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ENDOTRIVIAL MODULES FOR FINITE GROUPS VIA HOMOTOPY THEORY

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Contents

1. Introduction

In modular representation theory of finite groups, the indecomposable kG-modules M whose restriction to a Sylow p-subgroup S split as the trivial module k plus a free kS-module are basic yet somewhat mysterious objects. Such modules form a group $T_k(G,S)$ under tensor product, discarding projective kG-summands, with neutral element k and M^* the inverse of M (see §2.1 for details). It is also denoted $K(G)$ in the literature. It contains the one-dimensional characters $\mathrm{Hom}(G,k^\times)$ as a subgroup, but has been observed to sometimes also contain *exotic* elements, i.e., modules of dimension greater than one. The group $T_k(G,S)$ is an important subgroup of the larger group of all so-called endotrivial modules $T_k(G)$, i.e., kG-modules M where $M^* \otimes M \cong k \oplus P$, for P a projective kG-module. Namely $T_k(G,S) = \ker(T_k(G) \to T_k(S))$, i.e., the kernel of the restriction to S. The group of endotrivial modules has a categorical interpretation as $T_k(G) \cong \mathrm{Pic}(\mathrm{StMod}_{kG})$, the Picard group of the stable module category. Such modules occur in many parts of representation theory, e.g., as source modules (see the surveys [Thé07, Car12, Car17] and the papers quoted below).

Classifying endotrivial modules has been a long-running quest, which has been reduced to calculating $T_k(G,S)$, through a series of fundamental papers: The group $T_k(S)$ was determined in celebrated works of Dade [Dad78a, Dad78b], Alperin

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[Alp01], and Carlson–Thévenaz [CT04, CT05]. From this, Carlson–Mazza–Nakano–Thévenaz [CMN06, MT07, CMT13] worked out the image of the restriction $T_k(G) \to T_k(S)$, at least as an abstract abelian group, and showed that the restriction is split onto its image, except in well-understood cases with cyclic Sylow p-subgroup (see (1.1)). Subsequently there has been an intense interest in calculating $T_k(G,S)$, with contributors Balmer, Carlson, Lassueur, Malle, Mazza, Nakano, Navarro, Robinson, Thévenaz, and others.

In this paper we give an elementary and computable homological description of the group $T_k(G,S)$, as the first cohomology group of the orbit category on non-trivial p-subgroups of G, with constant coefficients in $k^\times$, for any finite group G and any field k (not assumed algebraically closed) of characteristic p dividing the order of G. Using homological methods, adapted from mod p homology decompositions (but now with $k^\times$-coefficients, hence "prime to p"), we deduce a range of structural and computational results on $T_k(G,S)$, with answers expressed in terms of standard p-local group theory: We write $T_k(G,S)$ as an inverse limit of homomorphisms from normalizers of chains of p-subgroups to $k^\times$, answering the main conjecture of Carlson–Thévenaz [CT15, Ques. 5.5] in the positive (see also [Car17, Ques. 1]). A related "centralizer decomposition" expresses it in terms of the p-fusion system of G and centralizers of elementary abelian p-subgroups (see §§1.2–1.3). We also get a formula for $T_k(G,S)$ in terms of π_0 and π_1 of the p-subgroup complex $|\mathcal{S}_p(G)|$ of G. It implies that $T_k(G,S) \cong \mathrm{Hom}(G,k^\times)$ when the p-subgroup complex of G is simply connected. The formula can be seen as a topological correction to the old hope (too naïve; see [Car12, p. 106]) that exotic modules could only occur in the presence of a strongly p-embedded subgroup, meaning $|\mathcal{S}_p(G)|$ disconnected (see §1.1). We get bounds on $T_k(G,S)$ in terms of the fundamental group of the p-fusion system of G, and see the contribution of specific p-subgroups in G (see §1.4). Lastly we provide consequences of these results for specific classes of groups, e.g., finite groups of Lie type and sporadic groups, obtaining new computations as well as recovering and simplifying many old ones in the vast literature. As an example we try out one of our formulas on the Monster sporadic simple group, and easily calculate $T_k(G,S)$ for p any of the harder primes $3, 5, 7, 11$, and 13, which had been left open in the literature [LM15b] (see §1.5).

Our proof of the identification of $T_k(G,S)$ is direct and self-contained, and provides "geometric" models for the module generator in $T_k(G,S)$ corresponding to a 1-cocycle: it is the class in the stable module category represented by the unreduced Steinberg complex of G twisted by the 1-cocycle. It has the further conceptual interpretation as the homotopy left Kan extension of the 1-cocycle from the orbit category on non-trivial p-subgroups, to all p-subgroups. Our identification was originally inspired by a characterization due to Balmer of $T_k(G,S)$ in terms of what he dubs "weak homomorphisms" [Bal13], and we also indicate another argument of how to deduce the identification using these.

Let us now describe our work in detail. Call a kG-module *Sylow-trivial* if it upon restriction to kS splits as the trivial module k plus a projective kS-module. $T_k(G,S)$ identifies with the group of equivalence classes of Sylow-trivial modules, identifying two if they become isomorphic after discarding projective kG-summands. Each equivalence class contains a unique indecomposable representative, up to isomorphism (see Proposition 3.1). Let $\mathcal{O}_p^*(G)$ denote the orbit category of G with objects

G/P, for P a non-trivial p-subgroup, and morphisms G-maps. The following is our main identification of $T_k(G,S)$.

Theorem A. *Fix a finite group G and k a field of characteristic p dividing the order of G. The group $T_k(G,S)$ is described via the following isomorphism of abelian groups*

$$\Phi\colon\ T_k(G,S) \xrightarrow{\cong} H^1(\mathcal{O}_p^*(G);k^\times)$$

The map Φ sends $[M]\in T_k(G,S)$ to the functor $\varphi\colon\ \mathcal{O}_p^*(G)\to k^\times$ defined as follows: First consider

$$k_\varphi\colon\ G/P \mapsto \hat{H}^0(P;M) = M^P/(\sum_{g\in P} g)M,$$

given by zeroth Tate cohomology, a functor from $\mathcal{O}_p^*(G)$ to the connected groupoid of one-dimensional k-modules and isomorphisms. Then identify the target with the group $k^\times$, regarded as a category with one object, via an equivalence of categories, to obtain a functor $\varphi\colon\ \mathcal{O}_p^*(G)\to k^\times$, well-defined up to natural isomorphism of functors.

The inverse $\Phi^{-1}(\varphi)$ is the class in $T_k(G,S)$ of the unaugmented "twisted Steinberg complex"

$$C_*(|\mathcal{S}_p(G)|;k_\varphi)\in D^b(kG)/D^{\mathrm{perf}}(kG) \xleftarrow{\simeq} \mathrm{stmod}_{kG}$$

where k_φ is the G-twisted coefficient system induced by $\varphi\colon\ \mathcal{O}_p^*(G)\to k^\times$, i.e., assigning $k_\varphi(G/P_0)$ ($\cong k$) to the n-simplex $(P_0\le\cdots\le P_n)$ and endowing the chain complex with the canonical G-action (see §3.2.2*).*

In fact we establish in Theorem 3.11 a more general correspondence between kG-modules that split as a sum of trivial and projective modules upon restriction to S, and representations of the fundamental group $\pi_1(\mathcal{O}_p^*(G))$, refining parts of Green correspondence.

The one-dimensionality of $\hat{H}^0(P;M)$ is by definition of Sylow-trivial, as P is subconjugate to S. Also, we used the equivalence of homotopy categories $\mathrm{stmod}_{kG}\xrightarrow{\simeq} D^b(kG)/D^{\mathrm{perf}}(kG)$ between stmod_{kG}, the full subcategory of the stable module category StMod_{kG} with objects finitely generated kG-modules, and the bounded derived category of finitely generated kG-modules, modulo perfect complexes, recalled in §3.2.1. We recall G-twisted coefficient systems on the p-subgroup complex $|\mathcal{S}_p(G)|$, the nerve of the poset of non-trivial p-subgroups of G, in §3.2.2.

The chain complex $C_*(|\mathcal{S}_p(G)|;k_\varphi)$ can be interpreted as the value on G/e of the homotopy left Kan extension of φ along $\mathcal{O}_p^*(G)^{\mathrm{op}}\to\mathcal{O}_p(G)^{\mathrm{op}}$, i.e., to the opposite orbit category on *all* p-subgroups, which has model the homotopy colimit in chain complexes $\mathrm{hocolim}_{P\in\mathcal{S}_p(G)^{\mathrm{op}}} k_\varphi$ (see Proposition 3.7). When $\varphi=1$, the complex is the Steinberg complex without the k-augmentation in degree -1, and the fact that $\Phi^{-1}(k)\cong k$ is equivalent to projectivity of the augmented Steinberg complex, proved by Quillen [Qui78, 4.5] and Webb [Web91].

Let us briefly describe how to obtain the whole group of endotrivial modules $T_k(G)$ from that of $T_k(G,S)$ together with results in the literature. Using the definitions (see §2.2) have an exact sequence

$$0\to T_k(G,S)\to T_k(G)\xrightarrow{\mathrm{res}_S} L\to 0 \tag{1.1}$$

with $L=\mathrm{im}(T_k(G)\to T_k(S))$. The torsion-free rank of L has been determined in [CMN06, §3], extending the work of Alperin [Alp01]. By the classification of endotrivial modules for finite p-groups [CT04, CT05], L is torsion-free except when

S is cyclic or a semi-dihedral or quaternionic 2-group, and in particular the above sequence is split outside those cases. The exceptions can be described explicitly by a case-by-case analysis carried out in [MT07, CMT13] and it turns out that in all those cases the torsion part of L equals that of $T_k(S)$. When S is a semi-dihedral or quaternionic 2-group the restriction is furthermore split by [CMT13, Thms. 6.4 and 4.5]. When S is cyclic, $T_k(\mathbb{Z}/2) = 0$, and for $|S| > 2$, $T_k(S) \cong \mathbb{Z}/2$. The 4-fold periodic resolution of the trivial $\mathbb{F}_3\mathfrak{S}_3$-module shows that the restriction is not always split, but the structure of the extension above can in all cases be described explicitly (see [MT07, Thm. 3.2 and Lem. 3.5]). We remark that while L hence is known as an abstract abelian group for any finite group G, explicit generators for the torsion-free part have in fact hitherto been elusive (see [CMT14]). Combining the methods of this paper with methods from higher algebra, we have recently, together with Tobias Barthel and Joshua Hunt [BGH], also been able to describe these, giving the precise image $L \subseteq \lim_{G/P \in \mathscr{O}_p^*(G)} T_k(P) \subseteq T_k(S)$, and obtaining answers to conjectures in [CMT14].

We now embark in putting the model for $T_k(G,S)$ of Theorem A to use, expressing $H^1(\mathscr{O}_p^*(G); k^\times)$ in terms of p-local information about the group. We start with some elementary observations: By standard algebraic topology (recalled in §2.5)

$$H^1(\mathscr{O}_p^*(G); k^\times) \cong \mathrm{Rep}(\mathscr{O}_p^*(G), k^\times) \cong \mathrm{Hom}(\pi_1(\mathscr{O}_p^*(G)), k^\times) \cong \mathrm{Hom}(H_1(\mathscr{O}_p^*(G)), k^\times) \tag{1.2}$$

where Rep means isomorphism classes of functors, viewing $k^\times$ as a category with one object. Let

$$G_0 = \langle N_G(Q) | 1 < Q \leq S \rangle \tag{1.3}$$

also called the 1-generated core $\Gamma_{S,1}(G)$ by group theorists, which, if proper in G, is the *smallest strongly p-embedded subgroup containing S* (see Remark A.18). Recall that a Frattini argument implies that we have surjections $N_G(S)/S \twoheadrightarrow (G_0)_{p'} \twoheadrightarrow G_{p'}$, where we throughout the paper adopt the *convention* that

$$G_{p'} = G/\langle\, g \in G \,|\, g \text{ is of finite } p\text{-power order} \,\rangle. \tag{1.4}$$

(Hence, $G_{p'} = G/O^{p'}(G)$, when G is finite, and $M_{p'} = M/\operatorname{Tors}_p(M)$ when M is abelian.) An application of Alperin's fusion theorem [Alp67, §3] shows that $\mathscr{O}_p^*(G)$ and $\mathscr{O}_p^*(G_0)$ are equivalent categories and the Frattini surjections above refine to

$$N_G(S)/S \twoheadrightarrow \pi_1(\mathscr{O}_p^*(G)) \twoheadrightarrow (G_0)_{p'} \twoheadrightarrow G_{p'} \tag{1.5}$$

displaying $\pi_1(\mathscr{O}_p^*(G))$ as a finite p'-group (see Proposition 4.1). Via Theorem A, this encodes the classical bounds on $T_k(G,S)$ (cf. [CMN06, Prop. 2.6], [MT07, Lem. 2.7], and Proposition 3.2):

$$\mathrm{Hom}(G, k^\times) \leq \mathrm{Hom}(G_0, k^\times) \leq T_k(G,S) \leq \mathrm{Hom}(N_G(S)/S, k^\times) \tag{1.6}$$

using that $k^\times$ does not contain p-torsion.

We describe the precise kernel of $N_G(S)/S \twoheadrightarrow \pi_1(\mathscr{O}_p^*(G))$ in terms of p-local group theory in Theorem 4.10—it already directly implies a list of structural properties of $T_k(G,S)$ via Theorem A and (1.2) (see e.g., Corollary 4.14).

In the rest of this section we present our further descriptions of $\pi_1(\mathscr{O}_p^*(G))$ and its abelianization $H_1(\mathscr{O}_p^*(G))$, each highlighting different structural properties. We divide this into 5 subsections: §1.1 Subgroup categories, §1.2 Decompositions, §1.3 The Carlson–Thévenaz conjecture, §1.4 Fusion systems, and §1.5 Computations.

1.1. **Descriptions in terms of subgroup complexes.** Let $\mathscr{T}_p^*(G)$ denote the transport category of G with objects the non-trivial p-subgroups of G, and

$$\mathrm{Hom}_{\mathscr{T}_p^*(G)}(P, Q) = \{g \in G | {}^g P \leq Q\}.$$

We have a functor $\mathscr{T}_p^*(G) \to \mathscr{O}_p^*(G)$ sending g to the G-map $G/P \to G/Q$ specified by $eP \mapsto g^{-1}Q$. Since $\mathscr{O}_p^*(G)$ is a quotient of $\mathscr{T}_p^*(G)$ by morphisms in p-groups $Q \leq \mathrm{Aut}_{\mathscr{T}_p^*(G)}(Q)$,

$$\pi_1(\mathscr{T}_p^*(G))_{p'} \xrightarrow{\cong} \pi_1(\mathscr{O}_p^*(G)) \tag{1.7}$$

(see Proposition 4.5). As $k^\times$ contains no elements of finite p-power order, (1.7) and Theorem A imply that 1-dimensional characters of $\pi_1(\mathscr{T}_p^*(G))$ also parametrize Sylow-trivial modules for G, i.e.,

$$T_k(G, S) \cong \mathrm{Hom}(\pi_1(\mathscr{T}_p^*(G)), k^\times) \tag{1.8}$$

By definition $\mathscr{T}_p^*(G)$ equals the transport category (or Grothendieck construction) of the G-action on the poset of nontrivial p-subgroups $\mathcal{S}_p(G)$, under inclusion. Hence

$$|\mathscr{T}_p^*(G)| \simeq |\mathcal{S}_p(G)|_{hG} \tag{1.9}$$

by Thomason's theorem [Tho79, Thm. 1.2], where $X_{hG} = EG \times_G X$ denotes the Borel construction (see also Lemma 2.3). Hence we can also conclude the following.

Corollary 1.1. *For any finite group G and k any field of characteristic p,*

$$T_k(G, S) \cong H^1(|\mathcal{S}_p(G)|_{hG}; k^\times)$$

In particular if $H_1(|\mathcal{S}_p(G)|_{hG})_{p'} \xrightarrow{\cong} H_1(G)_{p'}$ then $T_k(G, S) \cong \mathrm{Hom}(G, k^\times)$.

From this perspective, exotic Sylow-trivial modules parametrize the failure of the collection of non-trivial p-subgroups to be '$H_1(-;\mathbb{Z})$-ample' generalizing Dwyer's definition [Dwy97, 1.2] for mod p cohomology (see Remark 4.8 and Theorem 4.35). It also allows us to deduce a very recent result of Balmer [Bal18], as we explain in Remark 4.9.

To further describe the group $\pi_1(\mathscr{T}_p^*(G))$, recall that

$$|\mathcal{S}_p(G)| \cong G \times_{G_0} |\mathcal{S}_p(G_0)| \tag{1.10}$$

with $|\mathcal{S}_p(G_0)|$ connected, as observed by Quillen [Qui78, §5] (see Proposition A.16). Hence $|\mathcal{S}_p(G)|_{hG} \cong EG \times_G (G \times_{G_0} |\mathcal{S}_p(G_0)|) \cong EG \times_{G_0} |\mathcal{S}_p(G_0)| \simeq |\mathcal{S}_p(G_0)|_{hG_0}$, and we have a fibration sequence

$$|\mathcal{S}_p(G_0)| \to |\mathcal{S}_p(G)|_{hG} \to BG_0 \tag{1.11}$$

On fundamental groups it induces an exact sequence

$$1 \to \pi_1(\mathcal{S}_p(G_0)) \to \pi_1(\mathscr{T}_p^*(G)) \to G_0 \to 1 \tag{1.12}$$

displaying $\pi_1(\mathscr{T}_p^*(G))$ as an extension of G_0 by another group, possibly infinite.

Using the identification (1.8), the low-degree cohomology sequence of the group extension (1.12) (see [HS71, VI.8]) furthermore induces an exact sequence as follows.

Theorem B (Subgroup complex sequence)**.** *Let G be a finite group, k a field of characteristic p, and $G_0 \leq G$ as in* (1.3)*. We have an exact sequence*

$$0 \to \mathrm{Hom}(G_0, k^\times) \to T_k(G, S) \to H^1(\mathcal{S}_p(G_0); k^\times)^{G_0} \xrightarrow{\partial} H^2(G_0; k^\times)$$

where superscript G_0 *means invariants. If* $(H_1(\mathcal{S}_p(G))_G)_{p'} = 0$ *then* $T_k(G,S) \cong \mathrm{Hom}(G_0, k^\times)$ *and if* $|\mathcal{S}_p(G)|$ *is simply connected, then* $T_k(G,S) \cong \mathrm{Hom}(G, k^\times)$.

In words, $T_k(G,S)$ is an extension of $\mathrm{Hom}(G_0, k^\times)$ (producing kG-modules via induction and discarding projective summands, cf. Lemma 3.6) by a "truly exotic" part, not induced from 1-dimensional kG_0-modules, described above as the kernel of the boundary map $\partial\colon H^1(\mathcal{S}_p(G_0); k^\times)^{G_0} \to H^2(G_0; k^\times)$ (see also Remark 4.7). The group $H^2(G_0; k^\times)$ identifies with the p'-part of the Schur multiplier of G_0 if k is algebraically closed.

There is already an extensive literature on when $|\mathcal{S}_p(G)|$ is simply connected; see e.g., [Smi11, §9]. It is known to hold for "sufficiently large" symmetric groups and finite groups of Lie type at the characteristic, as well as some finite group of Lie type away from the characteristic and certain sporadic groups. It is conjectured to hold for many more (see also §1.5).

The description of $T_k(G,S)$ as 1-dimensional representations of $\pi_1(\mathscr{T}_p^*(G))$ in (1.8) combined with manipulations with subgroup complexes also enables us to see precisely how $T_k(G,S)$ behaves under for instance passage to p'-index subgroups or p'-central extensions (see Corollaries 4.16 and 4.18). Furthermore the groups $\pi_1(\mathscr{O}_p^*(G))$ and $\pi_1(\mathscr{T}_p^*(G))$ only depend on very few of the p-subgroups of G, as we analyze in detail in Appendix A—see in particular Theorems A.10 and A.15.

1.2. **Homology decomposition descriptions.** We now use homology decomposition techniques to get formulas for $T_k(G,S)$. These techniques have a long history for providing results about mod p group cohomology (see e.g., [Dwy97, Gro02, GS06] and their references), but here we are interested in coefficients in $k^\times$, an abelian group with no p-torsion. We can however still describe the low-degree p'-homology of $|\mathcal{S}_p(G)|_{hG}$, by examining the bottom corner of spectral sequences, even if they do not collapse. More precisely, given an arbitrary *collection* $\mathcal{C}$ of subgroups (i.e., a set of subgroups closed under conjugation), there are 3 homology decompositions one usually considers associated to the G-action on $|\mathcal{C}|$: the subgroup decomposition, the normalizer decomposition, and the centralizer decomposition. The subgroup decomposition does not provide new information, if $\mathcal{C}$ is a collection of p-subgroups, but the two others do. Let us start with the normalizer decomposition.

Theorem C (Normalizer decomposition)**.** *Let G be a finite group, k a field of characteristic p, and $\mathcal{C} \subseteq \mathcal{S}_p(G)$ a subcollection such that the inclusion is a G-homotopy equivalence, e.g., $\mathcal{C}$ the collection of non-trivial p-radical subgroups or of non-trivial elementary abelian p-subgroups (see* §A.3*). Then*

$$T_k(G,S) \cong \lim_{[P_0 < \cdots < P_n]} \mathrm{Hom}(N_G(P_0 < \cdots < P_n), k^\times)$$

inside $\mathrm{Hom}(N_G(S)/S, k^\times)$*, with the limit taken over conjugacy classes of chains in $\mathcal{C}$ ordered by refinement. Explicitly:*

$$\begin{aligned} T_k(G,S) &\cong \ker\left(\oplus_{[P]} \mathrm{Hom}(N_G(P), k^\times) \to \oplus_{[P<Q]} \mathrm{Hom}(N_G(P) \cap N_G(Q), k^\times)\right) \\ &\leq \mathrm{Hom}(N_G(S), k^\times) \end{aligned}$$

Different possibilities for $\mathcal{C}$ are given in Theorem A.8 (see also [GS06, Thm. 1.1]). Appendix A, along with Proposition 5.3 and Theorem 5.6, provides a precise analysis of which subgroups are needed to make the conclusion of Theorem C hold. We note that the data going into calculating the righthand side, normalizers of chains of say p-radical subgroups, or elementary abelian p-subgroups, has been tabulated for a large number of groups, and this relates to a host of problems in local group theory and representation theory, such as the classification of finite simple groups and conjectures of Alperin, McKay, Dade, etc. To obtain Theorem C from Theorem A, the only extra input, apart from the isotropy spectral sequence, is a result of Symonds [Sym98], formerly known as Webb's conjecture, which we provide a short proof of in Proposition A.3, that appears to be new.

We now explain the centralizer decomposition, which ties into fusion systems. Let $\mathcal{F}_{\mathcal{C}}(G)$ denote the restricted p-fusion system of G with objects $P \in \mathcal{C}$ and $\mathrm{Hom}_{\mathcal{F}_{\mathcal{C}}(G)}(P,Q) = \mathrm{Hom}_{\mathscr{T}_p^*(G)}(P,Q)/C_G(P)$ i.e., monomorphisms induced by G-conjugation, writing $\mathcal{F}_p^*(G)$ when $\mathcal{C} = \mathcal{S}_p(G)$.

Theorem D (Centralizer decomposition). *For G a finite group and k a field of characteristic p, we have an exact sequence*

$$0 \to H^1(\mathcal{F}_p^*(G); k^\times) \to T_k(G,S) \to \lim_{V \in \mathcal{F}_{\mathcal{A}_p^2}(G)} \mathrm{Hom}(C_G(V), k^\times) \to H^2(\mathcal{F}_p^*(G); k^\times)$$

where $\mathcal{A}_p^2$ denotes the collection of elementary abelian p-subgroups of rank one or two.

In particular, if $H_1(C_G(x))_{p'} = 0$ for all elements x of order p, and $H_1(N_G(S)/S)$ is generated by elements in $N_G(S)$ that commute with some non-trivial element in S, then $T_k(G,S) = 0$.

This breaks $T_k(G,S)$ up into parts depending on the underlying fusion system and parts calculated from centralizers. It may be illuminating to note that it specializes to the sequence in cohomology with $k^\times$ coefficients induced by $1 \to C_G(V) \to N_G(V) \to N_G(V)/C_G(V) \to 1$ in the very special case where G has p-rank 1, and hence a unique non-trivial elementary abelian p-subgroup (see Corollary 4.14(2)). In particular $H^1(\mathcal{F}_p^*(G); k^\times)$ is a subgroup of $T_k(G,S)$ depending *only* on the fusion system. We describe how to calculate cohomology of $\mathcal{F}_p^*(G)$ in §4.3 and Appendix A; the first cohomology group is in fact zero in many, but not all, cases. The assumptions of the 'in particular' are often satisfied for the sporadic groups, e.g., for the Monster at the more difficult primes up to 13.

1.3. **The Carlson–Thévenaz conjecture.** Theorem C implies the Carlson–Thévenaz conjecture, which predicts an algorithm for calculating $T_k(G,S)$ from p-local information, essentially by a change of language, taking $\mathcal{C} = \mathcal{S}_p(G)$. Set $A^{p'}(G) = O^{p'}(G)[G,G]$, the smallest normal subgroup of G such that the quotient is an abelian p'-group.

Theorem E (The Carlson–Thévenaz conjecture, [CT15, Ques. 5.5]). *Let G be a finite group with non-trivial Sylow p-subgroup S, and define $\rho^r(S) \leq N_G(S)$ (depending on G and S) via the following definition (cf.* [CMN14, Prop. 5.7], [CT15, §4]*): $\rho^1(Q) = A^{p'}(N_G(Q))$, $\rho^i(Q) = \langle N_G(Q) \cap \rho^{i-1}(R) | 1 < R \leq S\rangle \supseteq \rho^{i-1}(Q)$. Then*

$$H_1(\mathscr{O}_p^*(G)) \cong N_G(S)/\rho^r(S)$$

for any r at least either the nilpotency class of S plus 1 or the number of groups in the longest proper chain of non-trivial p-radical subgroups. Hence by Theorem A, *for any field k of characteristic p,*

$$T_k(G,S) \cong \mathrm{Hom}(N_G(S)/\rho^r(S), k^\times)$$

This in fact strengthens the Carlson–Thévenaz conjecture, by providing a rather manageable bound on r (see also Theorem 5.8). (The original conjecture was only a prediction about the union $\rho^\infty(S)$, and also had an algebraically closed assumption on k.) Theorem E is well adapted to implementation on a computer, and indeed Carlson has already made one such implementation calculating $\rho^i(S)$, and to use this for proofs a theoretical bound on when the $\rho^i(S)$ stabilize is obviously also necessary.

As already noted, the inverse limit in Theorem C identifies with a subset of $\mathrm{Hom}(N_G(S), k^\times)$. One may naïvely ask if the limit could simply be described as the elements in $\mathrm{Hom}(N_G(S), k^\times)$ whose restriction to $\mathrm{Hom}(N_G(P) \cap N_G(S), k^\times)$ is zero on $A^{p'}(N_G(P)) \cap N_G(S)$ for all $[P] \in \mathcal{C}/G$, an obvious necessary condition to lie in the limit. In other words, one may ask if one, in the language of Theorem E, could always take $r = 2$. Computer calculations announced in [CT15] say this is not the case for $G_2(5)$ when $p = 3$. The main theorem of that paper [CT15, Thm. 5.1] shows that this naïve guess *is* true when S is abelian, as also follows from Theorem C. Our proof allows for a stronger statement:

Corollary 1.2. *If all non-trivial p-radical subgroups in G with $P < S$ are normal in S, then*

$$T_k(G, S) \cong \ker\Big(\mathrm{Hom}(N_G(S), k^\times) \to \oplus_{[P]} \mathrm{Hom}(N_G(S) \cap A^{p'}(N_G(P)), k^\times)\Big)$$

where $[P]$ runs through G-conjugacy classes of non-trivial p-radical subgroups with $P < S$. In particular in the notation of Theorem E

$$T_k(G, S) \cong \mathrm{Hom}(N_G(S)/\rho^2(S), k^\times).$$

More generally if we for each $[P]$ can pick $P \leq S$ with $Q = N_S(P)$ Sylow in $N_G(P)$ and $N_G(P \leq Q \leq S)A^{p'}(N_G(P \leq Q)) = N_G(P \leq Q)$, then the same formulas hold, choosing such P.

See also Corollaries 5.12 and 5.13. The last part of Corollary 1.2 provides a strengthening of Carlson–Thévenaz's more technical [CT15, Thm. 7.1], which instead of abelian assumes that $N_G(S)$ controls p-fusion along with extra conditions (see Remark 5.14). Corollary 1.2 however moves beyond these cases with limited fusion, and e.g., also holds for finite groups of Lie type in characteristic p. To illustrate the failure in general we calculate $T_k(G, S)$ for $G = G_2(5)$ and $p = 3$ in Proposition 6.3, using Theorem C. Remark 5.11 contains a more detailed discussion of bounds on r.

1.4. **Further relations to fusion systems.** Recall that a p-subgroup P is said to be centric if $Z(P) = C_G(P)$ and p-centric if $Z(P)$ is a Sylow p-subgroup in $C_G(P)$. We denote by a superscript c full subcategories with objects the p-centric subgroups. Subgroups of $\pi_1(\mathcal{F}^c)$ parametrize sub-fusion systems of p'-index of a fusion system $\mathcal{F}$ by [BCG+07, §5.1], and calculating $\pi_1(\mathcal{F}^c)$ is of current interest within p-local group theory—see e.g., [AOV12, §4], [Rui07], and [Asc11, Ch. 16]. The condition that $\pi_1(\mathcal{F}^c) = 1$ is one of the conditions for a fusion system to be reduced [AOV12, Def. 2.1]. We here state a rough, yet useful, relation between Sylow-trivial modules and $\pi_1(\mathcal{F}^c)$–much more precise information is given in the paper proper.

Theorem F. *We have the following commutative diagram of monomorphisms*

$$\begin{tikzcd} T_k(G,S) \arrow[r, hook] & \mathrm{Hom}(\pi_1(\mathscr{O}_p^c(G)),k^\times) \arrow[r, hook] & \mathrm{Hom}(N_G(S)/S,k^\times) \\ \mathrm{Hom}(\pi_1(\mathcal{F}_p^*(G)),k^\times) \arrow[r, hook] \arrow[u, hook] & \mathrm{Hom}(\pi_1(\mathcal{F}_p^c(G)),k^\times) \arrow[r, hook] \arrow[u, hook] & \mathrm{Hom}(N_G(S)/SC_G(S),k^\times) \arrow[u, hook] \end{tikzcd}$$

If all p-centric p-radical subgroups are centric then $\pi_1(\mathscr{O}_p^c(G)) \cong \pi_1(\mathcal{F}_p^c(G))$ and

$$\mathrm{Hom}(\pi_1(\mathcal{F}_p^*(G)),k^\times) \leq T_k(G,S) \leq \mathrm{Hom}(\pi_1(\mathcal{F}_p^c(G)),k^\times)$$

The underlying maps are elaborated in (4.8). The condition that p-radical p-centric subgroups are centric is satisfied for finite groups of Lie type at the characteristic, but also holds e.g., for many sporadic groups. In general the inclusions in the diagram of Theorem F may all be strict: for $\mathrm{GL}_n(q)$ and p not dividing q, the main theorem in Ruiz [Rui07] states that $\pi_1(\mathcal{F}_p^c(\mathrm{GL}_n(q))) \cong \mathbb{Z}/e$ for e the multiplicative order of q mod p and $n \geq ep$, whereas $\pi_1(\mathscr{O}_p^*(\mathrm{GL}_n(q))) = 1$ when the p-rank of $\mathrm{GL}_n(q))$ is at least 3 by §1.1 combined with [Qui78, Thm. 12.4], [Das95, Thm. A]. Section 4 and Appendix A analyse $\pi_1(\mathscr{O}_\mathcal{C}(G))$ and $\pi_1(\mathcal{F}_\mathcal{C}(G))$, for $\mathcal{C}$ an arbitrary collection of p-subgroups.

1.5. **Computational results.** We have already described how the results of this paper can be used to obtain a range of structural and computational results about $T_k(G,S)$. To further illustrate the computational potential we will in §6 go through different classes of groups: symmetric, groups of Lie type, sporadic, p-solvable, and others, obtaining new results, and reproving a range of old results. We briefly summarize this:

For sporadic groups $|\mathcal{S}_p(G)|$ is sometimes known to agree with a building, where simple connectivity has been studied extensively (see [Smi11, §9]). However it is often easier to apply Theorems C, D, F, or 4.10 directly, in particular since the necessary p-local data has already been tabulated, due to interest arising from counting conjectures in modular representation theory. We demonstrate this in §6.1 by showing that $T_k(G,S) = 0$ for G the Monster and k a field of characteristic $p = 3, 5, 7, 11$, or 13, the primes left open in the recent paper [LM15b] (see Theorem 6.1). It should be possible to fill in the remaining gaps in the existing sporadic group computations using similar arguments, though this is outside the scope of the present paper. (This has subsequently been carried out by Craven [Cra21].)

For finite groups of Lie type in characteristic p the p-subgroup complex is just the Tits building [Qui78], which is simply connected when the rank is at least 3, again recovering results of Carlson–Mazza–Nakano [CMN06]. For finite groups of Lie type in arbitrary characteristic, the p-subgroup complex is also believed to generically be a wedge of high dimensional spheres, which would imply that generically there were no exotic Sylow-trivial modules by Theorem B. It has been verified in a number of cases [Das95, Das98, Das00], and this way, we recover very recent results of Carlson–Mazza–Nakano for the general linear group for any characteristic [CMN14, CMN16], again using Theorem B, and for symplectic groups we get the following new result (see §6.2).

Theorem G. *Let $G = \mathrm{Sp}_{2n}(q)$, and k a field of characteristic p. If the multiplicative order of q mod p is odd, and G has an elementary abelian p-subgroup of rank 3, then $T_k(G,S) = 0$.*

In joint work in progress with Carlson, Mazza, and Nakano, we determine the group of endotrivial modules for all finite groups of Lie type using the methods of this paper combined with the "Φ_d-local" approach to finite groups of Lie type.

For symmetric and alternating groups, the p-subgroup complex is known to be simply connected (except known small exceptions) [Kso03, Kso04], so we recover results of Carlson–Hemmer–Mazza–Nakano [CMN09, CHM10], using Theorem B (see §6.3, where we also correct a small mistake in the literature about alternating groups).

For p-solvable groups above p-rank one, it has been shown that there are no exotic Sylow-trivial modules by works of Carlson–Mazza–Thévenaz [CMT11] and Robinson–Navarro [NR12], using an inductive argument that at one inductive step indirectly relies on the classification of finite simple groups. We link this inductive step to stronger conjectures by Quillen and Aschbacher about the connectivity of the p-subgroup complex for p-solvable groups (see §6.4).

As explained above, a calculation of $\pi_1(\mathscr{O}_p^*(G))$ and $H_1(\mathscr{O}_p^*(G))$ for all finite simple groups should be within reach. To pass to arbitrary finite groups one may then hope to use the structural properties of $\pi_1(\mathscr{O}_p^*(G))$ given in this paper (in particular §4 and Appendix A) to reduce the question of the existence of exotic Sylow-trivial modules to the simple case, using the generalized Fitting subgroup $F^*(G)$ from finite group theory. We note in this connection that Aschbacher [Asc93] has, in a certain sense, reduced the question of simple connectivity of $|\mathcal{S}_p(G)|$ to simple groups, modulo his conjecture for p-solvable groups alluded to above. Based on available data, it may be that $\pi_1(\mathscr{O}_p^*(G)) \xrightarrow{\cong} (G_0)_{p'}$ when the p-rank is at least three? This would imply $T_k(G,S) \cong \mathrm{Hom}(G_0, k^\times)$ under that assumption. See Remarks 5.11 and A.18, and §6, for more information.

Further vistas. In addition to the structural and computational consequences described so far, it is natural to wonder about further representation theoretic significance of the finite p'-groups $\pi_1(\mathscr{O}_p^*)$, $\pi_1(\mathscr{O}_p^c)$, $\pi_1(\mathcal{F}_p^c)$, and $\pi_1(\mathcal{F}_p^*)$, and the higher homotopy and homology groups, yet to be found. In a similar vein one may wonder if the method for constructing representations in Theorems A and 3.11, when applied to more general coefficient systems on $|\mathcal{S}_p(G)|$ (see §2.6), could enable one to describe a larger slice of the stable module category, potentially shedding light on standard counting conjectures and their derived generalizations. Remarks 3.17, 3.20, 4.12, and 4.40 give some hints in these directions. Further afield, one may look for a contribution of the homotopy type of the orbit or fusion category on collections of p-subgroups for problems involving any other 'p-local' symmetric monoidal (infinity) category that depend on G, whether in algebra or topology, similar to the one discovered here for StMod_{kG} (see also [Mat16] for infinity categorical considerations).

Organization of the paper. In §2 we state conventions and introduce the categories and constructions needed for the main results, providing a fair amount of detail in the hope of making the paper accessible to both group representation theorists and topologists. In §3 we prove Theorem A, only relying on the recollections in §2. In §4 we prove a number of results on $\pi_1(\mathscr{O}_p^*(G))$ and $\pi_1(\mathcal{F}_p^*(G))$, and among other things deduce the consequences described in §§1.1, 1.4, including Corollary 1.1, and Theorems B and F. In §5 the decompositions and the Carlson–Thévenaz conjecture are established as stated in §§1.2, 1.3, proving Theorems C, D, and E, and Corollary 1.2. In §6 we go through the computational consequences and results, including Theorem G. Finally Appendix A contains a number of results about changing the collection of subgroups, which are used throughout §§4–6, and should also be of independent interest.

2. Notation and preliminaries

This short section collects some conventions and definitions, giving some detail, in the hope to make the paper accessible to both group theorists and topologists. We defer certain parts of the discussion of coefficient systems and derived categories to §3.2.2 and §3.2.1 respectively.

2.1. **Conventions.** In this paper G will always be an arbitrary finite group and p an arbitrary prime dividing the order of G (to avoid having to make special statements in the trivial case where this is not so). We use the notation S for its Sylow p-subgroup and set $G_0 = \langle N_G(Q) | 1 < Q \leq S \rangle$, which when $G_0 < G$ is the smallest strongly p-embedded subgroup of G containing S (see also (1.3) and §A.5). By k we will always mean a field of characteristic p, where p divides $|G|$, but subject to no further restrictions. Thus k is not assumed to be algebraically closed. Note that the units $k^\times$ cannot have p-torsion, as the Frobenius map is injective, and it will be uniquely p-divisible if k is perfect (see also §4.4). Our kG-modules will not be assumed finitely generated, though everything could also be phrased inside the smaller category of finitely generated modules with the same result. Tensor products are over k.

By a collection of p-subgroups $\mathcal{C}$, we mean a set of p-subgroups of G, closed under conjugation, which we view as a poset under inclusion, hence as a category. We use standard notation for various specific collections of p-subgroups, like $\mathcal{A}_p(G)$ for the non-trivial elementary abelian p-subgroups, $\mathcal{B}_p(G)$ for the non-trivial p-radical subgroups, etc., which we also recall in Appendix A.

We use standard group theoretic notation, plus that

$$G_{p'} = G/\langle\, g \in G \,|\, g \text{ is of finite } p\text{-power order} \,\rangle$$

and $A^{p'}(G) = O^{p'}(G)[G, G]$, as also mentioned in §1.

