-homotopic, if and only if there exists a $y \in G_{n+1}$ such that $d_i y = 1$ for $i \leq n$ and $d_{n+1} y = f(i_n)$, i.e., if and only if $f(i_n) \in B_n(G)$, so the canonical isomorphism (3.1.1) induces an isomorphism $\pi_i(G, 1) \xrightarrow{\cong} Z_n(G)/B_n(G)$ as wanted.

The statement about basepoints is clear by functoriality, by considering the map of simplicial sets $x : G \rightarrow G$ given by left multiplication by $x \in G_0$ (viewed as an element in G_n for all n via degeneracy), which has inverse x^{-1} . \square

EXAMPLE 3.1.4. $N_0(\mathbb{Z}\Delta^1) = \mathbb{Z}(0) \oplus \mathbb{Z}(1)$, $N_1(\mathbb{Z}\Delta^1) = \mathbb{Z}((0 \leq 1) - (1 \leq 1))$

$$N_2(\mathbb{Z}\Delta^2) = \mathbb{Z}((0 \leq 1 \leq 2) - \text{????})$$

EXAMPLE 3.1.5 (Chain homotopies via the interval $N_*(\mathbb{Z}\Delta^1)$). By Example 3.1.4 we see that $N_*(\mathbb{Z}\Delta^1)$ is isomorphic to the chain complex I of Section 1.2.2, which plays the role of the unit interval for parametrizing chain homotopies.

Hence we see that, for $f_0, f_1 : C \rightarrow D$ chain maps, specifying a chain homotopy h between f_0 , and f_1 , i.e., a degree one map $H : C \rightarrow D$ such that $H\partial + \partial H = f_0 - f_1$ is equivalent to specifying a map $C \otimes N_*\mathbb{Z}\Delta^1 \rightarrow D$ such that we recover f_0 , and f_1 by precomposing with $N_*(d^i) : \mathbb{Z} \rightarrow N_*\mathbb{Z}\Delta^1$, $i = 0, 1$.

3.2. (Normalized) chains on simplicial abelian groups

In this section we will study simplicial abelian groups, i.e., functors $A : \Delta^{\text{op}} \rightarrow \text{Ab}$. These occur naturally in many connections, as "linearized versions" of simplicial set. In particular, to a simplicial set X we can associate canonical simplicial abelian group $\mathbb{Z}X$ obtained as the composite functor

$$\mathbb{Z}X : \Delta^{\text{op}} \xrightarrow{X} \text{Set} \xrightarrow{\mathbb{Z}} \text{Ab},$$

where \mathbb{Z} is the free functor, left adjoint to the forgetful functor $\text{Ab} \rightarrow \text{Set}$.

To a simplicial abelian group, we can associate several closely related chain complexes. The first is the following:

DEFINITION 3.2.1. For a simplicial abelian group $A : \Delta^{\text{op}} \rightarrow \text{Ab}$, we define the chain complex $C_*(A, \partial)$ of A by $C_n(A) := A_n$ for $n \geq 0$ and $\partial_n : C_n(A) = A_n \rightarrow C_{n-1}(A) = A_{n-1}$ given by $\partial_n = \sum_i (-1)^i d_i$.

This complex $C_*(A)$ is sometimes called the *Moore complex* of A , after John Moore. This is indeed a chain complex:

LEMMA 3.2.2. $\partial_{n-1} \circ \partial_n = 0$ for all n .

PROOF. This is a consequence of the simplicial identities, via the usual calculation which you saw for $\mathbb{Z} \text{Sing}_\bullet X$ in AlgTop, or whatever your first topology course was. As it is the mother

of all sign cancellations, we repeat it for your reading pleasure:

$$\begin{aligned}
\partial_{n-1} \circ \partial_n &= \sum_{i=0}^{n-1} (-1)^i d_i \left(\sum_{j=0}^n (-1)^j d_j \right) \\
&= \sum_{i < j \leq n} (-1)^{i+j} d_i d_j + \sum_{j \leq i \leq n-1} (-1)^{i+j} d_i d_j \\
&= \sum_{i < j \leq n} (-1)^{i+j} d_{j-1} d_i + \sum_{j \leq i \leq n-1} (-1)^{i+j} d_i d_j \\
&= \sum_{i \leq j \leq n-1} (-1)^{i+j+1} d_j d_i + \sum_{j \leq i \leq n-1} (-1)^{i+j} d_i d_j \\
&= 0
\end{aligned}$$

where the third equality uses the simplicial identity $d_i d_j = d_{j-1} d_i$ for $i < j$, and the rest are just rewriting the sum. The signs cancel! ¹ \square

With these definitions, we see that the singular homology of a topological space Y was (secretly) defined as the homology of the complex associated to the simplicial abelian group $\mathbb{Z} \text{Sing}_\bullet Y$.

$$(3.2.1) \quad H_*(Y; \mathbb{Z}) = H_*(\mathbb{Z} \text{Sing}_\bullet Y) .$$

It is hence well-motivated to introduce the following definition of the homology of an arbitrary simplicial set:

DEFINITION 3.2.3. For any simplicial set X , we define the *homology of X (with integer coefficients)* by

$$H_*(X; \mathbb{Z}) := H_*(\mathbb{Z}X) .$$

LEMMA 3.2.4. Let A be a simplicial abelian group.

(1) The subgroups

$$D_n(A) := \sum_i s_i(A_{n-1}) \subseteq A_n$$

for $n \geq 0$ form a sub-chain complex $D_*(A) \subset C_*(A)$.

(2) The subgroups

$$N_n(A) = \bigcap_{i=0}^{n-1} \ker(d_i : A_n \rightarrow A_{n-1})$$

form a sub-chain complex $N_*(A) \subset C_*(A)$.

We call $N_*(A)$ the *normalized chain complex* of A .

(3) The two previous inclusions of chain complexes combine into an isomorphism of chain complexes

$$N_*(A) \oplus D_*(A) \cong C_*(A) ,$$

and in particular we have a natural isomorphism

$$N_*(A) \xrightarrow{\cong} C_*(A)/D_*(A) .$$

(4) The complex $D_*(A)$ has zero homology, $H_*(D_*(A)) = 0$. Hence, the inclusion $N_*(A) \rightarrow C_*(A)$ is a quasi-isomorphism.

REMARK 3.2.5. The lemma says that $N_*(A)$ provides a natural splitting of the quotient $C_*(A) \rightarrow C_*(A)/D_*(A)$. In particular, when finitely generated, the rank of $N_n(A)$ is equal to the rank of the non-degenerate quotient of $C_*(A)$. When $A = \mathbb{Z}X$ this is the number of non-degenerate simplices.

¹Is there also a slick "no hands" argument for this?

SKETCH OF PROOF OF LEMMA 3.2.4. For (1):

$$\partial d_j x = (-1)^j d_j s_j x + (-1)^{j+1} d_{j+1} s_j x + \text{degenerate} = (1 - 1)x + \text{degenerate}$$

The argument for (2) is similar, and let us also skip (3)—we refer to e.g., Section 8.3 in [Wei94] for details.²

The most non-trivial part is (4): Let us abbreviate $D := D_*(A)$ and let us introduce, for $p \geq 0$, a sub-chain complex $F_p D \subseteq D$ by in degree n letting setting $(F_0 D)_n = 0$, $(F_p D)_n = \sum_{i=0}^p s_i(A_{n-1}) \subseteq D_n$ for $p \leq n-1$ and $(F_p D)_n = D_n$ for $p \geq n$. Then $F_* D$ gives us a bounded filtration³ of the complex D , with $(F_p D / F_{p-1} D)_n$ is a quotient of $s_p A_{n-1}$ for $p \leq n$ and zero otherwise.

The convergent spectral sequence $E_{pq}^1 = H_{p+q}(F_p D / F_{p-1} D) \Rightarrow H_{p+q}(D)$ tells us that it is sufficient to prove that $F_p D / F_{p-1} D$ has vanishing homology. A direct computation shows that $h_n := (-1)^{p+1} s_p$ induces a chain contraction of $F_p D / F_{p-1} D$, i.e. a chain homotopy from the identity of the chain complex $F_p D / F_{p-1} D$ to the zero chain self-map. In every degree n

$$\partial h_n + h_{n-1} \partial = 1$$

This implies the assertion. □

3.3. The Dold–Kan correspondence

We can now state the main result relating simplicial abelian groups and chain complexes of abelian groups.

THEOREM 3.3.1 (Dold–Kan-correspondence). *The normalized chain functor*

$$(3.3.1) \quad N_* : \mathbf{sAb} \xrightarrow{\cong} \mathbf{Ch}_{\geq 0}(\mathbb{Z})$$

of Lemma 3.2.4(2) is an equivalence of categories.

Its inverse equivalence

$$K_* : \mathbf{Ch}_{\geq 0}(\mathbb{Z}) \xrightarrow{\cong} \mathbf{sAb} = \mathbf{Fun}(\mathbf{\Delta}^{\text{op}}, \mathbf{Ab})$$

is given by

$$C \mapsto ([n] \mapsto K_n(C) = \bigoplus_{[n] \rightarrow [m]} C_m)$$

obtained by "freely adding degeneracies", or in slightly more functorial language

$$C \mapsto \mathbf{Hom}_{\mathbf{Ch}_{\geq 0}(\mathbb{Z})}(N_*(\mathbb{Z}\mathbf{\Delta}^\bullet), C)$$

where N_ means normalized chains⁴.*

Under the Dold–Kan correspondence $\pi_(A, 0) \cong H(N_*(A)) \cong H(C_*(A))$ by a natural isomorphism.*

We will also see in Proposition 3.5.3 that simplicial homotopy corresponds to chain homotopy, so we get an equivalence on homotopy categories.

REMARK 3.3.2. Recall that there are $\binom{n}{m}$ surjective maps $[n] \twoheadrightarrow [m]$ in $\mathbf{\Delta}$. (And $\binom{n+1}{m+1}$ maps $[m] \hookrightarrow [n]$, by the way.)

REMARK 3.3.3. Note that the more straightforward functor $C_* : \mathbf{sAb} \rightarrow \mathbf{Ch}_{\geq 0}$ does *not* induce an equivalence of categories, as any non-trivial chain complex in the image for instance will be unbounded, because of degeneracies. We'll however see below, that if one is only interested in things "up to homotopy", C_* will also do the job.

²Please note that there was a mistake in the original printing of Weibel's book. There is a correction on his website.

³In each chain degree we have finitely many non-trivial filtration quotients.

⁴Note that we also have a nerve functor floating around somewhere as $\mathbf{\Delta}^n$ is really $\mathbf{\Delta}^n = N([n])$, so adding a star in N_* reduces the confusion in this annoying clash of notation between N_* and N

SKETCH OF THE PROOF. Note that the adjunction looks like the adjunction between geometric realization and the singular functor. We just sketch the argument, and explore this analogy and fill in the missing bits in Exercise 3.9.6 and 3.9.7.

Consider the cosimplicial chain complex $N_*(\mathbb{Z}\Delta^\bullet) : \Delta \rightarrow \text{Ch}_{\geq 0}(\mathbb{Z})$ sending $[n]$ to $N_*(\mathbb{Z}\Delta^n)$.

Step 1: The functor $N_* : \text{sAb} \rightarrow \text{Ch}_{\geq 0}(\mathbb{Z})$ from (3.3.1) is the left Kan extension of the cosimplicial object $N_*(\mathbb{Z}\Delta^\bullet) : \Delta \rightarrow \text{Ch}_{\geq 0}(\mathbb{Z})$ along the composite functor

$$\Delta \rightarrow \text{Fun}(\Delta^{\text{op}}, \text{Set}) \xrightarrow{\mathbb{Z}(-)} \text{Fun}(\Delta^{\text{op}}, \text{Ab}) = \text{sAb},$$

i.e., the functor N_* is uniquely specified by what it does on $\mathbb{Z}\Delta^n$ by universal properties. This is a " \mathbb{Z} -linearized free cocompletion" as in Proposition ??.

Step 2: Now by abstract properties of this extension it has right adjoint K_* given by the above formula. Again see the exercises. Summing up, we have an adjunction $N_* \dashv K_*$, where K_* sends a chain complex C to the simplicial abelian group

$$[n] \mapsto \text{Hom}_{\text{Ch}_{\geq 0}(\mathbb{Z})}(N_*(\mathbb{Z}\Delta^n), C)$$

which, when spelled out, gives us the "free adding simplices" description of K_* from above.

Step 3: We now proceed to show that N_* and K_* induces equivalences of categories. For this it is enough to prove that $K_*N_* \cong \text{id}$ and that K_* is conservative, by Proposition 1.6.4. We start by proving that K_* is conservative. **[add]**

Step 4: We now prove that $K_*N_* \cong \text{id}$: We claim that $\bigoplus_{[n] \rightarrow [m]} N_m A \xrightarrow{\cong} A_n$ via $N_m A \hookrightarrow A_m \xrightarrow{\theta^*} A_n$ where $\theta : [n] \rightarrow [m]$. This follows by induction on n : $N_n A$ is clearly hit. By induction and the decomposition $N_k A \oplus D_k A \cong A_k$ so is $D_n A$.

It is surjective as both

[this is an induction on m]

For the direct verification that $K_*N_* \cong \text{id}$ and $N_*K_* \cong \text{id}$, we refer to Section 8.4 of [Wei94]. \square

REMARK 3.3.4. The Dold-Kan correspondence can be formulated for *any* abelian category instead of abelian groups; this includes in particular categories of modules over a ring R . Moreover, there is a dual version featuring *cosimplicial* objects in an abelian category and *cochain* complexes in that abelian category.

EXERCISE 3.3.5 (Exercise 8.4.2 in [Wei94]). Recall that a semisimplicial object in a category \mathcal{C} is a functor $\Delta_+^{\text{op}} \rightarrow \mathcal{C}$, where Δ_+ is the subcategory of the simplex category Δ containing all objects $[n]$, but only injective maps $[n] \hookrightarrow [m]$. Generating morphisms are the face inclusions $d^i : [n-1] \rightarrow [n]$, for $0 \leq i \leq n$; conversely, a semisimplicial object in \mathcal{C} can be described as a collection X_n of objects in \mathcal{C} , together with face maps $d_i : X_n \rightarrow X_{n-1}$, for $0 \leq i \leq n$, satisfying the semisimplicial identities $d_j d_i = d_i d_{j+1}$ for $i \leq j$.

- (1) Given a chain complex $(C, d^C) \in \text{Ch}_{\geq 0}(\mathbb{Z})$, show that the assignment $[n] \mapsto (\kappa C)_n := C_n$, together with the face maps $d_i : (\kappa C)_n \rightarrow (\kappa C)_{n-1}$ given by d^C for $i = n$ and by 0 for $i < n$, defines a semisimplicial abelian group κC . Show that this gives a functor $\kappa : \text{Ch}_{\geq 0}(\mathbb{Z}) \rightarrow \text{semisAb}$.
- (2) There is a forgetful functor $U : \text{sAb} \rightarrow \text{semisAb}$, induced by the inclusion of categories $\Delta_+ \hookrightarrow \Delta$. Show that U admits a left adjoint $F : \text{semisAb} \rightarrow \text{sAb}$, given on objects by sending $X \in \text{semisAb}$ to the simplicial abelian group $[n] \mapsto \bigoplus_{[n] \rightarrow [m]} X_m$.
- (3) Prove that $K : \text{Ch}_{\geq 0}(\mathbb{Z}) \rightarrow \text{sAb}$ from the Dold-Kan correspondence is naturally isomorphic to the composite functor $F \circ \kappa$.

3.4. Monoidality of the Dold-Kan correspondence and the Eilenberg-Zilber Theorem

Recall that we introduced a symmetric monoidal structure on $\text{Ch}_{\geq 0}$ in Section 1.2 as the well-known way of tensoring chain complexes together.

Similarly, \mathbf{sAb} also carries a symmetric monoidal structure by considering the level-wise monoidal product, which furthermore makes $\mathbb{Z}(-) : \mathbf{sSet} \rightarrow \mathbf{sAb}$ into a symmetric monoidal functor, i.e. there are natural isomorphisms

$$\mathbb{Z}(X \times Y) \cong \mathbb{Z}X \otimes \mathbb{Z}Y.$$

Note however that monoidal structures on $\mathbf{Ch}_{\geq 0}(\mathbb{Z})$ and \mathbf{sAb} are defined in a somewhat different way, so it is not obvious to what extent the Dold-Kan correspondence is compatible with these monoidal structures. It turns out that it is not totally compatible, but close...

First recall some definitions.

DEFINITION 3.4.1. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between monoidal categories is said to be (strict) monoidal if there exists an isomorphism $F(A \otimes B) \cong F(A) \otimes F(B)$, natural in A and B . It is said to be lax monoidal if there just exists a natural map $F(A) \otimes F(B) \rightarrow F(A \otimes B)$, and it is said to be oplax monoidal if a natural map exists in the other direction. It is said to be (strong, lax, or oplax) *symmetric* monoidal if the map can be chosen compatible with the symmetry isomorphism on both sides.

We now proceed to investigate the monoidality of Dold-Kan. Let A and B be simplicial abelian groups. We want define a map $C_*(A) \otimes C_*(B) \rightarrow C_*(A \otimes B)$.

For $a \in A_p$ and $b \in B_q$, define

$$\nabla_{A,B}(a \otimes b) := \sum_{(p,q)\text{-shuffle } (\mu,\nu)} \text{sgn}(\mu, \nu) s_\nu(a) \otimes s_\mu(b),$$

where the sum runs over strictly increasing maps $(s_\mu, s_\nu) : [n] \rightarrow [p] \times [q]$ with $n = p+q$ i.e., non-degenerate n -simplices $(s_\mu, s_\nu) : \Delta^n \rightarrow \Delta^p \times \Delta^q$, which are also called (p, q) -shuffles, and they identify with permutations $(\mu, \nu) = (\mu_1, \dots, \mu_p, \nu_1, \dots, \nu_q)$ of the n -element set $\{1, \dots, p+q\}$ with $\mu_1 < \dots < \mu_p$ and $\nu_1 < \dots < \nu_q$, and $\text{sgn}(\mu, \nu)$ denotes the sign of the permutation.

The formula for ∇ is given here for unnormalized chains, but it induces a map $\nabla_{A,B} : N_*(A) \otimes N_*(B) \rightarrow N_*(A \otimes B)$ on normalized chains that is called the *Eilenberg-Zilber map*.

PROPOSITION 3.4.2 (Lax symmetric monoidality of Dold-Kan correspondence). *The Eilenberg-Zilber maps*

$$(3.4.1) \quad \nabla_{A,B} : N_*(A) \otimes N_*(B) \rightarrow N_*(A \otimes B)$$

make the normalized chain functor $N_ : \mathbf{sAb} \rightarrow \mathbf{Ch}_{\geq 0}$ into a lax symmetric monoidal functor.*

We can also define maps in the opposite direction

$$\Delta_{A,B} : N_*(A \otimes B) \rightarrow N_*(A) \otimes N_*(B),$$

the so-called Alexander-Whitney maps given by

$$a_n \otimes b_n \mapsto \sum_{p+q=n} \text{front}_p(a_n) \otimes \text{back}_q(b_n)$$

where front_p is the "front p -face" map induced by precomposing with $[p] \hookrightarrow [n]$ sending $i \mapsto i$ and back_q is the "back q -face" map induced by precomposing with $[q] \hookrightarrow [n]$ sending $i \mapsto i+p$.

PROPOSITION 3.4.3 (OpLax monoidality of Dold-Kan correspondence). *The Alexander-Whitney maps*

$$\Delta_{A,B} : N_*(A \otimes B) \rightarrow N_*(A) \otimes N_*(B)$$

exhibit N_ as oplax monoidal (but not oplax symmetric monoidal!).*

The fact that N_* is lax monoidal implies that it translates an algebra A in simplicial abelian groups to a differential graded algebra $N_*(A)$. If A is commutative, so is $N_*(A)$: this is because N_* is also symmetric.

By an elementary result that carries the pompous name *doctrinal adjunctions* this implies automatically that the right adjoint $K : \mathbf{Ch}_{\geq 0}(\mathbb{Z}) \rightarrow \mathbf{sAb}$ inherits a lax monoidal structure and an oplax symmetric monoidal structure [ref!]. Again, K will not be lax *symmetric* monoidal, but the symmetry will be satisfied *up to all higher coherent homotopy*, see [Ric03] (it’s E_∞ for those who know what that means). [**where in Richter is this, and how can something be implied by a concept? Explain!**]

The Eilenberg-Zilber and Alexander-Whitney maps satisfy the following:

PROPOSITION 3.4.4. *For simplicial abelian groups A and B , the composition*

$$N_*(A) \otimes N_*(B) \xrightarrow{\text{Eilenberg-Zilber}} N_*(A \otimes B) \xrightarrow{\text{Alexander-Whitney}} N_*(A) \otimes N_*(B)$$

is the identity, while the composition

$$N_*(A \otimes B) \xrightarrow{\text{Alexander-Whitney}} N_*(A) \otimes N_*(B) \xrightarrow{\text{Eilenberg-Zilber}} N_*(A \otimes B)$$

is chain homotopic to the identity.

In particular N_ is strong symmetric monoidal on the homotopy category.*

We’ll skip the proof. [ref?]

3.5. Homotopy properties of the Dold–Kan correspondence

DEFINITION 3.5.1. Two maps $f_0, f_1 : A \rightarrow B$ of simplicial abelian groups are called *path homotopic* if there exists map $H : A \otimes \mathbb{Z}\Delta^1 \rightarrow B$ such that $A \cong A \otimes \mathbb{Z}\Delta^0 \xrightarrow{d^1} A \otimes \mathbb{Z}\Delta^1 \xrightarrow{H} B$ equals f_i for $i = 0, 1$. A map $f : A \rightarrow B$ of simplicial abelian groups is a *homotopy equivalence* if it has a homotopy inverse.

A map $f : A \rightarrow B$ of simplicial abelian groups is called a *weak equivalence* if the map of the underlying simplicial sets is a weak equivalence.

One can directly observe by means of the Dold–Kan correspondence:

COROLLARY 3.5.2. *A map $f : A \rightarrow B$ of simplicial abelian groups is a weak equivalence if and only if $N_*(f) : N_*(A) \rightarrow N_*(B)$ is a quasi-isomorphism.*

PROOF. This follows from the natural isomorphisms $\pi_*(A, 0) \cong H(N_*(A))$ for any simplicial abelian group A , and the fact that there are isomorphisms $\pi_*(A, 0) \cong \pi_*(A, a)$, for any vertex a of A (any element $a \in A_0$) induced by the “addition with a ” (the map of simplicial sets $A \rightarrow A$ given on n -simplices by $(-) + s_0^n(a) : A_n \rightarrow A_n$). \square

PROPOSITION 3.5.3 (Addendum to the Dold–Kan correspondence). *Let A, B be simplicial abelian groups and let $f, g : A \rightarrow B$ be simplicial group homomorphisms. Then there is a canonical correspondence between the sets of:*

- *homotopies of maps of simplicial abelian groups from f to g , i.e., maps $H : A \otimes \mathbb{Z}\Delta^1 \rightarrow B$ restricting to f and g , respectively, on the sub-simplicial abelian groups $A \otimes d^1\mathbb{Z}\Delta^0$ and $A \otimes d^0\mathbb{Z}\Delta^0$.*
- *chain homotopies between the chain maps $N_*(f), N_*(g) : N_*(A) \rightarrow N_*(B)$.*

i.e.,

$$[A, B]_{\mathbf{sAb}/\text{path h.e.}} \xrightarrow{\cong} [N_*A, N_*B]_{\mathcal{K}(\mathbb{Z})}$$

PROOF. Giving a simplicial homotopy $H : A \otimes \mathbb{Z}\Delta^1 \rightarrow B$ is equivalent to a chain map $N_*(H) : N_*(A \otimes \mathbb{Z}\Delta^1) \rightarrow N_*(B)$, as N_* is fully faithful. By Example 3.1.5 giving a chain map between is the same as specifying $N_*(A) \otimes N_*(\mathbb{Z}\Delta^1) \rightarrow N_*(B)$, restricting to f and g on $N_*(A) \otimes N_0(\mathbb{Z}\Delta^1)$.

However these notions are in bijective correspondence: To go from a simplicial homotopy to a chain homotopy precompose with the Eilenberg-Zilber map $N_*(A) \otimes N_*(\mathbb{Z}\Delta^1) \xrightarrow{EZ} N_*(A \otimes \mathbb{Z}\Delta^1)$.

To go the other way we use the Alexander-Whitney map $N_*(A \otimes \mathbb{Z}\Delta^1) \xrightarrow{AW} N_*(A) \otimes N_*(\mathbb{Z}\Delta^1)$. Unravelling the definitions show that these operations are each others inverses. \square

REMARK 3.5.4. By these correspondences, we see that just like for chain complexes, path homotopy equivalence and weak equivalence are different concepts in general. E.g., $0 \rightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \rightarrow 0$ is quasi-isomorphic to $0 \rightarrow 0 \rightarrow \mathbb{Z}/n \rightarrow 0$ but not chain homotopy equivalent, when n in \mathbb{Z} is not a unit.

Note that $\mathbb{Z}(-)$ sends simplicial homotopies to simplicial homotopies, as it is monoidal. The following is slightly less clear:

PROPOSITION 3.5.5. *The functor $\mathbb{Z}(-) : \mathbf{sSet} \rightarrow \mathbf{sAb}$ preserves weak equivalences.*

It has an important special case that the homology of a simplicial set agrees with the homology of its geometric realization, which we start by establishing:

PROPOSITION 3.5.6. *The weak equivalence $X \rightarrow \mathrm{Sing}_\bullet |X|$ for $X \in \mathbf{sSet}$ induces an isomorphism*

$$H_*(X; \mathbb{Z}) \cong \pi_*(\mathbb{Z}X) \xrightarrow{\cong} \pi_*(\mathbb{Z} \mathrm{Sing}_\bullet(|X|)) \cong H_*(|X|; \mathbb{Z}).$$

PROOF. The main idea is as follows: We can use a contracting homotopy $h : \Delta^n \times \Delta^1 \rightarrow \Delta^n$ of the n -simplex to one of its vertices (by giving a functor $[n] \times [1] \rightarrow [n]$) in order to obtain a contracting homotopy $\mathbb{Z}\Delta^n \otimes \mathbb{Z}\Delta^1 \rightarrow \mathbb{Z}\Delta^n$ and then via Proposition 3.5.3 a chain homotopy equivalence between $N_*(\mathbb{Z}\Delta^n)$ and $N_*(\mathbb{Z}\Delta^0) = \mathbb{Z}$. Together with (3.2.1), this tells us that the unit map $\Delta^n \rightarrow \mathrm{Sing}_\bullet |\Delta^n|$ induces a weak equivalence $\mathbb{Z}\Delta^n \rightarrow \mathbb{Z} \mathrm{Sing}_\bullet |\Delta^n|$ (because for this we just need that it induces a quasi-isomorphism after taking N_* by Corollary 3.5.2 — which we have just seen). Using this, one shows inductively that $\mathbb{Z}\mathrm{sk}_n X \rightarrow \mathbb{Z} \mathrm{Sing}_\bullet |\mathrm{sk}_n X|$ is a weak equivalence and then, finally, that the same is true for $\mathbb{Z}X \rightarrow \mathbb{Z} \mathrm{Sing}_\bullet |X|$. **[Can we do this easier, by using that weak equivalence implies homotopy equivalence in ss]** \square

PROOF OF PROPOSITION 3.5.5. Suppose that $f : X \rightarrow Y$ is a weak equivalence. Then the square

$$(3.5.1) \quad \begin{array}{ccc} \mathbb{Z}X & \longrightarrow & \mathbb{Z} \mathrm{Sing}_\bullet |X| \\ \downarrow f & & \downarrow \mathbb{Z} \mathrm{Sing}_\bullet |f| \\ \mathbb{Z}Y & \longrightarrow & \mathbb{Z} \mathrm{Sing}_\bullet |Y| \end{array}$$

commutes, and the horizontal maps are weak equivalences by Proposition 3.5.6. Since $H_*(\mathbb{Z} \mathrm{Sing}_\bullet -)$ is the functor taking singular homology of a space and since $|f|$ is a weak equivalence (both is true by definition), it suffices to see in the *topological* setting that weak equivalences induce homology isomorphisms — and this is a statement that we have proven in AlgTop2. (For a direct proof, see [GJ99] around page 173.) \square

PROPOSITION 3.5.7. *The free forgetful adjunction $\mathbb{Z} \dashv U$ passes to the homotopy category to induce bijections*

$$[\mathbb{Z}X, A]_{\mathrm{Ho}(\mathbf{sAb})} \cong [\mathbb{Z}X, A]_{\mathbf{sAb}/\text{path h.e.}} \cong [X, UA]_{\text{simp. homotopy}} \cong [X, UA]_{\mathrm{Ho}(\mathbf{sSet})}$$

PROOF. The middle bijection follows as $\mathbb{Z}(X \times \Delta^1) \cong \mathbb{Z}X \otimes \mathbb{Z}\Delta^1$. The left-most isomorphism follows as morphisms in the homotopy category agrees with morphisms up to path homotopy when the source is levelwise free as this holds for chain complexes, using the Dold-Kan correspondence. The right-most isomorphism follows as UA is Kan as simplicial set by Lemma 3.1.2 \square

3.6. Dold-Kan correspondence for bisimplicial groups

For later use, we'll now formulate the so-called *generalized Eilenberg-Zilber Theorem* using bisimplicial abelian groups. If you are eager to move on skip it now, and return to it when you need it.

DEFINITION 3.6.1. A bisimplicial abelian group is a functor $A : \Delta^{\text{op}} \times \Delta^{\text{op}} \rightarrow \text{Ab}$.

Via the Dold-Kan correspondence applied to the simplicial objects in the abelian category of simplicial abelian groups, the category ssAb of bisimplicial abelian groups is equivalent to the category $\text{Ch}_{\geq 0}(\text{Ch}_{\geq 0}(\mathbb{Z}))$ of first quadrant double chain complexes of abelian groups. We denote by $C(A)$ the unnormalized version of this first quadrant double complex associated to a bisimplicial abelian group; concretely, $C_{p,q}(A) = A_{p,q}$. The alternating sums of the face maps coming from the two copies of Δ give us the horizontal and vertical differential, respectively.

Any bisimplicial abelian group A (more generally: any bisimplicial object in *any* category) can be turned into a simplicial one by precomposing with the diagonal functor $\Delta^{\text{op}} \rightarrow \Delta^{\text{op}} \times \Delta^{\text{op}}$. We call this simplicial object the *diagonal* $d(A)$ of A .

THEOREM 3.6.2 (Generalized Eilenberg-Zilber Theorem of Dold-Puppe). *For any bisimplicial abelian group A , there is a natural chain homotopy equivalence*

$$C_*(d(A)) \simeq \text{Tot}_{\oplus} C_*(A)$$

where Tot_{\oplus} denotes the \oplus -totalization of the double complex $C_*(A)$.

SKETCH OF THE PROOF. We give the idea and refer to Theorem IV.2.5 in [GJ99] for the details. For $p, q \geq 0$, we have the bisimplicial set $\Delta^{p,q} = \text{Hom}_{\Delta}(-1, [p]) \times \text{Hom}_{\Delta}(-2, [q])$.

Every bisimplicial abelian group can be built from the bisimplicial abelian groups $\mathbb{Z}\Delta^{p,q}$ via colimits, and both functors $C_*(d(-)), \text{Tot}_{\oplus} C_*(-) : \text{ssAb} \rightarrow \text{Ch}_{\geq 0}(\mathbb{Z})$ preserve colimits, hence the proof of the statement can be reduced to the case $A = \mathbb{Z}\Delta^{p,q}$.

Proposition 3.4.4 gives us for simplicial sets X and Y a homotopy equivalence $C_*(X \times Y) \simeq C_*(X) \otimes C_*(Y)$ — this is the ‘ordinary’ Eilenberg-Zilber Theorem. For $X = \Delta^p$ and $Y = \Delta^q$, this will give us, together with the identification $d(\Delta^{p,q}) = \Delta^p \times \Delta^q$, a chain homotopy equivalence

$$C_*(d\Delta^{p,q}) \simeq C_*(\Delta^p) \otimes C_*(\Delta^q).$$

Moreover, one can directly observe an isomorphism of chain complexes $C_*(\Delta^p) \otimes C_*(\Delta^q) \cong \text{Tot}_{\oplus} C_*(\Delta^{p,q})$ (which is natural in the two simplicial variables, i.e. as a functor out of $\Delta \times \Delta$). This implies the assertion in the special case $A = \mathbb{Z}\Delta^{p,q}$. \square

3.7. Eilenberg MacLane spaces

For any abelian group G , we can consider the chain complex whose only non-trivial term is G is degree n , where $n \geq 0$ is fixed; we denote this complex by $G[n]$. By applying the functor $K_* : \text{Ch}_{\geq 0}(\mathbb{Z}) \rightarrow \text{sAb}$ we get a simplicial abelian group whose underlying simplicial set we denote by $K(G, n)$, i.e.

$$K(G, n) = UK_*(G[n])$$

for the forgetful functor U from simplicial abelian groups to simplicial sets. Thanks to the Dold-Kan correspondence, we know that it satisfies $\pi_n(K(G, n)) \cong G$ and $\pi_k(K(G, n)) = 0$ for $k \neq n$ (this is always with respect to the canonical basepoint 0 that any simplicial abelian group has). In other words, the Kan complexes $K(G, n)$ are simplicial analogues of the Eilenberg-MacLane spaces that we have constructed as topological spaces in AlgTop2. Note that for $n = 1$, we can get a $K(G, 1)$ also for a non-abelian group, but not via the Dold-Kan correspondence; instead, we can take the nerve of the category with one object and automorphism group G .

DEFINITION 3.7.1. For any simplicial set X we abbreviate $C_*(X) = C_*(\mathbb{Z}X)$ and for an abelian group G define

$$C^*(X; G) = \text{Hom}(C_*(X), G)$$

T and cohomology of X with coefficients in G as

$$H^*(X; G) = H^*(C^*(X; G)).$$

The following result is a simplicial version of the fact that Eilenberg–MacLane spaces represent cohomology:

THEOREM 3.7.2. *For any simplicial set X and any abelian group G , there are natural isomorphisms*

$$H^n(X; G) \cong [X, K(G, n)] , \quad n \geq 0.$$

PROOF. Via the (Quillen) adjunction between $\mathbb{Z}(-)$ and the forgetful functor U from simplicial abelian groups to simplicial sets (Remark 3.7.5), the homotopy classes $[X, K(G, n)]$ of maps $X \rightarrow K(G, n)$ can be identified with homotopy classes of maps $\mathbb{Z}X \rightarrow K_*(G[n])$ of simplicial abelian groups; this uses the definition $K(G, n) := UK_*(G[n])$. Using the Quillen equivalence $N_* \dashv K_*$, the homotopy classes of maps $\mathbb{Z}X \rightarrow K_*(G[n])$ are in bijection to chain maps $N_*(X) \rightarrow G[n]$ up to chain homotopy; this follows from Proposition 3.5.3. In summary,

$$[X, K(G, n)]_{\text{sSet}} \cong [\mathbb{Z}X, K_*(G[n])]_{\text{sAb}} \cong [N_*(X), G[n]]_{\text{Ch}_{\geq 0}} \cong [C_*(X), G[n]]_{\text{Ch}_{\geq 0}} \cong H^n(X; G)$$

(it is suppressed here that we take the homotopy classes of maps first in sSet , then in sAb and finally in $\text{Ch}_{\geq 0}(\mathbb{Z})$). As a last step, we realize that $[N_*(X), G[n]] \cong [C_*(X), G[n]] \cong H^n(X; G)$.
[rewrite without model category language] \square

The Dold–Kan correspondence can be used to decompose a simplicial abelian group into a product of Eilenberg–MacLane spaces:

THEOREM 3.7.3 (Simplicial Dold–Thom theorem). *Suppose A is a simplicial abelian group. Then as a simplicial set $A \simeq \prod_i K(\pi_i(A), i)$, non-canonically.*

We follow the proof given in [GJ99], page 175. The key step is the following lemma, which uses crucially that \mathbb{Z} has global dimension 1.

LEMMA 3.7.4. *Any chain complex of abelian groups is quasi-isomorphic to its homology*

PROOF. Let C be a chain complex of abelian groups, and denote by Z_n and B_n the subgroups of C_n given by cycles and boundaries, respectively. We can pick an epimorphism $r_n : F_n \rightarrow Z_n$, where F_n is a free abelian group, and define

$$K_n := \ker \left(F_n \xrightarrow{r_n} Z_n \rightarrow H_n(C) \right) .$$

By definition the restriction of r_n to K_n factors through a map $r'_n : K_n \rightarrow B_n$. Note that we have an epimorphism $C_{n+1} \rightarrow B_n$. Since K_n , as subgroup of a free abelian group, is also free, we can lift r'_n to a map $K_n \rightarrow C_{n+1}$. We denote by $F_n C$ the chain complex

$$\cdots \rightarrow 0 \rightarrow K_n \rightarrow F_n \rightarrow 0 \rightarrow \cdots$$

with K_n in degree $n+1$ and F_n in degree n . The epimorphism $F_n C \rightarrow H_n C$ yields a quasi-isomorphism $q_n : F_n C \rightarrow H_n C[n]$, where $H_n C[n]$ is the notation for the abelian group $H_n C$ seen as chain complex supported in degree n . The maps $K_n \rightarrow C_{n+1}$ and $F_n \rightarrow Z_n \subseteq C_n$ previously constructed give us a map $p_n : F_n C \rightarrow C$ that induces an isomorphism $H_n(F_n C) \cong H_n C$. As a result, we have quasi-isomorphisms

$$\bigoplus_{n \geq 0} H_n C[n] \xleftarrow{\cong} \bigoplus_{n \geq 0} F_n C \xrightarrow{\cong} C .$$

Moreover, one can observe that the canonical map $\bigoplus_{n \geq 0} H_n C[n] \rightarrow \prod_{n \geq 0} H_n C[n]$ from the coproduct to the product of chain complexes is an isomorphism. \square

PROOF OF THEOREM 3.7.3. The theorem essentially follows from Lemma 3.7.4, together with our preparations, taking $C = N_*(A)$, i.e. to

$$\prod_{n \geq 0} \pi_n(A)[n] \xleftarrow{\simeq} \bigoplus_{n \geq 0} F_n N_*(A) \xrightarrow{\simeq} N_*(A)$$

and apply the weak inverse functor $K_* : \text{Ch}_{\geq 0}(\mathbb{Z}) \rightarrow \text{sAb}$ to N_* . This translates the quasi-isomorphisms above to weak equivalences and yields

$$\prod_{n \geq 0} K(\pi_n(A), n) \xleftarrow{\simeq} \bigoplus_{n \geq 0} K(F_n N_*(A)) \xrightarrow{\simeq} K N_*(A) \cong A .$$

□

REMARK 3.7.5 (Model structures on sAb and $\text{Ch}_{\geq 0}(\mathbb{Z})$). For the people familiar with model categories, let us just add what some of this means in that language:

There is a model structure on simplicial abelian groups in which the weak equivalences are the ones from Definition 3.5.1 and fibrations are the maps that are Kan fibrations on the underlying simplicial sets (cofibrations are then determined by the choice of weak equivalences and of fibrations), thereby making the adjunction between $\mathbb{Z}(-) : \text{sSet} \rightarrow \text{sAb}$ and the forgetful functor $\text{sAb} \rightarrow \text{sSet}$ into a Quillen adjunction.

There is also a model structure on $\text{Ch}_{\geq 0}(\mathbb{Z})$ in which weak equivalences are quasi-isomorphisms and fibrations are degree-wise epimorphisms (cofibrations are again determined by that; it turns out that they are exactly degree-wise monomorphisms with degree-wise *projective* cokernel⁵). Then the adjunction $N_* \dashv K$ becomes a Quillen equivalence.

3.8. The Hurewicz map

The Hurewicz map that we have discussed in AlgTop2 relates the homology of a space to its homotopy groups. We will now discuss a simplicial version of this map.

If X is a simplicial set and $A \subset X$ a simplicial subset, we can define the *homology of X relative to A* by $H_*(X, A; \mathbb{Z}) := H_*(\mathbb{Z}X/\mathbb{Z}A)$. For a simplicial set X with base point $*$, we will use the shorthand $\tilde{\mathbb{Z}}X := \mathbb{Z}X/\mathbb{Z}*$ and refer to $\tilde{H}_*(X; \mathbb{Z}) := H_*(X, *; \mathbb{Z})$ as the *reduced homology*. By the Dold–Kan correspondence, we have

$$(3.8.1) \quad \pi_*(\mathbb{Z}X/\mathbb{Z}*, 0) \cong \tilde{H}_*(X; \mathbb{Z}) .$$

LEMMA 3.8.1. *For pointed simplicial sets X and Y , we have a natural isomorphism of simplicial abelian groups*

$$\tilde{\mathbb{Z}}(X \vee Y) \xrightarrow{\cong} \tilde{\mathbb{Z}}X \times \tilde{\mathbb{Z}}Y .$$

PROOF. The functor $\tilde{\mathbb{Z}}$ from pointed simplicial sets to simplicial abelian groups is a left adjoint and hence takes coproducts to coproducts. But finite coproducts in simplicial abelian groups sAb are the same as finite products (by the additive structure), so the map is an isomorphism. (As a sanity check, we can also observe directly that the map induces an isomorphism on π_i by noting that this is just the isomorphism $\tilde{H}_i(X \vee Y) \xrightarrow{\cong} \tilde{H}_i(X) \oplus \tilde{H}_i(Y)$ that we have encountered in topology.) □

For any simplicial set X , we denote by $h : X \rightarrow \mathbb{Z}X$ the unit of the adjunction $\mathbb{Z}(-) \dashv U$ (we suppress the forgetful functor $U : \text{sAb} \rightarrow \text{sSet}$ in the notation here). This map is also called the *Hurewicz map*. It sends an n -simplex $\sigma \in X_n$ to the corresponding basis element σ in the free abelian group $\mathbb{Z}X_n = (\mathbb{Z}X)_n$.

⁵For abelian groups/ \mathbb{Z} -modules, “projective” just means *free*.

DEFINITION 3.8.2. Let X be a connected Kan complex with base point $*$. Then for $n \geq 1$ we define the *Hurewicz morphism* as the group homomorphism

$$h_n : \pi_n(X, *) \xrightarrow{\text{Hurewicz map from above}} \pi_n(\mathbb{Z}X, *) \rightarrow \pi_n(\tilde{\mathbb{Z}}X, 0) \stackrel{(3.8.1)}{\cong} \tilde{H}_*(X; \mathbb{Z}) .$$

One can now formulate and prove the following version of the Hurewicz Theorem(s); for the details we refer to Section III.3 in [GJ99].

PROPOSITION 3.8.3. *Let X be a connected pointed Kan complex. Then the Hurewicz morphism $h_1 : \pi_1(X, *) \rightarrow \tilde{H}_1(X; \mathbb{Z})$ induces an isomorphism*

$$\pi_1(X, *) / [\pi_1(X, *), \pi_1(X, *)] \xrightarrow{\cong} \tilde{H}_1(X; \mathbb{Z}) ,$$

where $\pi_1(X, *) / [\pi_1(X, *), \pi_1(X, *)]$ is the abelianization of the group $\pi_1(X, *)$.

THEOREM 3.8.4. *Let $n \geq 1$ and let X be a pointed n -connected Kan complex, i.e. $\pi_i(X, *) = 0$ for $0 \leq i \leq n$. Then the Hurewicz morphism $h_i : \pi_i(X, *) \rightarrow \tilde{H}_i(X; \mathbb{Z})$ is an isomorphism for $i = n + 1$ and an epimorphism for $i = n + 2$.*

3.9. Exercises

PS5.

EXERCISE 3.9.1 (Symmetric monoidal structure, chain complexes). (1) Recall the monoidal structure on $\text{Ch}_{\geq 0}(\mathbb{Z})$ given by $(C \otimes D)_n = \bigoplus_{p+q=n} C_p \otimes D_q$, with differential $\partial(c \otimes d) = (\partial c) \otimes d + (-1)^{|c|} c \otimes (\partial d)$.

(2) Show that $C \otimes D \rightarrow D \otimes C$ given by $c \otimes d \mapsto (-1)^{|c||d|} d \otimes c$ is a natural isomorphism, where $|c|$ denotes the degree of c , i.e. the n such that $c \in C_n$. Show that $c \otimes d \mapsto d \otimes c$ is *not* even a chain morphism in general.

(3) Explain what a commutative algebra in $\text{Ch}_{\geq 0}(\mathbb{Z})$ is, in more concrete terms. They are called *commutative differential graded algebras*, CDGAs for short.

EXERCISE 3.9.2 (Symmetric monoidal structure, simplicial abelian groups). (1) Recall the monoidal structure on \mathbf{sAb} given by $(A \otimes B)_n = A_n \otimes B_n$.

(2) Show that $A \otimes B \rightarrow B \otimes A$ given by $a \otimes b \mapsto b \otimes a$ is a natural isomorphism.

(3) Explain what a commutative algebra in \mathbf{sAb} is, in more concrete terms. They are called simplicial commutative rings. Is there a possible conflict in terminology ?

EXERCISE 3.9.3 (Semi-additivity). A category C is called *pointed* if it has an initial object and a terminal object, and the unique morphism from the former to the latter is an isomorphism. We call such an object a zero object, and denote it by 0 . In this case, any pair of objects x, y has a *zero morphism* $x \rightarrow y$ given by $x \rightarrow 0 \rightarrow y$.

(1) Suppose C is pointed and has finite coproducts and products. Explain how to define a natural morphism $X \coprod Y \rightarrow X \times Y$. If this morphism is an isomorphism, we say C is *semi-additive*, and call \times and \coprod the direct sum, denoted \oplus .

(2) Explain how, in a semi-additive category, a morphism $X_1 \oplus \dots \oplus X_n \rightarrow Y_1 \oplus \dots \oplus Y_m$ can be represented by a matrix of morphisms. Convince yourself that matrix multiplication (where instead of multiplying, you compose) corresponds to composition. ⁶

(3) Let C be a semi-additive category and $x, y \in C$. Define a commutative monoid structure on $\text{hom}(x, y)$. Show that if you want a lift $C^{\text{op}} \times C \rightarrow \mathbf{CMon}$ of the hom-functor, then it has to be the one you just defined.

(4) Using the (ordinary) Yoneda lemma, observe that any natural morphism $\text{hom}(-, c) \rightarrow \text{hom}(-, d)$ has to be a morphism of commutative monoids.

(5) Let C, D be semi-additive categories, and $F : C \rightarrow D$ a functor. Show that F preserves finite products (including the empty product) if and only if it preserves finite coproducts,

⁶You can recover from this the fact that for vector spaces, matrices represent all endomorphisms.

if and only if for all x, y , $\text{hom}(x, y) \rightarrow \text{hom}(F(x), F(y))$ is a monoid morphism. You may want to observe that in $x \oplus y$, if p_x, p_y denote the projections and i_x, i_y the inclusions, then $i_x p_x + i_y p_y = \text{id}_{x \oplus y}$, $p_x i_y = 0$, $p_y i_x = 0$.

(6) Give lots of examples of semi-additive categories.

EXERCISE 3.9.4 (Additivity). Exercise 3.9.3 is crucial for this exercise. Let C be a semi-additive category, and recall that $\text{hom}(x, y)$ is canonically a commutative monoid. C is called *additive* if $\text{hom}(x, y)$ is a group for all x, y .

(1) Give examples of additive categories. Can you find an example of a semi-additive category which is not additive?

(2) C is still assumed to be semi-additive. Show that it is additive if and only if for all x , the following morphism, called the *shear map*, is an isomorphism: $x \oplus x \rightarrow x \oplus x$, represented by the matrix $\begin{pmatrix} \text{id}_x & \text{id}_x \\ 0 & \text{id}_x \end{pmatrix}$ (you may want to first prove a similar result for commutative monoids).

EXERCISE 3.9.5 (Additivity 2). Let C be a semi-additive category, and D a category with finite products.

(1) Let $F : C \rightarrow D$ be a finite product-preserving functors. Show that it factors through the category of commutative monoids in D (can you define “the category of commutative monoids in D ”?). Call \tilde{F} this lift.

(2) Let $G : C \rightarrow D$ be another such functor, and \tilde{G} a lift such as above. Show that any morphism $F \rightarrow G$ lifts to a morphism $\tilde{F} \rightarrow \tilde{G}$.

(3) Prove that the forgetful functor $\text{Fun}^\times(C, \mathbf{CMon}(D)) \rightarrow \text{Fun}^\times(C, D)$ is an equivalence, where Fun^\times is the category of finite product preserving functors.

(4) Prove that if C is additive, the same holds when replacing $\mathbf{CMon}(D)$ with $\mathbf{Ab}(D)$.

(5) Prove that the same holds for limit-preserving functors instead of finite-product preserving functors. For exercise 3.9.6, we *accept* that if D is nice enough and C is cocomplete, the same holds for right adjoints, i.e. $\text{Fun}^R(C, \mathbf{CMon}(D)) \rightarrow \text{Fun}^R(C, D)$ is an equivalence, where Fun^R is the category of functors that are right adjoint (that have a left adjoint).

EXERCISE 3.9.6 (The Dold-Kan correspondance). (1) Recall from the Homework problem on Sheet 1 that any functor $f : \Delta \rightarrow C$ with values in a cocomplete category induces an essentially unique colimit-preserving functor $\mathbf{sSet} \rightarrow C$ which restricts to f along the Yoneda embedding, and that this functor has a right adjoint given by $c \mapsto \text{hom}(f(-), c)$.

(2) Using exercise 3.9.5, show that if C is cocomplete and additive, this functor factors uniquely through a colimit-preserving functor $\mathbf{sAb} \rightarrow C$ (along the free abelian group functor $\mathbf{sSet} \rightarrow \mathbf{sAb}$), with right adjoint given by the same formula with the “natural” abelian group structure (recall what that natural abelian group structure is).

(3) Prove that if we start from $N_* : \Delta \rightarrow \text{Ch}_{\geq 0}(\mathbb{Z})$, $[n] \mapsto N_* \mathbb{Z} \Delta^n$, then the functor $\mathbf{sAb} \rightarrow \text{Ch}_{\geq 0}(\mathbb{Z})$ we get is indeed N_* (explain why it suffices to show that N_* preserves colimits, and prove that).

EXERCISE 3.9.7 ((Finally) a proof of the Dold-Kan correspondance). Recall the notations $N_* \dashv K$ of the Dold-Kan correspondance. We wish to prove it’s an equivalence.

(1) Explain why it suffices to show that the unit $A \rightarrow K N_* A$ is an isomorphism, and that K is conservative (i.e. if $K(f)$ is an isomorphism, so is f).

(2) We first show that K is conservative. For this, let $D^m = \cdots \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0 \rightarrow \cdots$ where the leftmost \mathbb{Z} is in degree m , except for $m = 0$ where we put $D^0 = \mathbb{Z}$ concentrated in degree 0. Show that $\text{hom}(D^m, C) \cong C_m$, and that $N_* \mathbb{Z} \Delta^m$ retracts onto D^m . Deduce that K is conservative.

(3) We now prove that the unit is an isomorphism: let A be a simplicial abelian group, and consider $A \rightarrow \text{hom}(N_* \mathbb{Z} \Delta^\bullet, N_* A)$ be the unit.

(a) Prove that $A_m = NA_m \oplus DA_m$, where $NA_m = \bigcap_{i < m} \ker(d_i)$, $DA_m = \sum_i \operatorname{im}(s_i)$. We can prove this by induction on $k < m$, by proving that $A_m = N_k A_m \oplus D_k A_m$, where $N_k A_m = \bigcap_{i \leq k} \ker(d_i)$, $D_k A_m = \sum_{i \leq k} \operatorname{im}(s_i)$ (note that there are degeneracies only up to $m-1$).

(b) Prove that the unit is an isomorphism in dimension m by induction on m . At level m , you might want to use the decomposition $NA_m \oplus DA_m$, which is natural in A , and use induction on k for injectivity on the DA_m part.

EXERCISE 3.9.8 (Symmetry). (1) Find an example of a simplicial commutative ring R , an integer n and an $x \in R_{2n+1}$ such that $x^2 \neq 0$.

(2) Using the Eilenberg-Zilber maps, explain how to define a multiplication on $N_*(R)$. You do not have to show that it is associative and unital (but it is). Show that if $x \in N_{2n+1}(R)$, then $x^2 = 0$.

(3) Suppose A is a CDGA (see exercise 3.9.1), and $x \in A_{2n+1}$. What can you say about x^2 ? Find an example where it's not 0. Conclude that the Alexander-Whitney maps are *not* symmetric.⁷

EXERCISE 3.9.9 (Alexander-Whitney). In this exercise, we construct the Alexander-Whitney maps, which provide an *oplax monoidal structure* on the functor N_* ⁸. We construct $\Delta_{A,B} : N_*(A \otimes B) \rightarrow N_*(A) \otimes N_*(B)$.

(1) Explain how to get, from this, a lax monoidal structure on the right adjoint K , in particular a natural map⁹ $K(C) \otimes K(D) \rightarrow K(C \otimes D)$.

(2) The formula for $\Delta_{A,B}$ is

$$\Delta_{A,B}(a \otimes b) = \oplus_{p+q=n} \tilde{d}^p(a) \otimes d^q(b)$$

where \tilde{d}^p is induced by the “front face” $[p] \rightarrow [p+q], i \mapsto i$, while d^q is induced by the “back face” $[q] \rightarrow [p+q], i \mapsto p+i$.

(a) Show that this is a morphism of chain complexes $C_*(A \otimes B) \rightarrow C_*(A) \otimes C_*(B)$.

(b) Show that it descends to a morphism $N_*(A \otimes B) \rightarrow N_*(A) \otimes N_*(B)$.

(3) Show that the composite $\Delta_{A,B} \circ \nabla_{A,B}$ is the identity of $N_*(A) \otimes N_*(B)$ ¹⁰.

EXERCISE 3.9.10 (Group homology). In this exercise, we compare two definitions of group homology. For a simplicial abelian group A , we let $C_*(A)$ denote the *un-normalized* chain complex associated to A .

(1) Recall from the lecture notes that the morphism $H_*(BG; \mathbb{Z}) \rightarrow H_*(|BG|; \mathbb{Z})$ induced by $BG \rightarrow \operatorname{Sing}(|BG|)$ is an isomorphism, where $BG = N(\mathbf{B}G)$. (Bonus: can you sketch the argument?)

(2) Show that $\mathbb{Z}[BG] = \mathbb{Z}[EG]_G$, where for an abelian group M with linear G -action, M_G denotes the abelian group of orbits of M , i.e. $M/\langle gx - x, g \in G, x \in M \rangle$. Show that $C_*(\mathbb{Z}[BG]) \cong C_*(\mathbb{Z}[EG])_G$.

(3) Show that $\mathbb{Z}[EG_n]$ is free as a $\mathbb{Z}[G]$ -module. Deduce that $C_*(\mathbb{Z}[EG])$ is a complex of free $\mathbb{Z}[G]$ -modules.

(4) Deduce that $H_*(BG; \mathbb{Z})$ is group homology, i.e. it's the left derived functors of $M \mapsto M_G$ evaluated on \mathbb{Z} .¹¹

⁷In fact, this shows that there cannot be a (lax) symmetric monoidal equivalence of categories between the category of non-negatively graded chain complexes and that of simplicial abelian groups

⁸One of the points of homework problem 1 is to see that there cannot be a symmetric one

⁹A lax monoidal structure is in principle slightly more than this, and there are axioms to check. I'm only asking for this map here

¹⁰The other composite is *homotopic* to the identity of $N_*(A \otimes B)$, but not equal.

¹¹More generally, one can show that for a G -abelian group M , group homology evaluated on M is $H_*(|BG|; \underline{M})$, where \underline{M} is a local coefficients system on $|BG|$ induced by the G -action on M .

This week, the homework problems rely more on the exercises than usual. It's ok to use the results from the exercises without proving them, and just as usual, it's completely ok (and encouraged) to ask questions about the exercises.

EXERCISE 3.9.11 (Eilenberg-Zilber). [1+2+2+2+3] In this exercise, we study the Eilenberg-Zilber maps, which provide a *lax symmetric monoidal structure* on the functor N_* .

This is a natural morphism $\nabla_{A,B} : N_*(A) \otimes N_*(B) \rightarrow N_*(A \otimes B)$.

(1) Assuming such a natural morphism exists and is suitably compatible with the monoidal structure, explain how this yields the structure of an algebra on $N_*(R)$, for any simplicial ring R . Can you work out what “suitably compatible” means ?

(2) Recall from the lecture notes that

$$\nabla_{A,B}(a \otimes b) = \sum_{(\mu,\nu) \in sh(p,q)} \text{sgn}(\mu,\nu) s_\nu(a) \otimes s_\mu(b)$$

for $a \in A_p, b \in B_q$. We show here that this is well-defined.

(a) Do you understand the geometric intuition of this definition ? Recall, if necessary, the simplices of $\Delta^p \times \Delta^q$.

(b) Show that this is a chain map $C_*(A) \otimes C_*(B) \rightarrow C_*(A \otimes B)$ between the un-normalized chain complexes.

(c) Show that it descends to the normalized chain complexes $N_*(A) \otimes N_*(B) \rightarrow N_*(A \otimes B)$.

(3) Show that $\nabla_{A,B}$ is *symmetric*, i.e. $\nabla_{A,B} \circ \tau = N_*(\tau') \circ \nabla_{B,A}$, where τ, τ' are the symmetry isomorphisms from exercises 3.9.1 and 3.9.2 .

