Smith-Treumann Theory

Jesper Grodal

The goal of my talk at the Arbeitsgemeinschaft was to give an account of Smith theory for sheaves, as initiated by D. Treumann, with literature [Tre19], [RW19] and [Wil19]. In the following couple of pages, I'll summarize some highlights from this theory. I've tried to kept it as "generic" and non-technical as possible, to highlight structural features, rather than particular technical issues of a given model, perhaps at the slight expense of precision. The version I present here also differ somewhat from the sources. I'll remark on this as we go along, and at the end.

Classical Smith theory, named after P. A. Smith, whose works date from the 1930's, is a collection of results stating relations between mod ℓ invariants of a space X and those of the fixed-points X^{μ_ℓ}, where μ_ℓ is a finite group of prime order ℓ acting on X. (There are related characteristic zero results where μ_ℓ is replaced by the circle $T = S^1$, or, for the daring, its ℓ-torsion points μ_{ℓ^∞}.)

Smith's ideas have been very influential in the intervening 80+ years. It was recast in the 1960s and 1970s in the work of A. Borel, D. Quillen, and others, e.g., through the localization theorem. This again fed into fundamental conjectures in homotopy theory by G. Segal and D. Sullivan relating μ_ℓ–fixed-points and homotopy μ_ℓ–fixed-points, proved in the mid 80s by G. Carlsson and H. Miller, after 15 years of intense interest.

Let X be, say, a finite T-CW complex, for T the circle S^1, and k a field of characteristic ℓ. Write $H_T^*(-;k)$ for Borel equivariant cohomology with coefficients in k. Recall that a version of the classical localization theorem says that the restriction map

$$H_T^*(X;k) \to H_T^*(X^{\mu_\ell};k),$$

which is a map of $H_T^*(pt;k)$–algebras via the map p to a point, becomes an isomorphism after inverting the degree 2 class $u \in k[u] \cong H_T^*(pt;k)$, i.e.,

$$H_T^*(X;k)[u^{-1}] \xrightarrow{\cong} H_T^*(X^{\mu_\ell};k)[u^{-1}] \cong H^*(X^{\mu_\ell};k) \otimes_k k[u,u^{-1}].$$

Many statements of Smith theory follow from this formula. For example one sees that if X is mod ℓ acyclic, i.e., $H^*(X;k)$ one-dimensional over k, then the same holds for X^{μ_ℓ}, as $H_T^*(X;k) \xleftarrow{\cong} H_T^*(pt;k) \cong k[u]$. Similarly one sees that the μ_ℓ–fixed-points of a mod ℓ homology sphere is again a mod ℓ homology sphere or the empty set ("the (-1)-sphere"), by considering the pair (CX, X), for CX the cone on X.

The Treumann version of the localization theorem, as further refined by Riche-Williamson, is a similar result for the whole T–equivariant bounded derived category of sheaves $D_T^b(X;k)$, or alternatively T–equivariant constructible sheaves. Here X can either be as before, or a real subanalytic variety (the setup of Treumann [Tre19]) or a finite type T-scheme over a field $\mathbb{F}$ of characteristic $p \neq \ell$, with $T = \mathbb{G}_m$, and equipped with the étale topology (the setup of [RW19]).

For this, consider the restriction map

$$D^b_T(X;k) \xrightarrow{i^*} D^b_T(X^{\mu_\ell};k)$$

between triangulated (or ∞–) categories. Note that this is a morphism under $D^b_T(pt;k)$ via p^*, where p is the map to a point. We can similarly consider a morphism $(u : k \to k[2]) \in D^b_T(pt;k)$ representing the class u from before. Inverting u in this setting translates into "killing the cone cofib(u)", i.e., forming the Verdier quotient with respect to the thick tensor ideal generated by cofib(u) in $D^b_T(X;k)$ and $D^b_T(X^{\mu_\ell};k)$ respectively, via p^*. Let us denote forming this quotient by $(-)[u^{-1}]$, and abbreviate "thick tensor ideal" to just ideal. In this formulation the theorem becomes:

Theorem 1 ("The localization theorem for sheaves")**.** *Restriction induces an equivalence of triangulated (or ∞–) categories*

$$i^* : D^b_T(X;k)[u^{-1}] \xrightarrow{\simeq} D^b_T(X^{\mu_\ell};k)[u^{-1}].$$

Likewise $i^!$ also descends to the quotient, where it agrees with i^*.