By a space we will for convenience mean a simplicial set, $|\cdot|$ denotes the nerve functor from categories to simplicial sets, and homotopy equivalence means homotopy equivalence after geometric realization. Group theorists not familiar with simplicial sets are largely free to think of them as simplicial complexes, or topological spaces, and can find a quick introduction in [Ben91b, Ch. 1.8], and more information e.g., in [DH01]. A space is simply connected if it is connected with trivial fundamental group, and connected spaces are assumed to be non-empty.

We use $[\cdot]$ for conjugacy and equivalence classes, $\cong$ for isomorphism, and $\simeq$ for equivalence.

2.2. **Sylow-trivial modules.** As stated in §1 we use the term *Sylow-trivial* for our basic objects: kG-modules that when restricted to a Sylow p-subgroup S split as the trivial module k direct sum a projective module. Two Sylow-trivial modules M and N are called *equivalent* if there exist projective kG-modules P, Q such that $M \oplus P \cong N \oplus Q$. We denote by $T_k(G, S)$ the set of equivalence classes of Sylow-trivial modules. By Proposition 3.1 each Sylow-trivial module has an, up to isomorphism unique, indecomposable representative, and this is isomorphic to a summand of

$k[G/S]$. We claim that tensor product over k endows $T_k(G,S)$ with an abelian group structure with neutral element k. First it is clear that the tensor product of two Sylow-trivial modules is again Sylow-trivial, as the tensor product of a projective kS-module with any kS-module is again projective. The same fact (now over kG) implies that the multiplication descends to equivalence classes. Also if $M\downarrow_S \cong k \oplus (\mathrm{proj})$, then the cokernel of the unit kG-map $k \to M^* \otimes M$ is a projective kG-module, since projectivity is detected on kS, and as kS-modules, $M^* \otimes M \cong k^* \otimes k \oplus (\mathrm{proj})$, with the map $k \to k^* \otimes k \cong k$ the identity. Thus M^* is an inverse to M in $T_k(G,S)$, and $T_k(G,S) \le T_k(G)$, the group of endotrivial modules.

2.3. **The stable module category StMod_{kG}.** Recall that the stable module category StMod_{kG} is the category with objects kG-modules and morphisms from M to N the quotient of $\mathrm{Hom}_{kG}(M,N)$ where we identify two maps if their difference factors through a projective kG-module. We denote by stmod_{kG} the full subcategory with objects *finitely generated* kG-modules. The relevance of the stable module category for endotrivial modules stems from the following well-known fact.

Lemma 2.1. *Two kG-modules M, N are isomorphic in StMod_{kG} if and only if there exist projective kG-modules P, Q such that $M \oplus P \cong N \oplus Q$. In particular $T_k(G) \xrightarrow{\cong} \mathrm{Pic}(\mathrm{StMod}_{kG})$.*

Proof. It is clear that equivalent modules are isomorphic in StMod_{kG}. Conversely (following an online argument by Rickard), assume we have maps $f\colon M \to N$ and $g\colon N \to M$ such that $fg-1$ and $gf-1$ factor through projectives, and let $M \xrightarrow{\varphi} Q \xrightarrow{\psi} M$ be a factorization of $gf-1$. Let $\tilde f = (f,\varphi)\colon M \to N \oplus Q$ and $\tilde g = (g-\psi)\colon N \oplus Q \to M$. Then $\tilde g \tilde f = 1$ and so $M \oplus \ker(\tilde g) \cong N \oplus Q$. As $\tilde f \tilde g - 1\colon N \oplus Q \to N \oplus Q$ also factors through a projective (using that $fg-1$ does), and is the identity on $\ker(\tilde g)$, we conclude that $P = \ker(\tilde g)$ is a retract of a projective and hence projective as wanted.

By the first part $T_k(G) \hookrightarrow \mathrm{Pic}(\mathrm{StMod}_{kG})$, where $\mathrm{Pic}(\mathrm{StMod}_{kG})$ is defined as isomorphism classes of invertible objects under tensor product in StMod_{kG}. It is also surjective as any invertible object M in StMod_{kG} has inverse M^* by a small calculation, true in any closed symmetric monoidal category (see e.g., [HPS97, Prop. A.2.8]). ☐

As for Sylow-trivial modules, modules in StMod_{kG} in fact have a representative without projective summands, unique up to isomorphism of kG-modules (see Proposition 3.1 and [Ric97, Lem. 3.1]).

2.4. **Categorical constructions.** Define the transport category $\mathscr{T}(G)$ as the category with objects all subgroups of G and morphisms $\mathrm{Mor}(P,Q) = \{g \in G | {}^gP \le Q\}$, i.e., the transport category, or Grothendieck construction, of the left conjugation action of G on the poset of all subgroups (see also Lemma 2.3). We have a quotient functor $\mathscr{T}(G) \to \mathscr{O}(G)$ which on objects sends H to G/H and assigns to $(g, {}^gH \le K)$ the G-map $G/H \xrightarrow{[g^{-1}]} G/K$ that sends the trivial coset eH to $g^{-1}K$. This induces $K\backslash \mathrm{Hom}_{\mathscr{T}(G)}(H,K) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{O}(G)}(G/H, G/K)$ (see also e.g., [tD87, I.10]). Denote by $\mathscr{O}_{\mathcal{C}}(G)$ and $\mathscr{T}_{\mathcal{C}}(G)$ the full subcategories with objects G/H and H respectively, for $H \in \mathcal{C}$, and we continue to use the notation $\mathscr{O}_p^*(G)$

and $\mathscr{T}_p^*(G)$ for these categories when $\mathcal{C} = \mathcal{S}_p(G)$ is the collection of all non-trivial p-subgroups. We also introduce the fusion category $\mathcal{F}_\mathcal{C}(G)$ and fusion-orbit category $\bar{\mathcal{F}}_\mathcal{C}(G)$ both with objects $P \in \mathcal{C}$ and morphisms

$$\operatorname{Hom}_{\mathcal{F}_\mathcal{C}(G)}(P,Q) = \operatorname{Hom}_{\mathscr{T}_\mathcal{C}(G)}(P,Q)/C_G(P) \qquad \text{and}$$
$$\operatorname{Hom}_{\bar{\mathcal{F}}_\mathcal{C}(G)}(P,Q) = Q\backslash \operatorname{Hom}_{\mathscr{T}_\mathcal{C}(G)}(P,Q)/C_G(P)$$

respectively, i.e., monomorphisms induced by conjugation in G and ditto modulo conjugation in the target. (The fusion-orbit category is called the exterior quotient $\tilde{\mathcal{F}}$ of $\mathcal{F}_\mathcal{C}(G)$ by Puig [Pui06], and is also sometimes denoted $\mathscr{O}(\mathcal{F})$.) All four categories hence have object-sets identifiable with $\mathcal{C}$, and morphisms related via quotients

$$\begin{tikzcd} & \mathscr{T}_\mathcal{C}(G) \arrow[dl, two heads] \arrow[dr, two heads] & \\ \mathscr{O}_\mathcal{C}(G) \arrow[r, two heads] & \bar{\mathcal{F}}_\mathcal{C}(G) & \mathcal{F}_\mathcal{C}(G) \arrow[l, two heads] \end{tikzcd}$$

Remark 2.2 (On op's and inverses). Since op's and inverses are a common source of light confusion, we make a few remarks about their presence in the formulas: A group G viewed as a category with one object is isomorphic as a category to G^{op} via the map $g \mapsto g^{-1}$. In particular $\mathscr{T}(G)$ is isomorphic to the category with morphism set $\operatorname{Mor}'(P,Q) = \{g \in G | P^g \leq Q\}$, which is the Grothendieck construction $\mathcal{S}_p(G)_{G^{\mathrm{op}}}$. Redefining the transport category this way would get rid of the inverse appearing in the formula for the projection map $\mathscr{T}(G) \to \mathscr{O}(G)$; alternatively one could reparametrize the orbit category, the choice made e.g., in [AKO11, III.5.1]. Notice also that when we are considering functors to an abelian group such as $k^\times$, viewed as a category with one object, covariant functors naturally *equal* contravariant functors. (The identification using the isomorphism between G and G^{op} produces the automorphism given by "pointwise inverse".)

2.5. **Low dimensional cohomology and homotopy of categories.** The cohomology of a small category $\mathbf{D}$ with constant coefficients in an abelian group A is defined as the cohomology of the simplicial set $|\mathbf{D}|$ with constant coefficients A. In particular $H^1(\mathbf{D}; A)$ identifies with functors $\mathbf{D} \to A$, up to natural isomorphism of functors, where A is viewed as a category with one object. Indeed, a 1-cocycle is a function $F\colon \operatorname{Mor}(\mathbf{D}) \to A$ such that $F(\beta \circ \alpha) = F(\beta) + F(\alpha)$ (hence $F(\mathrm{id}) = 0$), in other words a functor $F\colon \mathbf{D} \to A$. And a 1-coboundary is a 1-cocycle of the form $F(\alpha) = g(\operatorname{cod}(\alpha)) - g(\operatorname{dom}(\alpha))$, for a function $g\colon \operatorname{Ob}(\mathbf{D}) \to A$, i.e., a functor that admits a natural transformation (hence isomorphism) to the zero functor. Furthermore $H^1(\mathbf{D}; A) \xrightarrow{\cong} \operatorname{Hom}(H_1(\mathbf{D}), A)$, where $H_1(\mathbf{D})$ is the abelian group of cycles of morphisms in $\mathbf{D}$, modulo the equivalence relation coming from composition, a special case of the universal coefficient theorem [Hat02, Thm. 3.2]. (See also e.g., [Web07].)

Homotopy groups are defined as $\pi_i(\mathbf{D}, d) = \pi_i(|\mathbf{D}|, d)$, and also have a categorical description for $i = 1$: the fundamental groupoid $\pi(|\mathbf{D}|)$ identifies as $\mathbf{D}[\operatorname{Mor}(\mathbf{D})^{-1}]$ the category obtained by formally inverting all morphisms in $\mathbf{D}$ (left adjoint to the inclusion functor from groupoids to categories), providing a canonical isomorphism $\pi_1(|\mathbf{D}|, d) \cong \operatorname{Aut}_{\mathbf{D}[\operatorname{Mor}(\mathbf{D})^{-1}]}(d)$. (See also e.g., [Qui73, §1].)

Assume that $\mathbf{D}$ is connected, i.e., that all objects x, y can be connected by a finite zig-zag $x = x_0 \to x_1 \leftarrow \cdots \to x_n = y$ of morphisms. The universal properties give us isomorphisms

$$\operatorname{Hom}(\pi_1(\mathbf{D},d),A) \xrightarrow{\cong} H^1(\pi_1(\mathbf{D},d);A) \xleftarrow{\cong} H^1(\mathbf{D}[\operatorname{Mor}(\mathbf{D})^{-1}];A) \xrightarrow{\cong} H^1(\mathbf{D};A)$$

and one also sees directly that the canonical map $\pi_1(\mathbf{D},d) \to H_1(\mathbf{D})$, viewing loops as 1-cycles, is abelianization, a special case of the Hurewicz theorem [Hat02, Thm. 2A.1].

It is often convenient to have a concrete way of writing elements in $\pi_1(\mathbf{D},d)$. For this, pick for each object $x \in \mathbf{D}$ a path i_x from x to d, producing a functor $\mathbf{D} \to \pi_1(\mathbf{D},d)$ via $(\varphi\colon x \to y) \mapsto i_y \circ \varphi \circ i_x^{-1}$. This induces a functor $\omega\colon \mathbf{D}[\operatorname{Mor}(\mathbf{D})^{-1}] \to \pi_1(\mathbf{D},d)$, which is manifestly an inverse equivalence of categories to the inclusion, and we can think of elements of $\pi_1(\mathbf{D},d)$ as finite zig-zags of morphisms in $\mathbf{D}$ in this way. Recall also that we may replace $\mathbf{D}$ by an equivalent category without changing the result. In particular for $\mathbf{D} = \mathscr{O}_p^*(G)$, we can, up to equivalence of categories, replace it with the full subcategory $\mathscr{O}_S^*(G)$ with objects G/Q for $1 < Q \leq S$, and take basepoint G/S, so that we have canonical maps $i_{G/Q}\colon G/Q \to G/S$. Hence $\omega(G/P \to G/Q) = 1$ for $P \leq Q$, allowing us to effectively ignore morphisms induced by inclusions; similarly for the other standard categories from §2.4. We will often suppress the basepoint from the notation, and the above shows why we can do this without ambiguity. Note also that any basepoint-dependence disappears after abelianization.

We used at various points, e.g., in (1.9), that the Borel construction on the nerve of a small category can be expressed as the nerve of the transport category (or Grothendieck construction). For convenience of the reader, let us prove this special case of Thomason's theorem [Tho79, Thm. 1.2].

Lemma 2.3 (Thomason's theorem for Borel constructions). *For a small category $\mathbf{D}$ with an action of G, let $\mathbf{D}_G$ denote the transport category with objects the objects of $\mathbf{D}$ and morphisms from x to y given by a pair $(g, f\colon gx \to y)$, where $g \in G$ and $f \in \operatorname{Hom}_{\mathbf{D}}(gx,y)$. Then*

$$|\mathbf{D}_G| \cong |\mathbf{D}|_{hG}$$

where $|\mathbf{D}|_{hG}$ denotes the Borel construction. In particular $|\mathscr{T}_{\mathcal{C}}| = |\mathcal{C}_G| \cong |\mathcal{C}|_{hG}$ for any collection $\mathcal{C}$.

Proof. Define the category $E\mathbf{G}$ to be the category with objects the elements of G and a unique morphism between all elements, so that $|E\mathbf{G}| = EG$, the universal free contractible G-space. Our group G acts freely on the product category $E\mathbf{G}\times\mathbf{D}$, on objects given by $g\cdot(h,x) = (hg^{-1}, gx)$. The quotient $E\mathbf{G}\times_G \mathbf{D}$ identifies with $\mathbf{D}_G$. Since the nerve functor commutes with products and free G-actions we have identifications $|\mathbf{D}|_{hG} = EG\times_G|\mathbf{D}| \cong |E\mathbf{G}\times_G\mathbf{D}| = |\mathbf{D}_G|$ as wanted. □

2.6. **Coefficient systems.** Finally we recall the notion of coefficient system, to be specialized and elaborated in later sections. A general homological G-coefficient system on a G-space X is just a functor $A\colon (\Delta X)_G \to R$-mod, to R-modules, for R a ground ring. Here ΔX is the *category of simplices* with objects the simplices of X, and morphisms given by iterated face and degeneracy maps [GJ99, I.2], and $(\Delta X)_G$ is the associated transport category of the left G-action on ΔX. The chain complex $C_*(X;A)$, with $C_n(X;A) = \oplus_\sigma A(\sigma)$, and the standard simplicial differential, is a chain complex of RG-modules via the induced G-action, $A((g,\mathrm{id}_{g\sigma}))\colon A(\sigma) \to A(g\sigma)$ (see also [Gro02, §2]). In this paper two more restrictive types of coefficient systems play a special role, namely G-twisted coefficient systems, used in §3, and

Bredon G-isotropy coefficient systems, used in §5. These special systems enjoy homotopy invariance properties, not enjoyed by general G-coefficient systems, as we explain in those sections.

3. Proof of Theorem A

In this section we prove Theorem A, just using the preparations from the preceding section.

3.1. **Injectivity of the map Φ.** We start with some elementary facts about Sylow-trivial modules, including dealing with the finite-dimensionality issue once and for all.

Proposition 3.1. *Any Sylow-trivial kG-module M is of the form $N \oplus P$, where N is an indecomposable direct summand of $k[G/S]$ and P is projective, and N is uniquely determined, up to isomorphism. If $O_p(G) \neq 1$, then any indecomposable Sylow-trivial kG-module is one-dimensional.*

Proof. This is well known, and follows by [BBC09, Thm. 2.1] and [MT07, Lem. 2.6], but since it is among the few "classical" representation theory facts used, we give a direct proof. Let M be our Sylow-trivial module, and recall that M is a direct summand of $M \downarrow_S^G \uparrow_S^G$, since the composite of the kG-map $M \to M \downarrow_S \uparrow^G = kG \otimes_{kS} M$, $m \mapsto \sum_{g_i \in G/S} g_i \otimes g_i^{-1} m$, and the kG-map $M \downarrow_S \uparrow^G \to M$, $g \otimes m \mapsto gm$, is multiplication by $|G : S|$, which is a unit in k. (See also e.g., [Ben91a, Cor. 3.6.10], where the standing finitely generated assumption is not being used.) By assumption $M \downarrow_S \cong k \oplus (\mathrm{proj})$, so M is a summand of $k \uparrow_S^G \oplus (\mathrm{proj})$. As explained in [Ric97, Lem. 3.1], any kG-module, also infinite dimensional, can be written as a direct sum of a projective module and a module without projective summands, and the non-projective part is unique up to isomorphism. Furthermore, by [Ric97, Lem. 3.2], this decomposition respects direct sums. This shows that $M \cong N \oplus P$, with N a non-projective direct summand of $k \uparrow_S^G$ which is uniquely determined, up to isomorphism. Furthermore, N has to be indecomposable, since otherwise it cannot be Sylow-trivial.

Now assume $O_p(G) \neq 1$. Since M is a direct summand of $k \uparrow_S^G$ by the first part, then $M \downarrow_S^G$ is a direct summand of $k \uparrow_S^G \downarrow_S^G \cong \oplus_{g \in S\backslash G/S} k \uparrow_{S \cap {}^g S}^S$, which does not contain any projective summands, since $S \cap {}^g S \geq O_p(G) \neq 1$. Hence $M \downarrow_S^G \cong k$ as wanted. □

Let us also give the following well known special case of Green correspondence [Ben91a, Thm. 3.12.2] (see also [CMN06, Prop. 2.6(a)]), used for injectivity of the map Φ of Theorem A.

Proposition 3.2. *Let M be a kG-module such that $M \downarrow_{N_G(S)} \cong k \oplus (\mathrm{proj})$. Then $M \cong k \oplus (\mathrm{proj})$. In particular restriction provides an inclusion*

$$T_k(G, S) \hookrightarrow T_k(N_G(S), S) \cong \mathrm{Hom}(N_G(S)/S, k^\times).$$

Proof. As mentioned this is a special case of Green correspondence, but let us extract a direct argument: By Proposition 3.1 it is enough to see that if M is indecomposable, then $M \cong k$. So, set $N = N_G(S)$, and assume that M is an indecomposable kG-module such that $M \downarrow_N^G \cong k \oplus (\mathrm{proj})$. As in Proposition 3.1, M will be a summand of $M \downarrow_N^G \uparrow_N^G \cong k \uparrow_N^G \oplus (\mathrm{proj})$, and hence a summand of $k \uparrow_N^G \cong k \oplus L$, for L a complement of $k(\sum_{g \in G/N} gN)$. But $L \downarrow_S^G \cong \oplus_{g \in S\backslash G/N, g \notin N} k \uparrow_{S \cap {}^g N}^S$, and in particular it does not contain k as a direct summand, so M has to be a direct summand of k. Thus restriction provides an inclusion $T_k(G,S) \hookrightarrow T_k(N_G(S),S)$, and furthermore $T_k(N_G(S),S) \cong \mathrm{Hom}(N_G(S)/S, k^\times)$ by Proposition 3.1. $\square$

We can already now prove a part of Theorem A.

Proposition 3.3. *For any Sylow-trivial module M,*

$$G/P \mapsto \hat{H}^0(P;M) = M^P/(\sum_{g\in P} g)M$$

defines a functor from $\mathscr{O}_p^*(G)^{\mathrm{op}}$ to one-dimensional k-modules and isomorphisms, which we can identify with a functor $\mathscr{O}_p^*(G) \to k^\times$. The assignment that sends a Sylow-trivial module M to the above functor defines an injective group homomorphism $\Phi\colon\ T_k(G,S) \to H^1(\mathscr{O}_p^*(G);k^\times)$.

Proof. It is clear that $\hat{H}^0(P;M) = M^P/(\sum_{g\in P} g)M$ is one-dimensional, since $M\downarrow_P \cong k \oplus (\mathrm{proj})$ by assumption, and the construction kills the projective part. It furthermore defines a functor on $\mathscr{O}_p^*(G)^{\mathrm{op}}$ that sends $G/P \xrightarrow{g} G/P'$ to the morphism $M^{P'}/(\sum_{g\in P'} g)M \to M^P/(\sum_{g\in P} g)M$ given by multiplication by g. Picking a fixed one-dimensional k-vector space, say $M^S/(\sum_{g\in S} g)M$, and for each one-dimensional k-vector space a fixed isomorphism to $M^S/(\sum_{g\in S} g)M$, identifies this with a functor $\mathscr{O}_p^*(G) \to k^\times$ (concretely we may model $\mathscr{O}_p^*(G)^{\mathrm{op}}$, up to equivalence of categories, by the full subcategory $\mathscr{O}_S^*(G)^{\mathrm{op}}$ with objects G/P for $1 < P \le S$, and use the identifications induced by canonical morphisms $G/P \to G/S$). (See also Remark 2.2 for a discussion of variance.) The resulting functor is uniquely defined, up to isomorphism of functors, and hence defines a unique element in $H^1(\mathscr{O}_p^*(G);k^\times)$.

It is also clear that $T_k(G,S) \to H^1(\mathscr{O}_p^*(G);k^\times)$ is a group homomorphism, where the group structure on the right is pointwise multiplication, as

$$(M\otimes N)^P/(\sum_{g\in P} g)(M\otimes N) \xleftarrow{\cong} (M^P/(\sum_{g\in P} g)M) \otimes (N^P/(\sum_{g\in P} g)N)$$

and tensoring two 1-dimensional $kN_G(P)$-modules amount to multiplying their characters.

Finally we check injectivity: By Proposition 3.1, $M\downarrow_{N_G(S)} \cong \hat{H}^0(S;M) \oplus (\mathrm{proj})$. If $\Phi([M])$ is the identity, then the action of $N_G(S)$ on $\hat{H}^0(S;M)$ is trivial, i.e., $M\downarrow_{N_G(S)} \cong k \oplus (\mathrm{proj})$. Hence $M \cong k \oplus (\mathrm{proj})$ by Proposition 3.2 as wanted. $\square$

Remark 3.4. For the modules we consider in Theorem A,

$$\hat{H}^0(P;M) \cong M^P/\mathrm{Im}(\oplus_{Q<P}(\sum_{[g]\in P/Q} g)M^Q),$$

the Brauer quotient, and this may be another way to view this construction.

3.2. ***G*-twisted coefficient systems and the Buchweitz–Rickard equivalence.** Before continuing with the rest of the proof of Theorem A in §3.2.1, we now recall the Buchweitz–Rickard equivalence, used in Theorem A, and also explain G-twisted coefficient systems in some detail.

3.2.1. *The Buchweitz–Rickard equivalence.* We recall the equivalence of homotopy categories

$$\mathrm{stmod}_{kG} \xrightarrow{\simeq} D^b(kG)/D^{\mathrm{perf}}(kG), \tag{3.1}$$

given by viewing a module as a chain complex concentrated in degree zero, going back to [Ric89, Thm. 2.1] and [Buc, Thm. 4.4.1]. Here stmod_{kG} is the "small" stable module category with objects finitely generated kG-modules and morphisms homomorphisms of kG-modules modulo the relation that two morphisms are equivalent if their difference factors through a projective kG-module. The category $D^b(kG)$ is the "small" bounded derived category, with objects unbounded chain complexes of finitely generated kG-modules, with homology concentrated in a bounded range of degrees. A morphism in the underlying category is a kG-linear chain map. It induces an isomorphism in the homotopy category if it induces isomorphisms on homology (i.e., is a quasi-isomorphism). (We shall not describe precisely the set of morphisms in the homotopy category as it shall not use it here, but see e.g., [Huy06, §2] for an elementary treatment, and [Lur17, §1] for an ∞-categorical perspective.) The category $D^{\mathrm{perf}}(kG)$ is the full subcategory of complexes quasi-isomorphic to a finite complex of finitely generated projective kG-modules. The quotient $D^b(kG)/D^{\mathrm{perf}}(kG)$ is the Verdier quotient, inverting morphisms in $D^b(kG)$ with cofiber in $D^{\mathrm{perf}}(kG)$. (Recall that the cofiber can be obtained as the cokernel of an underlying monomorphism of chain complexes.)

Let us construct the inverse used in Theorem A, displaying how an object in $D^b(kG)/D^{\mathrm{perf}}(kG)$ is isomorphic to one in the image under (3.1): By choosing a projective resolution, represent an isomorphism class by a bounded below complex P_* of finitely generated projective kG-modules. In $D^b(kG)/D^{\mathrm{perf}}(kG)$ this complex is canonically equivalent to its truncation $\bar{P}_* = (\cdots \to P_{r+1} \to P_r \to 0 \to \cdots)$, for any r. Taking r to be the degree of the top non-trivial homology class of P_*, the complex $\bar{P}_*$ has homology only in degree r, and is hence equivalent in $D^b(kG)/D^{\mathrm{perf}}(kG)$ to $\Omega^{-r}(P_r/\operatorname{im}(d_{r+1}))$, the $-r$th Heller shift of the rth homology group, viewed as a chain complex in degree 0. (Recall that the inverse Heller shift $\Omega^{-1}(M)$ is the cokernel of the map from M to its injective hull, the "suspension" in the triangulated structure.) Hence it is in the image of $\Omega^{-r}(P_r/\operatorname{im}(d_{r+1})) \in \mathrm{stmod}_{kG}$ under (3.1).

3.2.2. *G-twisted coefficient systems on subgroup complexes.* A *G-twisted coefficient system* over k on a space X is a G-coefficient system $A\colon (\Delta X)_G \to k\text{-mod}$, as in §2.6, with the added feature that it sends all morphisms to isomorphisms. It hence factors through fundamental groupoid of the category $(\Delta X)_G$, or equivalently the fundamental groupoid of X_{hG} (see §2.5). For X_{hG} connected, a G-twisted coefficient system over k is thus equivalent to a $k\pi_1(X_{hG}, x)$-module M, for a choice of basepoint $x \in X_{hG}$. Such coefficient systems are hG-homotopy invariants in the sense that if $Y \to X$ is a G-equivariant map and a homotopy equivalence, and Y is given the coefficient system induced by a G-twisted coefficient system A on X, then $C_*(Y; A) \to C_*(X; A)$ is a kG-homomorphism and a homology equivalence. See e.g., [Qui78, §7],[Ben91b, 6.2.3], or [Hat02, Ch. 3.H] for more information.

In particular for $X = |\mathcal{C}|$ a G-twisted coefficient system is the same as a functor from $\mathscr{T}_{\mathcal{C}}(G)$ to k-vector spaces sending all morphisms to isomorphisms. If furthermore $\mathscr{T}_{\mathcal{C}}(G)$ is connected (e.g., $\mathcal{C}$ is a collection of p-subgroups containing S), then a G-twisted coefficient system can be identified with a $k\pi_1(\mathscr{T}_{\mathcal{C}}(G), P)$-module, for $P \in \mathcal{C}$. A one-dimensional $k\pi_1(\mathscr{T}_{\mathcal{C}}(G), P)$-module is just a homomorphism $\pi_1(\mathscr{T}_{\mathcal{C}}(G), P) \to k^\times$. So, for $\varphi\colon \mathscr{O}_p^*(G) \to k^\times$ a functor (where $k^\times$ is viewed as a category with one object) we consider the corresponding functor k_φ from $\mathscr{O}_p^*(G)$ to one dimensional k-vector spaces and isomorphisms, and view this as a G-twisted coefficient system on $|\mathcal{S}_p(G)|$, still denoted k_φ, in the canonical way via

$$(\Delta|\mathcal{S}_p(G)|)_G \to \mathcal{S}_p(G)_G = \mathscr{T}_p^*(G) \to \mathscr{O}_p^*(G) \tag{3.2}$$

Here $\Delta|\mathcal{S}_p(G)| \to \mathcal{S}_p(G)$ sends $(P_0 \leq \cdots \leq P_n) \mapsto P_0$ and $\mathcal{S}_p(G)_G \to \mathscr{O}_p^*(G)$ sends $({}^gP \leq P', g) \mapsto (G/P \xrightarrow{[g^{-1}]} G/P')$. Note that, if one ignores the group action, this is a twisted coefficient system in the ordinary (non-equivariant) sense, depending on the fundamental groupoid of $|\mathcal{S}_p(G)|$.

3.3. **Surjectivity of Φ.** With the above preliminaries in place, we are ready for surjectivity of Φ.

Proposition 3.5. *Let $\varphi\colon \pi_1(\mathscr{O}_p^*(G)) \to k^\times$ be a homomorphism and equip $|\mathcal{S}_p(G)|$ with the corresponding G-twisted coefficient system k_φ via* (3.2)*. Then for any non-trivial p-subgroup P we have the equivalences*

$$k_\varphi(P) \xrightarrow{\simeq} C_*(|\{P\}|; k_\varphi) \xrightarrow{\simeq} C_*(|\mathcal{S}_p(G)|^P; k_\varphi) \xrightarrow{\simeq} C_*(|\mathcal{S}_p(G)|; k_\varphi)$$

in $D^b(kN_G(P))/D^{\mathrm{perf}}(kN_G(P))$.

Consequently $C_*(|\mathcal{S}_p(G)|; k_\varphi)$ gives a Sylow-trivial module M via $\mathrm{stmod}_{kG} \xrightarrow{\simeq} D^b(kG)/D^{\mathrm{perf}}(kG)$ *(cf.* §3.2.1*) and $\Phi([M]) = \varphi$.*

Proof. The first map is a chain homotopy equivalence, indeed an isomorphism onto the normalized chain complex concentrated in degree 0.

For the second map, recall that for a non-trivial p-subgroup P, $|\mathcal{S}_p(G)|^P$ is $N_G(P)$-equivariantly contractible by Quillen's argument: $|\mathcal{S}_p(G)|^P = |\mathcal{S}_p(G)^P|$, which is contractible via a contracting homotopy induced by $Q \leq PQ \geq P$ (see [Qui78, 1.3], [GS06, Rec. 2.1]). Thus the second map, induced by the $N_G(P)$-homotopy equivalence $|\{P\}| \to |\mathcal{S}_p(G)|^P$, is an equivalence in $D^b(kN_G(P))$, as G-twisted coefficient systems are hG-homotopy invariant, recalled in §3.2.2.

We show that the right map is an equivalence in $D^b(kN_G(P))/D^{\mathrm{perf}}(kN_G(P))$ by generalizing Quillen's proof that the (reduced) Steinberg complex is isomorphic to a finite complex of projectives [Qui78, Prop 4.1 and 4.5] (see also [Web91, Thm. 2.7.4] and [Sym08]): Choose a Sylow p-subgroup S' of $N_G(P)$, and note that it is enough to prove that the map is an equivalence in $D^b(kS')/D^{\mathrm{perf}}(kS')$ (since an element in the bounded derived category is perfect if, in the notation of §3.2.1, $P_r/\operatorname{im}(d_{r+1})$ is projective, which is detected on a Sylow p-subgroup). Also since $|\mathcal{S}_p(G)|^{S'} \to |\mathcal{S}_p(G)|^P$ is an S'-homotopy equivalence, it is enough to prove the statement with P replaced by S'. Now, set $\Delta = |\mathcal{S}_p(G)|$ and let Δ_s denote the singular set under the S'-action, i.e., the union of all simplices where S' does not act freely. By definition we have an exact sequence of chain complexes

$$0 \to C_*(\Delta_s, \Delta^{S'}; k_\varphi) \to C_*(\Delta, \Delta^{S'}; k_\varphi) \xrightarrow{f} C_*(\Delta, \Delta_s; k_\varphi) \to 0$$

As observed in [Qui78, Prop. 4.1] (though only stated for S' the Sylow p-subgroup), the singular set Δ_s is contractible. (To see this, one can also note that Δ_s is covered by the contractible subcomplexes Δ^Q, for $1 \neq Q \leq S'$, all of whose intersections are also contractible; see [Seg68, §4] or [tD08, Thm. 6.7.11].) Hence $\Delta^{S'} \to \Delta_s$ is a homotopy equivalence, so $C_*(\Delta_s, \Delta^{S'}; k_\varphi)$ is acyclic, using the homotopy invariance of non-equivariant homology with twisted coefficient systems (see [Hat02, Sec. 3.H] and §3.2.2). Therefore f is an equivalence in $D^b(kS')$. By definition $C_*(\Delta, \Delta_s; k_\varphi)$ is a complex of free kS'-modules, so $C_*(\Delta, \Delta^{S'}; k_\varphi)$ is in $D^{\mathrm{perf}}(kS')$ as wanted, establishing the first part of the proposition.

To see that $C_*(|\mathcal{S}_p(G)|; k_\varphi)$ is Sylow-trivial in stmod_{kG}, it is enough to prove that $C_*(|\mathcal{S}_p(G)|; k_\varphi)$ is equivalent to the trivial module k in $D^b(kS)/D^{\mathrm{perf}}(kS)$, as the equivalence $\mathrm{stmod}_{kG} \to D^b(kG)/D^{\mathrm{perf}}(kG)$ is compatible with restriction. However this follows from the first part upon taking $P = S$.

We finally observe that $\Phi([C_*(|\mathcal{S}_p(G)|; k_\varphi)]) = \varphi$ in $H^1(\mathscr{O}_p^*(G); k^\times)$. Namely, by Proposition 4.1 it is enough to see that the two functors agree as $kN_G(S)$-modules when evaluated on G/S, which follows by the first part, finishing the proof. (In fact the above argument shows that for any G/P the identification of $C_*(|\mathcal{S}_p(G)|; k_\varphi)$ with k_φ in $\mathrm{stmod}_{kN_G(P)}$ is compatible with restriction and conjugation, and hence defines an isomorphism of functors on $\mathscr{O}_p^*(G)$, avoiding the reference to Proposition 4.1.) □

Proof of Theorem A. By Proposition 3.3, Φ is a group monomorphism. Proposition 3.5 shows that $C_*(|\mathcal{S}_p(G)|; k_\varphi)$ does define a Sylow-trivial module via the equivalence of categories $\mathrm{stmod}_{kG} \xrightarrow{\simeq} D^b(kG)/D^{\mathrm{perf}}(kG)$, and this assignment is a right inverse to Φ. So Φ is surjective as well. □

We note that Theorem A includes the bijection between Sylow-trivial kG_0-modules and Sylow-trivial kG-modules is given by induction modulo projectives [MT07, Lem. 2.7(2)], using (1.10).

Lemma 3.6 (Sylow-trivial modules for groups with a strongly p-embedded subgroup)**.**

$$C_*(|\mathcal{S}_p(G)|; k_\varphi) \cong C_*(|\mathcal{S}_p(G_0)|; k_\varphi) \uparrow_{G_0}^{G}$$

□

As mentioned in §1 the chain complex $C_*(|\mathcal{S}_p(G)|; k_\varphi)$ may be interpreted as a homotopy colimit of k_φ over $\mathcal{S}_p(G)^{\mathrm{op}}$. We detail this as a proposition for the interested reader.

Proposition 3.7 (Homotopy Kan extensions)**.** *We have equivalences of kG-chain complexes*

$$C_*(|\mathcal{S}_p(G)|; k_\varphi) \cong \operatorname{hocolim}_{P \in \mathcal{S}_p(G)^{\mathrm{op}}} k_\varphi \xleftarrow{\simeq} \operatorname{hocolim}_{\eta \downarrow G/e} k_\varphi \cong (\mathbb{L}Kan_\eta\, k_\varphi)(G/e)$$

where the homotopy colimits are taken in chain complexes over k, using the standard model from e.g., [Hir03, Ch. 18.1.1], *$\mathbb{L}Kan_\eta\, k_\varphi \colon \mathscr{O}_p(G)^{\mathrm{op}} \to k$-(chain complexes) is the homotopy left Kan extension of k_φ along $\eta \colon \mathscr{O}_p^*(G)^{\mathrm{op}} \to \mathscr{O}_p(G)^{\mathrm{op}}$, and $\eta \downarrow G/e$ is the overcategory of G/e.*

Proof. The left isomorphism is by the model for hocolim, and the right isomorphism is also by the definitions. As the overcategory $\eta \downarrow G/e$ admits a canonical G-equivariant functor to $\mathcal{S}_p(G)^{\mathrm{op}}$, which is an equivalence of categories (see (A.2)), this produces the middle equivalence in $D^b(kG)$. □

The naïve guess for the inverse in Theorem A might have been the non-derived $\operatorname{colim}_{P \in \mathcal{S}_p(G)^{\mathrm{op}}} k_\varphi$. This is however zero, unless φ corresponds to a Sylow-trivial module induced from a 1-dimensional kG_0-module, as we see next—it was this viewpoint that led us to the formula in Theorem A.

Proposition 3.8 (Kan extensions)**.** *We have isomorphisms of kG-modules*

$$H_0(|\mathcal{S}_p(G_0))|;k_\varphi){\uparrow}_{G_0}^{G} \cong H_0(|\mathcal{S}_p(G)|;k_\varphi) \cong \\ \operatorname{colim}_{P\in\mathcal{S}_p(G)^{\mathrm{op}}} k_\varphi \xleftarrow{\cong} \operatorname{colim}_{\eta\downarrow G/e} k_\varphi \cong (LKan_\eta\, k_\varphi)(G/e)$$

with LKan the left Kan extension. The kG_0-module $H_0(|\mathcal{S}_p(G_0))|;k_\varphi)$ is 0 unless the action of $kN_G(S)$ on $k_\varphi(G/S)$ extends to kG_0, where it is the unique 1-dimensional kG_0-module with this property.

Proof. The first isomorphism is by Lemma 3.6, using that induction is exact. The second and fourth are by definition, while the third follows as in Proposition 3.7.

As $\mathcal{S}_p(G_0)$ is connected by (1.10), $k_\varphi(G_0/S) \twoheadrightarrow \operatorname{colim}_{P\in\mathcal{S}_p(G_0)^{\mathrm{op}}} k_\varphi(G_0/P)$ as $kN_G(S)$-modules. Hence if the quotient is non-zero this means that the $kN_G(S)$-action extends to kG_0. Likewise if $k_\varphi(G/S)$ extends to a kG_0-module, then the colimit is obviously 1-dimensional. Finally note furthermore that any extension is necessarily unique by the Frattini argument [Gor68, Thm. I.3.7]. □

Using the above remarks, Theorem A gives a more explicit model for the Sylow-trivial module when $|\mathcal{S}_p(G)|$ is G-homotopy equivalent to one-dimensional complex. This in fact appears to cover all currently known examples of exotic Sylow-trivial modules! Recall the Heller shift Ω from §3.2.1.