EXERCISE 3.9.12 (General Dold-Kan). [2+1+3+4] The goal of this exercise is to generalize the Dold-Kan correspondence to other categories than \mathbf{Ab} . We will use the results of exercises 3.9.3, 3.9.4, 3.9.6 and 3.9.7, you *do not* need to reprove them. We will also use the fact that the functor K from exercises 3.9.6, 3.9.7 agrees with the one defined in the lecture notes, i.e. $K(C)_n = \bigoplus_{[n] \twoheadrightarrow [m]} C_m$.

(1) Let C be a small additive category. Show that the functor $c \mapsto \text{hom}(-, c) : C \rightarrow \text{Fun}(C^{\text{op}}, \mathbf{Ab})$ is fully faithful, and preserves finite products and co-products.

(2) Explain in a few words why $K : \text{Ch}_{\geq 0}(C) \rightarrow \text{Fun}(\Delta^{\text{op}}, C)$ can be defined, and explain why the following diagram commutes up to natural isomorphism:

$$\begin{array}{ccc} \text{Ch}_{\geq 0}(C) & \xrightarrow{\quad\quad\quad} & \text{Fun}(\Delta^{\text{op}}, C) \\ \downarrow & & \downarrow \\ \text{Ch}_{\geq 0}(\text{Fun}(C^{\text{op}}, \mathbf{Ab})) & \xrightarrow{\quad\quad\quad} & \text{Fun}(\Delta^{\text{op}}, \text{Fun}(C^{\text{op}}, \mathbf{Ab})) \end{array}$$

(3) Show that the bottom functor in that diagram is an equivalence. You may want to observe that for any small category I , $\text{Ch}_{\geq 0}(\text{Fun}(I, \mathbf{Ab})) \simeq \text{Fun}(I, \text{Ch}_{\geq 0}(\mathbf{Ab}))$, and that adjunctions are preserved by passing to functor categories.

We call an additive category C *idempotent-complete* if for any $x \in C$, and $e : x \rightarrow x$ such that $e \circ e = e$ (a *projector*¹²), there exists a y and a retraction $r : x \rightarrow y, i : y \rightarrow x$ such that $i \circ r = e$.

(4) Deduce that the top functor is fully faithful. Show that if C is idempotent-complete, then it is also essentially surjective.

¹²You have most likely seen this notion in vector spaces.

CHAPTER 4

Homotopy categories and derived functors

In this chapter we will introduce the notion of a category \mathcal{C} equipped with a class of weak equivalences \mathcal{W} . Given this data, one wants to define a localization $\mathcal{C}[\mathcal{W}^{-1}]$, or homotopy category $\mathrm{Ho}(\mathcal{C})$, i.e. a new category in which these weak equivalences are *formally inverted*. We will study how to construct this in practice, and calculate the maps between any two object, including how to ensure that the maps actually form a set, which is not a priori guaranteed is the objects in \mathcal{C} do not form a set. We'll also see how our classical examples such as spaces and simplicial sets fit into this framework. All this will be covered in Section 4.1.

Next, in Section 4.2, given a functor $f : \mathcal{C} \rightarrow \mathcal{D}$ between categories with weak equivalences we can ask if a dotted arrow exists making the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ \downarrow & & \downarrow \\ \mathrm{Ho}(\mathcal{C}) & \cdots\cdots\cdots & \mathrm{Ho}(\mathcal{D}) \end{array}$$

commute, i.e., if the functor f is homotopy invariant. Many functors we care about will not be homotopy invariant. However there still exists a canonical functor $\mathrm{Ho}(\mathcal{C}) \rightarrow \mathrm{Ho}(\mathcal{D})$, with a universal property. Namely we may ask if the left or right Kan extension of $\mathcal{C} \xrightarrow{f} \mathcal{D} \rightarrow \mathrm{Ho}(\mathcal{D})$ along $\mathcal{C} \rightarrow \mathrm{Ho}(\mathcal{C})$ exists. If it does, we call this the respectively left or right derived functor of f . E.g., the *homotopy limit* will be the right derived functor of limit and the *homotopy colimits* will be the left derived functor of colimit. Concrete constructions of these, and how to calculate them, will then be the subject of the next chapter...

[ch1 has undergone revisions since this was written. It would be good to restructure material here and connect more directly to ch1.]

4.1. Homotopy categories and localizations

DEFINITION 4.1.1. A *relative category* is a pair $(\mathcal{C}, \mathcal{W})$ where \mathcal{C} is a category and \mathcal{W} is a collection of “weak equivalences” in \mathcal{C} ; more precisely, \mathcal{W} is a subcategory of \mathcal{C} containing all objects and all isomorphisms. A *homotopical category* is a relative category where \mathcal{W} satisfies the *2-of-3 property*: given two composable morphisms f, g in \mathcal{C} , if any two out of f, g, gf are in \mathcal{W} , then so is the third.¹ We will often leave the class of weak equivalences implicit in the notation and just talk about a homotopical category \mathcal{C} .

DEFINITION 4.1.2. Let $(\mathcal{C}, \mathcal{W})$ be a relative category. We say that a functor $L : \mathcal{C} \rightarrow \mathcal{C}[\mathcal{W}^{-1}]$ exhibits a category $\mathcal{C}[\mathcal{W}^{-1}]$ as the *localization of \mathcal{C} at \mathcal{W}* if for every category \mathcal{D} , the induced map

$$L^* : \mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

is fully faithful with essential image the functors which send all morphisms in \mathcal{W} to isomorphisms.

¹Sometimes it is better to assume the stronger *2-of-6* property: if we have composable maps $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$ where gf and hg are in \mathcal{W} , then f, g, h and hgf are also in \mathcal{W} . (Note that this is sort of saying that g has both a left and a right “inverse”.) This stronger condition is assumed in the definition of homotopical category in [Rie14] or [Dwy+04], but here we will just need 2-of-3.

REMARK 4.1.3. One can check that the category $\mathcal{C}[\mathcal{W}^{-1}]$ is uniquely determined up to equivalence. Therefore, it makes sense to talk about *the* localization of a category at a class of weak equivalences.

The next proposition says that we can always invert a set of morphisms in a small category:

PROPOSITION 4.1.4 (Gabriel-Zisman). *Suppose $(\mathcal{C}, \mathcal{W})$ is a relative category where \mathcal{C} is small. Then there exists a localization of \mathcal{C} at \mathcal{W} .*

SKETCH OF PROOF. Define a new (small) category $\mathcal{C}[\mathcal{W}^{-1}]$ with objects the objects of \mathcal{C} and morphisms given by formal zig-zags of maps in \mathcal{C} , where maps in \mathcal{W} are allowed to go the wrong way, modulo the obvious identifications given by composition, removing identities, and cancelling $a \xrightarrow{w} b \xleftarrow{w} a$ and $b \xleftarrow{w} a \xrightarrow{w} b$ where w is in \mathcal{W} . The construction is spelled out in [GZ67]. \square

If $(\mathcal{C}, \mathcal{W})$ is a homotopical category, we also use the notation $\mathrm{Ho}(\mathcal{C})$ for $\mathcal{C}[\mathcal{W}^{-1}]$, leaving \mathcal{W} implicit. We call this the *homotopy category* of \mathcal{C} with respect to \mathcal{W} .

REMARK 4.1.5. We would like to apply this construction to categories such as Top and $\mathrm{Top}^{\mathcal{I}}$ (where \mathcal{I} is an indexing category), and such categories are not small. However, there is a set-theoretical issue involved in doing this: as soon as \mathcal{C} contains a proper class of objects and morphisms (i.e. a class that is possibly not a set), also formal zig-zags and their equivalence classes might form only a proper class. We will generally ignore this issue, but to avoid consternation, let us point out that we can avoid this issue by assuming that inside our set theory there is a sub-class of *small* sets that themselves form a model of set theory; then we can use the large (i.e. not necessarily small) sets instead of worrying about classes. (The precise version of this idea is called a *Grothendieck universe*.) We then use the following terminology:

- a category is the datum of a (potentially large) *set* of objects and a (potentially large) *set* of morphisms, together with source, target, identity and composition maps satisfying the usual requirements;
- a category is *small* if its set of objects and all its Hom-sets are small;
- a category is *locally small* if its Hom-sets are all small but its set of objects is (potentially) large;
- a category is *large* if its set of objects and its Hom-sets are all (potentially) large.

We can then take Set to be the (locally small) category of small sets, Top to be the (locally small) category of topological spaces whose underlying sets are small, etc.

The construction of $\mathcal{C}[\mathcal{W}^{-1}]$ now works just as well for locally small but not small homotopical categories $(\mathcal{C}, \mathcal{W})$, it just yields in general a large category $\mathcal{C}[\mathcal{W}^{-1}]$; so we can for example define the homotopy category $\mathrm{Ho}(\mathrm{Top})$ by starting with the category Top with objects all topological spaces and then formally inverting the weak homotopy equivalences. Similarly, we can define the (bounded) derived category of R -modules by taking all bounded chain complexes over R and formally inverting the quasi-isomorphisms (homology isomorphisms). We can also construct homotopy categories of diagrams:

DEFINITION 4.1.6. Suppose \mathcal{I} is a small category and $(\mathcal{C}, \mathcal{W})$ is a (possibly large) homotopical category. Then $\mathcal{C}^{\mathcal{I}}$ is a homotopical category if we equip it with the *natural weak equivalences*, i.e. the weak equivalences $\mathcal{W}_{\mathcal{I}}$ are those natural transformations $\eta: F \rightarrow G$ such that $\eta_i: F(i) \rightarrow G(i)$ is in \mathcal{W} for all $i \in \mathcal{I}$. Then we can define $\mathrm{Ho}(\mathcal{C}^{\mathcal{I}})$ as $\mathcal{C}^{\mathcal{I}}[\mathcal{W}_{\mathcal{I}}^{-1}]$.

REMARK 4.1.7. There is a less formal set-theoretical issue with these localizations, however: although the categories we start with (Top , $\mathrm{Top}^{\mathcal{I}}$, etc.) are all locally small, a priori the localized categories are just large categories — the Hom-sets are not necessarily small. In practice, to work with these categories it is useful to know that the localizations are also locally small. We will not deal with this issue here, however, as it will not actually affect us. That being said,

local smallness of the homotopy category is part of the package you get from a *model structure*, and all the categories we will consider do have model structures, so in any case there is nothing to worry about.

REMARK 4.1.8. We stress right away that the category $\mathrm{Ho}(\mathcal{C}^{\mathcal{I}})$ is in general *not* the same as the category $\mathrm{Ho}(\mathcal{C})^{\mathcal{I}}$. $\mathrm{Ho}(\mathcal{C}^{\mathcal{I}})$ is a much richer category since diagrams are required to strictly commute, whereas $\mathrm{Ho}(\mathcal{C})^{\mathcal{I}}$ is generally too weak to be of much interest (see e.g. Example 4.1.21 below). Another problem with $\mathrm{Ho}(\mathcal{C})^{\mathcal{I}}$ is that the functor $\delta : \mathrm{Ho}(\mathcal{C}) \rightarrow \mathrm{Ho}(\mathcal{C})^{\mathcal{I}}$ forming constant diagrams in general does not have an adjoint, i.e. limits and colimits do not exist in the homotopy category (see e.g. Example 4.1.24 below). If we instead work with $\mathrm{Ho}(\mathcal{C}^{\mathcal{I}})$, then δ often *does* have an adjoint, and this gives one definition of the homotopy colimit: the *homotopy colimit* will be the left adjoint of $\delta : \mathrm{Ho}(\mathcal{C}) \rightarrow \mathrm{Ho}(\mathcal{C}^{\mathcal{I}})$.

Inside a homotopical category \mathcal{C} we can often find full subcategories \mathcal{C}' of “good” (or “cofibrant”) objects such that $\mathrm{Ho} \mathcal{C}'$ is equivalent to $\mathrm{Ho} \mathcal{C}$, but where $\mathrm{Ho} \mathcal{C}'$ is simpler to describe, or some functor we are interested in is better-behaved on the objects in \mathcal{C}' . Making this precise gives the notion of *deformation*:

DEFINITION 4.1.9. Let \mathcal{C} be a homotopical category. A *left deformation* of \mathcal{C} is a functor $Q : \mathcal{C} \rightarrow \mathcal{C}$ and a natural transformation $q : Q \rightarrow \mathrm{id}_{\mathcal{C}}$ such that $q_c : Qc \rightarrow c$ is a weak equivalence for all $c \in \mathcal{C}$. We write \mathcal{C}_Q for the full subcategory of \mathcal{C} spanned by the essential image of Q .

A *right deformation* is defined dually.

REMARK 4.1.10. Note that by the 2-of-3 property Q takes weak equivalences to weak equivalences.

LEMMA 4.1.11. Suppose (Q, q) is a left deformation of a homotopical category \mathcal{C} . If \mathcal{C}' is a full subcategory containing the image of Q (for instance \mathcal{C}_Q), then the functor $i : \mathrm{Ho} \mathcal{C}' \rightarrow \mathrm{Ho} \mathcal{C}$ induced by the inclusion is an equivalence of categories.

SKETCH PROOF. Since the natural map $q_c : Qc \rightarrow c$ is a weak equivalence, and hence an isomorphism in $\mathrm{Ho} \mathcal{C}$, the functor i is essentially surjective. Suppose x and y are objects of \mathcal{C}' , then we need to prove that $\mathrm{Hom}_{\mathrm{Ho} \mathcal{C}'}(x, y) \rightarrow \mathrm{Hom}_{\mathrm{Ho} \mathcal{C}}(ix, iy)$ is a bijection. Using q we can replace any zig-zag of morphisms from x to y in \mathcal{C} by an equivalent zig-zag in $\mathcal{C}_Q \subseteq \mathcal{C}'$, so this map is surjective. Similarly if two zig-zags become equivalent in \mathcal{C} we can use q to show they are equivalent also in \mathcal{C}_Q , giving injectivity. \square

For example, if \mathcal{C} is Top we can take $Q = |\mathrm{Sing}_{\bullet}(-)|$ and \mathcal{C}_Q to be the full subcategory of CW complexes. In fact we can do this also for diagrams:

PROPOSITION 4.1.12. Let \mathcal{I} be a (small) indexing category. Then $|\mathrm{Sing}_{\bullet}(-)|$ is a left deformation of $\mathrm{Top}^{\mathcal{I}}$ into $\mathrm{CW}^{\mathcal{I}}$, and hence the inclusion $\mathrm{Ho}(\mathrm{CW}^{\mathcal{I}}) \rightarrow \mathrm{Ho}(\mathrm{Top}^{\mathcal{I}})$ is an equivalence of categories.

PROOF. The counit $q : |\mathrm{Sing}_{\bullet}(-)| \rightarrow \mathrm{id}(-)$ induces for each $F \in \mathrm{Top}^{\mathcal{I}}$ and $i \in \mathcal{I}$ a weak homotopy equivalence $|\mathrm{Sing}_{\bullet}(F(i))| \rightarrow F(i)$, so the result follows from Lemma 4.1.11. \square

While Proposition 4.1.12 shows that we can without any real restriction assume that our functors take values in CW complexes, it does not show that such functors have any special properties, as we elaborate in Remark 4.1.14 below. Only when \mathcal{I} is a point, does this deformation allow us to identify the homotopy category with the usual construction:

PROPOSITION 4.1.13. $|\mathrm{Sing}_{\bullet}(-)|$ induces an equivalence $\mathrm{Ho}(\mathrm{Top}) \rightarrow \mathrm{hCW}$, the category with objects CW complexes and morphisms homotopy classes of maps.

PROOF. First note that $\mathrm{Ho}(\mathrm{CW}) \xrightarrow{\simeq} \mathrm{Ho}(\mathrm{Top})$, as a special case of Lemma 4.1.12, with \mathcal{I} a point.

It remains to show that $\text{Ho}(\text{CW})$ is equivalent to hCW . By Whitehead's Theorem weak homotopy equivalences between CW complexes are homotopy equivalences, so the functor $\text{Ho}(\text{CW}) \rightarrow \text{hCW}$ is well-defined. Likewise the functor $\text{CW} \rightarrow \text{Ho}(\text{CW})$ descends to $\text{hCW} \rightarrow \text{Ho}(\text{CW})$: Namely the two inclusion $i_0 : X \rightarrow X \times I$ and $i_1 : X \rightarrow X \times I$ are both weak equivalences and right inverses to the projection $X \times I \rightarrow X$, and hence they become equal isomorphisms in $\text{Ho}(\text{CW})$. If $f_0, f_1 : X \rightarrow Y$ are homotopy homotopic, then there exists $F : X \times I \rightarrow Y$ so that $f_0 = F \circ i_0$ and $f_1 = F \circ i_1$. But then f_0 and f_1 are equal in $\text{Ho}(\text{CW})$.

We have hence seen that the identity functor on CW induces inverse equivalences of categories between $\text{Ho}(\text{CW})$ and hCW . \square

REMARK 4.1.14. Note that in $\text{CW}^{\mathcal{I}}$ we invert maps that are *objectwise* homotopy equivalences. We have not shown that such maps have *natural* homotopy inverse. In particular we have *not* shown in Proposition 4.1.12 an equivalence of $\text{Ho}(\text{CW}^{\mathcal{I}})$ with $\text{h}(\text{CW}^{\mathcal{I}})$, where the latter means the homotopy category obtained using the notion of homotopy obtained by crossing with the unit interval. In fact this is *false*, this requires much stronger assumptions on the diagram than just being objectwise given by CW complexes. We will see in Proposition 4.1.17 what this stronger condition is for pushout diagrams, and generalizing this for arbitrary diagrams is the subject of Chapter 5.

REMARK 4.1.15. Note that $\text{Ho}(\mathcal{C}) = \mathcal{C}[\mathcal{W}^{-1}]$, as defined here, is not a homotopical object in our definition, as $\text{Ho}(\mathcal{C})$ is just an ordinary 1-category, not a category with extra structure such as a collection of weak equivalences, since we have set the weak equivalences in \mathcal{W} to be *isomorphisms* in $\mathcal{C}[\mathcal{W}^{-1}]$.

This why one sometimes means something more refined by “localization” so that the localized category also has the higher order structure, i.e., the localization “is” the category $(\mathcal{C}, \mathcal{W})$ up to a suitable notion of equivalence, or one interprets all this ∞ -categorically, where weak equivalences become isomorphisms in the ∞ -categorical sense.

We also mention the following example, well known from homological algebra, without proof.

PROPOSITION 4.1.16. *Let R be a ring, and suppose $\mathcal{C} = \text{Ch}_R^{\geq 0}$ be the category of non-negatively graded R -chain complexes, with weak equivalences given by quasi-isomorphisms. Take \mathcal{C}' to be the full subcategory $\text{Proj}_R^{\geq 0}$ of chain complexes of projective R -modules. Then $\text{Ho} \text{Ch}_R^{\geq 0} \simeq \text{hProj}_R^{\geq 0}$ where the right-hand side means we take chain homotopy equivalences of maps.*

Use the dictionary “topological spaces \leftrightarrow non-negatively graded R -chain complexes” and “CW complexes \leftrightarrow non-negatively graded R -chain complexes of projectives” to compare the previous example with Proposition 4.1.13.

We will make use of deformations for general diagram categories later. For now we will just describe how such replacements can work in some fundamental examples:

PROPOSITION 4.1.17. *Let $\mathcal{I} = (0 \leftarrow 1 \rightarrow 2)$ be the pushout category. Then $\text{Ho}(\text{Top}^{\mathcal{I}})$ is equivalent to the category with objects \mathcal{I} -diagrams of CW complexes and CW complex inclusions, and morphisms homotopy classes of maps between diagrams.*

We need a technical lemma that will be useful again later. Recall that we call a map a *cofibration* (or sometimes Hurewicz or h-cofibration) if it has the homotopy extension property (cf. [Hat02, Ch. 0]). An example is an inclusion of a CW subcomplex into a CW complex.

LEMMA 4.1.18. *Suppose we are given a commutative square*

$$\begin{array}{ccc} X & \xhookrightarrow{i} & Y \\ \downarrow f & & \downarrow g \\ X' & \xhookrightarrow{i'} & Y' \end{array}$$

of topological spaces, where i and i' are cofibrations and f and g are homotopy equivalences. Then every homotopy inverse of f can be extended to a homotopy inverse of g (meaning more precisely that the homotopies to the identity map on Y and Y' respectively can be chosen such as to extend any given homotopies on X and X').

PROOF. See [May99, §6.5]. \square

PROOF OF PROPOSITION 4.1.17. Any diagram is weakly equivalent to a diagram of this form: The map $|\mathrm{Sing}_\bullet(X)| \rightarrow X$ allows us to replace our diagram by a diagram of CW complexes. Then turn the two maps into CW inclusions using mapping cylinders. We conclude that for all diagrams X there exist a diagram QX of CW complexes and CW inclusions, and a natural weak equivalence $QX \rightarrow X$. By Lemma 4.1.11 this means inverting the weak equivalences in $\mathrm{Top}^{\mathcal{I}}$ is equivalent to inverting them in the subcategory of such diagrams. It remains to show that a weak equivalence $\phi: F \rightarrow G$ between two of these is a homotopy equivalence. By Whitehead's Theorem the maps $F(i) \rightarrow G(i)$ are all homotopy equivalences, so we need to show we can choose a natural homotopy inverse. This follows from Lemma 4.1.18. It then follows by the same argument as for Top that inverting homotopy equivalences is equivalent to taking homotopy classes of maps. \square

As we mentioned already, the categories $\mathrm{Ho}(\mathcal{C}^{\mathcal{I}})$ and $\mathrm{Ho}(\mathcal{C})^{\mathcal{I}}$ are quite different in general. We can now give an explicit example of this:

EXAMPLE 4.1.19. Let $\delta: \mathrm{Top} \rightarrow \mathrm{Top}^{\mathcal{I}}$ be the constant-diagram functor. With \mathcal{I} as above and $n \geq 1$, we have bijections of sets

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{Top})^{\mathcal{I}}}([* \leftarrow S^{n-1} \rightarrow *], \delta S^n) \cong *;$$

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{Top}^{\mathcal{I}})}([* \leftarrow S^{n-1} \rightarrow *], \delta S^n) \cong \mathbb{Z}.$$

As an exercise, try to verify this: The first claim is easy to verify. The second claim takes a bit more work (replacing $[* \leftarrow S^{n-1} \rightarrow *]$ with the weakly equivalent diagram $[D^n \leftarrow S^{n-1} \rightarrow D^n]$ is a good start), but will follow at once later when we get our model for homotopy colimit.

The next example is also good to keep in mind, but we will not give a full proof here:

PROPOSITION 4.1.20 (Sketch of Proof). *For G a (finite) group, let $\mathcal{B}G$ denote the category with one object and G as morphisms. Then $\mathrm{Top}^{\mathcal{B}G}$ is the category of spaces with a G -action, and $\mathrm{Ho}(\mathrm{Top}^{\mathcal{B}G})$ is the category where we invert G -maps that are ordinary weak equivalences. This is equivalent to the category whose objects are CW complexes with free G -action and morphisms G -homotopy equivalence classes of G -maps.*

PROOF. The idea is analogous to the previous proposition. As before, we can functorially replace any G -space X by a G -CW complex, namely $|\mathrm{Sing}_\bullet(X)|$ (check that it has a (non-free) cellular G -action!). Now consider the space EG , the classifying space of the category with objects the elements of G and exactly one morphism between any two objects. This space is contractible and naturally carries a free G -action induced by the translation action on the category. Hence the map

$$QX := EG \times |\mathrm{Sing}_\bullet(X)| \rightarrow X$$

is a weak equivalence in $\mathrm{Top}^{\mathcal{B}G}$ and replaces X by a CW complex QX with a free G -action.

We can now apply Lemma 4.1.11 to the subcategory $\mathrm{CW}^{\mathcal{B}G, \mathrm{free}} \subset \mathrm{Top}^{\mathcal{B}G}$ of CW-complexes with free G -action, together with the deformation

$$Q = EG \times |\mathrm{Sing}_\bullet(-)|: \mathrm{Top}^{\mathcal{B}G} \rightarrow \mathrm{CW}^{\mathcal{B}G, \mathrm{free}} \subset \mathrm{Top}^{\mathcal{B}G}.$$

We would now like to see that given a G -equivariant homotopy equivalence $f: X \rightarrow Y$ between free G -CW complexes, there exists a G -equivariant homotopy inverse $g: Y \rightarrow X$, such that the two composites $g \circ f$ and $f \circ g$ are G -equivariantly homotopic to the identities of X and Y , respectively. This is easy with a bit of equivariant homotopy theory, but we will not give

all details here: we just suggest that a G -equivariant map $g: Y \rightarrow X$ can be constructed inductively over skeleta of Y , and that each time, trying to extend g over a G -equivariant *free* cell (which consists of $|G|$ cells of the CW complex Y), we extend g suitably over one cell and extend G -equivariantly over the entire G -equivariant cell in the unique possible way (see e.g. [Bre67]).

Once this is done, the proof proceeds similarly as for Proposition 4.1.13. □

EXAMPLE 4.1.21. Specifying an object in $\mathrm{Ho}(\mathrm{Top}^{\mathcal{B}G})$ amounts to giving a space with a group action. Note how this is quite different from $(\mathrm{Ho}(\mathrm{Top}))^{\mathcal{B}G}$ which is just a topological space X and a homomorphism $G \rightarrow [X, X]^{\mathrm{inv}}$ from G to the group of homotopy classes of self-homotopy equivalences of X . For example, if $X = S^n$, $[X, X]^{\mathrm{inv}} = \mathbb{Z}/2$. This example tells us that, generally, specifying a homomorphism $G \rightarrow [X, X]^{\mathrm{inv}}$ will not describe how the group G acts on X in any real way.

An example to keep in mind is when $X = BH$ for another group H . Then the genuine group actions of G on BH (i.e. the objects of $\mathrm{Ho}(\mathrm{Top}^{\mathcal{B}G})$ with underlying space BH , up to G -weak equivalences) would correspond to extensions $0 \rightarrow H \rightarrow \hat{G} \rightarrow G \rightarrow 0$ (up to isomorphisms of extensions that are the identity on G), whereas the other notion would just be a morphism $G \rightarrow \mathrm{Out}(H)$ to the outer automorphisms of H (up to global conjugation by elements in $\mathrm{Out}(H)$). Exercise: Convince yourself that these two types of structures are *not* in bijection.

We can also give another model for $\mathrm{Ho}(\mathrm{Top}^{\mathcal{B}G})$:

PROPOSITION 4.1.22. *Denote by BG the classifying space of a group G . Consider the category $\mathrm{Top} \downarrow BG$ with objects topological spaces together with a map to BG and with morphisms being maps between topological spaces that commute with the map to BG . Endow this category with a notion of weak equivalence by saying that a map $f: (X \rightarrow BG) \rightarrow (Y \rightarrow BG)$ is a weak equivalence if the underlying map of spaces $f: X \rightarrow Y$ is a weak equivalence.*

We have a functor $\mathrm{Top}^{\mathcal{B}G} \rightarrow \mathrm{Top} \downarrow BG$ sending an object $X \in \mathrm{Top}^{\mathcal{B}G}$ to the object $EG \times_G X = (EG \times X)/G \rightarrow BG$ in $\mathrm{Top} \downarrow BG$. Likewise we have a functor $\mathrm{Top} \downarrow BG \rightarrow \mathrm{Top}^{\mathcal{B}G}$ sending an object $X \rightarrow BG$ to the pullback $EG \times_{BG} X \subset EG \times X$, equipped with the restriction of the G -action on $X \times EG$ obtained as product of the trivial action on X and the canonical, free action on EG . These functors define equivalences of homotopy categories.

We will not give the proof here. The necessary techniques are similar to the ones used in Section 2.11.

As it is good to have a set of homotopy categories to keep in mind, we also cannot resist mentioning the following example.

THEOREM 4.1.23 (Elmendorf). *Let G be a finite group. Say that a map of G -CW complexes $f: X \rightarrow Y$ is a G -homotopy equivalence if there is a G -equivariant map $g: Y \rightarrow X$ such that gf and fg are G -equivariantly homotopic to the identities of X and Y respectively (this is stronger than just asking f to be a weak equivalence between the spaces X and Y). Then*

$$\mathrm{Ho}(G\text{-CW complexes}, G\text{-homotopy equivalence}) \xrightarrow{\sim} \mathrm{Ho}(\mathrm{Top}^{\mathcal{O}(G)^{\mathrm{op}}})$$

where $\mathcal{O}(G)$ is the orbit category of G with objects transitive G -sets and morphisms G -maps. The functor is given by assigning to X the functor $\mathcal{O}(G)^{\mathrm{op}} \rightarrow \mathrm{Top}$ which sends G/H to the fixed points X^H .

We will not prove this now, but may return to it later, when we have the language to easily describe the inverse (as a teaser: how can we construct out of a functor $\mathcal{O}(G)^{\mathrm{op}} \rightarrow \mathrm{Top}$ a single G -space with prescribed H -fixed points for all $H \subset G$?).

EXAMPLE 4.1.24. Limits and colimits do not exist in the homotopy category $\mathrm{Ho}(\mathcal{C})^{\mathbb{I}}$ in general, except in very special cases like products and coproducts. Let's for instance see that

$* \rightarrow K(\mathbb{Z}, 3) \xleftarrow{p} K(\mathbb{Z}, 3)$ does not have a limit in the homotopy category, where p is a prime number and we denote by p also the self-map of $K(\mathbb{Z}, 3)$ (unique up to homotopy) whose action on $\pi_3 = \mathbb{Z}$ is multiplication by p . The diagram has a limit if and only if the functor $F(-) = \ker(H^3(-; \mathbb{Z}) \xrightarrow{p} H^3(-; \mathbb{Z}))$ is representable in the form $[-, P]$ for some space P . However, F is not exact in the middle on cofibration sequences, so this is impossible.

[Exercise: Construct a concrete example. Hint: Try to realize the sequence $\mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^2$ on H^3 of a cofibration sequence.]

EXERCISE 4.1.25. Can you say something about the homotopy type of $B\mathcal{O}(G)$? Hint: the transitive G -set $*$ is a terminal object in $\mathcal{O}G$.

EXERCISE 4.1.26. As a warm-up to Theorem 4.1.23 prove the equivariant Whitehead Theorem: a G -map between G -CW complexes is a G -homotopy equivalence iff it induces a homotopy equivalence on H -fixed-points for all $H \leq G$. (If you get stuck, you can e.g., look up [Ben91].)

4.2. Derived functors via deformations

There is a natural notion of morphism between homotopical categories:

DEFINITION 4.2.1. Suppose \mathcal{C} and \mathcal{D} are homotopical categories. We say a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is *homotopical* if it takes weak equivalences in \mathcal{C} to weak equivalences in \mathcal{D} . A homotopical functor induces a unique functor $\mathrm{Ho}\mathcal{C} \rightarrow \mathrm{Ho}\mathcal{D}$ (which we also call F) that fits in a commutative square

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow \gamma_{\mathcal{C}} & & \downarrow \gamma_{\mathcal{D}} \\ \mathrm{Ho}\mathcal{C} & \xrightarrow{F} & \mathrm{Ho}\mathcal{D} \end{array},$$

where $\gamma_{\mathcal{C}}$ and $\gamma_{\mathcal{D}}$ are the localizations.

However, many functors we are interested in are *not* homotopical — this is why we want derived functors.

DEFINITION 4.2.2. Let \mathcal{C} be a homotopical category, and let \mathcal{E} be some other category (playing the role of $\mathrm{Ho}(\mathcal{D})$); consider \mathcal{E} as a homotopical category by declaring only isomorphisms in \mathcal{E} to be weak equivalences. Let $F: \mathcal{C} \rightarrow \mathcal{E}$ be any functor. The *total left derived functor* $\mathbb{L}F$ of F is a homotopical functor $\mathbb{L}F: \mathcal{C} \rightarrow \mathcal{E}$ that best approximates F from the left.

More precisely, the total left derived functor is a pair $(\mathbb{L}F, \lambda)$ consisting of a homotopical functor $\mathbb{L}F: \mathcal{C} \rightarrow \mathcal{E}$ and a natural transformation $\lambda: \mathbb{L}F \rightarrow F$ satisfying the following: whenever (G, η) is a pair of a homotopical functor $G: \mathcal{C} \rightarrow \mathcal{E}$ and a natural transformation $\eta: G \rightarrow F$, there is a unique natural transformation $\theta: G \rightarrow \mathbb{L}F$ factoring η through λ , i.e. $\eta = \lambda \circ \theta$.

The given universal property ensures that if a total left derived functor of F exists, then it is unique up to unique natural isomorphism.

REMARK 4.2.3. Note that specifying a homotopy invariant functor $\mathcal{C} \rightarrow \mathcal{E}$ (i.e. a functor sending weak equivalences in \mathcal{C} to isomorphisms in \mathcal{E}) is equivalent to specifying a functor on $\mathrm{Ho}(\mathcal{C})$. Hence, with this identification, the total *left* derived functor is simply the *right* Kan extension $\mathrm{Ran}_{\gamma_{\mathcal{C}}} F$ of the functor $\mathcal{C} \rightarrow \mathcal{E}$ along $\gamma_{\mathcal{C}}: \mathcal{C} \rightarrow \mathrm{Ho}(\mathcal{C})$ (cf. Section ??).

If $F: \mathcal{C} \rightarrow \mathcal{D}$ is a functor between two homotopical categories, we call the left derived functor

$$\mathbb{L}F = \mathbb{L}(\gamma_{\mathcal{D}} \circ F): \mathrm{Ho}(\mathcal{D}) \longrightarrow \mathrm{Ho}(\mathcal{D})$$

of the composition of F with the localization of $\gamma_{\mathcal{D}}: \mathcal{D} \rightarrow \mathrm{Ho}(\mathcal{D})$ the left derived functor of F .

DEFINITION 4.2.4. A *left derived functor* of a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between homotopical categories is a homotopical functor $F': \mathcal{C} \rightarrow \mathcal{D}$ together with a natural transformation $F' \rightarrow F$ such that the induced functor $\gamma_{\mathcal{D}} \circ F': \mathcal{C} \rightarrow \mathrm{Ho}\mathcal{D}$ and natural transformation $\gamma_{\mathcal{D}} F' \rightarrow \gamma_{\mathcal{D}} F$ is a total left derived functor of $\gamma_{\mathcal{D}} \circ F$.

Note that, a priori, there is no guarantee that a left derived functor for F be *unique* up to unique isomorphism. In fact, in the case $\mathcal{C} = \mathcal{D}$ and $F = \text{id}_{\mathcal{C}}$, we see that $\text{id}_{\mathcal{C}}$ as well as any left deformation of \mathcal{C} are left derived functors of F . The homotopy colimit functor we will construct will be a left derived functor of the colimit functor.

REMARK 4.2.5. There is a natural dual notion of *right* derived functors. Given a random functor, you might wonder whether we should care about its left or right derived functor (assuming both exist, which is rare). Vaguely speaking, we typically want left derived functors of left adjoints and right derived functors of right adjoints — for one thing, as we will see later, we then get derived adjunctions.

We can use left deformations to construct left derived functors, provided the deformation is *compatible* with the functor in the following sense:

DEFINITION 4.2.6. A *left deformation* for a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between homotopical categories is a left deformation (Q, q) for \mathcal{C} with the additional property that the restriction of F to some full subcategory of \mathcal{C} containing all objects in the image of Q is homotopical. If a left deformation for F exists, we call F *left deformable*.

EXAMPLE 4.2.7. The category $\text{Ch}_{\geq 0}(R)$ of non-negatively graded chain complexes of modules over a ring R becomes a homotopical category with quasi-isomorphisms as weak equivalences. An additive functor F from R -modules to S -modules induces a functor $F_{\bullet}: \text{Ch}_{\geq 0}(R) \rightarrow \text{Ch}_{\geq 0}(S)$. It is possible to build a functorial projective resolution, i.e. a functor $Q: \text{Ch}_{\geq 0}(R) \rightarrow \text{Ch}_{\geq 0}(R)$ sending each complex $X \in \text{Ch}_{\geq 0}(R)$ to a complex QX which is degree-wise a projective R -module and comes with a quasi-isomorphism $q_X: QX \rightarrow X$ natural in X . This is a deformation for $\text{Ch}_{\geq 0}(R)$ and, in fact, it is adapted to F_{\bullet} and hence a *deformation for F_{\bullet}* . The reason for this is that F_{\bullet} preserves chain homotopy equivalences (because F is additive) and that any quasi-isomorphism between non-negatively graded complexes of projective R -modules is a chain homotopy equivalence.

PROPOSITION 4.2.8 (Dwyer–Hirschhorn–Kan–Smith, cf. [Rie14, Thm. 2.2.8]). *Let (Q, q) be a left deformation for a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between homotopical categories. Then (FQ, Fq) is a left derived functor of F .*

PROOF. We must show that given any homotopical functor $G: \mathcal{C} \rightarrow \text{Ho } \mathcal{D}$ equipped with a natural transformation $\alpha: G \rightarrow \gamma_{\mathcal{D}} F$, the natural transformation α factors uniquely through $\gamma_{\mathcal{D}} Fq: \gamma_{\mathcal{D}} FQ \rightarrow \gamma_{\mathcal{D}} F$. To see that such a factorization exists, consider the commutative diagram

$$\begin{array}{ccc} GQ & \xrightarrow{\alpha_Q} & \gamma_{\mathcal{D}} FQ \\ \downarrow Gq & & \downarrow \gamma_{\mathcal{D}} Fq \\ G & \xrightarrow{\alpha} & \gamma_{\mathcal{D}} F. \end{array}$$

Here the left vertical arrow is an isomorphism, since q is a natural weak equivalence and G is homotopical. Thus α factors as $\gamma_{\mathcal{D}} Fq \circ \bar{\alpha}$ where $\bar{\alpha} := \alpha_Q \circ (Gq)^{-1}$.

Now suppose we are given any factorization of α as $\gamma_{\mathcal{D}} Fq \circ \alpha'$. We first show that $\alpha'_Q = \bar{\alpha}_Q$. To see this, consider the diagram

$$\begin{array}{ccc} GQ & \xrightarrow{\alpha'_Q} & \gamma_{\mathcal{D}} FQq \\ & \searrow \alpha_Q & \downarrow \gamma_{\mathcal{D}} FqQ \\ & & \gamma_{\mathcal{D}} FQ. \end{array}$$

Here $\gamma_{\mathcal{D}} FqQ$ is an isomorphism (objectwise) because FqQ is a weak equivalence (objectwise), because (Q, q) is a deformation for F . So $\alpha'_Q = (\gamma_{\mathcal{D}} FqQ)^{-1} \circ \alpha_Q$. This holds for any α' , so in particular $\alpha'_Q = \bar{\alpha}_Q$.

Next consider the commutative square

$$\begin{array}{ccc} GQ & \xrightarrow{\alpha'_Q} & \gamma_{\mathcal{D}} FQ \\ \downarrow Gq & & \downarrow \gamma_{\mathcal{D}} FQq \\ G & \xrightarrow{\alpha'} & \gamma_{\mathcal{D}} FQ. \end{array}$$

This implies $\alpha' = (\gamma_{\mathcal{D}} Fq) \circ \alpha'_Q \circ (Gq)^{-1} = (\gamma_{\mathcal{D}} Fq) \circ \bar{\alpha}_Q \circ (Gq)^{-1} = \bar{\alpha}$, as required. \square

PROPOSITION 4.2.9. *Suppose $F: \mathcal{C} \rightleftarrows \mathcal{D} : G$ is an adjunction between homotopical categories, Q is a left deformation of \mathcal{C} compatible with F (in the sense of Definition 4.2.6), and, dually, R is a right deformation of \mathcal{D} compatible with G (this is called a deformable adjunction). Then the total derived functors $\mathbb{L}F := \mathbb{L}(\gamma_{\mathcal{D}} \circ F): \mathcal{C} \rightarrow \text{Ho } \mathcal{D}$ and $\mathbb{R}G := \mathbb{R}(\gamma_{\mathcal{C}} \circ G): \mathcal{D} \rightarrow \text{Ho } \mathcal{C}$ (which can be computed, for example, via $F \circ Q$ and $G \circ R$), form an adjunction*

$$\mathbb{L}F: \text{Ho } \mathcal{C} \rightleftarrows \text{Ho } \mathcal{D} : \mathbb{R}G.$$

For the proof, we will need the following Lemma which can be extracted from the proof of Lemma 2.2.13 in [Rie14]:

LEMMA 4.2.10. *Let \mathcal{C} be a homotopical category and let \mathcal{D} be a category (considered as homotopical by taking isomorphisms as weak equivalences). The total left derived functor of a left deformable functor $F: \mathcal{C} \rightarrow \mathcal{D}$ (i.e. a functor for which a compatible left deformation (Q, q) of \mathcal{C} exists) is an absolute right Kan extension (of $F \circ Q$) along $\gamma_{\mathcal{C}}: \mathcal{C} \rightarrow \text{Ho } \mathcal{C}$, i.e. a right Kan extension which is preserved by the postcomposition with any functor: for any functor of categories $G: \mathcal{D} \rightarrow \mathcal{E}$, we have that $\text{Ran}_{\gamma_{\mathcal{C}}}(G \circ F)$ is naturally isomorphic to $G \circ (\text{Ran}_{\gamma_{\mathcal{C}}} F)$. The dual statement holds for right deformable functors.*

PROOF OF PROPOSITION 4.2.9. We can obtain the counit of the adjunction $\mathbb{L}F \dashv \mathbb{R}G$ by

$$\begin{aligned} \mathbb{L}F \circ \mathbb{R}G &:= \mathbb{L}(\gamma_{\mathcal{D}} \circ F) \circ \mathbb{R}(\gamma_{\mathcal{C}} \circ G) \\ &\cong \mathbb{L}(\gamma_{\mathcal{D}} \circ F) \circ \text{Lan}_{\gamma_{\mathcal{D}}}(\gamma_{\mathcal{C}} \circ G) \quad (\text{Lemma 4.2.10}) \\ &\cong \square \text{Lan}_{\gamma_{\mathcal{D}}}(\mathbb{L}(\gamma_{\mathcal{D}} \circ F) \circ \gamma_{\mathcal{C}} \circ G) \quad (\text{Lemma 4.2.10}) \\ &\xrightarrow{\text{transformation that is part of the total left derivative}} \text{Lan}_{\gamma_{\mathcal{D}}}(\gamma_{\mathcal{D}} \circ F \circ G) \\ &\xrightarrow{\text{counit of } F \dashv G} \text{Lan}_{\gamma_{\mathcal{D}}} \gamma_{\mathcal{D}} \\ &\xrightarrow{\text{transformation given by universal property of } \text{Lan}_{\gamma_{\mathcal{D}}} \text{ and by equality isomorphism } \gamma_{\mathcal{D}} \cong \text{id}_{\text{Ho } \mathcal{D}} \circ \gamma_{\mathcal{D}}} \text{id}_{\text{Ho } \mathcal{D}}. \end{aligned}$$

Dually, one obtains the unit. One then needs to verify that these are really unit and counit of an adjunction, see Proposition 1.4.5 for this. \square

PROPOSITION 4.2.11. *Let $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ be composable functors between homotopical categories. Suppose that F and G are left deformable by (Q, q) and (R, r) , respectively, and let $\mathcal{C}_Q \subset \mathcal{C}$ and $\mathcal{D}_R \subset \mathcal{D}$ be full subcategories containing the images of Q and R and on which F and G are homotopical. Assume also $F(\mathcal{C}_Q) \subset \mathcal{D}_R$. Then there is a natural isomorphism of total left derived functors*

$$\mathbb{L}(G \circ F) \cong \mathbb{L}G \circ \mathbb{L}F.$$

PROOF. For $x \in \mathcal{C}$ and the (non-total) left derived functors $F' = FQ$ and $G' = GR$, we have a natural transformation

$$(4.2.1) \quad G'F'X = GRFQx \xrightarrow{G(r_{FQx})} GFQx.$$

By assumption $r_{FQx}: RFQx \rightarrow FQx$ is a weak equivalence between objects in \mathcal{D}_R and hence it is sent to a weak equivalence by G . This implies that (4.2.1) is actually a natural weak equivalence.

We next show that Q is a deformation for GF : for this, it suffices to show that for any objects $x, y \in \mathcal{C}$ and any weak equivalence $f : Qx \rightarrow Qy$ in \mathcal{C} , also GFf is a weak equivalence in \mathcal{E} . First, since (Q, q) is a deformation for F , we have that $Ff : FQx \rightarrow FQy$ is a weak equivalence. Second, by the previous argument together with 3-of-4 on the following diagram

$$\begin{array}{ccc} GRFQx & \xrightarrow{G(r_{FQx})} & GFQx \\ \downarrow GRFf & & \downarrow GFf \\ GRFQy & \xrightarrow{G(r_{FQy})} & GFQy, \end{array}$$

we conclude that also GFf is a weak equivalence.

Postcomposing (4.2.1) with $\gamma_{\mathcal{E}}$ and then applying $\text{Ran}_{\gamma_{\mathcal{C}}}(-)$ yields the desired natural isomorphism. \square

REMARK 4.2.12. Any model category can be seen as homotopical category with the weak equivalences being the weak equivalences of the model structure (this means that we are forgetting fibrations and cofibrations). For a *Quillen adjunction*

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{D}$$

between model categories (see Remark 2.10.6), we can obtain the total left derived functor through *cofibrant replacements*: A cofibrant replacement of an object X in a model category is a trivial fibration $\tilde{X} \rightarrow X$, where \tilde{X} is cofibrant. This gives us a left deformation for the Quillen left adjoint F (we will ignore here the question whether such a cofibrant replacement can be chosen in a functorial way). The total left derived functor of F is therefore given by $\mathbb{L}F(X) = F(\tilde{X})$ with a cofibrant replacement $\tilde{X} \rightarrow X$. Dually, $\mathbb{R}G(Y) = G(\tilde{Y})$, where $Y \rightarrow \tilde{Y}$ is a *fibrant replacement*, i.e. a trivial cofibration with \tilde{Y} fibrant. For more details, we refer to Section II.8 of [GJ99] and also [Hov99].

4.3. Exercises

PS6.

EXERCISE 4.3.1 (The easy part of Elmendorf's theorem). Let X, Y be G -spaces, and $f, g : X \rightarrow Y$ equivariant maps. A G -homotopy between them is an equivariant map $H : X \times [0, 1] \rightarrow Y$ satisfying the obvious boundary conditions, where $[0, 1]$ is given the trivial G -action, and $X \times [0, 1]$ the product action.

Show that if $f : X \rightarrow Y$ has a G -homotopy inverse, then it induces a homotopy equivalence on all fixed point subspaces $X^H \rightarrow Y^H$.

EXERCISE 4.3.2 (Deformations). (1) Show the following 2-universal property of the localization $C[W^{-1}]$: for all D , restriction along the localization functor $C \rightarrow C[W^{-1}]$ induces an equivalence $\text{Fun}(C[W^{-1}], D) \rightarrow \text{Fun}_W(C, D)$, where $\text{Fun}_W(C, D)$ is the full subcategory of $\text{Fun}(C, D)$ on those functors which send every morphism in W to an isomorphism.

(2) How is this different from the definition in the lecture notes ?

(3) Deduce lemma 4.1.10. from the lecture notes : if $q : Q \rightarrow \text{id}$ is a left deformation of C into C_Q , then $Ho(C) \simeq Ho(C_Q)$ (state this more precisely !).

EXERCISE 4.3.3 (Absoluteness lemma). Let $(M, W_M), (N, W_N)$ be relative categories, $F : M \rightarrow N$ a functor, and $q : Q \rightarrow \text{id}$ a left deformation of F . We wish to show that the total left derived functor $\mathbf{L}F$ of F is an absolute right Kan extension (in particular, pointwise).

(1) Explain why it suffices to show the following: for an arbitrary category D , and functor $F : M \rightarrow D$, if $q : Q \rightarrow \text{id}$ is a left deformation of F , then the unique \tilde{F} such that $F \circ Q \cong \tilde{F} \delta$ (where $\delta : M \rightarrow Ho(M)$ is the localization functor) is a right Kan extension of F along δ .

- (2) Prove the statement from (1).
 (3) Dualize the absoluteness lemma we just proved.

EXERCISE 4.3.4 (Adjunctions). Show that if $F : M \rightarrow N, G : N \rightarrow M$ are adjoints between relative categories, and both have total (left, resp. right) derived functors that are absolute (right, resp. left) Kan extensions, then $\mathbf{L}F$ is left adjoint to $\mathbf{R}G$.

EXERCISE 4.3.5 (Monoidal structures on homotopy categories). (1) Let M, N be relative categories. Explain how to define weak equivalences on $M \times N$, and show that $Ho(M \times N) \simeq Ho(M) \times Ho(N)$.

(2) Explain how to use this to define, e.g. the derived tensor product \otimes_R^L on $Ho(Ch_{\geq 0}(R))$, when R is a commutative ring. Show that it is associative up to isomorphism, and symmetric up to isomorphism.

(3) Compute some derived tensor products - what are the homology groups of $\mathbb{Z}/n \otimes^L \mathbb{Z}/m$? Of $\mathbb{Q}/\mathbb{Z} \otimes^L \mathbb{Q}/\mathbb{Z}$? $\mathbb{Q}/\mathbb{Z} \otimes^L \mathbb{Z}/n$? $\mathbb{Q} \otimes^L A$? Over a field k , compute $k[x, y]/(x) \otimes_{k[x, y]}^L k[x, y]/(x)$.

(4) (Bonus) Things can really go wrong in the unbounded case: let $k[\epsilon] := k[x]/(x^2)$ and consider the following chain complex of $k[\epsilon]$ -modules: $\cdots \rightarrow k[\epsilon] \xrightarrow{\epsilon} k[\epsilon] \xrightarrow{\epsilon} k[\epsilon] \xrightarrow{\epsilon} \cdots$. Observe that it is an exact complex of projective modules. Observe that tensoring it with k over $k[\epsilon]$ returns something non-exact. Where did we use boundedness before?

EXERCISE 4.3.6 (Hom sets in derived categories). (1) Let P be a non-negatively graded complex of projective R -modules. Show that the functor $K : M \mapsto \text{hom}(P, M)/\text{chain homotopy}$ is homotopical on $Ch_{\geq 0}(R)$.

(2) Let F be a homotopical functor $Ch_{\geq 0}(R) \rightarrow \mathbf{Ab}$ with a morphism from $\text{hom}(P, -)$. Prove that this morphism factors (necessarily uniquely) through K from question (1). Deduce that K is the left Kan extension of $\text{hom}(P, -)$ along the localization $Ch_{\geq 0}(R) \rightarrow Ho(Ch_{\geq 0}(R))$.

(3) Deduce that K is isomorphic to $\text{hom}_{Ho(Ch_{\geq 0}(R))}(P, -)$. Use this to compute $\text{hom}_{Ho(Ch_{\geq 0}(R))}(A[n], B[m])$ where A, B are abelian groups, $n, m \geq 0$ are integers and $A[n]$ means a complex concentrated in degree n with value A .

EXERCISE 4.3.7 (Stability of the derived category). We use again the notation I for the category that looks like $\bullet \rightarrow \bullet \leftarrow \bullet$.

(1) Recall the long exact sequence associated to a short exact sequence of chain complexes. Deduce that the pullback functor is homotopical on the full subcategory of $Ch(R)^I$ where one of the legs is a surjection. Deduce a formula for the homotopy pullback of chain complexes, as well as a long exact sequence of homology groups associated to a pullback.

(2) Do the analogous question for pushouts in place of pullbacks.

(3) For any chain complex C , there is a canonical diagram $0 \rightarrow C \leftarrow 0$. Describe its homotopy pullback in elementary terms - we call it ΩC . Do the same for the pushout of $0 \leftarrow C \rightarrow 0$, which we call ΣC . Show in particular that Σ is left adjoint to Ω , and that they form an adjoint equivalence.² (actually, first prove that it is an inverse pair of equivalences, adjointness is maybe a bit harder)

EXERCISE 4.3.8 (Homotopy pullbacks of spaces). We observed in the homework of Sheet 4 that the pullback functor was homotopical on the subcategory of \mathbf{Top}^I consisting of those diagrams $X \rightarrow Y \leftarrow Z$ such that $Z \rightarrow Y$ was a fibration (we did the simplicial analogue, but this is true too and not much harder to prove) - here, I is the category that looks like $\bullet \rightarrow \bullet \leftarrow \bullet$. We take this for granted here.

Construct a right deformation of \mathbf{Top}^I onto this subcategory, to deduce a formula for the homotopy pullback of spaces. Deduce a formula for the homotopy fiber of pointed spaces.

Describe for instance the pullback $\{x\} \times_X \{x\}$ for $x \in X$.

²This is related to stable ∞ -categories.

EXERCISE 4.3.9 (Uncoherent actions). [2+2+2+2+2] Let G, H be groups.

(1) Prove that $\text{Aut}_{\text{Ho}(\mathbf{sSet})}(BH) \cong \text{Out}(H) := \text{Aut}(H)/\text{Inn}(H)$, where $\text{Inn}(H)$ is the set of inner automorphisms of H , i.e. automorphisms of the form $x \mapsto h x h^{-1}$. Deduce a description of elements of $\text{Ho}(\mathbf{sSet})^{BG}$ whose underlying homotopy type is BH .

(2) Suppose given $X \in \mathbf{sSet}^{BG}$ whose underlying simplicial set is a Kan complex homotopy equivalent to BH . Show that $X \times EG$ with the diagonal G -action is a free G -Kan complex with the same homotopy type. Using the classification of fibrations, explain how to construct from this a short exact sequence of the form $1 \rightarrow H \rightarrow E \rightarrow G \rightarrow 1$ (called an *extension* of G by H). Show that if $X \rightarrow Y$ is equivariant and a homotopy equivalence on underlying Kan complexes, it induces the same extension up to isomorphism.

(3) Suppose conversely given an extension $1 \rightarrow H \rightarrow E \rightarrow G \rightarrow 1$. Explain how to construct an $X \in \mathbf{sSet}^{BG}$ as before. Show that the extension you get from this X (by (2)) is isomorphic to the one you started with.

(4) Conversely, show that if X and Y (still Kan complexes) induce the same extension up to isomorphism, then they are isomorphic in $\text{Ho}(\mathbf{sSet}^{BG})$.

(5) Deduce that the forgetful functor $\text{Ho}(\mathbf{sSet}^{BG}) \rightarrow \text{Ho}(\mathbf{sSet})^{BG}$ is, in general, neither injective nor surjective on objects up to isomorphism. You're allowed to use, without justification, that any $X \in \mathbf{sSet}^{BG}$ is equivalent to some action on a Kan complex.

EXERCISE 4.3.10 (Derived functors in homological algebra). [3+4+3]

(1) Let $F : \mathbf{Mod}_R \rightarrow \mathbf{Mod}_S$ be a right exact functor. Show that it induces a functor $\text{Ch}_{\geq 0}(R) \rightarrow \text{Ch}_{\geq 0}(S)$ which is homotopical on the subcategory of chain complexes of projectives. Deduce that F has a left derived functor.

(2) Show that for any short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in \mathbf{Mod}_R , there is a homotopy pushout square in $\text{Ch}(S)$ ³

$$\begin{array}{ccc} \mathbf{LFA}[0] & \longrightarrow & \mathbf{LFB}[0] \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbf{LFC}[0] \end{array}$$

Observe, or prove, that it is also a homotopy pullback square. You may use the horseshoe lemma.⁴

(3) Deduce the long exact sequence in homology groups that you know from homological algebra.

³A homotopy pushout is by definition a possible value of the original pushout diagram (not necessarily unique in the strict sense) under a left derived functor of the pushout functor

⁴Recall that the horseshoe lemma says that given a projective resolution P of A and Q of C , one can find a projective resolution T of B fitting in a short exact sequence $0 \rightarrow P \rightarrow T \rightarrow Q \rightarrow 0$, in a way compatible with our original exact sequence.

CHAPTER 5

Homotopy limits and colimits

5.1. The homotopy colimit construction

In this section we will give a definition of the homotopy colimit as a functor $\mathrm{hocolim} : \mathrm{Top}^{\mathcal{I}} \rightarrow \mathrm{Top}$, using a certain explicit construction, and we will look at some examples. Later we will justify this construction by proving that it is a left derived functor of the colimit functor. We first work in the full subcategory CW of Top consisting of spaces homotopy equivalent to CW complexes. We will show:

- (1) There is a natural transformation $\iota : \mathrm{hocolim} \rightarrow \mathrm{colim}$ of functors $\mathrm{CW}^{\mathcal{I}} \rightarrow \mathrm{CW}$. (Proposition 5.1.18; see also Proposition 5.3.11.)
- (2) The functor $\mathrm{hocolim}$ is homotopical, i.e. it takes weak equivalences in $\mathrm{CW}^{\mathcal{I}}$ to weak equivalences in CW . (Theorem 5.2.1.)
- (3) For any homotopical functor $G : \mathrm{CW}^{\mathcal{I}} \rightarrow \mathrm{CW}$ with a natural transformation $\eta : G \rightarrow \mathrm{colim}$, then there is a unique factorization of $\gamma\eta$ as

$$\gamma G \xrightarrow{\theta} \gamma \mathrm{hocolim} \xrightarrow{\gamma\iota} \gamma \mathrm{colim},$$

where $\gamma = \gamma_{\mathrm{CW}}$ is the localization functor $\mathrm{CW} \rightarrow \mathrm{Ho}(\mathrm{CW})$, and $\theta : \gamma G \rightarrow \gamma \mathrm{hocolim}$ is a natural transformation. (Theorem 5.4.3.)