Note that the quotient category is 2-periodic, with periodicity induced by multiplication by u (an element in homological grading -2). In particular the quotient is not the bounded derived category of any ordinary ring. (It is one over the Tate fixed-point spectrum k^{tT}, though.) The quotient categories, should be thought of as a "Tate construction" applied to the original categories (and this can indeed be made precise!). E.g.,

$$\mathrm{Ext}^*_{D^b_T(pt;k)[u^{-1}]}(k,k) = k[u,u^{-1}]$$

It can also be described as a "singularity category" or "stable module category", of dualizable objects modulo compact objects, a sort of "category at infinity".

One of the wonderful things about the isomorphism in Theorem 1 is that it is as natural in X as can be, in the sense that it commutes with all the usual functors we consider:

Theorem 2 ("Localization commutes with all operations")**.** *Let $f : X \to Y$ be a T–equivariant morphism in one of the categories from earlier. Then f_*, f^*, $f_!$, $f^!$ and Verdier duality $\mathbb{D}$ preserve the property of being in the ideal generated by* cofib(u)*, so also induce functors after inverting u, and the usual adjunctions continue to hold in the quotient.*

Furthermore for F either f_*, f^*, $f_!$ or $f^!$, the following diagram commute

$$\begin{tikzcd}
D^b_T(X;k)[u^{-1}] \arrow[r, dash, "F"] \arrow[d, "i^*"] & D^b_T(Y;k)[u^{-1}] \arrow[d, "i^*"] \\
D^b_T(X^{\mu_\ell};k)[u^{-1}] \arrow[r, dash, "F"] & D^b_T(Y^{\mu_\ell};k)[u^{-1}]
\end{tikzcd}$$

(with F going in the appropriate direction), and likewise for $F = \mathbb{D}$ with $Y = X$.

Before sketching the proof of these two theorems, let us make a note about the ideal generated by $\mathrm{cofib}(u)$. Upon restriction to $k\mu_\ell$, the chain complex $\mathrm{cofib}(u)$ can be modelled by

$$\cdots \to 0 \to k\mu_\ell \xrightarrow{1-g} k\mu_\ell \to 0 \to \cdots ,$$

non-zero in degree 1 and 0, and g a generator of μ_ℓ. In particular, the stalk of $p^*(\mathrm{cofib}(u))$ at $x \in X^{\mu_\ell}$ will be a perfect complex of $k\mu_\ell$–modules. As this is perserved under tensor products and summands, the same is true for any sheaf in the ideal generated $\mathrm{cofib}(u)$. Theorems 1 and 2 thus imply similar statements where we quotient out by the, a priori larger, subcategory given by all sheaves $\mathcal{F}$ where the stalk at $x \in X^{\mu_\ell}$ is a perfect complex of $k\mu_\ell$–modules. This was what was considered in [Tre19] and [RW19]. $D^b_T(X^{\mu_\ell}; k)$ modulo this, a priori larger, subcategory is called $\mathrm{Perf}(X^{\mu_\ell}; \mathcal{T})$ in [Tre19] and $\mathrm{Sm}(X^{\mu_\ell}, k)$ in [RW19]. Under suitable niceness assumptions on X the two ideals coincide, but the best-possible result in this direction is not entirely clear to us.

We will now sketch the proofs of the two theorems.

Sketch of proof of Theorem 1. We first prove the claim when $X^{\mu_\ell} = \emptyset$, i.e., if the T–action is "free at ℓ". Here the statement becomes that any sheaf $\mathcal{F} \in D^b_T(X; k)$ lies in the ideal generated by $\mathrm{cofib}(u)$. For this, one first observes that in this case $D^b_T(X; k) \cong D^b(X/T; k)$, as the action is "free at ℓ" (here one appeals to a property equivariant derived category in the relevant setting). In particular $\mathrm{Ext}^*_{D^b_T(X;k)}(\mathcal{F}, \mathcal{F})$ is concentrated in only finitely many dimensions (a property of X/T). Hence, for some n, u^n maps to zero under the natural map

$$\mathrm{Ext}^*_{D^b_T(pt;k)}(k, k) \xrightarrow{p^*} \mathrm{Ext}^*_{D^b_T(X;k)}(\mathcal{F}, \mathcal{F})$$

Said in other words, the map $\mathcal{F} \xrightarrow{u^n} \mathcal{F}[2n]$ is zero, and hence $\mathrm{cofib}(u^n) \otimes \mathcal{F} \cong \mathcal{F}[1] \oplus \mathcal{F}[2n]$. But $\mathrm{cofib}(u^n)$ lies in the ideal (as it can be constructed by iterated cofibers starting with $\mathrm{cofib}(u)$), and hence so does $\mathcal{F}$, as wanted.