Corollary 3.9. *Suppose that G is a finite group such that $|\mathcal{S}_p(G)|$ is G-homotopy equivalent to a one-dimensional complex (e.g., G has p-rank at most 2, or at most one proper inclusion between p-radicals), and suppose $\varphi \in \operatorname{Hom}(\pi_1(\mathscr{O}_p^*(G)), k^\times)$ is not in the subgroup $\operatorname{Hom}(G_0, k^\times)$, cf. (1.5). Then the corresponding Sylow-trivial module is given as*

$$\Omega^{-1}(H_1(|\mathcal{S}_p(G_0)|;k_\varphi))\uparrow_{G_0}^{G}$$

Proof. By Proposition 3.8, $H_0(|\mathcal{S}_p(G_0)|;k_\varphi)$ is trivial if φ is not in $\operatorname{Hom}(G_0,k^\times)$. Hence $C_*(|\mathcal{S}_p(G_0)|;k_\varphi)$ is isomorphic in $D^b(kG_0)$ to $H_1(|\mathcal{S}_p(G_0)|;k_\varphi)$, viewed as a chain complex concentrated in degree 1. The corollary now follows from Theorem A and Lemma 3.6 (see also §3.2.1). □

Remark 3.10. If $\mathcal{C}$ is a 1-dimensional collection in G_0, G_0-homotopy equivalent to $|\mathcal{S}_p(G_0)|$, then

$$H_1(|\mathcal{S}_p(G_0)|;k_\varphi) \cong \\ \ker\left(\oplus_{[P<Q]\in|\mathcal{C}|_1/G_0} k_\varphi(P)\uparrow_{N_{G_0}(P<Q)}^{G_0} \xrightarrow{d_0-d_1} \oplus_{[P]\in|\mathcal{C}|_0/G_0} k_\varphi(P)\uparrow_{N_{G_0}(P)}^{G_0}\right) \tag{3.3}$$

as kG_0-modules, by definition, where $k_\varphi(P)$ is the 1-dimensional $N_G(P)$-module given by $N_G(P) \to \pi_1(\mathscr{O}_p^*(G)) \to k^\times$. Often $\mathcal{C}$ can be chosen so that $|\mathcal{C}|_1/G$ is small, maybe even a single element (see §6.3 for an example). (The relationship between collections in G and G_0 is explained in §A.5.)

Lastly we remark that $\Omega^{-1}M$ is isomorphic to $\Omega^{-1}k\otimes M$, modulo projectives, so in terms of finding generators for the group of endotrivial modules, $\Omega^{-1}M$ works as well as M.

One may wonder what happens if one applies the map $\Phi^{-1}(-) = [C_*(|\mathcal{S}_p(G)|;-)]$ of Theorem A to an arbitrary $k\pi_1(\mathscr{O}_p^*(G))$-module (semi-simple since $\pi_1(\mathscr{O}_p^*(G))$ is a

p'-group by (1.5)). The proof of Theorem A in fact shows that Φ will still be a left inverse, and one can identify the image. The following more precise theorem generalizes Theorem A, and arose as response to questions by Radha Kessar (Remark 3.16) and David Craven. Call a kG-module M Sylow-semi-simple if $M\downarrow_S \cong k^r \oplus (kS)^s$ for non-negative integers r, s.

Theorem 3.11 (Classification of "Sylow-semi-simple" modules)**.** *We have a bijection*

$$\left\{ \begin{array}{c} \textit{Sylow-semi-simple } kG\textit{-modules} \\ \textit{without projective summands,} \\ \textit{up to isomorphism} \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} \textit{finitely generated} \\ k\pi_1(\mathscr{O}_p^*(G))\textit{-modules,} \\ \textit{up to isomorphism} \end{array} \right\}$$

given by restriction to $N_G(S)$, discarding projective summands, and viewing the rest as a $k\pi_1(\mathscr{O}_p^*(G)))$-module, using that $\ker(N_G(S) \twoheadrightarrow \pi_1(\mathscr{O}_p^*(G)))$ *acts trivially.*

Under this bijection indecomposable modules correspond to simple modules, giving the bijection between Sylow-trivial modules and 1-dimensional $k\pi_1(\mathscr{O}_p^*(G))$-modules of Theorem A*, a restriction of Green correspondence. Furthermore*

(1) *The forward map in the bijection identifies with the functor Φ sending a kG-module M to the $k\pi_1(\mathscr{O}_p^*(G))$-module corresponding to the functor from $\mathscr{O}_p^*(G)^{\mathrm{op}}$ to the connected groupoid of k-vector spaces and isomorphisms, given by $G/P \mapsto \hat{H}^0(P; M)$, as in Theorem* A*.*

(2) *The inverse map is described as assigning to a $k\pi_1(\mathscr{O}_p^*(G))$-module N the element $C_*(|\mathcal{S}_p(G)|; N) \in D^b(kG)/D^{\mathrm{perf}}(kG)$, with G-twisted coefficient system on $|\mathcal{S}_p(G)|$ via $\mathscr{T}_p^*(G) \to \mathscr{O}_p^*(G) \to \pi_1(\mathscr{O}_p^*(G))$ as in Theorem* A*, and identifying this with a unique kG-module M without projective summands, via the Buchweitz–Rickard equivalence of* §3.2.1*.*

Proof. This is shown by modifying the argument of the proof of Theorem A slightly, and we follow the setup there: Suppose that M is a kG-module as above. Then it is clear that Φ, via the functor $G/P \mapsto \hat{H}^0(P; M)$, produces a $k\pi_1(\mathscr{O}_p^*(G))$-module, which when inflated along $N_G(S) \twoheadrightarrow \pi_1(\mathscr{O}_p^*(G))$ agrees with the module N of the decomposition $M\downarrow_{N_G(S)} \cong N \oplus P$, as stated in (1).

By Green correspondence [Ben91a, Thm. 3.12.2] there is a bijection between indecomposable trivial-source kG-modules with vertex S and simple $N_G(S)/S$-modules, given by restriction and disposing summands not with vertex S. In particular the map in the theorem is injective. As restriction preserves direct sum it is also clear that indecomposable modules correspond to simple modules under the bijection, once we have seen surjectivity.

For surjectivity, with inverse as described in (2), suppose that N is a $k\pi_1(\mathscr{O}_p^*(G))$-module, and let M be the kG-module without projective summands corresponding to $C_*(|\mathcal{S}_p(G)|; N)$. Observe that the argument given in the first half of Proposition 3.5 still gives equivalences in $D^b(kN_G(S))/D^{\mathrm{perf}}(kN_G(S))$

$$N \to C_*(|\mathcal{S}_p(G)|^S; N) \to C_*(|\mathcal{S}_p(G)|; N)$$

which again implies that N and $M\downarrow_{N_G(S)}$ are isomorphic after throwing away projective summands, by the Buchweitz–Rickard equivalence §3.2.1. $\square$

Remark 3.12. Subsequent sections give many ways of computing $\ker(N_G(S) \to \pi_1(\mathscr{O}_p^*(G))$. In particular Theorem 4.10 gives a group theoretic description in terms of generators and relations.

Remark 3.13. As noted, the map in Theorem 3.11 is a restriction of Green correspondence [Ben91a, Thm. 3.12.2] which provides a bijection $M \mapsto N$ between indecomposable kG-modules M that split $M\downarrow_{N_G(S)} \cong N \oplus N'$ where N is simple and N' is a sum of indecomposable modules with vertex a proper subgroup of S, and all simple $kN_G(S)/S$-modules.

Remark 3.14. Since the restriction map preserves direct sum and tensor product, Theorem 3.11 in fact gives an isomorphism of semi-rings between Sylow-semi-simple modules without projective summands and finitely generated $\pi_1(\mathscr{O}_p^*(G))$-modules, where the tensor product on Sylow-semi-simple modules means tensoring and discarding projective summands. Sylow-trivial modules constitute the units in the semi-ring of Sylow-semi-simple modules.

Corollary 3.15. *Suppose M is a kG-module that arises from the correspondence of Theorem 3.11 with N an absolutely simple $k\pi_1(\mathscr{O}_p^*(G))$-module. Then $\mathrm{End}_k(M) \cong k \oplus M'$ as kG-modules, where M' is a kG-module without k in its socle.*

Proof. If N is absolutely simple, its dimension divides $|\pi_1(\mathscr{O}_p^*(G))|$ (see [Ser79, §6.5 Cor. 2]), and is in particular prime to p, since $\pi_1(\mathscr{O}_p^*(G))$ is a p'-group by (1.5). Hence the dimension of M is prime to p, since the dimension of M is congruent to the dimension of N modulo $|S|$, by Theorem 3.11. In particular $k \to \mathrm{End}_k(M) \xrightarrow{\mathrm{Tr}} k$ is an isomorphism, so k splits off $\mathrm{End}_k(M)$. To see that k is not in the socle of M' note that

$$\underline{\mathrm{Hom}}_G(M,M) \subseteq \underline{\mathrm{Hom}}_{N_G(S)}(M,M) \cong \underline{\mathrm{Hom}}_{N_G(S)}(N,N) \cong \underline{\mathrm{Hom}}_{\pi_1(\mathscr{O}_p^*(G))}(N,N) \cong k$$

since N is absolutely simple. □

Remark 3.16. As pointed out to us by Radha Kessar, modules as in Corollary 3.15 are interesting since they are candidates for the image of simple modules under self-equivalences of the stable module category. Carlson proved in [Car98] that for p-groups, modules N satisfying $\underline{\mathrm{Hom}}_G(N,N) \cong k$ are in fact endotrivial, but for general finite groups the class is bigger, e.g., it contains all simple modules. Its size in general, and the precise image given via Theorem 3.11, is at present unclear.

Remark 3.17 (More general modules from p-local information). The process of constructing modules from p-local information via a homotopy left Kan extension as in Theorems A and 3.11 should be of interest also for more general families of modules on p-local subgroup $N_G(P)$, even when they are not translates of a fixed module on $N_G(S)$. Or, said differently, when one considers more general G-local coefficient systems on $|\mathcal{S}_p(G)|$, as explained in §2.6. For instance it would be worthwhile to understand the work of Wheeler [Whe02] from this point of view (see also [Mat16]). Fundamental counting conjectures in representation theory, such as Alperin's [Alp87], predict a relationship between kG-modules and modules on normalizers of p-subgroups, and it is not unnatural to expect that the glue between these subgroups provided by the transport and orbit categories should play a role in categorifiying, and potentially proving, these conjectures.

Remark 3.18 (Discrete valuation rings). Our model for Sylow-trivial modules can also be lifted to characteristic 0: Suppose that (K,R,k) is a p-modular system, with R a complete rank one discrete valuation ring with residue field k of characteristic p [Ben91a, §1.9]. Then reduction modulo the maximal ideal $R \to k$ induces an isomorphism on finite roots of unity in R and k by Hensel's lemma. Hence

$H^1(\mathscr{O}_p^*(G); R^\times) \xrightarrow{\cong} H^1(\mathscr{O}_p^*(G); k^\times)$, so we can uniquely define the twisted Steinberg complex $C_*(|\mathcal{S}_p(G)|; R_\varphi)$ over R. By the RG-lattice version of the Buchweitz–Rickard theorem [Pou, Prop. 3.4] it defines an object in the stable module category of RG-lattices, lifting the kG-module corresponding to $\varphi \in H^1(\mathscr{O}_p^*(G); k^\times)$. This makes explicit the lift known to exist by virtue of Sylow-trivial modules being trivial source [Ben91a, Cor. 3.11.4(i)].

3.4. **Addendum:** $A_k(G, S) \cong H^1(\mathscr{O}_p^*(G); k^\times)$**.** In this addendum we relate Balmer's notion of a weak homomorphism to $H^1(\mathscr{O}_p^*(G); k^\times)$, providing a different proof of Theorem A. This version is less direct as it uses the main result of [Bal13], where he identifies $T_k(G, S)$ with a group he calls $A_k(G, S)$ of weak S-homomorphisms from G to $k^\times$, but may nevertheless be instructive for readers familiar with that work. By [Bal13, Def. 2.2], a weak S-homomorphism is a map from G to $k^\times$ such that

(WH1) $\varphi(g) = 1$ for $g \in S$,
(WH2) $\varphi(g) = 1$ when $S \cap S^g = 1$, and
(WH3) $\varphi(g)\varphi(h) = \varphi(gh)$ when $S \cap S^h \cap S^{gh} \neq 1$,

where as usual $H^g = g^{-1}Hg$. It would be interesting to play off the construction of endotrivial modules in Theorem A with [Bal13, Constr. 2.5 and Thm. 2.9].

Proposition 3.19. *For any finite group and field k, $A_k(G, S) \cong H^1(\mathscr{O}_p^*(G); k^\times)$, where $A_k(G, S)$ is the group of weak S-homomorphisms from G to $k^\times$.*

Proof. We prove that $A_k(G, S)$ identifies with $\operatorname{Hom}(\pi_1(\mathscr{O}_p^*(G)), k^\times)$, by observing that there are canonical group homomorphisms in both directions, that we check are well defined and inverses to each other. Recall the bijection

$$\operatorname{Hom}_{\mathscr{O}_p^*(G)}(G/P, G/Q) \cong \{g \in G | P^g \leq Q\}/Q$$

described in §2.4, and that, by §2.5, we have an isomorphism of sets which is an isomorphism of abelian groups, where the group structure on the left is pointwise multiplication in the target, and Rep means isomorphism classes of functors. Up to equivalence of categories (which does not change Rep) we can furthermore replace $\mathscr{O}_p^*(G)$ by the equivalent full subcategory $\mathscr{O}_S^*(G)$ with objects G/P for $1 < P \leq S$, for our fixed Sylow p-subgroup S (see §2.5).

We are thus just left with verifying that isomorphism classes of functors $\mathscr{O}_S^*(G) \to k^\times$ agree with the group $A_k(G, S)$ that Balmer introduced: Given $\varphi \in A_k(G, S)$ define $\Phi\colon \mathscr{O}_S^*(G) \to k^\times$ by sending a morphism $G/Q \xrightarrow{[g]} G/Q'$ to $\varphi(g)$. This is well defined, since replacing g by gq, for $q \in Q$, yields $\varphi(gq) = \varphi(g)\varphi(q) = \varphi(g)$ by (WH3) and (WH1). It is likewise a functor: By (WH1), $\Phi([\mathrm{id}_{G/Q}]) = \varphi(1) = 1$ and given a composite $G/Q \xrightarrow{[g]} G/Q' \xrightarrow{[h]} G/Q''$ we have $Q^g \leq Q' \leq S$ and $(Q')^h \leq Q'' \leq S$, so

$$S^{gh} \cap S^h \cap S = (S^g \cap S)^h \cap S \geq (Q^g)^h \cap S = (Q^g)^h > 1$$

Hence by (WH3), writing composition in categories from right to left,

$$\Phi([h] \circ [g]) = \Phi([gh]) = \varphi(gh) = \varphi(g)\varphi(h) = \varphi(h)\varphi(g) = \Phi([h])\Phi([g])$$

as wanted.

Conversely given a functor $\Phi\colon \mathscr{O}_S^*(G) \to k^\times$, up to isomorphism, we construct a weak homomorphism $\varphi\colon G \to k^\times$ as follows: By §2.5, Φ is isomorphic to a unique functor $\tilde{\Phi}\colon \mathscr{O}_S^*(G) \to k^\times$ that factors through $\mathscr{O}_S^*(G) \to \pi_1(\mathscr{O}_S^*(G))$, with the model for $\pi_1(\mathscr{O}_S^*(G))$ of §2.5. This model in particular sends morphisms induced by inclusions to the identity. We may without restriction replace Φ by this functor $\tilde{\Phi}$. Set

$$\varphi(g) = \begin{cases} \Phi(G/({}^gS \cap S) \xrightarrow{[g]} G/(S \cap S^g)) & \text{if } S \cap S^g \neq 1 \\ 1 & \text{otherwise} \end{cases}$$

It is clear that (WH1) and (WH2) are satisfied. For (WH3) suppose that $S \cap S^h \cap S^{gh} \neq 1$ and consider the diagram

$$\begin{tikzcd} G/({}^{gh}S \cap {}^gS \cap S) \arrow[rr, "{[gh]}"] \arrow[dr, "{[g]}"'] & & G/(S \cap S^h \cap S^{gh}) \\ & G/({}^hS \cap S \cap S^g) \arrow[ur, "{[h]}"'] & \end{tikzcd}$$

where the top map is the quotient of $G/({}^{gh}S \cap S) \xrightarrow{[gh]} G/(S \cap S^{gh})$, and similarly for the two other maps. Hence applying $\Phi(-)$ to this diagram, and using that morphisms $G/Q \to G/Q'$ induced by inclusions $Q \leq Q'$ go to the identity we see that $\varphi(gh) = \Phi([gh]) = \Phi(h)\Phi(g) = \varphi(h)\varphi(g) = \varphi(g)\varphi(h)$ as wanted.

As we have now given maps between $A_k(G,S)$ and $\mathrm{Rep}(\mathscr{O}_S^*(G), k^\times)$ that are group homomorphisms under pointwise multiplication in $k^\times$, and mutual inverses, we have finished the proof. $\square$

Remark 3.20. Another perspective on "weak homomorphisms" can be given by showing that they correspond to morphisms of partial groups in the sense of Chermak [Che13] from a locality of G based on all non-trivial subgroups p-subgroups to $k^\times$.

4. Fundamental groups of orbit and fusion categories

In this section we describe how to calculate and manipulate our basic invariants $\pi_1(\mathscr{O}_p^*(G))$ and $\pi_1(\mathcal{F}_p^*(G))$, and related groups. In §4.1 we establish basic properties of $\pi_1(\mathscr{O}_p^*(G))$ and establish Corollary 1.1 and Theorem B. In §4.2 we expand on its properties and in §4.3 we carry out a similar analysis for $\pi_1(\mathcal{F}_p^*(G))$. In §4.4 we look at higher homotopy groups—these occur naturally in the analysis, even if one is ultimately only interested in π_1. Some results are stated for an arbitrary collection of p-groups $\mathcal{C}$, appealing to Appendix A to get minimal hypothesis on $\mathcal{C}$–the reader may take $\mathcal{C} = \mathcal{S}_p(G)$ at first reading. We refer to Appendix A for much additional information. The categories discussed were introduced in §2.4.

Recall that $H_{p'}$ means the quotient of H by the subgroup generated by elements of p-power order (so $H_{p'} = H/O^{p'}(H)$ when H is finite). For a fixed Sylow p-subgroup S and a collection $\mathcal{C}$, set

$$G_{0,\mathcal{C}} = \langle N_G(Q) | Q \leq S,\ Q \in \mathcal{C} \rangle \quad \text{and} \quad \mathcal{C}_0 = \{Q \in \mathcal{C} | Q \leq G_{0,\mathcal{C}}\}. \tag{4.1}$$

Hence $\mathcal{C}_0$ is a collection in $G_{0,\mathcal{C}}$, and when $\mathcal{C} = \mathcal{S}_p(G)$, $G_{0,\mathcal{C}} = G_0$ of (1.3). A p-subgroup is called *p-essential* if $\mathcal{S}_p(N_G(P)/P)$ is disconnected (hence non-empty). See §A.5 for an elaboration.

4.1. Fundamental groups of orbit categories: Proof of Corollary 1.1 and Theorem B.

Proposition 4.1 (Bounds on $\pi_1(\mathscr{O}_{\mathcal{C}}(G))$)**.** *Let $\mathcal{C}$ be a collection of p-subgroups closed under passage to p-essential and Sylow overgroups. Then the categories $\mathscr{O}_{\mathcal{C}}(G)$ and $\mathscr{O}_{\mathcal{C}_0}(G_{0,\mathcal{C}})$, as well as $\mathscr{T}_{\mathcal{C}}(G)$ and $\mathscr{T}_{\mathcal{C}_0}(G_{0,\mathcal{C}})$, are equivalent, all connected, and hence $\pi_1(\mathscr{O}_{\mathcal{C}}(G)) \xleftarrow{\cong} \pi_1(\mathscr{O}_{\mathcal{C}_0}(G_{0,\mathcal{C}}))$ and $\pi_1(\mathscr{T}_{\mathcal{C}}(G)) \xleftarrow{\cong} \pi_1(\mathscr{T}_{\mathcal{C}_0}(G_{0,\mathcal{C}}))$. Furthermore we have a sequence of surjections*

$$N_G(S)/S \twoheadrightarrow \pi_1(\mathscr{O}_{\mathcal{C}}(G)) \twoheadrightarrow (G_{0,\mathcal{C}})_{p'} \twoheadrightarrow G_{p'}$$

and in particular $\pi_1(\mathscr{O}_{\mathcal{C}}(G))$ is a finite p'-group.

Proof. The categories are connected since $S \in \mathcal{C}$. That $\mathscr{O}_{\mathcal{C}}(G)$ and $\mathscr{O}_{\mathcal{C}_0}(G_{0,\mathcal{C}})$ are equivalent categories follows from Alperin's fusion theorem: By Sylow's theorem they are both equivalent to their full subcategories $\mathscr{O}_{\mathcal{C},S}(G)$ and $\mathscr{O}_{\mathcal{C}_0,S}(G_{0,\mathcal{C}})$ with objects G/Q and $G_{0,\mathcal{C}}/Q$ respectively, with $Q \le S$ and $Q \in \mathcal{C}$, for some fixed Sylow p-subgroup S. Now Alperin's fusion theorem [Alp67, §3], in the version of Goldschmidt–Miyamoto–Puig [Miy77, Cor. 1], says that for any conjugation $G/P \xrightarrow{[g]} G/P^g$ with $P, P^g \le S$ we can write $g = g_1 \cdots g_r n$ where $g_i \in N_G(P_i)$ with $P \le P_1$, $P_i \le S$ p-essential, $P^{g_1\cdots g_i} \le P_{i+1}$, and $n \in N_G(S)$. In particular $g \in G_{0,\mathcal{C}}$. Hence the two subcategories $\mathscr{O}_{\mathcal{C},S}(G)$ and $\mathscr{O}_{\mathcal{C}_0,S}(G_{0,\mathcal{C}})$ are isomorphic, and thus $\mathscr{O}_{\mathcal{C}}(G)$ and $\mathscr{O}_{\mathcal{C}_0}(G_{0,\mathcal{C}})$ are equivalent. The same argument applies verbatim to $\mathscr{T}$. (See also [Gro02, §10] for information on versions of the fusion theorem.) As the categories are equivalent, their fundamental groups are isomorphic.

To see the stated surjections, recall that $G = N_G(S)O^{p'}(G)$, for any finite group G, by the Frattini argument [Gor68, Thm. I.3.7]. In particular we have surjections $N_G(S)/S \twoheadrightarrow G_{p'}$ and $N_G(S)/S \twoheadrightarrow (G_{0,\mathcal{C}})_{p'}$. Thus we have established the proposition if we show the surjection $N_G(S)/S \twoheadrightarrow \pi_1(\mathscr{O}_{\mathcal{C}}(G))$. For this, first note that by Lemma A.11 we can without restriction assume that $\mathcal{C}$ is closed under passage to all p-subgroups, not just p-essential and Sylow subgroups. Next, recall the model for $\pi_1(\mathscr{O}_{\mathcal{C}}(G))$ of §2.5: Take G/S as basepoint and consider the functor $\omega\colon \mathscr{O}_{\mathcal{C},S}(G) \to \pi_1(\mathscr{O}_{\mathcal{C}}(G))$ from §2.5 given by sending $G/P \xrightarrow{[g]} G/Q$, for $P, Q \le S$, to the loop $G/S \leftarrow G/P \xrightarrow{[g]} G/Q \to G/S$. We have $\omega(G/P \to G/Q) = 1$ for $P \le Q$, and the image of ω generates $\pi_1(\mathscr{O}_{\mathcal{C}}(G))$. Again by the fusion theorem, $\pi_1(\mathscr{O}_{\mathcal{C}}(G))$ is in fact generated by $N_G(P)/P$ for $P \le S$, $P \in \mathcal{C}$ (compare also [BLO03b, Pf. of Prop. 1.12]). Furthermore, the fusion theorem in Alperin's version [Alp67, §3] says that any conjugation $G/P \xrightarrow{[g]} G/P^g$ can be obtained as a sequence of conjugations by elements of *p-power* order in $N_G(P_i)$ for p-subgroups for a sequence of p-subgroups $P_i \le S$ containing a conjugate of P, related as above (but now not necessarily p-essential), and an element in $N_G(S)$. However, any element $[x] \in N_G(P)/P$ of p-power order is trivial in the fundamental group, since it will be conjugate to an element in S, which is zero: To see this explicitly, pick g which conjugates $\langle x, P\rangle$ into S; then we can consider the diagram

$$\begin{tikzcd}
G/P \arrow[r, "{[g^{-1}]}"] \arrow[d, "{[x]}"] & G/{}^gP \arrow[r] \arrow[d, "{[gxg^{-1}]}"] & G/S \arrow[d, "{[1]}"] \\
G/P \arrow[r, "{[g^{-1}]}"] & G/{}^gP \arrow[r] & G/S
\end{tikzcd}$$

which commutes since $gxg^{-1} \in S$, hence showing that $G/P \xrightarrow{[x]} G/P$ maps to the identity in $\pi_1(\mathscr{O}_{\mathcal{C}}(G))$. This shows that $N_G(S)/S \to \pi_1(\mathscr{O}_{\mathcal{C}}(G))$ is surjective as wanted. $\square$

Remark 4.2. For $\mathcal{C} = \mathcal{A}_2(G)$ and $G = \mathbb{Z}/4$, $|\mathscr{O}_{\mathcal{C}}(G)| \simeq B\mathbb{Z}/2$, so $\pi_1(\mathscr{O}_{\mathcal{C}}(G))$ is not a $2'$-group.

Remark 4.3. The maps in Proposition 4.1 are natural in the collection, so if $\mathcal{C}' \leq \mathcal{C}$ we have a surjection $\pi_1(\mathscr{O}_{\mathcal{C}'}(G)) \twoheadrightarrow \pi_1(\mathscr{O}_{\mathcal{C}}(G))$, introducing a natural filtration on $\pi_1(\mathscr{O}_p^*(G))$.

Remark 4.4. For $\mathcal{C}$ a collection of p-subgroups of G, closed under passage to p-overgroups, and $N_G(S) \leq H \leq G$, then by Proposition 4.1 we have surjections

$$N_G(S)/S \twoheadrightarrow \pi_1(\mathscr{O}_{\mathcal{C}'}(H)) \twoheadrightarrow \pi_1(\mathscr{O}_{\mathcal{C}}(G))$$

for $\mathcal{C}'$ the elements of $\mathcal{C}$ that are subgroups of H. Via Theorem A this can be seen as a refinement of the fact that restriction to H is injective on Sylow-trivial modules, as is usually seen via Green correspondence [CMN06, Prop. 2.6(a)].

Proposition 4.5 (Quotienting out by p-torsion)**.** *For any collection $\mathcal{C}$ of p-subgroups, and any basepoint $P \in \mathcal{C}$, the natural surjections of categories induce isomorphisms*

$$\pi_1(\mathscr{T}_{\mathcal{C}}(G))_{p'} \xrightarrow{\cong} \pi_1(\mathscr{O}_{\mathcal{C}}(G))_{p'} \text{ and } \pi_1(\mathcal{F}_{\mathcal{C}}(G))_{p'} \xrightarrow{\cong} \pi_1(\bar{\mathcal{F}}_{\mathcal{C}}(G))_{p'}$$

In particular $H^1(\mathscr{O}_{\mathcal{C}}(G); A) \xrightarrow{\cong} H^1(\mathscr{T}_{\mathcal{C}}(G); A)$ if A is a p-torsion-free abelian group (e.g., $A = k^\times$) and $H_1(\mathscr{T}_{\mathcal{C}}(G); A) \xrightarrow{\cong} H_1(\mathscr{O}_{\mathcal{C}}(G); A)$ if A instead is assumed to be p-divisible.

Proof. Note that we include the basepoint P in the formulation, since without further assumptions on $\mathcal{C}$ the categories could be disconnected (though this is never the case for the $\mathcal{C}$ we are interested in) (see §2.5 for more detail). We first prove $\pi_1(\mathscr{T}_{\mathcal{C}}(G))_{p'} \xrightarrow{\cong} \pi_1(\mathscr{O}_{\mathcal{C}}(G))_{p'}$. Note that $\mathscr{T}_{\mathcal{C}}$ and $\mathscr{O}_{\mathcal{C}}$ have the same path components, since the quotient functor is a bijection on objects and a surjection on morphisms. Choose for each $Q \in \mathcal{C}$ which lie in the same path component as P, a preferred path in $\mathscr{T}_{\mathcal{C}}$ from Q to P, as explained in §2.5, which induces a corresponding path in $\mathscr{O}_{\mathcal{C}}$. Now, the morphisms in $\mathscr{T}_{\mathcal{C}}(G)$ surject onto the morphisms of $\mathscr{O}_{\mathcal{C}}(G)$, and if two morphisms in $\mathscr{T}_{\mathcal{C}}(G)$ are mapped to the same morphism in $\mathscr{O}_{\mathcal{C}}(G)$, then they differ by an automorphism of p-power order. Since the morphisms in the category generate the fundamental group, we conclude that $\pi_1(\mathscr{T}_{\mathcal{C}}(G))_{p'} \xrightarrow{\cong} \pi_1(\mathscr{O}_{\mathcal{C}}(G))_{p'}$ as wanted. The case $\pi_1(\mathcal{F}_{\mathcal{C}}(G))_{p'} \xrightarrow{\cong} \pi_1(\bar{\mathcal{F}}_{\mathcal{C}}(G))_{p'}$ is identical.

The consequences for cohomology and homology now follow, using the universal coefficient theorem and the Hurewicz theorem explained in §2.5. $\square$

We have now justified all the ingredients to prove Corollary 1.1 and Theorem B from §1.1:

Proof of Corollary 1.1. We have isomorphisms

$$T_k(G,S) \xrightarrow{\cong} H^1(\mathscr{O}_p^*(G); k^\times) \xrightarrow{\cong} H^1(\mathscr{T}_p^*(G); k^\times) \cong H^1(|\mathcal{S}_p(G)|_{hG}; k^\times)$$

by Theorem A, Proposition 4.5, and Lemma 2.3 respectively. $\square$

Proof of Theorem B. As mentioned in (1.11) and (1.12), we have a fibration sequence $|\mathcal{S}_p(G_0)| \to |\mathcal{S}_p(G)|_{hG} \to BG_0$, whose long exact sequence in homotopy groups produces the short-exact sequence

$$1 \to \pi_1(\mathcal{S}_p(G_0)) \to \pi_1(|\mathcal{S}_p(G)|_{hG}) \to G_0 \to 1,$$

as BG_0 has no higher homotopy groups and $|\mathcal{S}_p(G_0)|$ is connected. The exact sequence of Theorem B identifies with first four terms of the five-term exact sequence in group cohomology with $k^\times$-coefficients (see [HS71, VI.8]) arising from the above group extension, and using $T_k(G,S) \xrightarrow{\cong} H^1(|\mathcal{S}_p(G)|_{hG}; k^\times)$ by Corollary 1.1. (Alternatively apply the five-term exact sequence of the fibration (1.11) directly.)

The first of the two final statements follows from the exact sequence in the first part, together with the Universal Coefficient Theorem and Frobenius reciprocity (see also Proposition 4.6). The second now also follows, as $|S_p(G)|$ simply connected implies $G = G_0$ by (1.10). $\square$

Let us also spell out the homology version of Theorem B, as this is often useful in practice.

Proposition 4.6. *We have an exact sequence*

$$H_2(\mathscr{O}_p^*(G)) \to H_2(G_0)_{p'} \xrightarrow{\partial} (H_1(\mathcal{S}_p(G_0))_{G_0})_{p'} \to H_1(\mathscr{O}_p^*(G)) \to H_1(G_0)_{p'} \to 0$$

(where also $H_1(\mathcal{S}_p(G_0))_{G_0} \cong H_1(\mathcal{S}_p(G))_G$ *by* (1.10) *and Frobenius reciprocity).*

Proof. The five-term exact sequence in homology with $\mathbb{Z}[\frac{1}{p}]$-coefficients for the extension (1.12) is

$$H_2(\mathscr{T}_p^*(G);\mathbb{Z}[\tfrac{1}{p}]) \to H_2(G_0;\mathbb{Z}[\tfrac{1}{p}]) \to H_1(\mathcal{S}_p(G_0);\mathbb{Z}[\tfrac{1}{p}])_{G_0}$$
$$\to H_1(\mathscr{T}_p^*(G);\mathbb{Z}[\tfrac{1}{p}]) \to H_1(G_0;\mathbb{Z}[\tfrac{1}{p}]) \to 0$$

(see again [HS71, VI.8]). We have $H_i(\mathscr{T}_p^*(G);\mathbb{Z}[\frac{1}{p}]) \xrightarrow{\cong} H_i(\mathscr{O}_p^*(G))$ for $i = 1,2$ by Proposition 4.34 and Theorem 4.35 so we can rewrite the sequence as

$$H_2(\mathscr{O}_p^*(G)) \to H_2(G_0)\otimes\mathbb{Z}[\tfrac{1}{p}] \to (H_1(\mathcal{S}_p(G_0))_{G_0})\otimes\mathbb{Z}[\tfrac{1}{p}]$$
$$\to H_1(\mathscr{O}_p^*(G)) \to H_1(G_0)\otimes\mathbb{Z}[\tfrac{1}{p}] \to 0$$

using also exactness of inverting p. All terms except the middle are known to be finite, so the middle is as well, and we can hence replace $(-)\otimes\mathbb{Z}[\frac{1}{p}]$ with $(-)_{p'}$ everywhere as wanted. $\square$

Remark 4.7. The boundary map ∂ in Theorem B sends an element

$$f \in H^1(\mathcal{S}_p(G_0); k^\times)^{G_0} \hookrightarrow H^1(\mathcal{S}_p(G_0); k^\times) \cong \mathrm{Hom}(H_1(\mathcal{S}_p(G_0)), k^\times)$$

to $-f_*([\alpha]) \in H^2(G_0; k^\times)$, where $[\alpha] \in H^2(G_0; H_1(\mathcal{S}_p(G_0)))$ is the extension class of the abelianization of the extension (1.12) (see e.g., [HS53, Thm. 4] or [Eve91, Thm. 7.3.1]). Dually for ∂ in Proposition 4.6. This extension class may deserve closer study.

Remark 4.8. Propositions 4.1 and 4.5 should be compared to the situation at the prime p, where

$$H_*(\mathscr{T}_p^*(G))_{(p)} \xrightarrow{\cong} H_*(G)_{(p)} \qquad (4.2)$$

by a classical result of Brown [Bro94, Thm. X.7.3] (translated via Lemma 2.3). The theory of 'ample collections' describes for which $\mathcal{C}$ this continues to hold (see

[Dwy97, 1.3], [Gro02, §9]). This combines to show that $H_1(\mathscr{T}_p^*(G)) \xrightarrow{\cong} H_1(G)$ if and only if $H_1(\mathscr{O}_p^*(G)) \xrightarrow{\cong} H_1(G)_{p'}$. See Theorem 4.35 for a more general version.

Remark 4.9 (Equivariant complex line bundles on $\mathcal{S}_p(G)$). Via a remark of Totaro [Bal18, Rem. 2.7], Corollary 1.1 easily implies a very recent theorem of Balmer [Bal18, Thm. 1.1] that identifies $T_k(G,S)$ with the p'-torsion part of the group of G-equivariant complex line bundles on $|\mathcal{S}_p(G)|$, under the assumption that k is algebraically closed. Namely, in this case we have an embedding $\mathrm{Tors}_{p'}(\mathbb{Q}/\mathbb{Z}) \cong \mu_\infty(k) \subseteq k^\times$, where $\mathrm{Tors}_{p'}$ means the subgroup of elements of finite order prime to p, and $\mu_\infty(k)$ are the units of finite order. Hence we have isomorphisms

$$\mathrm{Tors}_{p'}(H^2(|\mathcal{S}_p(G)|_{hG};\mathbb{Z})) \xleftarrow{\cong} \mathrm{Tors}_{p'}(H^1(|\mathcal{S}_p(G)|_{hG};\mathbb{Q}/\mathbb{Z})) \xrightarrow{\cong} H^1(|\mathcal{S}_p(G)|_{hG};k^\times)$$

where the first is induced by the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ and the second uses that $H_1(|\mathcal{S}_p(G)|_{hG})$ is finite, e.g., by (1.5), (1.7), and (1.9). But now the left-hand term identifies with the p'-torsion part of the G-equivariant complex line bundles on $|\mathcal{S}_p(G)|$, as remarked by Totaro [Bal18, Rem. 2.7], and the right-hand side identifies with $T_k(G,S)$ by Corollary 1.1. More generally, since (4.2) shows that $H^2(|\mathcal{S}_p(G)|_{hG};\mathbb{Z})$ is a finite abelian group with p-torsion part $H^2(G)_{(p)}$, we can describe *all* G-equivariant complex line bundles on $|\mathcal{S}_p(G)|$ as

$$\mathrm{Pic}^G(|\mathcal{S}_p(G)|) \cong H^2(G)_{(p)} \oplus T_k(G,S). \tag{4.3}$$

Here k should in fact just be large enough so that the one-dimensional representations of $\pi_1(\mathscr{O}_p^*(G))$ do not depend on k, e.g., containing all $|N_G(S):S|$th roots of unity.

4.2. **Fundamental groups of orbit categories: Further structural results.**

The fundamental group $\pi_1(\mathscr{O}_p^*(G))$ can be described in a purely group theoretic way.

Theorem 4.10 (Group theoretic description of $\pi_1(\mathscr{O}_p^*(G))$)**.** *Let $K_{\mathscr{O}}$ be the subgroup of $N_G(S)$ generated by elements $g \in N_G(S)$ such that there exist nontrivial subgroups $1 < Q_0, \dots, Q_r \leq S$ and a factorization $g = x_1 \cdots x_r$ in G, where $x_i \in O^{p'}(N_G(Q_i))$ and $Q_0^{x_1\cdots x_i} \leq Q_{i+1}$ for $i \geq 0$. Then*

$$N_G(S)/K_{\mathscr{O}} \xrightarrow{\cong} \pi_1(\mathscr{O}_p^*(G))$$

In particular $N_G(S) \cap O^{p'}(N_G(P)) \leq \ker\big(N_G(S) \twoheadrightarrow \pi_1(\mathscr{O}_p^*(G))\big)$ for $1 < P \leq S$.

Proof. The proof amounts to a careful study of the proof of Proposition 4.1. The canonical map $N_G(S) \to \pi_1(\mathscr{O}_p^*(G))$ is surjective by Proposition 4.1. We first justify that it factors through $K_{\mathscr{O}}$, to induce a homomorphism $\varphi\colon N_G(S)/K_{\mathscr{O}} \twoheadrightarrow \pi_1(\mathscr{O}_p^*(G))$. Let $g \in K_{\mathscr{O}}$ be a generator with a decomposition $g = x_1 \cdots x_r$ as in the theorem. Then $G/S \xrightarrow{[g]} G/S$ is equal to $G/Q_0 \xrightarrow{[g]} G/Q_0^g$ in $\pi_1(\mathscr{O}_p^*(G))$, as inclusions go to the identity, as explained in §2.5. Furthermore, for each i, we have a commutative diagram

$$\begin{tikzcd} G/Q_i \arrow[r, "{[x_i]}"] & G/Q_i \\ G/Q_0^{x_1\cdots x_{i-1}} \arrow[u] \arrow[r, "{[x_i]}"] & G/Q_0^{x_1\cdots x_i} \arrow[u] \end{tikzcd}$$

equating $G/Q_i \xrightarrow{[x_i]} G/Q_i$ and $G/Q_0^{x_1\cdots x_{i-1}} \xrightarrow{[x_i]} G/Q_0^{x_1\cdots x_i}$ in $\pi_1(\mathscr{O}_p^*(G))$. But $G/Q_i \xrightarrow{[x_i]} G/Q_i$ is zero in $\pi_1(\mathscr{O}_p^*(G))$ as it is a product of elements of p-power order in $N_G(Q_i)/Q_i$, and $\pi_1(\mathscr{O}_p^*(G))$ is a p'-group by Proposition 4.1. Hence $[g]$ is zero in $\pi_1(\mathscr{O}_p^*(G))$ as well, and $K_{\mathscr{O}} \leq \ker\big(N_G(S) \twoheadrightarrow \pi_1(\mathscr{O}_p^*(G))\big)$ as wanted. The last statement in the theorem follows from this.