It then follows Definition 4.2.4 that $\mathrm{hocolim}$ is a left derived functor of colim . As part of doing this we will devise a functorial way $Q : \mathrm{CW}^{\mathcal{I}} \rightarrow \mathrm{CW}^{\mathcal{I}}$ to replace any diagram by a “fattened-up” diagram, and we will then see that $\mathrm{hocolim} \cong \mathrm{colim} \circ Q$ (Proposition 5.1.24), that Q is a left deformation of $\mathrm{CW}^{\mathcal{I}}$ (Proposition 5.1.23), and that colim is homotopical on the image of Q .

Note that it follows from Proposition 4.2.9 that $\mathrm{hocolim} : \mathrm{Ho}(\mathrm{CW}^{\mathcal{I}}) \rightarrow \mathrm{Ho}(\mathrm{CW})$ is then the left adjoint to the constant diagram functor $\mathrm{Ho}(\mathrm{CW}) \rightarrow \mathrm{Ho}(\mathrm{CW}^{\mathcal{I}})$. (Theorem 5.4.3.) By the uniqueness of adjoints this property specifies the functor uniquely up to unique natural isomorphism as a functor between the homotopy categories.

REMARK 5.1.1. The statement above can be immediately propagated to Top by redefining our $\mathrm{hocolim}$ functor to be the functor given by first applying $|\mathrm{Sing}_{\bullet}(-)|$ to land in CW , and then the old $\mathrm{hocolim}$, cf. Proposition 4.1.12. Better yet, the above statement holds verbatim also for Top without applying $|\mathrm{Sing}_{\bullet}(-)|$, even with the model for $\mathrm{hocolim}$ we have in mind. The proof of this last refinement however requires more point set topology than we want to get into here—we refer the reader to [DI04, App. A] instead for a proof of this.

REMARK 5.1.2. We work out our $\mathrm{hocolim}$ construction with topological spaces. However, we might as well have carried out all our constructions in sSet , and there are similar formulas in chain complexes—we may expand on this in the Exercises.

And indeed there are similar constructions in model categories (see [Hir03]) or indeed in any infinity category (where $\mathrm{hocolim}$ is just the colimit, see [Lur09, §1.2.13 and Ch.4]).

REMARK 5.1.3. We again stress what we have said earlier: While one can choose many point-set models for homotopy colimit, the derived functor of colim is unique if it exists, so all models will agree in the homotopy category. We choose to work with “our” point-set version of $\mathrm{hocolim}$, because of its relative simplicity. And the fact that it is the geometric realization of a simplicial space in fact gives us a spectral sequence for computing the homology of a $\mathrm{hocolim}$, which we will set up in a later section.

REMARK 5.1.4. Just because we abstractly know that hocolim is the “best” approximation to colim in the sense above doesn’t mean that it is very good in concrete cases — it may be quite far from the actual colim . The homotopy colimit of a point under the (trivial) action of a finite group G is for example the infinite dimensional space BG . We will provide loads of examples after giving the definition.

Before giving the definition of hocolim , we first need to say a few words about simplicial spaces.

5.1.1. Simplicial spaces.

DEFINITION 5.1.5. Recall that if $X_\bullet : \Delta^{\text{op}} \rightarrow \text{Top}$ is a simplicial space, then the *geometric realization* $|X_\bullet|$ is the coequalizer

$$\text{coeq} \left(\coprod_{(\phi: [n] \rightarrow [m]) \in \Delta} X_n \times \Delta_{\text{top}}^m \rightrightarrows \coprod_{[k] \in \Delta} X_k \times \Delta_{\text{top}}^k \right),$$

where the two maps are given on the factor indexed by ϕ by $X_n \times \Delta_{\text{top}}^m \xrightarrow{\phi^* \times \text{id}} X_m \times \Delta_{\text{top}}^m$ and $X_n \times \Delta_{\text{top}}^m \xrightarrow{\text{id} \times \phi_*} X_n \times \Delta_{\text{top}}^n$. (Here Δ_{top}^n denotes the *topological* n -simplex.) More explicitly, this is the quotient space obtained from the disjoint union $\coprod_n X_n \times \Delta_{\text{top}}^n$ by identifying $(\phi^*(x), y) \in X_n \times \Delta_{\text{top}}^n$ with $(x, \phi_* y) \in X_m \times \Delta_{\text{top}}^m$ for all $\phi: [n] \rightarrow [m]$, $x \in X_m$, and $y \in \Delta_{\text{top}}^n$.

REMARK 5.1.6. Note that there is a natural map

$$|X_\bullet| \rightarrow \text{colim}_{\Delta^{\text{op}}} X_\bullet$$

induced by the taking coequalizers of the rows of the following diagram.

$$\begin{array}{ccc} \coprod_{\phi: [m] \rightarrow [n]} X_n \times \Delta_{\text{top}}^m & \rightrightarrows & \coprod_k X_k \times \Delta_{\text{top}}^k \\ \downarrow & & \downarrow \\ \coprod_{\phi: [m] \rightarrow [n]} X_n & \rightrightarrows & \coprod_k X_k, \end{array}$$

where the vertical maps are given by projecting off the Δ_{top}^n -factors. By definition the coequalizer of the top row is $|X_\bullet|$, whereas the coequalizer in the bottom row is $\text{colim}_{\Delta^{\text{op}}} X_\bullet$. In this way the geometric realization is a “fattened” version of the colimit of X_\bullet over Δ^{op} , where we have stuck in some contractible spaces.

The following more economical description of $\text{colim}_{\Delta^{\text{op}}} X_\bullet$ will also be used repeatedly later:

PROPOSITION 5.1.7.

$$\text{colim}_{\Delta^{\text{op}}} X_\bullet \cong \text{coeq}(d_0, d_1 : X_1 \rightrightarrows X_0)$$

SKETCH OF PROOF. You can probably convince yourself directly that this is true. A formal argument uses cofinality: First note that by cofinality (see Proposition 5.5.4),

$$\text{colim}_{\Delta^{\text{op}}} X_\bullet \cong \text{colim}_{(\Delta_{\leq 1})^{\text{op}}} X_\bullet$$

Now observe directly that the colimit over $(\Delta_{\leq 1})^{\text{op}}$ can be further replaced by the colimit over $(\Delta_{\leq 1}^{\text{inj}})^{\text{op}}$, i.e. we can leave out the degeneracy s_0 without changing the colimit. This is now the wanted coequalizer. \square

5.1.2. The definition of the homotopy colimit.

DEFINITION 5.1.8. For a functor $F: \mathcal{I} \rightarrow \mathbf{Top}$ define the associated simplicial space F_{\bullet}^{Δ} by $F_n^{\Delta} := \coprod_{i_0 \rightarrow \dots \rightarrow i_n} F(i_0)$; for $\phi: [m] \rightarrow [n]$ in Δ , the map $\phi^*: F_n^{\Delta} \rightarrow F_m^{\Delta}$ is given on the component indexed by $i_0 \rightarrow \dots \rightarrow i_n$ by $F(i_0 \rightarrow \dots \rightarrow i_{\phi(0)}): F(i_0) \rightarrow F(i_{\phi(0)})$ where the target is in the component indexed by $i_{\phi(0)} \rightarrow i_{\phi(1)} \rightarrow \dots \rightarrow i_{\phi(m)}$.

We then define the *homotopy colimit* of F as

$$\mathrm{hocolim} F = \mathrm{hocolim}_{\mathcal{I}} F = |F_{\bullet}^{\Delta}| \in \mathbf{Top}.$$

We can unwind this definition to see that $\mathrm{hocolim} F$ is the quotient space

$$\mathrm{hocolim} F = \left(\coprod_n \coprod_{i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_n} F(i_0) \times \Delta_{top}^n \right) / \sim$$

where \sim is generated by the usual identities $(d_i x, t) \sim (x, d^i t)$ and $(s_i x, t) \sim (x, s^i t)$, with $x \in \coprod_{i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_n} F(i_0)$ and $d_i x$ defined by:

$$d_0(i_0 \xrightarrow{k} i_1 \rightarrow \dots \rightarrow i_n, x \in F(i_0)) = (i_1 \rightarrow \dots \rightarrow i_n, k(x) \in F(i_1));$$

$$d_j(i_0 \rightarrow \dots \rightarrow i_n, x \in F(i_0)) = (i_0 \rightarrow \dots \rightarrow i_j \rightarrow \dots \rightarrow i_n, x \in F(i_0)) \text{ for } j \neq 0.$$

REMARK 5.1.9. For arbitrary simplicial spaces, geometric realization is not so homotopically well-behaved, and does not agree with the homotopy colimit over Δ^{op} . However, we will see that this is true under an extra point-set topological condition on a simplicial space called “Reedy cofibrancy”

REMARK 5.1.10. Note that the simplicial space F_{\bullet}^{Δ} is really defined by the formula $F_n^{\Delta} := \coprod_{i: [n] \rightarrow \mathcal{I}} F(i(0))$, where $[n]$ is the ordered set $0 < \dots < n$; hence the set parametrising the disjoint union is the same as the set $(N\mathcal{I})_n$ appearing in the nerve of \mathcal{I} .

The following proposition is immediate, and its proof is left as exercise.

PROPOSITION 5.1.11. *Let \mathcal{I} be a small category and let $F: \mathcal{I} \rightarrow \mathbf{Top}$ be a functor.*

- (1) *If $G: \mathcal{I}' \rightarrow \mathcal{I}$ is a functor between small categories, then there is a natural map of spaces $\mathrm{hocolim}_{\mathcal{I}'} (F \circ G) \rightarrow \mathrm{hocolim}_{\mathcal{I}} F$, induced by sending the copy of $F \circ G(i'(0))$ corresponding to $i': [n] \rightarrow \mathcal{I}'$ to the copy of $F(G \circ i'(0))$ corresponding to $G \circ i'$ via the identity map of spaces.*
- (2) *If $F': \mathcal{I} \rightarrow \mathbf{Top}$ is another functor and $\eta: F \rightarrow F'$ is a natural transformation, then there is a natural map $\mathrm{hocolim}_{\mathcal{I}} F \rightarrow \mathrm{hocolim}_{\mathcal{I}} F'$ induced by sending the copy of $F(i(0))$ corresponding to i to the copy of $F'(i(0))$ corresponding to i via the map $\eta_{i(0)}$.*

Now let us look at some examples:

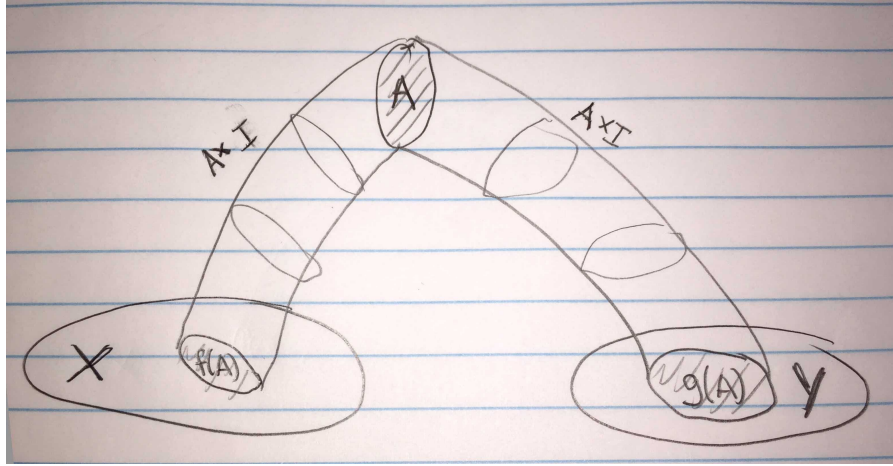
EXAMPLE 5.1.12. Suppose $F: \mathcal{I} \rightarrow \mathbf{Top}$ is the constant functor with value $*$. Then F^{Δ} is precisely the nerve $N\mathcal{I}$ (viewed as a discrete simplicial space) and so $\mathrm{hocolim}_{\mathcal{I}} F$ is the classifying space $B\mathcal{I}$. More generally if F is constant with value X then $\mathrm{hocolim}_{\mathcal{I}} F \cong X \times B\mathcal{I}$.¹

EXAMPLE 5.1.13. Consider the pushout diagram: $X \xleftarrow{f} A \xrightarrow{g} Y$. The definition reveals that the homotopy colimit is a double mapping cylinder construction. In more details:

$$\mathrm{hocolim}(X \xleftarrow{f} A \xrightarrow{g} Y) \cong ((X \amalg A \amalg Y) \amalg (A \times I \amalg A \times I)) / \sim$$

where \sim identifies certain points as in the following picture:

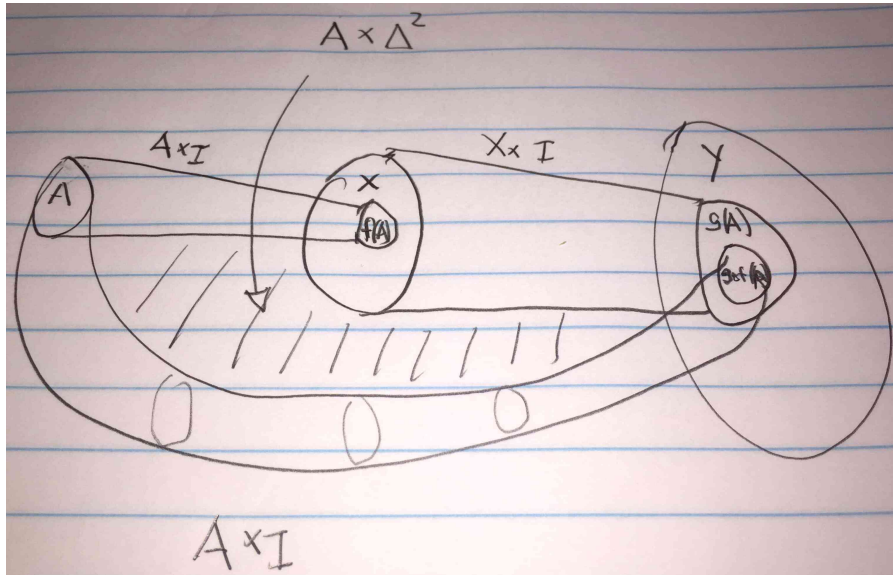
¹Perhaps we need to assume X locally compact, or work within compactly generated Hausdorff spaces...



To spell this out, in the left copy of $A \times I$, the “face” $A \times 0$ is glued to A and the other “face” $A \times 1$ is glued to X via f , i.e., $A \times I \ni (a, 1) \sim f(a) \in X$. In the right copy of $A \times I$, $A \times 0$ is glued to A and $A \times 1$ is glued to Y via $A \times I \ni (a, 1) \sim g(a) \in Y$.

Note that we have obtained $\operatorname{hocolim}(X \xleftarrow{f} A \xrightarrow{g} Y)$ as the quotient of the disjoint union of only 5 spaces of the form $S \times \Delta_{top}^m$; and on the other hand, the nerve of the pushout category $N(\bullet \leftarrow \bullet \rightarrow \bullet)$, as a simplicial set, contains precisely 5 non-degenerate simplices. Can you find an explanation (amenable to generalisations) for this?

EXERCISE 5.1.14. For the diagram $A \rightarrow X \rightarrow Y$, check that the homotopy colimit looks as in the following picture:



In particular, show that $\operatorname{hocolim}(A \rightarrow X \rightarrow Y)$ as a quotient of $A \times \Delta_{top}^2 \amalg X \times \Delta_{top}^1 \amalg Y$, and show that $\operatorname{hocolim}(A \rightarrow X \rightarrow Y)$ deformation retracts onto Y .

EXERCISE 5.1.15. Check that the homotopy colimit of a group G acting on a space X is homotopy equivalent to the so-called Borel construction, i.e.

$$\operatorname{hocolim}_G X \simeq (X \times EG)/G.$$

EXERCISE 5.1.16. (1) Give the precise definition of mapping cylinder $\operatorname{Cyl}(F)$ of F , as a quotient of $\coprod_{n \geq 0} F_n \times I$ by gluing $F_n \times \{1\}$ to $F_{n+1} \times \{0\}$ along the map $F(n < n+1): F_n \rightarrow F_{n+1}$.

(2) Identify $\operatorname{Cyl}(F)$ as a subspace of $\operatorname{hocolim}_{\mathbb{N}} F$.

(3) Prove that $\mathrm{hocolim}_{\mathbb{N}} F$ deformation retracts onto $\mathrm{Cyl}(F)$.

5.1.3. hocolim as a deformation. We now embark on establishing the basic properties of $\mathrm{hocolim}$. Our first item on the agenda is to relate the homotopy colimit to the ordinary colimit.

First note that Proposition 5.1.7 allows us to reexpress the colimit of F as a colimit of F^Δ :

PROPOSITION 5.1.17. $\mathrm{colim}_{\mathcal{I}} F \cong \mathrm{colim}_{\Delta^{\mathrm{op}}} F^\Delta$. \square

With this, it is easy to see that the first requirement for a derived functor from Definition 4.2.4 holds for our definition of $\mathrm{hocolim}$:

PROPOSITION 5.1.18. *There is a natural transformation $\mathrm{hocolim} \rightarrow \mathrm{colim}$.*

PROOF. By Remark 5.1.6 we have for any simplicial space $X \in \mathrm{Top}^{\Delta^{\mathrm{op}}}$ the map $|X| \rightarrow \mathrm{colim}_{\Delta^{\mathrm{op}}} X$. Applying this to F_\bullet^Δ we obtain the desired map

$$\mathrm{hocolim}_{\mathcal{I}} F = |F_\bullet^\Delta| \rightarrow \mathrm{colim}_{\Delta^{\mathrm{op}}} F^\Delta \cong \mathrm{colim}_{\mathcal{I}} F.$$

\square

An important special case of homotopy invariance is when the indexing category has a terminal object, and we start with that case.

PROPOSITION 5.1.19. *Let \mathcal{I} be a small category with a terminal object t . Then for any functor $F: \mathcal{I} \rightarrow \mathrm{Top}$, the natural map $\mathrm{hocolim}_{\mathcal{I}} F \rightarrow \mathrm{colim}_{\mathcal{I}} F \cong F(t)$ is a homotopy equivalence, in particular a weak equivalence.*

REMARK 5.1.20. We have already seen an instance of this situation in Exercise 5.1.14, and looking at the associated picture should give a good idea of why this is true. The formal proof is similar to the proof that the realization of the nerve of a category with a terminal object is contractible, but now keeping an extra “space-coordinate” around: Recall that that proof goes by defining a functor $\mathcal{I} \times [1] \rightarrow \mathcal{I}$ as the identity on $\mathcal{I} \times 0$, constant t on $\mathcal{I} \times 1$, and sending the morphism $(i, 0) \rightarrow (i, 1)$ to the unique map $i \rightarrow t$. Upon realization of nerve this gives the wanted homotopy.

PROOF OF PROPOSITION 5.1.19. We want to make a deformation retraction onto $F(t)$, i.e. a homotopy

$$\mathrm{hocolim}_{\mathcal{I}} F \times I \rightarrow \mathrm{hocolim}_{\mathcal{I}} F$$

contracting onto $F(t) \subset \mathrm{hocolim}_{\mathcal{I}} F$. The geometric realization functor $|-|: \mathrm{Top}^{\Delta^{\mathrm{op}}} \rightarrow \mathrm{Top}$ commutes with products; in particular, since $I \cong |\Delta^1|$, we can describe the homotopy colimit $\mathrm{hocolim}_{\mathcal{I}} F \times I$ as the realization of the product simplicial space $F^\Delta \times \Delta^1$,² remembering the definition of F^Δ , we can describe $F^\Delta \times \Delta^1$ as

$$(F^\Delta \times \Delta^1)_{n+1} := \coprod_{i: [n] \rightarrow (\mathcal{I} \times [1])} F(\mathrm{pr}_{\mathcal{I}}(i(0))),$$

where $[n] = (0 < \dots < n)$ (hence functors $F: [n] \rightarrow \mathcal{I} \times [1]$ correspond to n -simplices in $N(\mathcal{I} \times [1])$), and $\mathrm{pr}_{\mathcal{I}}: \mathcal{I} \times [1] \rightarrow \mathcal{I}$ is the projection functor

Let T be the functor $T: \mathcal{I} \times [1] \rightarrow \mathcal{I} \times [1]$ restricting to the identity on $\mathcal{I} \cong \mathcal{I} \times 0 \rightarrow \mathcal{I}$ and restricting to the constant functor on $\mathcal{I} \times 1 \rightarrow \{t\}$. Note that there is a natural transformation $\eta: \mathrm{id}_{\mathcal{I} \times [1]} \rightarrow T$ which takes an identity morphism on an object of the form $x \times 0$ and takes the terminal map to t on an object of the form $x \times 1$.

We now define a map of simplicial spaces $F^\Delta \times \Delta^1 \rightarrow F^\Delta$ as follows: given $n \geq 0$ and $i: [n] \rightarrow \mathcal{I} \times [1]$, we map the copy of $F(\mathrm{pr}_{\mathcal{I}}(i(0)))$ corresponding to i to the copy of $F(\mathrm{pr}_{\mathcal{I}}(T \circ i(0)))$ corresponding to $\mathrm{pr}_{\mathcal{I}} \circ T \circ i$, along the map $F(\mathrm{pr}_{\mathcal{I}}(\eta_{i(0)}))$.

²Here we consider the simplicial set Δ^1 as a discrete simplicial space.

Check that this gives a map of simplicial spaces, and induces the desired deformation retract on realizations. \square

REMARK 5.1.21. The above proof can be generalized to the situation called “having an extra degeneracy”; this is explained in homework.

We can consider Proposition 5.1.19 as an instance of how the homotopy colimit construction first replaces a diagram by a homotopy equivalent and “good” diagram, and then takes the ordinary colimit of this second diagram; this generalizes the mapping cylinder construction and the examples/exercises in the beginning of the section.

DEFINITION 5.1.22. For a functor $F: \mathcal{I} \rightarrow \text{Top}$, we define a new functor $QF: \mathcal{I} \rightarrow \text{Top}$ by

$$(QF)(i) := \text{hocolim}_{\mathcal{I}/i} F = \left| n \mapsto \coprod_{i_0 \rightarrow \dots \rightarrow i_n \rightarrow i} F(i_0) \right|.$$

PROPOSITION 5.1.23. The map $QF(i) = \text{hocolim}_{\mathcal{I}/i} F \xrightarrow{\sim} F(i)$ induced by

$$F(i_0)_{i_0 \rightarrow \dots \rightarrow i_n \rightarrow i} \xrightarrow{F(i_0 \rightarrow i)} F(i)$$

gives a natural transformation $q_F: QF \rightarrow F$, and is an objectwise homotopy equivalence by Proposition 5.1.19. In particular (Q, q) is a left deformation of $\text{Top}^{\mathcal{I}}$, and it restricts to a left deformation of $\text{CW}^{\mathcal{I}}$, in the sense of Definition 4.1.9. \square

PROPOSITION 5.1.24. For any functor F , we have a natural isomorphism

$$\text{colim}_{\mathcal{I}} QF = \text{hocolim}_{\mathcal{I}} F.$$

SKETCH PROOF. We unravel the definitions, using the notation $i_0 \rightarrow \dots \rightarrow i_n \downarrow i$ for an n -fold composition in \mathcal{I}/i :

$$\begin{aligned} \text{colim}_{\mathcal{I}} QF &= \text{colim}_{i \in \mathcal{I}} \left(\left(\coprod_{i_0 \rightarrow \dots \rightarrow i_n \downarrow i} F(i_0) \times \Delta_{\text{top}}^n \right) / \sim \right) \\ &= \left(\coprod_{i \in \mathcal{I}} \left(\left(\coprod_{i_0 \rightarrow \dots \rightarrow i_n \downarrow i} F(i_0) \times \Delta_{\text{top}}^n \right) / \sim \right) \right) / \approx \\ &= \left(\coprod_{i_0 \rightarrow \dots \rightarrow i_n} F(i_0) \times \Delta_{\text{top}}^n \right) / \sim \\ &= \text{hocolim}_{\mathcal{I}} F \end{aligned}$$

where \sim is the equivalence relations coming from the simplicial identities and \approx is the equivalence relation identifying each point in $\left(\coprod_{i_0 \rightarrow \dots \rightarrow i_n \downarrow j} F(i_0) \times \Delta_{\text{top}}^n \right) / \sim$ with the corresponding point in $\left(\coprod_{i_0 \rightarrow \dots \rightarrow i_n \downarrow k} F(i_0) \times \Delta_{\text{top}}^n \right) / \sim$, for all morphisms $f: j \rightarrow k$ in \mathcal{I} . One can check that \approx is generated by just identifying each factor $F(i_0) \times \Delta_{\text{top}}^n$ indexed by $i_0 \rightarrow \dots \rightarrow i_n \downarrow i$ with the corresponding factor $F(i_0) \times \Delta_{\text{top}}^n$ indexed by $i_0 \rightarrow \dots \rightarrow i_n \downarrow i_n$, i.e. a factor in whose index the map “ \downarrow ” is the identity of i_n (so that $i_n \downarrow i_n$ is terminal in \mathcal{I}/i_n). Points coming from such special factors have no further identifications due to \approx , so we can compute $\text{colim}_{\mathcal{I}} QF$ as the quotient of $\coprod_{i_0 \rightarrow \dots \rightarrow i_n \downarrow i_n} F(i_0) \times \Delta_{\text{top}}^n \cong \coprod_{i_0 \rightarrow \dots \rightarrow i_n} F(i_0) \times \Delta_{\text{top}}^n$ by \sim , which is the definition of $\text{hocolim}_{\mathcal{I}} F$. \square

We will see another slick proof of Proposition 5.1.24 later, in Proposition 5.3.11 after we have introduced the two-sided bar construction.

5.2. Homotopy invariance of the homotopy colimit

Our goal in this section is to prove the homotopy invariance of the homotopy colimit. More precisely, we will show:

THEOREM 5.2.1. *Suppose F and F' are functors $\mathcal{I} \rightarrow \text{CW}$ and $\eta: F \rightarrow F'$ is a natural transformation such that for every $i \in \mathcal{I}$ the morphism $\eta_i: F(i) \rightarrow F'(i)$ is a weak homotopy equivalence. Then the induced map $\text{hocolim}_{\mathcal{I}} F \rightarrow \text{hocolim}_{\mathcal{I}} F'$ is a weak equivalence.*

We will deduce Theorem 5.2.1 from a general result on homotopy invariance for the geometric realization of simplicial spaces. To state it we need some definitions:

DEFINITION 5.2.2. If X_{\bullet} is a simplicial space, its n^{th} *latching object* $L_n X$ is the colimit of the composite functor

$$((\Delta_{<n}^{\text{surj}})_{[n]/})^{\text{op}} \rightarrow \Delta^{\text{op}} \xrightarrow{X} \text{Top},$$

where Δ^{surj} is the subcategory of Δ spanned by all objects and containing only surjective maps, and the awkward looking $(\Delta_{<n}^{\text{surj}})_{[n]/}$ is the (full) subcategory of the under category $(\Delta^{\text{surj}})_{[n]/}$ spanned by all surjections $[n] \twoheadrightarrow [i]$ with $i < n$ (that is $i \neq n$). In other words $L_n X$ is just the colimit of X_i over all the degeneracies $[n] \twoheadrightarrow [i]$, $i < n$.

REMARK 5.2.3. We can think of $L_n X$ as the collection of all *degenerate* simplices inside X_n . If X is a simplicial set, this is literally true. In general we only have a well-defined (and injective) map $L_n X \rightarrow X_n$, adjoint to the natural transformation $X \rightarrow \delta(X_n)$ of functors $(\Delta_{<n}^{\text{surj}})_{[n]/} \rightarrow \text{Top}$ given on the object $(\theta: [n] \twoheadrightarrow [i])$ by $\theta^*: X_i \rightarrow X_n$.

REMARK 5.2.4. Notice that $s_i: X_{n-1} \rightarrow X_n$ has a left inverse d_i , by the simplicial identities, so it is an injection and a homeomorphism onto its image $s_i(X_{n-1})$ (since for any open $U \subseteq X_{n-1}$, $U = s_i^{-1} d_i^{-1}(U)$). Thus we can identify the colimit $L_n X$ as the union $\bigcup_{i=0}^n s_i(X_{n-1})$ inside X_n , at least as a set.

Moreover, if X_n is Hausdorff, then the equality of subsets $s_i(X_{n-1}) = \{x \in X_n : s_i d_i x = x\}$ implies that $s_i(X_{n-1})$ is a closed subset of X_n .³ Similarly, one can show that for all $\theta: [n] \twoheadrightarrow [i]$ with $n > i$ one has that $\theta^*(X_i) \subset X_n$ is closed. Now $L_n X$ has the topology of a *finite* colimit of closed inclusions of topological spaces, all of which are closed subspaces of X_n : it then follows that $L_n X \subset X$ is a closed embedding (homeomorphism with image + closed image).

DEFINITION 5.2.5. A simplicial space X_{\bullet} is *Reedy cofibrant* if for every n the map $L_n X \rightarrow X_n$ is a (closed) cofibration of topological spaces (i.e. a closed embedding with the homotopy extension property).

Let us check that the main example we have in mind is indeed Reedy cofibrant.

LEMMA 5.2.6. *For any functor $F: \mathcal{I} \rightarrow \text{Top}$, the simplicial space F^{Δ} is Reedy cofibrant.*

PROOF. The latching object $L_n(F^{\Delta})$ can be identified with the part of the disjoint union $F_n^{\Delta} = \coprod_{i_0 \rightarrow \dots \rightarrow i_n} F(i_0) \times \Delta_{\text{top}}^n$ corresponding to sequences $i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_n$ where at least one of the maps is an identity. Thus F_n^{Δ} is a coproduct of $L_n(F^{\Delta})$ and the coproduct over the remaining n -simplices⁴ in $N\mathcal{I}$, and the inclusion is obviously a closed cofibration. \square

The result about simplicial spaces that we are aiming at for now is:

THEOREM 5.2.7. *Let X_{\bullet} and Y_{\bullet} be Reedy cofibrant simplicial spaces such that X_n and Y_n are CW complexes for all n , and let $f_{\bullet}: X_{\bullet} \rightarrow Y_{\bullet}$ be a levelwise weak equivalence. Then $|f|: |X| \rightarrow |Y|$ is a weak equivalence.*

³Recall that if $f, g: X \rightarrow Y$ are continuous and Y is Hausdorff, then $\{x : f(x) = g(x)\}$ is always a closed subset of X .

⁴...which are precisely the non-degenerate n -simplices in $N\mathcal{I}$.

Before embarking on the proof, let us see that it implies Theorem 5.2.1.

PROOF OF THEOREM 5.2.1 ASSUMING THEOREM 5.2.7. We have already seen that F_\bullet^Δ is Reedy cofibrant in Lemma 5.2.6. Furthermore by assumption

$$F^\Delta([n]) = \coprod_{i_0 \rightarrow \dots \rightarrow i_n} F(i_0) \rightarrow \coprod_{i_0 \rightarrow \dots \rightarrow i_n} F'(i_0) = F'^\Delta([n])$$

is a weak homotopy equivalence, so it follows from Theorem 5.2.7 that

$$\operatorname{hocolim}_{\mathcal{I}} F = |F_\bullet^\Delta| \rightarrow |F'_\bullet^\Delta| = \operatorname{hocolim}_{\mathcal{I}} F'$$

is a weak equivalence as wanted. \square

To prove Theorem 5.2.7 we need some input from more classical homotopy theory, which we summarize in a lemma:

LEMMA 5.2.8.

- (i) Suppose $X \hookrightarrow Y$ is a cofibration and $X \rightarrow X'$ is a homotopy equivalence. Then the pushout $Y \rightarrow Y \cup_X X'$ is also a homotopy equivalence.
- (ii) Suppose we are given a commutative diagram

$$\begin{array}{ccccc} X & \longleftarrow & A & \xleftarrow{i} & Y \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longleftarrow & A' & \xleftarrow{i'} & Y' \end{array},$$

where i and i' are cofibrations and the vertical maps are homotopy equivalences. Then the induced map on pushouts $X \cup_A Y \rightarrow X' \cup_{A'} Y'$ is a homotopy equivalence.

- (iii) Suppose we are given a map of sequences

$$\begin{array}{ccccccc} X_0 & \hookrightarrow & X_1 & \hookrightarrow & X_2 & \hookrightarrow & \dots \\ \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \\ Y_0 & \hookrightarrow & Y_1 & \hookrightarrow & Y_2 & \hookrightarrow & \dots \end{array},$$

where the maps $X_i \hookrightarrow X_{i+1}$ and $Y_i \hookrightarrow Y_{i+1}$ are all cofibrations, and the maps $f_i: X_i \rightarrow Y_i$ are all homotopy equivalences. Then the induced map on colimits $f: \operatorname{colim}_{n \rightarrow \infty} X_n \rightarrow \operatorname{colim}_{n \rightarrow \infty} Y_n$ is a homotopy equivalence.

- (iv) Suppose $A \hookrightarrow A'$ and $B \hookrightarrow B'$ are closed cofibrations. Then the induced map $A \times B' \amalg_{A \times B} A' \times B \rightarrow A' \times B'$ is also a closed cofibration.

PROOF. For (i) see [Hat02, Exercise 0.27]. (ii) follows formally from (i) — see [MP12, Proposition 15.4.4]. For (iii) we can use Lemma 4.1.18 to inductively build a sequence of compatible homotopy inverses $Y_n \rightarrow X_n$ of f_n , in the limit these then give a homotopy inverse to f . For (iv), see [May99, §6.4]. \square

We also need to introduce the notion of n -skeleton for simplicial spaces, and to prove some basic properties of these:

DEFINITION 5.2.9. For X_\bullet a simplicial space, let $\operatorname{sk}_n |X|$ denote the analogue of the geometric realization where we only consider the objects $[k] \in \Delta$ where $k \leq n$, i.e.

$$\operatorname{coeq} \left(\coprod_{\substack{([m] \rightarrow [k]) \in \Delta \\ m, k \leq n}} X_k \times \Delta_{\operatorname{top}}^m \rightrightarrows \coprod_{\substack{[l] \in \Delta \\ l \leq n}} X_l \times \Delta_{\operatorname{top}}^l \right).$$

LEMMA 5.2.10. Suppose X_\bullet is a simplicial space.

(i) The geometric realization $|X|$ is the colimit in \mathbf{Top} of the natural maps

$$\mathrm{sk}_0|X| \rightarrow \mathrm{sk}_1|X| \rightarrow \cdots$$

(ii) For every n there is a natural pushout square in \mathbf{Top}

$$\begin{array}{ccc} L_n X \times \Delta_{top}^n \amalg_{L_n X \times \partial \Delta_{top}^n} X_n \times \partial \Delta_{top}^n & \longrightarrow & \mathrm{sk}_{n-1}|X| \\ \downarrow & & \downarrow \\ X_n \times \Delta_{top}^n & \longrightarrow & \mathrm{sk}_n|X|. \end{array}$$

(iii) If X is Reedy cofibrant, then the maps $\mathrm{sk}_n|X| \rightarrow \mathrm{sk}_{n+1}|X|$ are all closed cofibrations.

PROOF. (i) is formal since colimits commute and $\mathbf{\Delta}$ is the union of the subcategories with objects $\leq n$. It is also geometrically intuitive since every point in $|X|$ lies in $X_n \times \Delta_{top}^n$ for some (unique) minimal n , and so $|X|$ is the union of the subspaces $\mathrm{sk}_n|X|$.

(ii) makes sense geometrically since the n -skeleton is obtained from the $(n-1)$ -skeleton by gluing on $X_n \times \Delta_{top}^n$. This meets the $(n-1)$ -skeleton precisely along the images of $X_{n-1} \times \Delta_{top}^n$ along a degeneracy in the first factor, and the images of $X_n \times \Delta_{top}^{n-1}$ along the inclusion of a face of Δ_{top}^n in the second factor. These are glued to the $(n-1)$ -skeleton via the corresponding codegeneracy $X_{n-1} \times \Delta^n \rightarrow X_{n-1} \times \Delta^{n-1}$ in the second factor and the corresponding face map $X_n \times \Delta^{n-1} \rightarrow X_{n-1} \times \Delta^{n-1}$ in the first factor, respectively. Taking the union of these over all degeneracies and faces we get $L_n X \times \Delta^n$ and $X_n \times \partial \Delta^n$, which meet in $L_n X \times \partial \Delta^n$, giving the pushout in the top left corner. See Remark 5.2.11 below for a formal argument in simplicial spaces.

(iii) follows from (ii) and the assumption of Reedy cofibrancy, since closed cofibrations are preserved under pushouts and the map $L_n X \times \Delta^n \amalg_{L_n X \times \partial \Delta^n} X_n \times \partial \Delta^n \rightarrow X_n \times \Delta^n$ is a closed cofibration by Lemma 5.2.8(iv). \square

REMARK 5.2.11. Let us give a more formal argument for the pushout square (ii). Let $\mathbf{\Delta}_{\leq n}$ denote the full subcategory of $\mathbf{\Delta}$ with objects $[k]$ where $k \leq n$, and write $i_n: \mathbf{\Delta}_{\leq n} \rightarrow \mathbf{\Delta}$ for the inclusion functor. If $X \in \mathbf{Top}^{\mathbf{\Delta}^{\mathrm{op}}}$ is a simplicial space, we can define its n -skeleton $\mathrm{sk}_n X \in \mathbf{Top}^{\mathbf{\Delta}^{\mathrm{op}}}$ as the left Kan extension $i_{n,!} i_n^* X$ (where $i_n^* X$ is the restriction of X to the subcategory $\mathbf{\Delta}_{\leq n}^{\mathrm{op}}$). Notice that everything in $(\mathrm{sk}_n X)_i$ is degenerate for $i > n$, so the realization $|\mathrm{sk}_n X|$ is homeomorphic to $\mathrm{sk}_n|X|$ as we defined this above. Note also that $(\mathrm{sk}_n X)_{n+1}$ is precisely $L_{n+1} X$. Since geometric realization is a left adjoint, it preserves colimits; and as it also preserves products, it is enough to show that we have a pushout of simplicial spaces

$$\begin{array}{ccc} L_n X \times \Delta^n \amalg_{L_n X \times \partial \Delta^n} X_n \times \partial \Delta^n & \longrightarrow & \mathrm{sk}_{n-1} X \\ \downarrow & & \downarrow \\ X_n \times \Delta^n & \longrightarrow & \mathrm{sk}_n X, \end{array}$$

where Δ^n and $\partial \Delta^n$ here denote simplicial sets (viewed as discrete simplicial spaces) rather than spaces. These simplicial spaces are all n -skeletal (i.e. are left Kan extensions of their restrictions to $\mathbf{\Delta}_{\leq n}^{\mathrm{op}}$), so as $i_{n,!}$ preserves colimits (being a left adjoint) it suffices to prove we have a levelwise pushout when evaluated at $[k]$ for $k \leq n$. Evaluating at $k < n$ we have $(\Delta^n)_k = (\partial \Delta^n)_k$ so the top left space is

$$L_n X \times (\Delta^n)_k \amalg_{L_n X \times (\Delta^n)_k} X_n \times (\Delta^n)_k \cong X_n \times (\Delta^n)_k;$$

hence both vertical arrows are isomorphisms, and the square is indeed a pushout. Evaluating at $k = n$, we have $(\Delta^n)_n = \{*\} \amalg (\partial \Delta^n)_n$ so the top left object is $L_n X \amalg Y$ where $Y := X_n \times (\partial \Delta^n)_n$.

On the other hand $X_n \times (\Delta^n)_n = X_n \amalg Y$ and so the diagram is

$$\begin{array}{ccc} L_n X \amalg Y & \longrightarrow & L_n X \\ \downarrow & & \downarrow \\ X_n \amalg Y & \longrightarrow & X_n, \end{array}$$

which is clearly a pushout.

PROOF OF THEOREM 5.2.7. Using Lemma 5.2.8(iii) and Lemma 5.2.10(iii) we see that it is enough to prove that $\text{sk}_n|f|: \text{sk}_n|X| \rightarrow \text{sk}_n|Y|$ is a homotopy equivalence for all n . Now applying Lemma 5.2.8(ii) to the natural pushout squares from Lemma 5.2.10(ii) (where by Lemma 5.2.8(iv) the left vertical maps are closed cofibrations) we see that $\text{sk}_n|f|$ is a homotopy equivalence provided the maps $\text{sk}_{n-1}|f|$, $f_n \times \Delta^n_{\text{top}}$, and $L_n f \times \Delta^n_{\text{top}} \amalg_{L_n f \times \partial \Delta^n_{\text{top}}} f_n \times \partial \Delta^n_{\text{top}}$ are all homotopy equivalences.⁵ For $f_n \times \Delta^n_{\text{top}}$ this holds by assumption (since X_n and Y_n are assumed to be CW complexes), so we can complete the proof by induction if we can show that the map $L_n f \times \Delta^n \amalg_{L_n f \times \partial \Delta^n} f_n \times \partial \Delta^n$ is a homotopy equivalence. Using Lemma 5.2.8(ii) again, we see this will be true if $L_n f: L_n X \rightarrow L_n Y$ is a homotopy equivalence. But $L_n X$ can be naturally written as an iterated pushout of copies of X_{n-1} along inclusions of $L_{n-1} X$, so again this follows by induction and Lemma 5.2.8(ii). \square

As we already showed that Theorem 5.2.7 implies Theorem 5.2.1, this concludes the proof of homotopy invariance.

5.3. Coends and homotopy coends

To prove the other properties of the homotopy colimits we are interested in, it turns out to be convenient to put our construction of the homotopy colimit into a slightly more general context. We start by the non-homotopical version of this, which is the notion of *coends*.

5.3.1. Coends and functor tensor products. Recall our expression for the colimit of a functor $F: \mathcal{I} \rightarrow \mathcal{C}$ as

$$\text{colim } F \cong \text{coeq} \left(\coprod_{(f: i \rightarrow j) \in \text{Mor}(\mathcal{I})} F(i) \rightrightarrows \coprod_{k \in \text{Ob}(\mathcal{I})} F(k) \right).$$

Now we introduce a variant of this construction: given a functor $\Phi: \mathcal{I}^{\text{op}} \times \mathcal{I} \rightarrow \mathcal{C}$, its *coend* is the coequalizer

$$\text{coend } \Phi := \text{coeq} \left(\coprod_{(f: i \rightarrow j) \in \text{Mor}(\mathcal{I})} \Phi(j, i) \rightrightarrows \coprod_{k \in \text{Ob}(\mathcal{I})} \Phi(k, k) \right).$$

where the two morphisms are given on the component $\Phi(j, i)$ corresponding to $f: i \rightarrow j$ by $\Phi(f, \text{id}): \Phi(j, i) \rightarrow \Phi(i, i)$ and $\Phi(\text{id}, f): \Phi(j, i) \rightarrow \Phi(j, j)$. The coend of Φ is sometimes denoted $\int^{\mathcal{I}} \Phi$; we will generally avoid this notation though.

REMARK 5.3.1. You might ask what this construction “means”, or more precisely whether it has a universal property. In the literature this is usually discussed in terms of a rather obscure notion of “extranatural transformations”. However, there is a very natural way to understand coends as ordinary colimits: If \mathcal{I} is a category, the *twisted arrow category* $\text{Tw}(\mathcal{I})$ of \mathcal{I} is the

⁵It is important to notice that Lemma 5.2.8(ii) requires some maps to be actual homotopy equivalences, and not just weak equivalences.

category whose objects are the morphisms of \mathcal{I} and whose morphisms from $i \rightarrow j$ to $i' \rightarrow j'$ are the commutative diagrams of the form

$$\begin{array}{ccc} i & \longrightarrow & i' \\ \downarrow & & \downarrow \\ j & \longleftarrow & j'. \end{array}$$

There is an obvious forgetful functor $\mathrm{Tw}(\mathcal{I}) \rightarrow \mathcal{I}^{\mathrm{op}} \times \mathcal{I}$, given by taking target and source of morphisms, and the coend of a functor Φ can be identified with the colimit of the composite functor

$$\mathrm{Tw}(\mathcal{I}) \rightarrow \mathcal{I}^{\mathrm{op}} \times \mathcal{I} \xrightarrow{\Phi} \mathcal{C}.$$

(This is not completely obvious, it requires a cofinality argument.)

A key special case of the coend construction is the so-called *functor tensor product*: given functors $W: \mathcal{I}^{\mathrm{op}} \rightarrow \mathcal{C}$ and $F: \mathcal{I} \rightarrow \mathcal{C}$, and assuming that \mathcal{C} admits products, we write

$$W \otimes_{\mathcal{I}} F = \mathrm{coend}(W \times F)$$

where $W \times F: \mathcal{I}^{\mathrm{op}} \times \mathcal{I} \rightarrow \mathcal{C}$.

EXAMPLES 5.3.2. The following hold.

- (i) If $F: \mathcal{I} \rightarrow \mathrm{Top}$ is any functor, then $* \otimes_{\mathcal{I}} F$ is the colimit of F — this is the description of the colimit we have repeatedly used before.
- (ii) If X_{\bullet} is a simplicial space, then the geometric realization $|X|$ is nothing but the tensor $\Delta^{\bullet} \otimes_{\Delta^{\mathrm{op}}} X_{\bullet}$, or else $X_{\bullet} \otimes_{\Delta} \Delta^{\bullet}$.
- (iii) If F takes values in Top , then for $d \in \mathcal{I}$ we have a functor $\mathcal{I}(-, d): \mathcal{I}^{\mathrm{op}} \rightarrow \mathrm{Set} \subset \mathrm{Top}$, and then we have $\mathcal{I}(-, d) \otimes_{\mathcal{I}} F \cong \mathrm{colim}_{\mathcal{I}/d} F \xrightarrow{\cong} F(d)$, as the expression for $\mathcal{I}(-, d) \otimes_{\mathcal{I}} F$ can be rewritten as

$$\mathrm{coeq} \left(\coprod_{i \rightarrow j \rightarrow d} F(i) \rightrightarrows \coprod_{k \rightarrow d} F(k) \right).$$

- (iv) More generally, given functors $F: \mathcal{I} \rightarrow \mathrm{Top}$ and $\phi: \mathcal{I} \rightarrow \mathcal{J}$, the left Kan extension $\phi_! F: \mathcal{J} \rightarrow \mathrm{Top}$ (cf. Section ??) can by a similar argument be described as

$$d \mapsto \mathcal{J}(\phi(-), d) \otimes_{\mathcal{I}} F.$$

We will need the fact that coends, just like colimits, commute with each other — Using the integral notation for coends this looks as follows, and is sometimes called the “Fubini theorem”:

PROPOSITION 5.3.3. *For a functor $\Phi: \mathcal{I}^{\mathrm{op}} \times \mathcal{J}^{\mathrm{op}} \times \mathcal{J} \times \mathcal{I} \rightarrow \mathcal{C}$ there are natural isomorphisms*

$$\int^{\mathcal{I}} \int^{\mathcal{J}} \Phi \cong \int^{\mathcal{I} \times \mathcal{J}} \Phi \cong \int^{\mathcal{J}} \int^{\mathcal{I}} \Phi.$$

In particular, if \mathcal{C} admits products, given $W: \mathcal{I}^{\mathrm{op}} \rightarrow \mathcal{C}$, $\Phi: \mathcal{I} \times \mathcal{J}^{\mathrm{op}} \rightarrow \mathcal{C}$, and $F: \mathcal{J} \rightarrow \mathcal{C}$, we have a natural isomorphism

$$(W \otimes_{\mathcal{I}} \Phi) \otimes_{\mathcal{J}} F \cong W \otimes_{\mathcal{I}} (\Phi \otimes_{\mathcal{J}} F).$$

PROOF. We have a commutative diagram⁶

$$\begin{array}{ccc} \coprod_{\substack{i \rightarrow i' \\ j \rightarrow j'}} \Phi(i', j', j, i) & \rightrightarrows & \coprod_{i, j \rightarrow j'} \Phi(i, j', j, i) \\ \Downarrow & & \Downarrow \\ \coprod_{i \rightarrow i', j} \Phi(i', j, j, i) & \rightrightarrows & \coprod_{i, j} \Phi(i, j, j, i). \end{array}$$

Taking the colimit of this in three different ways gives the three expressions: First, we can do the coequalizers of the two columns, to get

$$\coprod_{i \rightarrow i'} \int^{\mathcal{J}} \Phi(i', -, -, i) \rightrightarrows \coprod_i \int^{\mathcal{J}} \Phi(i, -, -, i),$$

and then take the coequalizer of this to get $\int^{\mathcal{I}} \int^{\mathcal{J}} \Phi$. (The first step is a left Kan extension along the functor from our original diagram shape to $\bullet \rightrightarrows \bullet$ that collapses the columns, and the colimit of a left Kan extension is the colimit of the original diagram.) Second, we can do the same thing but interchanging rows and columns to get $\int^{\mathcal{J}} \int^{\mathcal{I}} \Phi$. And third, we can take the coequalizer of the four diagonal maps (by a cofinality argument), which gives $\int^{\mathcal{I} \times \mathcal{J}} \Phi$. \square

ALTERNATIVE PROOF. Recall the twisted arrow category construction from Remark 5.3.1. Then there is a natural isomorphism of categories $\mathrm{Tw}(\mathcal{I}) \times \mathrm{Tw}(\mathcal{J}) \cong \mathrm{Tw}(\mathcal{I} \times \mathcal{J})$, compatible along the target-source functors with the isomorphism $(\mathcal{I}^{\mathrm{op}} \times \mathcal{I}) \times (\mathcal{J}^{\mathrm{op}} \times \mathcal{J}) \cong (\mathcal{I} \times \mathcal{J})^{\mathrm{op}} \times (\mathcal{I} \times \mathcal{J})$. We can then pullback Φ to a functor $\Phi': \mathrm{Tw}(\mathcal{I}) \times \mathrm{Tw}(\mathcal{J})$, then the statement reduces to an instance of commutativity of colimits:

$$\mathrm{colim}_{\mathrm{Tw}(\mathcal{I})} \mathrm{colim}_{\mathrm{Tw}(\mathcal{J})} \Phi' \cong \mathrm{colim}_{\mathrm{Tw}(\mathcal{J})} \mathrm{colim}_{\mathrm{Tw}(\mathcal{I})} \Phi' \cong \mathrm{colim}_{\mathrm{Tw}(\mathcal{I} \times \mathcal{J})} \Phi'.$$

\square

5.3.2. Homotopy coends aka the two-sided bar construction. Now we introduce a homotopical version of coends, analogous to our construction of the homotopy colimit: Given $\Phi: \mathcal{I}^{\mathrm{op}} \times \mathcal{I} \rightarrow \mathrm{Top}$ we define a simplicial space, denoted abusively⁷ $\Phi^\Delta = \Phi_\bullet^\Delta$, by setting

$$\Phi_n^\Delta = \coprod_{i_0 \rightarrow \cdots \rightarrow i_n} \Phi(i_n, i_0) = \coprod_{i: [n] \rightarrow \mathcal{I}} \Phi(i(n), i(0)).$$

For $\phi: [m] \rightarrow [n]$ in Δ , the structure map $\phi^*: \Phi_n^\Delta \rightarrow \Phi_m^\Delta$ is given on the component $\Phi(i_n, i_0)$ indexed by $i_0 \rightarrow \cdots \rightarrow i_n$ by the map $\Phi(i_n, i_0) \rightarrow \Phi(i_{\phi(m)}, i_{\phi(0)})$ with the target in the component indexed by $i_{\phi(0)} \rightarrow \cdots \rightarrow i_{\phi(m)}$. In other words, we use the map

$$\Phi(i(\phi(m)) \leq n, i(0 \leq \phi(0))) : \Phi(i(n), i(0))_{i: [n] \rightarrow \mathcal{I}} \rightarrow \Phi(i(\phi(m)), i(\phi(0)))_{i \circ \phi: [m] \rightarrow \mathcal{I}}.$$

The *homotopy coend* of Φ is then the realization

$$\mathrm{hocoend}(\Phi) = |\Phi_\bullet^\Delta|.$$

Just as the homotopy colimit is a “fattened” version of the ordinary colimit, this is a “fattened” version of the colimit of the simplicial diagram Φ_\bullet^Δ , which is the coend of Φ by Proposition 5.1.7.

We are interested in the homotopy version of the functor tensor product; for historical reasons this has a special name:

⁶The indexing category of this diagram is $(\bullet \rightrightarrows \bullet) \times (\bullet \rightrightarrows \bullet)$, in particular it has 4 objects and 12 non-identity morphisms, 4 of which (the “diagonal” ones) are not represented in the picture.

⁷For a generic functor F we have already introduced another simplicial space F_\bullet^Δ , without using that the source category of F has the form $\mathcal{I}^{\mathrm{op}} \times \mathcal{I}$.

DEFINITION 5.3.4. For functors $F: \mathcal{I} \rightarrow \text{Top}$ and $W: \mathcal{I}^{\text{op}} \rightarrow \text{Top}$, the *two-sided simplicial bar construction* $\mathbb{B}_{\bullet}(W, \mathcal{I}, F)$ is the simplicial space $(W \times F)^{\Delta}$. The *two-sided bar construction* $B(W, \mathcal{I}, F)$ is the geometric realization $|\mathbb{B}_{\bullet}(W, \mathcal{I}, F)|$, i.e.,

$$B(W, \mathcal{I}, F) = |(W \times F)^{\Delta}|$$

EXAMPLE 5.3.5. $B(*, \mathcal{I}, *) = B\mathcal{I}$. More generally $B(*, \mathcal{I}, F) = \text{hocolim } F$, as $\mathbb{B}_{\bullet}(*, \mathcal{I}, F)$ is the simplicial space we previously denoted F^{Δ} .

DEFINITION 5.3.6. As suggested by Example 5.3.2(iv), we define the *homotopy Kan extension* of $F: \mathcal{I} \rightarrow \text{Top}$ along $\phi: \mathcal{I} \rightarrow \mathcal{J}$ using the two-sided bar construction as

$$d \mapsto B(\mathcal{J}(\phi(-), d), \mathcal{I}, F).$$

We will abbreviate this by $\phi_! F: \mathcal{J} \rightarrow \text{Top}$.

We will now prove some formal properties of the two-sided bar construction. First of all, from our result on the homotopy invariance of geometric realizations we immediately get:

PROPOSITION 5.3.7 (Homotopy invariance of the bar construction). *For any functor $W: \mathcal{I}^{\text{op}} \rightarrow \text{CW}$, the functor $B(W, \mathcal{I}, -): \text{CW}^{\mathcal{I}} \rightarrow \text{CW}$ preserves weak equivalences, and similarly in the other variable.*

PROOF. This follows from Theorem 5.2.7, just as in the proof of Theorem 5.2.1: $\mathbb{B}_{\bullet}(W, \mathcal{I}, F)$ is always Reedy cofibrant, and for a (objectwise) weak equivalence $\eta: F \rightarrow F'$ of functors $F, F' \in \text{CW}^{\mathcal{I}}$, the map of simplicial spaces $\mathbb{B}_{\bullet}(W, \mathcal{I}, \eta)$ is a level-wise weak equivalence. \square

LEMMA 5.3.8. *The two-sided bar construction commutes with functor tensor products in each variable, i.e. given functors $W: \mathcal{J} \times \mathcal{I}^{\text{op}} \rightarrow \text{Top}$, $F: \mathcal{I} \times \mathcal{K}^{\text{op}} \rightarrow \text{Top}$, $\Phi: \mathcal{J}^{\text{op}} \rightarrow \text{Top}$, and $\Psi: \mathcal{K} \rightarrow \text{Top}$, there is a canonical isomorphism*

$$\Phi \otimes_{\mathcal{J}} B(W, \mathcal{I}, F) \otimes_{\mathcal{K}} \Psi \cong B(\Phi \otimes_{\mathcal{I}} W, \mathcal{I}, F \otimes_{\mathcal{K}} \Psi).$$

Here we consider, for instance, $B(W, \mathcal{I}, F)$ as a functor $\mathcal{J} \times \mathcal{K}^{\text{op}} \rightarrow \text{Top}$, and $\Phi \otimes_{\mathcal{I}} W$ as a functor $\mathcal{I}^{\text{op}} \rightarrow \text{Top}$.

PROOF. The geometric realization commutes with colimits and products, so this amounts to expanding everything out and commuting some colimits, and is left as exercise. \square

DEFINITION 5.3.9. For convenience, we will use the standard (but perhaps slightly confusing!) notation $B(\mathcal{I}, \mathcal{I}, F): \mathcal{I} \rightarrow \text{Top}$ for the functor $i \mapsto B(\mathcal{I}(-, i), \mathcal{I}, F)$, and $B(W, \mathcal{I}, \mathcal{I}): \mathcal{I}^{\text{op}} \rightarrow \text{Top}$ for $i \mapsto B(W, \mathcal{I}, \mathcal{I}(i, -))$.⁸

We use similar notation for (non-derived) functor tensor products, and note that $\mathcal{I} \otimes_{\mathcal{I}} F \cong F$, and $W \otimes_{\mathcal{I}} \mathcal{I} \cong W$.