For the general case, we consider the recollement

$$\begin{tikzcd} D^b_T(X^{\mu_\ell}; k) \arrow[r, "i_*"] & D^b_T(X; k) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & D^b_T(X \setminus X^{\mu_\ell}; k) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

Here we need to see that the image under j_* of the right-hand term lies in the ideal generated by $\mathrm{cofib}(u)$. We have already seen that $D^b_T(X \setminus X^{\mu_\ell}; k)$ equals the ideal, so this statement is covered by the first part of Theorem 2, which says that all the functors preserve the property of being in this ideal—we will sketch the proof of Theorem 2 below.

The last statement is similarly a consequence of the first part of Theorem 2. We need to see that the cofiber of $i^!\mathcal{F} \to i^*\mathcal{F}$ lies in the ideal. Recall that by six functor formalism 'localization triangle' we have the cofibration sequence $i_!i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F}$, which upon applying i^* gives a cofibration sequence $i^!\mathcal{F} \to i^*\mathcal{F} \to i^*j_*j^*\mathcal{F}$, as $i^*i_! = i^*i_* = 1$. But $i^*j_*j^*\mathcal{F}$ lies in the ideal generated generated by $\mathrm{cofib}(u)$, for the same reason as before: $j^*\mathcal{F}$ lies in $D^b_T(X \setminus X^{\mu_\ell}; k)$, and is particular in the ideal, and being in the ideal is preserved by i^*j_* by the first half of Theorem 2. Hence $i^!$ and i^* agree on the quotient. (Compare also [Tre19, Sec. 4.2] and [RW19, Prop. 2.6].) $\square$

Sketch of proof of Theorem 2. First note that $\mathbb{D}$ preserves the ideal generated by $\mathrm{cofib}(u)$. Namely, for $p : X \to pt$, we have that

$$\mathbb{D}(\mathrm{cofib}(u)\otimes\mathcal{F}) = \mathrm{map}(\mathrm{cofib}(u)\otimes\mathcal{F}, p^!(k)) \cong \mathrm{map}(\mathrm{cofib}(u), \mathbb{D}\mathcal{F}) \cong \mathrm{cofib}(u)\otimes\mathbb{D}\mathcal{F}[-1]$$

It is obvious that f^* preserves the ideal generated by $\mathrm{cofib}(u)$, as pullback is functorial, and hence the same is true for $f^!$ using that $= \mathbb{D}f^! = f^*\mathbb{D}$. That $f_!$ preserves the ideal follows from the projection formula $f_!(\mathcal{F} \otimes f^*(\mathcal{G})) \cong f_!(\mathcal{F}) \otimes \mathcal{G}$, with $\mathcal{G} = \mathrm{cofib}(u)$, under the assumptions when this formula holds, e.g., finite covering dimension of Y, and again this implies the same for f_*, by Verdier duality. This concludes the proof of the first part of Theorem 2 (used in the proof of Theorem 1).

Let us now check that the diagram commutes in all cases. For $F = \mathbb{D}$, we need to see that the cofiber of $\mathbb{D}i^*\mathcal{F} \cong i^!\mathbb{D}\mathcal{F} \to i^*\mathbb{D}\mathcal{F}$ is in the ideal generated by $\mathrm{cofib}(u)$, which follows by the last part of Theorem 1.

That the diagram commutes for $F = f^*$ is obvious as $(-)^*$ distributes over composition. The diagram also commutes for $F = f_!$ as $i^*f_! \xrightarrow{\cong} f_!i^*$ by base change. The statements for $F = f^!$ and $F = f_*$ now follows by Verdier duality, as Verdier duality transforms f^* into $f^!$ and $f_!$ into f_*. $\square$

We end with a few remarks. As noted I've stated things a bit different from the original papers in this note, to get formulations closer to the original localization theorem in equivariant cohomology. In particular I've stated the main result as an equivalence of categories. Furthermore, as explained above, I'm also quotienting by a different (potentially smaller) subcategory than the one used in [Tre19] and [RW19] (though in practice probably often equivalent)—in my talk at the Arbeitsgemeinschaft I only gave a vague comment that something like that should be true, mumbling something about finiteness. This led to some confusion and follow-up conversations with Gurbir Dhillon and Geordie Williamson. A version indeed turns out to the true, and the above proof sketch follows that rute, thanks to those conversations.

References

[Tre19] D. Treumann. *Smith theory and geometric Hecke algebras.* Math. Ann., **375** (2019), 595–628.

[Wil19] G. Williamson. *Modular representations and reflection subgroups.* Current Developments in Mathematics, International Press, Vol. 2019, 113–184.

[RW19] S. Riche and G. Williamson. *Smith-Treumann theory and the linkage principle.* arxiv:2003.08522.