To show the whole theorem, i.e., φ is an isomorphism, we construct an inverse $\psi\colon \pi_1(\mathscr{O}_p^*(G)) \to N_G(S)/K_{\mathscr{O}}$ using Alperin's fusion theorem. For this, recall that $\mathscr{O}_p^*(G)$ is equivalent to a full category $\mathscr{O}_S^*(G)$ with objects G/P for $P \leq S$, and consider $G/P \xrightarrow{[g]} G/P^g$, with $P,\ P^g \leq S$. Now, by Alperin's fusion theorem [Alp67, §3], we can find $x_1, \ldots, x_r$ and non-trivial p-subgroups $Q_1, \ldots, Q_r \leq S$ such that $P = Q_0 \leq Q_1$, $x_i \in N_G(Q_i)$ is of p-power order, $P^{x_1\cdots x_i} \leq Q_{i+1} \leq S$, and $S^g = S^{x_1\cdots x_r}$, i.e., $g = x_1\cdots x_r n$ with $n \in N_G(S)$. (Our Q_i correspond to "$Q_i \cap P$", when $1 \leq i \leq r$, in the notation of [Alp67, §3].) We claim that the map sending $[g]$ to $nK_{\mathscr{O}}$ gives a well defined functor $\mathscr{O}_S^*(G) \to N_G(S)/K_{\mathscr{O}}$. If $g = x_1\cdots x_r n = y_1\cdots y_s m$ then $mn^{-1} = y_s^{-1}\cdots y_1^{-1}x_1\cdots x_r$, where the right-hand side lies in $K_{\mathscr{O}}$, so n equals m in the quotient. Also note that changing g by a different coset representative will not change this image. Likewise the map is a functor since if we have a composite $G/P \xrightarrow{[g]} G/P^g \xrightarrow{[h]} G/P^{gh}$ with $g = x_1\cdots x_r n$ and $h = y_1\cdots y_s m$, then $gh = x_1\cdots x_r n y_1 n^{-1} n y_2 \cdots y_{s-1} n^{-1} n y_s n^{-1} nm$ and hence is sent to nm as wanted. By the universal property of the fundamental group (see §2.5), we hence get an induced group homomorphism $\psi\colon\ \pi_1(\mathscr{O}_p^*(G)) \to N_G(S)/K_{\mathscr{O}}$, which is clearly a left and a right inverse to φ, as both maps commute with the surjection from $N_G(S)$. □

Remark 4.11. Let us briefly discuss the assumptions in Theorem 4.10. First note that the relationship between the subgroups Q_i is an important part of the statement, i.e., we cannot just consider the bigger subgroup $N_G(S) \cap \langle O^{p'}(N_G(P)) | 1 < P \leq S\rangle$ as examples such as $\mathfrak{S}_7$ at $p = 3$ show. Second, one could ask if the elements x_i could be assumed to lie in $N_G(S)$, i.e., if the subgroups of the 'in particular' generate the kernel. This often holds but fails e.g., for $G_2(5)$ at $p = 3$, as we shall analyze in connection with the Carlson–Thévenaz conjecture, Theorem E (see Proposition 6.3 and its proof). Finally we remark that one cannot just assume that all Q_i are p-essential, i.e., $N_G(Q_i)$ cannot be replaced by $O^{p'}(N_G(Q_i))$ in the Goldschmidt–Miyamoto–Puig [Miy77, Cor. 1] version of the fusion theorem, as $G_2(5)$ at $p = 3$ is again a counterexample by Example A.12. Which subgroups are needed is analyzed in detail in Appendix A in terms of homotopy properties of the collection $\mathcal{C}$.

Remark 4.12. The subcategories of $\mathscr{T}_p^*(G)$ and $\mathscr{O}_p^*(G)$ obtained as preimages of subgroups of $\pi_1(\mathscr{O}_p^*(G))$ can be thought of as subcategories "of p'-index" analogous to the results in [BCG⁺07, §5] on fusion system and linking systems, and the above proof may be compared to [BCG⁺07, Pf. of Prop. 5.2]. Furthermore $\mathscr{T}_p^*(G)$ is an example of an abstract transporter system as defined in [OV07, Def. 3.1], so in light of (1.8), it would be interesting to further understand the group theoretic significance of these subcategories.

Let us describe the relationship between $\pi_1(\mathscr{O}_p^*(G))$ and $G_{p'} = G/O^{p'}(G)$ in simple cases.

Corollary 4.13. *If $|\mathcal{S}_p(G)|$ is simply connected then*

$$\pi_1(\mathscr{T}_p^*(G)) \xrightarrow{\cong} G \quad \text{and} \quad \pi_1(\mathscr{O}_p^*(G)) \xrightarrow{\cong} G_{p'}.$$

If we only assume that $|\mathcal{S}_p(G_0)|$ is simply connected, then $\pi_1(\mathscr{T}_p^*(G)) \xrightarrow{\cong} G_0$ and $\pi_1(\mathscr{O}_p^*(G)) \xrightarrow{\cong} (G_0)_{p'}$.

Proof. By (1.12) we have an exact sequence $1 \to \pi_1(\mathcal{S}_p(G_0)) \to \pi_1(\mathscr{T}_p^*(G)) \to G_0 \to 1$. This gives the statements about $\mathscr{T}_p^*(G)$ and $\mathscr{T}_p^*(G_0)$ and the statements about $\pi_1(\mathscr{O}_p^*(G))$ and $\pi_1(\mathscr{O}_p^*(G_0))$ now follow using Propositions 4.1 and 4.5. ☐

Corollary 4.14 recovers and extends "classical" facts about $T_k(G,S)$, by Carlson, Mazza, Nakano, and Thévenaz, via Theorem A. (See Remark 4.15 for some historical references.)

Corollary 4.14 (Basic calculations of $T_k(G,S)$ via $\pi_1(\mathscr{O}_p^*(G))$)**.**

(1) *If for all $g \in G$, $S \cap S^g \neq 1$ (e.g., if $O_p(G) \neq 1$) then $\pi_1(\mathscr{O}_p^*(G)) \xrightarrow{\cong} G_{p'}$ and hence $T_k(G,S) \cong \mathrm{Hom}(G,k^\times)$.*

(2) *If S has p-rank one, then $\pi_1(\mathscr{O}_p^*(G)) \xrightarrow{\cong} (G_0)_{p'}$, with $G_0 = N_G(\Omega_p(Z(S)))$, and hence $T_k(G,S) \cong \mathrm{Hom}(N_G(\Omega_p(Z(S))),k^\times)$, with $\Omega_p(Z(S))$ the elements of order at most p in $Z(S)$.*

(3) *If G is a trivial intersection (T.I.) group, i.e., a group where unequal Sylow p-subgroups intersect trivially, then $G_0 = N_G(S)$, $\pi_1(\mathscr{O}_p^*(G)) \cong N_G(S)/S$, and hence $T_k(G,S) \cong \mathrm{Hom}(N_G(S),k^\times)$.*

(4) *If $G \trianglelefteq H \circ_C K$ is a normal subgroup of a central product, with $p||H \cap G|$ and $p||K \cap G|$, then $\pi_1(\mathscr{O}_p^*(G)) \xrightarrow{\cong} G_{p'}$ and $T_k(G,S) \cong \mathrm{Hom}(G,k^\times)$.*

(5) *If $G = H \wr \mathfrak{S}_n$ is a wreath product, with $p||H|$ and $n \geq 2$, then $\pi_1(\mathscr{O}_p^*(G)) \xrightarrow{\cong} G_{p'}$ and $T_k(G,S) \cong \mathrm{Hom}(G,k^\times)$.*

Proof. (1) Suppose $G/P \xrightarrow{[g]} G/Q$ goes to $1 \in G_{p'}$, so g can be written as a product in G of elements of p-power order. We can thus without loss of generality assume that g is itself of p-power order. Furthermore, by changing g up to conjugacy we can assume that $Q \leq S$, and it is enough to prove that $G/S \xrightarrow{[g]} G/S^g$ is the identity in $\pi_1(\mathscr{O}_p^*(G))$. However here it factors as $G/S \leftarrow G/(S\cap S^g) \xrightarrow{[g]} G/(S\cap S^g) \to G/S^g$, which is the identity in $\pi_1(\mathscr{O}_p^*(G))$, as g has p-power order in $N_G(S\cap S^g)/S\cap S^g$.

(2) Since $|\mathcal{A}_p(G)| = G/N_G(\Omega_p(Z(S)))$, a discrete G-space, we have $G_0 = N_G(\Omega_p(Z(S)))$. In particular $O_p(G_0) \neq 1$ and the claim follows from (1), using Proposition 4.1.

(3) Since $\mathcal{S}_p(G)$ is G-homotopy equivalent to the collection of non-trivial Sylow-intersections, e.g., by [GS06, Thm. 1.1], $|\mathcal{S}_p(G)| \simeq G/N_G(S)$ and $G_0 = N_G(S)$ and the claim again follows from (1).

(4) Let S be a Sylow p-subgroup of G, and note that $Z(S) \cap H \neq 1$, by a basic property of p-groups, and similarly for K. We claim that $O^{p'}(G) \cap N_G(S)$ is generated by $O^{p'}(N_G(Z(S)\cap K)) \cap N_G(S)$ and $O^{p'}(N_G(Z(S)\cap H)) \cap N_G(S)$, which will finish the proof by the 'in particular' part of Theorem 4.10. To see this, set $\tilde{H} = H \cap G$, $\hat{H} = O^{p'}(\tilde{H}S) \cap \tilde{H}$, and define $\tilde{K}$ and $\hat{K}$ symmetrically. Note that for $h \in H$, $s \in S$, we have $hsh^{-1}s^{-1} \in \tilde{H}$ so $hsh^{-1} \in \tilde{H}S$ and hence $hsh^{-1} \in O^{p'}(\tilde{H}S)$ as s is of p-power order. Thus $hsh^{-1}s^{-1} \in \hat{H}$ and ${}^hS \leq \hat{H}S$. In particular ${}^h(\tilde{H}S) \leq \tilde{H}({}^hS) \leq \tilde{H}S$ and hence ${}^h(O^{p'}(\tilde{H}S)) \leq O^{p'}(\tilde{H}S)$, so $\hat{H} \trianglelefteq H$. As the similar statements hold for $\hat{K}$ we conclude that $\hat{H}\hat{K} \trianglelefteq HK$ and $\hat{H}\hat{K}S \trianglelefteq HK$.

In particular $\hat{H}\hat{K}S = O^{p'}(G)$ as it, by the above, is a normal subgroup generated by elements of p-power order, and of p' index. The factorization $O^{p'}(G) = \hat{H}\hat{K}S$ also holds in $N_G(S)$: If $hk \in \hat{H}\hat{K} \cap N_G(S)$, then ${}^kS = S^h \leq \hat{H}S \cap \hat{K}S$. But $\hat{H}S \cap \hat{K}S = (\hat{H} \cap \hat{K})S \leq CS$, since if $x \in \hat{H}S \cap \hat{K}S$ of order prime to p then $x \in \hat{H} \cap \hat{K} \leq C$, as $\hat{H}$ and $\hat{K}$ are normal. Hence $h, k \in N_G(S)$, so

$$h \in \hat{H} \cap N_G(S) \leq O^{p'}(N_G(Z(S) \cap K)) \cap N_G(S),$$

and symmetrically for k as wanted.

(5) Suppose $G = H \wr \mathfrak{S}_n$. Let H_i denote the ith copy of H and S_i its Sylow p-subgroup, and let Δ denote Sylow p-subgroup of H embedded diagonally. Since $\mathfrak{S}_n \leq C_G(\Delta)$, $O^{p'}(\mathfrak{S}_n) \leq O^{p'}(N_G(\Delta))$. Likewise since $H_i \leq C_G(H_j)$ for $i \neq j$, $O^{p'}(H_i) \leq O^{p'}(N_G(S_j))$. But

$$\begin{aligned} N_G(S) \cap O^{p'}(G) &= N_G(S) \cap (\prod_i O^{p'}(H_i) \rtimes O^{p'}(\mathfrak{S}_n)) \\ &= (\prod_i N_G(S) \cap O^{p'}(H_i)) \rtimes (N_G(S) \cap O^{p'}(\mathfrak{S}_n)). \end{aligned}$$

So the result again follows from the 'in particular' in Theorem 4.10, as $N_G(S) \cap O^{p'}(G)$ is generated by groups that are trivial in $\pi_1(\mathscr{O}_p^*(G))$ by the above rewriting. $\square$

Remark 4.15. The statement about $T_k(G,S)$ in (1) above is [MT07, Lem. 2.6]. (Note also that if $O_p(G) \neq 1$, $|\mathcal{S}_p(G)|$ is contractible by [Qui78, Prop. 2.4], and compare also to Proposition 3.1.) For the statement about $T_k(G,S)$ in (2) see [MT07, Lem. 3.5] (noting that the proof there works also for S of p-rank one, not just cyclic) and compare also [CT15, §6]. For (3) see [CMN06, Prop. 2.8 and Rem. 2.9] and also [LM15b, §3.3]. The statement about $T_k(G,S)$ in (4) is a slight strengthening of the recent [CMN16, Thm. 2.4]. Note furthermore that $|\mathcal{S}_p(H \times K)|$ is simply connected unless H and K both contain strongly p-embedded subgroups by [Qui78, Prop. 2.6] and Lemma A.5.

Corollary 4.16 (Subgroups of p'-index)**.** *If $H \triangleleft G$ is of p' index, there is a diagram with exact rows*

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(\mathscr{O}_p^*(H)) \arrow[r] \arrow[d, two heads] & \pi_1(\mathscr{O}_p^*(G)) \arrow[r] \arrow[d, two heads] & G/H \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & H_{p'} \arrow[r] & G_{p'} \arrow[r] & G/H \arrow[r] & 1 \end{tikzcd} \tag{4.4}$$

In particular $\pi_1(\mathscr{O}_p^*(H)) \xrightarrow{\cong} H_{p'}$ if and only if $\pi_1(\mathscr{O}_p^*(G)) \xrightarrow{\cong} G_{p'}$, an isomorphism $H_1(\mathscr{O}_p^*(H)) \xrightarrow{\cong} H_1(H)_{p'}$ implies $H_1(\mathscr{O}_p^*(G)) \xrightarrow{\cong} H_1(G)_{p'}$, and $T_k(H,S) \cong \mathrm{Hom}(H,k^\times)$ implies $T_k(G,S) \cong \mathrm{Hom}(G,k^\times)$.

Proof. Note that $|\mathcal{S}_p(H)| = |\mathcal{S}_p(G)|$. Hence taking Borel constructions and using (1.9) produces the following diagram, where the rows are fibration sequences

$$\begin{tikzcd} {|\mathscr{T}_p^*(H)|} \arrow[r] \arrow[d] & {|\mathscr{T}_p^*(G)|} \arrow[r] \arrow[d] & B(G/H) \arrow[d, equal] \\ BH \arrow[r] & BG \arrow[r] & B(G/H) \end{tikzcd} \tag{4.5}$$

Diagram (4.4) is the bottom of the associated long exact sequence of homotopy groups, after dividing out elements of finite p-power order. Diagram (4.4) implies the consequence about π_1 by the 5-lemma. The homology consequence also follows from the 5-lemma, now applied to the associated ladder of exact sequences coming from (4.4)

$$\begin{tikzcd} H_2(G/H) \arrow[r] \arrow[d, equal] & H_1(\mathscr{O}_p^*(H))_{G/H} \arrow[r] \arrow[d] & H_1(\mathscr{O}_p^*(G)) \arrow[r] \arrow[d] & H_1(G/H) \arrow[r] \arrow[d, equal] & 0 \\ H_2(G/H) \arrow[r] & (H_1(H)_{p'})_{G/H} \arrow[r] & H_1(G)_{p'} \arrow[r] & H_1(G/H) \arrow[r] & 0 \end{tikzcd}$$

Finally the consequence about $T_k(G,S)$ follows from the dual sequence in cohomology with $k^\times$-coefficients, and using Theorem A. □

Remark 4.17. The converse to the last two implications in Corollary 4.16 fail for $G = \mathrm{SL}_2(\mathbb{F}_8) \rtimes C_3$, with C_3 acting via the Frobenius, $H = \mathrm{SL}_2(\mathbb{F}_8)$, and $p = 2$, where $\pi_1(\mathscr{O}_2^*(G)) = C_7 \rtimes C_3$ and $\pi_1(\mathscr{O}_2^*(H)) = C_7$ (see also Remark A.20). Corollary 4.16 says that a full calculation of $\pi_1(\mathscr{O}_p^*(G))$ not only allows the determination of the Sylow-trivial modules for G but also those of its normal subgroups of p' index, which is not possible from just knowing H_1. We illustrate this in §6.3 with the symmetric and alternating groups, in fact correcting a small mistake in the literature.

Corollary 4.18 (Central p'-extensions)**.** *Suppose Z is a central p'-subgroup of G. Then there is a diagram of spaces, with horizontal maps fibration sequences*

$$\begin{tikzcd} BZ \arrow[r] \arrow[d] & {|\mathscr{T}_p^*(G)|} \arrow[r] \arrow[d] & {|\mathscr{T}_p^*(G/Z)|} \arrow[d] \\ BZ \arrow[r] & BG_0 \arrow[r] & B(G_0/Z) \end{tikzcd}$$

and hence a ladder of exact sequences

$$\begin{tikzcd} H_2(\mathscr{O}_p^*(G)) \arrow[r] \arrow[d] & H_2(\mathscr{O}_p^*(G/Z)) \arrow[r, "\partial"] \arrow[d] & Z \arrow[r] \arrow[d, equal] & H_1(\mathscr{O}_p^*(G)) \arrow[r] \arrow[d, "\varphi"] & H_1(\mathscr{O}_p^*(G/Z)) \arrow[r] \arrow[d, "\bar\varphi"] & 0 \\ H_2(G_0)_{p'} \arrow[r] & H_2(G_0/Z)_{p'} \arrow[r, "\partial'"] & Z \arrow[r] & H_1(G_0)_{p'} \arrow[r] & H_1(G_0/Z)_{p'} \arrow[r] & 0 \end{tikzcd}$$

In particular $0 \to \mathrm{im}(\partial')/\mathrm{im}(\partial) \to \ker(\varphi) \to \ker(\bar\varphi) \to 0$ *is exact. And if* $H_2(\mathscr{O}_p^*(G/Z)) \to H_2(G/Z)_{p'}$ *is surjective, e.g., if* $\oplus_{[P]\in\mathcal{S}_p(G)/G} H_2(N_{G/Z}(P))_{p'} \twoheadrightarrow H_2(G/Z)_{p'}$, *then* $\ker(\varphi) \xrightarrow{\cong} \ker(\bar\varphi)$, *i.e., G and G/Z have isomorphic groups of "truly exotic" Sylow-trivial modules, as defined after Theorem* B.

Proof. Since Z is central, we have a principal fibration sequence

$$BZ \to (|\mathcal{S}_p(G)| \times EG)/G \to (|\mathcal{S}_p(G)| \times E(G/Z))/G$$

with the canonical action of BZ on $(|\mathcal{S}_p(G)| \times EG)/G$ (see e.g., [GJ99, V.3]). This fibration sequence identifies with the top fibration sequence of the corollary using (1.9) and the fact that the projection map $\mathcal{S}_p(G) \xrightarrow{\cong} \mathcal{S}_p(G/Z)$ is a bijection. The map to the standard bottom fibration sequence follows as $\mathscr{T}_p^*(G)$ is equivalent to $\mathscr{T}_p^*(G_0)$ by Proposition 4.1 and $(G/Z)_0 = G_0/Z$. The ladder of exact sequences is now the five-term exact sequence in homology of a fibration with coefficients in $\mathbb{Z}[\frac{1}{p}]$, using that we can replace $\mathscr{T}_p^*$ by $\mathscr{O}_p^*$ when considering p'-coefficients by

Proposition 4.34 and Theorem 4.35. The stated consequences follow from the snake lemma. □

Remark 4.19. Note that the kernel of φ is described via Proposition 4.6, and this can be used for an alternative derivation of the last part of Corollary 4.18.

Example 4.20. Let us illustrate Corollary 4.18 by the symmetric groups: Recall that by homological stability, $\mathbb{F}_2 \cong H_2(\mathfrak{S}_4) \xrightarrow{\cong} H_2(\mathfrak{S}_{n+4})$ for any $n \geq 0$ [Ker05, Thm. 2], so $H_2(N_{\mathfrak{S}_n}((1\cdots p))) \twoheadrightarrow H_2(\mathfrak{S}_n)$ when $n \geq p+4$. Hence $T_k(2^{\pm}\mathfrak{S}_n, S) \xrightarrow{\cong} T_k(\mathfrak{S}_n, S)$ for $n \geq p+4$ and p odd, by Corollary 4.18, where $2^{\pm}\mathfrak{S}_n$ denotes the two double covers of $\mathfrak{S}_n$, following $\mathbb{ATLAS}$ [CCN+85] notation. This was first proved in [LM15a, Thm. B(2)] (under an algebraically closed assumption), by lifting to characteristic zero and examining the list of possible characters.

Remark 4.21. In [LT17, Thm. 1.1] a reduction of the problem describing Sylow-trivial modules for arbitrary p'-extensions to that of central p'-extensions, at least as far as obtaining an upper bound on Sylow-trivial modules of the extension. The proof relies, through reference to earlier results, on the classification of finite simple groups. It would be interesting to rework and extend this result in light of the methods of the present paper.

4.3. **Fundamental groups of fusion categories: Proof of Theorem F.** We now analyze $\pi_1(\mathcal{F}_p^*(G))$ and related categories, and use this to prove Theorem F, and set the stage for later calculations involving the centralizer decomposition, Theorem D. The starting point is bounds on $\pi_1(\mathcal{F}_p^*(G))$ analogously to (1.5) for $\pi_1(\mathscr{O}_p^*(G))$:

$$N_G(S)/C^{p'}(N_G(S)) \twoheadrightarrow \pi_1(\mathcal{F}_p^*(G)) \twoheadrightarrow G_0/C^{p'}(G_0) \twoheadrightarrow G/C^{p'}(G) \tag{4.6}$$

where

$$C^{p'}(G) = \langle x \in G |\, p \text{ divides } |C_G(x)|\rangle, \tag{4.7}$$

the group generated by "positive defect" elements. This is a special case of Proposition 4.22.

Recall that a p-subgroup P is called *p-centric* if $Z(P)$ is a Sylow p-subgroup of $C_G(P)$ (hence $C_G(P) \cong Z(P) \times O_{p'}(C_G(P))$ by Schur-Zassenhaus). It is called *$\mathcal{F}$-essential* if it is not Sylow and $W_0PC_G(P)/P$ is a proper subgroup of $W = N_G(P)/P$, or equivalently if it is p-centric and $\mathcal{S}_p(N_G(P)/PC_G(P))$ is disconnected (see §A.4.2 and Lemma A.17). Continuing the notation from (4.1), the analog of Proposition 4.1 states:

Proposition 4.22 (Bounds on $\pi_1(\mathcal{F}_\mathcal{C}(G))$). *Suppose that $\mathcal{C}$ is a collection of p-subgroups closed under passage to $\mathcal{F}$-essential and Sylow overgroups. Then $\mathcal{F}_\mathcal{C}(G)$ is equivalent to $\mathcal{F}_{\mathcal{C}_0}(G_{0,\mathcal{C}})$ and with $\bar{C}_\mathcal{C}^{p'}(H) = \langle PC_H(P) | P \leq H, P \in \mathcal{C}\rangle$, for $S \leq H \leq G$, we have canonical surjections*

$$N_G(S)/SC_G(S) \twoheadrightarrow N_G(S)/\bar{C}_\mathcal{C}^{p'}(N_G(S)) \xrightarrow{\cong} \pi_1(\bar{\mathcal{F}}_\mathcal{C}(N_G(S)))$$
$$\twoheadrightarrow \pi_1(\bar{\mathcal{F}}_\mathcal{C}(G)) \twoheadrightarrow G_{0,\mathcal{C}}/\bar{C}_\mathcal{C}^{p'}(G_{0,\mathcal{C}})$$

If $\mathcal{C}$ contains all minimal p-centric subgroups, then $\pi_1(\mathcal{F}_\mathcal{C}(G)) \xrightarrow{\cong} \pi_1(\bar{\mathcal{F}}_\mathcal{C}(G))$, and thus $\pi_1(\mathcal{F}_\mathcal{C}(G))$ is a finite p'-group.

Proof. First note that $\mathcal{F}_{\mathcal{C}_0}(G_{0,\mathcal{C}}) \to \mathcal{F}_{\mathcal{C}}(G)$, and hence also $\bar{\mathcal{F}}_{\mathcal{C}_0}(G_{0,\mathcal{C}}) \to \bar{\mathcal{F}}_{\mathcal{C}}(G)$, are equivalences of categories by Alperin's fusion theorem, either in the version of Goldschmidt–Miyamoto–Puig [Miy77, Cor. 1] or more generally for fusion systems [AKO11, Thm. I.3.5] (see again [Gro02, §10] for a discussion of versions of the fusion theorem). For the rest of the proof we can hence without loss of generality assume that $G = G_{0,\mathcal{C}}$ so the claimed sequence becomes

$$N_G(S)/SC_G(S) \twoheadrightarrow N_G(S)/\bar{C}^{p'}_{\mathcal{C}}(N_G(S)) \xrightarrow{\cong} \pi_1(\bar{\mathcal{F}}_{\mathcal{C}}(N_G(S))) \twoheadrightarrow \pi_1(\bar{\mathcal{F}}_{\mathcal{C}}(G)) \twoheadrightarrow G/\bar{C}^{p'}_{\mathcal{C}}(G).$$

We model $\pi_1(\bar{\mathcal{F}}_{\mathcal{C}}(G))$ analogously to the proof of Proposition 4.1 with generators maps in $\bar{\mathcal{F}}_{\mathcal{C}}(G)$ between subgroups P, Q of S with $P, Q \in \mathcal{C}$, via the functor $\bar{\mathcal{F}}_{\mathcal{C}}(G) \to \pi_1(\bar{\mathcal{F}}_{\mathcal{C}}(G))$, taking S as basepoint, cf. again §2.5. In this notation the right-most epimorphism $\pi_1(\mathcal{F}_{\mathcal{C}}(G)) \twoheadrightarrow G/\bar{C}^{p'}_{\mathcal{C}}(G)$ is the map sending $(c_g\colon P \to Q)$ to $[g] \in G/\bar{C}^{p'}_{\mathcal{C}}(G)$. Since $\pi_1(\bar{\mathcal{F}}_{\mathcal{C}}(G))$ is a quotient of $\pi_1(\mathscr{O}_{\mathcal{C}}(G))$, we have a surjection $N_G(S) \twoheadrightarrow \pi_1(\bar{\mathcal{F}}_{\mathcal{C}}(G))$ by Proposition 4.1, and by the same argument $N_G(S) \twoheadrightarrow \pi_1(\bar{\mathcal{F}}_{\mathcal{C}}(N_G(S)))$. We have hence shown that the map from $N_G(S)$ to all the terms in the sequence is surjective, so all maps in the sequence are surjections as claimed. Likewise $SC_G(S) \le C^{p'}_{\mathcal{C}}(N_G(S)) = \langle PC_{N_G(S)}(P) | P \le N_G(S), P \in \mathcal{C}\rangle$, as $S \in \mathcal{C}$, so the first map is well-defined.

To finish showing that we have the stated sequence of maps we therefore just have to show that $\bar{C}^{p'}_{\mathcal{C}}(N_G(S))$ is exactly the kernel of $N_G(S) \to \pi_1(\bar{\mathcal{F}}_{\mathcal{C}}(N_G(S)))$. It is in the kernel since if $g \in PC_{N_G(S)}(P)$ then we have a commutative diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d, "id"] & S \arrow[d, "c_g"] \\ P \arrow[r] & S \end{tikzcd}$$

in $\bar{\mathcal{F}}_{\mathcal{C}}(G)$ and hence $c_g\colon S \to S$ represents the identity in $\pi_1(\mathcal{F}^*_p(N_G(S)))$. However, then the kernel has to be exactly $\bar{C}^{p'}_{\mathcal{C}}(N_G(S))$, since taking $G = N_G(S)$, the second and fifth terms of the sequence of the proposition agree and the map is the identity.

To finish the proof assume now that $\mathcal{C}$ contains all minimal p-centric subgroups. We will show that $\pi_1(\mathcal{F}_{\mathcal{C}}(G)) \xrightarrow{\cong} \pi_1(\bar{\mathcal{F}}_{\mathcal{C}}(G))$. This in fact follows easily from results about fusion theorems (see Remark 4.30), but let us include a stand-alone argument for completeness: By Proposition A.14 we may assume that $\mathcal{C}$ is closed under passage to all p-overgroups, and in particular contains all p-centric subgroups. By the fusion theorem, again, $\pi_1(\mathcal{F}_{\mathcal{C}}(G))$ is generated by self-maps of elements $P \in \mathcal{C}$, $P \le S$, so we just need to see that all elements of p-power order in $\mathrm{Aut}_{\mathcal{F}}(P) \cong N_G(P)/C_G(P)$ are trivial in $\pi_1(\mathcal{F}_{\mathcal{C}}(G))$. We can without loss of generality assume that $N_S(P)$ is a Sylow p-subgroup of $N_G(P)$ (i.e., P is "fully G-normalized" in S), G-conjugating P if necessary. By conjugation in $\mathrm{Aut}_{\mathcal{F}}(P)$ it is furthermore enough to prove that all elements in the image of $N_S(P)$ in $\mathrm{Aut}_{\mathcal{F}}(P)$ are trivial in $\pi_1(\mathcal{F}_{\mathcal{C}}(G))$. In $\pi_1(\mathcal{F}_{\mathcal{C}}(G))$, such elements are equal to elements of $\mathrm{Inn}(S) \le \mathrm{Aut}_{\mathcal{F}}(S)$, since inclusions are trivial in $\pi_1(\mathcal{F}_{\mathcal{C}}(G))$. The claim is hence reduced to seeing that $\mathrm{Inn}(S) \le \mathrm{Aut}_{\mathcal{F}}(S)$ map to the identity in $\pi_1(\mathcal{F}_{\mathcal{C}}(G))$. Now, any element $x \in S$ is G-conjugate to an element $x' \in S$ such that $C_S(x')$ is a Sylow p-subgroup of $C_G(x')$ (i.e., x' is "fully G-centralized" in S). Note that $Q = C_S(x')$ is p-centric in G, since $C_G(Q) \le C_G(x')$, and Q is obviously p-centric in $C_G(x')$, it being a Sylow p-subgroup. The image of $x' \in S$ in $\mathrm{Aut}_{\mathcal{F}}(S)$ identifies with the image of x' in $\mathrm{Aut}_{\mathcal{F}}(Q)$, as elements of $\pi_1(\mathcal{F}_{\mathcal{C}}(G))$, again since inclusions map to the identity. But x' goes to the identity in $\mathrm{Aut}_{\mathcal{F}}(Q)$, so $x' \in S$ represents the identity in $\pi_1(\mathcal{F}_{\mathcal{C}}(G))$. Hence so does the G-conjugate element x, as wanted. □

Let us spell out what Proposition 4.22 says for $\mathcal{C} = \mathcal{S}_p(G)$, in classical group-theoretic terms, as used in the proof of Theorem D in §5.

Corollary 4.23 (A vanishing condition for $\pi_1(\mathcal{F}^*_p(G))$)**.** *Let L be a complement to S in $N_G(S)$, and set $L_0 = \langle C_L(x) | x \in S \setminus 1\rangle \trianglelefteq L$. Then*

$$L/L_0 \xrightarrow{\cong} \pi_1(\mathcal{F}^*_p(N_G(S))) \twoheadrightarrow \pi_1(\mathcal{F}^*_p(G))$$

In particular if L is generated by elements that commute with at least one non-trivial element in S, then $\pi_1(\mathcal{F}^*_p(G)) = 1$. If $H_1(L)$ is generated by such elements then $H_1(\mathcal{F}^*_p(G)) = 0$. □

We also get Corollary 4.24, analogous to Corollary 4.13.

Corollary 4.24. *If $|\mathcal{S}_p(G)|$ is simply connected, e.g., if $O_p(G) \neq 1$, then*

$$\pi_1(\mathcal{F}^*_p(G)) \xrightarrow{\cong} G/C^{p'}(G).$$

If just $|\mathcal{S}_p(G_0)|$ is simply connected then $\pi_1(\mathcal{F}^*_p(G)) \xrightarrow{\cong} G_0/C^{p'}(G_0)$.

Proof. The second case reduces to the first since $\mathcal{F}^*_p(G_0) \to \mathcal{F}^*_p(G)$ is an equivalence of categories by Proposition 4.22, also using (1.10). Now, consider the diagram obtained also using (1.12)

$$\begin{tikzcd}
\pi_1(\mathscr{T}^*_p(G)) \arrow[r, "\simeq"] \arrow[d, two heads] & G \arrow[d, two heads] \\
\pi_1(\mathcal{F}^*_p(G)) \arrow[r, two heads] & G/C^{p'}(G)
\end{tikzcd}$$

We just need to see that any element $x \in G$ which lies in $C^{p'}(G)$ goes to zero in $\pi_1(\mathcal{F}^*_p(G))$ under the composite given by the top isomorphism and the left-hand epimorphism. Assume that x is a generator of $C^{p'}(G)$, i.e., we can find a non-trivial p-subgroup P such that $x \in C_G(P)$. Lift x to $x\colon P \to P$ in $\pi_1(\mathscr{T}^*_p(G))$, which maps to the identity in $\pi_1(\mathcal{F}^*_p(G))$ as wanted. □

Remark 4.25. Note that Corollary 4.24 can also be applied to subgroups H of our group G. And if $N_G(S) \leq H \leq G$ then $\pi_1(\mathcal{F}_p(H)) \twoheadrightarrow \pi_1(\mathcal{F}^*_p(G))$ by naturality, as in Remark 4.4.

We also state Corollary 4.26, suggested to us by Ellen Henke.

Corollary 4.26. *If $N_G(S)/SC_G(S)$ is abelian, then $\pi_1(\mathcal{F}^*_p(G))$ is cyclic.*

Proof. Set $H = N_G(S)/SC_G(S)$ and choose an irreducible H-submodule V of $\Omega_p(Z(S))$, the elements of order at most p in $Z(S)$. Then $H/C_H(V)$ is cyclic by elementary representation theory [Gor68, Thm. 3.2.3], which implies the claim by Corollary 4.23. □

Remark 4.27. As hinted above, the quotient $G/C^{p'}(G)$ is often trivial or a rather small group, and bounding its size is an interesting and so far apparently unexplored group theoretical question about p'-actions. We study this in joint work in progress with Geoffrey Robinson.

Analogous to Theorem 4.10 we can also describe $\pi_1(\mathcal{F}_p^*(G))$ purely group theoretically.

Theorem 4.28 (Group theoretic description of $\pi_1(\mathcal{F}_p^*(G))$)**.** *Let $K_{\mathcal{F}}$ be the subgroup of $N_G(S)$ generated by elements $g \in N_G(S)$ such that there exist nontrivial subgroups $1 < Q_0, \ldots, Q_r \le S$ and a factorization $g = x_1 \cdots x_r$ in G, where $x_i \in C^{p'}(N_G(Q_i))$ and $Q_0^{x_1 \cdots x_i} \le Q_{i+1}$ for $i \ge 0$. Then*

$$N_G(S)/K_{\mathcal{F}} \xrightarrow{\cong} \pi_1(\mathcal{F}_p^*(G))$$

In particular $N_G(S) \cap C^{p'}(N_G(P)) \le \ker\big(N_G(S) \twoheadrightarrow \pi_1(\mathcal{F}_p^*(G))\big)$ for $1 < P \le S$.

Proof. Recall that in the proof of Theorem 4.10 we showed the isomorphism

$$\varphi\colon\ N_G(S)/K_{\mathscr{O}} \xrightarrow{\cong} \pi_1(\mathscr{O}_p^*(G)),$$

by constructing a left inverse ψ. And by Proposition 4.22 we have a surjection $\pi_1(\mathscr{O}_p^*(G)) \twoheadrightarrow \pi_1(\mathcal{F}_p^*(G))$. As $K_{\mathscr{O}} \le K_{\mathcal{F}}$ by definition we can thus establish the theorem by verifying that φ and ψ induce well-defined maps $\bar\varphi$ and $\bar\psi$ on the quotients

$$\begin{tikzcd}
N_G(S)/K_{\mathscr{O}} \arrow[r, two heads, "\varphi", "\cong"'] \arrow[d] & \pi_1(\mathscr{O}_p^*(G)) \arrow[l, bend left, "\psi"] \arrow[d] \\
N_G(S)/K_{\mathcal{F}} \arrow[r, dotted, two heads, "\bar\varphi"] & \pi_1(\mathcal{F}_p^*(G)) \arrow[l, dotted, bend left, "\bar\psi"]
\end{tikzcd}$$

which will then necessarily be inverse equivalences. This amounts to unravelling the definitions:

To see that $\bar\varphi$ is well-defined, we need to check that $K_{\mathcal{F}}$ lies in the kernel of $N_G(S) \to \pi_1(\mathcal{F}_p^*(G))$. Given a generator g of $K_{\mathcal{F}}$ as in the theorem, we have the following commutative diagram

$$\begin{tikzcd}
S \arrow[rrrr, "c_{g^{-1}}"] & & & & S \\
Q_0 \arrow[u] \arrow[r, "c_{x_1^{-1}}"] & Q_1 \arrow[r] & \cdots \arrow[r] & Q_{r-1} \arrow[r, "c_{x_r^{-1}}"] & Q_r \arrow[u]
\end{tikzcd}$$

in $\mathcal{F}_p^*(G)$, and $c_{x_i^{-1}}\colon Q_{i-1} \to Q_i$ is equivalent to $c_{x_i^{-1}}\colon Q_i \to Q_i$ in $\pi_1(\mathcal{F}_p^*(G))$. The kernel of $N_G(Q_i) \to \pi_1(\mathcal{F}_p^*(N_G(Q_i)))$ is $C^{p'}(N_G(Q_i))$ by Corollary 4.24, so the factorization $N_G(Q_i)/C_G(Q_i) \to \pi_1(\mathcal{F}_p^*(N_G(Q_i))) \to \pi_1(\mathcal{F}_p^*(G))$ shows that $c_{x_i^{-1}}\colon Q_i \to Q_i$, and hence $c_g\colon S \to S$, is zero in $\pi_1(\mathcal{F}_p^*(G))$ as wanted. This shows that we have a well-defined map $\bar\varphi\colon N_G(S)/K_{\mathcal{F}} \twoheadrightarrow \pi_1(\mathcal{F}_p^*(G))$. It also establishes the last part of the theorem.