By the above, we can think of the functors $B(\mathcal{I}, \mathcal{I}, F)$ and $B(W, \mathcal{I}, \mathcal{I})$ as “fattened up” versions of F and W . These functors have an alternative description that will be important to us:

LEMMA 5.3.10. *Let $F: \mathcal{I} \rightarrow \text{Top}$. There are natural isomorphisms of spaces*

$$B(\mathcal{I}(-, i), \mathcal{I}, F) \cong B(*, \mathcal{I}_{/i}, F) \cong \text{hocolim}_{\mathcal{I}_{/i}} F$$

and hence an equivalence of functors

$$B(\mathcal{I}, \mathcal{I}, F) \cong (i \mapsto \text{hocolim}_{\mathcal{I}_{/i}} F) = QF$$

⁸A better but heavier notation could be $B(\text{Hom}_{\mathcal{I}}, \mathcal{I}, F)$ and $B(W, \mathcal{I}, \text{Hom}_{\mathcal{I}})$, considering $\text{Hom}_{\mathcal{I}}$ as a functor $\mathcal{I}^{\text{op}} \times \mathcal{I} \rightarrow \text{Set} \subset \text{Top}$.

where Q was introduced in Definition 5.1.22. Similarly, for $W: \mathcal{I}^{\text{op}} \rightarrow \text{Top}$,

$$B(W, \mathcal{I}, \mathcal{I}) \cong (i \mapsto B(W, \mathcal{I}_i, *)) \cong (i \mapsto \text{hocolim}_{(\mathcal{I}_i)^{\text{op}}} W) = (i \mapsto \text{hocolim}_{(\mathcal{I}^{\text{op}})_i} W).$$

In particular $B(\mathcal{I}, \mathcal{I}, *) \cong B(\mathcal{I}_{/-}) = |N(\mathcal{I}_{/-})|$ as functors $\mathcal{I} \rightarrow \text{Top}$, and similarly $B(*, \mathcal{I}, \mathcal{I}) \cong B(\mathcal{I}_{-/})$.

PROOF. Expanding out the definitions, we observe that the simplicial spaces $\mathbb{B}_\bullet(\mathcal{I}(-, i), \mathcal{I}, F)$ and $\mathbb{B}_\bullet(*, \mathcal{I}_i, F)$ are isomorphic, as the space of n -simplices in both cases is $\coprod_{i_0 \rightarrow \dots \rightarrow i_n \rightarrow i} F(i_0)$ (and structure maps are compatible). Similarly in the other case. The final statements are obtained by taking $F = W = *$. \square

Using our preparation, we can now give several alternative descriptions of the homotopy colimit, including the expression $\text{hocolim}_{\mathcal{I}} F \cong \text{colim}_{\mathcal{I}} QF$, first sketched in Proposition 5.1.24.

PROPOSITION 5.3.11. *For a functor $F: \mathcal{I} \rightarrow \text{Top}$, there are canonical isomorphisms*

$$\text{hocolim}_{\mathcal{I}} F \cong B(\mathcal{I}_{-/}) \otimes_{\mathcal{I}} F \cong \text{colim}_{\mathcal{I}} B(\mathcal{I}, \mathcal{I}, F) \cong \text{colim}_{i \in \mathcal{I}} \left(\text{hocolim}_{\mathcal{I}_i} F \right).$$

and the canonical map $\text{hocolim}_{\mathcal{I}} F \rightarrow \text{colim}_{\mathcal{I}} F$ identifies with the projection map

$$\text{hocolim}_{\mathcal{I}} F \cong B(\mathcal{I}_{-/}) \otimes_{\mathcal{I}} F \rightarrow * \otimes_{\mathcal{I}} F = \text{colim}_{\mathcal{I}} F$$

PROOF. We have a chain of isomorphisms

$$\text{hocolim} F \cong B(*, \mathcal{I}, F) \cong B(*, \mathcal{I}, \mathcal{I} \otimes_{\mathcal{I}} F) \cong B(*, \mathcal{I}, \mathcal{I}) \otimes_{\mathcal{I}} F \cong B(\mathcal{I}_{-/}) \otimes_{\mathcal{I}} F.$$

Here the first isomorphism is by comparing the original definition of F^Δ with the abused definition of $(* \times F)^\Delta$, where $*$: $\mathcal{I}^{\text{op}} \rightarrow \text{Top}$ is the constant functor: the two simplicial spaces are isomorphic. The second is the observation in Definition 5.3.9. The third uses Lemma 5.3.8. And the fourth is Lemma 5.3.10. This establishes the first isomorphism in the proposition.

On the other hand, we also have a chain of isomorphisms

$$B(*, \mathcal{I}, F) \cong B(* \otimes_{\mathcal{I}} \mathcal{I}, \mathcal{I}, F) \cong * \otimes_{\mathcal{I}} B(\mathcal{I}, \mathcal{I}, F) \cong \text{colim}_{\mathcal{I}} B(\mathcal{I}, \mathcal{I}, F)$$

now using the observation in Definition 5.3.9, Lemma 5.3.8, and Example 5.3.2(i). This establishes the second isomorphism of the proposition, from which the third follows, using Lemma 5.3.10.

That the map $\text{hocolim}_{\mathcal{I}} F \rightarrow \text{colim}_{\mathcal{I}} F$ has the prescribed form also follows from the definitions (check it as exercise!). \square

We now prove an associativity result for two-sided bar construction, analogous to the associativity of the functor tensor product of Proposition 5.3.3, that we will use several times later:

PROPOSITION 5.3.12. *Suppose we are given functors $F: \mathcal{J} \rightarrow \text{Top}$, $\Phi: \mathcal{I} \times \mathcal{J}^{\text{op}} \rightarrow \text{Top}$ and $W: \mathcal{I}^{\text{op}} \rightarrow \text{Top}$. Then there is a natural isomorphism*

$$B(W, \mathcal{I}, B(\Phi, \mathcal{J}, F)) \cong B(B(W, \mathcal{I}, \Phi), \mathcal{J}, F).$$

PROOF. For a generic bisimplicial space $Y: \Delta^{\text{op}} \times \Delta^{\text{op}} \rightarrow \text{Top}$, one can define two simplicial spaces Y' and Y'' by taking geometric realization along one of the two simplicial coordinates, i.e. $Y': [n] \mapsto |Y([n], -)|$ and $Y'': [m] \mapsto |Y(-, [m])|$. One can then check

$$|Y'| \cong |Y''| \cong \left(\coprod_{n, m \geq 0} Y([n], [m]) \times \Delta_{\text{top}}^n \times \Delta_{\text{top}}^m \right) / \sim,$$

where \sim is generated by identifying, for all $\phi: [n] \rightarrow [n']$, $\psi: [m] \rightarrow [m']$, $y \in Y([n'], [m'])$, $s \in \Delta_{\text{top}}^n$ and $t \in \Delta_{\text{top}}^m$, the points $((\phi, \psi)^*(y), (s, t)) \sim (y, \phi_*(s), \psi_*(t))$. This is an instance of

the “Fubini theorem”, Proposition 5.3.3, with $\mathcal{I} = \mathcal{J} = \Delta$ and $\Phi: \mathcal{I}^{\text{op}} \times \mathcal{J}^{\text{op}} \times \mathcal{J} \times \mathcal{I} \rightarrow \text{Top}$ given by $\Delta_{\text{top}}^\bullet \times Y \times \Delta_{\text{top}}^\bullet$.

We apply the previous to the bisimplicial space $X: \Delta^{\text{op}} \times \Delta^{\text{op}} \rightarrow \text{Top}$ given by

$$\begin{aligned} X_{n,m} = X([n], [m]) &= \coprod_{\substack{i:[n] \rightarrow \mathcal{I} \\ j:[m] \rightarrow \mathcal{J}}} W(i(n)) \times \Phi(i(0), j(m)) \times F(j(0)) \\ &= \coprod_{\substack{i_0 \rightarrow \dots \rightarrow i_n \\ j_0 \rightarrow \dots \rightarrow j_m}} W(i_n) \times \Phi(i_0, j_m) \times F(j_0). \end{aligned}$$

The pair of functors $\phi: [n] \rightarrow [n']$ and $\psi: [m] \rightarrow [m']$ is sent to the map of spaces restricting to the map between the components indexed by (i, j) and $(i \circ \phi, j \circ \psi)$:

$$\begin{aligned} X(\phi, \psi) &:= W(i(\phi(n) \leq n')) \times \Psi(i(0 \leq \phi(0)), j(\psi(m) \leq m')) \times F(j(0 \leq \psi(0))) \\ &\quad : W(i(n')) \times \Phi(i(0), j(m')) \times F(j(0)) \\ &\rightarrow W(i \circ \phi(n)) \times \Phi(i \circ \phi(0), j \circ \psi(m)) \times F(j \circ \psi(0)). \end{aligned}$$

The two sides in the isomorphism of the statement are $|X'|$ and $|X''|$, respectively. \square

As a final preliminary, we record the following restatement and extension of Proposition 5.1.23:

PROPOSITION 5.3.13.

- (i) For a functor $F: \mathcal{I} \rightarrow \text{Top}$, there is a natural transformation $B(\mathcal{I}, \mathcal{I}, F) \rightarrow F$ that is a levelwise homotopy equivalence; moreover we have a natural isomorphism $QF \cong B(\mathcal{I}, \mathcal{I}, F)$.
- (ii) For a functor $W: \mathcal{I}^{\text{op}} \rightarrow \text{Top}$, there is a natural transformation $B(W, \mathcal{I}, \mathcal{I}) \rightarrow W$ that is a levelwise homotopy equivalence.

PROOF. For (i), by Lemma 5.3.10 we can identify $B(\mathcal{I}(-, i), \mathcal{I}, F)$ with $B(*, \mathcal{I}_{/i}, F)$ and with $\text{hocolim}_{\mathcal{I}_{/i}} F$, naturally in $i \in \mathcal{I}$, and the map we want is just the canonical map $\text{hocolim}_{\mathcal{I}_{/i}} F \rightarrow F(i)$, which is a homotopy equivalence by Proposition 5.1.19. In (ii), we can similarly use the identification of $B(W, \mathcal{I}, \mathcal{I}(i, -))$ with $B(W, \mathcal{I}_{i/}, *)$ and $\text{hocolim}_{(\mathcal{I}_{i/})^{\text{op}}} W$, note that $(\mathcal{I}_{i/})^{\text{op}}$ is the same as $(\mathcal{I}^{\text{op}})_{/i}$, and use again Proposition 5.1.19 to conclude that the canonical map $\text{hocolim}_{(\mathcal{I}^{\text{op}})_{/i}} W \rightarrow W(i)$ is a homotopy equivalence. \square

REMARK 5.3.14. Suppose X is a simplicial space. Then we now have a description of the homotopy colimit of X as $B(\Delta_{/-}) \otimes_{\Delta^{\text{op}}} X$. On the other hand, we can write the geometric realization as $\Delta_{\text{top}}^\bullet \otimes_{\Delta^{\text{op}}} X$.

A variant of our proof of homotopy invariance for geometric realizations implies that if X is Reedy cofibrant then the functor $- \otimes_{\Delta^{\text{op}}} X$ preserves weak equivalences between cosimplicial CW complexes that satisfy an analogue of the Reedy cofibrancy condition: the map from the n^{th} “matching object” (which is obtained as the colimit over all the face maps into $[n]$; e.g. for $\Delta_{\text{top}}^\bullet$ we get $\partial \Delta_{\text{top}}^n$) into the n^{th} space is a cofibration. This holds for the cosimplicial spaces $\Delta_{\text{top}}^\bullet$ and $B(\Delta_{/-})$; using this we can show that if X is a Reedy cofibrant simplicial space then $\text{hocolim}_{\Delta^{\text{op}}} X$ is weakly equivalent to $|X|$.

5.4. The homotopy colimit as a derived functor

We are now ready to prove that the homotopy colimit is indeed a derived functor. Recall that we proved in Proposition 5.1.23 (and again in Proposition 5.3.13) that (Q, q) is a deformation of $\text{CW}^{\mathcal{I}}$. Recall also that we proved in Lemma 5.3.10, that the functor $Q: \text{Top}^{\mathcal{I}} \rightarrow \text{Top}^{\mathcal{I}}$ (and hence its restriction to $\text{CW}^{\mathcal{I}}$) identifies with $B(\mathcal{I}, \mathcal{I}, -)$. And we proved in Theorem 5.2.1 that $\text{hocolim}_{\mathcal{I}} = \text{colim}_{\mathcal{I}} \circ Q: \text{CW}^{\mathcal{I}} \rightarrow \text{CW}$ respects weak equivalences, i.e., that $\text{hocolim}_{\mathcal{I}}$ is a homotopical functor.

To prove that $\mathrm{hocolim}_{\mathcal{I}}$ is a left derived functor of $\mathrm{colim}_{\mathcal{I}}$, by Proposition 4.2.8 we are just left with establishing the following proposition:

PROPOSITION 5.4.1. *The map $\mathrm{colim}_{\mathcal{I}}(q_{QF}): \mathrm{colim}_{\mathcal{I}} Q(QF) \rightarrow \mathrm{colim}_{\mathcal{I}}(QF)$ is a weak equivalence of spaces for any $F \in \mathrm{CW}^{\mathcal{I}}$.*

Once this proposition is proved, it is easy to check that $\mathrm{colim}_{\mathcal{I}}$ is homotopical on the essential image of Q : given $F, G \in \mathrm{CW}^{\mathcal{I}}$ and a weak equivalence $\eta: QF \rightarrow QG$, we have that $\mathrm{colim}_{\mathcal{I}}(\eta)$ is a weak equivalence by 3-of-4 in the following commutative square, where we use that $\mathrm{colim}_{\mathcal{I}} \circ Q$ is known to be homotopical:

$$\begin{array}{ccc} \mathrm{colim}_{\mathcal{I}} Q(QF) & \xrightarrow[\simeq]{\mathrm{colim}_{\mathcal{I}} q_{QF}} & \mathrm{colim}_{\mathcal{I}}(QF) \\ \simeq \downarrow \mathrm{colim}_{\mathcal{I}} Q(\eta) & & \downarrow \mathrm{colim}_{\mathcal{I}}(\eta) \\ \mathrm{colim}_{\mathcal{I}} Q(QG) & \xrightarrow[\simeq]{\mathrm{colim}_{\mathcal{I}} q_{QG}} & \mathrm{colim}_{\mathcal{I}}(QG) \end{array}$$

PROOF OF PROPOSITION 5.4.1. This will also follow from what we have proven so far, by writing out what this map is. By Proposition 5.3.11, we are considering the map

$$\mathrm{hocolim}_{\mathcal{I}} QF = B(\mathcal{I}_{-/}) \otimes_{\mathcal{I}} QF \rightarrow * \otimes QF = \mathrm{colim}_{\mathcal{I}} QF$$

induced by the projection $B(\mathcal{I}_{-/}) \rightarrow *$. But this can now be rewritten as

$$\begin{array}{ccccc} B(\mathcal{I}_{-/}) \otimes_{\mathcal{I}} QF & \cong & B(\mathcal{I}_{-/}) \otimes_{\mathcal{I}} B(\mathcal{I}, \mathcal{I}, F) & \cong & B(B(\mathcal{I}_{-/}), \mathcal{I}, F) \\ \downarrow & & \downarrow & & \downarrow \\ * \otimes_{\mathcal{I}} QF & \cong & * \otimes_{\mathcal{I}} B(\mathcal{I}, \mathcal{I}, F) & \cong & B(*, \mathcal{I}, F) \end{array}$$

using the observations of Lemma 5.3.8 and Definition 5.3.9. However, the right-hand vertical map is a homotopy equivalence by Proposition 5.3.7, as $B\mathcal{I}_{i/} \rightarrow *$ is a weak equivalence for all i , showing the claim. \square

REMARK 5.4.2. The same argument shows that for any $W: \mathcal{I}^{\mathrm{op}} \rightarrow \mathrm{Top}$, the functor

$$B(W, \mathcal{I}, -) \cong W \otimes_{\mathcal{I}} B(\mathcal{I}, \mathcal{I}, -)$$

is a left derived functor of $W \otimes_{\mathcal{I}} -$. (Here we use in the last step that $B(W, \mathcal{I}, \mathcal{I}) \rightarrow W$ is a natural weak equivalence for any W , not just for $W = *$.)

Let us summarize what we have proved:

THEOREM 5.4.3. *The functor $\mathrm{hocolim}_{\mathcal{I}}(-) = \mathrm{colim}_{\mathcal{I}}(Q(-))$, is a left derived functor of $\mathrm{colim}_{\mathcal{I}}$, and the induced functor $\mathrm{Ho}(\mathrm{CW}^{\mathcal{I}}) \rightarrow \mathrm{Ho}(\mathrm{CW})$ is left adjoint to the diagonal functor $\delta: \mathrm{Ho}(\mathrm{CW}) \rightarrow \mathrm{Ho}(\mathrm{CW}^{\mathcal{I}})$.*

Replacing Q by $Q \circ |\mathrm{Sing}_{\bullet}(-)|$ the same statements hold with Top in place of CW .

PROOF. To see that it is a derived functor we want to check that the assumptions of Proposition 4.2.8 are satisfied:

- By Proposition 5.1.23 (or Proposition 5.3.13), (Q, q) is a deformation of $\mathrm{CW}^{\mathcal{I}}$.
- By Theorem 5.2.1, $\mathrm{hocolim}_{\mathcal{I}} = \mathrm{colim}_{\mathcal{I}} Q: \mathrm{CW}^{\mathcal{I}} \rightarrow \mathrm{CW}$ respects weak equivalences, i.e., it is homotopical.
- By Proposition 5.4.1 the map “ $\mathrm{colim}_{\mathcal{I}} q_{QF}$ ” is a weak equivalence of CW complexes for all $F \in \mathrm{CW}^{\mathcal{I}}$, and this in turn implies that $\mathrm{colim}_{\mathcal{I}}$ is homotopical on the essential image of Q . This proves by Proposition 4.2.8 that $\mathrm{hocolim}$ is a left derived functor of colim . That the induced functor on homotopy categories is left adjoint to the diagonal follows formally from Proposition 4.2.9, noting that $\delta: \mathrm{CW} \rightarrow \mathrm{CW}^{\mathcal{I}}$ is already homotopy invariant, so we can take “ R ” to be the identity.

We have also already remarked in Remark 5.1.1 that $Q \circ |\mathrm{Sing}_\bullet(-)|$ is a deformation on $\mathrm{Top}^{\mathcal{I}}$ with the wanted properties. (And in fact, with more effort in the proof, we can keep Q unchanged.) \square

5.5. Cofinality for homotopy colimits and Quillen's Theorem A

The final standard property about homotopy colimits that we want to mention is cofinality. Given a functor $\phi : \mathcal{I} \rightarrow \mathcal{J}$ it relates homotopy colimits indexed by \mathcal{J} to those indexed by \mathcal{I} . For this we recall the definition of the undercategory, already given in Definition 1.10.3.

DEFINITION 5.5.1. For $\phi : \mathcal{I} \rightarrow \mathcal{J}$ and $j \in \mathcal{J}$ we define the undercategory of ϕ under j to be $\mathcal{I}_{j/\phi} := \mathcal{I} \times_{\mathcal{J}} \mathcal{J}_{j/}$. In other words, the category $\mathcal{I}_{j/\phi}$ has objects pairs $(i \in \mathcal{I}, (j \rightarrow \phi(i)) \in \mathcal{J})$ and morphisms $(i, j \rightarrow \phi(i)) \rightarrow (i', j \rightarrow \phi(i'))$ are given by morphisms $f : i \rightarrow i'$ such that the following diagram commutes

$$\begin{array}{ccc} & j & \\ \swarrow & & \searrow \\ \phi(i) & \xrightarrow{\phi(f)} & \phi(i'). \end{array}$$

We may sometimes suppress ϕ from the notation $\mathcal{I}_{j/\phi}$, in particular when ϕ is an inclusion of categories, hoping that this will not cause confusion with the ordinary undercategory, and write $\mathcal{I}_{j/}$.

Overcategories $\mathcal{I}_{\phi/j}$ are defined dually, for a functor $\phi : \mathcal{I} \rightarrow \mathcal{J}$ and $j \in \mathcal{J}$.

Before getting to homotopy cofinality, we discuss cofinality for ordinary colimits.

5.5.1. Cofinality for colimits.

DEFINITION 5.5.2. A functor $\phi : \mathcal{I} \rightarrow \mathcal{J}$ between small categories is *cofinal* if for every $j \in \mathcal{J}$ the undercategory $\mathcal{I}_{j/\phi}$ is non-empty and connected, i.e. we can connect any two objects by a finite zig-zag of morphisms; or, equivalently, the classifying space $B\mathcal{I}_{j/\phi}$ is non-empty and connected.

EXERCISE 5.5.3. Prove that for a small (non-empty!) category \mathcal{C} it is indeed equivalent to require either that (1) $B\mathcal{C}$ is a connected topological space, or (2) the equivalence relation on $\mathrm{Ob}(\mathcal{C})$ generated by $x \sim y$ whenever there is a morphism $x \rightarrow y$ admits a unique equivalence class.

PROPOSITION 5.5.4. If a functor $\phi : \mathcal{I} \rightarrow \mathcal{J}$ is cofinal, then for every diagram $F : \mathcal{J} \rightarrow \mathcal{C}$ the natural map $\mathrm{colim}_{\mathcal{I}} F\phi \rightarrow \mathrm{colim}_{\mathcal{J}} F$ is an isomorphism (assuming either colimit exists, in which case both colimits do exist).

PROOF. The reader should easily be able to convince themselves that this is true, by directly manipulating the definitions, but we give a formal proof in preparation for homotopy cofinality; for the following argument we assume that \mathcal{C} is equal to Top (or at least it has finite products, coproducts, and is endowed with a preferred functor $\mathrm{Set} \rightarrow \mathcal{C}$, all satisfying convenient properties that are implicit in our manipulations):

Recall from Example 5.3.2(iii) that $\mathcal{J}(-, \phi(i)) \otimes_{\mathcal{J}} F \cong F(\phi(i))$ for any $i \in \mathcal{I}$. Thus we can write the functor $F \circ \phi$ as $\mathcal{J}(-, \phi(-)) \otimes_{\mathcal{J}} F$, giving

$$\mathrm{colim}_{\mathcal{I}} F\phi \cong * \otimes_{\mathcal{I}} F\phi \cong * \otimes_{\mathcal{I}} (\mathcal{J}(-, \phi(-)) \otimes_{\mathcal{J}} F),$$

by Example 5.3.2(i).

Now using associativity for the tensor product of functors, Proposition 5.3.3, we see that this is isomorphic to $(* \otimes_{\mathcal{I}} \mathcal{J}(-, \phi(-))) \otimes_{\mathcal{J}} F$. For $j \in \mathcal{J}$, the colimit $\mathrm{colim}_{\mathcal{I}} \mathcal{J}(j, \phi(-)) = * \otimes_{\mathcal{I}} \mathcal{J}(j, \phi(-))$ is given by the coequalizer of

$$\coprod_{i \rightarrow i'} \mathcal{J}(j, \phi(i)) \rightrightarrows \coprod_i \mathcal{J}(j, \phi(i)).$$

THEOREM 5.5.9 (Cofinality for hocolim). *Suppose $\phi: \mathcal{I} \rightarrow \mathcal{J}$ is a homotopy cofinal functor. Then for every functor $F: \mathcal{J} \rightarrow \text{Top}$ taking values in CW complexes, the natural map*

$$\text{hocolim}_{\mathcal{I}} F\phi \rightarrow \text{hocolim}_{\mathcal{J}} F$$

is a weak equivalence.

Before the proof let us note a very important special case, obtained by applying Theorem 5.5.9 to the constant functor $\mathcal{J} \rightarrow \text{Top}$ with value $*$:

COROLLARY 5.5.10 (Quillen's Theorem A). *Suppose $F: \mathcal{I} \rightarrow \mathcal{J}$ is a homotopy cofinal functor. Then the induced map on classifying spaces $BF: B\mathcal{I} \rightarrow B\mathcal{J}$ is a weak equivalence.* \square

PROOF OF THEOREM 5.5.9. Our proof will be a homotopical version of the analogous proof for colim:

Recall first that we have a weak equivalence $B(\mathcal{J}(-, j), \mathcal{J}, F) \rightarrow F(j)$, natural in $j \in \mathcal{J}$, by Lemma 5.3.10. Hence we get a weak equivalence

$$\text{hocolim}_{\mathcal{I}} F\phi \cong B(*, \mathcal{I}, F\phi) \xrightarrow{\sim} B(*, \mathcal{I}, B(\mathcal{J}(-, \phi(-)), \mathcal{J}, F)).$$

by homotopy invariance of the two-sided bar construction, Proposition 5.3.7. Now using associativity for the bar construction, Proposition 5.3.12, we get an isomorphism between this and the space $B(B(*, \mathcal{I}, \mathcal{J}(-, \phi(-))), \mathcal{J}, F)$. But by Lemma 5.5.8 we can again rewrite this as $B(B\mathcal{I}_{-/ \phi}, \mathcal{J}, F)$. So we can describe the map in the theorem as

$$\text{hocolim}_{\mathcal{I}} F\phi \cong B(B\mathcal{I}_{-/ \phi}, \mathcal{J}, F) \rightarrow B(*, \mathcal{J}, F) \cong \text{hocolim}_{\mathcal{J}} F,$$

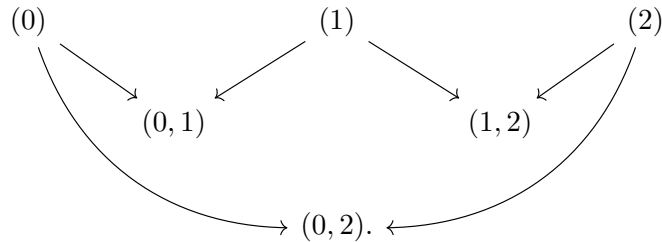
But now by homotopy invariance of the bar construction again (Proposition 5.3.7), the map is a weak equivalence if ϕ is homotopy cofinal, since this means that we have a weak equivalence $B\mathcal{I}_{-/ \phi} \simeq *$ of functors $\mathcal{J}^{\text{op}} \rightarrow \text{CW}$. \square

REMARK 5.5.11. Again this is an if and only if statement: the functors that induce weak equivalences on all homotopy colimits are precisely the homotopy cofinal functors. As before, we see this by taking F to be $\mathcal{J}(j, -)$; then we have $\text{hocolim}_{\mathcal{I}} \mathcal{J}(j, \phi(-)) \cong B\mathcal{I}_{j/ \phi}$ whereas $\text{hocolim}_{\mathcal{J}} \mathcal{J}(j, -) \cong B\mathcal{J}_j \cong *$, by Lemma 5.5.8.

EXAMPLE 5.5.12. Consider the simplex category $\Delta\mathcal{I}$ with objects n -simplices $i_0 \rightarrow i_1 \rightarrow \cdots \rightarrow i_n$ and morphisms given by face and degeneracy maps. Formally, $\Delta\mathcal{I}$ is the category of elements $\text{el}(\text{Fun}(-, \mathcal{I}))$ associated with the functor $\text{Fun}(-, \mathcal{I}): \Delta^{\text{op}} \rightarrow \text{Set}$ which in turns sends $[n]$ to the set of functors $\text{Fun}([n], \mathcal{I})$: see Definition 1.11.1.

The functor $\phi: \Delta\mathcal{I} \rightarrow \mathcal{I}$ given by $(i_0 \rightarrow i_1 \rightarrow \cdots \rightarrow i_n) \mapsto i_0$ is cofinal, and also homotopy cofinal, as for all $i \in \mathcal{I}$ the undercategory $(\Delta\mathcal{I})_{i/ \phi}$ has as initial object “ $i \rightarrow i$ ”, which is formally the pair $(([0] \rightarrow \mathcal{I}, 0 \mapsto i), i \xrightarrow{\text{id}_i} i)$.

REMARK 5.5.13. The inclusion $\Delta_{\leq 1}^{\text{op}} \hookrightarrow \Delta^{\text{op}}$ is *not* homotopy cofinal — the slice category $(\Delta_{\leq 1})_{/[2]}$ is connected, but not simply connected: we get a non-trivial loop in $B(\Delta_{\leq 1})_{/[2]}$ via the diagram⁹



⁹Do you notice a similarity between this diagram and the barycentric subdivision of $\partial\Delta^2$?

More generally, it can be shown that for $\Delta_{\leq n}^{\text{op}} \hookrightarrow \Delta^{\text{op}}$ the slice categories are $(n-1)$ -connected, but *not* n -connected, so in order to compute homotopy colimits we cannot restrict to any finite part of Δ^{op} .

EXERCISE 5.5.14. Identify the homotopy type of $(\Delta_{\leq 1}^{\text{op}})_{/[2]}$ and more generally of $(\Delta_{\leq n}^{\text{op}})_{/[m]}$.

5.6. The Grothendieck construction as a homotopy colimit

As an application of the machinery we have set up, in this section we will give a more explicit description of homotopy colimits for diagrams of spaces $\mathcal{I} \rightarrow \text{Top}$ that come from diagrams of categories $\mathcal{I} \rightarrow \text{Cat}$ by taking classifying spaces. First we must introduce some notation:

DEFINITION 5.6.1. Suppose $F: \mathcal{I} \rightarrow \text{Cat}$ is a functor. The *Grothendieck construction* $\text{Gr}(F)$ is the category with objects given by pairs $(i \in \mathcal{I}, x \in F(i))$, and morphisms $(i, x) \rightarrow (i', x')$ given by pairs of a map $f: i \rightarrow i'$ in \mathcal{I} and a map $\phi: F(f)(x) \rightarrow x'$ in $F(i')$. Note that there is an obvious projection functor $\text{Gr}(F) \rightarrow \mathcal{I}$ that takes an object (i, x) to i .

REMARK 5.6.2. A special case of the Grothendieck construction is when the functor lands in Set (viewed as categories with no non-identity morphisms), in which case the morphism ϕ above has to be the identity. In this case it is a classical construction known as the “translation category” or “category of elements” (see Definition 1.11.1). The objects consists of pairs $(i \in \mathcal{I}, x \in F(i))$, and a morphism is a map $f: i \rightarrow i'$ such that $F(f)(x) = x'$.

We already saw an example of this special case, namely the simplex category $\Delta\mathcal{I}$ of a category \mathcal{I} . More generally, for a simplicial set $X: \Delta^{\text{op}} \rightarrow \text{Set}$, one can define $\Delta X = \text{el}(X)$ (and $\Delta\mathcal{I}$ is nothing but $\Delta(N\mathcal{I})$).

REMARK 5.6.3. It is possible to characterize the functors $p: \mathcal{E} \rightarrow \mathcal{I}$ that arise as projections of Grothendieck constructions: namely, if p is a so-called “Grothendieck opfibration”, then there exists an essentially unique functor $F: \mathcal{I} \rightarrow \text{Cat}$ such that \mathcal{E} is equivalent to $\text{Gr}(F)$ over \mathcal{I} .

THEOREM 5.6.4 (Thomason [Tho79]). *Suppose we are given a functor $F: \mathcal{I} \rightarrow \text{Cat}$. Then the spaces $\text{hocolim}_{i \in \mathcal{I}} BF(i)$ and $B\text{Gr}(F)$ are weakly equivalent.*

EXAMPLE 5.6.5. Consider a group G acting on a collection of subgroups \mathcal{C} . We see that the classifying space of \mathcal{C}_G identifies with $(BC)_{hG}$. [**Explain this example. Maybe \mathcal{C} is a category? What is \mathcal{C}_G ?**]

We will prove Theorem 5.6.4 as a special case of a more general result that describes homotopy colimits indexed by Grothendieck constructions:

THEOREM 5.6.6. *Suppose we are given a functor $F: \mathcal{I} \rightarrow \text{Cat}$ and a functor $\Phi: \text{Gr}(F) \rightarrow \text{Top}$. Then we have a weak equivalence*

$$\text{hocolim}_{\text{Gr}(F)} \Phi \simeq \text{hocolim}_{i \in \mathcal{I}} \text{hocolim}_{F(i)} \Phi|_{F(i)}.$$

(Here we abusively denote “ $\Phi|_{F(i)}$ ” the composition of Φ and the obvious functor $F(i) \rightarrow \text{Gr}(F)$ sending $x \mapsto (i, x)$ and $(\phi: x \rightarrow x') \mapsto (\text{id}_i, \phi)$.)

For the proof of Theorem 5.6.6, we first need an observation about homotopy left Kan extensions. Recall that for $\phi: \mathcal{I} \rightarrow \mathcal{J}$ and $F: \mathcal{I} \rightarrow \mathcal{C}$ the left Kan extension $\phi_! F$ of F along ϕ is given by

$$j \mapsto \mathcal{J}(\phi(-), j) \otimes_{\mathcal{I}_{\phi/j}} F,$$

where $\mathcal{I}_{\phi/j}$ denotes the fiber product category $\mathcal{I} \times_{\mathcal{J}} \mathcal{J}_{/j}$ (ϕ is implicit in the last notation). Similarly we define:

DEFINITION 5.6.7 (Homotopy left Kan extension). For $F: \mathcal{I} \rightarrow \text{Top}$ we define the *homotopy left Kan extension* $\phi_!^{\mathbb{L}} F$ of F along ϕ by

$$j \mapsto B(\mathcal{J}(\phi(-), j), \mathcal{I}, F) \cong \text{hocolim}_{\mathcal{I}_{\phi/j}} F$$

Here the last rewriting is justified by expanding the definitions as in Lemma 5.5.8.

LEMMA 5.6.8. *Given functors $\mathcal{I} \xrightarrow{\phi} \mathcal{J} \xrightarrow{\psi} \mathcal{K}$ and $F: \mathcal{I} \rightarrow \text{Top}$, there is a natural weak equivalence*

$$\psi_!^{\mathbb{L}} \phi_!^{\mathbb{L}} F \simeq (\psi\phi)_!^{\mathbb{L}} F.$$

PROOF. For $k \in \mathcal{K}$, the left-hand side is the iterated bar construction

$$B(\mathcal{K}(\psi(-), k), \mathcal{J}, B(\mathcal{J}(\phi(-), -), \mathcal{I}, F)).$$

By associativity this is naturally isomorphic to

$$B(B(\mathcal{K}(\psi(-), k), \mathcal{J}, \mathcal{J}(\phi(-), -)), \mathcal{I}, F).$$

But recall that for any $j \in \mathcal{J}$ and for any $W: \mathcal{J}^{\text{op}} \rightarrow \text{Top}$ we have a natural weak equivalence $B(W, \mathcal{J}, \mathcal{J}(j, -)) \rightarrow W(j)$, so in particular for $W = \mathcal{K}(\psi(-), k)$ and putting $j = \phi(i)$, we have a weak equivalence of spaces (the second being discrete), natural in $i \in \mathcal{I}^{\text{op}}$ and $k \in \mathcal{K}$

$$B(\mathcal{K}(\psi(-), k), \mathcal{J}, \mathcal{J}(\phi(i), -)) \xrightarrow{\simeq} \mathcal{K}(\psi\phi(i), k).$$

By homotopy invariance of the bar construction, this gives a natural weak equivalence

$$B(B(\mathcal{K}(\psi(-), k), \mathcal{J}, \mathcal{J}(\phi(-), -)), \mathcal{I}, F) \rightarrow B(\mathcal{K}(\psi\phi(-), k), \mathcal{I}, F),$$

where the right-hand side is precisely $(\psi\phi)_!^{\mathbb{L}} F$. \square

PROOF OF THEOREM 5.6.6. For \mathcal{C} a category, let us write \mathcal{C}_+ for the category obtained from \mathcal{C} by freely adjoining a terminal object ∞ . More precisely, \mathcal{C}_+ has objects $\text{Ob}(\mathcal{C}) \amalg \{\infty\}$ and its morphisms are defined by

$$\text{Hom}_{\mathcal{C}_+}(x, y) = \begin{cases} \text{Hom}_{\mathcal{C}}(x, y) & x, y \in \text{Ob}(\mathcal{C}), \\ *, & y = \infty, \\ \emptyset, & x = \infty, y \neq \infty. \end{cases}$$

For $F: \mathcal{I} \rightarrow \text{Cat}$, we similarly write F_+ for the functor that takes $i \in \mathcal{I}$ to $F(i)_+$. There is a fully faithful inclusion $A: \text{Gr}(F) \hookrightarrow \text{Gr}(F_+)$ coming from the obvious inclusions $F(i) \hookrightarrow F(i)_+$. And let $*$ $\in \text{Cat}$ denote the category with one object and one identity morphism.

Applying Lemma 5.6.8 to the homotopy Kan extensions obtained from diagram

$$\begin{array}{ccc} \text{Gr}(F) & \xrightarrow{\Phi} & \text{Top} \\ \downarrow A & & \\ \text{Gr}(F_+) & & \\ \downarrow & & \\ * & & \end{array}$$

we get a weak equivalence

$$\text{hocolim}_{\text{Gr}(F)} \Phi \simeq \text{hocolim}_{\text{Gr}(F_+)} A_!^{\mathbb{L}} \Phi.$$

Next, define $\Omega: \mathcal{I} \rightarrow \text{Gr}(F_+)$ by $i \mapsto (i, \infty \in F(i)_+)$. Note that there exists exactly one functor Ω with the given behaviour on objects. We claim that the functor Ω is homotopy cofinal: Namely, for $(i, x \in F(i)_+) \in \text{Gr}(F_+)$ the undercategory $\mathcal{I}_{(i,x)/\Omega}$, with objects $(i', (i, x) \rightarrow (i', \infty))$, has an initial object $(i, (i, x) \rightarrow (i, \infty))$, so it is in particular a contractible category.

Thus we have

$$\operatorname{hocolim}_{\operatorname{Gr}(F)} \Phi \simeq \operatorname{hocolim}_{\mathcal{I}} A_i^{\mathbb{L}} \Phi \circ \Omega$$

by cofinality, Theorem 5.5.9. The functor $A_i^{\mathbb{L}} \Phi \circ \Omega$ takes $i \in \mathcal{I}$ to $\operatorname{hocolim}_{\operatorname{Gr}(F)_{A/(i,\infty)}} \Phi$.

There is a natural inclusion $\iota_i: F(i) \rightarrow \operatorname{Gr}(F)_{A/(i,\infty)}$ given by $x \in F(i) \mapsto (i, x)$. This functor is also homotopy cofinal: Objects of $\operatorname{Gr}(F)_{A/(i,\infty)}$ are essentially given by triples $(i' \in \mathcal{I}, x \in F(i'), f: i' \rightarrow i)$ (we do not need to specify the terminal map $F(f)(x) \rightarrow \infty \in F(i)_+$), and the undercategory $F(i)_{(i',x,f)/\iota_i}$ has objects consisting of a pair $(y \in F(i), F(f)(x) \rightarrow y)$, such that the following triangle commutes in $F(i) \subset \operatorname{Gr}(F)$ (where t denotes any terminal morphism)

$$\begin{array}{ccc} (i, F(f)(x)) & \longrightarrow & (i, y) \\ \downarrow t & & \downarrow t \\ (i, F(f)(\infty)) & \xrightarrow{t} & (i, \infty); \end{array}$$

clearly, the diagram commutes for any chosen morphism $F(f)(x) \rightarrow y$, i.e. an object in $F(i)_{(i',x,f)/\iota_i}$ is just given by a pair $(y \in F(i), F(f)(x) \rightarrow y)$. We see that $(F(f)(x), F(f)(x) \xrightarrow{\operatorname{id}} F(f)(x))$ is an initial object of this category, hence $F(i)_{(i',x,f)/\iota_i}$ is weakly contractible.

It follows that

$$\operatorname{hocolim}_{\operatorname{Gr}(F)} \Phi \simeq \operatorname{hocolim}_{i \in \mathcal{I}} \operatorname{hocolim}_{F(i)} \Phi|_{F(i)},$$

as required. \square

PROOF OF THEOREM 5.6.4. Taking Φ in Theorem 5.6.6 to be the constant functor with value $*$, we get $\operatorname{BGr}(F) \xrightarrow{\sim} \operatorname{hocolim}_{\mathcal{I}} BF(i)$. \square

5.7. Quillen's Theorem B

In this section we indicate how our results so far lead to a proof of Quillen's Theorem B. Recall that this says:

THEOREM 5.7.1 (Quillen's Theorem B). *Suppose $p: \mathcal{E} \rightarrow \mathcal{B}$ is a functor such that for all maps $f: b \rightarrow b'$ in \mathcal{B} , the natural map $B(\mathcal{E}_b) \rightarrow B(\mathcal{E}_{b'})$ is a (weak¹⁰) homotopy equivalence. Then the natural map from $B(\mathcal{E}_b)$ to the homotopy fibre of $B\mathcal{E} \rightarrow B\mathcal{B}$ at $b \in \mathcal{B}$ is a weak equivalence.*

We will deduce this from our results in the last section together with the following result of Quillen [Qui73]:

PROPOSITION 5.7.2. *Suppose $F: \mathcal{I} \rightarrow \operatorname{Top}$ is a functor such that for every morphism $\phi: i \rightarrow j$ in \mathcal{I} , the map $F(\phi): F(i) \rightarrow F(j)$ is a homotopy equivalence. Then the homotopy fibre of the natural map*

$$\operatorname{hocolim}_{\mathcal{I}} F \rightarrow \operatorname{hocolim}_{\mathcal{I}} * \cong B\mathcal{I}$$

at i is weakly equivalent to $F(i)$.

WORDS ABOUT THE (SKIPPED) PROOF. Quillen's proof of this result is not hard, but we will not give it here as it uses some results about *quasifibrations* that we do not have time to discuss. A map of spaces is called a *quasifibration* if the natural maps from its strict fibres to its homotopy fibres are all weak equivalences. It is easy to see that each strict fibre of the above map is homeomorphic to the space $F(i)$ for some $i \in \mathcal{I}$ (for example, using that geometric realization commutes with pullbacks). Quillen shows that the map is a quasifibration using an induction over the skeleta of $B\mathcal{I}$ and some basic results on quasifibrations from [DT58] (which incidentally is probably the most well-known paper in homotopy theory written in German). \square

¹⁰These are anyway CW complexes...

REMARK 5.7.3. Proposition 5.7.2 is a vast generalization of the fibration sequence $X \rightarrow X_{hG} \rightarrow BG$ arising from the action of a group G on a space X via the Borel construction. The key point is that all the morphisms in BG induce homeomorphisms, hence weak equivalences.

Combining Proposition 5.7.2 with Theorem 5.6.4, we immediately get:

COROLLARY 5.7.4. *Suppose $F: \mathcal{I} \rightarrow \text{Cat}$ is a functor such that for every map $f: i \rightarrow i'$ the map $BF(f): BF(i) \rightarrow BF(i')$ is a (weak) homotopy equivalence. Then the homotopy fibre of $BGr(F) \rightarrow B\mathcal{I}$ at i is $BF(i)$.* \square

From this corollary, Theorem B follows easily:

PROOF OF THEOREM B. Let $F: \mathcal{B} \rightarrow \text{Cat}$ be the functor $b \mapsto \mathcal{E}/_b = \mathcal{E} \times_{\mathcal{B}} \mathcal{B}/_b$, and set $\bar{\mathcal{E}} := Gr(F)$. Then the objects of $\bar{\mathcal{E}}$ are triples $(b \in \mathcal{B}, e \in \mathcal{E}, \phi: p(e) \rightarrow b)$ and a morphism $(b, e, \phi) \rightarrow (b', e', \phi')$ is given by maps $f: b \rightarrow b'$ and $g: e \rightarrow e'$ such that the square

$$\begin{array}{ccc} p(e) & \xrightarrow{p(g)} & p(e') \\ \downarrow \phi & & \downarrow \phi' \\ b & \xrightarrow{f} & b' \end{array}$$

commutes. Let \bar{p} denote the projection $\bar{\mathcal{E}} \rightarrow \mathcal{B}$ that takes (b, e, ϕ) to b ; there is also a projection $q: \bar{\mathcal{E}} \rightarrow \mathcal{E}$ that takes this to e . Moreover, we have an inclusion $i: \mathcal{E} \rightarrow \bar{\mathcal{E}}$ given by $i(e) = (p(e), e, p(e) \xrightarrow{\text{id}} p(e))$ and a commutative triangle

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{i} & \bar{\mathcal{E}} \\ & \searrow p & \swarrow \bar{p} \\ & \mathcal{B} & \end{array}$$

Clearly $qi = \text{id}_{\mathcal{E}}$, and there is also a natural transformation $iq \rightarrow \text{id}_{\bar{\mathcal{E}}}$ given at (b, e, ϕ) by the map $(pe, e, \text{id}_{p(e)}) \rightarrow (b, e, \phi)$ determined by ϕ and id_e . But then Bq is a homotopy inverse to Bi , so $Bi: B\mathcal{E} \rightarrow B\bar{\mathcal{E}}$ is a homotopy equivalence. Thus the homotopy fibres of \bar{p} and p are also (weakly) homotopy equivalent, and now applying Corollary 5.7.4 to F completes the proof. \square

5.8. The homology spectral sequence of a simplicial space

In this section we will discuss a spectral sequence that computes the homology of the geometric realization of a simplicial space, and derive a description of its E^2 -page. This is originally due to Segal [Seg68]; there is a more detailed presentation of Segal's result on the first differential in the book [KT06] (but unfortunately some details are still omitted). For precise statements, see Propositions 5.8.5 and 5.8.8.

Let us start by briefly recalling the spectral sequence of a filtered chain complex.

5.8.1. Recollections on the spectral sequence of a filtered chain complex and filtered space. A *filtered chain complex* is a sequence of inclusions of chain complexes

$$C(0) \hookrightarrow C(1) \hookrightarrow C(2) \hookrightarrow \dots$$

(One can also consider more general kinds of filtration, e.g. indexed by \mathbb{Z} , but this suffices for us.) Recall that from a filtered chain complex we get a spectral sequence of the form

$$E_{s,t}^1 = H_{s+t}(F(s)) \Rightarrow H_{s+t}(C)$$

where $F(s)$ is the quotient $C(s)/C(s-1)$ and $C = \text{colim}_{i \rightarrow \infty} C(i)$ (which is an increasing union). The differentials are of the form

$$d_r: E_{s,t}^r \rightarrow E_{s-r, t+r-1}^r.$$

The spectral sequence converges if, for example, we have $H_{s+t}(F(s)) = 0$ for $t < 0$ — this condition implies that the spectral sequence lives in the first quadrant.

REMARK 5.8.1. The first differential d_1 can be described as follows: Given a class $[x] \in H_{s+t}(F(s))$ let $x \in F(s)_{s+t}$ be a cycle representing $[x]$. The map $C(s) \rightarrow F(s)$ is surjective, so we can choose $\bar{x} \in C(s)$ mapping to x . We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(s-1)_{s+t} & \longrightarrow & C(s)_{s+t} & \longrightarrow & F(s)_{s+t} & \longrightarrow & 0 \\ \downarrow & & \downarrow d & & \downarrow d & & \downarrow d & & \downarrow \\ 0 & \longrightarrow & C(s-1)_{s+t-1} & \longrightarrow & C(s)_{s+t-1} & \longrightarrow & F(s)_{s+t-1} & \longrightarrow & 0. \end{array}$$

Thus, as $dx = 0$ in $F(s)_{s+t-1}$, the element $d\bar{x}$ must be the image of a (unique) $x' \in C(s-1)_{s+t-1}$ (and $dx' = 0$ since it maps to $d^2\bar{x} = 0$ under an injective map). Also the image x'' of x' in $F(s-1)_{s+t-1}$ is a cycle. We define $d_1[x] \in H_{s+t-1}F(s-1)$ to be class represented by the cycle x'' .

EXERCISE 5.8.2. Check that the class $d_1[x]$ in $H_{s+t-1}(F(s-1))$ is independent of the choices of x and \bar{x} we made.

REMARK 5.8.3. If $d_1[x] = 0 \in H_{s+t-1}(F(s-1))$ then this means we can choose $y \in F(s-1)_{s+t}$ such that $dy = x'' \in F(s-1)_{s+t-1}$. Choose a $\bar{y} \in C(s-1)_{s+t}$ mapping to y under quotient projection, then $d\bar{y} \in C(s-1)_{s+t-1}$ maps to x'' in $F(s-1)_{s+t-1}$ under quotient projection, and so $d\bar{y} - x'$ maps to 0 and by exactness lies in the image of $C(s-2)_{s+t-1}$. Let $y' \in C(s-2)_{s+t-1}$ be the unique element that maps to $d\bar{y} - x'$; y' is a cycle, and the image y'' of y' in $F(s-2)_{s+t-1}$ is also a cycle; moreover $d_1[y''] = 0$. We define $d_2[x]$ to be the class in $E_{s,t}^2$ represented by y'' . Iterating this process gives d_r for all r .

Now suppose we have a sequence of cofibrations of topological spaces

$$X(0) \hookrightarrow X(1) \hookrightarrow X(2) \hookrightarrow \cdots.$$

Let $X = \operatorname{colim}_{i \rightarrow \infty} X(i)$.¹¹ Then taking singular chains gives a filtered chain complex

$$C_*(X(0)) \hookrightarrow C_*(X(1)) \hookrightarrow C_*(X(2)) \hookrightarrow \cdots.$$

The associated spectral sequence is of the form

$$E_{s,t}^1 = H_{s+t}(X(s), X(s-1)) \Rightarrow H_{s+t}(X);$$

it converges if, for instance, we assume that $H_{s+t}(X(s), X(s-1))$ vanishes for $t < 0$, i.e. the quotient $X(s)/X(s-1)$ has no *reduced* homology in degrees $\leq (s-1)$.

LEMMA 5.8.4. *In this spectral sequence the first differential d_1 is given by*

$$H_{s+t}(X(s), X(s-1)) \xrightarrow{\partial} H_{s+t-1}(X(s-1)) \rightarrow H_{s+t-1}(X(s-1), X(s-2)),$$

i.e. the composite of the connecting homomorphism in the LES of $(X(s), X(s-1))$ and the relativisation morphism of $(X(s-1), X(s-2))$.

PROOF. Immediate from the definition of d_1 above and the definition of the boundary map ∂ . \square

¹¹This colimit is taken in Top: it is the space whose underlying set is the increasing union of the $X(i)$'s, and the topology is the weak topology on the union.

5.8.2. The homology spectral sequence of a simplicial space. We now want to apply this to the geometric realization $|X|$ of a simplicial space X_\bullet . Suppose X is Reedy cofibrant, so the inclusion $L_n X \hookrightarrow X_n$ is a closed cofibration for all $n \geq 1$. Then, as we saw before, in the pushout square

$$\begin{array}{ccc} L_n X \times \Delta_{top}^n \amalg_{L_n X \times \partial \Delta_{top}^n} X_n \times \partial \Delta_{top}^n & \longrightarrow & \text{sk}_{n-1}|X| \\ \downarrow & & \downarrow \\ X_n \times \Delta_{top}^n & \longrightarrow & \text{sk}_n|X|. \end{array}$$

from Lemma 5.2.10(ii) the left vertical map is also a cofibration, hence so is the right vertical map. Thus the skeletal filtration

$$\text{sk}_0|X| \hookrightarrow \text{sk}_1|X| \hookrightarrow \cdots \hookrightarrow |X|$$

is given by cofibrations. We then have a spectral sequence of the form

$$E_{s,t}^1 = H_{s+t}(\text{sk}_s|X|, \text{sk}_{s-1}|X|) \Rightarrow H_{s+t}(|X|).$$

The above pushout square also implies that we have isomorphisms

$$\begin{aligned} H_{s+t}(\text{sk}_s|X|, \text{sk}_{s-1}|X|) &\cong H_{s+t}(X_s \times \Delta_{top}^s, L_s X \times \Delta_{top}^s \amalg_{L_s X \times \partial \Delta_{top}^s} X_s \times \partial \Delta_{top}^s) \\ &\cong H_{s+t}((X_s, L_s X) \wedge (\Delta_{top}^s, \partial \Delta_{top}^s)). \end{aligned}$$

Here we use the notation $(A, B) \wedge (A', B') = (A \times A', A \times B' \amalg_{B \times A'} B \times A')$ — note that $(A \times A') / (A \times B' \amalg_{B \times A'} B \times A') \cong A/B \wedge A'/B'$ for the usual smash product of pointed spaces; in particular, using the isomorphism $H_*(A, B) \cong \tilde{H}_*(A/B)$ for a cofibration $B \hookrightarrow A$, we obtain an isomorphism $H_{s+t}((X_s, L_s X) \wedge (\Delta_{top}^s, \partial \Delta_{top}^s))$. In our case we thus have

$$\begin{aligned} H_{s+t}(\text{sk}_s|X|, \text{sk}_{s-1}|X|) &\cong H_{s+t}((X_s, L_s X) \wedge (\Delta_{top}^s, \partial \Delta_{top}^s)) \\ &\cong \tilde{H}_{s+t}((X_s/L_s X) \wedge (\Delta_{top}^s/\partial \Delta_{top}^s)) \\ &\cong \tilde{H}_{s+t}(\Sigma^s(X_s/L_s X)) \\ &\cong \tilde{H}_t(X_s/L_s X) \\ &\cong H_t(X_s, L_s X). \end{aligned}$$

In particular this clearly vanishes for $t < 0$, so the spectral sequence converges. In summary, we have:

PROPOSITION 5.8.5. *Suppose X_\bullet is a Reedy cofibrant simplicial space. Then there is a convergent spectral sequence of the form*

$$E_{s,t}^1 = H_t(X_s, L_s X) \Rightarrow H_{s+t}(|X|).$$

□

Our next goal is to identify the first differential in this spectral sequence: we will show that it is given by the alternating sum of the maps induced in homology by the face maps of X_\bullet .

As a warm-up to proving this, let us consider the case where X is a simplicial set. Then writing X_s^{nd} for the set of non-degenerate s -simplices, we have a pushout square

$$\begin{array}{ccc} X_s^{\text{nd}} \times \partial \Delta_{top}^s & \longrightarrow & \text{sk}_{s-1}|X| \\ \downarrow & & \downarrow \\ X_s^{\text{nd}} \times \Delta_{top}^s & \longrightarrow & \text{sk}_s|X|. \end{array}$$

Consider $X_s^{\text{nd}} \times \Delta_{\text{top}}^s$ equipped with the skeletal filtration coming from Δ_{top}^s , i.e. let $X_s^{\text{nd}} \times \text{sk}_i \Delta_{\text{top}}^s$ be the i^{th} filtration space, for $i \geq 0$. With this filtration the map $X_s^{\text{nd}} \times \Delta_{\text{top}}^s \rightarrow |X|$ is a map of filtered spaces. This gives a commutative diagram

(5.8.1)

$$\begin{array}{ccccc} H_i(X_s^{\text{nd}} \times \Delta_{\text{top}}^s, X_s^{\text{nd}} \times \partial \Delta_{\text{top}}^s) & \xrightarrow{\partial} & H_{i-1}(X_s^{\text{nd}} \times \partial \Delta_{\text{top}}^s) & \xrightarrow{\theta} & H_{i-1}(X_s^{\text{nd}} \times \partial \Delta_{\text{top}}^s, X_s^{\text{nd}} \times \text{sk}_{s-2} \Delta_{\text{top}}^s) \\ \downarrow \cong & & \downarrow & & \downarrow \rho \\ H_i(\text{sk}_s |X|, \text{sk}_{s-1} |X|) & \xrightarrow{\partial} & H_{i-1}(\text{sk}_{s-1} |X|) & \longrightarrow & H_{i-1}(\text{sk}_{s-1} |X|, \text{sk}_{s-2} |X|). \end{array}$$

Here the bottom row gives the E^1 -differential d_1 . On the other hand, we have

$$(X_s^{\text{nd}} \times \partial \Delta_{\text{top}}^s) / (X_s^{\text{nd}} \times \text{sk}_{s-2} \Delta_{\text{top}}^s) \cong \bigvee_{j=0}^s (X_s^{\text{nd}} \times \Delta_{\text{top}}^{s-1}) / (X_s^{\text{nd}} \times \partial \Delta_{\text{top}}^{s-1}),$$

inducing in homology an isomorphism

$$\begin{aligned} H_{i-1}(X_s^{\text{nd}} \times \partial \Delta_{\text{top}}^s, X_s^{\text{nd}} \times \text{sk}_{s-2} \Delta_{\text{top}}^s) &\cong \bigoplus_{j=0}^s H_{i-1}(X_s^{\text{nd}} \times \Delta_{\text{top}}^{s-1}, X_s^{\text{nd}} \times \partial \Delta_{\text{top}}^{s-1}) \\ (5.8.2) \qquad \qquad \qquad &\cong \bigoplus_{j=0}^s H_{i-s}(X_s^{\text{nd}}). \end{aligned}$$

We can identify $H_i(X_s^{\text{nd}} \times \Delta_{\text{top}}^s, X_s^{\text{nd}} \times \partial \Delta_{\text{top}}^s) \cong H_{i-s}(X_s^{\text{nd}})$ using the homology cross product with the relative fundamental class of the pair $(\Delta_{\text{top}}^s, \partial \Delta_{\text{top}}^s)$; under this identification, the map $\theta \circ \partial$, i.e. the top row of diagram (5.8.1), becomes the direct product¹²

$$\theta = \prod_{j=0}^s (-1)^j \text{id}: H_{i-s}(X_s^{\text{nd}}) \rightarrow \bigoplus_{j=0}^s H_{i-s}(X_s^{\text{nd}})$$

On the other hand, we can identify $H_{i-1}(\text{sk}_{s-1} |X|, \text{sk}_{s-2} |X|) \cong H_{i-s}(X_{s-1})$, and the map ρ , under this identification and the one from (5.8.2), is given by the direct sum

$$\rho = \bigoplus_{j=0}^s (d_j)_*: \bigoplus_{j=0}^s H_{i-s}(X_s^{\text{nd}}) \rightarrow H_{i-s}(X_{s-1}).$$

Composing the top row of diagram (5.8.1) with ρ we obtain the desired formula $\sum_{j=0}^s (-1)^j (d_j)_*$ for the E^1 -differential d_1 .

For a simplicial set X we have of course an isomorphism

$$H_i(X_s, L_s X) \cong H_{i-s}(X_s^{\text{nd}}) \cong \begin{cases} \mathbb{Z} X_s^{\text{nd}} & t = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the E^1 -page of the spectral sequence is 0 except for the row $t = i - s = 0$ where we have a chain complex

$$\cdots \rightarrow \mathbb{Z} X_s^{\text{nd}} \rightarrow \mathbb{Z} X_{s-1}^{\text{nd}} \rightarrow \cdots \rightarrow \mathbb{Z} X_0^{\text{nd}} = \mathbb{Z} X_0.$$

Since the spectral sequence converges, the homology of this chain complex is isomorphic to the homology $H_*(|X|)$ of the realization. We can interpret this using the chain complex associated to the simplicial set X ; this uses a standard result on simplicial abelian groups, which is a reformulation of Lemma 3.2.4:

¹²Here we use that a finite direct sum is both a coproduct and a product in abelian groups.

LEMMA 5.8.6. *Suppose A_\bullet is a simplicial abelian group, and write $C(A)$ for the associated chain complex with $C(A)_n = A_n$ and $d = \sum_i (-1)^i d_i$. Let $D_k(A)$ be the subgroup generated by degenerate elements. Then the differential on $C(A)$ restricts to a differential on $D(A)$. Moreover, the chain complex $D(A)$ is contractible, so the projection $C(A) \rightarrow \bar{C}(A) := C(A)/D(A)$ is a quasi-isomorphism.*

The chain complex $E_{s,0}^1$ found above is isomorphic to $\bar{C}(\mathbb{Z}X)$, which is by the previous lemma quasi-isomorphic to the usual chain complex $C(\mathbb{Z}X)$. Thus, if we want we can add back in the degenerate simplices and as the spectral sequence converges we have proved an isomorphism

$$H_*(|X|) \cong H_*(C(\mathbb{Z}X)).$$

Before we turn to the case of general simplicial spaces, we state a lemma:

LEMMA 5.8.7. *Let X_\bullet be a Reedy cofibrant simplicial space, and for $0 \leq r \leq s$ let $F_{-r}X_s$ denote the union of all subspaces of X_s of the form $s_{i_1} \cdots s_{i_r} X_{s-r}$; let moreover $F_{-r}H_*X_s \subseteq H_*(X_s)$ be generated by the images of H_*X_{s-r} under the degeneracies. Then*

- (1) *The natural map $H_*(F_{-r}X_s) \rightarrow H_*(F_{1-r}X_s)$ is injective.*
- (2) *The image of $H_*(F_{-r}X_s)$ in H_*X_s is precisely $F_{-r}H_*X_s$. In particular, $H_*(L_sX) \cong D(H_*(X_\bullet))_s$, considering $H_*(X_\bullet)$ as a graded simplicial abelian group.*

This is supposed to follow from the simplicial identities, but we will not give a complete proof. In analogy with the proof of Lemma 2.1.14, one can show that $H_*(X_s)$ splits as a direct sum

$$H_*(X_s) \cong \bigoplus_{\substack{m \leq s \\ \theta: [s] \rightarrow [m]}} H_*(\theta^*(X_m), \theta^*(L_m(X))),$$

using essentially the map $(d_{i_1} \cdots d_{i_r})_*: H_*(X_s) \rightarrow H_*(X_{s-r}, L_{s-r}X)$ to witness the summand corresponding to $\theta = s^{i_1} \cdots s^{i_r}: [s] \rightarrow [s-r]$. Once this is proved, Lemma 5.8.7 follows from identifying $H_*(F_{-r}X_s) \subset H_*(X_s)$ with the part of the direct sum corresponding to pairs (m, θ) with $m \leq s-r$.

PROPOSITION 5.8.8. *Let X_\bullet be a Reedy cofibrant simplicial space. Then the t^{th} row in the E^1 -page of the spectral sequence is $\bar{C}(H_t(X_\bullet))$. Thus we have*

$$E_{s,t}^2 \cong H_s(H_t(X_\bullet)),$$

where we consider $H_t(X_\bullet)$ as a simplicial abelian group, and define its homology as in Definition 3.2.1.

PROOF. Filter X_s as before, and filter $X_s \times \Delta_{top}^s$ by taking

$$F_k := F_k(X_s \times \Delta^s) = \bigcup_{i+j=k} F_i X_s \times \text{sk}_j \Delta_{top}^s \quad 0 \leq k \leq s.$$

Thus $F_s = X_s \times \Delta_{top}^s$ and $F_{s-1} = L_s X \times \Delta_{top}^s \amalg_{L_s X \times \partial \Delta_{top}^s} X_s \times \partial \Delta_{top}^s$. With this filtration the map $X_s \times \Delta_{top}^s \rightarrow |X|$ is a map of filtered spaces. This means we get a commutative diagram

$$(5.8.3) \quad \begin{array}{ccccc} H_i(F_s, F_{s-1}) & \xrightarrow{\partial} & H_{i-1}(F_{s-1}) & \xrightarrow{\bar{\theta}} & H_{i-1}(F_{s-1}, F_{s-2}) \\ \downarrow \cong & & \downarrow & & \downarrow \bar{\rho} \\ H_i(\text{sk}_s |X|, \text{sk}_{s-1} |X|) & \xrightarrow{\partial} & H_{i-1}(\text{sk}_{s-1} |X|) & \longrightarrow & H_{i-1}(\text{sk}_{s-1} |X|, \text{sk}_{s-2} |X|), \end{array}$$

and the left vertical map already identifies, for $i = s+t$, the entry $E_{s,t}^1$ in the spectral sequence with $H_{s+t}(F_s, F_{s-1})$, which by Lemma 5.8.7 is isomorphic to $\bar{C}_s(H_t(X_\bullet)) = H_t(X_s)/D(H_t(X_\bullet))_s$.

We then note that the space F_{s-1}/F_{s-2} splits as a wedge sum

$$((L_s X / F_{-2} X_s) \wedge (\Delta_{top}^s / \partial \Delta_{top}^s)) \vee ((X_s / L_s X) \wedge (\partial \Delta_{top}^s / \text{sk}_{s-2} \Delta_{top}^s)).$$

Correspondingly, we have a decomposition in homology

$$\begin{aligned} H_{i-1}(F_{s-1}, F_{s-2}) &\cong H_{i-1}((L_s X, F_{-2} X_s) \wedge (\Delta_{top}^s, \partial \Delta_{top}^s)) \oplus H_{i-1}((X_s, L_s X) \wedge (\partial \Delta_{top}^s, \text{sk}_{s-2} \Delta_{top}^s)) \\ &\cong H_{i-1-s}(L_s X, F_{-2} X_s) \oplus \bigoplus_{j=0}^s H_{i-s}(X_s, L_s X). \end{aligned}$$

The composition of the first row of diagram (5.8.3) can be identified with the help of the following commutative diagram, in which the comparison between middle and bottom row comes from a map of filtered spaces, the comparison between top and middle row comes from the homology cross product with the fundamental class of $(\Delta_{top}^s, \partial \Delta_{top}^s)$, and we abbreviate the pair $(\Delta_{top}^s, \partial \Delta_{top}^s)$ by $(\Delta_{top}^s, \partial)$:

$$\begin{array}{ccccc} H_{i-s}(X_s, L_s X) & \xrightarrow{\partial} & H_{i-s-1}(L_s X) & \longrightarrow & H_{i-s-1}(L_s X, F_{-2} X_s) \\ \downarrow -\times[\Delta_{top}^s, \partial] \cong & & \downarrow -\times[\Delta_{top}^s, \partial] \cong & & \downarrow -\times[\Delta_{top}^s, \partial] \cong \\ H_i((X_s, L_s X) \wedge (\Delta_{top}^s, \partial)) & \xrightarrow{\partial} & H_{i-1}(L_s X \times (\Delta_{top}^s, \partial)) & \longrightarrow & H_{i-1}((L_s X, F_{-2} X_s) \wedge (\Delta_{top}^s, \partial)) \\ \downarrow \cong & & \downarrow & & \downarrow \\ H_i(F_s, F_{s-1}) & \xrightarrow{\partial} & H_{i-1}(F_{s-1}, L_s X \times \partial \Delta_{top}^s) & \longrightarrow & H_{i-1}(F_{s-1}, F_{s-2}) \\ & \searrow \partial & \uparrow & \nearrow \bar{\theta} & \\ & & H_{i-1}(F_{s-2}) & & \end{array}$$

But here the top left boundary map $H_t(X_s, L_s X) \rightarrow H_{t-1}(L_s X)$ is zero, since by Lemma 5.8.7 the map $H_*(L_s X) \rightarrow H_*(X_s)$ is injective. Thus the composition $\bar{\theta} \circ \partial$, i.e. the top row of diagram (5.8.3), has image inside the summand $\bigoplus_{j=0}^s H_{i-s}(X_s, L_s X)$.

We can then use the following diagram, where we abbreviate $\Delta = \Delta_{top}^s$, $\partial = \partial \Delta_{top}^s$ and $\text{sk} = \text{sk}_{n-2} \partial \Delta_{top}^s$

$$\begin{array}{ccccc} H_i((X_s, L_s X) \wedge (\Delta, \partial)) & \xrightarrow{\partial} & H_{i-1}((X_s, L_s X) \times \partial) & \longrightarrow & H_{i-1}((X_s, L_s X) \wedge (\partial, \text{sk})) \\ \downarrow \cong & & \downarrow & & \downarrow \\ H_i(F_s, F_{s-1}) & \xrightarrow{\partial} & H_{i-1}(F_{s-1}, L_s X \times \partial) & \longrightarrow & H_{i-1}(F_{s-1}, F_{s-2}) \\ & \searrow \partial & \uparrow & \nearrow \bar{\theta} & \\ & & H_{i-1}(F_{s-2}) & & \end{array}$$

The composite of the top row of the last diagram is the identity of $H_{i-s}(X_s, L_s X)$ tensored with the map $H_s(\Delta_{top}^s, \partial \Delta_{top}^s) \rightarrow H_{s-1}(\partial \Delta_{top}^s, \text{sk}_{n-2} \partial \Delta_{top}^s)$ sending $[\Delta_{top}^s, \partial \Delta_{top}^s]$ to $\sum_{j=0}^s (-1)^j d_*^j [\Delta^{s-1}, \partial \Delta^{s-1}]$. The composition of τ with the map $\bar{\rho}$ from diagram (5.8.3) is induced on the quotient $H_{i-1}((X_s, L_s X) \wedge (\partial, \text{sk})) \cong H_{i-1}(X_s \times (\partial, \text{sk})) / H_{i-1}(L_s X \times (\partial, \text{sk}))$ by the map

$$H_{i-1}(X_s \times (\partial \Delta_{top}^s, \text{sk}_{s-2} \partial \Delta_{top}^s)) \cong \bigoplus_{j=0}^s H_{i-1}(X_s \times (\Delta_{top}^{s-1}, \partial \Delta_{top}^{s-1})) \rightarrow H_{i-1}(\text{sk}_{s-1} |X|, \text{sk}_{s-2} |X|)$$

which on the j^{th} summand is induced by the composite

of $\bigoplus_{j=0}^s H_{i-s}(X_s, L_s X)$ to the map induced in homology by the composite

$$X_s \times (\Delta^{s-1} / \partial \Delta^{s-1}) \xrightarrow{d_j \times \text{id}} X_{s-1} \times (\Delta^{s-1} / \partial \Delta^{s-1}) \rightarrow \text{sk}_{s-1} |X| / \text{sk}_{s-2} |X|.$$

This concludes the identification of the E^1 -differential. \square

5.9. The homology spectral sequence of a homotopy colimit

We now want to give a more conceptual (and potentially more computable) description of the E^2 -page of the homology spectral sequence for the simplicial space $\mathbb{B}_\bullet(*, \mathcal{I}, F)$ whose geometric realization is the homotopy colimit of F . This will be in terms of the left derived functors of the colimit functor for abelian groups. We will now introduce these, and then derive the required expression for relating them to the E^2 -page.

First of all, observe that for any small category \mathcal{I} , the functor category $\text{Ab}^{\mathcal{I}}$ is an abelian category. To define left derived functors (of covariant functors), we first need to know this category has enough projectives:

DEFINITION 5.9.1. Let \mathcal{I} be a small category. For $i \in \mathcal{I}$, write L_i for the functor $\mathcal{I} \rightarrow \text{Ab}$ given by $i \mapsto \mathbb{Z}\mathcal{I}(i, -)$.

EXAMPLE 5.9.2. If $\mathcal{I} = 1 \leftarrow 0 \rightarrow 2$, then $L_0 = (\mathbb{Z} \leftarrow \mathbb{Z} \rightarrow \mathbb{Z})$, $L_1 = (\mathbb{Z} \leftarrow 0 \rightarrow 0)$, and $L_2 = (0 \leftarrow 0 \rightarrow \mathbb{Z})$.

LEMMA 5.9.3. *The object $L_i \in \text{Ab}^{\mathcal{I}}$ is projective.*

PROOF. This follows since L_i is “free”: For $F \in \text{Ab}^{\mathcal{I}}$ we have

$$\text{Hom}_{\text{Ab}^{\mathcal{I}}}(L_i, F) \cong \text{Hom}_{\text{Set}^{\mathcal{I}}}(\mathcal{I}(i, -), F) \cong F(i)$$

by the adjunction $\mathbb{Z}(-): \text{Set} \rightleftarrows \text{Ab}: U$ (which induces an adjunction between diagram categories) and by the Yoneda Lemma. The previous is in fact a chain of isomorphisms of abelian groups, natural in $F \in \text{Ab}^{\mathcal{I}}$. Hence $\text{Hom}_{\text{Ab}^{\mathcal{I}}}(L_i, -): \text{Ab}^{\mathcal{I}} \rightarrow \text{Ab}$ preserves exact sequences, so L_i is projective. \square

EXERCISE 5.9.4. This exercise is somewhat paranthetical, but some may enjoy it: Show that the argument in Lemma 5.9.3 can be seen directly as an instance of the Yoneda lemma in enriched category theory, where the enrichment is over abelian groups (and find out what these words mean). [Hint: Introduce the linear category $\mathbb{Z}\mathcal{I}$ with objects \mathcal{I} and morphisms the \mathbb{Z} -linear span of the morphisms in \mathcal{I} ie, $\text{Hom}_{\mathbb{Z}\mathcal{I}}(i, j) = \mathbb{Z}\text{Hom}_{\mathcal{I}}(i, j)$, and observe that $L_i(j) = \text{Hom}_{\mathbb{Z}\mathcal{I}}(i, j)$.]

LEMMA 5.9.5. *The category $\text{Ab}^{\mathcal{I}}$ has enough projectives.*

PROOF. Given $F \in \text{Ab}^{\mathcal{I}}$ we can take $P = \bigoplus_{i \in \mathcal{I}} \bigoplus_{x \in F(i)} L_i$; this is projective and has a surjective map $P \rightarrow F$ given on the component corresponding to $x \in F(i)$ by the map $L_i \rightarrow F$ corresponding to x . \square

The functor $\text{colim}: \text{Ab}^{\mathcal{I}} \rightarrow \text{Ab}$ is clearly right exact (since it commutes with all colimits) and by Lemma 5.9.5 the category $\text{Ab}^{\mathcal{I}}$ has enough projectives. We can therefore make the following definition:

DEFINITION 5.9.6. For $s \geq 0$ define colim_s as the s^{th} left derived functor of colim . In other words, for a functor $F \in \text{Ab}^{\mathcal{I}}$, the abelian group $\text{colim}_s F \in \text{Ab}$ is $H_s(\text{colim } P_\bullet)$ where P_\bullet is a projective resolution of F in $\text{Ab}^{\mathcal{I}}$.

EXAMPLE 5.9.7. If $\mathcal{I} = \mathcal{B}G$, meaning a group G viewed as a category with one object $*$, then the functor L_* is just the group algebra $\mathbb{Z}G$ viewed as a free module of rank one, and $\text{colim}_s M$ identifies with the group homology $H_s(G; M)$ (defined, for instance, as $\text{Tor}_s^{\mathbb{Z}G}(\mathbb{Z}, M)$).

EXAMPLE 5.9.8. [This will probably be done in the exercise session.] Suppose \mathcal{I} is the category $\bullet \leftarrow \bullet \rightarrow \bullet$, and consider $M = A \xleftarrow{f} C \xrightarrow{g} B$. Then

$$\operatorname{colim}_s M \cong \begin{cases} \operatorname{colim} M \cong \operatorname{coker} \left(C \xrightarrow{(f, -g)} A \oplus B \right) & s = 0, \\ \ker \left(C \xrightarrow{(f, -g)} A \oplus B \right) & s = 1, \\ 0 & s > 1. \end{cases}$$

To see this, consider the exact sequence

$$0 \rightarrow (A \leftarrow 0 \rightarrow B) \rightarrow (A \leftarrow C \rightarrow B) \rightarrow (0 \leftarrow C \rightarrow 0) \rightarrow 0$$

By the long exact sequence of derived functors, the claim will follow from the following claims:

$$\operatorname{colim}_s((A \leftarrow 0 \rightarrow B)) = \begin{cases} A \oplus B & s = 0 \\ 0 & s > 0 \end{cases}$$

$$\operatorname{colim}_s((0 \leftarrow C \rightarrow 0)) = \begin{cases} 0 & s \neq 1 \\ C & s = 1 \end{cases}$$

For the first claim, we pick as projective resolution of $(A \leftarrow 0 \rightarrow B)$ one of the form $(P_\bullet^A \leftarrow 0 \rightarrow P_\bullet^B)$, obtained by using projective resolutions of A and B in \mathbf{Ab} .

For the second claim, let $P \xrightarrow{p} C$ be a projective cover in \mathbf{Ab} , and let $K = \ker(p) \in \mathbf{Ab}$; note that K is also a free/projective abelian group. Let $C_0 = (P \leftarrow P \rightarrow P)$ which maps onto $(0 \leftarrow C \rightarrow 0)$ with kernel $(P \leftarrow K \rightarrow P)$.¹³ Set

$$C_1 = ((K \leftarrow K \rightarrow K) \oplus (0 \leftarrow 0 \rightarrow P) \oplus (P \leftarrow 0 \rightarrow 0))$$

which maps onto $(P \leftarrow K \rightarrow P)$ with kernel $C_2 := (K \leftarrow 0 \rightarrow 0) \oplus (0 \leftarrow 0 \rightarrow K)$

We have hence produced a projective resolution of length 2. Taking colim gives the chain complex of abelian groups

$$\dots 0 \rightarrow K \oplus K \rightarrow (K \oplus K)/\Delta(K) \oplus P \oplus P \rightarrow (P \oplus P)/\Delta(P) \rightarrow 0 \dots$$

which has homology groups

$$\dots 0; \quad 0; \quad C; \quad 0; \quad 0 \dots$$

[Alternative proof of the second part: First observe that the claim holds for diagrams $(0 \leftarrow P \rightarrow 0)$ with P projective, and use the long the long exact sequence for

$$(0 \leftarrow K \rightarrow 0) \rightarrow (0 \leftarrow P \rightarrow 0) \rightarrow (0 \leftarrow C \rightarrow 0)$$

to prove it in general.]

REMARK 5.9.9. Notice that colim_s , as a functor from pushouts of abelian groups to abelian groups, vanishes for $s \geq 2$, despite the fact that the projective dimension of the category of pushouts of abelian groups is 2. An explanation of this fact is given in Proposition 5.9.11, and has to do with considering $\operatorname{colim}_{\mathcal{I}} F = \mathbb{Z} \otimes_{\mathcal{I}} F$ as a bi-functor (together with the fact that $\mathbb{Z} = \delta(\mathbb{Z}) \in \mathbf{Ab}^{\mathcal{I}^{\text{op}}}$ admits a projective resolution of length 1 in $\mathbf{Ab}^{\mathcal{I}^{\text{op}}}$, where \mathcal{I} is the pushout category).

DEFINITION 5.9.10. For $F: \mathcal{I} \rightarrow \mathbf{Ab}$, let F^Δ denote the simplicial abelian group where

$$F_n^\Delta := \bigoplus_{i_0 \rightarrow \dots \rightarrow i_n} F(i_0) = \bigoplus_{i: [n] \rightarrow \mathcal{I}} F(i(0)),$$

¹³Don't be tempted to think that $(P \leftarrow K \rightarrow P)$ is projective in $\mathbf{Ab}^{\mathcal{I}}$, just because P and K are projective in \mathbf{Ab} !

with the by now familiar structure maps. We write $E(F) := C(F^\Delta)$ for the associated chain complex.

PROPOSITION 5.9.11. *For $F : \mathcal{I} \rightarrow \mathbf{Ab}$, we have a natural isomorphism*

$$\operatorname{colim}_s F \cong H_s E(F).$$

PROOF. For $W : \mathcal{I}^{\text{op}} \rightarrow \mathbf{Ab}$ and $F : \mathcal{I} \rightarrow \mathbf{Ab}$, in this proof we will write $W \otimes_{\mathcal{I}} F$ for the coend of $W \otimes F : \mathcal{I}^{\text{op}} \times \mathcal{I} \rightarrow \mathbf{Ab}$. Then, writing $\mathbb{Z} = \delta(\mathbb{Z}) \in \mathbf{Ab}^{\mathcal{I}^{\text{op}}}$, we have

$$\operatorname{colim}_{\mathcal{I}} F \cong \mathbb{Z} \otimes_{\mathcal{I}} F,$$

since $\mathbb{Z} \in \mathbf{Ab}$ is the unit for the tensor product of \mathbf{Ab} .

Moreover, $-\otimes_{\mathcal{I}}-$ is right exact in both variables (since it preserves colimits in both), and by the same double complex argument as for the derived functors of $-\otimes_R-$: $\mathbf{Mod}_R \times_R \mathbf{Mod} \rightarrow \mathbf{Ab}$ we can use a projective resolution in either variable to compute derived functors.

We claim that the functor $i \mapsto E(L_i)$ is a projective resolution of \mathbb{Z} in $\mathbf{Ab}^{\mathcal{I}^{\text{op}}}$. Note that $i \mapsto L_i(x) = \mathbb{Z}\mathcal{I}(i, x) = \mathbb{Z}\mathcal{I}^{\text{op}}(x, i)$ is projective in $\mathbf{Ab}^{\mathcal{I}^{\text{op}}}$, so $E(L_{(-)})$ is a chain complex of projectives since each term is a direct sum of such functors: we have indeed $E(L_{(-)})_n = \bigoplus_{i: [n] \rightarrow \mathcal{I}} L_{(-)}(i(0))$.

Now by the usual rewriting we have $E(L_i)_n = \bigoplus_{i \rightarrow i_0 \rightarrow \dots \rightarrow i_n} \mathbb{Z} \cong \mathbb{Z}(\mathcal{N}\mathcal{I}_i)_n$. Thus $E(L_i) \cong C(\mathbb{Z}\mathcal{N}\mathcal{I}_i)$ and so

$$H_* E(L_i) \cong H_* B\mathcal{I}_i \cong \begin{cases} \mathbb{Z}, & * = 0 \\ 0, & * \neq 0, \end{cases}$$

since $B\mathcal{I}_i$ is contractible. A morphism $i \rightarrow i'$ in \mathcal{I} induces a map of spaces $B\mathcal{I}_{i'} \rightarrow B\mathcal{I}_i$ which on H_0 induces the identity of \mathbb{Z} : thus by naturality we have that $H_0(E(L_{(-)})) \cong \delta\mathbb{Z} \in \mathbf{Ab}^{\mathcal{I}^{\text{op}}}$.

It follows that $\operatorname{colim}_s F$ is computed by the chain complex $E(L_{(-)}) \otimes_{\mathcal{I}} F$. Now we need to identify this with $E(F)$. We have a chain of isomorphisms in $\mathbf{Ab}^{\mathcal{I}^{\text{op}}}$, natural in $[n] \in \mathbf{\Delta}^{\text{op}}$

$$\begin{aligned} (L_{(-)})_n^\Delta \otimes_{\mathcal{I}} F &\cong \operatorname{coeq} \left(\bigoplus_{i \rightarrow i' \rightarrow i_0 \rightarrow \dots \rightarrow i_n} \mathbb{Z}\mathcal{I}(i', i_0) \otimes F(i) \rightrightarrows \bigoplus_i \bigoplus_{i_0 \rightarrow \dots \rightarrow i_n} \mathbb{Z}\mathcal{I}(i, i_0) \otimes F(i) \right) \\ &\cong \operatorname{coeq} \left(\bigoplus_{i \rightarrow i' \rightarrow i_0 \rightarrow \dots \rightarrow i_n} F(i) \rightrightarrows \bigoplus_{i \rightarrow i_0 \rightarrow \dots \rightarrow i_n} F(i) \right) \\ &\cong \bigoplus_{i_0 \rightarrow \dots \rightarrow i_n} \operatorname{coeq} \left(\bigoplus_{i \rightarrow i' \rightarrow i_0} F(i) \rightrightarrows \bigoplus_{i \rightarrow i_0} F(i) \right) \\ &\cong \bigoplus_{i_0 \rightarrow \dots \rightarrow i_n} \operatorname{colim}_{\mathcal{I}/i_0} F \\ &\cong \bigoplus_{i_0 \rightarrow \dots \rightarrow i_n} F(i_0) \\ &= F_n^\Delta. \end{aligned}$$

Passing to associated chain complexes and to homology, we get $\operatorname{colim}_s F \cong H_s(E(L_{(-)}) \otimes_{\mathcal{I}} F) \cong H_s(E(F))$, as required. \square

THEOREM 5.9.12. *Given $F : \mathcal{I} \rightarrow \mathbf{Top}$ there is a convergent spectral sequence of the form*

$$E_{s,t}^2 = \operatorname{colim}_s H_t F \Rightarrow H_{s+t}(\operatorname{hocolim} F).$$

PROOF. We consider the homology spectral sequence of the simplicial space $\mathbb{B}_\bullet(*, \mathcal{I}, F)$. By Proposition 5.8.5 this converges, and by Proposition 5.8.8 it has E^2 -term given by

$$E_{s,t}^2 \cong H_s(H_t(\mathbb{B}_\bullet(*, \mathcal{I}, F))).$$

But here $H_t(\mathbb{B}_n(*, \mathcal{I}, F)) \cong \bigoplus_{i_0 \rightarrow \dots \rightarrow i_n} H_t F(i_0) \cong E(H_t F)_n$, and the differentials clearly agree too. So

$$E_{s,t}^2 \cong H_s E(H_t F) \cong \operatorname{colim}_s H_t F. \quad \square$$

EXERCISE 5.9.13. By Example 5.9.8, the E_2 -term of the spectral sequence for a homotopy pushout $(X \leftarrow A \rightarrow Y)$ degenerates to two lines:

$$\operatorname{colim}_0(H_i(-)) = \operatorname{coker}(H_i(C) \rightarrow H_i(X) \oplus H_i(Y))$$

$$\operatorname{colim}_1(H_i(-)) = \operatorname{ker}(H_i(C) \rightarrow H_i(X) \oplus H_i(Y))$$

Check that the corresponding long exact sequence identifies with the Mayer-Vietoris sequence.

5.10. The Čech decomposition of a space as a homotopy colimit

Notes from CatTop 2009.

5.11. Exercises

PS7.

EXERCISE 5.11.1 (Homotopy Orbits I). Let G be a group, and X a space with a G -action, viewed as a functor $BG \rightarrow \mathbf{Spaces}$, where \mathbf{Spaces} is either **Top** or **sSet**.

(1) Show that the homotopy colimit (called the *homotopy orbits* of X) of this functor is the quotient $(X \times EG)/G$.

(2) Show that if the action of G on X is free, then this is homotopy equivalent to X/G .

(3) Let S^1 have the C_2 -action given by multiplication by -1 (inside the complex numbers, say). Compute its homotopy orbits.

(4) Compute the homotopy quotient of S^1 by the trivial C_2 -action. Deduce that the homotopy orbits of a G -space don't only depend on the object of $Ho(\mathbf{Spaces})^{BG}$ in general.

EXERCISE 5.11.2 (Homotopy Orbits II). Let $H \leq G$ be a subgroup inclusion, and let Ind_H^G denote the induction functor, i.e. the left adjoint to the restriction functor $\mathbf{Spaces}^{BG} \rightarrow \mathbf{Spaces}^{BH}$.

(1) Can you describe this a bit more explicitly? In particular, can you prove that it preserves weak equivalences?

(2) Deduce that the G -homotopy orbits of $\operatorname{Ind}_H^G X$ are homotopy equivalent to the H -homotopy orbits of X .

(3) Deduce, e.g. that $(G/H)_{hG} \simeq BH$.

EXERCISE 5.11.3 (Sequential colimits). Consider \mathbb{N} with its usual order, viewed as a category. (1) Give an explicit description of homotopy colimits indexed over this category, as an "infinite mapping telescope".

(2) Use this to compute the homology groups of the homotopy colimit of

$$S^n \xrightarrow{2} S^n \xrightarrow{3} S^n \xrightarrow{4} S^n \xrightarrow{5} S^n \rightarrow \dots$$

where $k : S^n \rightarrow S^n$ is the unique (up to homotopy) degree k self-map of the sphere.

EXERCISE 5.11.4 (A spectral sequence). Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a short exact sequence of non-negatively graded chain complexes.

Explain how to construct a filtered chain complex out of this, and how to recover the long exact sequence in homology from the associated spectral sequence.¹⁴

¹⁴This exercise is mainly here to get a better feel for spectral sequences, and to see how they're a generalization of long exact sequences.

EXERCISE 5.11.5 (Quillen's theorem A). Recall the definition of homotopy cofinality from Sheet 3 : a functor $f : I \rightarrow J$ is homotopy cofinal if for each $j \in J$, the comma category $I_{j/}$ is contractible.

We consider the following property of the functor f : f is said to be hocolim-cofinal if for any functor $\varphi : J \rightarrow \text{Spaces}$, the canonical transformation $\text{hocolim}_I(\varphi \circ f) \rightarrow \text{hocolim}_J \varphi$ is a weak equivalence.

- (1) Explain where this canonical transformation comes from.
- (2) Assume that homotopy cofinal implies hocolim-cofinal (we prove this afterwards). Prove that a homotopy cofinal functor induces an equivalence $|I| \rightarrow |J|$ - this was Quillen's original statement of his theorem A.¹⁵
- (3) Let f^* denote the precomposition functor $\text{Spaces}^J \rightarrow \text{Spaces}^I$. Show that it preserves weak equivalences. We want to show that $\mathbf{L}(\text{colim}_I) \circ f^* \cong \mathbf{L}(\text{colim}_J)$. Explain why it suffices to show $\mathbf{R}(\text{Ran}_f) \circ \Delta \cong \Delta$, where Ran_f is the right Kan extension functor, \mathbf{R} means right derived functor, and Δ is the appropriate diagonal functor (we're abusing notation here by using the same letter).
- (4) Explain why it suffices to show this isomorphism after evaluating at each $j \in J$, and find a formula for $ev_j \circ \mathbf{R}(\text{Ran}_f)$. Conclude from there that a homotopy cofinal functor is hocolim-cofinal.¹⁶
- (5) Show that if f is hocolim-cofinal, then it is homotopy cofinal. (Hint : you might want to consider representable functors, and use Thomason's theorem)

EXERCISE 5.11.6 (Subgroup complexes). Let G be a finite group. Consider the poset $S_p(G)$ of non-trivial p -subgroups of G , ordered by inclusion, and the subposet $A_p(G)$ of p -subgroups which are isomorphic to C_p^r for some r , where C_p is the cyclic group of order p (call them " p -tori"). We view them as simplicial sets via the nerve.

- (1) Using Quillen's theorem A, prove that the inclusion $i : A_p(G) \rightarrow S_p(G)$ is a homotopy equivalence.
- (2) Show that if G has a normal p -subgroup, then $S_p(G)$ is contractible.¹⁷

EXERCISE 5.11.7 (The Lyndon-Hochschild-Serre spectral sequence). Let $1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$ be a short exact sequence of groups.

- (1) Recall from Homework problem 1 from the previous sheet how this defines a Q -action on some BH , and explain how the homotopy orbits of this Q -action are BG .
- (2) Using the homotopy colimit spectral sequence together with homework problem 5.11.8, deduce the existence of the so-called Lyndon-Hochschild-Serre spectral sequence whose E^2 -page and abutment is $E_{p,q}^2 = H_p(BQ; H_q(BH)) \Rightarrow H_{p+q}(BG)$.¹⁸
- (3) Use this to compute some group homology, e.g. $H_*(S_3)$, the symmetric group on three letters..

EXERCISE 5.11.8 (Homotopy Orbits III). Let R be a ring, we consider $Ch_{\geq 0}(R)$, the category of non-negatively graded chain complexes over a given ring R .

- (1) Describe a subcategory of $Ch_{\geq 0}(R)^{BG}$ where the G -orbits functor is homotopical. Deduce a description of its total left derived functor. (Hint : we saw something similar for tensor products in the previous sheet - you can take homological algebra for granted).

¹⁵He initially proved it for applications in algebraic K -theory. For instance, both the additivity theorem and the devissage theorem are consequences of this.

¹⁶In fact, this is true even if we consider homotopy colimits in arbitrary combinatorial model categories.

¹⁷Quillen conjectured that the converse was true - he proved it in some special cases.

¹⁸There is a more general spectral sequence for fibrations, called the Serre spectral sequence - you probably saw it in AlgTop2. We can't state it that way with our tools, but it is also a homotopy colimit spectral sequence.

(2) Show that the following square of functors commutes up to natural isomorphism :

$$\begin{array}{ccc} Ho(\mathbf{Spaces}^{BG}) & \xrightarrow{\text{chains}} & Ho(Ch_{\geq 0}(R)^{BG}) \\ \text{hocolim} \downarrow & & \downarrow \text{hocolim} \\ Ho(\mathbf{Spaces}) & \xrightarrow[\text{chains}]{} & Ho(Ch_{\geq 0}(R)) \end{array}$$

where "chains" is the singular chains functor. (Hint : You may want to consider the corresponding square of right adjoints, and you may want to use the Dold-Kan correspondance).

(3) Deduce a topological interpretation for the homotopy orbits of the constant functor at R (concentrated in degree 0). Deduce the homology groups of $\text{hocolim}_{B\mathbb{Z}} R$ (so the homotopy orbits for $G = \mathbb{Z}$).

(4) Conversely, observe that (3) gives you an algebraic interpretation of the homology of (the space) BG . Compute this way the homology of BC_n , where C_n is the cyclic group of order n . You may want to observe that, letting g denote a generator of C_n , the kernel of $x \mapsto (g-1)x$ is the ideal generated by $\sum_{i=0}^{n-1} g^i$, and conversely, the ideal generated by $g-1$ is the kernel of multiplication by this sum. ¹⁹

EXERCISE 5.11.9 (Thomason's theorem). [2+3+2+3] We will only prove a weak version of Thomason's theorem (the one that's needed for the converse of Quillen's theorem A).

Let $F : I \rightarrow \mathbf{Set}$ be a functor. Its *category of elements* $\int F$ is defined as follows : objects are pairs (i, x) where $i \in I$ and $x \in F(i)$, and an arrow $(i, x) \rightarrow (j, y)$ is an arrow $f : i \rightarrow j$ in I such that $F(f)(x) = y$.

In particular, there is a functor $\pi : \int F \rightarrow I$ sending (i, x) to i and f to f . You do not have to prove that this is a functor (You can have a look at the first exercise sheet for more details on the category of elements).

(1) Let $(\int F)_i$ denote the fiber of π at i , i.e. the subcategory of $\int F$ on objects x such that $\pi(x) = i$, and morphisms between those that induce the identity of i . Show that the natural inclusion $(\int F)_i \rightarrow (\int F)_{/i}$ has a left adjoint (recall that the latter is the pullback $\int F \times_I I_{/i}$).

(2) Using the $\pi_0 \dashv$ "discrete simplicial set" adjunction, show that the composite functor $\mathbf{Set}^I \rightarrow \mathbf{sSet}^I \rightarrow Ho(\mathbf{sSet}^I)$ is fully faithful, and show that its essential image is precisely the subcategory on those diagrams whose every term is weakly homotopy equivalent to a discrete set.

(3) Let Lan_π denote the functor of left Kan extension along π . Take for granted the following fact: there is a left deformation for Lan_π on \mathbf{sSet}^{fF} . Deduce, using exercise 3 from Sheet 6 (which you do not have to reprove), that

$$\text{ev}_i \circ \mathbf{L}(\text{Lan}_\pi) \cong \text{hocolim}_{\int F_{/i}}$$

Note that ev_i is well-defined on the homotopy category because it preserves weak equivalences.

(4) Prove that $\text{hocolim}_I \circ \mathbf{L}(\text{Lan}_\pi) \cong \text{hocolim}_{\int F}$, and finally use example 5.1.12. from the script to deduce Thomason's theorem: $\text{hocolim}_I F \cong |\int F|$.

EXERCISE 5.11.10 (Quillen's theorem B). [2+2+4+2] The goal of this exercise is to give a proof of Quillen's theorem B. We will first prove Proposition 5.7.2. from the lecture notes and discuss in the exercise session how one obtains Theorem B from 5.7.2.

PROPOSITION 5.11.11. *Let $F : \mathcal{I} \rightarrow \mathbf{Top}$ be a functor where \mathcal{I} is a small category such that for every morphism $\varphi : i \rightarrow j$, the induced map $F(\varphi) : F(i) \rightarrow F(j)$ is a homotopy equivalence. Then the homotopy fiber of the natural map*

$$(5.11.1) \quad \text{hocolim}_{\mathcal{I}} F \longrightarrow \text{hocolim}_{\mathcal{I}} * = B\mathcal{I}$$

¹⁹This interplay between topology and algebra is really nice in group co/homology, one can really go in both directions.

at $i \in \mathcal{I}$ is weakly equivalent to $F(i)$

A *quasi-fibration* is a map $f: X \rightarrow Y$ of topological spaces such that the natural map $f^{-1}(x) \hookrightarrow F(f, x)$ from the fiber into the homotopy fiber is a weak equivalence for all $x \in Y$.

- (1) Explain why it is enough to prove that the map (5.11.1) is a quasi-fibration.
- (2) Let $U, V \subset Y$ be two open subsets. Show that $f: X \rightarrow Y$ is a quasi-fibration if its restriction to $f^{-1}(U)$, $f^{-1}(V)$ and $f^{-1}(U \cap V)$ is a quasi-fibration.
- (3) Recall that the homotopy colimit is obtained from the geometric realization of the simplicial set

$$[n] \longmapsto \coprod_{i_0 \mapsto \cdots \mapsto i_n} F(i_0).$$

Show that the restriction of the map (5.11.1) to the n -skeleton is a quasi-fibration for every n .

Hint: Use induction on n . First prove that the map also is a quasi-fibration in a certain open neighborhood of the $(n-1)$ -skeleton. Then use the previous exercise.

- (4) Deduce the proposition.

CHAPTER 6

Localization and completion

In this chapter, we will construct localizations and completions for simply connected spaces via a quick-and-dirty explicit construction, working only in the homotopy category. After this we will discuss more sophisticated approaches.

6.1. Localizations and reflective subcategories

Before we start we will need to do a bit of abstract yoga. Recall from Proposition 4.1.4, that given a small category \mathcal{C} we can form a category $\mathcal{C}[\mathcal{W}^{-1}]$, which is the category where we formally invert the morphisms in \mathcal{W} , i.e., turn them into isomorphisms. We can also do this when \mathcal{C} is not small, although in this case it is not a priori clear that $\mathcal{C}[\mathcal{W}^{-1}]$ is locally small (i.e., the “hom sets” may not be sets).

DEFINITION 6.1.1. Let \mathcal{W} be a collection of morphisms (not necessarily a set) in a category \mathcal{C} . We say an object $c \in \mathcal{C}$ is \mathcal{W} -local if for every morphism $\phi: x \rightarrow y$ in \mathcal{W} , the map

$$\mathrm{Hom}_{\mathcal{C}}(y, c) \xrightarrow{\phi^*} \mathrm{Hom}_{\mathcal{C}}(x, c)$$

is an isomorphism.

REMARK 6.1.2. It does not harm to assume that \mathcal{W} satisfies 2-of-3. To see this, if \mathcal{W} is any collection of morphisms in \mathcal{C} , we can define \mathcal{W}' to be the smallest collection of morphisms in \mathcal{C} that contains \mathcal{W} and satisfies the 2-of-3 property. Then it is immediate to see that an object of \mathcal{C} is \mathcal{W} -local if and only if it is also \mathcal{W}' -local.

We will henceforth assume that \mathcal{W} satisfies the 2-of-3 property.

Moreover, for a collection \mathcal{W} satisfying 2-of-3, we can define a new, possibly larger collection \mathcal{W}'' containing all morphisms $f: x \rightarrow y$ inducing for all \mathcal{W} -local c a bijection $\mathrm{Hom}_{\mathcal{C}}(y, c) \cong \mathrm{Hom}_{\mathcal{C}}(x, c)$. Then it is immediate to see that an object of \mathcal{C} is \mathcal{W} -local if and only if it is also \mathcal{W}'' -local.

We will henceforth assume that \mathcal{W} is “saturated” in the sense that it coincides with \mathcal{W}'' .

DEFINITION 6.1.3. For $x \in \mathcal{C}$, a \mathcal{W} -localization of x is a morphism $\lambda_x: x \rightarrow Lx$ where Lx is a \mathcal{W} -local object of \mathcal{C} and the map λ_x lies in \mathcal{W} .

We first note that maps in \mathcal{W} between \mathcal{W} -local objects are isomorphisms:

PROPOSITION 6.1.4. *If x, y are \mathcal{W} -local and $\phi: x \rightarrow y$ is in \mathcal{W} , then ϕ is an isomorphism. In particular, if x is \mathcal{W} -local, then a \mathcal{W} -localization $\lambda_x: x \rightarrow Lx$ of x is an isomorphism.*

PROOF. By the isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(y, x) \xrightarrow{\phi^*} \mathrm{Hom}_{\mathcal{C}}(x, x)$$

there exists $f: y \rightarrow x$ such that $f \circ \phi = 1_x$. Hence under the isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(y, y) \xrightarrow{\phi^*} \mathrm{Hom}_{\mathcal{C}}(x, y),$$

both $\phi \circ f$ and 1_y go to ϕ , and are thus equal; so f is a right inverse of ϕ as well.

(Or slightly less directly one may appeal to our old friend the Yoneda lemma, observing that $\mathrm{Hom}_{\mathcal{C}_W}(y, -) \xrightarrow{\phi^*} \mathrm{Hom}_{\mathcal{C}_W}(x, -)$ is an isomorphism of representable functors, with \mathcal{C}_W the full subcategory of \mathcal{C} generated by \mathcal{W} -local objects.) \square

LEMMA 6.1.5. *Suppose $x \rightarrow Lx$ is a \mathcal{W} -localization.*

(i) *For any $f: x \rightarrow y$ with y \mathcal{W} -local there exists a unique $f': Lx \rightarrow y$ such that*

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ \lambda_x \downarrow & \nearrow f' & \\ Lx & & \end{array}$$

commutes.

(ii) *For any map $f: x \rightarrow y$ in \mathcal{W} there exists a unique map $\ell: y \rightarrow Lx$ such that*

$$\begin{array}{ccc} x & \xrightarrow{\lambda_x} & Lx \\ f \downarrow & \nearrow \ell & \\ y & & \end{array}$$

commutes.

PROOF. Since y is \mathcal{W} -local and λ_x is in \mathcal{W} , we have an isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(Lx, y) \xrightarrow{\lambda_x^*} \mathrm{Hom}_{\mathcal{C}}(x, y).$$

Then f has a unique preimage f' , which gives (i).

For (ii), observe that since Lx is \mathcal{W} -local and f is in \mathcal{W} , we have an isomorphism $\mathrm{Hom}_{\mathcal{C}}(y, Lx) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{C}}(x, Lx)$. Then λ_x has a unique preimage ℓ . \square

REMARK 6.1.6. We can reformulate Lemma 6.1.5 as: a \mathcal{W} -localization $\lambda_x: x \rightarrow Lx$ is the initial map from x to a \mathcal{W} -local object, and the terminal map in \mathcal{W} out of x . In particular, we see that the \mathcal{W} -localization is unique (up to unique isomorphism) if it exists.

PROPOSITION 6.1.7. *Suppose every object of \mathcal{C} has a \mathcal{W} -localization, and let $\mathcal{C}_{\mathcal{W}}$ denote the full subcategory of \mathcal{C} spanned by the \mathcal{W} -local objects. Then the inclusion $i: \mathcal{C}_{\mathcal{W}} \hookrightarrow \mathcal{C}$ has a left adjoint $L: \mathcal{C} \rightarrow \mathcal{C}_{\mathcal{W}}$, with unit of the adjunction given by $\lambda_x: x \rightarrow Lx$ for a \mathcal{W} -localization of x . The composite*

$$\mathcal{C}_{\mathcal{W}} \rightarrow \mathcal{C} \rightarrow \mathcal{C}[\mathcal{W}^{-1}]$$

is an equivalence of categories with inverse equivalence given by the functor $\mathcal{C}[\mathcal{W}^{-1}] \rightarrow \mathcal{C}_{\mathcal{W}}$ induced by $L: \mathcal{C} \rightarrow \mathcal{C}_{\mathcal{W}}$. In particular $\mathcal{C}[\mathcal{W}^{-1}]$ is a locally small category in this case.¹

PROOF. If $x \rightarrow Lx$ is a \mathcal{W} -localization of x , then the induced isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(x, iy) \cong \mathrm{Hom}_{\mathcal{C}}(Lx, y) \cong \mathrm{Hom}_{\mathcal{C}_{\mathcal{W}}}(Lx, y)$$

is natural in $y \in \mathcal{C}_{\mathcal{W}}$, so the functor $\mathrm{Hom}_{\mathcal{C}}(x, i(-)): \mathcal{C}_{\mathcal{W}} \rightarrow \mathrm{Set}$ is representable (by Lx) for every x . By the Yoneda Lemma it follows that i has a left adjoint with $x \rightarrow Lx$ as the unit map at x .

More explicitly, we can define the functor L on morphisms by taking $L(f)$ for $f: x \rightarrow y$ to be the unique map $Lx \rightarrow Ly$ that factors $\lambda_y \circ f: x \rightarrow Ly$ through Lx . Note also that L sends morphisms in \mathcal{W} to isomorphisms: if $f \in \mathcal{W}$, then by 2-of-3 also the composite $\lambda_y \circ f = Lf \circ \lambda_x$ lies in \mathcal{W} , and again by 2-of-3 also Lf lies in \mathcal{W} ; by Proposition 6.1.4 we then obtain that Lf is an isomorphism.

Now suppose $F: \mathcal{C} \rightarrow \mathcal{D}$ is a functor that takes the morphisms in \mathcal{W} to isomorphisms. Let $F' := F \circ i: \mathcal{C}_{\mathcal{W}} \rightarrow \mathcal{D}$. Then $F \cong F'L = FiL$: the unit transformation $\lambda: \mathrm{id} \rightarrow iL$ is given by maps in \mathcal{W} , so since F takes these maps to isomorphisms, $F\lambda: F \rightarrow FiL = F'L$ is a natural isomorphism. Thus F factors through L up to natural isomorphism.

¹If you do not want to consider “large” categories at all, you can rephrase the statement as: if every object in \mathcal{C} has a \mathcal{W} -localization, then $\mathcal{C}_{\mathcal{W}}$ has the universal property for being $\mathcal{C}[\mathcal{W}^{-1}]$.

To see that this factorization is unique up to *unique* natural isomorphism, suppose we have a natural isomorphism $F \cong GL$ for some $G: \mathcal{C}_W \rightarrow \mathcal{D}$. Then we get $F' = Fi \cong GLi \cong G$ since $Li \cong \text{id}$.

Moreover, if $\eta: F' \xrightarrow{\cong} F'$ is a natural automorphism of F' such that $\eta L: F'L \xrightarrow{\cong} F'L$ is the *identity* natural automorphism of $F'L$, then for $x \in \mathcal{C}_W$ we have $\eta_{Lix} = \text{id}_{F'Lix}$, and the following commutative square, together with the fact that $\epsilon_x: Lix \rightarrow x$ is an isomorphism, implies that $\eta_x = \text{id}_{F'x}$:

$$\begin{array}{ccc} F'Lix & \xrightarrow{\eta_{Lix} = \text{id}_{F'Lix}} & F'Lix \\ \downarrow F'\epsilon_x & & \downarrow F'\epsilon_x \\ F'x & \xrightarrow{\eta_x} & F'x. \end{array}$$

This shows that \mathcal{C}_W satisfies the universal property of $\mathcal{C}[\mathcal{W}^{-1}]$. \square

REMARK 6.1.8. Note that the above identifies the localization functor $\mathcal{C} \rightarrow \mathcal{C}$ as the *composite* of a left adjoint L followed by a right adjoint i . This means that good properties are in general not just “formal”.

REMARK 6.1.9. When $\mathcal{C} = \text{Ho}(\mathcal{D})$ is a homotopy category, we can sometimes construct the localization as a deformation of the “underlying” category \mathcal{D} , in the sense of Chapter 4. This allows us to get a strict functor, rather than just a functor on the homotopy category.

DEFINITION 6.1.10. A full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ such that the inclusion has a left adjoint is called a *reflective* subcategory.

EXAMPLE 6.1.11. Let $S \subseteq \mathbb{Z}$ be a set of primes, and take \mathcal{W} to be the collection of maps $f: A \rightarrow B$ in Ab such that the kernel and cokernel of f is S -torsion (equivalently, $S^{-1}f: S^{-1}A \rightarrow S^{-1}B$ is an isomorphism). Then \mathcal{W} -localizations of objects in Ab exist and are given by the natural map $A \rightarrow S^{-1}A$. (This is just a reformulation of the usual universal property of $S^{-1}A$.)

REMARK 6.1.12. If \mathcal{C} is a very well-behaved category, namely a so-called *presentable* (or sometimes “locally presentable”) category, then \mathcal{W} -localizations always exist for classes of morphisms \mathcal{W} that are in a suitable sense generated by a *set* of morphisms.

6.2. R -localization of spaces

We are interested in \mathcal{W} -localizations in $\text{Ho}(\text{Top})$ of the following type:

DEFINITION 6.2.1. Let R be a commutative ring. We denote by W_R the class of maps $f: X \rightarrow Y$ in $\text{Ho}(\text{Top})$ such that $f_*: H_*(X; R) \rightarrow H_*(Y; R)$ is an isomorphism. We will call the maps in W_R *R -equivalences*, and we will call W_R -local spaces *R -local spaces* and talk about *R -localization* of spaces.

LEMMA 6.2.2. *The class W_R satisfies the assumptions from Remark 6.1.2.*

PROOF. It is clear that $H_*(-; R)$ -equivalences satisfy 2-of-3. For “saturation”, note that Lemma 6.2.6 in the following implies that $K(M, n)$ is R -local for all R -module M and $n \geq 1$. Let $f: X \rightarrow Y$ induce an isomorphism $[Y, Z] \cong [X, Z]$ for all R -local Z ; then in particular f induces an isomorphism $H^*(Y; M) \cong H^*(X; M)$; this is equivalent to the vanishing of $H^*(Y, X; M)$ for all $M \in \text{Mod}_R$ (we replace f by an inclusion via mapping cylinder), i.e. the chain complex of free R -modules $C_*(Y, X; R)$ becomes acyclic under each functor $\text{Hom}_R(-; M)$; this in turn implies that $C_*(Y, X; R)$ is already acyclic (and being levelwise free and bounded below, it is actually chain null-homotopic), so that $H_*(X; R) \cong H_*(Y; R)$ under f , i.e. $f \in W_R$. \square

The category $\mathrm{Ho}(\mathrm{Top})$ is far from being presentable (one requirement is the existence of colimits), so the existence of R -localizations is not formal. Nevertheless, we have the following result of Bousfield:

THEOREM 6.2.3 (Bousfield [Bou75]). *Any space $X \in \mathrm{Ho}(\mathrm{Top})$ admits an R -localization.*

We will not prove this here. Instead, we will give explicit constructions of R -localizations for nice (more precisely, simply connected²) spaces in two important cases:

- $R = \mathbb{Z}_{(p)} := \mathbb{Z}[q^{-1} : q \neq p \text{ prime}]$, for a fixed prime number p ; the $\mathbb{Z}_{(p)}$ -localization is called p -localization, and we write $X_{(p)}$ for the p -localization of a space X . More generally we will consider subrings $R \subseteq \mathbb{Q}$; this gives for example the *rationalization* $X_{\mathbb{Q}}$ of a space, i.e. its \mathbb{Q} -localization.
- $R = \mathbb{F}_p$; \mathbb{F}_p -localization is often called p -completion, and we write X_p^\wedge for the p -completion of X .

The general existence result gives an inexplicit description of the localization that is not useful for computations, so the explicit construction is in any case important. Note that with more work everything we do can be extended from simply connected spaces to *nilpotent* spaces.

REMARK 6.2.4.

- With more care one can construct R -localization as a functor on Top rather than $\mathrm{Ho}(\mathrm{Top})$, which is (very) often convenient since it is much better to have diagrams that commute strictly than only up to homotopy, as we have seen in the previous sections.
- The R -local objects can be shown to be the collection of fibrant objects in a model category, and this is often a convenient way to construct the localization.

REMARK 6.2.5. The name “ p -completion” will make more sense when we understand \mathbb{F}_p -local objects. Note that there is a map from the p -localization to the p -completion (as well as to the rationalization) induced by the maps $\mathbb{Z}_{(p)} \rightarrow \mathbb{F}_p$ and $\mathbb{Z}_{(p)} \rightarrow \mathbb{Q}$.

In general, for $X \rightarrow Y$ a map of spaces and $R \rightarrow S$ a map of commutative rings, if $H_*(X; R) \xrightarrow{\cong} H_*(Y; R)$ then $H_*(X; S) \xrightarrow{\cong} H_*(Y; S)$; this implies in turn that an S -local space is also R -local, and hence, for a space X , the initial map $X \rightarrow X_S$ in $\mathrm{Ho}(\mathrm{Top})$ to an S -local space factors through the initial map $X \rightarrow X_R$ in $\mathrm{Ho}(\mathrm{Top})$ to an R -local space. See also Lemma 6.6.12 in the following.

The p -localization is easier to define than the p -completion, but the p -completion is in general more computable and therefore more useful. This is related to the fact that we have more methods for calculating maps into p -complete spaces — they are “more algebraic” in the sense that p -completions can be expressed as limits of $K(\mathbb{F}_p, n)$ ’s for different values of n (in fact in several different ways). This ultimately gives us ways of understanding the maps into p -complete spaces in terms of mod p cohomology.

We will also examine how a space X can be recovered from these localizations. This is called Sullivan’s arithmetic square or, as Sullivan writes, a Hasse principle for spaces. There is both a Sullivan square for p -localization and one for p -completion.

We will now consider some examples of R -local spaces:

LEMMA 6.2.6. *Let R be a commutative ring. Then the Eilenberg-MacLane space $K(M, n)$ is R -local for any R -module M .*

PROOF. Let $f: X \rightarrow Y$ be an R -local map of spaces, i.e. $f_*: H_*(X; R) \rightarrow H_*(Y; R)$ is an isomorphism; we want to prove that f induces a bijection $[Y, K(M, n)] \xrightarrow{\cong} [X, K(M, n)]$ between homotopy classes of maps into $K(M, n)$. We have $[X, K(M, n)] \cong H^n(X; M)$, so our goal is to prove that f induces a bijection on $H^n(-; M)$.

²Most of our arguments work in fact without changes for generic *simple* spaces, of which simply connected ones are a subclass. For the even larger class of *nilpotent* spaces more work is needed.

Up to replacing Y by the mapping cylinder of f , we can assume that f is a cofibration, and hence we can consider (Y, X) as a pair of spaces. The hypothesis and the LES in $H_*(-; R)$ -homology imply that $H_*(Y, X; R) = 0$ in all degrees.

Now $H_*(Y, X; R)$ is computed by $C_*(Y, X; R)$, i.e. the relative, singular chain complex of (Y, X) with coefficients in R : this is a non-negatively graded chain complex of *free* R -modules (note also that the boundary maps are R -linear), and since it is acyclic, it has to be chain null-homotopic.