To see that $\bar\psi$ is well-defined we have to justify that for any $g \in C_G(P)$, the element $G/P \xrightarrow{[g]} G/P$ in $\pi_1(\mathscr{O}_p^*(G))$ is mapped to an element in $K_{\mathcal{F}}/K_{\mathscr{O}}$ under ψ. We can without restriction assume that $P \le S$. As explained in the proof of Theorem 4.10, the map ψ is given by sending $[g]$ to $nK_{\mathscr{O}}$, where we express

$g = x_1 \cdots x_r n$, with $x_i \in O^{p'}(N_G(Q_i))$ and $n \in N_G(S)$, where $1 < Q_0, \ldots, Q_r \leq S$, $P = Q_0$, $Q_0^{x_1 \cdots x_i} \leq Q_{i+1}$ for $i \geq 0$, using Alperin's fusion theorem. But then $n^{-1} = g^{-1} x_1 \cdots x_r$, with $P^{g^{-1}} = P$ and $g^{-1} \in C_G(P) \leq C^{p'}(N_G(P))$, which shows that $n^{-1} \in K_{\mathcal{F}}$ as wanted. □

Proof of Theorem F. Consider the following commutative diagram

$$\begin{tikzcd}
N_G(S)/S \arrow[r, two heads] \arrow[d, two heads] & \pi_1(\mathscr{O}_p^c(G)) \arrow[r, two heads] \arrow[d, two heads] & \pi_1(\mathscr{O}_p^*(G)) \arrow[d, two heads] \\
N_G(S)/C_G(S)S \arrow[r, two heads] & \pi_1(\bar{\mathcal{F}}_p^c(G)) \arrow[r, two heads] & \pi_1(\bar{\mathcal{F}}_p^*(G)) \\
 & \pi_1(\mathcal{F}_p^c(G)) \arrow[r, two heads] \arrow[u, "\simeq"] & \pi_1(\mathcal{F}_p^*(G)) \arrow[u, "\simeq"]
\end{tikzcd} \tag{4.8}$$

The top horizontal maps in (4.8) are epimorphisms by Proposition 4.1, as the collection of p-centric subgroups is closed under passage to p-overgroups. The surjections between the top and middle rows follow by definition, and the properties of the maps in and between the second and third rows follow by Proposition 4.22. Applying $\mathrm{Hom}(-, k^{\times})$ and Theorem A now gives the first part of Theorem F.

Now suppose that all p-radical p-centric subgroups are centric: We claim that then in fact *all* p-centric subgroups are centric. Namely suppose that P is p-centric so $C_G(P) \cong ZP \times K$ where $K = O_{p'}(C_G(P))$. Then K is normalized by $O_p(N_G(P))$ and vice versa. But as K and $O_p(N_G(P))$ are of coprime order they then have to commute. If $O_p(N_G(P))$ is not p-radical, we can repeat the process of taking $O_p(N_G(-))$, until we arrive at a p-radical subgroup (the p-radical closure, which will also be used in §5). Hence K centralizes a p-centric p-radical subgroup and is hence trivial by assumption, which is what we wanted. Hence $\mathscr{O}_p^c(G) \cong \bar{\mathcal{F}}_p^c(G)$ by definition, and thus $\pi_1(\mathscr{O}_p^c(G)) \cong \pi_1(\bar{\mathcal{F}}_p^c(G)) \xleftarrow{\cong} \pi_1(\mathcal{F}_p^c(G))$, using Proposition 4.1 again. With this isomorphism, the last part of the theorem now follows from the first part. □

Remark 4.29. Appendix A, e.g., Theorem A.10 and Proposition A.14, can be used to further analyze which p-centric subgroups need to be centric, for $\pi_1(\mathscr{O}_p^c(G))$ and $\pi_1(\mathcal{F}_p^c(G))$ to agree.

Remark 4.30. The p'-quotient groups of $\pi_1(\mathcal{F}^c(G))$ were originally studied in [BCG⁺07, §5.1], where they were related to subsystems of the fusion system of p'-index. It was remarked to the authors by Aschbacher that the group itself is a finite p'-group; see [BCG⁺07, p. 3839], [Asc11, Ch. 11] and [AKO11, Prop. III.4.19].

Let us round off our discussion of $\pi_1(\mathcal{F}_{\mathcal{C}}(G))$ for now by giving a few computational examples—see also Appendix A for more information on how it depends on $\mathcal{C}$.

Example 4.31. For φ an automorphism, of order prime to p, of a finite p-group S, $\pi_1(\mathcal{F}_p^*(S \rtimes \langle\varphi\rangle)) \cong \mathbb{Z}/r$, generated by φ, with r the greatest common divisor of all natural numbers s such that φ^s acts with a fixed-point on $S \setminus 1$, by Corollary 4.23.

Proposition 4.32. *Suppose that $\mathcal{F}$ is a fusion system over $S{=}3_+^{1+2}$. Then $\pi_1(\mathcal{F}^*){=}1$ unless $\mathcal{F}{=}\mathcal{F}_3(G)$ for $G{=}3_+^{1+2}{:}8$, in which case $\pi_1(\mathcal{F}^*){\cong}\mathbb{Z}/2$. The sporadic group J_2*

is the unique finite simple group with 3-fusion system $\mathcal{F}_3(3_+^{1+2}:8)$, and hence for all other finite simple groups G with Sylow 3-subgroup 3^{1+2}, $\pi_1(\mathcal{F}_3^*(G)) = 1$.

Proof. We have that $\mathrm{Out}(S) \cong \mathrm{GL}_2(\mathbb{F}_3)$ of order 48, with the canonical action on $S/Z(S) \cong (\mathbb{F}_3)^2$ and action on $Z(S)$ through the determinant map, described in detail in [Win72]. As we can lift $S/Z(S)$ to an $\mathrm{Out}(S)$ invariant subset of S, an element of $\mathrm{Out}(S)$ acts freely on $S \setminus 1$ iff it acts freely on $S/Z(S) \setminus Z(S)$ and on $Z(S) \setminus 1$. We have that $L = N_{\mathcal{F}}(S)/S$ is a subgroup of the Sylow 2-subgroup SD_{16}, which identifies with the semi-linear automorphisms of $\mathbb{F}_9$, generated by a generator σ of $\mathbb{F}_9^\times$ and the Frobenius τ, subject to the relations $\sigma^8 = \tau^2 = 1$, and $\tau\sigma\tau = \sigma^3$. Both σ and τ have determinant -1 inside $\mathrm{GL}_2(\mathbb{F}_3)$. We claim that elements of the form σ^{2k+1} are the only elements of SD_{16} that act freely on $S \setminus 1$. Such elements act freely, as they act freely on $\mathbb{F}_9^\times$ and on $Z(S) \setminus 1$. The elements σ^{2k} and $\sigma^{2k-1}\tau$, $k \geq 1$, act with a fixed-point, since they act trivially on $Z(S)$, and so does τ as it fixes $\mathbb{F}_3^\times \subseteq \mathbb{F}_9^\times$. The element $\sigma^{2k}\tau$ has a fixed-point as well, as it is conjugate to τ via $\sigma^k(\sigma^{2k}\tau)\sigma^{-k} = \tau$. The only two subgroups that contain σ^{2k+1} are $\langle\sigma\rangle$ and SD_{16}. For $L = SD_{16}$ we have $\sigma = (\sigma\tau)\tau$, in the notation of Corollary 4.23, so $L = L_0$. For $L = \langle\sigma\rangle$, $L/L_0 \cong \mathbb{Z}/2$ by Example 4.31.

Now by [RV04, Thm. 1.1] all abstract fusion systems on $S = 3_+^{1+2}$ arise from groups, and by [RV04, Tables 1.1 and 1.2] $\mathcal{F}_3(3_+^{1+2} : 8)$ is the only fusion system on 3_+^{1+2} with $N_{\mathcal{F}}(S)/S \cong \mathbb{Z}/8$. Hence Corollary 4.23, combined with the analysis above, shows that $\mathcal{F} = \mathcal{F}_3(3_+^{1+2} : 8)$ is the only fusion system with $\pi_1(\mathcal{F}^*) \neq 1$, and that $\pi_1(\mathcal{F}^*) \cong \mathbb{Z}/2$ in that case. Furthermore J_2 is the unique finite simple group realizing $\mathcal{F}$ by [RV04, Rem. 1.4]. □

Example 4.33. By [CCN+85] the centralizers of 3-elements in J_2 are $C_{J_2}(3A) = 3 \cdot \mathrm{PSL}_2(9)$ and $C_{J_2}(3B) = 3 \times A_4$. In particular they satisfy $H_1(-)_{3'} = 0$. Hence Proposition 4.32 combined with Theorem D gives

$$T_{\mathbb{F}_3}(J_2, S) \cong \mathrm{Hom}(\pi_1(\mathcal{F}_3^*(J_2)), \mathbb{F}_3^\times) \cong \mathbb{Z}/2$$

in this case. See [LM15b, 6.4] for a very different derivation of this result.

4.4. **Higher homology groups.** We conclude the section by briefly considering the higher homology of $|\mathscr{T}_p^*(G)|$ and $|\mathscr{O}_p^*(G)|$. Higher homology groups occur naturally, as obstructions to extending compatible elements, as in Theorem D. We already made a forward reference to this subsection in the proof of Corollary 4.18, but otherwise the main results of the paper do not rely in them.

Proposition 4.34. *For $\mathcal{C}$ any collection of p-subgroups of G,*

$$H_*(\mathscr{T}_{\mathcal{C}}(G)) \otimes \mathbb{Z}[\tfrac{1}{p}] \xrightarrow{\cong} H_*(\mathscr{O}_{\mathcal{C}}(G)) \otimes \mathbb{Z}[\tfrac{1}{p}],$$
$$H_*(\mathcal{F}_{\mathcal{C}}(G)) \otimes \mathbb{Z}[\tfrac{1}{p}] \xrightarrow{\cong} H_*(\bar{\mathcal{F}}_{\mathcal{C}}(G)) \otimes \mathbb{Z}[\tfrac{1}{p}],$$
$$\text{and } H_*(|\mathcal{C}|/G) \otimes \mathbb{Q} \xleftarrow{\cong} H_*(\mathscr{T}_{\mathcal{C}}(G)) \otimes \mathbb{Q} \xrightarrow{\cong} H_*(\mathcal{F}_{\mathcal{C}}(G)) \otimes \mathbb{Q}.$$

Proof. The statements follow by a Grothendieck composite functor spectral sequence argument, since the morphisms differ by finite p-groups or finite groups. More precisely, [BLO03a, Lem. 1.3] implies that the two first maps are equivalences in homology with $\mathbb{F}_\ell$-coefficients for all primes $\ell \neq p$. Since the spaces are of finite type this implies equivalence in homology with $\mathbb{Z}_{(\ell)}$-coefficients, for all primes $\ell \neq p$, and hence an isomorphism in homology with $\mathbb{Z}[\frac{1}{p}]$-coefficients, and after tensoring with $\mathbb{Z}[\frac{1}{p}]$. The third isomorphism holds since the isotropy spectral sequence, Proposition 5.1, for the G-action on $|\mathcal{C}|$ converging to $|\mathcal{C}|_{hG} \simeq |\mathscr{T}_{\mathcal{C}}(G)|$, collapses, and the fourth isomorphism again follows from the proof of [BLO03a, Lem. 1.3], now using that $\tilde{H}_*(C_G(P);\mathbb{Q}) = 0$. $\square$

Theorem 4.35. *Let $\mathcal{C}$ be collection of p-subgroups, closed under passage to p-radical overgroups. For $i > 0$ the groups $H_i(\mathscr{O}_{\mathcal{C}}(G))$, $H_i(\mathscr{T}_{\mathcal{C}}(G))$, and $H_i(\mathcal{F}_{\mathcal{C}}(G))$ are finite, and $H_i(\mathscr{O}_{\mathcal{C}}(G))$ and $H_i(\mathcal{F}_p^*(G))$ are of order prime to p. When $\mathcal{C}$ is ample (i.e., $H_*(\mathscr{T}_{\mathcal{C}}(G))_{(p)} \xrightarrow{\cong} H_*(G)_{(p)}$; see Remark 4.8),*

$$H_i(\mathscr{T}_{\mathcal{C}}(G)) \xrightarrow{\cong} H_i(\mathscr{O}_{\mathcal{C}}(G)) \oplus H_i(G)_{(p)}, \quad i > 0.$$

Proof. First note that all the groups are finitely generated, since there are finitely many simplices in each dimension, the spaces being nerves of finite categories. Furthermore, by Proposition A.3, $H_i(|\mathcal{C}|/G) = 0$ for $i > 0$. Now the finiteness of all the groups follows from Proposition 4.34.

The statements about absence of p-torsion are well known consequences of the theory of mod p homology decompositions: To see that $H_i(\mathscr{O}_{\mathcal{C}}(G))$ is of order prime to p we just have to see that $H^i(\mathscr{O}_{\mathcal{C}}(G);\mathbb{Z}_{(p)}) = 0$ for $i > 0$. By definition $H^i(\mathscr{O}_{\mathcal{C}}(G);\mathbb{Z}_{(p)}) \cong \lim^i_{\mathscr{O}_{\mathcal{C}}(G)} \mathbb{Z}_{(p)}$ (see [Gro02, Prop. 2.6]). But [Gro02, Thm. 1.3] gives a spectral sequence for calculating $\lim^i$, whose E^1-term in this case, by [Gro02, Cor. 5.4], is zero except for one $\mathbb{Z}_{(p)}$, coming from the Sylow p-subgroup, responsible for $\lim^0_{\mathscr{O}_{\mathcal{C}}(G)} \mathbb{Z}_{(p)} \cong \mathbb{Z}_{(p)}$, showing the claim. Likewise $H_i(\mathcal{F}_p^*(G);\mathbb{Z}_{(p)}) = 0$, $i > 0$, by [Dwy98, 7.3] and [Gro02, Ex. 8.6] (variants on the classical [JM92, Prop. 2.1]).

Finally, if $H_*(\mathscr{T}_{\mathcal{C}}(G))_{(p)} \xrightarrow{\cong} H_*(G)_{(p)}$ then $H_i(\mathscr{T}_{\mathcal{C}}(G)) \xrightarrow{\cong} H_i(\mathscr{O}_{\mathcal{C}}(G)) \oplus H_i(G)_{(p)}$ for $i > 0$ by Proposition 4.34. $\square$

Remark 4.36. In the degenerate case when $\mathcal{C}$ is the collection of all p-subgroups, including the trivial one, Theorem 4.35 says that $H_i(\mathscr{O}_p(G)) \cong H_i(G)_{p'}$, $i > 0$, as $|\mathscr{T}_{\mathcal{C}}(G)| \simeq BG$.

Remark 4.37. The homology groups of $\mathscr{T}_{\mathcal{C}}(G)$, $\mathscr{O}_{\mathcal{C}}(G)$, or $\mathcal{F}_{\mathcal{C}}(G)$ will generally not be finite without assumptions on $\mathcal{C}$. For example if G is abelian then $\mathcal{F}_{\mathcal{C}}(G) = \mathcal{C}$ and $|\mathcal{C}|/G = |\mathcal{C}|$, so examples can be constructed using Proposition 4.34. Taking $G = (\mathbb{Z}/p)^r$ and $\mathcal{C}$ the collection of proper non-trivial subgroups of G, then $|\mathcal{C}|$ has homotopy type a wedge of spheres (e.g., $p+1$ points and a wedge of p^3 circles, for $r = 2, 3$ respectively).

Corollary 4.38. *For any field k of characteristic p,*

$$H^i(\mathscr{T}_p^*(G);k^\times) \xleftarrow{\cong} H^i(\mathscr{O}_p^*(G);k^\times) \oplus H^i(G;k^\times)_{(p)} \text{ for } i > 0.$$

If k is perfect, then $k^\times$ is uniquely p-divisible, and

$$H^*(\mathscr{T}_p^*(G);k^\times) \xleftarrow{\cong} H^*(\mathscr{O}_p^*(G);k^\times).$$

Proof. The first claim is a consequence of Theorem 4.35, the Universal Coefficient Theorem, and the five-lemma, using that $H_i(\mathscr{T}_p^*(G))_{(p)} \xrightarrow{\cong} H_i(G)_{(p)}$ by (4.2). That k is perfect means that the Frobenius map $(-)^p \colon k^\times \to k^\times$ is not only injective, but also surjective, i.e., $k^\times$ is uniquely p-divisible, and hence $H^i(G;k^\times)_{(p)} = 0$, for $i > 0$ by an application of the transfer. $\square$

Remark 4.39. Any finite field or any algebraically closed field is of course perfect. For any field of characteristic p we have $H^1(G;k^\times)_{(p)} = 0$, as $k^\times$ is p-torsion-free, but the higher groups are non-trivial in general for non-perfect fields. E.g., if $k = \mathbb{F}_p(x)$, rational functions in one variable over $\mathbb{F}_p$, the unit $k^\times$ is isomorphic to $\mathbb{F}_p^\times \times \mathbb{Z}^{(\mathbb{N})}$, as an abelian group, with a basis for the torsion-free part given by monic irreducible polynomials, so $H^i(G;k^\times)_{(p)} \cong (H^i(G;\mathbb{Z})_{(p)})^{(\mathbb{N})}$ in that case.

Remark 4.40 (Interpretation of $H^2(\mathscr{O}_p^*(G);k^\times)$). The group $H^2(\mathscr{O}_p^*(G);k^\times)$ may be thought of as a "p-local Schur multiplier", analogous to $H^2(G;k^\times)$. One may also ask if it also has a representation theoretic interpretation, as a suitable Brauer group. Note in this connection that $H^2(\mathcal{F}^c;k^\times)$ occurs in connection with the so-called gluing problem for blocks; see [Lin04, Lin05, Lin09].

The next two remarks, Remarks 4.41 and 4.42, explain the underlying picture on the level of homotopy.

Remark 4.41 (Higher homotopy groups). That $H_i(\mathscr{O}_\mathcal{C}(G))$, $i > 0$, is finite of order prime to p, for $\mathcal{C}$ a collection of p-subgroups closed under passage to p-radical overgroups, in fact has a strengthening, which also has the finiteness in Proposition 4.1 as a special case: For such $\mathcal{C}$, $\pi_i(\mathscr{O}_\mathcal{C}(G))$ is a finite p'-group for *all* i. This follows by a slight modification of the argument above: As $\pi_1(\mathscr{O}_\mathcal{C}(G))$ is a finite p'-group by Proposition 4.1, it is sufficient to show that $H^i(X;\mathbb{Z}_{(p)}) = 0$ for all $i > 0$, for X the universal cover of $|\mathscr{O}_\mathcal{C}(G)|$, by the Hurewicz theorem modulo Serre classes [tD08, Thm. 20.6.1]. But $H^*(X;\mathbb{Z}_{(p)}) \cong H^*(|\mathscr{O}_\mathcal{C}(G)|;\mathbb{Z}_{(p)}[\pi_1(\mathscr{O}_\mathcal{C}(G))])$, equipping $|\mathscr{O}_\mathcal{C}(G)|$ with the canonical twisted coefficient system (since on chains

$$\begin{aligned} C^*(|\mathscr{O}_\mathcal{C}(G)|;\mathbb{Z}_{(p)}[\pi_1(\mathscr{O}_\mathcal{C}(G))]) &\cong \mathrm{Hom}_{\pi_1(\mathscr{O}_\mathcal{C}(G))}(C_*(X),\mathbb{Z}_{(p)}[\pi_1(\mathscr{O}_\mathcal{C}(G))]) \\ &\cong \mathrm{Hom}(C_*(X),\mathbb{Z}_{(p)}), \end{aligned}$$

by definition [Hat02, Sec. 3.H] and the finiteness of $\pi_1(\mathscr{O}_\mathcal{C}(G))$). This again equals $\lim^*_{\mathscr{O}_\mathcal{C}(G)} \mathbb{Z}_{(p)}[\pi_1(\mathscr{O}_\mathcal{C}(G))]$, which vanishes in positive degree as in the proof of Theorem 4.35 (all elements of order p in $N_G(P)/P$ act trivially on $\mathbb{Z}_{(p)}[\pi_1(\mathscr{O}_\mathcal{C}(G))]$, as $\pi_1(\mathscr{O}_\mathcal{C}(G))$ is a finite p'-group). Note that this is in contrast to $\mathscr{T}_\mathcal{C}(G)$, where $\pi_i(\mathscr{T}_\mathcal{C}(G)) \xleftarrow{\cong} \pi_i(\mathcal{C})$, for $i \geq 2$, which is in general not finite; e.g., $|\mathcal{S}_p(G)|$ is homotopy equivalent to a wedge of spheres when G is a finite group of Lie type in characteristic p.

Remark 4.42 (Inverting p on $\mathscr{T}_\mathcal{C}(G)$). The relationship between the homotopy (or homology) groups of $\mathscr{T}_\mathcal{C}(G)$ and $\mathscr{O}_\mathcal{C}(G)$ from above can be stated on the level of spaces: For $\mathcal{C}$ as in Theorem 4.35,

$$|\mathscr{O}_\mathcal{C}(G)| \simeq L_{p'}|\mathscr{T}_\mathcal{C}(G)| \tag{4.9}$$

where $L_{p'}$ denotes localization with respect to the multiplication by p map $S^1 \xrightarrow{p} S^1$ [Far96]. Namely, recall that $|\mathscr{T}_\mathcal{C}(G)| \simeq \mathrm{hocolim}_{G/P\in\mathscr{O}_\mathcal{C}(G)} EG \times_G G/P$; see [Dwy97, §§1.7,3.2]. Hence

$$\begin{aligned} L_{p'}|\mathscr{T}_\mathcal{C}(G)| &\simeq L_{p'}(\mathrm{hocolim}_{G/P\in\mathscr{O}_\mathcal{C}(G)} L_{p'}(EG\times_G G/P)) \\ &\xrightarrow{\simeq} L_{p'}(\mathrm{hocolim}_{G/P\in\mathscr{O}_\mathcal{C}(G)} *) \simeq L_{p'}|\mathscr{O}_\mathcal{C}(G)|. \end{aligned} \tag{4.10}$$

Here we used that $L_{p'}$ is a left adjoint [Far96, Prop. 1.D.3] for the first homotopy equivalence and that $L_{p'}(BP) \simeq *$ for a finite p-group P (as is seen from the definition or [CP93, Thm. 3.2]) for the second. Now the claim follows by observing that $L_{p'}|\mathscr{O}_\mathcal{C}(G)| \simeq |\mathscr{O}_\mathcal{C}(G)|$, as $|\mathscr{O}_\mathcal{C}(G)|$ is a space whose homotopy groups are finite p'-groups by Remark 4.41, which implies that it is $L_{p'}$-local (see e.g., [CP93, Cor. 2.13]). A very special case is if $\mathcal{C} = \mathcal{S}_p^e(G)$ where $|\mathscr{T}_{\mathcal{C}}(G)| \simeq BG$, and thus

$$|\mathscr{O}_p(G)| \simeq L_{p'}BG \tag{4.11}$$

elaborating $H_i(\mathscr{O}_p(G)) \cong H_i(G)_{p'}$, $i > 0$, from Remark 4.36.

5. Homology decompositions and the Carlson–Thévenaz conjecture

In this section we establish the results about homology decompositions stated in §1, and show how they imply the Carlson–Thévenaz conjecture. The key tool is the isotropy spectral sequence, recalled below. Applied to the space $|\mathcal{C}|$ this gives us the normalizer decomposition (Theorem C). For the centralizer decomposition (Theorem D) we instead use the space $|E\mathbf{A}_{\mathcal{C}}|$, where $E\mathbf{A}_{\mathcal{C}}$ is the overcategory $\iota \downarrow G$ for $\iota\colon \mathcal{F}_{\mathcal{C}} \to \mathcal{F}_{\mathcal{C}\cup G}$ (see §A.1 for details). (There is also a third decomposition, the subgroup decomposition, based on a space $|E\mathscr{O}_{\mathcal{C}}|$, but since the isotropy subgroups are p-groups, it does not provide us with new information when taking coefficients prime to p.) We work in both homology and cohomology—these are essentially equivalent, but from a practical viewpoint it may feel more convenient to work in homology, only mapping into $k^\times$ at the end, so we give both versions.

5.1. Homology decompositions: Proof of Theorems C and D.

A Bredon G-*isotropy coefficient systems* is a functor $\mathfrak{F}\colon \mathscr{O}(G) \to R$-mod. It induces a G-coefficient system, as in §2.6, on X via the canonical functor $(\Delta X)_G \to \mathscr{O}(G)$ on objects sending $\sigma \mapsto G/G_\sigma$. Such coefficient systems are G-homotopy invariants, in the sense that a G-homotopy equivalence $Y \to X$ induces an RG-chain homotopy equivalence $C_*(Y;\mathfrak{F}) \to C_*(X;\mathfrak{F})$. Let $H_*^G(X;\mathfrak{F}) = H(C_*(X;\mathfrak{F})^G)$ denote Bredon homology equipped with an isotropy coefficient system $\mathfrak{F}$ (see e.g., [Bre67] for more details).

Proposition 5.1 (The isotropy spectral sequence)**.** *Let G be a finite group, X a G-space, and A an abelian group. We have a homological isotropy spectral sequence for the action of G on X*

$$E^2_{i,j} = H_i^G(X; H_j(-;A)) \Rightarrow H_{i+j}(X_{hG};A)$$

The bottom right-hand corner produces an exact sequence

$$\begin{aligned} H_2(X_{hG};A) \to H_2(X/G;A) \to H_0^G(X;H_1(-;A)) \\ \to H_1(X_{hG};A) \to H_1(X/G;A) \to 0 \end{aligned}$$

The dual spectral sequence in cohomology produces

$$\begin{aligned} 0 \to H^1(X/G;A) \to H^1(X_{hG};A) \to H^0_G(X;H^1(-;A)) \\ \to H^2(X/G;A) \to H^2(X_{hG};A). \end{aligned}$$

If $H_1(X/G;A) = H_2(X/G;A) = 0$ then this degenerates to

$$H_1(X_{hG};A) \cong H_0^G(X;H_1(-;A)).$$

Dually if $H^1(X/G;A) = H^2(X/G;A) = 0$ then $H^1(X_{hG};A) \cong H^0_G(X;H^1(-;A))$. By definition

$$H_0^G(X;H_1(-;A)) = \operatorname{coker}\left(\oplus_{\sigma\in X_1/G} H_1(G_\sigma;A) \xrightarrow{d_0-d_1} \oplus_{\sigma\in X_0/G} H_1(G_\sigma;A)\right)$$

for X_i the non-degenerate i-simplices, and dually for cohomology.

Proof. As explained in standard references such as [Dwy98, §2.3], [Bro94, VII(5.3)], the (homology) isotropy spectral sequence is constructed as the spectral sequence of the double complex $C_*(EG) \otimes_G C_*(X;A)$, filtered via the skeletal filtration of X. Hence $E^1_{*j} = H_j(G; C_*(X;A))$, and the E^2-term is obtained by taking homology induced by the differential on $C_*(X;A)$. The stated properties now follow from the definitions. □

We would like to alternatively view the H_0^G in Proposition 5.1 as a colimit, so we also recall the general principle behind this: Recall from §2.6 that a general (covariant) coefficient system on X is just a functor $\Delta X \to R$-mod, where ΔX is the category of simplices.

It is convenient to say that a space is *complex-like* if every non-degenerate simplex $\Delta[n] \to X$ is an injection on sets, i.e., if it "looks like" an ordered simplicial complex [Tho80, p. 311]. For a complex-like space, the subdivision category $\operatorname{sd} X$ is the full subcategory of ΔX on the non-degenerate simplices; it has a unique morphism $\sigma \to \tau$ if τ can be obtained from σ via face maps, and no other morphisms (see [DK83, §5]). The following classical proposition gives the relationship we need, stated also for higher homology for clarity:

Proposition 5.2. *Let* $\mathbf{D}$ *be a small category, and* R *a commutative ring.*

(1) *For any functor* $F\colon \mathbf{D} \to R$-mod, $\operatorname{colim}_*^{\mathbf{D}} F \cong H_*(|\mathbf{D}|; \mathscr{F})$*, where* $\mathscr{F}$ *is the coefficient system induced via* $\Delta|\mathbf{D}| \to \mathbf{D}$*,* $(d_0 \to \cdots \to d_n) \mapsto d_0$*.*
(2) *For any functor* $F\colon \mathbf{D}^{\mathrm{op}} \to R$-mod, $\operatorname{colim}_*^{\mathbf{D}^{\mathrm{op}}} F \cong H_*(|\mathbf{D}|; \mathscr{F})$*, where* $\mathscr{F}$ *is the coefficient system induced via* $\Delta|\mathbf{D}| \to \mathbf{D}^{\mathrm{op}}$*,* $(d_0 \to \cdots \to d_n) \mapsto d_n$*.*
(3) *Suppose* X *is a complex-like space. For any functor* $F\colon \operatorname{sd} X \to R$-mod, $\operatorname{colim}_*^{\operatorname{sd} X} F \cong H_*(X; \mathscr{F})$*, where* $\mathscr{F}$ *is induced from* F *via* $\Delta X \to \operatorname{sd} X$*, the functor sending all degeneracies to identities (see* [DK83, §5]*).*

Proof. We shall only need non-derived $* = 0$ part of these statements, which follows easily by writing down the definitions (for the last point also using cofinality), which we invite the reader to do. For (1) and (2), in the general case, see [GZ67, App. II.3.3], and also [Gro02, Prop. 2.6]. (The point is that both sides can be seen as homology of $C_*(|- \downarrow \mathbf{D}|) \otimes_{\mathbf{D}} F$ respectively $F \otimes_{\mathbf{D}} C_*(|\mathbf{D} \downarrow -|)$.) For (3), notice that both sides can be seen as the homology of $C_*(|\sigma|) \otimes_{\operatorname{sd} X} F$, where $|\sigma|$ is the n-simplex defined by the vertices of σ (by assumption distinct). It is a contravariant functor on $\operatorname{sd} X$ by $(\sigma \to \tau)$ assigning the map induced by the unique face inclusion of τ in σ (i.e., the extra structure on $\operatorname{sd} X$ allows us to 'avoid a subdivision'; see also [Gro02, Prop. 7.1]). □

Proposition 5.3. *Let* $\mathcal{C}$ *be a collection of* p*-subgroups such that* $H_1(|\mathcal{C}|/G)_{p'} = H_2(|\mathcal{C}|/G)_{p'} = 0$*; then*

$$\begin{aligned} H_1(\mathscr{O}_{\mathcal{C}}(G))_{p'} &\xleftarrow{\cong} \operatorname{coker}\left(d_0 - d_1\colon \oplus_{[P<Q]} H_1(N_G(P<Q))_{p'} \to \oplus_{[P]} H_1(N_G(P))_{p'}\right) \\ &\cong \operatorname{colim}_{[P_0 < \cdots < P_n]} H_1(N_G(P_0 < \cdots < P_n))_{p'} \end{aligned}$$

where the colim is over G*-conjugacy classes of strict chains in* $\mathcal{C}$*, ordered by reverse refinement.*

The conditions on $\mathcal{C}$ *are satisfied if it is closed under passage to* p*-radical overgroups, or just abstractly* G*-homotopy equivalent to such a collection (e.g.,* $\mathcal{C} = \mathcal{A}_p(G)$*).*

Proof. First note that the conditions are indeed satisfied if $\mathcal{C}$ is closed under passage to p-radical overgroups as $|\mathcal{C}|/G$ is then contractible by Symonds' theorem, Proposition A.3. As the condition only depends on the G-equivariant homotopy type of $|\mathcal{C}|$, it also holds if $|\mathcal{C}|$ is only abstractly G-homotopy equivalent to $|\mathcal{C}'|$ for another collection $\mathcal{C}'$ which is closed under passage to p-radical overgroups. This is the case for $\mathcal{A}_p(G)$, as $|\mathcal{A}_p(G)| \to |\mathcal{S}_p(G)|$ is a G-homotopy equivalence (see Theorem A.8(2)).

Now, if $H_1(|\mathcal{C}|/G)_{p'} = H_2(|\mathcal{C}|/G)_{p'} = 0$ then the isotropy spectral sequence, Proposition 5.1, applied to the G-space $|\mathcal{C}|$ with $A = \mathbb{Z}[\frac{1}{p}]$ gives $H_1(|\mathcal{C}|_{hG}; \mathbb{Z}[\frac{1}{p}]) \cong H_0^G(|\mathcal{C}|; H_1(-;\mathbb{Z}[\frac{1}{p}]))$. Since the right-hand side is obviously finite so is the left-hand side; hence taking coefficients in $\mathbb{Z}[\frac{1}{p}]$ is the same as applying $(-)_{p'}$. Furthermore $H_1(\mathscr{O}_\mathcal{C}(G))_{p'} \cong H_1(\mathscr{T}_\mathcal{C}(G))_{p'} \cong H_1(|\mathcal{C}|_{hG})_{p'}$ by Proposition 4.5 and Lemma 2.3. The first formula now follows by definition of H_0^G. The second rewriting as a colimit over conjugacy classes of strict chains follows from Proposition 5.2(3), noting that $|\mathcal{C}|/G$ is complex-like. □

Proof of Theorem C. We have that $T_k(G,S) \cong \mathrm{Hom}(H_1(\mathscr{O}_p^*(G)), k^\times)$, by Theorem A and (1.2). Combining this with Proposition 5.3 for $\mathcal{C} = \mathcal{S}_p(G)$ gives the wanted expressions, using that $\mathrm{Hom}(-,k^\times)$ sends colimits to limits. Again, it is a subset of $\mathrm{Hom}(N_G(S)/S, k^\times)$ by Proposition 4.1. □

We now prove the centralizer version.

Proposition 5.4. *For $\mathcal{C}$ a collection of p-subgroups and A an abelian group, we have exact sequences*

$$H_2(\mathcal{F}_\mathcal{C}(G);A) \to \operatorname*{colim}_{P\in\mathcal{F}_\mathcal{C}(G)^{\mathrm{op}}} H_1(C_G(P);A) \to H_1(\mathscr{T}_\mathcal{C}(G);A) \to H_1(\mathcal{F}_\mathcal{C}(G);A) \to 0$$

and

$$0 \to H^1(\mathcal{F}_\mathcal{C}(G);A) \to H^1(\mathscr{T}_\mathcal{C}(G);A) \to \lim_{P\in\mathcal{F}_\mathcal{C}(G)} H^1(C_G(P);A) \to H^2(\mathcal{F}_\mathcal{C}(G);A)$$

Here we may replace $\mathscr{T}_\mathcal{C}$ by $\mathscr{O}_\mathcal{C}$ if A satisfies the assumptions in Proposition 4.5.

Furthermore to calculate the limit (or colimit over the opposite category) we may replace $\mathcal{F}_\mathcal{C}(G)$ by a final subcategory, e.g., if $\mathcal{C}$ is a collection of non-trivial p-subgroups containing the elementary abelian p-subgroups of rank one or two $\mathcal{A}_p^2(G)$, then we may replace $\mathcal{C}$ by $\mathcal{A}_p^2(G)$.

Proof. Consider the isotropy exact sequence in homology from Proposition 5.1, with X the G-space $|E\mathbf{A}_\mathcal{C}|$ introduced in the beginning of this section (and in more detail in §A.1):

$$H_2(|E\mathbf{A}_\mathcal{C}|/G;A) \to H_0^G(|E\mathbf{A}_\mathcal{C}|; H_1(-;A)) \to H_1(|E\mathbf{A}_\mathcal{C}|_{hG};A) \\ \to H_1(|E\mathbf{A}_\mathcal{C}|/G;A) \to 0$$

We want to identify this sequence with the first sequence of the proposition. As remarked in §A.1, $|E\mathbf{A}_\mathcal{C}|/G = |\mathcal{F}_\mathcal{C}|$, which identifies the first and the fourth term. For the third term, remark that, again by §A.1, the G-map $|E\mathbf{A}_\mathcal{C}| \to |\mathcal{C}|$ is a homotopy equivalence, and hence induces $|E\mathbf{A}_\mathcal{C}|_{hG} \xrightarrow{\simeq} |\mathcal{C}|_{hG}$, and the space $|\mathcal{C}|_{hG}$ again identifies with $|\mathscr{T}_\mathcal{C}(G)|$ by Lemma 2.3. For the second term, notice that the stabilizer of an n-simplex $i\colon V_0 \to V_1 \to \cdots \to V_n \to G$ is $C_G(i(V_n))$. Hence Proposition 5.2(2) also identifies $H_0^G(|E\mathbf{A}_\mathcal{C}|; H_1(-;A))$ with the colimit as stated.

The sequence in cohomology follows dually using the isotropy exact sequence in cohomology from Proposition 5.1, and the dual version of Proposition 5.2(2) stated e.g., as [Gro02, Prop. 2.6].

The addendum about replacing $\mathscr{T}_{\mathcal{C}}$ by $\mathscr{O}_{\mathcal{C}}$ follows directly from Proposition 4.5, and the replacement of categories by a (co)final subcategory is a general fact about calculation of (co)limits, as explained e.g., in [Mac71, XI.3]. The stated example is easily seen to be final. □

Proof of Theorem D. This follows from the cohomological sequence in Proposition 5.4, taking $\mathcal{C}$ to be the collection of all non-trivial p-subgroups and $A = k^\times$, and utilizing the two additions at the end of Proposition 5.4: Using Proposition 4.5 and Theorem A we may replace $H^1(\mathscr{T}_p^*(G); k^\times) \xleftarrow{\cong} H^1(\mathscr{O}_p^*(G); k^\times) \xleftarrow{\cong} T_k(G, S)$, and also restrict to elementary abelian p-subgroups of rank one or two in the limit by finality.

For the final part, note that since the centralizers of elements x of order p are assumed to satisfy $H_1(C_G(x))_{p'} = 0$ (i.e., are "p'-perfect"), the inverse limit is obviously zero since the values are zero. The assumptions on the action of $N_G(S)$ imply that $H^1(\mathcal{F}_p^*(G); k^\times) = 0$ by Corollary 4.23. □

Remark 5.5 (Isotropy versus Bousfield–Kan spectral sequence). The above arguments in terms of the isotropy spectral sequence can also be recast in terms of the Bousfield–Kan spectral sequence of a homotopy colimit. As explained e.g., in [Dwy98, §3], [Dwy97, §3.3], the isotropy spectral sequence identifies with the Bousfield–Kan spectral sequence for the normalizer homology decomposition

$$|\mathcal{C}|_{hG} \simeq \operatorname{hocolim}_{\sigma \in |\mathcal{C}|/G} EG \times_G G/G_\sigma$$

It is also possible to work with $\mathscr{O}_{\mathcal{C}}(G)$ directly, instead of passing via $|\mathcal{C}|_{hG}$, since by [Sło91, Cor. 2.18] the orbit category admits a normalizer decomposition

$$|\mathscr{O}_{\mathcal{C}}(G)| \simeq \operatorname{hocolim}_{(P_0 < \cdots < P_n) \in |\mathcal{C}|/G} BN_G(P_0 < \cdots < P_n)/P_0 \tag{5.1}$$

(where $BN_G(P_0 < \cdots < P_n)/P_0$ has to be interpreted as $E(G/P_0) \times_G G/N_G(P_0 < \cdots < P_n)$, for $E(G/P_0)$ the translation groupoid of the G-set G/P_0, in order to get a strict functor to spaces). The associated spectral sequence for this homotopy colimit can also be obtained in a more low-tech way as the Leray spectral sequence of the projection map $|\mathscr{O}_{\mathcal{C}}(G)| = |E\mathscr{O}_{\mathcal{C}}|/G \to |\mathcal{C}|/G$.

5.2. **The Carlson–Thévenaz conjecture: Proof of Theorem E and Corollary 1.2.** We now prove the results in §1.3, and in particular deduce Theorem E from Theorem C. Via Theorem A, this will amount to describing how the colimit appearing in Proposition 5.3 can be calculated in certain cases, simplified using a Frattini argument. We define the *p-radical closure* $\overline{P}$ of a p-subgroup P in G as the subgroup obtained by successively applying $O_p(N_G(-))$, starting from P until the process stabilizes. Let $N_{G,p}(P)$ denote a Sylow p-subgroup of $N_G(P)$, well defined up to $N_G(P)$-conjugation, and write $\mathcal{B}_p^e(G)$ for the collection of all p-radical subgroups of G.