Applying the additive functor $\text{Hom}_R(-; M)$ to $C_*(Y, X; R)$, we obtain precisely the cochain complex $C^*(Y, X; M)$: in this step it is good to notice that $C_*(Y, X; R)$ is itself $C_*(Y, X; \mathbb{Z}) \otimes R$. Hence also $C^*(Y, X; M)$ is chain null-homotopic, and in particular $H^*(Y, X; M) = 0$ in all degrees. The LES for $H^*(-; M)$ implies in degree n the desired statement. \square

LEMMA 6.2.7. *The spaces $K(\mathbb{Z}/p^k, n)$ and $K(\mathbb{Z}_p^\wedge, n)$ are \mathbb{F}_p -local.*

PROOF. To see this for $K(\mathbb{Z}/p^k, n)$ we induct on k (note that the case $k = 1$ follows from Lemma 6.2.6) and use the long exact sequence in cohomology induced by the short exact sequence of groups

$$0 \rightarrow \mathbb{Z}/p^{k-1} \rightarrow \mathbb{Z}/p^k \rightarrow \mathbb{Z}/p \rightarrow 0$$

plus the 5-Lemma. For \mathbb{Z}_p^\wedge we use the Milnor \lim^1 -sequence, which is

$$0 \rightarrow \lim_k^1 H^{n-1}(X; \mathbb{Z}/p^k) \rightarrow H^n(X; \mathbb{Z}_p^\wedge) \rightarrow \lim_k H^n(X; \mathbb{Z}/p^k) \rightarrow 0,$$

to see we get an isomorphism on \mathbb{Z}_p^\wedge -cohomology. \square

6.3. p -localization of simply connected spaces

Suppose that $R \subseteq \mathbb{Q}$ — this implies that $R = S^{-1}\mathbb{Z}$ for some set S of primes. Homology with R -coefficients for such R is easy to describe: we have $H_*(-; R) = H_*(-) \otimes R$ by the Universal Coefficient Theorem, since the functor $- \otimes R$ is exact for R a torsion-free abelian group. Furthermore, we have an alternative description of R -equivalences between simply connected spaces:

LEMMA 6.3.1. *Suppose $R \subseteq \mathbb{Q}$. Let $f : X \rightarrow Y$ be a map between simply connected spaces. Then f is an R -equivalence if and only if it induces an isomorphism $\pi_*(X) \otimes R \rightarrow \pi_*(Y) \otimes R$.*

PROOF. This follows from the Whitehead theorem modulo Serre Classes, cf Theorem 6.3.2 below, taking as Serre class the class of abelian groups A such that $A \otimes R = 0$. \square

THEOREM 6.3.2 (Whitehead theorem modulo Serre classes). *Let $f : X \rightarrow Y$ be a map between simply connected spaces, and \mathcal{C} a Serre class. Then $\pi_*(f)$ is an isomorphism modulo \mathcal{C} if and only if $H_*(f)$ is an isomorphism modulo \mathcal{C} .*

PROOF. This is almost in Hatcher [Hat], (cf. [Hat, Thm. 5.8]) but not stated explicitly.

Recall that a Serre class \mathcal{C} is a class of abelian groups containing 0 and closed under isomorphisms, taking subgroups, quotients, extensions, arbitrary filtered colimits, tensor products against arbitrary abelian groups and Tor against arbitrary abelian groups. Examples of Serre classes are finite groups, and S -power torsion groups for a subset $S \subset \mathbb{Z}$ of prime numbers (this includes the class of *all* torsion abelian groups). We note that the intersection of a Serre class \mathcal{C} with finitely generated abelian groups can only be one of the following:

- all f.g. abelian groups;
- for a set of primes $S \subset \mathbb{Z}$, all finite abelian groups with only S -power torsion.

Using the classification of f.g. abelian groups, knowledge about homology of cyclic groups (including $\mathbb{Z} \dots$) and the Künneth theorem, one can prove the following statement (prove it yourself!): if $M \in \mathcal{C}$ is f.g., then the group homology $\tilde{H}_*(M; \mathbb{Z}) = \tilde{H}_*(K(M; 1); \mathbb{Z})$ is degree-wise in \mathcal{C} (we write $\tilde{H}_*(K(M; 1); \mathbb{Z}) \subset \mathcal{C}$).

We then note that a generic $M \in \mathcal{C}$ can be written as filtered colimit of its f.g. subgroups, which also lie in \mathcal{C} ; correspondingly $\tilde{H}_*(K(M; 1); \mathbb{Z})$ is the filtered colimit of groups in \mathcal{C} , hence is in \mathcal{C} as well.

We then prove for $n \geq 2$ and $M \in \mathcal{C}$ that $\tilde{H}_*(K(M; n); \mathbb{Z}) \subset \mathcal{C}$. For this we use the fibration sequence $K(M; n-1) \simeq \Omega K(M; n) \rightarrow * \simeq PK(M; n) \rightarrow K(M; n)$ and the associated Serre spectral sequence in (unreduced) homology. On the second page we have $H_p(K(M; n); H_q(K(M; n-1); \mathbb{Z}))$: this is $H_p(K(M; n); \mathbb{Z})$ for $q = 0$, and is in \mathcal{C} for $q \geq 1$ and any p , using homology universal coefficients, Tor and tensor stability of \mathcal{C} , and the inductive hypothesis $\tilde{H}_*(K(M; n-1); \mathbb{Z}) \subset \mathcal{C}$. The spectral sequence converges to $0 \in \mathcal{C}$ in all places except $p = q = 0$ (which is why the final statement only works for reduced homology); using this, one can exhibit each $H_p(K(M; n); \mathbb{Z})$ (with $p \geq 1$) as an iterated extension of groups in \mathcal{C} , hence a group in \mathcal{C} as well.

Once these preliminary steps are done, we are ready to start the actual proof. We say that a map of abelian groups $\phi: A \rightarrow B$ is an “isomorphism mod \mathcal{C} ” if $\ker(\phi)$ and $\operatorname{coker}(\phi)$ are in \mathcal{C} . Given a LES of abelian groups of the form

$$\cdots \rightarrow A_n \rightarrow B_n \rightarrow C_n \rightarrow A_{n-1} \rightarrow B_{n-1} \rightarrow C_{n-1} \rightarrow \cdots$$

we observe that asking all $A_n \in \mathcal{C}$ is equivalent to asking all maps $B_n \rightarrow C_n$ be isomorphisms mod \mathcal{C} .

Let now $f: X \rightarrow Y$ be a map of simply connected spaces, and assume first that $\pi_*(f)$ is an isomorphism mod \mathcal{C} ; let Z be the homotopy fibre of f (for a reminder on homotopy fibres, see Definition 6.3.14). The LES of homotopy groups tells us that Z is connected, $\pi_1(Z)$ is abelian and $\pi_*(Z) \subset \mathcal{C}$. We first want to prove that $\tilde{H}_*(Z; \mathbb{Z}) \subset \mathcal{C}$; to prove that $H_i(Z; \mathbb{Z}) \in \mathcal{C}$ for a fixed $i \geq 1$, we can actually replace Z by the Postnikov truncation $P_i Z$ (see Proposition 6.3.5 for a reminder on the Postnikov tower), using that $Z \rightarrow P_i Z$ is $(i+1)$ -connected, in particular it is an isomorphism on $H_i(-; \mathbb{Z})$; and $P_i Z$ enjoys all listed properties of Z , plus that $\pi_*(P_i Z) = 0$ for $* \geq i+1$.

Now we need the following lemma, left as exercise: if $F \rightarrow E \rightarrow B$ is a fibre sequence of connected spaces, with B simply connected, and $\tilde{H}_*(B), \tilde{H}_*(F) \subset \mathcal{C}$, then also $\tilde{H}_*(E) \subset \mathcal{C}$; this is an application of the Serre spectral sequence (notice that the very definition of Serre class is designed so that this argument goes through!). Since $P_i Z$ is a (finite) iterated tower of fibrations with fibres of the form $K(M; n)$ with $M \in \mathcal{C}$ and $n \leq i$, we obtain that $H_i(P_i Z; \mathbb{Z}) \cong H_i(Z; \mathbb{Z}) \in \mathcal{C}$ as desired.

Once we have proved that $\tilde{H}_*(Z; \mathbb{Z}) \subset \mathcal{C}$, we can run the Serre spectral sequence for $Z \rightarrow X \rightarrow Y$: on the E^2 -page we see $H_*(Y; \mathbb{Z})$ on the bottom line $E_{0,*}^2$, and above, on $E_{p,q}^2$, there is only some “noise” in \mathcal{C} ; the limit is $H_*(X; \mathbb{Z})$, and the effect of passing to the limit on the bottom row is, modulo noise in \mathcal{C} , the map $f_*: H_*(X; \mathbb{Z}) \rightarrow H_*(Y; \mathbb{Z})$, which is hence an equivalence mod \mathcal{C} .

Viceversa, suppose $f: X \rightarrow Y$ is an $H_*(-; \mathbb{Z})$ -isomorphism mod \mathcal{C} , replace f by an inclusion via mapping cylinder, and reinterpret the hypothesis as $H_*(Y, X) \subset \mathcal{C}$. Let $i \geq 2$ be the minimal degree (if any) in which $\pi_*(f)$ is not an isomorphism.

If $\pi_i(f)$ is not surjective, let $M_1 = \operatorname{coker}(\pi_i(f))$, and note that the relative Hurewicz theorem identifies this with a subgroup of $\pi_i(Y, X) \cong H_i(Y, X) \in \mathcal{C}$, so $M_1 \in \mathcal{C}$. We can then consider the natural map $Y \rightarrow K(M_1, i)$ and denote Y' its homotopy fibre; a Serre spectral sequence argument shows that $Y' \rightarrow Y$ is a H_* -isomorphism mod \mathcal{C} , and it is clearly also a π_* -isomorphism mod \mathcal{C} ; moreover $f: X \rightarrow Y$ factors up to homotopy through a map $f': X \rightarrow Y'$, which is a H_* -isomorphism mod \mathcal{C} by 2-of-3. We can thus just aim at proving that f' is a π_* -isomorphism mod \mathcal{C} , and we have basically reduced to the case in which $\pi_i(f)$ is surjective.

If $\pi_i(f)$ is not injective (but surjective), denote $M_2 = \ker(\pi_i(f))$, and note that relative Hurewicz identifies M_2 with a quotient of $\pi_{i+1}(Y, X) \cong H_{i+1}(Y, X) \in \mathcal{C}$. Let Y'' be the fibre product of Y and $K(\pi_i(X), i)$ over $K(\pi_i(Y), i)$, and note that $f: X \rightarrow Y$ factors through a

map $f'': X \rightarrow Y''$. Moreover the homotopy fibre of $Y'' \rightarrow Y$ is a $K(M_2, i)$, which again by a Serre spectral sequence argument implies that $Y'' \rightarrow Y$ is both a H_* -isomorphism mod \mathcal{C} and a π_* -isomorphism mod \mathcal{C} . This in turn implies that we can ask ourselves whether f'' , which we know is a H_* -isomorphism mod \mathcal{C} by 2-of-3, is also a π_* -isomorphism mod \mathcal{C} . And now we have reduced to the case in which $\pi_i(f)$ is also injective, hence an isomorphism.

By a sequence of such replacements, for any fixed $i \geq 2$ we can reduce to the case in which we assume that f is a π_* -isomorphism for $* \leq i$ and a H_* -isomorphism mod \mathcal{C} in all degrees, and we want to prove that $\pi_i(f)$ is an isomorphism mod \mathcal{C} : now this is really obvious! \square

Our first main result gives a characterization of R -local spaces:

THEOREM 6.3.3. *Suppose X is simply connected and $R \subseteq \mathbb{Q}$. Then X is R -local if and only if $\pi_n(X)$ is R -local for all n .*

Let us first prove the easy direction:

LEMMA 6.3.4. *Suppose $R \subseteq \mathbb{Q}$ and X is a simply connected R -local space. Then $\pi_n(X)$ is R -local for all n .*

PROOF. The degree- p map $S^n \xrightarrow{p} S^n$ is in W_R for all primes p that are invertible in R : We have $H_*(S^n; R) \cong H_*(S^n) \otimes R$, and on $H_*(S^n)$ this map is given by multiplication by p . Thus if X is R -local, this map induces an isomorphism on $[S^n, X] = \pi_n(X)$, i.e. $\pi_n(X)$ is R -local. \square

For the less trivial direction, we will use an induction going up the Postnikov tower, so we first briefly review this:

PROPOSITION 6.3.5 (Postnikov towers). *Suppose that X is a simply connected CW complex. Then there exists a tower of principal fibrations*

$$\cdots \rightarrow P_n X \rightarrow P_{n-1} X \rightarrow P_1 X$$

and a map from X into the tower, satisfying the following:

- $X \rightarrow \lim_n P_n X$ is a homotopy equivalence,
- $\pi_i(X) \rightarrow \pi_i(P_n X)$ is an isomorphism for $i \leq n$ and $\pi_i(P_n X) = 0$ for $i > n$.

See for example [Hat02, Theorem 4.69] for a proof. Here a fibration $F \rightarrow E \rightarrow B$ is called *principal* if we have a commutative diagram

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ \Omega X & \longrightarrow & E' & \longrightarrow & B' \longrightarrow X, \end{array}$$

where $B' \rightarrow X$ is a fibration, the vertical maps are weak equivalences, and the bottom row is the start of a Puppe sequence. For the principal fibration $P_n X \rightarrow P_{n-1} X$ it follows from the long exact sequence that the fibre is a $K(\pi_n X, n)$, so the “classifying space” X has to be $K(\pi_n X, n+1)$.³

Thus we have a homotopy pullback square

$$\begin{array}{ccc} P_n X & \longrightarrow & P_{n-1} X \\ \downarrow & & \downarrow k_{n-1} \\ * & \longrightarrow & K(\pi_n X, n+1). \end{array}$$

³As exercise, prove that the classifying space $B\mathrm{hAut}(K(A, n))$ for $K(A, n)$ -fibrations is homotopy equivalent to the homotopy quotient $K(A, n+1)//\mathrm{Aut}(A)$; then *principal* $K(A, n)$ -fibrations are precisely those whose classifying map lifts to $K(A, n+1)$.

The maps $k_{n-1}: P_{n-1}X \rightarrow K(\pi_n X, n+1)$ correspond to cohomology classes $k_{n-1} \in H^{n+1}(P_{n-1}X; \pi_n X)$ called the k -invariants of X .

REMARK 6.3.6. For a general space the Postnikov tower still exists as a tower of fibrations, but in general they do not have to be *principal*. In fact, it can be shown that the fibrations in the Postnikov tower are principal if and only if the space X is *simple*, which means that $\pi_1 X$ is abelian and acts trivially on $\pi_n X$. (The homotopy fibre of a map of simply connected spaces is always simple; this example will come up later.) More generally, Postnikov towers are also well-behaved for *nilpotent* spaces, meaning spaces for which the fundamental group is nilpotent (in the sense of group theory) and acts nilpotently on the higher homotopy groups; but this is a bit more complicated.

DEFINITION 6.3.7. Let \mathcal{W} be a class of maps in $\text{Ho}(\text{Top})$ as in Remark 6.1.2. We say that \mathcal{W} is *product invariant* if the following holds: whenever $f: X \rightarrow Y$ is a map in \mathcal{W} and S is a CW complex, then $S \times f: S \times X \rightarrow S \times Y$ is in \mathcal{W} as well.

For instance, by the Künneth theorem we have that for any commutative ring R the collection W_R of $H_*(-; R)$ -equivalences is product invariant. If you only know the basic Künneth statement saying that $H_*(X \times Y; R) \cong H_*(X; R) \otimes_R H_*(Y; R)$ if either $H_*(X; R)$ or $H_*(Y; R)$ is a free R -module, this is enough! Just filter S by skeleta, prove that if $f: X \rightarrow Y$ is a $H_*(-; R)$ -isomorphism then also $H_*(\text{sk}_i S \times X, \text{sk}_{i-1} S \times X; R) \cong H_*(\text{sk}_i S \times Y, \text{sk}_{i-1} S \times Y; R)$, and conclude via repeated 5-lemma. See also Lemma 6.6.7. And compare the following with Lemma 6.6.8.

LEMMA 6.3.8. *Let \mathcal{W} be a class of maps as in Remark 6.1.2. Then the following are equivalent:*

- (1) \mathcal{W} is product invariant;
- (2) for any $f: X \rightarrow Y$ in \mathcal{W} and any \mathcal{W} -local space Z , the induced map between mapping spaces $\text{map}(f, Z): \text{map}(Y, Z) \rightarrow \text{map}(X, Z)$ is a weak equivalence.

PROOF. (1) \Rightarrow (2). Fix $f: X \rightarrow Y$ in \mathcal{W} and a \mathcal{W} -local space Z . By the Yoneda lemma, it suffices to show that composing with f induces a natural isomorphism of functors $[-, \text{map}(Y, Z)]$ and $[-, \text{map}(X, Z)]$ from CW to Set. However, for S a CW complex we have $\text{map}(S, \text{map}(Y, Z)) \cong \text{map}(S \times Y, Z)$, and similarly for X , hence, taking components, we have a bijection $[S, \text{map}(Y, Z)] \cong [S \times Y, Z]$, and similarly for X ; we can now use the hypotheses on f and Z to get the bijection $[S \times Y, Z] \cong [S \times X, Z]$, which translates to the bijection $[S, \text{map}(Y, Z)] \cong [S, \text{map}(X, Z)]$.

(2) \Rightarrow (1). Fix again $f: X \rightarrow Y$ in \mathcal{W} and a \mathcal{W} -local space Z , and fix $S \in \text{CW}$. Then again we can convert the given bijection $[S, \text{map}(Y, Z)] \cong [S, \text{map}(X, Z)]$ into $[S \times Y, Z] \cong [S \times X, Z]$; since Z is arbitrary and \mathcal{W} is assumed to be “saturated” as in Remark 6.1.2, we obtain that $S \times f$ is also in \mathcal{W} . \square

Using this we have:

LEMMA 6.3.9 (Fibre Lemma). *Suppose $E \rightarrow B$ is a fibration where E and B are \mathcal{W} -local, and where \mathcal{W} is a product invariant class of maps. Then the fibre F is also \mathcal{W} -local.*

PROOF. By Lemma 6.3.8, given $f: X \rightarrow Y$ in \mathcal{W} , we have a commutative diagram whose vertical arrows are weak equivalences

$$\begin{array}{ccc} \text{map}(Y, E) & \xrightarrow{p_Y} & \text{map}(Y, B) \\ \downarrow \simeq & & \downarrow \simeq \\ \text{map}(X, E) & \xrightarrow{p_X} & \text{map}(X, B) \end{array}$$

It follows that, for any $* \in B$, the homotopy fibres of p_Y and p_X over the constant maps $Y \rightarrow * \hookrightarrow B$ and $X \rightarrow * \hookrightarrow B$, respectively, are homotopy equivalent: these homotopy fibres

are however also homotopy equivalent to the actual fibres of the fibrations p_Y and p_X , that is $\text{map}(Y, F)$ and $\text{map}(X, F)$, respectively, where F is the fibre of $E \rightarrow B$ over $*$. Using again Lemma 6.3.8, we conclude that F is \mathcal{W} -local. \square

LEMMA 6.3.10 (Tower Lemma). *Suppose $\cdots \rightarrow Z_2 \rightarrow Z_1 \rightarrow Z_0$ is a tower of fibrations between \mathcal{W} -local spaces, where \mathcal{W} is a product invariant class of maps. Then $\lim_{n \rightarrow \infty} Z_n$ is also \mathcal{W} -local.*

PROOF. Using the characterization from Lemma 6.3.8, it suffices to prove that, for $f: X \rightarrow Y$ in \mathcal{W} , we have a weak equivalence $\text{map}(Y, \lim_{n \rightarrow \infty} Z_n) \simeq \text{map}(X, \lim_{n \rightarrow \infty} Z_n)$. However we can replace $\text{map}(Y, \lim_{n \rightarrow \infty} Z_n) \cong \lim_{n \rightarrow \infty} \text{map}(Y, Z_n)$, and similarly for X . We conclude by noting that $\lim_{n \rightarrow \infty} \text{map}(Y, Z_n)$ is also the limit of a tower of fibrations, and $\lim_{n \rightarrow \infty} \text{map}(X, Z_n)$ is the limit of a levelwise weakly equivalent tower: the two limits are therefore weakly equivalent. \square

We can now complete the proof of the Theorem:

PROOF OF THEOREM 6.3.3. It remains to prove that if a simply connected space X has R -local homotopy groups, then X is R -local. We will first show by induction that the spaces $P_n X$ in the Postnikov tower of X are R -local. Since X is simply connected, the first of these is $P_2 X$ which is $K(\pi_2 X, 2)$; this is R -local by Lemma 6.2.6. Assuming $P_{n-1} X$ is R -local, we have a homotopy pullback square

$$\begin{array}{ccc} P_n X & \longrightarrow & P_{n-1} X \\ \downarrow & & \downarrow k_{n-1} \\ * & \longrightarrow & K(\pi_n X, n+1). \end{array}$$

Here $K(\pi_n X, n+1)$ is again R -local by Lemma 6.2.6, so $P_n X$ is R -local by Lemma 6.3.9. Then $X \simeq \lim_n P_n X$ is R -local by Lemma 6.3.10. \square

EXAMPLE 6.3.11. Let $J = \{p_1, p_2, \dots\}$ be the set of all primes except p . Then the space

$$S_{(p)}^n := \text{hocolim}(S^n \xrightarrow{p_1} S^n \xrightarrow{p_1 p_2} S^n \xrightarrow{p_1 p_2 p_3} \dots)$$

is a p -localization of S^n : We can describe this homotopy colimit by replacing the maps by cofibrations and then taking the sequential colimit. Since π_* commutes with sequential colimits along cofibrations (this by a compactness argument), we get

$$\pi_*(S_{(p)}^n) \cong \text{colim}_k \pi_*(S^n) \cong \pi_*(S^n)_{(p)}.$$

Thus $S_{(p)}^n$ is p -local by Theorem 6.3.3, and the first-inclusion map $S^n \rightarrow S_{(p)}^n$ is a $\mathbb{Z}_{(p)}$ -equivalence by Lemma 6.3.1.

THEOREM 6.3.12. *Suppose X is simply connected and $R \subseteq \mathbb{Q}$. Then the R -localization $X \rightarrow X_R$ exists and is characterized as the unique (up to homotopy) map $X \rightarrow Y$ to a simply connected Y that induces an isomorphism $\pi_*(X) \otimes R \xrightarrow{\sim} \pi_*(Y)$.*

We will again prove this by an induction using the Postnikov tower. The base case is the following:

LEMMA 6.3.13. *Suppose M is an abelian group and $R \subseteq \mathbb{Q}$. Then the map $K(M, n) \rightarrow K(M \otimes R, n)$ exhibits $K(M \otimes R, n)$ as the R -localization of $K(M, n)$.*

PROOF. The space $K(M \otimes R, n)$ is R -local by Theorem 6.3.3 (or just by Lemma 6.2.6, since $M \otimes R$ is an R -module), and the map $K(M, n) \rightarrow K(M \otimes R, n)$ is an R -equivalence by Lemma 6.3.1. \square

Since the universal property of R -localizations only produces homotopy commutative squares, for the induction we will need an observation about homotopy fibres:

DEFINITION 6.3.14. The homotopy fiber of a based map $f : (X, x) \rightarrow (Y, y)$ is defined as the space $F_f = X \times_Y PY$, where PY is the path space consisting of paths $p : I \rightarrow Y$ such that $p(0) = y$; the pullback is taken along f and the map $PY \rightarrow Y$ given by evaluation at 1. Note that the projection map $F_f \rightarrow X$ is a fibration, since $PY \rightarrow Y$ is, and that F_f can be viewed as the actual fiber of the fibration $X \times_Y Y^I \rightarrow Y$, given by evaluating p at 0; moreover the map $X \rightarrow X \times_Y Y^I$ induced by the constant-path map $Y \rightarrow Y^I$ is a homotopy equivalence.

(See also the discussion after [Hat02, Prop, 4.64].)

LEMMA 6.3.15. *Suppose that we have a homotopy-commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow \alpha & & \downarrow \beta \\ Z & \xrightarrow{g} & W \end{array}$$

Then there exists a map $h : F_f \rightarrow F_g$ such that we have a diagram

$$\begin{array}{ccccccc} \Omega Y & \longrightarrow & F_f & \longrightarrow & X & \xrightarrow{f} & Y \\ \downarrow \Omega \beta & & \downarrow h & & \downarrow \alpha & & \downarrow \beta \\ \Omega W & \longrightarrow & F_g & \longrightarrow & Z & \xrightarrow{g} & W \end{array}$$

where the first and third squares commute up to homotopy and the second square commutes strictly.

For a proof see for example [MP12, Lemma 1.2.3.].

PROOF OF THEOREM 6.3.12. Note that it suffices to construct a map of spaces $X \rightarrow X'$ such that $\pi_*(X) \otimes R \rightarrow \pi_*(X')$ is an isomorphism — then X' is R -local by Theorem 6.3.3, and the map $X \rightarrow X'$ is an R -equivalence by Lemma 6.3.1. We construct such a map by going up the Postnikov tower. Since X is simply connected, the base case is $P_2X = K(\pi_2X, 2)$ where $P_2X \rightarrow (P_2X)_R$ is given by $K(\pi_2X, 2) \rightarrow K(\pi_2X \otimes R, 2)$ by Lemma 6.3.13. Now suppose we have an R -localization $P_{n-1}X \rightarrow (P_{n-1}X)_R$. The composite $k_{n-1} : P_{n-1}X \rightarrow K(\pi_nX, n+1) \rightarrow K(\pi_nX \otimes R, n+1)$ is a map from $P_{n-1}X$ to an R -local space, so by the universal property of R -localization in $\text{Ho}(\text{Top})$ there is a unique commutative square

$$\begin{array}{ccc} P_{n-1}X & \longrightarrow & (P_{n-1}X)_R \\ \downarrow k_{n-1} & & \downarrow k_{n-1,R} \\ K(\pi_nX, n+1) & \longrightarrow & K(\pi_nX \otimes R, n+1). \end{array}$$

By Lemma 6.3.15 there is then a diagram

$$\begin{array}{ccccccc} K(\pi_nX, n) & \longrightarrow & F_{k_{n-1}} & \longrightarrow & P_{n-1} & \xrightarrow{k_{n-1}} & K(\pi_nX, n+1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K(\pi_nX \otimes R, n) & \longrightarrow & F_{k_{n-1,R}} & \longrightarrow & (P_{n-1}X)_R & \xrightarrow{k_{n-1,R}} & K(\pi_nX \otimes R, n+1), \end{array}$$

where the middle square commutes strictly. The homotopy fibre $F_{k_{n-1}}$ is homotopy equivalent to P_nX , and from the fibration sequence we see that $(P_nX)_R := F_{k_{n-1,R}}$ is an R -localization of P_nX . Now we define $X_R := \lim_n (P_nX)_R$. This is R -local by Lemma 6.3.10, and $\pi_i X_R \cong \pi_i((P_nX)_R) \cong \pi_i X \otimes R$ (for $n \geq i$; since the homotopy groups of $(P_nX)_R$ stabilize we have no \lim^1 -term). \square

6.4. Sullivan's arithmetic square for p -localization

In this section we construct Sullivan's arithmetic square for p -localization, which is easy.

THEOREM 6.4.1 (Sullivan's arithmetic square for p -localization). *Suppose X is simply connected. Then the canonical square*

$$\begin{array}{ccc} X & \longrightarrow & \prod_p X_{(p)} \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & \left(\prod_p X_{(p)} \right)_{\mathbb{Q}} \end{array}$$

is a homotopy pullback.

REMARK 6.4.2. The bottom horizontal map in the diagram is the \mathbb{Q} -localization of the top map, so the diagram commutes — strictly if we have a functorial model for \mathbb{Q} -localization, otherwise just in the homotopy category.

For the proof we need an algebraic lemma:

LEMMA 6.4.3. *Let M be an abelian group. Then the commutative square*

$$\begin{array}{ccc} M & \longrightarrow & \prod_p M \otimes \mathbb{Z}_{(p)} \\ \downarrow & & \downarrow \\ M \otimes \mathbb{Q} & \longrightarrow & \left(\prod_p M \otimes \mathbb{Z}_{(p)} \right) \otimes \mathbb{Q} \end{array}$$

gives isomorphisms on the kernels and cokernels of the rows.

PROOF. We first observe that it suffices to prove that this holds after localizing at all primes q : A map of abelian groups $f: A \rightarrow B$ is an isomorphism if and only if the maps $f \otimes \mathbb{Z}_{(q)}$ are isomorphisms for all primes q , and since $- \otimes \mathbb{Z}_{(q)}$ is exact this implies we get an isomorphism on (co)kernels in the square if and only if we get such isomorphisms after tensoring the square with $\mathbb{Z}_{(q)}$ for any q .

We can rewrite the square as

$$\begin{array}{ccc} M & \longrightarrow & (M \otimes \mathbb{Z}_{(q)}) \oplus \prod_{p \neq q} M \otimes \mathbb{Z}_{(p)} \\ \downarrow & & \downarrow \\ M \otimes \mathbb{Q} & \longrightarrow & (M \otimes \mathbb{Q}) \oplus \left(\prod_{p \neq q} M \otimes \mathbb{Z}_{(p)} \right) \otimes \mathbb{Q} \end{array}$$

Tensoring this with $\mathbb{Z}_{(q)}$ now gives

$$\begin{array}{ccc} M \otimes \mathbb{Z}_{(q)} & \longrightarrow & (M \otimes \mathbb{Z}_{(q)}) \oplus \left(\prod_{p \neq q} M \otimes \mathbb{Z}_{(p)} \right) \otimes \mathbb{Q} \\ \downarrow & & \downarrow \\ M \otimes \mathbb{Q} & \longrightarrow & (M \otimes \mathbb{Q}) \oplus \left(\prod_{p \neq q} M \otimes \mathbb{Z}_{(p)} \right) \otimes \mathbb{Q} \end{array}$$

(here we use that $\left(\prod_{p \neq q} M \otimes \mathbb{Z}_{(p)} \right) \otimes \mathbb{Z}_{(q)} \cong \left(\prod_{p \neq q} M \otimes \mathbb{Z}_{(p)} \right) \otimes \mathbb{Q}$ as q is already inverted in this product); this square is clearly both a pushout and a pullback, and so it gives isomorphisms on the kernels and cokernels of the rows. \square

We also make use of the following criterion for homotopy pullbacks:

LEMMA 6.4.4. *A commutative square of spaces*

$$\begin{array}{ccc} P & \longrightarrow & Q \\ \downarrow f & & \downarrow g \\ A & \xrightarrow{\phi} & B \end{array}$$

is a homotopy pullback square if and only if the induced map on homotopy fibres $F_f \rightarrow F_g$ is a weak equivalence for all choices of ϕ -related points in A and B .

PROOF. We can replace the maps f and g by fibrations. This reduces us to prove that the map $P \rightarrow \phi^*Q$ of fibrations over A is a weak equivalence if and only if each map of fibres is a weak equivalence, which is immediate from the long exact sequences on homotopy groups and the 5-Lemma, with all possible choices of basepoints. \square

PROOF OF THEOREM 6.4.1. We assume our square commutes strictly — otherwise we can replace it by a weakly equivalent square that does. Let F and F' denote the homotopy fibres of $\alpha: X \rightarrow \prod_p X_{(p)}$ and $\beta: X_{\mathbb{Q}} \rightarrow \left(\prod_p X_{(p)}\right)_{\mathbb{Q}}$, respectively. Then we want to show that the induced map $F \rightarrow F'$ is a weak equivalence. We consider the map of long exact sequences

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \pi_n F & \longrightarrow & \pi_n X & \xrightarrow{\pi_n \alpha} & \pi_n \left(\prod_p X_{(p)}\right) & \longrightarrow & \pi_{n-1} F & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \pi_n F' & \longrightarrow & \pi_n X_{\mathbb{Q}} & \xrightarrow{\pi_n \beta} & \pi_n \left(\prod_p X_{(p)}\right)_{\mathbb{Q}} & \longrightarrow & \pi_{n-1} F' & \longrightarrow & \cdots \end{array}$$

By Theorem 6.3.12 we can identify the middle square with

$$\begin{array}{ccc} \pi_n X & \longrightarrow & \prod_p \pi_n X \otimes \mathbb{Z}_{(p)} \\ \downarrow & & \downarrow \\ \pi_n X \otimes \mathbb{Q} & \longrightarrow & \left(\prod_p \pi_n X \otimes \mathbb{Z}_{(p)}\right) \otimes \mathbb{Q}, \end{array}$$

and by Lemma 6.4.3 we have isomorphisms on the kernels and cokernels of the rows, i.e. the induced maps $\ker \pi_n \alpha \rightarrow \ker \pi_n \beta$ and $\operatorname{coker} \pi_n \alpha \rightarrow \operatorname{coker} \pi_n \beta$ are isomorphisms.

For each n we have a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{coker} \pi_{n+1} \alpha & \longrightarrow & \pi_n F & \longrightarrow & \ker \pi_n \alpha & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \operatorname{coker} \pi_{n+1} \beta & \longrightarrow & \pi_n F' & \longrightarrow & \ker \pi_n \beta & \longrightarrow & 0. \end{array}$$

The 5-Lemma now implies that the map $\pi_n F \rightarrow \pi_n F'$ is an isomorphism, which completes the proof. \square

6.5. p -completion of simply connected spaces I (the easy case)

We now turn to p -completion of spaces, i.e. localization with respect to \mathbb{F}_p . In this section we focus on the case of spaces with finitely generated homotopy groups, which explains where the term “ p -completion” comes from. We will return to the general case below in §6.8 after constructing Sullivan’s arithmetic square in the next section. Our goal in this section is to prove

THEOREM 6.5.1.

- (i) *If X is a simply connected space such that $\pi_n X$ is a finitely generated \mathbb{Z}_p^\wedge -module for all n , then X is p -complete.*

- (ii) If X is a simply connected space with finitely generated homotopy groups, then the p -completion $X \rightarrow X_p^\wedge$ of X exists and is characterized by the map $\pi_* X \otimes \mathbb{Z}_p^\wedge \rightarrow \pi_*(X_p^\wedge)$ being an isomorphism.⁴
- (iii) If X and Y are simply connected spaces with finitely generated homotopy groups, then a map $f: X \rightarrow Y$ is an \mathbb{F}_p -equivalence if and only if $\pi_* f \otimes \mathbb{Z}_p^\wedge$ is an isomorphism.

WARNING 6.5.2. The finiteness hypotheses are essential here — for more general homotopy groups simply taking their p -completion does not give the \mathbb{F}_p -localization.

We can already prove (i):

PROOF OF THEOREM 6.5.1(i). We saw in Lemma 6.2.7 that the Eilenberg-MacLane spaces $K(\mathbb{Z}/p^k, n)$ and $K(\mathbb{Z}_p^\wedge, n)$ are p -complete. A finitely generated \mathbb{Z}_p^\wedge -module M is a finite direct sum of these modules (indeed \mathbb{Z}_p^\wedge is a PID), so it follows that $K(M, n)$ is a finite product of p -complete spaces and hence also p -complete.

We now consider the Postnikov tower of X . The space $P_2 X = K(\pi_2 X, 2)$ is p -complete since $\pi_2 X$ is a finitely generated \mathbb{Z}_p^\wedge -module. And if $P_{n-1} X$ is p -complete then we see that $P_n X$ is p -complete by applying Lemma 6.3.9 to the homotopy pullback square

$$\begin{array}{ccc} P_n X & \longrightarrow & P_{n-1} X \\ \downarrow & & \downarrow k_{n-1} \\ * & \longrightarrow & K(\pi_n X, n+1), \end{array}$$

where $K(\pi_n X, n+1)$ and $P_{n-1} X$ are both p -complete. Since X is weakly equivalent to $\lim_n P_n X$ it follows that X is p -complete using Lemma 6.3.10.⁵ \square

Before we prove part (ii) of the theorem we prove a criterion for a map to be an \mathbb{F}_p -equivalence in terms of the homotopy groups of the homotopy fibre:

THEOREM 6.5.3. *Let $f: X \rightarrow Y$ be a map between simply connected spaces, and let F be the homotopy fibre of f . Then the following are equivalent:*

- (1) f is an \mathbb{F}_p -equivalence;
- (2) $\tilde{H}_*(F; \mathbb{F}_p) = 0$;
- (3) $\tilde{H}_*(F; \mathbb{Z})$ is a $\mathbb{Z}[\frac{1}{p}]$ -module;
- (4) $\pi_*(F)$ is a $\mathbb{Z}[\frac{1}{p}]$ -module.

This will take some work to prove. We start with a 2-of-3 criterion for \mathbb{F}_p -equivalences:

PROPOSITION 6.5.4. *Suppose we are given a commutative square*

$$\begin{array}{ccc} X & \xrightarrow{\psi} & Y \\ \downarrow f & & \downarrow g \\ Z & \xrightarrow{\phi} & W \end{array}$$

where Z and W are simply connected, and let $\eta: F_f \rightarrow F_g$ be the induced map on homotopy fibres. If R is a principal ideal domain and two of the three morphisms ϕ , ψ , and η are R -equivalences, then so is the third.

We will not prove this here. The case where ϕ and η are R -equivalences is an immediate consequence of naturality for the Serre spectral sequences. The other two cases are also proved

⁴More precisely: there is an isomorphism $\pi_* X \otimes \mathbb{Z}_p^\wedge \cong \pi_*(X_p^\wedge)$ under $\pi_*(X)$.

⁵Notice that the same proof works if we replace the hypothesis “ X simply connected” by “ X simple and connected”: we just have to start from $P_1 X$, and use that there is a k_1 -invariant to $K(\pi_2(X), 3)$. Similarly for many of the proofs in this chapter.

using the Serre spectral sequences by more complicated arguments. See [Hat, Proposition 1.12] for an argument in one of these cases.

As a special case, we get:

COROLLARY 6.5.5. *Let*

$$\begin{array}{ccc} X & \xrightarrow{\psi} & Y \\ \downarrow f & & \downarrow g \\ Z & \xrightarrow{\phi} & W \end{array}$$

be a homotopy pullback square of spaces with Z and W simply connected. If R is a principal ideal domain, then ϕ is an R -equivalence if and only if ψ is an R -equivalence.

PROOF. Since the square is a homotopy pullback, the induced map of homotopy fibres $F_f \rightarrow F_g$ is a weak equivalence by Lemma 6.4.4, so this follows from Proposition 6.5.4. \square

This gives a first approximation to the Theorem:

LEMMA 6.5.6. *Let $f: X \rightarrow Y$ be a map between simply connected spaces, and let F be the homotopy fibre of f . Then the following are equivalent:*

- (1) *f is an \mathbb{F}_p -equivalence,*
- (2) *$\tilde{H}_*(F; \mathbb{F}_p) = 0$,*
- (3) *$\tilde{H}_*(F; \mathbb{Z})$ is a $\mathbb{Z}[\frac{1}{p}]$ -module.*

PROOF. The equivalence of (1) and (2) is a special case of Corollary 6.5.5. The equivalence of (2) and (3) follows from the Bockstein long exact sequence (i.e. the long exact sequence of homology groups induced by the short exact sequence of abelian groups $0 \rightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \rightarrow \mathbb{Z}/p \rightarrow 0$):

$$\cdots \rightarrow \tilde{H}_i(F; \mathbb{Z}) \xrightarrow{p} \tilde{H}_i(F; \mathbb{Z}) \rightarrow \tilde{H}_i(F; \mathbb{F}_p) \rightarrow \tilde{H}_{i-1}(F; \mathbb{Z}) \rightarrow \cdots$$

Here we see that $\tilde{H}_i(F; \mathbb{F}_p) = 0$ for all i , then multiplication by p is an isomorphism on $\tilde{H}_i(F; \mathbb{Z})$ for all i , and vice versa. \square

The fibre of a map of simply connected spaces is not necessarily simply connected, but it is a *simple* space, meaning its fundamental group is abelian and acts trivially on the higher homotopy groups. Lemma 6.5.6 therefore reduces the proof of Theorem 6.5.3 to showing:

PROPOSITION 6.5.7. *Suppose X is a simple space. Then $\tilde{H}_*(X; \mathbb{F}_p) = 0$ if and only if $\pi_n X$ is a $\mathbb{Z}[\frac{1}{p}]$ -module for all n .*

We first consider the case of Eilenberg-MacLane spaces:

LEMMA 6.5.8. *An abelian group A is a $\mathbb{Z}[\frac{1}{p}]$ -module if and only if $\tilde{H}_*(K(A, n); \mathbb{F}_p) = 0$ (for any $n \geq 1$).*

PROOF. First suppose $\tilde{H}_*(K(A, n); \mathbb{F}_p) = 0$. Then as we saw above, the Bockstein long exact sequence implies that $\tilde{H}_i(K(A, n); \mathbb{Z})$ is a $\mathbb{Z}[\frac{1}{p}]$ -module for all i . But since A is abelian we have $A \cong H_n(K(A, n); \mathbb{Z})$, so A is a $\mathbb{Z}[\frac{1}{p}]$ -module.

Now suppose A is a $\mathbb{Z}[\frac{1}{p}]$ -module. We first consider the case where $n = 1$. Since $\tilde{H}_*(-; \mathbb{F}_p)$ preserves filtered colimits it suffices to consider A a finitely generated $\mathbb{Z}[\frac{1}{p}]$ -module. Then A is a finite direct sum of copies of \mathbb{Z}/q^k , where q^k is some prime power with $q \neq p$, and copies of $\mathbb{Z}[\frac{1}{p}]$. We will not show that $\tilde{H}_*(K(\mathbb{Z}/q^k, 1); \mathbb{F}_p) = 0$; this can for example be done using group cohomology. [Exercise: Check this for $K(\mathbb{Z}/2, 1) = \mathbb{RP}^\infty$ using cellular homology.] In the case of $\mathbb{Z}[\frac{1}{p}]$ we have $\mathbb{Z}[\frac{1}{p}] = \text{colim}(\mathbb{Z} \xrightarrow{p} \mathbb{Z} \cdots)$ so $K(\mathbb{Z}[\frac{1}{p}], 1) \cong \text{colim}(K(\mathbb{Z}, 1) \xrightarrow{p} K(\mathbb{Z}, 1) \cdots)$ and so as homology commutes with filtered colimits we have $\tilde{H}_*(K(\mathbb{Z}[\frac{1}{p}], 1), \mathbb{Z}) \cong \tilde{H}_*(S^1; \mathbb{Z})[\frac{1}{p}]$, which is a $\mathbb{Z}[\frac{1}{p}]$ -module, so $\tilde{H}_*(K(\mathbb{Z}[\frac{1}{p}], 1), \mathbb{F}_p) = 0$.

We can now inductively extend this to $n > 1$: Applying Corollary 6.5.5 to the homotopy pullback square

$$\begin{array}{ccc} K(A, n-1) & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & K(A, n). \end{array}$$

gives that $\tilde{H}_*(K(A, n); \mathbb{F}_p) = 0$ if and only if $\tilde{H}_*(K(A, n-1); \mathbb{F}_p) = 0$. \square

PROOF OF PROPOSITION 6.5.7. We first consider the case where X is a simply connected space. For $n \geq 2$, by applying Corollary 6.5.5 to the homotopy pullback square

$$\begin{array}{ccc} K(\pi_n X, n) & \longrightarrow & P_n X \\ \downarrow & & \downarrow \\ * & \longrightarrow & P_{n-1} X \end{array}$$

we see that if $\tilde{H}_*(P_{n-1} X; \mathbb{F}_p) = 0$ then $\tilde{H}_*(K(\pi_n X, n); \mathbb{F}_p) \xrightarrow{\sim} \tilde{H}_*(P_n X; \mathbb{F}_p)$. In particular, by Lemma 6.5.8 we have that if $\tilde{H}_*(P_{n-1} X; \mathbb{F}_p) = 0$ then $\tilde{H}_*(P_n X; \mathbb{F}_p) = 0$ if and only if $\pi_n X$ is a $\mathbb{Z}[\frac{1}{p}]$ -module.

The map $X \rightarrow P_n X$ is $(n+1)$ -connected (meaning the homotopy fibre F_n is n -connected), and since $P_n X$ is simply connected the Serre spectral sequence for this map has E^2 -term

$$E_{s,t}^2 = H_s(P_n X; \mathbb{F}_p) \otimes_{\mathbb{F}_p} H_t(F_n; \mathbb{F}_p) \Rightarrow H_{s+t}(X; \mathbb{F}_p).$$

As $H_t(F_n; \mathbb{F}_p) = 0$ for $1 \leq t \leq n$ there is no room for differentials in the range $s+t \leq n$, and thus $H_i(X; \mathbb{F}_p) = E_{i,0}^\infty \xrightarrow{\sim} H_i(P_n X; \mathbb{F}_p) = E_{i,0}^2$ for $i \leq n$. Moreover, there are no differentials out of $H_{n+1}(P_n X; \mathbb{F}_p) = E_{n+1,0}^2$ so the map $H_{n+1}(X; \mathbb{F}_p) \rightarrow H_{n+1}(P_n X; \mathbb{F}_p)$ is surjective.⁶

We can therefore conclude that if $\pi_i X$ is a $\mathbb{Z}[\frac{1}{p}]$ -module for all $i \leq n$ then $\tilde{H}_*(P_n X; \mathbb{F}_p) = 0$ and so $\tilde{H}_i(X; \mathbb{F}_p) = 0$ for $i \leq n$. In particular, if all the homotopy groups of X are $\mathbb{Z}[\frac{1}{p}]$ -modules then $\tilde{H}_*(X; \mathbb{F}_p) = 0$.

Now suppose $\tilde{H}_*(X; \mathbb{F}_p) = 0$. Then for every n we have $\tilde{H}_i(P_n X; \mathbb{F}_p) = 0$ for $i \leq n+1$. Assume we know that $\pi_i X$ is a $\mathbb{Z}[\frac{1}{p}]$ -module for all $i < n$. Then we get that $\tilde{H}_i(K(\pi_n X, n); \mathbb{F}_p) = 0$ for $i \leq n+1$. But then from the Bockstein long exact sequence we see that multiplication by p on $H_n(K(\pi_n X, n); \mathbb{Z}) \cong \pi_n X$ is an isomorphism, i.e. $\pi_n X$ is a $\mathbb{Z}[\frac{1}{p}]$ -module. By inducting on n this completes the proof for X simply connected.

If X is not simply connected, we consider the Serre spectral sequence for the map $X \rightarrow B\pi_1 X$ whose homotopy fibre is the universal cover \tilde{X} . Since X is simple, $\pi_1(X)$ acts trivially on $H_*(\tilde{X}; \mathbb{F}_p)$, i.e. there is no non-trivial local system involved the E^2 -term, so this spectral sequence is of the form

$$E_{s,t}^2 = H_s(\tilde{X}; \mathbb{F}_p) \otimes_{\mathbb{F}_p} H_t(B\pi_1 X; \mathbb{F}_p) \Rightarrow H_{s+t}(X; \mathbb{F}_p).$$

If $\pi_n X$ is a $\mathbb{Z}[\frac{1}{p}]$ -module for all $n \geq 1$, then since \tilde{X} is simply connected we have $\tilde{H}_*(\tilde{X}; \mathbb{F}_p) = 0$, and $\tilde{H}_*(B\pi_1 X; \mathbb{F}_p) = 0$ by Lemma 6.5.8. Thus $E_{s,t}^2 = 0$ except when $s = t = 0$ and thus $\tilde{H}(X; \mathbb{F}_p) = 0$.

On the other hand, if $\tilde{H}(X; \mathbb{F}_p) = 0$, then we know $\tilde{H}(X; \mathbb{Z})$ is a $\mathbb{Z}[\frac{1}{p}]$ -module, hence $\pi_1 X \cong H_1(X; \mathbb{Z})$ is a $\mathbb{Z}[\frac{1}{p}]$ -module, and so $\tilde{H}(B\pi_1 X; \mathbb{F}_p) = 0$ by Lemma 6.5.8. Thus $E_{s,t}^2 = 0$ except when $t = 0$, so the spectral sequence collapses and we have $\tilde{H}_*(\tilde{X}; \mathbb{F}_p) \cong \tilde{H}_*(X; \mathbb{F}_p) = 0$. Then since \tilde{X} is simply connected we have that $\pi_n \tilde{X} = \pi_n X$ is a $\mathbb{Z}[\frac{1}{p}]$ -module for all $n \geq 2$. \square

⁶One can of course also use that $H_*(P_n X, X; \mathbb{F}_p) = 0$ for $* \leq n+1$ since $P_n X$ can be constructed from X by attaching cells of degree $\geq n+2$.

This completes the proof of Theorem 6.5.3, and we are ready to prove the base case of (ii) in Theorem 6.5.1:

LEMMA 6.5.9. *Suppose A is a finitely generated abelian group. Then $K(A, n) \rightarrow K(A_p^\wedge, n)$ is a p -completion, for $n \geq 1$.*

PROOF. We know from Theorem 6.5.1(i) that $K(A_p^\wedge, n)$ is p -complete, so it remains to show that the map is a \mathbb{F}_p -equivalence. Since A is finitely generated, it suffices to check separately the cases where A is \mathbb{Z}/p^k , \mathbb{Z}/q^k (q a prime $\neq p$), and \mathbb{Z} . In these cases A_p^\wedge is \mathbb{Z}/p^k , 0, and \mathbb{Z}_p^\wedge , respectively. For \mathbb{Z}/p^k the space $K(\mathbb{Z}/p^k, n)$ is already p -complete and there is nothing to prove. The group \mathbb{Z}/q^k is a $\mathbb{Z}[\frac{1}{p}]$ -module, so by Theorem 6.5.3 it follows that $K(\mathbb{Z}/q^k, n)$ is \mathbb{F}_p -equivalent to a point, as required.

The remaining case $A = \mathbb{Z}$ is the more interesting one. Here the map $\phi: \mathbb{Z} \rightarrow \mathbb{Z}_p^\wedge$ is injective, and we can complete it to a SES $\mathbb{Z} \rightarrow \mathbb{Z}_p^\wedge \rightarrow \text{coker}(\phi)$; correspondingly, we can exhibit $K(\mathbb{Z}, n)$ as the homotopy fibre of the map $K(\mathbb{Z}_p^\wedge, n) \rightarrow K(\text{coker}(\phi), n)$. Even when $n = 1$, note that the fibre sequence $K(\mathbb{Z}, n) \rightarrow K(\mathbb{Z}_p^\wedge, n) \rightarrow K(\text{coker}(\phi), n)$ is principal, and therefore we can compute $H_*(K(\mathbb{Z}_p^\wedge, n), \mathbb{F}_p)$ as limit of the Serre spectral sequence with $E_{s,t}^2 = H_s(K(\text{coker}(\phi), n); \mathbb{F}_p) \otimes_{\mathbb{F}_p} H_t(K(\mathbb{Z}, n); \mathbb{F}_p)$. If we show that $\text{coker } \phi$ is a $\mathbb{Z}[\frac{1}{p}]$ -module we are done: the E^2 -page vanishes for $s \neq 1$, and this implies that $K(\mathbb{Z}, n) \rightarrow K(\mathbb{Z}_p^\wedge, n)$ is an $H_*(-; \mathbb{F}_p)$ -equivalence.

It thus suffices to show that $\text{coker } \phi$ is a $\mathbb{Z}[\frac{1}{p}]$ -module, i.e. multiplication by p is an isomorphism. An element x of \mathbb{Z}_p^\wedge can be written as $\sum_{i=0}^{\infty} a_i p^i$, where $0 \leq a_i < p$. The image of \mathbb{Z} is precisely those sums with only finitely many non-zero a_i . Modulo \mathbb{Z} we then see that x equals $p(\sum_{i=1}^{\infty} a_i p^{i-1})$, so multiplication by p is surjective. On the other hand, if $p(\sum_{i=0}^{\infty} a_i p^i)$ is 0 modulo \mathbb{Z} then $a_i = 0$ except for finitely many i , i.e. $\sum a_i p^i$ is in the image of \mathbb{Z} . Hence multiplication by p is injective. This completes the proof. \square

PROOF OF THEOREM 6.5.1(II). We once again work our way up the Postnikov tower. First we can take $P_2 X_p^\wedge := K(\pi_2 X \otimes \mathbb{Z}_p^\wedge, 2)$, then $P_2 X \rightarrow P_2 X_p^\wedge$ is a p -completion by Lemma 6.5.9. Next if we have a p -completion $\phi: P_{n-1} X \rightarrow P_{n-1} X_p^\wedge$, we get from the universal property a homotopy-commutative square

$$\begin{array}{ccc} P_{n-1} X & \xrightarrow{\phi} & P_{n-1} X_p^\wedge \\ \downarrow k_{n-1} & & \downarrow (k_{n-1})_p^\wedge \\ K(\pi_n X, n+1) & \xrightarrow{\psi} & K(\pi_n X \otimes \mathbb{Z}_p^\wedge, n+1). \end{array}$$

On homotopy fibres we get a map $P_n X \rightarrow P_n X_p^\wedge := F_{(k_{n-1})_p^\wedge}$, where the space $P_n X_p^\wedge$ is p -complete by Lemma 6.3.9. Moreover, since ϕ and ψ are \mathbb{F}_p -equivalences it follows from Proposition 6.5.4 that this map is an \mathbb{F}_p -equivalence, so it is a p -completion.

Now define $X_p^\wedge := \text{holim}_n P_n X_p^\wedge$: here by “holim” (homotopy limit) we mean that we first replace the tower $\cdots \rightarrow P_3 X_p^\wedge \rightarrow P_2 X_p^\wedge$ by a tower of fibrations, and then take the limit. The result is p -complete by Lemma 6.3.10. It remains to show that $X \rightarrow X_p^\wedge$ is an \mathbb{F}_p -equivalence. To see this, let F_n be the homotopy fibre of the map $P_n X \rightarrow P_n X_p^\wedge$. Then the homotopy fibre of $X \rightarrow X_p^\wedge$ is weakly equivalent to $\text{holim}_n F_n$. Since the homotopy groups of $P_n X$ and $P_n X_p^\wedge$ stabilize (we have in fact $\pi_i(P_n X) \cong \pi_i(P_{n+1} X)$ and $\pi_i(P_n X_p^\wedge) \cong \pi_i(P_{n+1} X_p^\wedge)$ for $i \leq n-1$), we see from the long exact sequence that so do those of F_n . Thus $\pi_i(\lim_n F_n) \cong \lim_n \pi_i F_n$ with no \lim^1 -term, and the latter is in fact $\cong \pi_i F_k$ for k sufficiently large. Since the homotopy groups of the spaces F_n are $\mathbb{Z}[\frac{1}{p}]$ -modules, it follows that the same is true for those of $\lim_n F_n$ (we just use that a limit of abelian groups on which multiplication by p is invertible has the same property), hence the map $X \rightarrow X_p^\wedge$ is an \mathbb{F}_p -equivalence by Theorem 6.5.3. \square

PROOF OF THEOREM 6.5.1(III). Applying the 2-of-3 property for \mathbb{F}_p -equivalences to the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X_p^\wedge & \xrightarrow{f_p^\wedge} & Y_p^\wedge \end{array}$$

we see that f is an \mathbb{F}_p -equivalence if and only if f_p^\wedge is one. But the spaces X_p^\wedge and Y_p^\wedge are p -complete, so f_p^\wedge is an \mathbb{F}_p -equivalence if and only if it is an isomorphism in $\text{Ho}(\text{Top})$, i.e. if it gives an isomorphism $\pi_* X_p^\wedge \rightarrow \pi_* Y_p^\wedge$. Using Theorem 6.5.1(ii) we see from this that f is an \mathbb{F}_p -equivalence if and only if $\pi_*(f) \otimes \mathbb{Z}_p^\wedge$ is an isomorphism. \square

6.6. Sullivan's arithmetic square for p -completion

We now consider Sullivan's arithmetic square for p -completion:

THEOREM 6.6.1. *Suppose X is simply connected. Then we have a homotopy pullback square*

$$\begin{array}{ccc} X & \longrightarrow & \prod_p X_p^\wedge \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & \left(\prod_p X_p^\wedge \right)_{\mathbb{Q}}. \end{array}$$

We will prove this in 3 steps:

- (1) Let Y be the homotopy pullback in the square, then we will show that the induced map $\phi: X \rightarrow Y$ is a \mathbb{Z} -equivalence (isomorphism in \mathbb{Z} -homology).
- (2) Next, we will see that Y is a \mathbb{Z} -local space. Combined with (1), this shows that Y is the \mathbb{Z} -localization of X .
- (3) Finally, we will observe that X is itself \mathbb{Z} -local, so that (2) implies that $X \xrightarrow{\sim} Y$.

REMARK 6.6.2. One could also try to prove step (1) and then try to prove that Y is simply connected, concluding then by Hurewicz. However, in order to prove the vanishing of $\pi_1(Y)$, one needs to prove that the map $\pi_2(X_{\mathbb{Q}}) \oplus \pi_2(\prod_p X_p^\wedge) \rightarrow \pi_2((\prod_p X_p^\wedge)_{\mathbb{Q}})$ is surjective, and for this one needs to understand $\pi_2((\prod_p X_p^\wedge)_{\mathbb{Q}})$. This strategy can be pursued effectively when we know that $\pi_*(X)$ are finitely generated, using the results of the previous section, but is hostic otherwise.

For the proof of the first step we need the following result. With finite generation assumptions this follows from the results of the previous section, and the general case will follow from the results on p -completion for general simply connected spaces we will prove later.

PROPOSITION 6.6.3. *If X is a simply connected p -complete space, then $\pi_* X$ are $\mathbb{Z}[\frac{1}{q}]$ -modules for any prime $q \neq p$.*

To see that ϕ is a \mathbb{Z} -equivalence we will use the following criterion:

LEMMA 6.6.4. *A map of spaces $f: X \rightarrow Y$ induces an isomorphism on $H_*(-; \mathbb{Z})$ if and only if it induces an isomorphism on $H_*(-; \mathbb{Q})$ and on $H_*(-; \mathbb{F}_p)$ for all primes p .*

PROOF. The forward direction follows from the universal coefficient theorem. For the backward direction, we can assume that f is an inclusion, and the result follows if we can show that $H_*(Y, X) = 0$ under the stated assumptions. The universal coefficient sequence gives us for any abelian group A an exact sequence

$$0 \rightarrow H_n(Y, X) \otimes A \rightarrow H_n(Y, X; A) \rightarrow \text{Tor}_1^{\mathbb{Z}}(H_{n-1}(Y, X), A) \rightarrow 0.$$

If $H_i(Y, X; \mathbb{F}_p) = 0$ for $i = n-1, n$ then it follows that $H := H_n(Y, X)$ satisfies $H \otimes \mathbb{F}_p = 0$ and $\text{Tor}_1^{\mathbb{Z}}(H, \mathbb{F}_p) = 0$. This implies that H is uniquely p -divisible for all primes p , i.e. it is a \mathbb{Q} -vector space. But then $H \cong H \otimes \mathbb{Q}$, and if $H_n(Y, X; \mathbb{Q}) = 0$ the exact sequence implies that $H \otimes \mathbb{Q} = 0$. \square

PROPOSITION 6.6.5. *The map $\phi: X \rightarrow Y$ is a \mathbb{Z} -equivalence.*

PROOF. By Lemma 6.6.4 it is enough to prove that ϕ gives isomorphisms on $H_*(-; \mathbb{F}_p)$ for all primes p and on $H_*(-; \mathbb{Q})$.

Consider the following diagram, where the isomorphisms are ensured by the universal property of \mathbb{Q} -localization:

$$\begin{array}{ccccc} H_*(X; \mathbb{Q}) & \longrightarrow & H_*(Y; \mathbb{Q}) & \longrightarrow & H_*\left(\prod_p X_p^\wedge; \mathbb{Q}\right) \\ & \searrow \cong & \downarrow & & \downarrow \cong \\ & & H_*(X_{\mathbb{Q}}; \mathbb{Q}) & \longrightarrow & H_*\left(\left(\prod_p X_p^\wedge\right)_{\mathbb{Q}}; \mathbb{Q}\right), \end{array}$$

Corollary 6.5.5 implies that the left vertical map is also an isomorphism, which shows that $X \rightarrow Y$ is an isomorphism on $H_*(-; \mathbb{Q})$.

Now consider the following diagram:

$$\begin{array}{ccccc} & & \cong & & \\ & \nearrow & & \searrow & \\ H_*(X; \mathbb{F}_q) & \longrightarrow & H_*(Y; \mathbb{F}_q) & \longrightarrow & H_*\left(\prod_p X_p^\wedge; \mathbb{F}_q\right) \\ & & \downarrow & & \downarrow \\ & & H_*(X_{\mathbb{Q}}; \mathbb{F}_q) & \xrightarrow{\cong} & H_*\left(\left(\prod_p X_p^\wedge\right)_{\mathbb{Q}}; \mathbb{F}_q\right), \end{array}$$

Here the top map is an isomorphism since $\prod_p X_p^\wedge = X_q^\wedge \times \prod_{p \neq q} X_p^\wedge$ where the homotopy groups of the second space are $\mathbb{Z}[\frac{1}{q}]$ -modules by Proposition 6.6.3, and so its reduced \mathbb{F}_q -homology is trivial by Theorem 6.5.3. By the Künneth theorem this means

$$H_*\left(\prod_p X_p^\wedge; \mathbb{F}_q\right) \cong H_*(X_q^\wedge; \mathbb{F}_q) \otimes_{\mathbb{F}_q} H_*\left(\prod_{p \neq q} X_p^\wedge; \mathbb{F}_q\right) \cong H_*(X_q^\wedge; \mathbb{F}_q),$$

and so the map $H_*(X; \mathbb{F}_q) \rightarrow H_*(\prod_p X_p^\wedge; \mathbb{F}_q)$ is an isomorphism. The bottom horizontal map is also an isomorphism: since the homotopy groups of the spaces at the bottom are \mathbb{Q} -vector spaces, again by Theorem 6.5.3 the \mathbb{F}_q -homology in the bottom row is trivial, so the bottom arrow is trivially an isomorphism.

Now using Corollary 6.5.5 again the top horizontal map is also an isomorphism, which shows that $X \rightarrow Y$ gives an isomorphism on $H_*(-; \mathbb{F}_q)$ for all q . Thus $X \rightarrow Y$ gives an isomorphism on $H_*(-; \mathbb{Z})$, as required. \square

For step (2), we need the following result:

PROPOSITION 6.6.6. *If*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

is a homotopy pullback square and Y , Z , and W are R -local for some ring R , then X is R -local.

To see this we need some observations about R -equivalences and weak equivalences:

LEMMA 6.6.7. *If $f: X \rightarrow Y$ is an R -equivalence, then so is $f \times T: X \times T \rightarrow Y \times T$ for any space T .*

PROOF. We have a (convergent) Künneth spectral sequence

$$E_{p,q}^2 = \bigoplus_{i+j=p} \mathrm{Tor}_R^q(H_i(X; R), H_j(T; R)) \Rightarrow H_{i+j}(X \times T; R),$$

and similarly for Y . This is natural, and the map induced by f gives an isomorphism on the E^2 -terms. Therefore we get an isomorphism $H_*(X \times T; R) \xrightarrow{\cong} H_*(Y \times T; R)$, as required. See also the discussion after Definition 6.3.7. \square

LEMMA 6.6.8. *A space Z is R -local if and only if for every R -equivalence $f: X \rightarrow Y$ the induced map between mapping spaces*

$$\mathrm{map}(Y, Z) \rightarrow \mathrm{map}(X, Z)$$

is a weak equivalence.

To show this, we will use the following fact:

FACT 6.6.9. *A map of spaces $f: X \rightarrow Y$ is a weak equivalence if and only if the induced map on homotopy classes $[T, X] \rightarrow [T, Y]$ is an isomorphism of sets for all CW complexes T . (Equivalently, f induces an isomorphism in $\mathrm{Ho}(\mathrm{Top})$.)*

This is a special case of a standard fact about model categories, for example.

WARNING 6.6.10. These are unpointed homotopy classes. To show that $X \rightarrow Y$ is a weak equivalence it is *not* enough to know that $[S^n, X] \rightarrow [S^n, Y]$ is an isomorphism for all spheres. Think of the infinite symmetric group $G = \bigcup_{n \geq 1} \mathfrak{S}_n$, obtained as union of all finite symmetric groups, and construct a “shift” map $BG \rightarrow BG$ giving a counterexample.

PROOF OF LEMMA 6.6.8. If $\mathrm{map}(Y, Z) \rightarrow \mathrm{map}(X, Z)$ is a weak equivalence for all R -equivalences f , then on π_0 we get an isomorphism $[Y, Z] \rightarrow [X, Z]$ for every R -equivalence, i.e. Z is R -local.

Conversely, suppose Z is R -local and $f: X \rightarrow Y$ is an R -equivalence. The map $\mathrm{map}(Y, Z) \rightarrow \mathrm{map}(X, Z)$ is a weak equivalence if and only if for all spaces T the induced map

$$[T, \mathrm{map}(Y, Z)] \rightarrow [T, \mathrm{map}(X, Z)]$$

is an isomorphism. But this is isomorphic to the map

$$[T \times Y, Z] \rightarrow [T \times X, Z]$$

induced by $T \times f$, which is an R -equivalence by Lemma 6.6.7, and so this map is an isomorphism since Z is R -local. \square

LEMMA 6.6.11. *Given a commutative cube*

$$\begin{array}{ccccc} A & \xrightarrow{\quad} & B & & \\ & \searrow \alpha & \downarrow & \searrow \beta & \\ & & A' & \xrightarrow{\quad} & B' \\ & & \downarrow & & \downarrow \\ C & \xrightarrow{\quad} & D & & \\ & \searrow \gamma & \downarrow & \searrow \delta & \\ & & C' & \xrightarrow{\quad} & D' \end{array}$$

where the front and back faces are homotopy pullbacks, if the maps β , γ , and δ are weak equivalences, so is α .

PROOF. This is a special case of homotopy invariance for homotopy limits. \square

PROOF OF PROPOSITION 6.6.6. If $f: A \rightarrow B$ is an R -equivalence, by Lemma 6.6.8 we need to prove that $\text{map}(B, X) \rightarrow \text{map}(A, X)$ is a weak equivalence. This follows by applying Lemma 6.6.11 to the cube

$$\begin{array}{ccccc}
 \text{map}(B, X) & \xrightarrow{\quad} & \text{map}(B, Y) & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & \text{map}(A, X) & \xrightarrow{\quad} & \text{map}(A, Y) & \\
 & \downarrow & \downarrow & \downarrow & \\
 \text{map}(B, Z) & \xrightarrow{\quad} & \text{map}(B, W) & & \\
 & \searrow & \searrow & \searrow & \\
 & \text{map}(A, Z) & \xrightarrow{\quad} & \text{map}(A, W), &
 \end{array}$$

where the front and back faces are homotopy pullbacks since $\text{map}(A, -)$ preserves homotopy limits. \square

We need one more observation before we can prove Theorem 6.6.1:

LEMMA 6.6.12. *If a space X is R -local for some ring R , then X is \mathbb{Z} -local.*

PROOF. It suffices to show that if $f: Y \rightarrow Z$ is a \mathbb{Z} -equivalence then it is an R -equivalence for any ring R . From the universal coefficient theorem we have a map of short exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H_n(Y) \otimes R & \longrightarrow & H_n(Y; R) & \longrightarrow & \text{Tor}^1(H_{n-1}(Y), R) \longrightarrow 0 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & H_n(Z) \otimes R & \longrightarrow & H_n(Z; R) & \longrightarrow & \text{Tor}^1(H_{n-1}(Z), R) \longrightarrow 0,
 \end{array}$$

so the 5-Lemma implies that f is an R -equivalence if it is a \mathbb{Z} -equivalence. \square

PROOF OF THEOREM 6.6.1. The spaces $X_{\mathbb{Q}}$, X_p^{\wedge} and $(\prod_p X_p^{\wedge})_{\mathbb{Q}}$ are \mathbb{Z} -local by Lemma 6.6.12, and the space $\prod_p X_p^{\wedge}$ is also \mathbb{Z} -local since an arbitrary product of R -local spaces is always R -local (just consider that mapping spaces into a product are product of mapping spaces). Thus the homotopy pullback Y is \mathbb{Z} -local by Proposition 6.6.6. Since $X \rightarrow Y$ is also a \mathbb{Z} -equivalence by Proposition 6.6.5, this means that Y is the \mathbb{Z} -localization of X . But we know from Theorem 6.3.3 that a simply connected space is R -local for $R \subseteq \mathbb{Q}$ if and only if its homotopy groups are R -modules — taking $R = \mathbb{Z}$ this says that every simply connected space is \mathbb{Z} -local. Thus the \mathbb{Z} -localization of X is just X , and so $X \rightarrow Y$ is a weak equivalence, as required. \square

REMARK 6.6.13. Theorem 6.6.1 says that we can recover a (simply connected) space X from its rationalization $X_{\mathbb{Q}}$, its p -completions X_p^{\wedge} at all primes, and the map $X_{\mathbb{Q}} \rightarrow (\prod_p X_p^{\wedge})_{\mathbb{Q}}$. In fact, we can say a bit more: If we have p -complete spaces $Y(p)$ for all primes p and a rational space Q , all simply connected, together with a map $Q \rightarrow (\prod_p Y(p))_{\mathbb{Q}}$, let P be the homotopy pullback in

$$\begin{array}{ccc}
 P & \longrightarrow & \prod_p Y(p) \\
 \downarrow & & \downarrow \\
 Q & \longrightarrow & (\prod_p Y(p))_{\mathbb{Q}}.
 \end{array}$$

Then the same proof shows: Q is the rationalization of P and $Y(p)$ is the p -completion of P . Thus we can build a space with arbitrary rationalization and p -completions, provided we have the bottom map — this is the only “interaction” between the rational and p -complete “parts”

of the space. Note also that $\pi_1(P)$ may be non-trivial (but it is a quotient of $\pi_2((\prod_p Y(p))_{\mathbb{Q}})$, so for instance it is surely a divisible abelian group).

6.7. An algebraic interlude: derived p -completion

We saw above that p -completion of spaces is closely related to p -completion of abelian groups — but only for finitely generated abelian groups. The reason is that the naïve definition of the p -completion of an abelian group A as $A_p^\wedge := \lim_k A/p^k$ must be replaced by a “derived” p -completion in chain complexes of abelian groups. This involves using derived versions of both parts of the construction, i.e. quotienting by p^k and taking the limit:

- Instead of just taking the quotient A/p^k , i.e. the cokernel of the map $A \xrightarrow{p^k} A$ we take the “derived cokernel”, meaning the *mapping cone* of this map. This is the chain complex

$$\cdots \rightarrow 0 \rightarrow A \xrightarrow{p^k} A \rightarrow 0 \rightarrow \cdots$$

with the non-zero groups in degrees 0 and 1. We will denote this $A//p^k$. Note that $H_0(A//p^k)$ recovers A/p^k , whereas $H_1(A//p^k)$ is the kernel of multiplication by p^k , i.e. the group of p^k -torsion elements in A .

- Now we want to take the limit of the chain complexes $A//p^k$ over k — but since \lim is not an exact functor (though it is left exact) we must take the *right derived* limit. This means we first replace the diagram $\mathbb{N}^{\text{op}} \rightarrow \text{Ch}(\mathbb{Z})$ sending $k \mapsto A//p^k$ by a *fibrant* object in the category $\text{Ch}(\mathbb{Z})^{\mathbb{N}^{\text{op}}}$ of diagrams of chain complexes, and then take the limit of that. This, of course, requires fixing a model structure; alternatively, we just need a right deformation Q of the identity functor of $\text{Ch}(\mathbb{Z})^{\mathbb{N}^{\text{op}}}$ as in Definition 4.1.9, such that \lim is homotopy invariant on the essential image of Q . We will denote this derived limit of an inverse system of chain complexes $C(k)_{k \geq 0}$ by $\mathbb{R}\lim_k C(k)$.

REMARK 6.7.1. The derived limit $\mathbb{R}\lim_k C(k)$ of an inverse system $C: \mathbb{N}^{\text{op}} \rightarrow \text{Ch}(\mathbb{Z})$, $k \mapsto C(k)$, is well-defined up to quasi-isomorphism.

To compute it, we can for instance fix a model structure on $\mathbb{N}^{\text{op}} \rightarrow \text{Ch}(\mathbb{Z})$: one possible model structure has weak equivalences given by natural transformations $C(-) \rightarrow D(-)$ that for each k evaluate to a quasi-isomorphism $C(k) \xrightarrow{\sim} D(k)$; and it has cofibrations given by natural transformations $C(-) \rightarrow D(-)$ that for each k evaluate to an inclusion of chain complexes $C(k) \hookrightarrow D(k)$. A fibrant replacement of $C \in \mathbb{N}^{\text{op}} \rightarrow \text{Ch}(\mathbb{Z})$ can be then obtained by replacing each $C(k)$ by a fibrant object⁷ in $\text{Ch}(\mathbb{Z})$, in such a way that all maps $C(k) \rightarrow C(k-1)$ are fibrations in $\text{Ch}(\mathbb{Z})$. Examples of fibrant objects in $\text{Ch}(\mathbb{Z})$ are bounded above chain complexes of injective abelian groups (and if you want, this can be a motivation for the very definition of “injective abelian group”), and examples of fibrations are mapping cylinder projections $\text{Cyl}(A \rightarrow B) \rightarrow B$, if $A \rightarrow B$ is a chain map of fibrant chain complexes.

Note also that since we are working with chain complexes rather than cochain complexes the \lim^1 -term will contribute in degree -1 when we consider an inverse system $\cdots \rightarrow M_1 \rightarrow M_0$ of abelian groups (considered as chain complexes concentrated in degree 0): we have

$$H_i(\mathbb{R}\lim_k M_k) = \begin{cases} \lim_k M_k, & i = 0, \\ \lim_k^1 M_k, & i = -1, \\ 0, & \text{otherwise.} \end{cases}$$

More generally, for a sequence $C(k)$ of chain complexes we get short exact sequences

$$0 \rightarrow \lim_k^1 H_{i+1} C(k) \rightarrow H_i \mathbb{R}\lim_k C(k) \rightarrow \lim_k H_i C(k) \rightarrow 0.$$

⁷Again, we are considering a particular model structure on $\text{Ch}(\mathbb{Z})$, namely the one in which weak equivalences are quasi-isomorphisms and cofibrations are inclusions.

(This is a degenerate special case of Grothendieck's hyperhomology spectral sequence, for example.)

DEFINITION 6.7.2. The *derived p -completion* of an abelian group A is $\mathbb{R}\lim_k A//p^k$. We will denote this $\mathbb{D}A_p^\wedge$.

REMARK 6.7.3. From the short exact sequence for $\mathbb{R}\lim_k$ above, we see that

$$H_1\mathbb{D}A_p^\wedge \cong \lim_k \operatorname{Tor}(\mathbb{Z}/p^k, A)$$

and there is a short exact sequence

$$0 \rightarrow \lim_k^1 \operatorname{Tor}(\mathbb{Z}/p^k, A) \rightarrow H_0\mathbb{D}A_p^\wedge \rightarrow A_p^\wedge \rightarrow 0.$$

The group $\operatorname{Tor}(\mathbb{Z}/p^k, A)$ is naturally isomorphic to $A[p^k]$, i.e. the subgroup of A of p^k -torsion elements: just compute Tor by considering the resolution $\dots 0 \rightarrow \mathbb{Z} \xrightarrow{p^k} \mathbb{Z}$ of \mathbb{Z}/p^k and tensoring it with A . Moreover the natural map $\operatorname{Tor}(\mathbb{Z}/p^{k+1}, A) \rightarrow \operatorname{Tor}(\mathbb{Z}/p^k, A)$, induced by the quotient projection $\mathbb{Z}/p^{k+1} \rightarrow \mathbb{Z}/p^k$, corresponds to the map $A[p^{k+1}] \rightarrow A[p^k]$ given by multiplication by p inside A : just lift the quotient projection to the chain map

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{p^{k+1}} & \mathbb{Z} \\ \downarrow p & & \downarrow 1 \\ \mathbb{Z} & \xrightarrow{p^k} & \mathbb{Z} \end{array}$$

and again tensor with A and compute first homology. Thus if $A[p^k] = A[p^{k+1}]$ for k sufficiently large, then $H_1\mathbb{D}A_p^\wedge = 0$ and $H_0\mathbb{D}A_p^\wedge \cong A_p^\wedge$,⁸ so $\mathbb{D}A_p^\wedge$ is (up to quasi-isomorphism) just A_p^\wedge . This is, for example, always true if A is finitely generated, or if multiplication by p is injective on A .

DEFINITION 6.7.4. An abelian group A is *derived p -complete* (or *Ext- p -complete*) if the natural map $A \rightarrow \mathbb{D}A_p^\wedge$ is a quasi-isomorphism, i.e. if $A \xrightarrow{\sim} H_0\mathbb{D}A_p^\wedge$ and $H_1\mathbb{D}A_p^\wedge = 0$.

DEFINITION 6.7.5. Let \mathbb{Z}/p^∞ denote the abelian group obtained as the colimit of the sequence

$$\mathbb{Z}/p \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{p} \mathbb{Z}/p^3 \xrightarrow{p} \dots$$

REMARK 6.7.6. Our description of $H_1\mathbb{D}A_p^\wedge$ above can be interpreted as

$$\lim_k \operatorname{Hom}(\mathbb{Z}/p^k, A) \cong \operatorname{Hom}(\operatorname{colim}_k \mathbb{Z}/p^k, A) \cong \operatorname{Hom}(\mathbb{Z}/p^\infty, A).$$

We will now see that $H_0\mathbb{D}A_p^\wedge$ can similarly be interpreted as $\operatorname{Ext}(\mathbb{Z}/p^\infty, A)$. In fact, we will see that the chain complex $\mathbb{D}A_p^\wedge$ can be described as a shift of a derived Hom in the following sense:

DEFINITION 6.7.7. If C and D are chain complexes, we write $\mathbb{R}\operatorname{Hom}(C, D)$ for the chain complex obtained as either $\operatorname{Hom}(C', D)$ where C' is a cofibrant replacement of C , or as $\operatorname{Hom}(C, D')$ where D' is a fibrant replacement of D , after fixing a suitable model structure on $\operatorname{Ch}(\mathbb{Z})$.

Concretely, if C is bounded below we can replace it with C' , where $C' \rightarrow C$ is a quasi-isomorphism and C' is bounded below and levelwise a projective (free) abelian group. Or if D is bounded above, we can replace it with D' , where $D \rightarrow D'$ is a quasi-isomorphism and D' is bounded above and levelwise an injective abelian group.

LEMMA 6.7.8. *The chain complexes $\mathbb{D}A_p^\wedge$ and $\mathbb{R}\operatorname{Hom}(\mathbb{Z}/p^\infty, A)[1]$ are quasi-isomorphic. In particular we have*

$$\begin{aligned} H_0\mathbb{D}A_p^\wedge &\cong \operatorname{Ext}(\mathbb{Z}/p^\infty, A), \\ H_1\mathbb{D}A_p^\wedge &\cong \operatorname{Hom}(\mathbb{Z}/p^\infty, A). \end{aligned}$$

⁸The fact that in the given hypothesis $\lim_k^1 \operatorname{Tor}(\mathbb{Z}/p^k, A)$ vanishes is an instructive exercise.

SKETCH PROOF. We will use the fact that $\mathbb{R}\mathrm{Hom}(-, A)$ takes derived colimits (in the first variable) to derived limits.⁹ This is just because we can compute $\mathbb{R}\mathrm{Hom}(-, A)$ by using a fibrant replacement (aka injective resolution) of A , and if A is already a fibrant chain complex, then $\mathbb{R}\mathrm{Hom}(-, A): \mathrm{Ch}(\mathbb{Z}) \rightarrow \mathrm{Ch}(\mathbb{Z})$ sends cofibrations to fibrations.

The colimit defining \mathbb{Z}/p^∞ is already derived (all maps involved are already cofibrations, i.e. injective maps of abelian groups, seen as chain complexes concentrated in degree 0), so $\mathbb{R}\mathrm{Hom}(\mathbb{Z}/p^\infty, A) \simeq \mathbb{R}\mathrm{lim}_k \mathbb{R}\mathrm{Hom}(\mathbb{Z}/p^k, A)$. Now the chain complex $\mathbb{Z} // p^k$ is a projective resolution of \mathbb{Z}/p^k , so the chain complex $\mathbb{R}\mathrm{Hom}(\mathbb{Z}/p^k, A)$ is quasi-isomorphic to $\mathrm{Hom}(\mathbb{Z} // p^k, A)$, which can be identified with $A // p^k[-1]$. Thus $\mathbb{R}\mathrm{Hom}(\mathbb{Z}/p^\infty, A) \simeq \mathbb{R}\mathrm{lim}_k A // p^k[-1] \simeq \mathbb{D}A_p^\wedge[-1]$. \square

LEMMA 6.7.9. *There is a short exact sequence of abelian groups*

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[\frac{1}{p}] \rightarrow \mathbb{Z}/p^\infty \rightarrow 0.$$

PROOF. We have a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{p^k} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/p^k & \longrightarrow & 0 \\ \downarrow & & \downarrow \mathrm{id} & & \downarrow p & & \downarrow p & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{p^{k+1}} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/p^{k+1} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

Since $\mathbb{Z}[\frac{1}{p}] = \mathrm{colim}_k (\mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \rightarrow \cdots)$, taking colimits in this diagram gives the desired short exact sequences (as taking the colimit of a sequence is an exact functor). \square

As a consequence, if A' is an injective resolution of an abelian group A , we get a short exact sequence of chain complexes

$$0 \rightarrow \mathrm{Hom}(\mathbb{Z}/p^\infty, A') \rightarrow \mathrm{Hom}(\mathbb{Z}[\frac{1}{p}], A') \rightarrow A' \rightarrow 0,$$

or

$$0 \rightarrow \mathbb{R}\mathrm{Hom}(\mathbb{Z}/p^\infty, A) \rightarrow \mathbb{R}\mathrm{Hom}(\mathbb{Z}[\frac{1}{p}], A) \rightarrow \mathbb{R}\mathrm{Hom}(\mathbb{Z}, A) \rightarrow 0.$$

The associated long exact sequence in homology is

$$0 \rightarrow \mathrm{Hom}(\mathbb{Z}/p^\infty, A) \rightarrow \mathrm{Hom}(\mathbb{Z}[\frac{1}{p}], A) \rightarrow A \rightarrow \mathrm{Ext}(\mathbb{Z}/p^\infty, A) \rightarrow \mathrm{Ext}(\mathbb{Z}[\frac{1}{p}], A) \rightarrow 0.$$

From this we immediately see:

LEMMA 6.7.10. *An abelian group A is derived p -complete if and only if $\mathrm{Hom}(\mathbb{Z}[\frac{1}{p}], A) = 0$ and $\mathrm{Ext}(\mathbb{Z}[\frac{1}{p}], A) = 0$, i.e. if and only if $\mathbb{R}\mathrm{Hom}(\mathbb{Z}[\frac{1}{p}], A) \simeq 0$.*

We will also need the following characterization of derived p -complete groups:

PROPOSITION 6.7.11. *If $A \rightarrow \mathrm{Ext}(\mathbb{Z}/p^\infty, A)$ is an isomorphism, then all groups $\mathrm{Hom}(\mathbb{Z}/p^\infty, A)$, $\mathrm{Hom}(\mathbb{Z}[\frac{1}{p}], A)$ and $\mathrm{Ext}(\mathbb{Z}[\frac{1}{p}], A)$ vanish. In other words, A is derived p -complete if and only if $A \xrightarrow{\sim} \mathrm{Ext}(\mathbb{Z}/p^\infty, A)$.*

⁹Here we use the model structure on $\mathrm{Ch}(\mathbb{Z})$ for which weak equivalences are quasi-isomorphisms and cofibrations are inclusions.

PROOF. From the long exact sequence

$$0 \rightarrow \operatorname{Hom}(\mathbb{Z}/p^\infty, A) \rightarrow \operatorname{Hom}(\mathbb{Z}[\frac{1}{p}], A) \rightarrow A \rightarrow \operatorname{Ext}(\mathbb{Z}/p^\infty, A) \rightarrow \operatorname{Ext}(\mathbb{Z}[\frac{1}{p}], A) \rightarrow 0$$

we see that if $A \rightarrow \operatorname{Ext}(\mathbb{Z}/p^\infty, A)$ is an isomorphism, then the map $\operatorname{Hom}(\mathbb{Z}[\frac{1}{p}], A) \rightarrow A$ is zero. A homomorphism $\mathbb{Z}[\frac{1}{p}] \rightarrow A$ is determined by a sequence (a_0, a_1, \dots) such that $a_i = pa_{i+1}$. If the map to A is zero, then this means that for any such sequence we must have $a_0 = 0$. But (a_k, a_{k+1}, \dots) is a sequence in A of the same form, so we must have $a_k = 0$ for all k . Thus $\operatorname{Hom}(\mathbb{Z}[\frac{1}{p}], A) = 0$, and then from the exact sequence we get that $\operatorname{Hom}(\mathbb{Z}/p^\infty, A) = 0$ as well. \square

REMARK 6.7.12. The derived p -complete abelian groups are precisely the ones that are \mathcal{W} -local where \mathcal{W} is the class of maps $A \rightarrow B$ such that $\operatorname{Ext}(\mathbb{Z}/p^\infty, A) \xrightarrow{\sim} \operatorname{Ext}(\mathbb{Z}/p^\infty, B)$ is an isomorphism. The localization of an abelian group A with respect to \mathcal{W} is $LA = \operatorname{Ext}(\mathbb{Z}/p^\infty, A)$, with $A \rightarrow LA$ induced by the boundary map from the sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[\frac{1}{p}] \rightarrow \mathbb{Z}/p^\infty \rightarrow 0$.

6.8. p -completion of simply connected spaces II (the general case)

We now want to consider p -completion for general simply connected spaces. The theorem we want to prove is:

THEOREM 6.8.1.

- (i) *A simply connected space X is p -complete if and only if $\pi_n X$ is derived p -complete for all n .*
- (ii) *If X is a simply connected space then its p -completion $X \rightarrow X_p^\wedge$ exists, and for every n there is a short exact sequence*

$$0 \rightarrow \operatorname{Ext}(\mathbb{Z}/p^\infty, \pi_n X) \rightarrow \pi_n X_p^\wedge \rightarrow \operatorname{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1} X) \rightarrow 0.$$

The key step is understanding the p -completion of Eilenberg-MacLane spaces in terms of the algebraic derived p -completion we defined in the previous section. This requires introducing some notation:

DEFINITION 6.8.2. Suppose C is a non-negatively graded chain complex of abelian groups, with finitely many non-zero homology groups. For $n \geq 0$ we can define an Eilenberg-MacLane space from C by

$$K(C, n) := \prod_i K(H_i C, n + i).$$

REMARK 6.8.3. Using a more functorial version of this definition, it can be shown that this construction is compatible with derived limits. Moreover, if we have a short exact sequence of chain complexes

$$0 \rightarrow C \rightarrow C' \rightarrow C'' \rightarrow 0$$

then $K(C, n)$ is the homotopy fibre of the induced map¹⁰ $K(C', n) \rightarrow K(C'', n)$. We will not prove this, but it should be plausible since the long exact sequence in homology from the short exact sequence looks the same as the long exact sequence on homotopy groups.

DEFINITION 6.8.4. Let A be an abelian group. We define $K(A, n)_p^\wedge$ to be $K(\mathbb{D}A_p^\wedge, n)$.

More concretely, we can first define $K(A, n)/p^k$ to be $K(A//p^k, n)$, which is equivalently the homotopy fibre of $K(A, n+1) \xrightarrow{p^k} K(A, n+1)$. Then we can define $K(A, n)_p^\wedge$ as the homotopy limit of the maps $K(A, n)/p^{k+1} \rightarrow K(A, n)/p^k$ (where the homotopy limit is given by replacing these maps by fibrations and then taking the usual limit).

¹⁰This induced map is *not* just the product of the maps $K(H_i C', n + i) \rightarrow K(H_i C'', n + i)$ induced by the group homomorphisms $H_i C' \rightarrow H_i C''$, but something more complicated, built into the functorial construction

REMARK 6.8.5. From our description of $\mathbb{D}A_p^\wedge$ above, we see that

$$\pi_* K(A, n)_p^\wedge \cong \begin{cases} 0, & * \neq n, n+1, \\ \text{Ext}(\mathbb{Z}/p^\infty, A), & * = n, \\ \text{Hom}(\mathbb{Z}/p^\infty, A), & * = n+1. \end{cases}$$

PROPOSITION 6.8.6. *If A is an abelian group, then $K(A, n) \rightarrow K(A, n)_p^\wedge$ is a p -completion.*

Before we give the proof we need to make a simple observation:

LEMMA 6.8.7. *Suppose M is a \mathbb{Z}/p^k -module. Then $K(M, n)$ is p -complete.*

PROOF. We prove this by induction on k . For $k = 1$ the universal coefficient sequence gives (as there is no Ext term over the field \mathbb{F}_p) $H^*(X, M) \cong \text{Hom}_{\mathbb{F}_p}(H_*(X, \mathbb{F}_p), M)$, so an \mathbb{F}_p -equivalence induces an isomorphism on $H^n(-, M) \cong [-, K(M, n)]$.

Now suppose M is a \mathbb{Z}/p^k -module. Then there is a short exact sequence

$$0 \rightarrow pM \rightarrow M \rightarrow M/pM \rightarrow 0,$$

where pM and M/pM are \mathbb{Z}/p^{k-1} -modules. This induces a long exact sequence

$$\cdots \rightarrow H^n(X, pM) \rightarrow H^n(X, M) \rightarrow H^n(X, M/pM) \rightarrow H^{n+1}(X, pM) \rightarrow \cdots$$

Using the 5-Lemma this gives inductively that an \mathbb{F}_p -equivalence gives an isomorphism on $H^n(-, M) \cong [-, K(M, n)]$. \square

PROOF OF PROPOSITION 6.8.6. To see that $K(A, n)_p^\wedge$ is p -complete it suffices by Lemma 6.3.10 to see that $K(A, n)/p^k$ is p -complete for each k . But this space is weakly equivalent to $K(\text{Tor}(\mathbb{Z}/p^k, A), n+1) \times K(A/p^k, n)$. Here $\text{Tor}(\mathbb{Z}/p^k, A)$ and A/p^k are both \mathbb{Z}/p^k -modules, hence this space is p -complete by Lemma 6.8.7.

We saw in the previous section that we have a short exact sequence of chain complexes

$$0 \rightarrow \mathbb{R}\text{Hom}(\mathbb{Z}/p^\infty, A) \rightarrow \mathbb{R}\text{Hom}(\mathbb{Z}[\frac{1}{p}], A) \rightarrow \mathbb{R}\text{Hom}(\mathbb{Z}, A) \rightarrow 0,$$

which as $\mathbb{D}A_p^\wedge \simeq \mathbb{R}\text{Hom}(\mathbb{Z}/p^\infty, A)[1]$ gives a fibre sequence

$$K(A, n)_p^\wedge \rightarrow K(\mathbb{R}\text{Hom}(\mathbb{Z}[\frac{1}{p}], A), n+1) \rightarrow K(A, n+1).$$

Continuing this (as a Puppe sequence) we get a fibre sequence

$$K(\mathbb{R}\text{Hom}(\mathbb{Z}[\frac{1}{p}], A), n) \rightarrow K(A, n) \rightarrow K(A, n)_p^\wedge.$$

Thus the homotopy groups of the fibre are $\text{Hom}(\mathbb{Z}[\frac{1}{p}], A)$ in degree n and $\text{Ext}(\mathbb{Z}[\frac{1}{p}], A)$ in degree $n+1$. These are in particular both $\mathbb{Z}[\frac{1}{p}]$ -modules, and so the map $K(A, n) \rightarrow K(A, n)_p^\wedge$ is an \mathbb{F}_p -equivalence by Theorem 6.5.3. \square

COROLLARY 6.8.8. *An Eilenberg-MacLane space $K(A, n)$ is p -complete if and only if A is derived p -complete.*

PROOF. Since $K(A, n)_p^\wedge$ is the p -completion of $K(A, n)$, the space $K(A, n)$ is p -complete if and only if the natural map $K(A, n) \rightarrow K(A, n)_p^\wedge$ is a weak equivalence. From the computation of $\pi_* K(A, n)_p^\wedge$ we see this is equivalent to $\text{Hom}(\mathbb{Z}/p^\infty, A) = 0$ and $A \xrightarrow{\sim} \text{Ext}(\mathbb{Z}/p^\infty, A)$. \square

Now a Postnikov tower argument gives one direction of (i) in Theorem 6.8.1:

PROPOSITION 6.8.9. *If X is a simply connected space such that the groups $\pi_* X$ are derived p -complete, then X is p -complete.*

PROOF. Exactly as the proof of Theorem 6.5.1(i). \square

Using this we can prove part (ii) of the theorem:

PROOF OF THEOREM 6.8.1(ii). We construct the p -completion exactly as in the Proof of Theorem 6.5.1(ii): We take $P_2X_p^\wedge := K(\pi_2X, 2)_p^\wedge$, then $P_2X \rightarrow P_2X_p^\wedge$ is a p -completion by Proposition 6.8.6. Then if we have a p -completion $P_{n-1}X \rightarrow P_{n-1}X_p^\wedge$, we get from the universal property a homotopy-commutative square

$$\begin{array}{ccc} P_{n-1}X & \xrightarrow{\phi} & P_{n-1}X_p^\wedge \\ \downarrow k_{n-1} & & \downarrow (k_{n-1})_p^\wedge \\ K(\pi_nX, n+1) & \xrightarrow{\psi} & K(\pi_nX, n+1)_p^\wedge, \end{array}$$

which gives on homotopy fibres a map $P_nX \rightarrow P_nX_p^\wedge := F_{(k_{n-1})_p^\wedge}$, where the space $P_nX_p^\wedge$ is p -complete by Lemma 6.3.9. Finally, we take $X_p^\wedge := \lim_n P_nX_p^\wedge$, which is p -complete by Lemma 6.3.10. The maps $P_nX \rightarrow P_nX_p^\wedge$ and $X \rightarrow X_p^\wedge$ are \mathbb{F}_p -equivalences by the same arguments as in the finitely generated case.

It remains to show that we have the stated description of $\pi_nX_p^\wedge$. From the long exact sequence from the fibration $P_nX_p^\wedge \rightarrow P_{n-1}X_p^\wedge \rightarrow K(\pi_nX, n+1)_p^\wedge$ we see that there are isomorphisms

$$\begin{aligned} \pi_i P_nX_p^\wedge &\cong \pi_i P_{n-1}X_p^\wedge, \quad i < n, \\ \text{Hom}(\mathbb{Z}/p^\infty, \pi_nX) &\cong \pi_{n+1} P_nX_p^\wedge, \end{aligned}$$

and (using this isomorphism for $\pi_n P_{n-1}X_p^\wedge$) there is a short exact sequence

$$0 \rightarrow \text{Ext}(\mathbb{Z}/p^\infty, \pi_nX) \rightarrow \pi_n P_nX_p^\wedge \rightarrow \text{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1}X) \rightarrow 0.$$

The homotopy group $\pi_i P_nX_p^\wedge$ thus stabilizes for $n \geq i$ so there is no \lim^1 and we get $\pi_nX_p^\wedge \cong \pi_n P_nX_p^\wedge$, and so we have the desired description of this group. \square

REMARK 6.8.10. In fact, it can be shown that the short exact sequence

$$0 \rightarrow \text{Ext}(\mathbb{Z}/p^\infty, \pi_nX) \rightarrow \pi_nX_p^\wedge \rightarrow \text{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1}X) \rightarrow 0$$

always splits.

Finally, we end by proving the other direction in Theorem 6.8.1(i):

PROOF OF THEOREM 6.8.1(i). It remains to show that if X is simply connected and p -completed, then the abelian groups π_*X are derived p -complete. By the description of X_p^\wedge in Theorem 6.8.1(ii) we see if X is p -complete then there is a short exact sequence

$$0 \rightarrow \text{Ext}(\mathbb{Z}/p^\infty, \pi_nX) \rightarrow \pi_nX \rightarrow \text{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1}X) \rightarrow 0.$$

For $n = 2$ this says $\text{Ext}(\mathbb{Z}/p^\infty, \pi_2X) \xrightarrow{\sim} \pi_2X$. It follows that π_2X is derived p -complete, for example by Remark 6.7.12, or by using that the short exact sequence splits and then applying Proposition 6.7.11. Now if $\pi_{n-1}X$ is derived p -complete, then $\text{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1}X) = 0$ so the short exact sequence for π_nX gives $\pi_nX \cong \text{Ext}(\mathbb{Z}/p^\infty, \pi_nX)$. Thus we see by induction that all the homotopy groups of X are derived p -complete, as required. \square

6.9. Cosimplicial spaces, and the Bousfield–Kan model for p -completion

Notes from 2019.

6.10. Exercises

PS8.

EXERCISE 6.10.1 (Localizations in algebra). Let R be a ring, and let W_R be the class of morphisms of abelian groups f such that $R \otimes_{\mathbb{Z}} \ker(f) = 0$, $R \otimes_{\mathbb{Z}} \text{coker}(f) = 0$, and similarly $\text{Tor}_1^{\mathbb{Z}}(R, \text{co}/\ker(f)) = 0$.

(a) Let S be a set of primes, and $R = \mathbb{Z}[\frac{1}{p}, p \in S]$ ¹¹. Show that $M \rightarrow R \otimes_{\mathbb{Z}} M$ is a W_R -localization.

(b) Show that the same is not true for $R = \mathbb{F}_p$. Show, however, that $R \otimes_{\mathbb{Z}} M$ is always W_R -local.

(c) Let M be a finitely generated abelian group. Show that $\mathbb{Z}_p \otimes_{\mathbb{Z}} M \cong \lim_n M/p^n$. Deduce that the morphism $M \rightarrow \mathbb{Z}_p \otimes_{\mathbb{Z}} M$ is in $W_{\mathbb{F}_p}$, and deduce that it is so for every abelian group N .

(d) Prove further that if M is finitely generated, then $\mathbb{Z}_p \otimes_{\mathbb{Z}} M$ is $W_{\mathbb{F}_p}$ -local.

(d) Find an example of a non-finitely generated abelian group M where this isomorphism fails, and where $\mathbb{Z}_p \otimes_{\mathbb{Z}} M$ is not $W_{\mathbb{F}_p}$ -local.

(e) Prove that $\lim_n M/p^n$ is always $W_{\mathbb{F}_p}$ -local, but find an example where the canonical map $M \rightarrow \lim_n M/p^n$ is not a $W_{\mathbb{F}_p}$ -local equivalence.

EXERCISE 6.10.2 (Some p -adic stuff). Consider the forgetful functor $\mathbf{Mod}_{\mathbb{Z}_p} \rightarrow \mathbf{Mod}_{\mathbb{Z}}$.

(a) Show that it is fully faithful when restricted to finitely generated \mathbb{Z}_p -modules (Hint : you may want to use the fact that \mathbb{Z}_p is a PID). Deduce that the sentence “ A is a finitely generated \mathbb{Z}_p -module” makes sense, for an abelian group A .

(b) Show that it is not fully faithful in general. Deduce that the sentence “ A is a \mathbb{Z}_p -module” does not make sense.

EXERCISE 6.10.3 (The algebraic arithmetic squares). Show that the following two commutative squares are pullbacks :

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \prod_p \mathbb{Z}_{(p)} \\ \downarrow & & \downarrow \\ \mathbb{Q} & \longrightarrow & \mathbb{Q} \otimes_{\mathbb{Z}} (\prod_p \mathbb{Z}_{(p)}) \end{array} \quad \begin{array}{ccc} \mathbb{Z} & \longrightarrow & \prod_p \mathbb{Z}_p \\ \downarrow & & \downarrow \\ \mathbb{Q} & \longrightarrow & \mathbb{Q} \otimes_{\mathbb{Z}} (\prod_p \mathbb{Z}_p) \end{array}$$

What happens if we put $\prod_p(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})$, $\prod_p(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ instead ?

EXERCISE 6.10.4 (Homotopy limits and p -completion). (a) Prove that p -complete spaces are closed under homotopy pullbacks.

(b) Give an example of a space X with $\Omega(X_p^\wedge)$ is not equivalent to $(\Omega X)_p^\wedge$. Make sure you understand why that doesn't contradict (a).

EXERCISE 6.10.5 (Rational sphere). Recall from last week that $S_{\mathbb{Q}}^n \simeq K(\mathbb{Q}, n)$ if n is odd. Try to do the proof, and explain why we need n to be odd.

EXERCISE 6.10.6 (Milnor exact sequence). Let $\cdots \rightarrow C_{n+1} \rightarrow C_n \rightarrow \cdots \rightarrow C_0$ be an inverse system of chain complexes (each C_i is a chain complex). We want to understand the homology of the homotopy limit of the C_i 's.

(0) Let $\prod_i C_i \rightarrow \prod_i C_i$ be the map described by $(a_i)_{i \in \mathbb{N}} \mapsto (a_i - f_i(a_{i+1}))_{i \in \mathbb{N}}$, where $f_i : C_{i+1} \rightarrow C_i$ denotes the transition map. Prove that its kernel is naturally isomorphic to $\lim_i C_i$. We define $\lim_i^1 C_i$ to be its cokernel.

(1) Prove that the “inverse limit” functor is homotopical on the subcategory of inverse systems where each transition map is surjective (Hint : prove that the map from (0) is surjective if all the transition maps are surjective). Deduce a description of the homotopy limit.

(2) Assume now that our system has surjective transition maps. Prove that for each n , there is a natural short exact sequence of the form $0 \rightarrow \lim_i^1 H_{n+1}(C_i) \rightarrow H_n(\lim_i C_i) \rightarrow \lim_i H_n(C_i) \rightarrow 0$

(3) Deduce that in general, there is a short exact sequence of the form $0 \rightarrow \lim_i^1 H_{n+1}(C_i) \rightarrow H_n(\text{holim}_i C_i) \rightarrow \lim_i H_n(C_i) \rightarrow 0$

¹¹Can you show that these are the only subrings of \mathbb{Q} ?

EXERCISE 6.10.7 (Derived p -completion). (a) Deduce from exercise 6.10.6 a short exact sequence for the homology groups of the derived p -completion of an abelian group.

(b) Identify the \lim^1 -terms in more concrete terms.

(c) Compute the derived p -completion of \mathbb{Z}/p^∞ .

There's only one homework problem today. There are 3 bonus points.

EXERCISE 6.10.8 (p -completion of classifying spaces). [2+2+4+4+2+3+3+3 (3 bonus points)] In this exercise, we will try to understand some of $(BG)_p^\wedge$, for finite groups G . We will also do some general work around p -complete spaces.

Let $W_{\mathbb{F}_p}$ denote the class of morphisms of simplicial sets that induce an isomorphism on \mathbb{F}_p -homology. Recall that p -complete means $W_{\mathbb{F}_p}$ -local.

(a) Show that if $f \in W_{\mathbb{F}_p}$, then $f \times \text{id}_Z$ is also in $W_{\mathbb{F}_p}$ for any simplicial set Z . Deduce that, for a given Kan complex B , the following are equivalent :

I For any $f : X \rightarrow Y$ in $W_{\mathbb{F}_p}$, the morphism

$$\text{hom}_{\text{Ho}(\text{sSet})}(Y, B) \rightarrow \text{hom}_{\text{Ho}(\text{sSet})}(X, B)$$

induced by f is an isomorphism.

II For any $f : X \rightarrow Y$ in $W_{\mathbb{F}_p}$, the morphism

$$\text{map}(Y, B) \rightarrow \text{map}(X, B)$$

induced by f is an equivalence.

(recall that map is the mapping simplicial set)

(b) Deduce that if there is a homotopy fiber sequence $F \rightarrow E \rightarrow B$ of Kan complexes, and E, B are p -complete, then so is F . Deduce that if $E \rightarrow B$ is a principal fibration classified by a map from B to a p -complete space, and B is p -complete, then so is E .

(c) Suppose G is a p -group. Prove that BG is p -complete. (You can use freely that any p -group has a non-trivial center, and start with $G = C_p$). Suppose now that the order of G is prime to p . Show that the p -completion of BG is a point.

(d) Deduce from (c) and the Serre spectral sequence that if G is a finite group with a retraction onto its p -Sylow P , then $(BG)_p^\wedge \simeq BP$ (a p -Sylow is a subgroup of G such that $|G|/|P|$ is prime to p , a retraction means a morphism $r : G \rightarrow P$ which is the identity on P)

(e) Deduce from (d) the 2-completion of $B\Sigma_3$, where Σ_3 is the symmetric group on 3 letters. We now focus on its 3-completion.

(f) Let X be a p -complete space. Using the fact that a degree q map $S^n \rightarrow S^n$ is an isomorphism in \mathbb{F}_p -homology, prove that $\pi_n(X)$ has no q -torsion, for any q prime to p (we say it is " p -local"). (Hint : think of the fiber sequence $\text{map}_*(Y, X) \rightarrow \text{map}(Y, X) \rightarrow X$)

(g) Deduce that $\pi_1((B\Sigma_3)_3^\wedge) = 1$.

We will use cohomology to compute π_2, π_3 .¹²

Recall that, as an algebra $H^*(BC_3; \mathbb{F}_3) \cong \mathbb{F}_3[x, y]/(x^2)$, where $|x| = 1, |y| = 2$ (C_3 is the cyclic group on 3 elements). We accept that the C_2 -action induced on this cohomology by the extension $1 \rightarrow C_3 \rightarrow \Sigma_3 \rightarrow C_2 \rightarrow 1$ sends $x \mapsto -x$ and y to itself.

(h) Prove that the inclusion $C_3 \rightarrow \Sigma_3$ induces an injection on cohomology with \mathbb{F}_3 -coefficients, and prove further that the inclusion lands in C_2 -fixed points¹³. Deduce that $\pi_2((B\Sigma_3)_3^\wedge) \cong 0$, and that $\pi_3((B\Sigma_3)_3^\wedge) \cong H_3(B\Sigma_3; \mathbb{Z})$.

¹²The 3-completion of $B\Sigma_3$ is not trivial ! but it has no π_1 , so this shows how weirdly p -completion behaves away from simply-connected spaces

¹³In fact, in this case it is exactly the C_2 -fixed points

Smith theory and Sullivan's fixed point conjecture

[MORE HANDWRITTEN NOTES!!]

7.1. Equivariant homotopy theory and the theory of diagrams

Ref: [Die87; Die79; May96]

7.1.1. Assumptions and setup. Throughout the chapter, G will be a finite group. By a “collection” of subgroups \mathcal{C} of G we will mean a set of subgroups of G closed under conjugation by elements of G . By a “family” \mathcal{F} of subgroups we mean a set of subgroups G closed under conjugation and also closed under taking subgroups.

DEFINITION 7.1.1. A G -space is a topological space X with an action of G . A G -map (or G -equivariant map) $f : X \rightarrow Y$ between G -spaces is a map of spaces which respect the G -action, i.e., $f(g \cdot x) = g \cdot f(x)$ for all $x \in X$.

Two maps $f, f' : X \rightarrow Y$ are called G -homotopic, if there exists a G -map $F : X \times I \rightarrow Y$ such that $F(\cdot, 0) = f$ and $F(\cdot, 1) = g$. Here $X \times I$ is endowed with the G -action which is the product of the G -action on X and the trivial action on I .

DEFINITION 7.1.2. A G -CW complex is a space X with a G -action, which is built out of equivariant cells $G/H \times D^n$ for $H \leq G$ and $n \in \mathbb{N}_0$. More precisely, X is an increasing union $\bigcup_{n \geq 0} \text{sk}_n X$ of G -spaces sk_n (and is endowed with the weak topology of the union), by convention $\text{sk}_{-1} X = \emptyset$, and each $\text{sk}_n X$ is obtained from the previous $\text{sk}_{n-1} X$ by a pushout of the form

$$\begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod \phi_i} & \text{sk}_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \longrightarrow & \text{sk}_n, \end{array}$$

where I_n is a set (parametrising “equivariant n -cells”), with each $i \in I_n$ we associate a subgroup $H_i \leq G$, the space $G/H_i \times D^n$ and its “boundary” subspace $G/H_i \times S^{n-1}$ are endowed with the G -action coming from the trivial¹ action of G on D^n (resp. S^{n-1}) and the coset action of G on G/H_i , and with each $i \in I_n$ we associate a G -equivariant map $\phi_i : G/H_i \times S^{n-1} \rightarrow \text{sk}_{n-1} X$.

In these notes we will mainly consider G -spaces X which admit the structure of a G -CW complex. (Since G is finite this essentially amounts to assuming that X admits the structure of a CW complex on which G acts cellularly, i.e. permuting cells).

There are (at least) two different, natural notions of “ G -equivalence”:

DEFINITION 7.1.3. A G -map $f : X \rightarrow Y$ is called a G -equivalence (or sometimes strong G -equivalence) if there exists a G -map $g : Y \rightarrow X$, such that $f \circ g$ and $g \circ f$ are G -homotopic to the identity on Y and X respectively.

¹Some authors consider generic G -actions on (D^n, S^{n-1}) coming from n -dimensional representations of G ; this can have the advantage of reducing the number of needed cells, but prevents the cellular approximation theorem to hold.

A G -map $f : X \rightarrow Y$ is called an hG -equivalence (or sometimes weak G -equivalence) if f is a homotopy equivalence when considered as a non-equivariant map, i.e. as a plain map of spaces.

Note that “being G -equivalent” is an equivalence relation on G -spaces, by the G -equivariant analogue of the usual argument for spaces. Instead, being “ hG -equivalent” is not even a symmetric relation: for instance, if we consider a point $*$ with trivial G -action and a contractible space EG with free G -action, then the map $EG \rightarrow *$ is a hG -equivalence, but there exists no G -equivariant map $*$ $\rightarrow EG$, even less a hG -equivalence.

DEFINITION 7.1.4. Two G -spaces X and Y are said to be hG -equivalent if there is a zig-zag of spaces and hG -equivalences connecting them, of the form

$$X \leftarrow Z_1 \rightarrow Z_2 \leftarrow \cdots \leftarrow Z_{2n-1} \rightarrow Y.$$

THEOREM 7.1.5 (Equivariant Whitehead theorem). *A G -map $f : X \rightarrow Y$ between G -CW complexes is a (strong) G -equivalence if and only if the restricted map $f^H : X^H \rightarrow Y^H$ is a (non-equivariant) homotopy equivalence for all subgroups $H \leq G$. Here $X^H \subseteq X$ denotes the subspace of H -fixed points (points that are fixed by all elements of H).*

SKETCH OF PROOF. That G -equivalences induces equivalences on fixed-points follows directly from the definition: just observe that a G -homotopy $X \times I \rightarrow Y$ restricts for all $H \leq G$ to a homotopy $X^H \times I \rightarrow Y^H$.

Conversely, given a G -map $f : X \rightarrow Y$ between G -CW complexes, we can approximate it up to G -homotopy by a *cellular* G -map (prove this! You will see that you need that G acts trivially in the D^n -direction of an equivariant cell). If f is cellular, the mapping cylinder $\text{Cyl}(f)$ deformation retracts G -equivariantly onto Y ; so we can replace Y by $\text{Cyl}(f)$, and assume that f is an inclusion of G -CW complexes.

Proceeding inductively on skeleta, we can now construct a G -equivariant deformation retraction of Y onto X : for this we use repeatedly that if $H : Y \times I \rightarrow Y$ is a G -homotopy rel X from the identity of Y to a map $g_{n-1} : Y \rightarrow Y$ sending $X \cup \text{sk}_{n-1}Y$ inside X , for every attaching map $\phi_i : G/H_i \times S^{n-1} \rightarrow Y$ of a G -equivariant cell in $Y \setminus X$, the composite $g_{n-1} \circ \phi_i$ restricts to a map $D^n \cong H_i/H_i \times D^n \rightarrow Y^{H_i}$ which, by hypothesis, is homotopic rel $H_i/H_i \times S^{n-1}$ to a G -equivariant map $H_i/H_i \times D^n \rightarrow X^{H_i}$; this homotopy can be uniquely extended G -equivariantly over the entire G -equivariant cell, and we thus get a G -equivariant homotopy $X \cup \text{sk}_n Y \times I$ rel X starting from $g_{n-1}|_{\text{sk}_n Y}$ and ending with a map $X \cup \text{sk}_n Y \rightarrow X$.

We can then prove a G -equivariant version of the homotopy extension property for G -CW complexes (relative to their skeleta), and thus extend the previous to a homotopy $Y \times I \rightarrow Y$, ending with $g_n : Y \rightarrow Y$ which sends $X \cup \text{sk}_n Y$ inside X . The details are left as exercise; see also [Ben91, Thm. 6.4.2]. \square

REMARK 7.1.6. Note that, contrary to the above, a map being an hG -equivalence tells us nothing on what happens on the fixed-points for non-trivial subgroups H . E.g., for any G -space X , the projection $X \times EG \rightarrow X$ is an hG -equivalence, but $(X \times EG)^H = \emptyset$ for all $1 \subsetneq H \leq G$. (Where EG as usual denotes a free contractible G -space.)

In the next subsection, we will introduce a sort of derived version of the functor “taking fixed-points”, the homotopy fixed-points, which we can still work with in absence of meaningful actual fixed-points, and which by deep theorems agree with actual fixed-points in good cases (e.g. X finite, and “at a prime p ”).

PROPOSITION 7.1.7. *A G -map $f : X \rightarrow Y$ is an hG -equivalence if and only if $f \times EG : X \times EG \rightarrow Y \times EG$ is a G -equivalence.*

In particular two G -spaces X and Y are hG -equivalent if and only if $X \times EG$ and $Y \times EG$ are G -equivalent.

PROOF. The diagram

$$\begin{array}{ccc} X \times EG & \xrightarrow{f \times EG} & Y \times EG \\ \simeq \downarrow hG & & \simeq \downarrow hG \\ X & \xrightarrow{f} & Y \end{array}$$

shows that f is a homotopy equivalence of CW complexes if and only if $f \times EG$ is. Furthermore, if $H \leq G$ is a nontrivial subgroup then map $(f \times EG)^H$ is a map between empty spaces, so it is also a homotopy equivalence. Therefore, by Lemma 7.1.5, the map $f \times EG$ is a homotopy equivalence if and only if it is a G -equivalence.

For the “in particular”: If $X \times EG$ and $Y \times EG$ are G -equivalent, then X and Y are hG -equivalent via the zig-zag of G -equivariant homotopy equivalences

$$X \leftarrow X \times EG \simeq Y \times EG \rightarrow Y.$$

Conversely, if X and Y are hG -equivalent they are connected by a zig-zag of hG -equivalences.

$$X = X_0 \rightarrow X_1 \leftarrow X_2 \rightarrow \cdots \leftarrow X_n = Y$$

so applying $EG \times -$ shows that $X \times EG$ and $Y \times EG$ are G -equivalent, by the first part of the proposition and by the fact that G -equivalence is an actual equivalence relation. \square

REMARK 7.1.8. The localization of $G\text{Top}$ at the class of hG -equivalences, denoted $G\text{Top}[hG^{-1}]$ or briefly $hG - \text{Top}$, is thus equivalent to the G -equivariant homotopy category of spaces with a free G -action. The functor $EG \times -$ is a left deformation of $G\text{Top}$ at hG -equivalences, and its essential image is contained in free G -spaces.