Theorem 5.6. *For $\mathcal{C}$ a collection of p-subgroups closed under passage to p-radical overgroups*

$$H_1(\mathscr{O}_{\mathcal{C}}(G)) \cong$$

$$\operatorname{coker}\Big(d_0 - d_1 \colon \oplus_{[P] \in \mathcal{C}'^{<S}/G} H_1(N_G(P < \overline{N_{G,p}(P)}))_{p'} \to \oplus_{[P] \in \mathcal{C}'/G} H_1(N_G(P))_{p'}\Big)$$

with $\mathcal{C}' = \mathcal{C} \cap \mathcal{B}_p^e(G)$ *and* $\mathcal{C}'^{<S}/G$ *denoting* G*-conjugacy classes of* p*-radical subgroups in* $\mathcal{C}$ *except* $[S]$.

Proof. First note that $H_1(\mathscr{O}_{\mathcal{C}}(G))$ is a finite p'-group by Proposition 4.1. We want to see that the cokernel formula in Proposition 5.3 can be reduced to the simpler expression above. By Theorem A.8(1) we can without restriction assume that $\mathcal{C}$ is closed under passage to all p-overgroups. Before we start, also note that the domain in Proposition 5.3 runs over conjugacy classes of pairs $P < Q$, whose elements we can view as conjugacy classes of subgroups $P \in \mathcal{C}$ together with, for each P, $N_G(P)$-conjugacy classes of subgroups $Q \in \mathcal{C}$, with $P < Q$. (Recall that the formula is well defined by the identification of the result as a zeroth homology group; more naïvely one may note that $N_G(P)$ acts trivially on $H_1(N_G(P))_{p'}$.) Set $H(-) = H_1(-)_{p'}$ for short.

Also observe that $N_G(P) = N_G(P \leq \overline{P}) \leq N_G(\overline{P})$ for $\overline{P}$ the p-radical closure of P, and we hence have an induced map $H(N_G(P)) \to H(N_G(\overline{P}))$, which identifies a summand corresponding to $[P]$ with its image in the summand corresponding to $[\overline{P}]$. This enables us to view the cokernel in Proposition 5.3 as a quotient of $\oplus_{[P] \in \mathcal{C}'/G} H(N_G(P))$ via these maps.

Our task is thus reduced to showing that the image of $\oplus_{[P<Q] \in \mathcal{C}/G} H(N_G(P < Q))$, via the map from Proposition 5.3, agrees with the image of the subgroup defined by letting the sum run over just $[P < \overline{P}]$ and $[\overline{P} < \overline{N_{G,p}(\overline{P})}]$ for $P \in \mathcal{C}$. We will do this by downward induction on the size of P. If P has index p in S there is nothing to show as Q will necessarily be Sylow, which is included in the above. So assume that the statement is true for larger subgroups. We divide the induction into steps.

We first reduce to the case where P is normal in Q. Namely, if not set $Q' = N_Q(P)$ and note that $P < Q'$, as Q is a p-group. We claim that the image under $d_0 - d_1$ of the summand corresponding to $P < Q$ lies in the image under $d_0 - d_1$ generated by the summands $P \lhd Q'$ and $Q' < Q$. Namely consider the diagram

$$\begin{tikzcd}
 & & H(N_G(P < Q' < Q)) \arrow[dll] \arrow[d, equal] \arrow[drr] & & \\
H(N_G(P < Q')) \arrow[d] \arrow[drr] & & H(N_G(P < Q)) \arrow[dll] \arrow[drr] & & H(N_G(Q' < Q)) \arrow[dll] \arrow[d] \\
H(N_G(P)) & & H(N_G(Q')) & & H(N_G(Q))
\end{tikzcd}$$

and note that the image of any element $x \in H(N_G(P < Q))$ equals the image of a sum $x_1 + x_2$ where $x_1 \in H(N_G(P < Q'))$ and $x_2 \in H(N_G(Q' < Q))$ are the images of $x \in H(N_G(P < Q')) = H(N_G(P < Q' < Q))$ under the maps induced by inclusion of normalizers. As Q' is strictly bigger than P we are reduced to the case where P is normal in Q by induction.

Next consider the case of $P \lhd Q$ with $P \neq \bar{P}$. We claim that the image of the summand $P < Q$ is generated by the image of the summands corresponding to $P < \overline{P}$, $\overline{P} \leq \overline{P}Q$ and $Q \leq \overline{P}Q$, noting that $\overline{P}Q$ is again a p-group as Q normalizes P and hence $\overline{P}$. This will prove the claim, using the induction hypothesis, as $\overline{P}$ and Q are strictly larger than P. (Note that if the weak inequalities above are equalities, the map is just zero, so we do not have to separate out this case; topologically this corresponds to degenerate simplices.) The generation follows from the following diagram:

$$\begin{tikzcd}
H(N_G(P<Q)) \arrow[r] \arrow[rr, bend right] \arrow[rrr, bend left] \arrow[d] \arrow[dr] \arrow[drr] & H(N_G(P<\overline{P})) \arrow[dl] \arrow[d] & H(N_G(\overline{P}\leq \overline{P}Q)) \arrow[dl] \arrow[dr] & H(N_G(Q\leq \overline{P}Q)) \arrow[dl] \arrow[d] \\
H(N_G(P)) \arrow[r] & H(N_G(\overline{P})) & H(N_G(Q)) & H(N_G(\overline{P}Q))
\end{tikzcd}$$

Namely, the image under $d_0 - d_1$ of any element $x \in H(N_G(P < Q))$ equals the image of $x_1 + x_2 - x_3$ where $x_1 \in H(N_G(P < \overline{P}))$, $x_2 \in H(N_G(\overline{P} \leq \overline{P}Q))$, and $x_3 \in H(N_G(Q \leq \overline{P}Q))$ are the images of x under the maps induced by the inclusion of normalizers (the top horizontal arrows in the diagram).

Finally consider the case $P \lhd Q$ with $P = \overline{P}$, which we want to replace with $P < \overline{N_{G,p}(P)}$. Let R be a Sylow p-subgroup of $N_G(P < Q)$ (thus containing Q). We first claim that the image of the summand corresponding to $P < Q$ is generated by the summands for $P < R$ and $Q \leq R$. For this consider the diagram

$$\begin{tikzcd}
H(N_G(P<Q\leq R)) \arrow[d, two heads] \arrow[dr] \arrow[drr] & & \\
H(N_G(P<Q)) \arrow[d] \arrow[dr] & H(N_G(P<R)) \arrow[dl] \arrow[dr] & H(N_G(Q\leq R)) \arrow[dl] \arrow[d] \\
H(N_G(P)) & H(N_G(Q)) & H(N_G(R))
\end{tikzcd} \tag{5.2}$$

where $H(N_G(P < Q \leq R)) \twoheadrightarrow H(N_G(P < Q))$ is surjective by the Frattini argument, as $A^{p'}(N_G(P < Q))$ is normal in $N_G(P < Q)$ of p' index. Given any $x \in H(N_G(P < Q))$, we can lift it to $x' \in H(N_G(P < Q \leq R))$ and let x_1 and x_2 be the image of x' under the inclusion of normalizers in $H(N_G(P < R))$ and $H(N_G(Q < R))$. Then the image of x under $d_0 - d_1$ agrees with the image of $x_1 - x_2$ by the above diagram. If $Q = R$ then R is a Sylow p-subgroup of $N_G(P)$ and hence we may replace $P \lhd Q$ with $P < R = N_{G,p}(P)$ as the summand corresponding to $Q \leq R$ is taken care of by the induction hypothesis. If $Q < R$ then the argument shows that we may replace $P < Q$ with $P < R$, and we can hence repeat until Q is indeed Sylow in $N_G(P)$. Finally as $N_G(P < N_{G,p}(P)) \leq N_G(P < \overline{N_{G,p}(P)})$ we may replace $P < N_{G,p}(P)$ by $P < \overline{N_{G,p}(P)}$ as wanted, finishing the proof. □

Next we show how a cokernel such as in Theorem 5.6 can be described iteratively.

Lemma 5.7. *Suppose we have an increasing filtration $\{s\} = X_0 \subset X_1 \subset \cdots \subset X_n = X$, $n \geq 1$, of a finite set X, and a function $\varphi\colon X \setminus \{s\} \to X$, which strictly decreases filtration. Let $\oplus_{x\in X} A(x)$ and $\oplus_{x\in X} B(x)$ be abelian groups, and suppose we, for each $x \in X \setminus \{s\}$, are given two homomorphisms $f_x : A(x) \twoheadrightarrow B(x)$ and $g_x : A(x) \to B(\varphi(x))$, with f_x surjective. Set $B_0(x) = 0$ and $B_i(y) = \sum_{x\in\varphi^{-1}(y)} g_x(f_x^{-1}(B_{i-1}(x)))$. Then*

$$\operatorname{coker}\left(\oplus_{x\in X\setminus\{s\}} A(x) \xrightarrow{g_x - f_x} \oplus_{x\in X} B(x)\right) \cong B(s)/B_n(s)$$

Proof. We prove this by induction on n. The statement is true for $n = 1$ since any element of $B(x)$, for $x \neq s$, is identified with a unique element of $B(s)/B_1(s)$, by the surjectivity of f_x and the definition of $B_1(s)$. Suppose that it is true for $i < n$.

Notice that

$$\begin{aligned}\operatorname{coker}&\left(\oplus_{x\in X\setminus\{s\}}A(x)\xrightarrow{g_x-f_x}\oplus_{x\in X}B(x)\right)\\ &\cong\operatorname{coker}\left(\oplus_{x\in X\setminus\{s\}}A(x)\xrightarrow{g_x-f_x}\oplus_{x\in X}B(x)/B_1(x)\right)\\ &\cong\operatorname{coker}\left(\oplus_{x\in X_{n-1}\setminus\{s\}}A(x)\xrightarrow{g_x-f_x}\oplus_{x\in X_{n-1}}B(x)/B_1(x)\right)\end{aligned}$$

Here the first isomorphism is because elements of $B_1(x)$ are obviously zero in the cokernel and the second isomorphism follows as elements of $\oplus_{x\in X_n\setminus X_{n-1}}B(x)$ each get identified with unique elements of $\oplus_{x\in X_{n-1}}B(x)/B_1(x)$, by surjectivity of f_x and the definition of B_1. As $\oplus_{x\in X_{n-1}}A(x)$ and $\oplus_{x\in X_{n-1}}B(x)/B_1(x)$ satisfy the assumptions of the original setup, we are done by induction. □

For any conjugacy class of p-radical subgroups $[P]$ consider the chain constructed by taking $[P_0]=[P]$ and $[P_i]=[\overline{N_{G,p}(P_{i-1})}]$, well defined on conjugacy classes. Define the *normal-radical height* of $[P]$ in G as the smallest i such that $[P_i]=[S]$, with S a Sylow p-subgroup. Define the *normal-radical class* of a collection of p-subgroups $\mathcal{C}$ as the maximal normal-radical height of a p-radical $[P]\in\mathcal{C}/G$. (E.g., normal-radical class 0 means that only $[S]$ is p-radical.)

Theorem 5.8. *For a finite group G with Sylow p-subgroup S, let $\mathcal{C}$ be a collection of p-subgroups, closed under passage to p-radical overgroups. Set $\mathcal{C}'=\mathcal{C}\cap\mathcal{B}_p^e(G)$. For $Q\in\mathcal{C}$, let $\nu^1_{\mathcal{C}}(Q)=A^{p'}(N_G(Q))$ and define by induction*

$$\nu^i_{\mathcal{C}}(Q)=\langle(N_G(Q)\cap\nu^{i-1}_{\mathcal{C}}(P))A^{p'}(N_G(Q))\,|\,[P]\in\mathcal{C}'/G \text{ with } [Q]=[\overline{N_{G,p}(P)}]\rangle,$$

picking for each $[P]$ a representative P such that $N_Q(P)$ is Sylow in $N_G(P)$ and $\overline{N_Q(P)}=Q$. Then

$$H_1(\mathscr{O}_{\mathcal{C}}(G))\cong N_G(S)/\nu^r_{\mathcal{C}}(S)$$

for r at least the normal-radical class of $\mathcal{C}$ plus 1.

Proof. Theorem 5.6 provides a formula for $H_1(\mathscr{O}_{\mathcal{C}}(G))$. We claim that Lemma 5.7 allows us to reformulate that expression to the one given in the theorem. Namely take $X=\mathcal{C}'/G$, $A([P])=H_1(N_G(P<\overline{N_{G,p}(P)}))_{p'}$ and $B([P])=H_1(N_G(P))_{p'}$, and notice that the induced map $f_{[P]}\colon A([P])\to B([P])$ is surjective by the diagram

$$\begin{tikzcd}
H_1(N_G(P<N_{G,p}(P)\le\overline{N_{G,p}(P)}))_{p'} \arrow[r] \arrow[d, "="] & H_1(N_G(P<\overline{N_{G,p}(P)}))_{p'} \arrow[d, "f_{[P]}"] \\
H_1(N_G(P<N_{G,p}(P)))_{p'} \arrow[r, "d_0"] & H_1(N_G(P))_{p'}
\end{tikzcd}$$

together with the surjectivity of d_0, which follows from the Frattini argument with respect to the normal subgroup $A^{p'}(N_G(P))$. Let $X_i\subset X$ consist of those conjugacy classes of subgroups of normal-radical height at most i, as defined just before the theorem. Then by definition $B_i([P])=\nu^{i+1}_{\mathcal{C}}(P)/A^{p'}(N_G(P))$, and it now follows from Theorem 5.6 and Lemma 5.7 that $H_1(\mathscr{O}_{\mathcal{C}}(G))_{p'}\cong N_G(S)/\nu^r_{\mathcal{C}}(S)$, for r at least the normal-radical class of $\mathcal{C}$ plus 1. □

Lemma 5.9. *Let $\mathcal{C}\subseteq\mathcal{S}_p(G)$ be a collection closed under passage to p-radical overgroups, and let $\mathcal{C}'=\mathcal{C}\cap\mathcal{B}_p(G)$. The normal-radical class of $\mathcal{C}$ is bounded by* *the nilpotency class of S, as well as by* $\min\{\dim|\mathcal{C}'|, \dim|\mathcal{C}'^c|+1\}$*, where* $\mathcal{C}'^c$ *means the p-centric subgroups in* $\mathcal{C}'$ *(cf. §1.4).*

Proof. Let $P = P_0$ be an arbitrary p-radical subgroup, and let $[P_0], \ldots, [P_n] = [S]$ be the normal-radical series as above. For the nilpotency bound, let Z_i be the ith group in the lower central series of S, i.e., $Z_1 = Z(S)$ etc. We can assume $P_0 \leq S$. As Z_1 normalizes P_0, we can choose representative P_1 with $[P_1] = [\overline{N_{G,p}(P_0)}]$ and $Z_1 \leq P_1$. Assume by induction that we have chosen P_{i-1} with $Z_{i-1} \leq P_{i-1}$. Then Z_i/Z_{i-1} centralizes P_{i-1}/Z_{i-1}, and in particular Z_i normalizes P_{i-1}; we can choose P_i with $Z_i \leq P_i$, showing the claim. The $\dim|\mathcal{C}'|$ bound is obvious, and the $\dim|\mathcal{C}'^c| + 1$ bound holds as P_i will be p-centric for $i \geq 1$, as they contain a Sylow p-subgroup of $C_G(P_0)$. □

Proof of Theorem E. It follows from Theorem 5.8 that $\nu^r_{\mathcal{C}}(S) = \ker(N_G(S) \to H_1(\mathscr{O}^*_p(G)))$, for $\mathcal{C} = \mathcal{S}_p(G)$ and r at least the normal radical class of $\mathcal{C}$ plus 1. This shows a version of Theorem E, with ν^r instead of ρ^r, noting that the stated bounds are implied by Lemma 5.9. To be able to replace ν by ρ, notice first that $\nu^i_{\mathcal{C}}(Q) \leq \rho^i(Q)$ for any i and $Q \in \mathcal{C} \subseteq \mathcal{S}_p(G)$, by definition (for ρ, unlike ν, we do not assume that the subgroups are related by inclusion). Hence, to finish the proof, we just need to verify that also $\rho^i(S)$ lies in the kernel of $N_G(S) \to H_1(\mathscr{O}^*_p(G))$, since then $\nu^r_{\mathcal{C}}(S) \cong \rho^r(S)$. However $\rho^i(Q)$ lies in the kernel of $N_G(Q) \to H_1(\mathscr{O}^*_p(G))$ for any $1 < Q \leq S$ and any i follows essentially by definition (like for ν), as we now verify by induction on i: For $\rho^1(Q)$ it follows by the factorization $N_G(Q) \to H_1(N_G(Q))_{p'} \to H_1(\mathscr{O}^*_p(G))$, as $H_1(\mathscr{O}^*_p(G))$ is a p'-group by Proposition 4.1. And, if $g \in N_G(Q) \cap \rho^{i-1}(R) \subseteq \rho^i(Q)$, for $1 < R \leq S$, then we have a diagram

$$\begin{tikzcd} G/Q \arrow[r] \arrow[d, "{[g]}"] & G/QR \arrow[d, "{[g]}"] & G/R \arrow[l] \arrow[d, "{[g]}"] \\ G/Q \arrow[r] & G/QR & G/R \arrow[l] \end{tikzcd} \tag{5.3}$$

where QR denotes the subgroup generated by Q and R inside S. This shows that the image of $g \in N_G(Q) \cap \rho^{i-1}(R)$ in $H_1(\mathscr{O}^*_p(G))$ via

$$\rho^{i-1}(R) \to N_G(R) \to H_1(N_G(R)) \to H_1(\mathscr{O}^*_p(G))$$

equals the image of g via $\rho^i(Q) \to N_G(Q) \to H_1(N_G(Q)) \to H_1(\mathscr{O}^*_p(G))$, which is hence also zero by induction. As $\rho^i(Q)$ is generated by such g we conclude that $\rho^i(Q)$ maps to zero in $H_1(\mathscr{O}^*_p(G))$ as wanted. □

Remark 5.10. Note that the statement in the last proof that $\rho^i(S)$ lies in the kernel of $N_G(S) \to H_1(\mathscr{O}^*_p(G))$, via the dictionary of Theorem A, amounts to the statement that Sylow-trivial modules split as a trivial module k direct sum a projective module upon restriction to $\rho^i(S)$ which was already shown by Carlson–Thevenaz (see [CT15, Thm. 4.3]).

Remark 5.11 (The bound r in Theorem E and sparsity of Sylow-trivial modules). In Appendix A we provide a detailed analysis of how to find small collections $\mathcal{C} \subseteq \mathcal{S}_p(G)$ such that $H_1(\mathscr{O}^*_p(G))_{p'} \cong H_1(\mathscr{O}_{\mathcal{C}}(G))_{p'}$, and hence get other bounds on r in the Carlson–Thévenaz conjecture, Theorem E: By Theorem A.10 and Proposition A.3 we can take $\mathcal{C}$ to be the smallest collection closed under passage to p-radical overgroups and containing all the p-subgroups P where $N_G(P)/P$ admits an exotic Sylow-trivial module (a closure of the collection $\mathcal{E}_p(G)$ of §A.4). For a finite group of Lie type of characteristic p this subcollection of $\mathcal{B}_p(G)$ identifies with unipotent radicals of parabolic subgroups of rank at most one. As far as we know, this poset could have a uniform dimension bound in general, independent of the finite group G. The group $G_2(5)$ at $p = 3$ is an example where both bounds in Theorem E give $r = 3$ and $r = 2$ does not work (see the discussion before Proposition 6.3). We do not know of an example where one cannot take $r = 3$. In fact, to the best of our knowledge, in all finite groups where $T_k(G, S)$ has been calculated either $\rho^3(S) = N_{A^{p'}(G)}(S)$ (and hence $T_k(G, S) = \mathrm{Hom}(G, k^\times)$) or $\mathcal{S}_p(G)$ is G-homotopy equivalent to a 1-dimensional complex. The results of this paper indicate that finding bounds on r in general has links to many facets of p-local finite group theory (see also §§A.4, A.5).

Finally, let us finally address in more detail when one can take $r = 2$ as bound on the filtration.

Corollary 5.12. *Let $\mathcal{C}$ be a collection of p-subgroups, closed under passage to p-radical overgroups, and set $\mathcal{C}' = \mathcal{C} \cap \mathcal{B}_p(G)$. If each $[P] \in \mathcal{C}'/G$ satisfies that $[\overline{N_{G,p}(P)}] = [S]$ then*

$$H_1(\mathscr{O}_{\mathcal{C}}(G)) \cong N_G(S)/\langle N_G(P \leq S) \cap A^{p'}(N_G(P)) | [P] \in \mathcal{C}'/G \rangle$$

where each P is chosen in $[P]$ such that $N_S(P)$ is a Sylow p-subgroup of $N_G(P)$.

More generally, if one just, for each $[P] \in \mathcal{C}'/G$, can pick $P \leq S$ with $N_S(P)$ Sylow in $N_G(P)$ and $N_G(P \leq \overline{N_S(P)} \leq S)A^{p'}(N_G(P \leq \overline{N_S(P)})) = N_G(P \leq \overline{N_S(P)})$ then the same conclusion holds.

Proof. The first part follows directly from Theorem 5.8, as normal-radical class of $\mathcal{C}$ is at most 1. The 'more generally' part follows from its proof: Under the stated assumption, diagram (5.2) in the proof of Theorem 5.6 shows that we can replace $P \leq \overline{N_S(P)}$ with $P \leq S$ (taking $(Q = \overline{N_S(P)}$ and $R = S)$, so the cokernel in Theorem 5.6 can be calculated as claimed in this corollary. □

Proof of Corollary 1.2. If all p-radical subgroups $P \leq S$ are normal, then in particular $[\overline{N_{G,p}(P)}] = [S]$ and $S = N_S(P)$ is a Sylow p-subgroup of $N_G(P)$, so the first part follows from Corollary 5.12, together with Theorem A. For the 'more generally' part, we note that the assumption implies that of Corollary 5.12: By a Frattini argument $N_G(P \leq Q \leq S)A^{p'}(N_G(P \leq \overline{Q} \leq S)) = N_G(P \leq \overline{Q} \leq S)$ and $N_G(P \leq Q)A^{p'}(N_G(P \leq \overline{Q})) = N_G(P \leq \overline{Q})$, as Q is a Sylow p-subgroup and $N_G(Q) = N_G(Q \leq \overline{Q})$. Hence $N_G(P \leq \overline{Q} \leq S)A^{p'}(N_G(P \leq \overline{Q})) = N_G(P \leq Q \leq S)A^{p'}(N_G(P \leq \overline{Q})) = N_G(P \leq Q)A^{p'}(N_G(P \leq \overline{Q})) = N_G(P \leq \overline{Q})$ as wanted. □

Let us for completeness also state a variant of Corollary 5.12 valid for general collections.

Corollary 5.13. *Let $\mathcal{C}$ be a collection with $S \in \mathcal{C}$ and $H_1(|\mathcal{C}|/G)_{p'} = H_2(|\mathcal{C}|/G)_{p'} = 0$. Assume that for each G-conjugacy class of pairs $[P \leq Q]$ with $P, Q \in \mathcal{C}$, we can pick $P \leq Q \leq S$ such that $N_G(P \leq Q \leq S)A^{p'}(N_G(P \leq Q)) = N_G(P \leq Q)$ (e.g., if all subgroups in $\mathcal{C}_{\leq S}$ are normal in S); then $H_1(\mathscr{O}_{\mathcal{C}}(G))_{p'} \cong N_G(S)/\langle N_G(P \leq S) \cap A^{p'}(N_G(P)) | [P] \in \mathcal{C}/G \rangle$, with representative P picked as above.*

Proof. The formula for the cokernel in Proposition 5.3 reduces to the formula above via the diagram (5.2) (with $R = S$) as before, where the surjective map now is by assumption. $\square$

Remark 5.14 (A strong version of [CT15, Thm. 7.1]). Suppose that $N_G(S)$ controls p-fusion in G, and that for each nontrivial p-radical subgroup $Q \leq S$

$$(N_G(S) \cap C_G(Q))A^{p'}(C_G(Q)) = C_G(Q)$$

Then the general assumption of Corollary 5.12 is satisfied. Namely

$$\begin{aligned} N_G(P < Q) = N_G(P < Q < S)C_G(Q) &= N_G(P < Q < S)A^{p'}(C_G(Q)) \\ &\leq N_G(P < Q < S)A^{p'}(N_G(P < Q)) \end{aligned}$$

and the other inclusion is clear. Here the first equality is by control of fusion and the second by assumption. This provides a slightly stronger version of [CT15, Thm. 7.1], where the condition is only checked on p-radical subgroups.

6. Computations

By Theorem A calculating $T_k(G,S)$ amounts to calculating $H_1(\mathscr{O}_p^*(G))$, and we have developed a number of theorems and tools for this in the preceding sections. Formulas such as Theorems C and D make it computable for individual groups, since the input data has often already been tabulated, e.g., in connection with inductive approaches to the Alperin and McKay conjectures. Similarly Theorem B allows us to tap into the large preexisting literature on the fundamental group of subgroup complexes, which has been studied in topological combinatorics, due to its relationship to other combinatorial problems, as well as in finite group theory, where it is related to uniqueness question of a group given its p-local structure, and the classification of finite simple groups. Expanding on the summary in §1.5 we will in this section go through different classes of groups, and show how the strategy translates into explicit computations. We only pick some low-hanging fruit, but with a recipe for how to continue.

6.1. **Sporadic groups.** We complete the general discussion from §1.5 by using Theorem D to determine Sylow-trivial modules for the Monster finite simple group, as a computational example:

Theorem 6.1. *Let $G = \mathbb{M}$ be the Monster sporadic group, and k a field of characteristic p. Then*

$$T_k(G,S) \cong \begin{cases} 0 & \text{for } p \leq 13 \\ \operatorname{Hom}(N_G(S)/S, k^\times) & \text{for } p > 13 \end{cases}$$

The case $p = 2$ is clear since $N_G(S) = S$, and if $p > 13$, S is cyclic so the formula is standard, Corollary 4.14(2) (with values tabulated in [LM15b, Table 5]). We prove the remaining cases below, which were left open in the recent paper [LM15b, Table 3], using our formulas:

Proof of Theorem 6.1 *for $p = 3, 5, 7, 11, 13$.* There is some choice in methods, since several of our theorems can be used. For primes $p = 3, 5, 7, 11$ the easiest is probably to observe that in all cases $\operatorname{colim}^0_{V \in \mathcal{F}_{\mathcal{A}_p^2}(G)^{\mathrm{op}}} H_1(C_G(V))_{p'} = 0$ and $H_1(\mathcal{F}_p^*(G)) = 0$, and then appeal to the centralizer decomposition Theorem D, or more precisely the homological Proposition 5.4 and Theorem A. The vanishing statements will follow by a coup d'œil at the standard data about the Monster from [Wil88] and [AW10] (correcting [Yos05]), and [CCN+85]. In fact even the H_1's that appear in centralizer colimit vanish, and $H_1(\mathcal{F}_p^*(G)) = 0$ by the vanishing criterion of Corollary 4.23. In detail:

For $p = 3$: According to [CCN+85] there are 3 conjugacy classes of subgroups of order 3, with the following centralizers: $C_G(3A) = 3 \cdot Fi_{24}'$, $C_G(3B) = 3_+^{1+12} \cdot 2Suz$, $C_G(3C) = 3 \times Th$. All of these have zero $H_1(-)_{3'}$, since Fi_{24}', $2Suz$ and Th are perfect. Hence trivially colim $= 0$. We want to use Corollary 4.23 to see that also $\pi_1(\mathcal{F}_p^*(G)) = 1$. By [AW10, Table 2] $N_G(S) = S : (2^2 \times SD_{16})$ a subgroup of $N_G(3A^3) = 3^{3+2+6+6}.(L_3(3) \times SD_{16})$. Hence SD_{16} acts trivially on $3A^3$ and 2^2 acts as the diagonal matrices in $\mathrm{SL}_3(\mathbb{F}_3)$ and is generated by elements that fix a non-trivial element in $3A^3$. We conclude by Corollary 4.23 that $\pi_1(\mathcal{F}_p^*(G)) = 1$.

For $p = 5$: There are two conjugacy classes of subgroups of order 5 with centralizers $C_G(5A) = 5 \times HN$ and $C_G(5B) = 5_+^{1+6} : 4J_2$, which have zero $H_1(-)_{5'}$ since HN and $4J_2$ are perfect. Hence colim $= 0$. For $\pi_1(\mathcal{F}_5^*(G)) = 1$, note that $N_G(S) = S : (\mathfrak{S}_3 \times 4^2)$ inside $N_G(5B^2) = 5^{2+2+4}.(\mathfrak{S}_3 \times \mathrm{GL}_2(5))$. Hence $\mathfrak{S}_3$ acts trivially on $5B^2$, and 4^2 is generated by elements which act with a non-trivial fixed-point on $5B^2$, so the conclusion again follows by Corollary 4.23.

For $p = 7$: There are 2 conjugacy classes of subgroups of order 7 with centralizers $C_G(7A) = 7 \times He$ and $C_G(7B) = 7^{1+4} : 2\mathfrak{A}_7$, both with vanishing $H_1(-)_{7'}$, so colim $= 0$. By [Wil88, Thm. 7] and [AW10, Table 1], $N_G(S) = S : 6^2$ inside $N_G(7B^2) = 7^{2+1+2} : \mathrm{GL}_2(7)$, so again we can use Corollary 4.23.

For $p = 11$: We have just one conjugacy class of subgroups of order 11 with $C_G(11A) = 11 \times M_{12}$, which satisfy $H_1(C_G(11A))_{11'} = 0$. By [AW10, Table 1], $N_G(S) = N_G(11A^2) = 11^2 : (5 \times 2\mathfrak{A}_5)$. We want to see that $H_1(N_G(S)/S)$ is generated by elements which commute with a non-trivial element in S, so that we can apply Corollary 4.23. For this we describe the action more explicitly: Note that $(5 \times 2\mathfrak{A}_5)$ is not a subgroup of $\mathrm{SL}_2(11)$ (by the classification of maximal subgroups of $\mathrm{PSL}_2(11)$, say), so $(5 \times 2\mathfrak{A}_5) \cap \mathrm{SL}_2(11) = 2\mathfrak{A}_5$. Furthermore the 5-factor has to lie in the center of $\mathrm{GL}_2(11)$, since it commutes with $2\mathfrak{A}_5$, and otherwise the action of $2\mathfrak{A}_5$ on 11^2 would be reducible. In matrices we can hence write a generator of the 5-factor as $\mathrm{diag}(\alpha, \alpha)$, where α is a primitive 5th root of unity in $\mathbb{F}_{11}^\times$. However since $2\mathfrak{A}_5$ is a subgroup of $\mathrm{SL}_2(11)$ and has order divisible by 5, it contains up to conjugacy in $\mathrm{GL}_2(11)$ the element $\mathrm{diag}(\alpha, \alpha^{-1})$. Hence $\mathrm{diag}(\alpha^2, 1) \in N_G(S)/S \le \mathrm{GL}_2(11)$ generates $H_1(N_G(S)/S)$ and centralizes a non-trivial element is S. We conclude that $H_1(\mathcal{F}_{11}^*(G)) = 0$ by Corollary 4.23.

For $p = 13$: We use Theorem F. By [AW10, Table 1] all p-centric p-radicals are centric, so the assumptions of the last part of that theorem are satisfied, and by the same reference $N_G(S) = N_G(13B) = 13^{1+2} : (3 \times 4\mathfrak{S}_4)$. But $\pi_1(\mathcal{F}_{13}^c(\mathbb{M})) = 1$, since otherwise there would by [BCG+07, Thm. 5.4] need to exist a subsystem of index 3 or 2, but by [RV04, Thm. 1.1] no such subsystems exist. Hence Theorem F implies that $T_k(G, S) = 0$. (Alternatively apply the last part of Theorem D again; the centralizer condition holds since $C_G(13A) = 13 \times \mathrm{SL}_3(3)$ and $C_G(13B) = 13^{1+2} : 2\mathfrak{A}_4$ using [AW10, Table 1], and the $H_1(N_G(S)/S)$ condition is satisfied since $3 \times 4\mathfrak{S}_4 \cong N_G(S)/S \le \mathrm{Out}(S) \cong \mathrm{GL}_2(13)$ contains all diagonal elements in some basis, being the unique subgroup of order $2^5 3^2$.) □

(The remaining sporadic groups left open in [LM15b, Table 5] have subsequently also been dealt with by Craven [Cra21] using our methods, and in fact he redoes all sporadic groups this way.)

6.2. **Finite groups of Lie type.** The p-subgroup complex of a finite group of Lie type G at the characteristic is simply connected if the Lie rank is at least 3, since it is homotopy equivalent to the Tits building [Qui78, §3]. Theorem B hence implies that there are no exotic Sylow-trivial modules in that case, a result originally found in [CMN06]. (This is also true in rank two, by a small direct computation, using Theorem A and [Ste68, Thm. 12.6(b)].)

Away from the characteristic the p-subgroup complex is also expected to be simply connected if G is "large enough", but this is not known in general. Stronger yet, the p-subgroup complex appears often to be Cohen–Macaulay [Qui78, Thm. 12.4], [Das98], [Das00], [Smi11, §9.4]. Partial results in this direction imply by Theorem B that there are no exotic Sylow-trivial modules in those cases. We give two examples of this, for $\mathrm{GL}_n(q)$ and $\mathrm{Sp}_{2n}(q)$. The $\mathrm{GL}_n(q)$ case also follows from very recent work of Carlson–Mazza–Nakano, while the $\mathrm{Sp}_n(q)$ case is new.

Theorem 6.2 ([CMN14, CMN16])**.** *Let $G = \mathrm{GL}_n(q)$ with q prime to the characteristic p of k. If G has an elementary abelian p-subgroup of rank 3, then $T_k(G,S) \cong \mathrm{Hom}(G,k^\times)$.*

Proof. It was proved by Quillen [Qui78, Thm. 12.4] that $|\mathcal{S}_p(G)|$ is simply connected (and in fact Cohen-Macaulay) under the stated assumptions if $p|q-1$, and when $p\nmid q-1$ it is shown in [Das95, Thm. A]. Hence the result follows from Theorem B. ☐

Theorem G. *Let $G = \mathrm{Sp}_{2n}(q)$, and k a field of characteristic p. If the multiplicative order of q mod p is odd, and G has an elementary abelian p-subgroup of rank 3, then $T_k(G,S) = 0$.*

Proof of Theorem G. The main theorem in [Das98] states that $|\mathcal{S}_p(G)|$ is simply connected under these assumptions, so the result follows from Theorem B. ☐

That the p-rank of G is at least 3 is not enough to ensure simple connectivity in general. At the characteristic $\mathrm{SL}_2(\mathbb{F}_{p^r})$ or $\mathrm{SL}_3(\mathbb{F}_{p^r})$ shows this, and even away from the characteristic it is not true as $U_4(3) = O_6^-(3)$ has 2-rank 4, but the 2-subgroup complex is not simply connected (see [Smi11, Ex. 9.3.11]). In this case $H_1(\mathscr{O}_2^*(G)) \cong H_1(G)_{2'} = 0$ though, just by virtue of the Sylow 2-subgroup being its own normalizer. See e.g., [Asc93] and [Smi11, 9.3] for more on simple connectivity.

As mentioned in §1, in joint work in progress with Carlson, Mazza, and Nakano we classify all Sylow-trivial modules for finite groups of Lie type using the more detailed theorems of this paper together with the "Φ_d-local" approach to finite groups of Lie type. Here we will just do one more example, namely $G_2(5)$ at $p = 3$, which is one of the borderline cases where $H_1(\mathscr{O}_3(G)) = 0$, but $\mathcal{A}_p(G)$ is one-dimensional and not simply connected. It is furthermore interesting since Carlson–Thévenaz observed via computer that $\rho^2(S) \neq \rho^3(S) = \rho^\infty(S) = N_G(S)$ in [CT15]. As required, the assumptions of Corollary 1.2 are not satisfied, as all p-subgroups turn out to be p-radical, but one subgroup of order 3 is not normal in S. The group can however be easily calculated using either Theorem C (with $\mathcal{C}$either $\mathcal{B}_3(G)$ or

$\mathcal{A}_3(G)$) or Theorem D (using Proposition 4.32)—we choose the first, slightly more cumbersome, method, to see exactly how $\rho^2(S) \neq \rho^3(S) = \rho^\infty(S) = N_G(S)$.

Proposition 6.3. *For k of characteristic 3, $T_k(G_2(5), S) = 0$.*

Proof. Looking at [CCN+85], we see that $N_G(S) \cong 3_+^{1+2} : SD_{16}$ and that there are 2 conjugacy classes of elements of order 3, where $3A$ is a central element in 3_+^{1+2} and $3B$ is a non-conjugate non-trivial element. In fact $N_G(S)$ controls 3-fusion in G, as otherwise $3A$ and $3B$ would need to be conjugate (see e.g., [RV04, Lem. 4.1]). We hence have 3 conjugacy classes of proper, non-trivial subgroups $\langle 3A\rangle$, $\langle 3B\rangle$, and $V = \langle 3A, 3B\rangle$ and furthermore conjugacy classes of chains in this case coincide with chains of conjugacy classes. (The subgroups all turn out to be 3-radical, but $\langle 3B\rangle$ is not normal in S, so the assumptions of Corollary 1.2 are not satisfied.)

We want to see that any element in $H_1(N_G(S))_{3'}$ is equivalent to zero in the colimit. We do this by considering $H_1(\cdot)_{3'}$ of the following subdiagram:

$$\begin{tikzcd}
 & N_G(3B<V<S) \arrow[dl] \arrow[d] \arrow[dr, equal] & & \\
N_G(3B<V) \arrow[d] \arrow[dr] & N_G(V<S) \arrow[d] \arrow[dr] & N_G(3B<S) \arrow[d] \arrow[dll] & N_G(3A<S) \arrow[d] \arrow[dl, equal] \\
N_G(3B) & N_G(V) & N_G(S) & N_G(3A)
\end{tikzcd}$$

Note that $N_G(3B < V) = N(3B) \cap N(3A)$, by the description of fusion in S. By looking at the order of centralizers of elements and the list of maximal subgroups, as described by [CCN+85], we see that $N_G(V)$ and $N_G(S)$ are contained in $N_G(3A) \cong 3 \cdot U_3(5) : 2$, and that $N_G(3B)$ is contained in the maximal subgroup $N_G(2A) \cong 2 \cdot (\mathfrak{A}_5 \times \mathfrak{A}_5).2 = 2 \cdot A(\mathfrak{S}_5 \times \mathfrak{S}_5)$, the index 2 subgroup of $2\mathfrak{S}_5 \circ 2\mathfrak{S}_5$ of even permutations, for $2\mathfrak{S}_5$ the central extension with Sylow 2-subgroup Q_{16}. Recall also that $SD_{16} = \langle \sigma, \tau | \sigma^8 = \tau^2 = 1, \tau\sigma\tau = \sigma^3\rangle \leq \mathrm{Out}(3_+^{1+2}) \cong \mathrm{GL}_2(\mathbb{F}_3)$, with σ^4 identifying with the non-trivial central element in $\mathrm{Out}(3_+^{1+2})$ which acts as -1 on $3_+^{1+2}/Z(3_+^{1+2})$ and trivially on $Z(3_+^{1+2})$. In our model of 3_+^{1+2}, σ^4 will hence conjugate $3B$ to $-3B$ and commute with $3A$, and we can furthermore choose the generator τ so it commutes with $3B$ and conjugates $3A$ to $-3A$. Inside $2 \cdot A(\mathfrak{S}_5 \times \mathfrak{S}_5)$ we can hence represent $3B$ as (123), $3A$ as $(1'2'3')$, σ^4 as $(12)(4'5')$ and τ as $(45)(1'2')$. Hence $N_G(3B) = 2 \cdot A(\mathfrak{S}_3 \times \mathfrak{S}_2 \times \mathfrak{S}_5)$ and $N_G(3B < V) = 2 \cdot A(\mathfrak{S}_3 \times \mathfrak{S}_2 \times \mathfrak{S}_3 \times \mathfrak{S}_2)$.

Working inside these groups the diagram identifies as follows:

(6.1)

$$\begin{tikzcd}
 & V:(2\times 2) \arrow[dl] \arrow[d] \arrow[dr, equal] & & \\
2\cdot A(\mathfrak{S}_3\times\mathfrak{S}_2\times(\mathfrak{S}_3\times\mathfrak{S}_2)) \arrow[d] \arrow[dr] & 3_+^{1+2}:(2\times 2) \arrow[d] \arrow[dr] & V:(2\times 2) \arrow[d] \arrow[dll] & 3_+^{1+2}:SD_{16} \arrow[d] \arrow[dl, equal] \\
2\cdot A(\mathfrak{S}_3\times\mathfrak{S}_2\times\mathfrak{S}_5) & N_G(V) & 3_+^{1+2}:SD_{16} & 3\cdot U_3(5):2
\end{tikzcd}$$

(In fact $N_G(V) \cong 3 \cdot \mathfrak{A}_{3,4} : 2$, as is seen working inside $N_G(3A)$, but we shall not need this.) Taking $H_1(\cdot)_{3'}$ we obtain the following diagram:

$$
\begin{tikzcd}
& \mathbb{Z}/2\times\mathbb{Z}/2 \arrow[dl] \arrow[d, "\simeq"] \arrow[dr, two heads] & & \\
\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2 \arrow[d] \arrow[dr] & \underset{\sigma^4}{\mathbb{Z}/2}\times\underset{\tau}{\mathbb{Z}/2} \arrow[d] \arrow[dr] & \underset{\sigma^4}{\mathbb{Z}/2}\times\underset{\tau}{\mathbb{Z}/2} \arrow[dll] \arrow[d] & \mathbb{Z}/2\times\mathbb{Z}/2 \arrow[dl, two heads] \arrow[d] \\
\mathbb{Z}/2\times\mathbb{Z}/2 & \mathbb{Z}/2\times\mathbb{Z}/2 & \underset{\bar\sigma}{\mathbb{Z}/2}\times\underset{\bar\tau}{\mathbb{Z}/2} & \underset{\bar\tau}{\mathbb{Z}/2}
\end{tikzcd}
\tag{6.2}
$$

The double headed arrows are surjections by a Frattini argument, and we conclude from this that $H_1(N_G(V)) \cong \mathbb{Z}/2\times\mathbb{Z}/2$, since $N_G(V)/C_G(V) \cong \mathbb{Z}/2\times\mathbb{Z}/2$, by the G-fusion in V. The indicated generators follow since we know how σ^4 and τ act on $3A$ and $3B$ (and hence V). The right-hand part of the diagram immediately reveals that $\bar\sigma$ is zero in the colimit. The element $\bar\tau$ is also zero in the colimit: Namely, consider the element $(1'2')(4'5') \in N_G(3B < V)$. This maps to zero in $H_1(N_G(3B))$, by the above description. But in $H_1(N_G(V))$ it maps to the same element as τ, since it acts the same way on V. Hence $\bar\tau$ represents the zero element in the colimit as wanted. $\square$

6.3. **Symmetric groups.** The Sylow-trivial modules for the symmetric groups are understood via representation theoretic methods by the work of Carlson–Hemmer–Mazza–Nakano [CMN09, CHM10] (see also [LM15a] for extensions). Let us point out that simple connectivity of $\mathcal{S}_p(G)$ and our work directly implies their results, at least in the generic case.

Theorem 6.4. *If p is odd, and $3p+2 \le n < p^2$ or $n \ge p^2+p$, then*

$$T_k(\mathfrak{S}_n, S) \cong \operatorname{Hom}(\mathfrak{S}_n, k^\times) \cong \mathbb{Z}/2.$$

Proof. In [Kso04, Thm. 0.1] (building on [Kso03], [Bou92]) it is proved that $\mathcal{A}_p(\mathfrak{S}_n)$, and hence $\mathcal{S}_p(\mathfrak{S}_n)$, is simply connected if and only if n is in the above range, p odd. So the result follows from Theorem B. $\square$

When $p = 2$, $T_k(\mathfrak{S}_n, S) = 0$ since $N_{\mathfrak{S}_n}(S) = S$ for all n [Wei25, Cor. 2]. It is an interesting exercise to fill in the left-out cases, where p is odd and n is small relative to p, using the methods of this paper. Let us just quickly do this calculation where $n = 2p+b$, for p odd and $0 < b < p$, where $T_k(\mathfrak{S}_n, S) = \mathbb{Z}/2\times\mathbb{Z}/2$, as an illustration (one can in fact also do the general case directly this way, without much more effort). The case is of interest e.g., since $n = 7 = 2\cdot 3+1$ is the smallest case where $\mathcal{S}_p(G)$ is connected but not simply connected, and where the naïve guess [Car12, §5] that groups without strongly p-embedded subgroup should have no exotic Sylow-trivial modules fails: Pick $S = \langle (12\cdots p), ((p+1)\cdots 2p)\rangle$; there is just one $N_G(S)$-conjugacy class of non-trivial proper 3-radical subgroups, represented by $A = \langle (12\cdots p)\rangle$. By Proposition 5.3, $H_1(\mathcal{O}_p^*(G))$ equals the colimit of $H_1(-)_{p'}$ applied to the diagram $N_G(A) \leftarrow N_G(A < S) \to N_G(S)$, or

$$C_p \rtimes C_{p-1} \times \mathfrak{S}_{p+b} \leftarrow (C_p \rtimes C_{p-1})^2 \times \mathfrak{S}_b \to (C_p \rtimes C_{p-1}) \wr C_2 \times \mathfrak{S}_b,$$

which is seen to be $\mathbb{Z}/2\times\mathbb{Z}/2$. Hence by Theorem A,

$$T_k(\mathfrak{S}_{2p+b}, S) = \operatorname{Hom}(H_1(\mathcal{O}_p^*(\mathfrak{S}_{2p+b})), k^\times) \cong \mathbb{Z}/2\times\mathbb{Z}/2$$

as wanted. Alternatively one may use the centralizer decomposition, using that $|\mathcal{F}_p^*(G)|$ is contractible, since it is one-dimensional with $\pi_1(\mathcal{F}_p^*(G)) = 1$ by Corollary 4.23.

To come full circle in the above example, we can also use Proposition A.4 (or Theorem 4.10 directly) to determine $\pi_1(\mathscr{O}_p^*(\mathfrak{S}_{2p+b}))$ by describing the effect of adding the subgroup A–this reveals that

$$\begin{aligned}\pi_1(\mathscr{O}_p^*(\mathfrak{S}_{2p+b})) &\cong C_{p-1} \wr C_2 \times \mathfrak{S}_b / \overline{(C_{p-1} \wr C_2 \times \mathfrak{S}_b) \cap (1 \times \mathfrak{A}_{p+b})} \\ &\cong \begin{cases} D_8 & \text{if } b = 1 \\ C_2 \times C_2 & \text{if } 1 < b < p \end{cases}\end{aligned} \tag{6.3}$$

still with p odd. For $b = 0$, $G_0 = \mathfrak{S}_p \wr C_2$, and we likewise get $\pi_1(\mathscr{O}_p^*(\mathfrak{S}_{2p})) \cong D_8$.

By Corollary 4.16, the full π_1 calculation for $\mathfrak{S}_n$ enables us to do the calculation for $\mathfrak{A}_n$ as well. Let us dwell on this for a moment as it illustrates some of the preceding formulas, and lets us correct a small mistake in the literature: Notice that, in the above notation, the generator of a C_{p-1} and the wreathing C_2 correspond to odd elements inside $\mathfrak{S}_{2p+b}$. Hence their product is even and defines an element in $\ker(\pi_1(\mathscr{O}_p^*(\mathfrak{S}_{2p+b})) \to \mathbb{Z}/2)$, which is of order 4 when $b = 0, 1$. So by Corollary 4.16

$$\pi_1(\mathscr{O}_p^*(\mathfrak{A}_{2p+b})) \cong \begin{cases} C_4 & \text{if } b = 0, 1 \\ C_2 & \text{if } 1 < b < p \end{cases} \tag{6.4}$$

still with p odd. In particular

$$T_k(\mathfrak{A}_{2p+b}, S) \cong \operatorname{Hom}(\pi_1(\mathscr{O}_p^*(\mathfrak{A}_{2p+b})), k^\times) \cong \begin{cases} \mathbb{Z}/4 & \text{if } b = 0, 1 \\ \mathbb{Z}/2 & \text{if } 1 < b < p \end{cases} \tag{6.5}$$

when p is odd and k has a 4th root of unity. The formula (6.5) for $p > 3$ and $b = 0, 1$ corrects [CMN09, Thm. 1.2(c)], which states $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ (the mistake was that the given modules are not all self-dual).

Notice also that Corollary 3.9 gives a description of an exotic generator for, say, $T_k(\mathfrak{S}_{2p+b}, S)$ as $\Omega^{-1} H_1(|\mathcal{S}_p(\mathfrak{S}_{2p+b})|; k_\varphi)$, with

$$H_1(|\mathcal{S}_p(\mathfrak{S}_{2p+b})|; k_\varphi) \cong \ker\left(k_\varphi \uparrow_{N_G(V<S)}^G \twoheadrightarrow k_\varphi \uparrow_{N_G(V)}^G \oplus k_\varphi \uparrow_{N_G(S)}^G\right),$$

and $\varphi \in \operatorname{Hom}(\pi_1(\mathscr{O}_p^*(\mathfrak{S}_{2p+b})), k^\times)$ not coming from a one-dimensional representation of $\mathfrak{S}_{2p+b}$. E.g., taking $n = 7$ and $p = 3$, $H_1(|\mathcal{S}_3(\mathfrak{S}_7)|; k_\varphi)$ is of dimension $35 = 140 - (35 + 70)$ (cf. also [Bou92, §5.6]). This agrees on the level of dimensions with the description in [CMN09, Prop. 8.3] as the Young module $Y(4, 3)$, which is of dimension 28 with projective cover of dimension $28 + 35 = 63$, as explained to us by Anne Henke. For untwisted coefficients, the representation given by the top homology group of the p-subgroup complex has been studied in some generality by Shareshian–Wachs [Sha04, SW09].

6.4. **p-Solvable groups.** When G is a p-solvable group, it is proved in [NR12], building on [CMT11], that $T_k(G, S) \cong \operatorname{Hom}(G_0, k^\times)$, at least when k is algebraically closed, and $G = G_0$ when the p-rank is two or more by [Gol70, Thm. 2.2]. The proof in [NR12, CMT11] reduces to the case $G = AH$, where A is an elementary abelian p-group, and H is a normal p'-group, and appeals to the classification of the finite simple groups in the proof of the last statement, albeit in a mild way. By Theorem A the statement is equivalent to the isomorphism $H^1(\mathscr{O}_p^*(G); k^\times) \cong \operatorname{Hom}(G_0, k^\times)$. We will not reprove this isomorphism here, but would like to make two remarks: First, we have the following result.

Theorem 6.5. *Suppose that $G = AH$ with H a normal p'-group and A a non-trivial elementary abelian p-group. Then $T_k(G,S) \cong \lim^0_{V \in \mathcal{F}_{\mathcal{A}_p(A)}} \mathrm{Hom}(C_H(V), k^\times)$.*

Proof. Note that $\mathcal{F}_p^*(A) \cong \mathcal{F}_p^*(G)$, and in particular $|\mathcal{F}_p^*(G)|$ is contractible. so the result follows from Theorem D, also using that $(C_H(V))_{p'} \xrightarrow{\cong} (C_G(V))_{p'}$. $\square$

It would obviously be interesting to have an identification of the right-hand side with $\mathrm{Hom}(G, k^\times)$, when A has rank at least 2, via a proof which did not use the classification of finite simple groups.

Second, again $T_k(G,S) = \mathrm{Hom}(G, k^\times)$ would follow from the simply connectivity of $\mathcal{S}_p(G)$, by Theorem B. When $G = AH$ as above, Quillen [Qui78, Prob. 12.3] conjectures that $\mathcal{S}_p(G)$ should in fact be Cohen–Macaulay, and in particular simply connected when A has p-rank at least 3, and proved this when G is actually solvable [Qui78, Thm. 11.2(i)]. Aschbacher also conjectured the simple connectivity in [Asc93], and reduced the claim to where $H = F^*(G)$ is a direct product of simple components being permuted transitively by A; see also [Smi11, §9.3]—Aschbacher uses the commuting complex $K_p(G)$, but this is G-homotopy equivalent to $\mathcal{S}_p(G)$ (see e.g., [Gro02, p. 431] or [Smi11, §9.3]). (The related [Qui78, Thm. 11.2(ii)], giving non-zero homology in the top dimension, has in fact been generalized to the p-solvable case, though with a proof also using the classification of finite simple groups; see [Smi11, §8.2].) We remark though that when G is not of the form $G = AH$, the complex $\mathcal{S}_p(G)$ need not be simply connected even for large p-rank (see [PW00, Ex. 5.1]), so a direct proof in the p-solvable case would need one of our more precise theorems. (The homology of $\mathcal{S}_p(G)$ when G is solvable is described in [PW00, Prop. 4.2], and when G is p-solvable, the p-essential subgroups are those p-radical subgroups P where $N_G(P)/P$ has p-rank one by [Pui76, Cor. to Prop. II.4].) As noted in §1.5, one may hope to get vanishing results for $\pi_1(\mathscr{O}_p^*(G))$ for any finite group G by reducing to simple groups, by suitably generalizing the p-solvable case.

Appendix A. Varying the collection $\mathcal{C}$ of subgroups

In this appendix we describe how the homotopy and homology of $\mathscr{O}_\mathcal{C}(G)$ and the other standard categories of this paper behave under changing the collection $\mathcal{C}$. The results are often extensions of results from mod p homology decompositions [Dwy97, Gro02, GS06], but now working integrally or away from p, and also examining low-dimensional behavior. The results are referred to throughout the paper when moving to collections smaller than $\mathcal{S}_p(G)$. We start in §A.1 by recalling various G-categories associated to $\mathcal{C}$. In §A.2 we explain how the homotopy type of our categories change upon removing a given subgroup (postponing parts of the proof to §A.6). We then use it in §A.3 to show that the homotopy types agree for various collections, and in §A.4 to analyze the low-dimensional homotopy types, when more subgroups are removed. Finally §A.5 describes the set of components of $\mathcal{C}$ and its relation to group theoretic notions of strongly p-embedded subgroups.

A.1. **G-categories associated to collections of subgroups.** Recall the categories $\mathscr{O}_\mathcal{C}(G)$ and $\mathcal{F}_\mathcal{C}(G)$ from §2.4. We now introduce "fattened up" versions of $\mathcal{C}$ related to these, also used in e.g., [Dwy97, Gro02]—they will be preordered sets, meaning a category with at most one map between any two objects. Preordered sets are equivalent as categories to posets, but usually not equivariantly so. Let $E\mathscr{O}_\mathcal{C}$ be the category of "pointed G-sets", i.e., the G-category with objects $(G/P, x)$

for $P \in \mathcal{C}$, and $x \in G/P$, and morphisms maps of pointed G-sets. The group G acts on objects by $g \cdot (G/P, x) = (G/P, gx)$. Similarly $E\mathbf{A}_{\mathcal{C}}$ is the category with objects monomorphisms $i\colon P \to G$ induced by conjugation in G, with $P \in \mathcal{C}$, and morphisms $i \to i'$ given by the group homomorphisms $\varphi\colon P \to P'$, induced by conjugation in G, such that $i = i'\varphi$ (see also e.g., [Gro02, 2.8]). (As $\mathbf{A}_{\mathcal{C}}$ is an older name for $\mathcal{F}_{\mathcal{C}}$, $E\mathbf{A}_{\mathcal{C}}$ would most logically be called $E\mathcal{F}_{\mathcal{C}}$ in the current notation, but we keep the traditional name, to avoid confusion with other standard terminology.) By inspection of simplices, as already observed in e.g., [Gro02, Prop. 2.10],

$$|E\mathscr{O}_{\mathcal{C}}|/G = |\mathscr{O}_{\mathcal{C}}(G)| \text{ and } |E\mathbf{A}_{\mathcal{C}}|/G = |\mathcal{F}_{\mathcal{C}}(G)| \tag{A.1}$$

The advantage of such a description is that a G-homotopy equivalence induces a homotopy equivalence on orbit spaces, by elementary algebraic topology, which we will use to examine which subgroups can be removed from $\mathcal{C}$ without changing the homotopy type of $|\mathscr{O}_{\mathcal{C}}(G)|$ and $|\mathcal{F}_{\mathcal{C}}(G)|$.

Note also the G-equivariant functors

$$E\mathscr{O}_{\mathcal{C}} \to \mathcal{C} \leftarrow E\mathbf{A}_{\mathcal{C}} \tag{A.2}$$

given by $(G/P, x) \mapsto G_x$ and $(i\colon Q \to G) \mapsto i(Q)$. These are equivalences of categories but not in general G-equivalences (non-equivariant functors the other way are given by $P \mapsto (G/P, e)$ and $P \mapsto (P \to G)$). We can use the maps to $\mathcal{C}$ to describe the fixed-points, as one checks that we have equivalences of categories

$$E\mathscr{O}_{\mathcal{C}}^H \to \mathcal{C}_{\geq H} \quad \text{and} \quad E\mathbf{A}_{\mathcal{C}}^H \to \mathcal{C}_{\leq C_G(H)}. \tag{A.3}$$

(See also [GS06, (†)].) We record the following relationship:

Lemma A.1. *Let G be a finite group, and consider collections of p-subgroups $\mathcal{C}' \leq \mathcal{C}$.*

(a) *If $|\mathcal{C}'| \xrightarrow{\simeq} |\mathcal{C}|$, then $|\mathscr{T}_{\mathcal{C}'}(G)| \xrightarrow{\simeq} |\mathscr{T}_{\mathcal{C}}(G)|$ and $\pi_1(\mathscr{O}_{\mathcal{C}'}(G))_{p'} \xrightarrow{\cong} \pi_1(\mathscr{O}_{\mathcal{C}}(G))_{p'}$.*
(b) *If $|E\mathscr{O}_{\mathcal{C}'}| \to |E\mathscr{O}_{\mathcal{C}}|$ is a G-homotopy equivalence then $|\mathscr{O}_{\mathcal{C}'}(G)| \xrightarrow{\simeq} |\mathscr{O}_{\mathcal{C}}(G)|$.*
(c) *If $|E\mathbf{A}_{\mathcal{C}'}| \to |E\mathbf{A}_{\mathcal{C}}|$ is a G-homotopy equivalence then $|\mathcal{F}_{\mathcal{C}'}(G)| \xrightarrow{\simeq} |\mathcal{F}_{\mathcal{C}}(G)|$.*

Proof. For (a) consider the diagram

$$\begin{tikzcd}
|\mathscr{T}_{\mathcal{C}'}(G)| \arrow[r, equal] \arrow[d] & |\mathcal{C}'_G| \arrow[d] & |\mathcal{C}'|_{hG} \arrow[l, "\sim"'] \arrow[d, "\sim"] \\
|\mathscr{T}_{\mathcal{C}}(G)| \arrow[r, equal] & |\mathcal{C}_G| & |\mathcal{C}|_{hG} \arrow[l, "\sim"']
\end{tikzcd}$$

The horizontal equivalences are by Lemma 2.3 and the vertical equivalences follow since $|\mathcal{C}'| \to |\mathcal{C}|$ is a G-equivariant map, assumed to be a homotopy equivalence, and hence induces a homotopy equivalence between Borel constructions. Now the statement about fundamental groups follows from the first claim, using Proposition 4.5. Points (b) and (c) follow from A.1, since a G-homotopy equivalence induces a homotopy equivalence on orbits. □

A.2. **Removing a single conjugacy class.** The next result describes the effect of removing a conjugacy class of subgroups, called "pruning" in [Dwy98, §9] (see also [GS06, Lem. 2.5]). The result is phrased in terms of certain homotopy pushout squares [BK72, Ch. XII] (see e.g., [DH01, I.4.18] for an introduction). These give rise to Meyer-Vietoris sequences in homology, and a van Kampen theorem description for the fundamental groups. Some squares will furthermore be homotopy pushouts of G-spaces, which hence induce homotopy pushouts on both fixed-points and orbit spaces.

Denote by $(E\mathscr{O}_{\mathcal{C}})_{>P}$ the full subcategory of pairs $(G/Q, x)$ such that there exists a non-isomorphism $(G/Q, x) \to (G/P, e)$, i.e., $G_x > P$, and similarly for $(E\mathscr{O}_{\mathcal{C}})_{<P}$ and $E\mathbf{A}_{\mathcal{C}}$.

Proposition A.2 (Pruning collections)**.** *Let $\mathcal{C}'$ be a collection of p-subgroups obtained from a collection $\mathcal{C}$ by removing all G-conjugates of a p-subgroup $P \in \mathcal{C}$. We have the following five G-homotopy pushout squares, with corresponding quotient homotopy pushout squares:*

$$\begin{tikzcd}
G \times_N (|\mathcal{C}_{<P}| \star |\mathcal{C}_{>P}|) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(1a)}"] & |\mathcal{C}'| \arrow[d] \\
G/N \arrow[r] & |\mathcal{C}|
\end{tikzcd}
\qquad
\begin{tikzcd}
(|\mathcal{C}_{<P}|/P \star |\mathcal{C}_{>P}|)/W \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(1b)}"] & |\mathcal{C}'|/G \arrow[d] \\
pt \arrow[r] & |\mathcal{C}|/G
\end{tikzcd}$$

$$\begin{tikzcd}
G \times_N (EN \times (|\mathcal{C}_{<P}| \star |\mathcal{C}_{>P}|)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(2a)}"] & EG \times |\mathcal{C}'| \arrow[d] \\
G \times_N EN \arrow[r] & EG \times |\mathcal{C}|
\end{tikzcd}
\qquad
\begin{tikzcd}
(|\mathcal{C}_{<P}| \star |\mathcal{C}_{>P}|)_{hN} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(2b)}"] & |\mathscr{T}_{\mathcal{C}'}(G)| \arrow[d] \\
BN \arrow[r] & |\mathscr{T}_{\mathcal{C}}(G)|
\end{tikzcd}$$

$$\begin{tikzcd}
G \times_N (EW \times (|(E\mathscr{O}_{\mathcal{C}})_{<P}| \star |\mathcal{C}_{>P}|)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(3a)}"] & |E\mathscr{O}_{\mathcal{C}'}| \arrow[d] \\
G \times_N EW \arrow[r] & |E\mathscr{O}_{\mathcal{C}}|
\end{tikzcd}
\qquad
\begin{tikzcd}
(|(E\mathscr{O}_{\mathcal{C}})_{<P}|/P \star |\mathcal{C}_{>P}|)_{hW} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(3b)}"] & |\mathscr{O}_{\mathcal{C}'}(G)| \arrow[d] \\
BW \arrow[r] & |\mathscr{O}_{\mathcal{C}}(G)|
\end{tikzcd}$$

$$\begin{tikzcd}
G \times_N (EW' \times (|\mathcal{C}_{<P}| \star |(E\mathbf{A}_{\mathcal{C}})_{>P}|)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(4a)}"] & |E\mathbf{A}_{\mathcal{C}'}| \arrow[d] \\
G \times_N EW' \arrow[r] & |E\mathbf{A}_{\mathcal{C}}|
\end{tikzcd}
\qquad
\begin{tikzcd}
(|\mathcal{C}_{<P}| \star |(E\mathbf{A}_{\mathcal{C}})_{>P}|/C)_{hW'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(4b)}"] & |\mathcal{F}_{\mathcal{C}'}(G)| \arrow[d] \\
BW' \arrow[r] & |\mathcal{F}_{\mathcal{C}}(G)|
\end{tikzcd}$$

where $C = C_G(P)$, $N = N_G(P)$, $W = N/P$, $W' = N/C$, and $\star$ denotes join of spaces.

The proof of Proposition A.2 is not hard, but is postponed to §A.6 to not interrupt the flow. We move on to draw consequences, starting with Symonds' theorem, 'Webb's conjecture' [Web87, Conj. 4.2], on the contractibility of the orbit space of the p-subgroup complex, used in the proofs of Theorems C and E. To get minimal assumptions on the collection $\mathcal{C}$, rather than just closed under passage to p-overgroups, we make a forward reference Lemma A.6, allowing us to remove non-p-radical subgroups.

Proposition A.3 ([Sym98])**.** *If G is a finite group and $\mathcal{C}$ a non-empty collection of p-subgroups, closed under passage to p-radical overgroups, then $|\mathcal{C}|/G$ is contractible.*

Proof. By Lemma A.6 $|\mathcal{C}|$ is G-homotopy equivalent to $|\bar{\mathcal{C}}|$ where $\bar{\mathcal{C}}$ is obtained by adding all p-overgroups of subgroups in $\mathcal{C}$. We may thus without restriction assume that $\mathcal{C}$ is closed under passage to all p-overgroups, as G-homotopy equivalences induce homotopy equivalences on G-orbit spaces.

If $\mathcal{C}$ consists of just all Sylow (i.e., maximal) p-subgroups, then the claim amounts exactly to the part of Sylow's theorem saying that these subgroups are all conjugate. We will prove the claim in general by seeing that contractibility is preserved under adding conjugacy classes of subgroups in order of decreasing size, using Proposition A.2(1b) by an induction on the size of the group: Suppose the claim is true for all groups G of strictly smaller order, and that $\mathcal{C}$ is obtained from $\mathcal{C}'$ by adding G-conjugates of a subgroup P this way. We can also assume that P is non-trivial, since otherwise the claim is clear, as $\mathcal{C}$ is equivariantly contractible to the trivial subgroup in that case.

Now consider the pushout square of Proposition A.2(1b): The top left-hand corner becomes $|\mathcal{C}_{>P}|/W$, for $W = N_G(P)/P$. But $|\mathcal{C}_{>P}|$ has a W-equivariant deformation retraction onto $|\mathcal{S}_p(W)|$ via $Q \mapsto N_Q(P)/P$, as observed by Quillen [Qui78, Prop. 6.1] (see also Lemma A.6). But $|\mathcal{S}_p(W)|/W$ is contractible by our induction hypothesis, observing that W is of smaller order since P is non-trivial and that $\mathcal{S}_p(W)$ is non-empty as we have already added the Sylow p-subgroups. Hence $|\mathcal{C}'|/G \to |\mathcal{C}|/G$ is a homotopy equivalence, and we conclude that the claim is true for all collections $\mathcal{C}$ as in the proposition, proving the claim for G. $\square$

Let us spell out what Proposition A.2(3b) says when the subgroup P we remove is minimal.

Proposition A.4 (The effect of adding or removing a minimal subgroup)**.** *Suppose that $\mathcal{C}$ is a collection of p-subgroups in a finite group G, closed under passage to p-radical overgroups, and that $\mathcal{C}'$ is obtained from $\mathcal{C}$ by removing the conjugacy class of a minimal p-subgroup $P \in \mathcal{C}$. Set $W = N_G(P)/P$. Then $|\mathscr{O}_{\mathcal{C}}(G)|$ can be described via a homotopy pushout square*

$$\begin{tikzcd}
|\mathscr{T}_p^*(W)| \arrow[r] \arrow[d] & |\mathscr{O}_{\mathcal{C}'}(G)| \arrow[d] \\
BW \arrow[r] & |\mathscr{O}_{\mathcal{C}}(G)|
\end{tikzcd}$$

Here the top horizontal map identifies with the nerve of the composite $\mathscr{T}_p^*(W) \to \mathscr{O}_p^*(W) \to \mathscr{O}_{\mathcal{C}'}(G)$ where the first map is the natural one from §2.4 and the second sends a W-set X to $G/P \times_W X$ (replacing $\mathcal{C}$ by its closure under taking p-overgroups if necessary, using Lemma A.6(1)*). In particular*

(1) $\pi_1(\mathscr{O}_{\mathcal{C}}(G)) \xleftarrow{\cong} \pi_1(\mathscr{O}_{\mathcal{C}'}(G))/\overline{\mathrm{im}(K)}$, *with* $K = \ker(\pi_1(\mathscr{O}_p^*(W)) \to W_{p'})$ *and the overline denoting normal closure.*

(2) $H_1(\mathscr{O}_{\mathcal{C}}(G)) \xleftarrow{\cong} H_1(\mathscr{O}_{\mathcal{C}'}(G))/\mathrm{im}(K)$, *with* $K = \ker(H_1(\mathscr{O}_p^*(W)) \to H_1(W))$.

Hence

(3) *If* $\pi_1(\mathscr{O}_p^*(W)) \xrightarrow{\cong} W_{p'}$ *then* $\pi_1(\mathscr{O}_{\mathcal{C}'}(G)) \xrightarrow{\cong} \pi_1(\mathscr{O}_{\mathcal{C}}(G))$.

(4) *If* $H_1(\mathscr{O}_p^*(W)) \xrightarrow{\cong} H_1(W)_{p'}$ *then* $H_1(\mathscr{O}_{\mathcal{C}'}(G)) \xrightarrow{\cong} H_1(\mathscr{O}_{\mathcal{C}}(G))$.

Proof. We claim that square in the position identifies with the homotopy pushout square of Proposition A.2(3b), after making suitable identifications. Namely the left-hand corner in (3b) identifies with $|\mathscr{T}_p^*(W)|$ via the homotopy equivalences $|\mathcal{C}_{>P}|_{hW} \simeq |\mathcal{S}_p(W)|_{hW} \simeq |\mathscr{T}_p^*(W)|$, using again Lemma A.6 and (1.9), and the maps identify with the stated ones via this equivalence.

Now (1) is a consequence of van Kampen's theorem [Hat02, §1.2]: By Proposition 4.1 both fundamental groups on the right-hand side of the pushout square are finite p'-groups, so van Kampen's theorem and (1.7) produce a pushout of groups

$$\begin{tikzcd}
\pi_1(\mathscr{O}_p^*(W)) \arrow[r] \arrow[d, two heads] & \pi_1(\mathscr{O}_{\mathcal{C}'}(G)) \arrow[d, two heads] \\
W_{p'} \arrow[r] & \pi_1(\mathscr{O}_{\mathcal{C}}(G))
\end{tikzcd} \tag{A.4}$$

with vertical maps surjective, and (1) follows. Point (2) follows similarly, but using the Mayer–Vietoris sequence instead. Points (3) and (4) are now obvious. $\square$

For further applications of Proposition A.2, we recall the behavior of connectivity under joins. In general the join of an n-connected space with an m-connected space is $(n+m+2)$-connected (use cellular approximation [Hat02, §4.1]). The cases we will need are summarized in Lemma A.5.

Lemma A.5 (Connectivity of joins)**.** *The join $X \star Y$ is connected for X, Y non-empty or X connected. It is simply connected for X connected and Y non-empty, or for X simply connected.*

The join $X \star Y$ is (G-) contractible if X or Y is (G-) contractible.

Proof. This follows from the definition of the join, e.g., as the homotopy pushout of the diagram $X \leftarrow X \times Y \rightarrow Y$ (which identifies with the suspension of the smash $\Sigma(X \wedge Y)$ if picking a basepoint). □

We finally give the postponed lemma about removing subgroups, which will also be used several times in proofs of the subsequent theorems.

Lemma A.6. *Let $\mathcal{C}$ be a collection of p-subgroups in G, and let $\bar{\mathcal{C}}$ be the smallest collection containing $\mathcal{C}$, closed under passage to p-overgroups, and let P be a p-subgroup of G.*

(1) *If $\mathcal{C}$ contains all p-radical groups in $\bar{\mathcal{C}}_{>P}$, then $\mathcal{C}_{>P} \to \bar{\mathcal{C}}_{>P}$ is an $N_G(P)$-homotopy equivalence. Thus, if all p-radical overgroups of P are in $\mathcal{C}$ then we have $N_G(P)$-homotopy equivalences*

$$|\mathcal{C}_{>P}| \xrightarrow{\simeq} |\mathcal{S}_p(G)_{>P}| \xleftarrow{\simeq} |\mathcal{S}_p(N_G(P)/P)|.$$

(2) *If $\mathcal{C}$ contains all Sylow and p-essential overgroups in $\bar{\mathcal{C}}_{>P}$, then $\pi_0(\mathcal{C}_{>P}) \xrightarrow{\cong} \pi_0(\bar{\mathcal{C}}_{>P})$. Thus, if all Sylow and p-essential overgroups of P are in $\mathcal{C}$ then we have $N_G(P)$-equivariant bijections*

$$\pi_0(\mathcal{C}_{>P}) \xrightarrow{\cong} \pi_0(\mathcal{S}_p(G)_{>P}) \xleftarrow{\cong} \pi_0(\mathcal{S}_p(N_G(P)/P)).$$

Proof. Start by recalling that $\mathcal{S}_p(G)_{>P}$ is indeed $N_G(P)$-homotopy equivalent to $\mathcal{S}_p(N_G(P)/P)$ via the equivariant deformation retraction $R \mapsto N_R(P)/P$, as already observed by Quillen [Qui78, Prop. 6.1].

Now to see the first claim in (1), we add $N_G(P)$-conjugacy classes of p-subgroups to $\mathcal{C}_{>P}$ in order of decreasing size to reach $\bar{\mathcal{C}}_{>P}$ (see also [Gro02, Pf. of Thm. 1.2]). If $\tilde{\mathcal{C}}$ is obtained from $\mathcal{C}$ by adding an $N_G(P)$-conjugacy class of a p-subgroup Q, maximal in $\bar{\mathcal{C}} \setminus \mathcal{C}$, then we have an $N_G(Q)$-homotopy pushout square analogous to Proposition A.2(1a):

$$\begin{array}{ccc} N_G(P) \times_{N_G(P<Q)} (|\mathcal{S}_p(G)_{>Q}| \star |\mathcal{C}_{>P,<Q}|) & \longrightarrow & |\mathcal{C}_{>P}| \\ \downarrow & & \downarrow \\ N_G(P)/N_G(P<Q) & \longrightarrow & |\tilde{\mathcal{C}}_{>P}| \end{array} \tag{A.5}$$

By assumption Q is not p-radical, so that $\mathcal{S}_p(G)_{>Q}$ is $N_G(Q)$-contractible, via the standard contraction

$$R \geq N_R(Q) \leq N_R(Q)O_p(N_G(Q)) \geq O_p(N_G(Q)) \tag{A.6}$$

of Quillen [Qui78] and Bouc [Bou84]. Hence $|\mathcal{S}_p(G)_{>Q}| \star |\mathcal{C}_{>P,<Q}|$ is $N_G(P<Q)$-contractible by the initial observation and Lemma A.5. Hence (A.5) shows that $|\mathcal{C}_{>P}| \to |\tilde{\mathcal{C}}_{>P}|$ is an $N_G(P)$-homotopy equivalence. Continuing this way shows the first part of (1) by induction. The second half follows from the first, but applied to $\mathcal{C} \cup [P]$ so that $(\overline{\mathcal{C} \cup [P]})_{>P} = \mathcal{S}_p(G)_{>P}$.

The proof of (2) follows similarly. The assumption that $\mathcal{S}_p(N_G(Q)/Q)$ is connected still ensures that $|\mathcal{S}_p(G)_{>Q}| \star |\mathcal{C}_{>P,<Q}|$ is connected by Lemma A.5. Hence $|\mathcal{C}_{>P}| \to |\tilde{\mathcal{C}}_{>P}|$ is a bijection on π_0 by (A.5), and we again conclude $\pi_0(\mathcal{C}_{>P}) \xrightarrow{\cong} \pi_0(\bar{\mathcal{C}}_{>P})$ by induction. Again the second part follows by applying the first to $\mathcal{C} \cup [P]$. $\square$

Remark A.7. Lemma A.6 also has a generalization to higher homotopy groups: If $\mathcal{C}$ is closed under passage to all overgroups Q such that $|\mathcal{S}_p(N_G(Q)/Q)|$ is not i-connected, then ι is an isomorphism on π_j for $j \leq i$ and surjective on π_{i+1}. This follows by the same argument as above, but now in (A.5) appealing to the Blakers-Massey's excision theorem (see e.g., [tD08, Prop. 6.4.2]) instead.

A.3. **Varying the collection without changing homotopy types.** We now see how we can vary our collection $\mathcal{C}$ without changing the homotopy type of associated categories. The omnibus Theorem A.8 is mainly a translation of "classical" homotopy equivalences [GS06, Thm. 1.1], using the elementary Lemma A.1 (see also [GS06] for historical references).

As usual let $\mathcal{S}_p(G)$ and $\mathcal{A}_p(G)$ denote non-trivial p-groups and non-trivial elementary abelian p-subgroups $V \cong (\mathbb{Z}/p)^r$ respectively. The collection $\mathcal{B}_p(G)$ is the collection of non-trivial p-radical subgroups, i.e., non-trivial p-subgroups P such that $O_p(N_G(P)) = P$.

Here and elsewhere the superscript e means that we do not exclude the trivial subgroup.

Theorem A.8. *Let G be a finite group and $\mathcal{C}$ a collection of p-subgroups.*

(1) *Suppose $\mathcal{C}$ is closed under passage to p-radical overgroups. Then $|\mathcal{C} \cap \mathcal{B}_p^e(G)| \xrightarrow{\simeq} |\mathcal{C}|$ and $|E\mathscr{O}_{\mathcal{C}\cap\mathcal{B}_p^e(G)}(G)| \xrightarrow{\simeq} |E\mathscr{O}_{\mathcal{C}}|$ are G-homotopy equivalences and consequently*
$$|\mathscr{T}_{\mathcal{C}\cap\mathcal{B}_p^e(G)}(G)| \xrightarrow{\simeq} |\mathscr{T}_{\mathcal{C}}(G)| \quad \text{and} \quad |\mathscr{O}_{\mathcal{C}\cap\mathcal{B}_p^e(G)}(G)| \xrightarrow{\simeq} |\mathscr{O}_{\mathcal{C}}(G)|$$

(2) *Suppose $\mathcal{C}$ is closed under passage to non-trivial elementary abelian subgroups. Then*
$$|\mathcal{C} \cap \mathcal{A}_p^e(G)| \xrightarrow{\simeq} |\mathcal{C}| \quad \text{and} \quad |E\mathbf{A}_{\mathcal{C}\cap\mathcal{A}_p^e(G)}| \xrightarrow{\simeq} |E\mathbf{A}_{\mathcal{C}}|$$
are G-homotopy equivalences. Consequently
$$|\mathscr{T}_{\mathcal{C}\cap\mathcal{A}_p^e(G)}(G)| \xrightarrow{\simeq} |\mathscr{T}_{\mathcal{C}}(G)| \quad \text{and} \quad |\mathcal{F}_{\mathcal{C}\cap\mathcal{A}_p^e(G)}(G)| \xrightarrow{\simeq} |\mathcal{F}_{\mathcal{C}}(G)|.$$

Proof. Point (1): To see that $|\mathcal{C}\cap\mathcal{B}_p^e(G)| \xrightarrow{\simeq} |\mathcal{C}|$ is a G-homotopy equivalence, note that for an arbitrary $P \in \mathcal{C} \setminus \mathcal{B}_p^e(G)$, the space $|\mathcal{C}_{>P}| \star |\mathcal{C}_{<P}|$ is $N_G(P)$-contractible by Lemma A.6(1) and Lemma A.5, as $|\mathcal{S}_p(N_G(P)/P)|$ is since P is not p-radical. Hence the pushout square in Proposition A.2(1a) shows that we can remove the subgroups in $\mathcal{C} \setminus \mathcal{B}_p^e(G)$ one G-conjugacy class at a time (in some arbitrary order) without changing the G-homotopy type. That $|E\mathscr{O}_{\mathcal{C}\cap\mathcal{B}_p^e(G)}(G)| \xrightarrow{\simeq} |E\mathscr{O}_{\mathcal{C}}|$ is a G-homotopy equivalence follows from the analogous argument, but now using Proposition A.2(3a) instead. The homotopy equivalences on $\mathscr{T}$ and $\mathscr{O}$ follow from this and Lemma A.1((a)), ((b)), finishing the proof of (1). (For the last bit, one could also use Proposition A.2((2b), (3b)) directly.)