REMARK 7.1.9. The theorem has a generalization for all families \mathcal{C} of subgroups of G . We will prove this later; for the moment we just anticipate that EG has to be replaced by the space $E\mathbf{O}_{\mathcal{C}}$ constructed as follows: we consider the “orbit category” $\mathbf{O}_{\mathcal{C}}$, with objects the cosets G/H for $H \in \mathcal{C}$ and morphisms G -equivariant maps; we consider the functor $\Theta: \mathbf{O}_{\mathcal{C}} \rightarrow \text{Top}$, sending $\mathbf{O}_{\mathcal{C}} \ni G/H \mapsto G/H \in \text{Set} \subset \text{Top}$; and $E\mathbf{O}_{\mathcal{C}} := \text{hocolim } \Theta = B(*, \mathbf{O}_{\mathcal{C}}, \Theta)$. There is an action of G on $E\mathbf{O}_{\mathcal{C}}$ induced by an action of G on Θ by natural isomorphisms, and we immediately notice that for $H \leq G$ the fix-points $(E\mathbf{O}_{\mathcal{C}})^H$ are empty if $H \notin \mathcal{C}$, whereas $(E\mathbf{O}_{\mathcal{C}})^H \cong B(\mathbf{O}_{\mathcal{C}})_{(G/H)/} \simeq *$ if $H \in \mathcal{C}$.

Note that for \mathcal{C} containing only the trivial group we have that $E\mathbf{O}_{\mathcal{C}}$ is a contractible, free G -space.

7.1.2. Equivariant homotopy theory, and the diagram categories: Elmendorf’s theorem. Ref: [Elm83].

Proposition 7.1.5 shows that equivariant equivalences can be described as non-equivariant equivalences on fixed-points.

Note that the category of G -spaces and hG -equivalences can be described as a subcategory of a functor category, namely the category of functors from G (viewed as a category with one object, and G as morphisms) to spaces: the allowed morphisms are the natural equivalences which object-wise are weak equivalences.

In this subsection we will address two natural questions:

- Can the the category of G -spaces and G -equivalences be understood via non-equivariant homotopy theory as a certain functor category? (yes, Thm 7.1.12)
- Is there a relationship between the homotopy type of X and that of, say, $X^{\mathbb{Z}/p}$? (In general little by Remark 7.1.6, but under finiteness and “at p ” assumptions a lot, cf later).

To warm up to Elmendorf’s theorem, we do the following baby case first.

DEFINITION 7.1.10 (The orbit category and the p -orbit category). Let $\mathbf{O}(G)$ denote the orbit category of G , i.e., the category with objects G/P , where P runs through the subgroups of G , and morphisms G -maps.

The p -orbit category $\mathbf{O}_p(G)$ is the full subcategory of $\mathbf{O}(G)$, where P is assumed to be a p -subgroup.

Note that

$$\mathrm{Hom}_{\mathbf{O}(G)}(G/P, G/Q) = \{g \in G \mid g^{-1}Pg \leq Q\}/Q = (G/Q)^P$$

and in particular $\mathrm{Hom}_{\mathbf{O}(G)}(G/P, G/P) = N_G(P)/P$.

We will, in fact, most often be interested in the *opposite* orbit and p -orbit categories. Let us do a couple of examples:

EXAMPLE 7.1.11. (1) Suppose that $G = \mathbb{Z}/p$, then the p -orbit category $\mathbf{O}_p(G)$ coincides with the entire category $\mathbf{O}(G)$: it has two objects G/G and G/e ; moreover $\mathrm{Hom}_{\mathbf{O}(G)}(G/e, G/e) \cong G$, there is precisely one morphism in each hom set $\mathrm{Hom}_{\mathbf{O}(G)}(G/e, G/G)$ and $\mathrm{Hom}_{\mathbf{O}(G)}(G/G, G/G)$ (i.e. G/G is terminal), and $\mathrm{Hom}_{\mathbf{O}(G)}(G/G, G/e) \cong \emptyset$. Diagrammatically, the opposite p -orbit category $\mathbf{O}_p(G)^{\mathrm{op}}$ looks like:

$$\bullet_{G/G} \longrightarrow \bullet_{G/e} \begin{array}{c} \xrightarrow{G^{\mathrm{op}}} \\ \downarrow \end{array}$$

(2) Suppose $G = \mathfrak{S}_3$, the 3rd symmetric group. A skeleton of the opposite 2-orbit category $\mathbf{O}_2(G)^{\mathrm{op}}$, accounting for only one of the three (isomorphic) objects of the form G/P with $|P| = 2$, looks like

$$\bullet_{G/P} \xrightarrow{G/P} \bullet_{G/e} \begin{array}{c} \xrightarrow{G^{\mathrm{op}}} \\ \downarrow \end{array}$$

that is only the identity self-map of G/P , G worth of self-maps of G/e , and 3 maps from G/P to G , which naturally identify with G/P .

THEOREM 7.1.12 (Elmendorf [Elm83]). *The functor*

$$\Phi : G\mathrm{Top} \rightarrow \mathrm{Top}^{\mathbf{O}(G)^{\mathrm{op}}}, \quad X \mapsto \{X^H\}_{G/H \in \mathbf{O}(G)^{\mathrm{op}}}$$

induces a 1-to-1-correspondence between classes of objects

$$\{G\text{-spaces}\}/G\text{-equivalence} \longleftrightarrow \{\text{Functors } \mathbf{O}(G)^{\mathrm{op}} \rightarrow \mathrm{Top}\}/\text{objectwise h.e..}$$

In fact, with suitable definition of model category structures on the left- and right-hand side, this induces a Quillen equivalence of model categories.

In particular Φ induces an equivalence between localizations of categories

$$G\mathrm{Top}[\{G\text{-equivalences}\}^{-1}] \xrightarrow{\cong} \mathrm{Top}^{\mathbf{O}(G)^{\mathrm{op}}}[\{\text{objectwise homotopy equivalences}\}^{-1}].$$

/<empty citation>/

Before giving a sketch of proof of this theorem, let us do the special case $G = \mathbb{Z}/p$, which is already interesting.

EXAMPLE 7.1.13 (Elmendorf's theorem for $G = \mathbb{Z}/p$). Elmendorf's theorem for \mathbb{Z}/p claims that giving a G -space X , up to G -equivalence, is equivalent to giving a space Y_0 , a G -space Y_1 and a (non-equivariant) map $f : Y_0 \rightarrow Y_1^G$, up to natural equivalence of diagrams. We want to give an inverse functor. Given Y_0 and Y_1 , we can define a space X as the homotopy pushout of the diagram

$$\begin{array}{ccc} EG \times Y_0 & \xrightarrow{\mathrm{proj}} & Y_0 \\ \downarrow EG \times f & & \\ EG \times Y_1 & & \end{array}$$

The action of G on X is given by constructing the homotopy pushout as a double mapping cylinder and by noticing that, considering Y_0 as a G -space with trivial action, the previous is a diagram of G -spaces.

To show well-definedness up to equivalence of data, suppose that Y'_0, Y'_1 and $f : Y'_0 \rightarrow (Y'_1)^G$ is another set of data as Y_0, Y_1 and $f : Y_0 \rightarrow Y_1^G$, and suppose that there are maps $g_0 : Y_0 \rightarrow Y'_0$ and $g_1 : Y_1 \rightarrow Y'_1$ such that f_1 is G -equivariant, the composite $f' \circ g_0 : Y_0 \rightarrow (Y'_1)^G \subset Y'_1$ coincides with the composite $g_1 \circ f : Y_0 \rightarrow Y_1^G \subset Y_1 \rightarrow Y'_1$; finally, suppose that g_0 and g_1 are weak equivalences of spaces. Then we can use g_0 and g_1 to construct a natural transformation between the two pushout diagrams $EG \times Y_1 \leftarrow EG \times Y_0 \rightarrow Y_0$ and $EG \times Y'_1 \leftarrow EG \times Y'_0 \rightarrow Y'_0$; notice that this is a natural transformation of diagrams of G -spaces. Taking homotopy pushouts/double mapping cylinders, we obtain a weak equivalence $X \rightarrow X'$ which is also G -equivariant, showing that X and X' are hG -equivalent. Moreover $X^G \cong Y_0$ and $(X')^G \cong Y'_0$, and the restricted map $X^G \rightarrow (X')^G$ coincides with g_0 , so it is a weak equivalence. It follows from Theorem 7.1.5 that X is G -equivalent to X' .

Finally, one checks that these procedures are each other's inverses. Taking fixed-points on the homotopy pushout one easily sees that one recovers the diagram $Y_0 \rightarrow Y_1$, up to homotopy. For this, note also that X is homotopy equivalent (non- G -equivariantly) to the homotopy pushout of $Y_1 \leftarrow Y_0 \rightarrow Y_0$, i.e. to Y_1 .

Likewise, if one starts with a G -space, the homotopy colimit of $EG \times X \leftarrow EG \times X^G \rightarrow X^G$ maps to X , and this map is a G -equivalence, since it induces an equivalence on all fixed-points (on e -fixed-points and on G -fixed points).

SKETCH OF PROOF OF ELMENDORF'S THEOREM; GENERAL CASE. The inverse $\Psi : \text{Top}^{\mathbf{O}(G)^{\text{op}}} \rightarrow G\text{Top}$ is given by sending a functor $F : \mathbf{O}(G)^{\text{op}} \rightarrow \text{Top}$ to the geometric realization of the simplicial space whose space of n -simplices contains sequences

$$(G/e \rightarrow G/P_0 \rightarrow G/P_1 \rightarrow \cdots \rightarrow G/P_n, x \in F(G/P_n)),$$

and the topology and the simplicial maps are the obvious ones.

This is by definition the two-sided bar construction $B(F, \mathbf{O}(G), \Theta)$, where $\Theta : \mathbf{O}(G) \rightarrow G\text{Top}$ is the “identity functor”, sending the object $G/H \in \mathbf{O}(G)$ to the G -space (which is in fact a G -set) G/H ; another description of $\Psi(F)$, equivalent to $B(F, \mathbf{O}(G), \Theta)$, is $\text{hocolim}_{(\mathbf{O}(G)^{\text{op}}) \downarrow G/e} F$.

We can consider Θ also as a functor $\mathbf{O}(G) \rightarrow \text{Top}$ together with an action of G on Θ by natural isomorphisms; in this light we immediately obtain an action of G on $B(F, \mathbf{O}(G), \Theta)$, by naturality of the bar construction in the second functor.

Alternatively, we can let G act on the overcategory $(\mathbf{O}(G)^{\text{op}}) \downarrow G/e$ by letting it act on G/e , using naturality of the overcategory construction in the object “over which” the overcategory is taken. The functor $F : (\mathbf{O}(G)^{\text{op}}) \downarrow G/e$ is really F composed with the “projection-on-the-source” functor $(\mathbf{O}(G)^{\text{op}}) \downarrow G/e \rightarrow (\mathbf{O}(G)^{\text{op}})$, so F is invariant under the action of G on $(\mathbf{O}(G)^{\text{op}}) \downarrow G/e$. We then have an action of G on $\text{hocolim}_{(\mathbf{O}(G)^{\text{op}}) \downarrow G/e} F$. This gives another description of the action of G on $\Psi(F)$.

We now want to construct a natural transformation $\eta : \Phi\Psi \Rightarrow \text{Id}_{\text{Top}^{\mathbf{O}(G)^{\text{op}}}}$ which is objectwise a levelwise homotopy equivalence: that is, for all F and all G/H , $(\eta_F)_{G/H} : (\Phi\Psi(F))(G/H) \rightarrow F(G/H)$ is a homotopy equivalence of spaces.

For this, note that since the functor $F : (\mathbf{O}(G)^{\text{op}}) \downarrow G/e \rightarrow \text{Top}$ is G -invariant (and in particular H -invariant for all $H \leq G$), we have a natural identification

$$(\Phi\Psi(F))(G/H) = \left(\text{hocolim}_{(\mathbf{O}(G)^{\text{op}}) \downarrow G/e} F \right)^H \cong \text{hocolim}_{((\mathbf{O}(G)^{\text{op}}) \downarrow G/e)^H} F.$$

Moreover note that a morphism in $(\mathbf{O}(G)^{\text{op}}) \downarrow G/e$ is fixed by H if and only if its source and target objects are fixed by H : this is again a consequence of the fact that the action of G on $(\mathbf{O}(G)^{\text{op}}) \downarrow G/e$ is given by postcomposing with automorphisms of G/e ; so $((\mathbf{O}(G)^{\text{op}}) \downarrow G/e)^H$

is a full subcategory of $(\mathbf{O}(G)^{\text{op}}) \downarrow G/e$, and its objects are precisely those arrows $G/e \xrightarrow{f} G/P$ in $\mathbf{O}(G)$ that are invariant under precomposition by elements of $H \leq \text{Aut}_{\mathbf{O}(G)}(G/e)$; this translates into the requirement that f factors through the projection $G/e \rightarrow G/H$; but then we just have $((\mathbf{O}(G)^{\text{op}}) \downarrow G/e)^H \cong (\mathbf{O}(G)^{\text{op}}) \downarrow G/H$, which is a category with terminal object $G/H \xrightarrow{\text{id}_{G/H}} G/H$, hence we have a natural weak equivalence

$$\text{hocolim}_{((\mathbf{O}(G)^{\text{op}}) \downarrow G/e)^H} F \cong \text{hocolim}_{(\mathbf{O}(G)^{\text{op}}) \downarrow G/H} F \xrightarrow{\sim} F(G/H).$$

We declare $(\eta_F)_{G/H}$ to be the composite weak equivalence

Viceversa, we have a natural transformation $\sigma: \Psi\Phi \Rightarrow \text{Id}_{G\text{Top}}$: given a G -space X , we consider the composite map

$$\sigma_X: \Psi\Phi(X) = \text{hocolim}_{(f: G/e \rightarrow G/H) \in (\mathbf{O}(G)^{\text{op}}) \downarrow G/e} X^H \rightarrow \text{colim}_{(f: G/e \rightarrow G/H) \in (\mathbf{O}(G)^{\text{op}}) \downarrow G/e} X^H \rightarrow X$$

given by first mapping the homotopy colimit to the actual colimit, and then by mapping the copy of X^H corresponding to f to X along multiplication by f : here we use that a G -equivariant map $G/e \cong G \rightarrow G/H$ is given by first applying a *right* multiplication by an element $f \in G$, getting a G -equivariant map $- \cdot f: G \rightarrow G$, and then projecting onto G/H ; the final map only depends on the coset $fH \in G/H$ (and similarly the map $f: X^H \rightarrow X$ only depends on the coset $fH \in G/H$).

One checks that σ_X is G -equivariant, and that it induces for all $H \leq G$ a weak equivalence on H -fixed-points, by an argument analogue to the one above. \square

7.2. Homotopy limits and homotopy fixed-points

- Model for homotopy fixed points as space of sections.
- examples.

Notes from CatTop 2009.

————— For a subgroup $H \leq G$ we consider the functors $(-)^H: G\text{Top} \rightarrow \text{Top}$ and $(-)/H: G\text{Top} \rightarrow \text{Top}$, called “ H -fixed-points” and “ H -orbits” respectively. Both functors are homotopy invariant under G -equivalences, but they are not homotopy invariant under hG -equivalences when H is not the trivial group.

In this section we will introduce the functor “ H -homotopy-fixed-points”, denoted $(-H)^{hH}: G\text{Top} \rightarrow \text{Top}$, and show that this functor is homotopy invariant under hG -equivalence. We will furthermore discuss how to calculate fixed-points and homotopy fixed-points, and how to relate them. More precisely we want to see:

- When V is an *elementary abelian group* and X is a finite V -CW complex, the cohomology $H^*(X^V; \mathbb{F}_p)$ is determinable from cohomological information which only depends on the hV -homotopy type of X (i.e. we can replace X up to hV -equivalence and still be able to compute correctly $H^*(X^V; \mathbb{F}_p)$).
- More generally, when p is a prime, P is a p -group and X is a finite P -CW complex, there is a close relationship between the hP -homotopy type of X , and the homotopy type of the space X^P “at the prime p ” (read: after p -completion).
- Replacing fixed-points by homotopy fixed-points, we find a natural home for these theorems, and we can significantly weaken the finiteness assumption on X .

This will be used in the next section to explore how we can get strong invariants of X (at a prime p) by just considering all the spaces X^{hP} together (assembled as a functor out of the p -orbit category); these will be clearly invariants of X up to hG -equivalence.

7.2.1. Homotopy orbit space and homotopy fixed points. Ref: [DW94] Denote by BG a classifying space of G , i.e., $BG \simeq K(G, 1)$, and let EG denote a contractible G -space on which G acts freely. We have $BG \simeq EG/G$; moreover two different choices of BG as above are homotopy equivalent, and two choices of EG are G -equivalent.

DEFINITION 7.2.1. We define the Borel construction, or “ G -homotopy-orbits” as the functor $(-)_hG: G\text{Top} \rightarrow \text{Top}$, sending

$$X \mapsto X_{hG} := (X \times EG)/G,$$

where G acts diagonally on the product.

Dually we define the “ G -homotopy-fixed-points” as the functor $(-)^{hG}: G\text{Top} \rightarrow \text{Top}$, sending

$$X \mapsto X^{hG} := \text{map}_G(EG, X)$$

i.e., the space of G -equivariant maps, or equivalently the G -fixed-points of the mapping space $\text{map}(EG, X)$, which is a G -space by letting $g \in G$ act on $f: EG \rightarrow X$ by $(g \cdot f)(x) = gf(g^{-1}x)$.

Note that $*_{hG} = BG$. Note also that we have a map $X^G \rightarrow X^{hG}$ induced by the G -equivariant map $EG \rightarrow *$ (which is a hG -equivalence!). The homotopy fixed-points should be thought of as (right) derived fixed-points (in fact $\text{map}(EG, -): G\text{Top} \rightarrow G\text{Top}$ is a right deformation adapted to $(-)^G: G\text{Top} \rightarrow \text{Top}$, this should become clear later).

LEMMA 7.2.2. *Let X be a G -space; then the equivariant map $X \rightarrow *$ induces a map $\lambda: X_{hG} \rightarrow BG$, which is a fibre bundle with fibre X . Then X^{hG} can be identified with the space of sections of $\lambda: X_{hG} \rightarrow BG$.*

PROOF. The map $EG \times X \rightarrow EG \times *$ is a fibre bundle, and G acts freely (and properly discontinuously, up to choosing a good model for EG ...) on both spaces; it follows that $EG \rightarrow BG$ is a covering map, and a product trivialisation of $\lambda: X_{hG} \rightarrow BG$ can be found over any open set of BG for which $EG \rightarrow BG$ can be trivialised, pushing down the (global) trivialisation of $EG \times X \rightarrow EG \times *$.

We now describe the maps in the two directions: Given a G -equivariant map $f: EG \rightarrow X$, we get the commutative diagram of G -maps

$$\begin{array}{ccc} EG & \xrightarrow{(f, \text{id}_{EG})} & X \times EG \\ & \searrow & \downarrow \text{proj} \\ & & EG \end{array}$$

which upon quotienting out by G produces the section.

In the other direction, given a section $s: BG \rightarrow X_{hG}$ of λ we get a G -equivariant map $\tilde{s}: EG \rightarrow X \times EG$ by using that the trivial bundle $X \times EG \rightarrow EG$ is the pullback of the bundle λ along $EG \rightarrow BG$; we then consider the composite $EG \xrightarrow{\tilde{s}} X \times EG \xrightarrow{\text{proj}} X$. \square

PROPOSITION 7.2.3. *If $f: X \rightarrow Y$ is an hG -equivalence, then it induces homotopy equivalences $f_{hG}: X_{hG} \rightarrow Y_{hG}$ and $f^{hG}: X^{hG} \rightarrow Y^{hG}$.*

PROOF. If $X \rightarrow Y$ is an hG -equivalence, then $X \times EG \rightarrow Y \times EG$ is a G -equivalence, by Lemma 7.1.5, and hence $X_{hG} \rightarrow Y_{hG}$ is an equivalence (over BG) by the invariance property of orbit spaces under G -equivalence: in general, if A and B are G -equivalent G -spaces, then the G -maps and the G -homotopies witnessing this can be quotiented by G to get maps and homotopies witnessing that A/G is homotopy equivalent to B/G . This shows the first claim.

To see the second claim, note that by Lemma 7.2.2 the map $X^{hG} \rightarrow Y^{hG}$ identifies with the map from the space of sections of $\lambda_X: X_{hG} \rightarrow BG$ to the space of sections of $\lambda_Y: Y_{hG} \rightarrow BG$, given by composition with f_{hG} (here we use that f_{hG} is a map of bundles over BG). By elementary homotopy theory, we can choose an inverse equivalence *over* BG to the map $X_{hG} \rightarrow Y_{hG}$: for example we can first choose any homotopy inverse $i: Y_{hG} \rightarrow X_{hG}$ to f_{hG} ; we then have that λ_Y is homotopic to $\lambda_X \circ i$, and lifting such a homotopy along λ_X , starting with i , gives the desired homotopy inverse *over* BG . Similarly one can achieve that also the homotopies witnessing the homotopy equivalence of X_{hG} and Y_{hG} are *over* BG . But then we can use these maps and homotopies over BG to get maps and homotopies between the spaces of sections. \square

PROPOSITION 7.2.4. *The functors $EG \times -$ and $\text{map}(EG, -)$ turn an hG -equivalence $f : X \rightarrow Y$ into a G -equivalence.*

In particular two G -spaces X and Y are hG -equivalent if and only if $X \times EG$ and $Y \times EG$ are G -equivalent, and if and only if $\text{map}(EG, X)$ and $\text{map}(EG, Y)$ are G -equivalent.

PROOF. We have already seen the claim about $EG \times -$ in Proposition 7.1.7. The proof about $\text{map}(EG, -)$ is similar: The map $\text{map}(EG, f)$ is an hG -equivalence, since it is G -equivariant and a homotopy equivalence of spaces. Furthermore, if $H \leq G$ is a nontrivial subgroup then $\text{map}(EG, f)^H : \text{map}(EG, X)^H \rightarrow \text{map}(EG, Y)^H$ can be identified, up to using EG also as a model for EH (and indeed EG is contractible and admits a free H -action), with $\text{map}_H(EH, f) = f^{hG}$, which is a homotopy equivalence by Proposition 7.2.3. It follows from Lemma 7.1.5 that $\text{map}(EG, f)$ is in fact G -equivalences.

Likewise if X and Y are hG -equivalent they are connected by a zig-zag of hG -equivalences.

$$X = X_0 \rightarrow X_1 \leftarrow X_2 \rightarrow \cdots \leftarrow X_n = Y$$

Applying $\text{map}(EG, -)$ to this shows the one direction of the ‘in particular’. The other direction is immediate, noting that $X = \text{map}(*, X) \rightarrow \text{map}(EG, X)$ is an hG -equivalence. \square

REMARK 7.2.5. Examining this a bit more closely, one can prove that the category of G -spaces localised at hG -equivalences, is equivalent to the category of spaces over BG localised at homotopy equivalences; in symbols

$$G\text{Top}[\{hG\text{-equivalences}\}^{-1}] \cong \text{Top}_{/BG}[\{\text{homotopy equivalences (over } BG)\}^{-1}].$$

One can in fact construct two model structures on hG -spaces, where $EG \times -$ and $\text{map}(EG, -)$ are cofibrant respectively fibrant replacements.

7.3. Smith theory and the localization theorem

7.3.1. Smith theory: The fixed-points $X^{\mathbb{Z}/p}$ from X . We now want to discuss to which extent X^G can be determined from the hG -homotopy type of X .

7.3.2. The localization theorem.

DEFINITION 7.3.1. For a G -space X , we define the Borel equivariant cohomology of X as the ordinary cohomology of the Borel construction, i.e., $H_{hG}^*(X) = H^*(X_{hG})$. Similarly we define, for a commutative ring R , $H_{hG}^*(X; R) = H^*(X_{hG}; R)$; and if $Y \subset X$ is an inclusion of G -spaces, we get an inclusion $Y_{hG} \subset X_{hG}$ and define $H_{hG}^*(X, Y; R) = H^*(X_{hG}, Y_{hG}; R)$.

Borel cohomology is a G -equivariant cohomology theory (it is in particular a contravariant functor from $G\text{Top}$ to graded abelian groups), and obviously it only depends on X up to hG -equivalence. Note that $H_{hG}^*(*)$ is by definition $H^*(BG)$, which coincides with the group cohomology of G .

Notice also that $H_{hG}^*(-)$ is the composition of two functors, namely $(-)_hG : G\text{Top} \rightarrow \text{Top}$ and $H^*(-)$, which is in fact a contravariant functor from Top to graded rings; hence $H_{hG}^*(X)$ is naturally a graded ring; similarly in the relative case and with coefficients in R .

To understand the statement of the next theorem, note that the G -equivariant map $X \rightarrow *$ always gives rise to a map of rings $H^*(BG) \rightarrow H_{hG}^*(X)$. Moreover, for any couple of G -spaces (X, Y) there is a relative version of the cup product

$$H_{hG}^*(X, Y) \otimes H_{hG}^*(X) \rightarrow H_{hG}^*(X, Y)$$

which essentially is the standard, relative cup product

$$H^*(X_{hG}, Y_{hG}) \otimes H^*(X_{hG}) \rightarrow H^*(X_{hG}, Y_{hG}).$$

This makes $H_{hG}^*(X, Y)$ into a graded module over the graded ring $H_{hG}^*(X)$. In particular $H_{hG}^*(X, Y)$ is a $H^*(BG)$ -module.

THEOREM 7.3.2 (Localization theorem, Borel, Quillen,...). *Let $V = \mathbb{Z}/p$ and consider the cohomology ring $H^*(V; \mathbb{F}_p)$; let S denote the set of non-trivial elements in $H^1(V; \mathbb{F}_p)$ if $p = 2$, and the set of non-nilpotent elements in $H^2(V; \mathbb{F}_p)$ if p is odd.*

For a finite V -CW complex X and a V -subcomplex Y we have an isomorphism of localised modules, induced by the inclusion of pairs $(X^V, Y^V) \hookrightarrow (X, Y)$:

$$S^{-1}H_{hV}^*(X, Y; \mathbb{F}_p) \xrightarrow{\cong} S^{-1}H_{hV}^*(X^V, Y^V; \mathbb{F}_p) = S^{-1}H^*(V; \mathbb{F}_p) \otimes H^*(X^V, Y^V; \mathbb{F}_p)$$

There is also a more general version involving two elementary abelian p -groups $W \leq V$, which we leave it to the reader to state (and prove).²

SKETCH OF PROOF. The second equality is just the Künneth isomorphism together with localisation by S : the action of V on the pair (X^V, Y^V) is trivial, hence the pair $((X^V)_{hV}, (Y^V)_{hV})$ can be identified with the pair $(X^V \times BV, Y^V \times BV)$.

For the first isomorphism, by the long exact sequence in Borel cohomology, exactness of S -localisation and induction on cells, it is enough to prove the statement for relative cells, i.e. when (X, Y) is of the form $(V/V \times D^n, V/V \times S^{n-1})$ or $(V/e \times D^n, V/e \times S^{n-1})$.

For V/V there is nothing to prove, as $(X^V, Y^V) = (X, Y)$. For V/e instead we have $(X^V, Y^V) = (\emptyset, \emptyset)$, so that $S^{-1}H_{hV}^*(X^V, Y^V; \mathbb{F}_p)$ vanishes. However in this case $H_{hV}^*(X, Y; \mathbb{F}_p)$ can be identified with the ordinary, relative cohomology $H^*(D^n, S^{n-1}; \mathbb{F}_p)$, which is concentrated in the single degree n . Since S is non-empty and contains positive-degree elements, multiplication by an element in S is the zero map, and localising at S gives $S^{-1}H_{hV}^*(X, Y; \mathbb{F}_p) = 0$ as desired. \square

In the previous proof it is crucial that X is a finite V -CW complex, or at least it is finite-dimensional. Otherwise taking $X = EV$ and $Y = \emptyset$ gives an easy counterexample to the statement (at least if we know that $S^{-1}H^*(BV; \mathbb{F}_p)$ is non-trivial).

COROLLARY 7.3.3 (P.A.Smith). *Let X be a finite V -CW complex. If X is \mathbb{F}_p -acyclic, then so is X^V . If X is an \mathbb{F}_p -homology sphere, then so is X^V .*

To make the previous statement correct, we have to define a \mathbb{F}_p -homology sphere as a space having the same \mathbb{F}_p -homology as a sphere S^n for $n \dots$ at least -1 ! By convention $S^{-1} = \emptyset$, and we declare also \emptyset to be a \mathbb{F}_p -homology sphere.

PROOF. The map $X_{hG} \rightarrow BG$ can be regarded as a map of fibre bundles over BG , with map between fibres being $X \rightarrow *$. If X is \mathbb{F}_p -acyclic, the comparison map between second pages of Serre spectral sequences in \mathbb{F}_p -cohomology is an isomorphism, hence also the ∞ -pages are isomorphic and therefore we have $H_{hV}^*(X; \mathbb{F}_p) \cong H_{hV}^*(*; \mathbb{F}_p)$. This implies by Theorem 7.3.2 that $S^{-1}H^*(V; \mathbb{F}_p) \otimes H^*(X^V; \mathbb{F}_p)$ is isomorphic (along the natural map) to $S^{-1}H^*(V; \mathbb{F}_p) \otimes H^*(*; \mathbb{F}_p)$, and using that tensoring over \mathbb{F}_p with a non-zero graded \mathbb{F}_p -vector space such as $S^{-1}H^*(V; \mathbb{F}_p)$ detects isomorphisms, we obtain the first statement.

For the second statement we use similarly that X is a \mathbb{F}_p -homology sphere if and only if the pair (C, X) has relative \mathbb{F}_p -cohomology concentrated in a single degree, where $C = \text{Cone}(X)$; we then run a similar argument with spectral sequences to compute $H^*(C_{hV}, X_{hV}; \mathbb{F}_p)$ and identify it (as a graded \mathbb{F}_p -vector space) with $H^*(C, X; \mathbb{F}_p) \otimes H^*(V; \mathbb{F}_p)$: here it is crucial that the action of V on $H^*(C, X)$ is trivial (this is because a p -group cannot act non-trivially on \mathbb{F}_p , as $\text{Aut}(\mathbb{F}_p) = \mathbb{F}_p^* \cong C_{p-1}$), and that there is a single non-trivial row in the E_2 -page so the spectral sequence collapses. One concludes again using Theorem 7.3.2, using that $H^*(V; \mathbb{F}_p)$ is finite-dimensional in each degree, and noticing that $(C^V, X^V) \cong (\text{Cone}(X^V), X^V)$. \square

²An “elementary abelian p -group” is a (finite-dimensional) vector space over \mathbb{F}_p .

7.3.3. Smith theory, Dwyer-Wilkerson and Lannes style. The formula for the homology of the fixed-points, due to Dwyer-Wilkerson, where we'll explain the notation after the theorem.

THEOREM 7.3.4 (Dwyer-Wilkerson [DW88; DW91; LZ95]). *Let X be a finite V -CW complex. Then*

$$H_{hV}^*(X^V; \mathbb{F}_p) \xrightarrow{\cong} \mathrm{Un}(S^{-1}H_{hV}^*(X^V; \mathbb{F}_p)) \xleftarrow{\cong} \mathrm{Un}(S^{-1}(H_{hV}^*(X; \mathbb{F}_p)))$$

and in particular

$$H^*(X^V; \mathbb{F}_p) \cong \mathbb{F}_p \otimes_{H^*(V; \mathbb{F}_p)} \mathrm{Un}(S^{-1}H_{hV}^*(X; \mathbb{F}_p)).$$

Here Un is the largest unstable module, i.e.,

DEFINITION 7.3.5. For a module M over the Steenrod algebra \mathcal{A}_2 define

$$\mathrm{Un}(M) = \{x \in M \mid Sq^I(x) = 0 \text{ if } \mathrm{excess}(I) > |x|\}$$

i.e., the largest submodule which satisfies the instability condition. Similarly for a module over the Steenrod algebra \mathcal{A}_p , for p prime.

We also need to say how we get a Steenrod action after inverting S . This is described as follows.

OBSERVATION 7.3.6. Let R be an \mathbb{F}_p -algebra with an action of the Steenrod algebra \mathcal{A}_p , let M an unstable R -module over the Steenrod algebra (i.e., $R \otimes M \rightarrow M$ satisfies the Cartan formula), and let S a multiplicative subset of R ; then $S^{-1}M$ has a Steenrod algebra action given by the following formula

$$P_\xi(x/s) = P_\xi(x)(P_\xi(s))^{-1} = P_\xi(x)(s + Sq^1(s)\xi + Sq^2(s)\xi^2 + \cdots)^{-1}$$

For p prime, the formula is similar and only involves the power operations; the action of the Bockstein is much simpler: using that β is trivial on S , setting $\beta(x/s) = \beta(x)/s$ is meaningful.

SKETCH OF PROOF OF THEOREM 7.3.4. By the localization theorem we just need to see that

$$\mathrm{Un}(S^{-1}H^*(V) \otimes H^*(X^V)) = H^*(V) \otimes H^*(X^V)$$

By a small calculation this in fact holds for any unstable module M

$$H^*(V) \otimes M \xrightarrow{\cong} \mathrm{Un}(S^{-1}H^*(V) \otimes M)$$

This is a small calculation. The key reason is that, by the formula above, elements in the denominator have quite long (infinite?) Steenrod squares on them, and presence of these in hence not compatible with instability.... The short paper by Dwyer-Wilkerson well is worth reading! \square

REMARK 7.3.7. Note how Theorem 7.3.4 should seem surprising, in that, a priori, there would be no reason to believe that the cohomology $H^*(X^V; \mathbb{F}_p)$ should be determinable from the hV -homotopy type of X . (Historically, the first display of this was through the Sullivan conjecture; see below.)

EXERCISE 7.3.8. Use the Dwyer-Wilkerson theorem to prove the following: let X be a finite $\mathbb{Z}/2$ -CW complex with $H^*(X; \mathbb{F}_2) \cong H^*(\mathbf{RP}^n; \mathbb{F}_2) = \mathbb{F}_2[x]/x^{n+1}$; then $H^*(X^{\mathbb{Z}/2}; \mathbb{F}_2) \cong H^*(\mathbf{RP}^{i-1}) \oplus H^*(\mathbf{RP}^{j-1})$, for some i, j satisfying $i + j = n + 1$. (Each of these pairs is realized by the action of $\mathbb{Z}/2$ on \mathbf{RP}^n flipping i of the axes.)

7.4. The Sullivan conjecture and homotopy Smith theory

THEOREM 7.4.1 (Lannes, see [Lan92, §4.9]). *There exists a functor Fix (which is in fact the functor $Un(S^{-1}(-))$, derived from the so-called Lannes T -functor, such that for any space X (subject to some mild technical conditions)*

$$H^*(X^{hV}; \mathbb{F}_p) = \mathbb{F}_p \otimes_{H^*(V)} Fix(H_{hV}^*(X; \mathbb{F}_p))$$

Note that the right-hand side depends on $H_{hV}^*(X; \mathbb{F}_p)$, so ultimately on X_{hV} , whereas the left-hand side on X^{hV} .

COROLLARY 7.4.2 (Sullivan Conjecture, Miller, Carlsson, Lannes). *Let X be a finite P -CW complex, P a finite group, then*

$$(X^P)_p^\wedge \xrightarrow{\cong} (X_p^\wedge)^{hP}$$

In particular, under mild assumptions e.g., fixed-points simply connected,

$$H^*(X^P; \mathbb{F}_p) \cong H^*(X^{hP}; \mathbb{F}_p)$$

GUIDE TO VARIOUS PROOFS OF COR 7.4.2: Before you go, yeah, well, sure, note that in the case the action is trivial, the statement reads that $\text{map}(BP, X) \xrightarrow{\cong} X$, in particular $\text{map}_*(\mathbf{RP}^\infty, X) \cong *$, when X is a finite complex, which are not at all obvious—Sullivan considered it a test case for the conjecture. There are several proofs of the general statement, none of them easy!

There's a (technical) proof by Dwyer-Miller-Neisendorfer [DMN89], building on the original work by Miller in the trivial action case [Mil84]. There is a technical proof by Carlsson [Car91], deriving in from the Segal conjecture which he proved.

Finally there is Lannes proof which deduces it from Theorem 7.4.1. It is more “robust” than the others in that it builds up a lot of machinery of independent interest along the way, and then deduces the statement as a special case. \square

SKETCH OF PROOF OF THEOREM 7.4.1. Lannes proof (as simplified by Farjoun-Smith [DFS90]) proceeds in a number of steps, see [Lan92]: The steps in the proof:

- Let \mathcal{U} be the category of unstable modules over the Steenrod algebra \mathcal{A}_p . Observe that the functor $H^*(V; \mathbb{F}_p) \otimes_{\mathbb{F}_p} - : \mathcal{U} \rightarrow \mathcal{U}$ has a left adjoint T_V , i.e.,

$$\text{Hom}_{\mathcal{U}}(T_V A, B) \cong \text{Hom}_{\mathcal{U}}(A, H^*(V; \mathbb{F}_p) \otimes B)$$

- Prove that T_V is exact and commutes with tensor products (i.e. $T_V M \otimes_{\mathbb{F}_p} T_V N \cong T_V(M \otimes_{\mathbb{F}_p} N)$). [Note that exactness of T_V in degree 0 means that $H^*(V) \otimes -$ is exact, i.e. that $H^*(V)$ is injective in the category \mathcal{U} .]
- Prove under mild conditions on X

$$T_V H^*(X; \mathbb{F}_p) \xrightarrow{\cong} H^*(\text{map}(BV, X); \mathbb{F}_p)$$

by first verifying this for Eilenberg-MacLane spaces and do an induction on the Postnikov tower.

- Observe that $BV \times X^{hV} = BV \times \text{map}(BV, X_{hV})_1 = \text{map}(BV, X_{hV})_{(1)}$. Furthermore $\text{map}(BV, X_{hV})_{(1)}$ is a component of $\text{map}(BV, X_{hV})$

$$H^*(\text{map}(BV, X_{hV})) = T_V(H_{hV}^*(X)).$$

This can be given a componentwise version

$$H^*(\text{map}(BV, X_{hV})_{(1)}) = \mathbb{F}_p \otimes_{T_V^0(H_{hV}^*(X))} T_V(H_{hV}^*(X))$$

and we set $Fix = \mathbb{F}_p \otimes_{T_V^0(-)} T_V(-)$.

\square

REMARK 7.4.3. Note how the Sullivan conjecture relates to the Dwyer-Wilkerson formula: The formula of Dwyer-Wilkerson tells us that the homology of X^V can be extracted from the hV -homotopy type of X . The Sullivan conjecture then says that the homotopy type of X^V can be recovered from the hV -homotopy of X , up to p -completion, and even gives a computable formula for it as X^{hV} .

7.5. Loop structures on S^3

We will apply what we have learnt so far to prove the following, very concrete statement:

THEOREM 7.5.1. *There exist uncountably many connected spaces Y such that ΩY is homotopy equivalent to S^3 (the 3-sphere), that are pairwise not homotopy equivalent to each other.*

A good start towards Theorem 7.5.1 is to show that there exists a space whose loop space is $\simeq S^3$.

EXAMPLE 7.5.2. For $n \geq 0$, let $\mathbb{H}P^n$, the n^{th} quaternionic projective space, be the space of \mathbb{H} -lines in \mathbb{H}^{n+1} , where an \mathbb{H} -line is an \mathbb{H} -linear subspace of \mathbb{H}^{n+1} of \mathbb{H} -dimension 1. The space $\mathbb{H}P^n$ can also be described as the quotient $(\mathbb{H}^{n+1} \setminus \{0\})/\mathbb{H}^*$, or also as S^{4n+3}/S^3 , if we consider the group $S^3 \cong SU(2) \cong \text{Spin}(3)$ of norm-1 quaternions under multiplication, and we consider the unit sphere $S^{4n+3} \subset \mathbb{H}^{n+1}$.³ Here the blackboard notation “ S^3 ” is to emphasize that this is a Lie group, and not just a space. We have a fibre bundle whose base, fibre and total space are orientable manifolds $S^3 \hookrightarrow S^{4n+3} \rightarrow \mathbb{H}P^n$.

The inclusion $\mathbb{H}^{n+1} \subset \mathbb{H}^{n+2}$ induces an inclusion $\mathbb{H}P^n \subset \mathbb{H}P^{n+1}$, with complement a $(4n+4)$ -dimensional cell. It follows that $\mathbb{H}P^n$ has a cell decomposition with one cell in each even dimension between 0 and $4n$, and in particular its cohomology $H^*(\mathbb{H}P^n)$ is isomorphic as graded abelian group to the quotient ring $\mathbb{Z}[x]/x^{n+1}$, where x is a variable in degree 4. Poincaré duality implies moreover that this must be an isomorphism of rings (exercise for you!).

We then consider $\mathbb{H}P^\infty = \bigcup_{n \geq 0} \mathbb{H}P^n$, which is a CW complex having one cell in each dimension multiple of 4. The above considerations imply that $\mathbb{H}P^\infty$ has cohomology ring isomorphic to $\mathbb{Z}[x]$, the polynomial ring in one variable of degree 4; moreover we have a fibre bundle $S^3 \hookrightarrow S^\infty \rightarrow \mathbb{H}P^\infty$, and since the infinite-dimensional sphere S^∞ is contractible we obtain that S^3 is weakly equivalent to $\Omega \mathbb{H}P^\infty$. So $\mathbb{H}P^\infty$ is one of the spaces X in Theorem 7.5.1.

Compare this example with the probably more familiar one of $\mathbb{C}P^\infty$, a space whose loop space is homotopy equivalent to the Lie group S^1 .

In fact we can also think of S^∞ as an example of a contractible space ES^3 with a free action of the compact Lie group S^3 ; we can then have an equivalence $\mathbb{H}P^\infty \simeq BS^3$, meaning that $\mathbb{H}P^\infty$ is a classifying space for principal S^3 -bundles. And for any Lie group G we have indeed $\Omega BG \simeq G$.

We observe that if Y is a (connected) space with $\Omega Y \simeq S^3$, then Y must be simply connected (in fact, Y must be 3-connected); moreover we have an arithmetic square

$$\begin{array}{ccc} Y & \longrightarrow & \prod_p Y_p^\wedge \\ \downarrow & & \downarrow \\ Y_\mathbb{Q} & \longrightarrow & \left(\prod_p Y_p^\wedge \right)_\mathbb{Q}, \end{array}$$

which is a homotopy pullback whose vertical maps are rationalisations.

EXERCISE 7.5.3. Check that if $\Omega Y \simeq S^3$, then $Y_\mathbb{Q} \simeq K(\mathbb{Q}, 4)$: you can for instance prove that $S_\mathbb{Q}^3 \simeq \Omega(Y_\mathbb{Q})$, invoking Lemma 6.3.1 and Theorem 6.3.3, and then prove aside that $S_\mathbb{Q}^3 \simeq K(\mathbb{Q}, 3)$.

³You can also say: we first quotient \mathbb{H}^{n+1} by the action of the (central) subgroup $\mathbb{R}_+ \subset \mathbb{H}^*$, obtaining S^{4n+3} , and then quotient the latter by the residual action of $S^3 \cong \mathbb{H}^*/\mathbb{R}_+$.

LEMMA 7.5.4. *Let Y be a connected space with $\Omega Y \simeq S^3$, and let p be a prime. Then $\Omega(Y_p^\wedge) \simeq (S^3)_p^\wedge$.*

PROOF. By Theorem 6.8.1 we have that Y_p^\wedge is also 3-connected as Y , and its homotopy groups are derived p -complete. It follows that $\Omega(Y_p^\wedge)$ also has derived p -complete homotopy groups and is also simply connected, hence it is p -complete. Finally, by Proposition 6.5.4 we have that the map $\Omega Y \rightarrow \Omega(Y_p^\wedge)$ is a \mathbb{F}_p -equivalence. \square

The homotopy groups of $(S^3)_p^\wedge$ can be “computed” (i.e., understood to a sufficient level for our purposes) thanks to Theorem 6.5.1, together with Serre’s classical result that $\pi_*(S^3)$ are finitely generated: we have $\pi_3((S^3)_p^\wedge) \cong \mathbb{Z}_p$, whereas $\pi_{\geq 4}((S^3)_p^\wedge)$ is the p -power torsion part of $\pi_{\geq 4}(S^3)$. Therefore, if Y is a space with $\Omega Y \simeq S^3$, we have isomorphisms

$$\begin{aligned} \pi_4\left(\prod_p Y_p^\wedge\right) &\cong \prod_p \pi_4(Y_p^\wedge) \cong \prod_p \pi_3((S^3)_p^\wedge) \cong \prod_p \mathbb{Z}_p; \\ \pi_i\left(\prod_p Y_p^\wedge\right) &\cong \pi_{i-1}(S^3) \quad \text{for } i \neq 4 \quad (\text{a finite group}). \end{aligned}$$

We can now rationalise $\prod_p Y_p^\wedge$ to get a space $\left(\prod_p Y_p^\wedge\right)_\mathbb{Q}$ whose only non-zero homotopy group is π_4 , equal to $(\prod_p \mathbb{Z}_p) \otimes \mathbb{Q}$.

DEFINITION 7.5.5. We denote by \mathbb{A}_f the abelian group $(\prod_p \mathbb{Z}_p) \otimes \mathbb{Q}$, called the group of “finite adèles”. It has a ring structure, coming from tensor product of \mathbb{Q} and the product ring $\prod_p \mathbb{Z}_p$. Note also that \mathbb{A}_f is an infinite-dimensional vector space over \mathbb{Q} .

The previous discussion shows that whenever Y is connected and $\Omega Y \simeq S^3$, the space occurring in the bottom right corner of the arithmetic square for Y must be (homotopy equivalent to) $K(\mathbb{A}_f, 4)$.

At this point we restrict our attention to spaces Y such that $\Omega Y \simeq S^3$, and such that, moreover, there are equivalences $Y_\mathbb{Q} \simeq \mathbb{H}P_\mathbb{Q}^\infty$ and, for each prime p , $Y_p^\wedge \simeq (\mathbb{H}P^\infty)_p^\wedge$. Exercise 7.5.3 shows that assuming $\Omega Y \simeq S^3$ we immediately have $Y_\mathbb{Q} \simeq \mathbb{H}P_\mathbb{Q}^\infty \simeq K(\mathbb{Q}, 4)$; it is instead a more difficult fact (that we will not prove) that also the equivalence $Y_p^\wedge \simeq (\mathbb{H}P^\infty)_p^\wedge$ follows just from the assumption $\Omega Y \simeq S^3$.

To simplify notation, we denote in the following $X_p = (\mathbb{H}P^\infty)_p^\wedge$. The cohomology ring $H^*(X_p; \mathbb{F}_p)$ is isomorphic to $\mathbb{F}_p[x]$, with x in degree 4.

LEMMA 7.5.6. *Let p be an odd prime and let $f: X_p \rightarrow X_p$ be a self-map. Then the map $f^*: H^4(X_p; \mathbb{F}_p) \rightarrow H^4(X_p; \mathbb{F}_p)$ is the self-map of \mathbb{F}_p induced by multiplication by an element $d \in \mathbb{F}_p$ which is a square, i.e. $d = a^2$ for some $a \in \mathbb{F}_p$.*

PROOF. The argument follows closely another argument to be found in [Hat02, Example 4L.4]. There is a collapse map $\phi: \mathbb{C}P^\infty \rightarrow \mathbb{H}P^\infty$ inducing an isomorphism on $H^4(-; \mathbb{F}_p)$ (in fact, also on $H^4(-; \mathbb{Z})$); after p -completion we obtain a map $\phi_p^\wedge: (\mathbb{C}P^\infty)_p^\wedge \rightarrow X_p$ inducing again an isomorphism on $H^4(-; \mathbb{F}_p)$. If γ is a generator of $H^4(X_p, \mathbb{F}_p)$, then also $(\phi_p^\wedge)^*\gamma$ is a generator of $H^4((\mathbb{C}P^\infty)_p^\wedge; \mathbb{F}_p)$. The p -completion map $\iota_\mathbb{C}: \mathbb{C}P^\infty \rightarrow (\mathbb{C}P^\infty)_p^\wedge$ induces an isomorphism $\iota_\mathbb{C}^*: H^*((\mathbb{C}P^\infty)_p^\wedge; \mathbb{F}_p) \rightarrow H^*(\mathbb{C}P^\infty; \mathbb{F}_p)$ which is compatible with the action of the mod p Steenrod algebra \mathcal{A}_p ; in particular $P^1(\phi^*\gamma) = 2(\phi^*\gamma)^{(p+1)/2}$ since this is true (read Hatcher) after pulling back along $i_\mathbb{C}^*$.

Now the naturality of the Steenrod operations implies that $\phi^*(P^1(\gamma)) = P^1(\phi^*\gamma)$; on the other hand $\phi^*: H^{2p+2}(X_p; \mathbb{F}_p) \xrightarrow{\cong} H^{2p+2}((\mathbb{C}P^\infty)_p^\wedge; \mathbb{F}_p)$ is an isomorphism, implying $P^1(\gamma) = 2\gamma^{(p+1)/2}$.

And now, again by naturality, we have a chain of equalities

$$2d\gamma^{(p+1)/2} = dP^1(\gamma) = P^1(d\gamma) = P^1(f^*\gamma) = f^*(P^1(\gamma)) = f^*(2\gamma^{(p+1)/2}) = 2d^{(p+1)/2}\gamma^{(p+1)/2},$$

implying that d is a square mod p . \square

In particular, for p odd, if $f: X_p \rightarrow X_p$ is a self-homotopy equivalence, it induces on $\pi_4(X_p) \cong \mathbb{Z}_p$ an automorphism of the abelian group \mathbb{Z}_p ; such endomorphism is given by multiplication by an *invertible* element $\lambda \in \mathbb{Z}_p^\times$; Lemma 7.5.6 ensures that λ reduces to a square in \mathbb{F}_p , and this suffices to ensure that λ is the square of an element in \mathbb{Z}_p^\times , using that p is odd and Hensel's lemma. In fact the same holds also for the prime 2: a self-homotopy equivalence $f: X_2 \rightarrow X_2$ induces on $\pi_4(X_2) \cong \mathbb{Z}_2$ multiplication by a perfect square $\lambda \in (\mathbb{Z}_2)^\times$, but we will not prove this.

If now Y is a space such that $Y_p \simeq X_p$ and $Y_{\mathbb{Q}} \simeq X_{\mathbb{Q}} := K(\mathbb{Q}, 4)$, then we have an arithmetic square, which is a homotopy pullback square

$$\begin{array}{ccc} Y & \longrightarrow & \prod_p X_p \\ \downarrow & & \downarrow \phi \\ X_{\mathbb{Q}} & \xrightarrow{\psi} & \left(\prod_p X_p\right)_{\mathbb{Q}}, \end{array}$$

where ϕ is a \mathbb{Q} -equivalence and ψ is some map; viceversa, for any \mathbb{Q} -equivalence ϕ and any map ψ we can define Y as the above homotopy pullback, and then Y is an example of a space with ΩY fitting in a pullback square with vertical maps being \mathbb{Q} -equivalences

$$\begin{array}{ccc} \Omega Y & \longrightarrow & \prod_p (S^3)_p^\wedge \\ \downarrow & & \downarrow \Omega\phi \\ S_{\mathbb{Q}}^3 & \xrightarrow{\Omega\psi} & \left(\prod_p (S^3)_p^\wedge\right)_{\mathbb{Q}}, \end{array}$$

and this implies that ΩY is simply connected and has the same homology (rationally and mod p , hence integrally) as S^3 : it follows from Hurewicz that $\Omega Y \simeq S^3$.

Now we should avoid overcounting: how many essentially distinct choices of ϕ and ψ as above do we have? Clearly, if we fix homotopy automorphisms α, β and γ of $\prod_p X_p$, $X_{\mathbb{Q}}$ and $(\prod_p X_p)_{\mathbb{Q}}$ respectively, we can replace ϕ by $\phi' = \gamma \circ \phi \circ \alpha^{-1}$ and ψ by $\psi' = \gamma \circ \psi \circ \beta^{-1}$, and the new homotopy pullback Y' (obtained using ϕ' and ψ') is weakly equivalent to Y . Conversely, if Y and Y' are two spaces with rationalisation $X_{\mathbb{Q}}$ and p -completion X_p , and if $f: Y \rightarrow Y'$ is a weak equivalence, then the composites $X_{\mathbb{Q}} \simeq Y_{\mathbb{Q}} \xrightarrow{f_{\mathbb{Q}}} Y'_{\mathbb{Q}} \simeq X_{\mathbb{Q}}$, $\prod_p X_p \simeq \prod_p Y_p \xrightarrow{\prod_p f_p} \prod_p Y'_p \simeq \prod_p X_p$ and $(\prod_p X_p)_{\mathbb{Q}} \simeq (\prod_p Y_p)_{\mathbb{Q}} \xrightarrow{(\prod_p f_p)_{\mathbb{Q}}} (\prod_p Y'_p)_{\mathbb{Q}} \simeq (\prod_p X_p)_{\mathbb{Q}}$ are homotopy automorphisms of $X_{\mathbb{Q}}$, $\prod_p X_p$ and $(\prod_p X_p)_{\mathbb{Q}}$ respectively, and the maps ψ' and ϕ' for Y' are up to homotopy obtained from the maps ψ and ϕ for Y as described above.

We observe moreover that if V is a \mathbb{Q} -vector space, then the group $\pi_0(\text{hAut}(K(V, 4)))$ of homotopy automorphisms of $K(V, 4)$ up to homotopy is isomorphic to the group $GL_{\mathbb{Q}}(V)$ of \mathbb{Q} -linear automorphisms of V .

All in all, if we set $\mathcal{G} = \pi_0(\text{hAut}(\prod_p X_p))$, there is an action of $GL_{\mathbb{Q}}(\mathbb{A}_f) \times GL_{\mathbb{Q}}(\mathbb{Q}) \times \mathcal{G}$ on the set \mathcal{S} of choices of (ϕ, ψ) as above; moreover the action of the subgroup $GL_{\mathbb{Q}}(\mathbb{A}_f)$ on the set of choices of ϕ is simply transitive, since ϕ is itself required to be a rational equivalence, and hence can be defined uniquely by passing to the rationalisation of $\prod_p X_p$ and then taking an automorphism of $(\prod_p X_p)_{\mathbb{Q}}$. We can thus fix our favourite rational equivalence $\bar{\phi}: \prod_p X_p \rightarrow (\prod_p X_p)_{\mathbb{Q}}$ and i) restrict ourselves to the set $\bar{\mathcal{S}}$ of choices of (ϕ, ψ) with $\phi = \bar{\psi}$ and ii) restrict ourselves to the stabiliser of the set $\bar{\mathcal{S}}$ along the previous action: this is the subgroup

of $GL_{\mathbb{Q}}(\mathbb{A}_f) \times GL_{\mathbb{Q}}(\mathbb{Q}) \times \mathcal{G}$ of triples (α, β, γ) such that $\bar{\phi}$ is homotopic to $\gamma \circ \bar{\phi} \circ \alpha^{-1}$, and this is equivalent to requiring that γ is the rationalisation of α .

We can thus consider the residual action of $GL_{\mathbb{Q}}(\mathbb{Q}) \times \mathcal{G}$ on $\bar{\mathcal{S}}$, where $(\alpha, \beta) \in \mathcal{G}$ sends $(\bar{\phi}, \psi) \mapsto (\bar{\phi}, \alpha_{\mathbb{Q}} \circ \psi \circ \beta^{-1})$. We can now identify the homotopy classes of maps $\psi: K(\mathbb{Q}, 4) \rightarrow K(\mathbb{A}_f, 4)$ with the set $\text{Hom}(\mathbb{Q}, \mathbb{A}_f)$ of additive maps $\mathbb{Q} \rightarrow \mathbb{A}_f$. This set can be identified with \mathbb{A}_f itself, in such a way that the action of $GL_{\mathbb{Q}}(\mathbb{Q}) \times \mathcal{G}$ acquires the following description:

- $GL_{\mathbb{Q}}(\mathbb{Q})$ can be identified with \mathbb{Q}^{\times} , and acts on \mathbb{A}_f by scalar multiplication (using the ring structure of \mathbb{A}_f);
- if an element $\alpha \in \mathcal{G}$ acts on each $\pi_4(X_p)$ by multiplication by an element $\lambda_p \in (\mathbb{Z}_p^{\times})^2$, then it acts on \mathbb{A}_f by scalar multiplication by $\prod_p \lambda_p$.

We thus have a bijection of sets

$$\{\text{spaces } Y \text{ with } \Omega Y \simeq S^3, Y_p^{\wedge} \simeq X_p \text{ and } Y_{\mathbb{Q}} \simeq X_{\mathbb{Q}}\} / \text{w.e.} \leftrightarrow \mathbb{A}_f / \left(\mathbb{Q}^{\times} \times \prod_p (\mathbb{Z}_p^{\times})^2 \right).$$

THEOREM 7.5.7 (Dwyer-Miller-Wilkerson '84). *There is in fact a bijection*

$$\{\text{spaces } Y \text{ with } \Omega Y \simeq S^3\} / \text{w.e.} \leftrightarrow \mathbb{A}_f / \left(\mathbb{Q}^{\times} \times \prod_p (\mathbb{Z}_p^{\times})^2 \right).$$

We now notice that there is a surjection

$$\mathbb{A}_f \twoheadrightarrow \left(\prod_p \mathbb{F}_p \right) \otimes \mathbb{Q},$$

inducing a surjection

$$\mathbb{A}_f / \left(\mathbb{Q}^{\times} \times \prod_p (\mathbb{Z}_p^{\times})^2 \right) \twoheadrightarrow \left(\left(\prod_p \mathbb{F}_p \right) \otimes \mathbb{Q} \right) / \left(\mathbb{Q}^{\times} \times \prod_p (\mathbb{F}_p^{\times})^2 \right),$$

and we leave to the reader the pleasure of proving that the last set is more than countable!

7.6. p -compact groups

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