Point (2): We want to prove this by comparing $\mathcal{C}$ and $\mathcal{C} \cap \mathcal{A}_p^e(G)$ to the smallest collection $\bar{\mathcal{C}}$ containing $\mathcal{C}$ and closed under passage to non-trivial p-subgroups. We do this by adding G-conjugacy classes of subgroups to $\mathcal{C}$ in order of increasing size, and observe that this does not change the G-homotopy types of $|\mathcal{C}|$ and $E\mathbf{A}_{\mathcal{C}}$, and correspondingly for $\mathcal{C} \cap \mathcal{A}_p^e(G)$:

Let $\tilde{\mathcal{C}} = \mathcal{C} \cup [P]$, where $P \in \bar{\mathcal{C}} \setminus \mathcal{C}$ is minimal. Then Proposition A.2((1a), (4a)) shows that $|\mathcal{C}| \to |\tilde{\mathcal{C}}|$ and $|E\mathbf{A}_{\mathcal{C}}| \to |E\mathbf{A}_{\tilde{\mathcal{C}}}|$ are G-homotopy equivalences, once we see that $|\mathcal{C}_{<P}|$ is $N_G(P)$-equivariantly contractible (also using Lemma A.5). If $\mathcal{C}$ contains the trivial subgroup then $\mathcal{C}_{<P} = \mathcal{S}_p^e(G)_{<P}$ and the claim is obvious, so assume that this is not the case, i.e., $\mathcal{C}_{<P} = \mathcal{S}_p(G)_{<P}$. This is still $N_G(P)$-equivariantly contractible, as is seen using the standard contraction of Quillen

$$Q \leq Q\Phi(P) \geq \Phi(P), \tag{A.7}$$

where $\Phi(P)$ is the Frattini subgroup, generated by commutators and pth powers and $Q\Phi(P) < P$ since P is not elementary abelian.

By continuing to add subgroups this way we show that $|\mathcal{C}| \to |\bar{\mathcal{C}}|$ and $|E\mathbf{A}_{\mathcal{C}}| \to |E\mathbf{A}_{\bar{\mathcal{C}}}|$ are G-homotopy equivalences. The proof that $|\mathcal{C} \cap \mathcal{A}_p^e(G)| \to |\bar{\mathcal{C}}|$ and $|E\mathbf{A}_{\mathcal{C} \cap \mathcal{A}_p^e(G)}| \to |E\mathbf{A}_{\bar{\mathcal{C}}}|$ are G-homotopy equivalences is identical. The statements about $\mathscr{T}$ and $\mathcal{F}$ now follow from Lemma A.1((a)), ((c)). $\square$

Remark A.9. Theorem A.8(1) does not hold for $\mathcal{F}$, since $|\mathcal{F}_{\mathcal{B}_p(S)}(S)| \cong B(S/Z(S))$ (compare also Proposition 4.22). Theorem A.8(2) is not true for $\mathscr{O}$, since $|\mathscr{O}_{\mathcal{A}_2(C_4)}(C_4)| \cong B\mathbb{Z}/2$. See also [JM12] for results with $\bar{\mathcal{F}}$.

A.4. **Varying the collection without changing the low dimensional homotopy type.** In this subsection we continue to apply the formulas of §A.2 to remove subgroups, but now focusing on the first homology group and the fundamental group.

A.4.1. *Models for $H_1(\mathscr{O}_{\mathcal{C}}(G))$, $\pi_1(\mathscr{O}_{\mathcal{C}}(G))$ and $\pi_1(\mathscr{T}_{\mathcal{C}}(G))$.*

Theorem A.10 (Propagating fundamental groups, p-overgroup-closed version)**.** *Let $\mathcal{E}_p(G)$ denote the collection of p-subgroups P that are either Sylow, p-essential, or satisfy that $\psi_P : H_1(\mathscr{O}_p^*(N_G(P)/P)) \to H_1(N_G(P)/P)_{p'}$ is not an isomorphism. For any collection $\mathcal{C}$ of p-subgroups, closed under passage to p-overgroups in $\mathcal{E}_p(G)$, $H_1(\mathscr{O}_{\mathcal{C}\cap\mathcal{E}_p(G)}(G)) \xrightarrow{\cong} H_1(\mathscr{O}_{\mathcal{C}}(G))$.*

Likewise $\pi_1(\mathscr{O}_{\mathcal{C}\cap\mathcal{E}'_p(G)}(G)) \xrightarrow{\cong} \pi_1(\mathscr{O}_{\mathcal{C}}(G))$ and $\pi_1(\mathscr{T}_{\mathcal{C}\cap\mathcal{E}''_p(G)}(G)) \xrightarrow{\cong} \pi_1(\mathscr{T}_{\mathcal{C}}(G))$, where the collections $\mathcal{E}'_p(G)$ and $\mathcal{E}''_p(G)$ are defined analogously, replacing ψ_P by $\psi'_P \colon \pi_1(\mathscr{O}_p^*(N_G(P)/P)) \to (N_G(P)/P)_{p'}$ and $\psi''_P \colon \pi_1(\mathscr{T}_p^*(N_G(P)/P)) \to N_G(P)/P$, respectively. By construction $\mathcal{E}_p(G) \subseteq \mathcal{E}'_p(G) \subseteq \mathcal{E}''_p(G)$.

We remark that $\pi_1(\mathscr{T}_p^*(W)) \xrightarrow{\cong} W$ if and only if $\mathcal{S}_p(W)$ is simply connected by (1.12). The proof of Theorem A.10 needs a lemma, which can also be used to remove additional subgroups in it.

Lemma A.11. *Suppose that $\mathcal{C}$ is a collection of p-subgroups closed under passage to p-essential and Sylow p-overgroups, and let $\bar{\mathcal{C}} = \{P \in \mathcal{S}_p^e(G) |$ there exists $Q \leq P$ with $Q \in \mathcal{C}\}$. Then $\pi_1(\mathscr{T}_{\mathcal{C}}(G)) \xrightarrow{\cong} \pi_1(\mathscr{T}_{\bar{\mathcal{C}}}(G))$ and $\pi_1(\mathscr{O}_{\mathcal{C}}(G)) \xrightarrow{\cong} \pi_1(\mathscr{O}_{\bar{\mathcal{C}}}(G))$.*

Proof. Suppose $P \in \bar{\mathcal{C}} \setminus \mathcal{C}$. Then $\mathcal{C}_{>P}$ is connected by Lemma A.6(2) and $\mathcal{C}_{<P}$ is non-empty. Thus $|\mathcal{C}_{<P}| \star |\mathcal{C}_{>P}|$ and $|(E\mathscr{O}_{\mathcal{C}})_{<P}|/P \star |\mathcal{C}_{>P}|$ are simply connected by Lemma A.5. Proposition A.2((2b), (3b)), together with van Kampen's theorem, shows that we can add the conjugacy class of P to $\mathcal{C}$ without changing $\pi_1(\mathscr{T}_{\mathcal{C}}(G))$ or $\pi_1(\mathscr{O}_{\mathcal{C}}(G))$. Repeating this argument, now with $\mathcal{C} \cup [P]$ etc., allows us to add all the subgroups of $\bar{\mathcal{C}} \setminus \mathcal{C}$ as wanted. $\square$

Proof of Theorem A.10. By Lemma A.11 we can just as well compare groups relative to the two collections $\overline{\mathcal{C} \cap \mathcal{E}_p(G)} \subseteq \overline{\mathcal{C}}$. However all the subgroups in $\overline{\mathcal{C}} \setminus \overline{\mathcal{C} \cap \mathcal{E}_p(G)}$ can now be removed, in order of increasing size using Proposition A.4(4).

The statement for $\mathcal{E}'_p(G)$ follows by the same argument, but now appealing to Proposition A.4(3). The statement for $\mathcal{E}''_p(G)$ also follows from the same line of argument: As remarked after the statement of Theorem A.10, $P \notin \mathcal{E}''_p(G)$ means that $|\mathcal{S}_p(G)_{>P}|$ is simply connected. Hence Proposition A.2(2b), together with van Kampen's theorem, still allows us to remove subgroups in $\overline{\mathcal{C}} \setminus \overline{\mathcal{C} \cap \mathcal{E}_p(G)}$ without changing the fundamental group of $\mathscr{T}$. $\square$

Example A.12. For finite groups of Lie type in characteristic p, $\mathcal{E}_p(G) = \mathcal{E}'_p(G) = \{\text{Sylow}\} \cup \{p\text{-ess.}\}$ and are exactly the unipotent radicals of parabolic subgroups of rank at most one, whereas $\mathcal{E}''_p(G)$ identify with unipotent radicals of parabolic subgroups of rank at most 2 (see §6.2).

For $G = G_2(5)$ at $p = 3$, discussed in Proposition 6.3, the subgroup $\langle 3A \rangle$ is in $\mathcal{E}_p(G)$ but not p-essential. And for $\mathcal{C}$ the collection of non-trivial p-subgroups except $\langle 3A \rangle$, $\pi_1(\mathscr{O}_{\mathcal{C}}(G)) \cong C_2$ whereas $\pi_1(\mathscr{O}^*_p(G)) = 1$, so $\langle 3A \rangle$ is indeed necessary to control endotrivial modules.

Remark A.13. The collections from Theorem A.10 fit in a hierarchy involving higher homotopy groups. Note that $\pi_1(\mathscr{T}^*_p(N_G(P)/P)) \xrightarrow{\ncong} N_G(P)/P$ if and only if $|\mathcal{S}_p(N_G(P)/P)|$ is not simply connected. Say that P is "π_i-essential" if $|\mathcal{S}_p(N_G(P)/P)|$ is not i-connected. Then π_{-1}-essential is Sylow, π_0-essential is Sylow or p-essential, π_1-essential means in $\mathcal{E}''_p(G)$, and we have inclusions

$$\text{(A.8)} \quad \{\text{Sylow}\} \subseteq \{\text{Sylow}\} \cup \{p\text{-ess.}\} \subseteq \mathcal{E}_p(G) \subseteq \mathcal{E}'_p(G) \subseteq \mathcal{E}''_p(G) = \{\pi_1\text{-ess.}\} \subseteq \{\pi_2\text{-ess.}\} \cdots \subseteq \mathcal{B}^e_p(G)$$

It follows from Remark A.7 that the π_i-essential subgroups are the ones needed to describe the i-truncation of the homotopy type of $|\mathcal{S}_p(G)|$, and hence the i-truncation of $|\mathscr{T}^*_p(G)|$ and $|\mathscr{O}^*_p(G)|$. As an n-connected space of dimension n is contractible (see e.g., [Hat02, Exc. 4.12]) the filtration in (A.8) is finite, and furthermore Quillen's famous conjecture [Qui78, Conj. 2.9] predicts that p-radical implies π_n-essential, for n, say, the dimension of $\mathcal{S}_p(G)$ (see e.g., [AS93] for known cases). As mentioned in §6.2 and §6.4, several open questions about simply connectivity of $\mathcal{S}_p(G)$ may be shadows of stronger statements about Cohen–Macaulayness, also justifying an interest in π_i-essential subgroups.

A.4.2. *Models for* $\pi_1(\mathcal{F}_{\mathcal{C}}(G))$. A non–Sylow p-subgroup P is called $\mathcal{F}$*-essential* if $W_0 PC_G(P)/P$ is a proper subgroup of $W = N_G(P)/P$ (with W_0 as in (1.3)). It is obviously a subcollection of the p-essential subgroups and was introduced in [Pui76] (as "C-essential"). The $\mathcal{F}$-essential subgroups can also be described as the p-centric subgroups such that $N_G(P)/PC_G(P)$ contains a strongly p-embedded subgroup (see Lemma A.17). (Beware that some fusion literature such as [AKO11, Def. I.3.2], but not [Pui06, §5], takes "fully $\mathcal{F}$-normalized" in S as part of the definition, giving a smaller, but non-conjugacy invariant set, as it refers to a fixed Sylow p-subgroup S.)

Proposition A.14. *Suppose that $\mathcal{C}$ is closed under passage to $\mathcal{F}$-essential and Sylow p-overgroups, and that $\mathcal{C}'$ is a collection obtained from $\mathcal{C}$ by removing conjugacy classes of p-subgroups, which are neither Sylow p-subgroups, $\mathcal{F}$-essential, or minimal in $\mathcal{C}$. Then $\pi_1(\mathcal{F}_{\mathcal{C}'}(G)) \xrightarrow{\cong} \pi_1(\mathcal{F}_{\mathcal{C}}(G))$.*

Proof. It is enough to show that for both $\mathcal{F}_{\mathcal{C}}(G)$ and $\mathcal{F}_{\mathcal{C}'}(G)$ we get the same fundamental group as $\mathcal{F}_{\bar{\mathcal{C}}}(G)$, for $\bar{\mathcal{C}}$ the minimal collection containing $\mathcal{C}$ and closed under all p-overgroups. And for that, it is enough to show that the fundamental group of $\mathcal{F}_{\bar{\mathcal{C}}}(G)$ does not change when removing a p-subgroup P from $\bar{\mathcal{C}}$ which is neither minimal nor $\mathcal{F}$-essential nor a Sylow p-subgroup (by removing subgroups in order of increasing size). This will be a consequence of Proposition A.2(4b), if we see that $|(E\mathbf{A}_{\bar{\mathcal{C}}})_{>P}|/C_G(P)$ is connected. For this note, as in (A.2), that $|(E\mathbf{A}_{\bar{\mathcal{C}}})_{>P}| \to |\bar{\mathcal{C}}_{>P}|$ is an $N_G(P)$-equivariant map, which is a bijection on components, and hence $|(E\mathbf{A}_{\bar{\mathcal{C}}})_{>P}|/C_G(P) \to |\bar{\mathcal{C}}_{>P}|/C_G(P)$ is also a bijection on components. By assumption $|\bar{\mathcal{C}}_{>P}| = |\mathcal{S}_p(G)_{>P}|$ which is $N_G(P)$-homotopy equivalent to $|\mathcal{S}_p(N_G(P)/P)|$ by Lemma A.6. By (1.10), $\pi_0(|\mathcal{S}_p(N_G(P)/P)|) \cong W/W_0$, in the notation from above the proposition, and $W = W_0C_G(P)P/P$, as P is non-$\mathcal{F}$-essential. Putting this together we see that $|(E\mathbf{A}_{\bar{\mathcal{C}}})_{>P}|/C_G(P)$ is connected as wanted. □

A.4.3. *Fundamental groups for subgroup-closed collections.* We end with the dual case:

Theorem A.15 (Propagating fundamental groups, subgroup-closed version)**.** *Let $\mathcal{C}$ be a collection of p-subgroups closed under passage to non-trivial elementary abelian subgroups. Then $\pi_1(\mathscr{T}_{\mathcal{C}'}(G)) \xrightarrow{\cong} \pi_1(\mathscr{T}_{\mathcal{C}}(G))$, $\pi_1(\mathscr{O}_{\mathcal{C}'}(G))_{p'} \xrightarrow{\cong} \pi_1(\mathscr{O}_{\mathcal{C}}(G))_{p'}$, and $\pi_1(\mathcal{F}_{\mathcal{C}'}(G)) \xrightarrow{\cong} \pi_1(\mathcal{F}_{\mathcal{C}}(G))$ for $\mathcal{C}'$ the subgroups in $\mathcal{C}$ which are either maximal or elementary abelian of rank at most two. The same statements hold taking $\mathcal{C}'$ the elementary abelian subgroups of $\mathcal{C}$ of rank at most* 3.

Proof. First, the claim for $\mathscr{O}$ follows from $\mathscr{T}$ by Proposition 4.5. Second, the claims are obvious if the trivial subgroup is in $\mathcal{C}$, so we can assume that this is not the case. Now to prove the claim for $\mathscr{T}$ and $\mathcal{F}$, it is, as usual, enough to compare $\mathcal{C}$ and $\mathcal{C}'$ to a collection $\bar{\mathcal{C}}$ obtained from $\mathcal{C}$ by adding non-elementary abelian p-groups, to make it closed under passage to all non-trivial p-subgroups. By removing subgroups from $\bar{\mathcal{C}}$ in order of decreasing size, we just have to see that the fundamental groups do not change by removing a non-trivial subgroup P which is not elementary abelian of rank 1 or 2, and if elementary abelian of rank 3 not maximal in $\mathcal{C}$. However, this all follows from Proposition A.2((2b), (4b)) and van Kampen's theorem: If P is not elementary abelian then $\bar{\mathcal{C}}_{<P} = \mathcal{S}_p(G)_{<P}$ is contractible via the standard contraction (A.7). If P is elementary abelian of rank at least 4, then $\mathcal{S}_p(G)_{<P}$ is simply connected as it is homotopy equivalent to a simply connected Tits building. And if P is elementary abelian of rank 3 and not maximal in $\mathcal{C}$ then $\bar{\mathcal{C}}_{<P}$ is connected and is joined with a non-empty space giving something simply connected (cf. Lemma A.5). □

A.5. **Essential and strongly p-embedded subgroups: The connected components of $\mathcal{C}$.** We now describe the set of connected components of $\mathcal{C}$ in more detail, generalizing the description for $\mathcal{S}_p(G)$ due to Quillen [Qui78, §5], [Gro02, Prop. 5.8], explained in (1.10)—we need this in §4 to get the results in their optimal form, and it also has group theoretic significance; see Remark A.19.

Fix a Sylow p-subgroup S and set $G_{0,\mathcal{C}} = \langle N_G(Q) | Q \leq S, Q \in \mathcal{C}\rangle$ and $\mathcal{C}_0 = \{Q \in \mathcal{C} | Q \leq G_{0,\mathcal{C}}\}$ as in (4.1).

Proposition A.16 (Connected components of $\mathcal{C}$)**.** *Let $\mathcal{C}$ be a collection of p-subgroups in G, closed under passage to Sylow and p-essential overgroups. Then $G \times_{G_{0,\mathcal{C}}} |\mathcal{C}_0| \xrightarrow{\cong} |\mathcal{C}|$ is a G-equivariant isomorphism of simplicial sets, via $(g, P_0 \leq \cdots \leq P_n) \mapsto ({}^gP_0 \leq \cdots \leq {}^gP_n)$, and $|\mathcal{C}_0|$ is connected. On the set of components*

$$G/G_{0,\mathcal{C}} \xrightarrow{\cong} \pi_0(|\mathcal{C}|) \xleftarrow{\cong} \pi_0(|\mathcal{C}'|) \xleftarrow{\cong} G/G_{0,\mathcal{C}'}$$

with $\mathcal{C}'$ the Sylow or p-essential subgroups of $\mathcal{C}$.

Furthermore for a subgroup $H \leq G$ the following conditions are equivalent:

(1) *$G_{0,\mathcal{C}}$ is subconjugate to H.*
(2) *$p \nmid |G : H|$ and if, for any $g \in G$, $H \cap {}^gH$ contains an element of $\mathcal{C}$, then $g \in H$.*

In particular $G_{0,\mathcal{C}}$ is characterized as a minimal subgroup satisfying (2)*, containing S.*

Proof. Let us first show that the two conditions are equivalent. Suppose that (1) is satisfied. We can without loss of generality assume that $G_{0,\mathcal{C}} \leq H$, since the condition on H is conjugation invariant. Suppose now that $Q \leq H \cap {}^gH$, and $Q \in \mathcal{C}$. We want to show that $g \in H$. By Sylow's theorem in H, using that $p \nmid |G : H|$ we can upon changing g by an element in H assume that $Q \leq S$. By assumption ${}^{x^{-1}}Q \leq H$, and hence we can, again by Sylow's theorem, find $h \in H$ so that ${}^{hx^{-1}}Q \leq S$. Alperin's fusion theorem, in the version of Goldschmidt–Miyamoto–Puig [Miy77, Cor. 1] (see also [Gro02, §10]), now says that we can find p-essential subgroups $P_1, \ldots, P_r$, and elements $g_i \in N_G(P_i)$, and $n \in N_G(S)$, with $Q \leq P_1$ and ${}^{g_i \cdots g_1}Q \leq P_{i+1} \leq S$ for $i \leq r-1$, such that $hx^{-1} = ng_r \cdots g_1$. However since the right-hand side is in $G_{0,\mathcal{C}}$ by assumption, this shows that $x \in H$ as wanted, so (2) holds.

Now suppose that H is a subgroup satisfying (2). By Sylow's theorem, we can change H up to conjugation so that $S \leq H$. We want to show that $G_{0,\mathcal{C}} \leq H$. In fact we will prove the stronger statement that G_S, the stabilizer of $[S] \in \pi_0(\mathcal{C})$, is contained in H. As it is clear that $G_{0,\mathcal{C}} \leq G_S$, this will show (2) $\Rightarrow$ (1), and furthermore establish that $G_S = G_{0,\mathcal{C}}$ as claimed in the first part of the theorem. Hence suppose that $g \in G_S$, so that ${}^gS$ and S lie in the same component. By definition there exists a sequence of Sylow p-subgroups $S_0, \ldots, S_r$, so that $S = S_0$, $S_r = {}^gS$ such that $S_i \cap S_{i+1}$ contains an element of $\mathcal{C}$. Choose $g_i \in G$ such that ${}^{g_i}S_i = S_{i+1}$, so that ${}^{g_{r-1} \cdots g_0}S = {}^gS$. If $S_i \leq H$, then $S_{i+1} \leq H$ and $g_i \in H$ by our assumption, so by induction we conclude that $g_{r-1} \cdots g_0 \in H$, and hence also $g \in H$ since the two elements differ by an element of $N_G(S) \leq H$.

That $G \times_{G_{0,\mathcal{C}}} |\mathcal{C}_0| \xrightarrow{\cong} |\mathcal{C}|$ now follows: The map is surjective since for $P_0 \leq \cdots \leq P_n \in |\mathcal{C}|_n$ we can, by Sylow's theorem, find $g \in G$ so ${}^gP_n \leq S$, and hence $({}^gP_0 \leq \cdots \leq {}^gP_n) \in |\mathcal{C}_0|_n$. Furthermore if $({}^gP_0 \leq \cdots \leq {}^gP_n) = ({}^{g'}P'_0 \leq \cdots \leq {}^{g'}P'_n)$ then ${}^{g'^{-1}g}P_0 = P'_0 \leq G_{0,\mathcal{C}}$, and likewise $P_0 \leq G_{0,\mathcal{C}}$ by definition, i.e., P_0 and P'_0 lie in the same component so $g'^{-1}g \in G_S = G_{0,\mathcal{C}}$, by the first part. In other words $gG_{0,\mathcal{C}} = g'G_{0,\mathcal{C}}$ as wanted.

The last claim we need to justify is that $\pi_0(|\mathcal{C}'|) \xrightarrow{\cong} \pi_0(|\mathcal{C}|)$ which follows from Lemma A.6 with $P = e$, except the degenerate case where $e \in \mathcal{C}$. But here it is also true: this is clear if also $e \in \mathcal{C}'$ since both spaces are contractible, and if $e \notin \mathcal{C}'$

then Proposition A.2(1a) still says that we can add e to $\mathcal{C}'$ without changing the number of components, which hence has to be one. □

We now check that the two definitions of $\mathcal{F}$-essential from §A.4.2 agree (see also [Pui76, Cor. III.2]).

Lemma A.17. *$W_0PC_G(P)/P$ is a proper subgroup of $W = N_G(P)/P$ if and only if P is p-centric and $\mathcal{S}_p(N_G(P)/PC_G(P))$ is disconnected.*

Proof. Let H denote the preimage of $W_0PC_G(P)/P$ in $N_G(P)$. As $P \le PC_G(P) \le H \cap H^g$ for all $g \in N_G(P)$, Proposition A.16 (applied to $\mathcal{C} = \mathcal{S}_p(N_G(P)/P)$ in $N_G(P)/P$) shows that if $H \neq N_G(P)$, then $|PC_G(P) : P|$ is prime to p, as $|P : H \cap H^g|$ is, so P is p-centric.

If P is p-centric then $C_G(P) \cong Z(P) \times R$, for R a p'-group. Note that $|\mathcal{S}_p(N_G(P)/P)|/R \cong |\mathcal{S}_p(N_G(P)/PR)|$ (by [Gro02, Prop. 5.7]), a space with set of components W/W_0R. Hence W_0R is a proper subgroup of W if and only if $|\mathcal{S}_p(N_G(P)/PR)|$ is disconnected, showing the lemma. □

Remark A.18 (Groups with a strongly p-embedded subgroup). To use the theorems in this paper, it is useful to know when $G_0 = \langle N_G(Q) | 1 < Q \le S\rangle$ is proper in G, i.e., in group theoretic language, when G contains a strongly p-embedded subgroup. The answer to this question forms an important chapter in the classification of finite simple groups. The following is a theorem of Bender when $p = 2$ [Ben71], and only known as a consequence of the classification when p is odd: Either $\mathrm{rk}_p(G) = 1$ and $G_0 = N_G(\Omega_p(Z(S)))$, with $G_0 < G$ exactly when $O_p(G) = 1$, or $\mathrm{rk}_p(G) \ge 2$, $O_{p'}(G) \le G_0$, $\bar{G} = G/O_{p'}(G)$ has a unique minimal normal non-abelian simple subgroup $\bar{K} = F^*(\bar{G})$, and $\bar{G}/\bar{K} \le \mathrm{Out}(\bar{K})$. The group $\bar{K}$ is either a finite group of Lie type of rank 1 (possibly twisted) in defining characteristic, A_{2p} ($p \ge 3$), $(L_3(4), 3)$, $(M_{11}, 3)$, $(Fi_{22}, 5)$, $(Mc, 5)$, $(F_4(2)', 5)$, or $(J_4, 11)$. See [GLS98, 7.6.1, 7.6.2] and also [Qui78, §5], [Asc93, (6.2)] for more details.

Remark A.19. As noticed, when taking $\mathcal{C} = \mathcal{S}_p(G)$, Proposition A.16 is a strong version of Quillen's [Qui78, Prop. 5.2]. When taking $\mathcal{C}$ to be the collection of p-subgroups of p-rank at least k, $G_{0,\mathcal{C}} = \Gamma_{k,S}(G)$, the k-generated p-core from finite group theory, and one also makes geometric and extends a standard characterization of it [Asc00, (46.4)] (see also [GLS96, Sec. 22], [ALSS11, Sec. B.4]). Groups with proper 2-generated 2-core were famously classified by Aschbacher [Asc74].

Remark A.20. By examining the list in Remark A.18 one sees that very often when $G_0 < G$, $H_1(G_0)_{p'} \twoheadrightarrow H_1(G)_{p'}$ is not injective, providing exotic Sylow-trivial modules via (1.6) (take for instance $G = \mathrm{SL}_2(\mathbb{F}_{p^r})$ with $p^r \neq 2$). But it may also be injective: Let $p = 2$ and consider $K = \mathrm{SL}_2(\mathbb{F}_{2^r})$, $r > 1$ odd, and let C_r act on K via field automorphisms. Set $G = K \rtimes C_r$. Then K_0 consists of upper triangular matrices of determinant one, and $G_0 = H_0 \rtimes C_r$. Hence $H_1(G_0)_{2'} \cong H_1(G)_{2'} \cong C_r$. Other examples may be constructed along these lines, though perhaps limited to small primes. (We are grateful to Ron Solomon for consultations on these points.)

A.6. **Proof of Proposition A.2.** We now prove Proposition A.2, via general observations about links in preordered sets, an abstraction of observations in [Dwy98, GS06]. Let $\mathcal{X}$ be a preordered set, i.e., a small category with at most one morphism between any two objects. Note that our spaces $E\mathscr{O}_{\mathcal{C}}$, etc., are all examples of such. For $x \in \mathcal{X}$, let $\bar{x}$ denote the full subcategory of $\mathcal{X}$ on objects isomorphic to x, let

$\mathcal{X}_{<\bar{x}}$ denote the full subcategory of $\mathcal{X}$ on elements smaller than and not isomorphic to x, and define $\mathcal{X}_{>\bar{x}}$ similarly. For preordered sets $\mathcal{X}$ and $\mathcal{Y}$ the join $\mathcal{X} \star \mathcal{Y}$ is the preordered set obtained from the disjoint union of $\mathcal{X}$ and $\mathcal{Y}$ by adding a unique morphism from each object in $\mathcal{X}$ to each object in $\mathcal{Y}$. This has the property that $|\mathcal{X} \star \mathcal{Y}| \cong |\mathcal{X}| \star |\mathcal{Y}|$. The star $\mathbf{star}_{\mathcal{X}}(\bar{x})$ is the full subcategory of $\mathcal{X}$ on objects which admit a morphism to or from x, and the link $\mathbf{link}_{\mathcal{X}}(\bar{x})$ is the full subcategory on objects that admit a non-isomorphism to or from x. Note that

$$\mathbf{star}_{\mathcal{X}}(\bar{x}) = \mathcal{X}_{<\bar{x}} \star \bar{x} \star \mathcal{X}_{>\bar{x}} \text{ and } \mathbf{link}_{\mathcal{X}}(\bar{x}) = \mathcal{X}_{<\bar{x}} \star \mathcal{X}_{>\bar{x}}$$

If $\mathcal{X}$ has a G-action, these are all $G_{\bar{x}}$-subcategories, where $G_{\bar{x}}$ is the stabilizer of $\bar{x}$ as a set, and furthermore $\mathbf{star}_{\mathcal{X}}(\bar{x})$ is G_x-contractible to x (but generally not $G_{\bar{x}}$-contractible).

Proposition A.21. *Suppose $\mathcal{X}$ is a preordered set equipped with a G-action such that isomorphic objects are G-conjugate, and let $\mathcal{X}'$ denote the subcategory of $\mathcal{X}$ obtained by removing all G-conjugates of an element x.*

(1) *There is a pushout square of G-spaces, which is also a homotopy pushout square of G-spaces:*

$$\begin{tikzcd} G \times_{G_{\bar{x}}} |\mathbf{link}_{\mathcal{X}}(\bar{x})| \arrow[r] \arrow[d] & |\mathcal{X}'| \arrow[d] \\ G \times_{G_{\bar{x}}} |\mathbf{star}_{\mathcal{X}}(\bar{x})| \arrow[r] & |\mathcal{X}| \end{tikzcd}$$

where $\bar{x}$ denotes the subcategory of elements isomorphic to x and $G_{\bar{x}}$ its stabilizer as a set.

(2) *Assume in addition that the stabilizer G_x of any point $x \in \mathcal{X}$ is a normal subgroup in the stabilizer of its isomorphism class $\bar{x}$. Then the square*

$$\begin{tikzcd} G \times_{G_{\bar{x}}} (E(G_{\bar{x}}/G_x) \times |\mathbf{link}_{\mathcal{X}}(\bar{x})|) \arrow[r] \arrow[d] & |\mathcal{X}'| \arrow[d] \\ G \times_{G_{\bar{x}}} (E(G_{\bar{x}}/G_x) \times |\mathbf{star}_{\mathcal{X}}(\bar{x})|) \arrow[r] & |\mathcal{X}| \end{tikzcd}$$

obtained by collapsing $E(G_{\bar{x}}/G_x)$ and continuing as in (1) is again a pushout and homotopy pushout square of G-spaces, and remains a homotopy pushout of G-spaces after collapsing $|\mathbf{star}_{\mathcal{X}}(\bar{x})|$.

In particular, under these assumptions:

(3) *On G-orbits there is a homotopy pushout square*

$$\begin{tikzcd} ((|\mathcal{X}_{<\bar{x}}| \star |\mathcal{X}_{>\bar{x}}|)/G_x)_{hG_{\bar{x}}/G_x} \arrow[r] \arrow[d] & |\mathcal{X}'|/G \arrow[d] \\ B(G_{\bar{x}}/G_x) \arrow[r] & |\mathcal{X}|/G \end{tikzcd}$$

Proof. For (1), notice that it is a pushout of G-spaces by the fact that $|\mathcal{X}'|$ and the image of $G \times_{G_{\bar{x}}} |\mathbf{star}_{\mathcal{X}}(\bar{x})|$ cover $|\mathcal{X}|$, and the part of $G \times_{G_{\bar{x}}} |\mathbf{star}_{\mathcal{X}}(\bar{x})|$ that maps to $\mathcal{X}'$ is the subspace $G \times_{G_{\bar{x}}} |\mathbf{link}_{\mathcal{X}}(\bar{x})|$, just by the definitions, and that isomorphic elements are G-conjugate. It is homotopy pushout of G-spaces since $G \times_{G_{\bar{x}}} |\mathbf{link}_{\mathcal{X}}(\bar{x})| \to G \times_{G_{\bar{x}}} |\mathbf{star}_{\mathcal{X}}(\bar{x})|$ is an injective map of G-spaces.

For (2) it is enough to see that the projection square

$$\begin{tikzcd} G \times_{G_{\bar{x}}} E(G_{\bar{x}}/G_x) \times |\,\mathbf{link}_{\mathcal{X}}(\bar{x})| \arrow[r] \arrow[d] & G \times_{G_{\bar{x}}} \times |\,\mathbf{link}_{\mathcal{X}}(\bar{x})| \arrow[d] \\ G \times_{G_{\bar{x}}} E(G_{\bar{x}}/G_x) \times |\,\mathbf{star}_{\mathcal{X}}(\bar{x})| \arrow[r] & G \times_{G_{\bar{x}}} \times |\,\mathbf{star}_{\mathcal{X}}(\bar{x})| \end{tikzcd}$$

is a pushout and homotopy pushout of G-spaces, by transitivity of pushouts and homotopy pushouts. To see this it is again enough to see that

$$\begin{tikzcd} E(G_{\bar{x}}/G_x) \times |\,\mathbf{link}_{\mathcal{X}}(\bar{x})| \arrow[r] \arrow[d] & |\,\mathbf{link}_{\mathcal{X}}(\bar{x})| \arrow[d] \\ E(G_{\bar{x}}/G_x) \times |\,\mathbf{star}_{\mathcal{X}}(\bar{x})| \arrow[r] & |\,\mathbf{star}_{\mathcal{X}}(\bar{x})| \end{tikzcd}$$

is a pushout and homotopy pushout of $G_{\bar{x}}$-spaces, which we do by checking on all fixed-points for $H \le G_{\bar{x}}$. For $H \le G_x$ $E(G_{\bar{x}}/G_x)^H$ is contractible and the claim is clear. (Note, we use that $G_{\bar{x}}/G_x$ is a group, not just a coset, so the isotropy does not depend on the chosen point $x \in \bar{x}$.) For $H \not\le G_x$, the spaces on the left are empty, and $\mathbf{link}_{\mathcal{X}}(\bar{x})^H = \mathbf{star}_{\mathcal{X}}(\bar{x})^H$, so it is also a pushout and homotopy pushout square in that case. For the purposes of having a *homotopy* pushout square of G-spaces, we can replace $\mathbf{star}_{\mathcal{X}}(\bar{x})$ by a point, since $\mathbf{star}_{\mathcal{X}}(\bar{x})$ is G_x-contractible, and hence $E(G_{\bar{x}}/G_x) \times |\,\mathbf{star}_{\mathcal{X}}(\bar{x})| \to E(G_{\bar{x}}/G_x)$ a $G_{\bar{x}}$-homotopy equivalence.

For (3), note that since $G \times_{G_{\bar{x}}} E(G_{\bar{x}}/G_x) \times |\,\mathbf{link}_{\mathcal{X}}(\bar{x})| \to G \times_{G_{\bar{x}}} E(G_{\bar{x}}/G_x) \times |\,\mathbf{star}_{\mathcal{X}}(\bar{x})|$ is a cofibration of G-spaces, passing to G-orbits in (2) produces the homotopy pushout square in (3). □

Remark A.22. In Proposition A.21((2), (3)) we can replace $|\,\mathbf{link}_{\mathcal{X}}(\bar{x})| = |\mathcal{X}_{<\bar{x}}| \star |\mathcal{X}_{>\bar{x}}|$ by a G_x-equivalent space through a $G_{\bar{x}}$-equivariant map, without changing the conclusion, as any group element in $G_{\bar{x}} \setminus G_x$ acts freely on $E(G_{\bar{x}}/G_x)$.

Proof of Proposition A.2. This will be applications of Proposition A.21: (1a) is Proposition A.21(2) with $\mathcal{X} = \mathcal{C}$, noting that in this case $\mathcal{X}$ is a poset and $G_{\bar{x}} = G_x$. (1b) is Proposition A.21(3). For (2a) cross the diagram from (1a) with EG, noting that EG as an N-space is equivalent to EN. For (2b) take G-orbits and note that $|\mathcal{C}|_{hG}$ identifies with $|\mathscr{T}_{\mathcal{C}}|$ by Lemma 2.3. (Alternatively, more directly take $\mathcal{X} = E\mathscr{T}_{\mathcal{C}} = e \downarrow \iota$, the undercategory for $\iota\colon \mathscr{T}_{\mathcal{C}} \to \mathscr{T}_{\mathcal{C}\cup\{e\}}$.) For (3a) and (3b) take $\mathcal{X} = E\mathscr{O}_{\mathcal{C}}$, so for $x = (G/Q, y)$, $G_x = G_y$, and $G_{\bar{x}} = N_G(G_y)$. The diagrams now follow from Proposition A.21((2)–(3)) using that we have an $N_G(P)$-equivariant map $|(E\mathscr{O}_{\mathcal{C}})_{>P}| \to |\mathcal{C}_{>P}|$ which is a P-homotopy equivalence by (A.3), so we can replace $(E\mathscr{O}_{\mathcal{C}})_{>P}$ by $\mathcal{C}_{>P}$ by Remark A.22. Finally for (4a)-(4b) take $\mathcal{X} = E\mathbf{A}_{\mathcal{C}}$ and again use Proposition A.21((2)–(3)) (where $G_x = C_G(i(Q))$ and $G_{\bar{x}} = N_G(i(Q))$, for $x = (i\colon Q \to G)$), together with the simplification that $|(E\mathbf{A}_{\mathcal{C}})_{<P}| \to |\mathcal{C}_{<P}|$ is a $C_G(P)$-homotopy equivalence (which holds as $|(E\mathbf{A}_{\mathcal{C}})_{<P})|^H \xrightarrow{\simeq} |\mathcal{C}_{<P}|$ by (A.3) when $H \le C_G(P)$, or equivalently $P \le C_G(H)$). □

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