THE TRANSCENDENCE DEGREE OF THE MOD p COHOMOLOGY OF FINITE POSTNIKOV SYSTEMS

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Abstract. We examine the transcendence degree of the mod p cohomology of a finite Postnikov system E. We prove that, under mild assumptions on E, the transcendence degree of $H^*(E;\mathbf{F}_p)$ is always positive, and give a complete classification of the Postnikov systems where the transcendence degree of $H^*(E;\mathbf{F}_p)$ is finite. More precisely we prove that $H^*(E;\mathbf{F}_p)$ is of finite transcendence degree iff E is $\mathbf{F}_p$-equivalent to the classifying space of a p-toral group. To obtain the results we establish a general formula for determining the transcendence degree of an unstable algebra given in terms of the growth of certain 'unstable Betti numbers'. As an application of these results we derive statements about the n-connected cover $X\langle n\rangle$ of a finite complex X. We show for instance that, under suitable connectivity assumptions on X, the LS category of $X\langle n\rangle$ is always infinite assuming $X\langle n\rangle \neq X$. Finally we discuss generalizations of the obtained results to polyGEMs.

1. Introduction

In 1953 Serre showed his celebrated result that a 1-connected finite Postnikov system E with finitely generated homotopy always has homology in infinitely many dimensions, using his newly invented spectral sequence [31]. His methods, however, although revealing the asymptotic size of the Betti numbers (the coefficients in the Poincaré series) of $H^*(E;\mathbf{F}_p)$, did not in general give information about the ring or $\mathcal{A}$-module structure of $H^*(E;\mathbf{F}_p)$ (here $\mathcal{A}$ denotes the Steenrod algebra). Serre's theorem has since then been generalized in several ways by a number of people (Dwyer-Wilkerson [9], Lannes-Schwartz [20, 22], McGibbon-Neisendorfer [24]) all utilizing the theory of unstable modules over the Steenrod algebra as developed by Lannes, Schwartz and others. One of the main advantages of this approach is that, by relating certain properties of the cohomology to questions about mapping spaces, it gives a grip on how these properties behave with respect to fibrations—i.e. it turns traditional spectral sequence questions into long exact sequence questions. This paper is a contribution along these lines.

We offer the following two main theorems:

Theorem 1.1. *Let E be a connected nilpotent finite Postnikov system with finite $\pi_1(E)$. Assume that $H^*(E;\mathbf{F}_p)$ is of finite type and that $\bar{H}^*(E;\mathbf{F}_p) \neq 0$. Then $\bar{H}^*(E;\mathbf{F}_p)$ contains an element of infinite height.*

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Theorem 1.2. *Let E be a connected nilpotent finite Postnikov system with finite $\pi_1(E)$. Assume that $H^*(E;\mathbf{F}_p)$ is of finite type. Then $H^*(E;\mathbf{F}_p)$ has finite transcendence degree iff E is $\mathbf{F}_p$-equivalent to a space E' fitting into a principal fibration sequence of the form*

$$\mathbf{CP}^\infty \times \cdots \times \mathbf{CP}^\infty \to E' \to K(P,1)\,,$$

where P is a finite p-group.

Here we define the transcendence degree, $d(K)$, of a graded algebra K as the maximal number of homogeneous algebraically independent elements in K. When K is connected noetherian, $d(K)$ is equal to the Krull dimension of K. Note that Theorem 1.1 can be reformulated as saying that the transcendence degree of $H^*(E;\mathbf{F}_p)$ is always positive. Theorem 1.1 was previously known in the case $p=2$ for E 1-connected by work of Lannes and Schwartz [22], but was actually rediscovered independently by the author and formed the starting point for this work.

From now on $H^*(X)$ denotes the mod p cohomology of X for some fixed but arbitrary prime p. In the rest of this introduction we use some standard notation concerning unstable modules over the Steenrod algebra. In the next section we briefly introduce these concepts, for a general reference see e.g. Schwartz [30].

The key topological result needed in proving the results about $H^*(E)$ is the following asymptotic growth formula in $\operatorname{rk}_p V$, where V is an elementary abelian group:

$$\log_p |[BV, K(\pi_n(E),n)]| \lesssim \log_p |[BV,E]| \lesssim C \log_p |[BV, K(\pi_n(E),n)]|\,,$$

where n is the dimension of the highest homotopy group which is not uniquely p-divisible.

The above growth formula relates to the algebra structure of $H^*(E)$ by the following theorem of Lannes, generalizing earlier work of Miller [25]:

Theorem 1.3. [18, 19] *Let X be a connected space. Suppose that X is nilpotent with finite $\pi_1(X)$ and $H^*(X)$ of finite type. Then the natural map $f \mapsto f^*$, $[BV,X] \to \operatorname{Hom}_{\mathcal{K}}(H^*(X), H^*(V))$ is a bijection.*

Here $\mathcal{K}$ denotes the category of unstable algebras over the Steenrod algebra.

We have the following theorem of Lannes and Schwartz:

Theorem 1.4. [29, 30] *Let M be an unstable module over the Steenrod algebra. Then the following two conditions are equivalent:*

1. *M is nilpotent.*
2. *$\operatorname{Hom}_{\mathcal{U}}(M, H^*(V)) = 0$ for all elementary abelian p-groups V.*

Here $\mathcal{U}$ denotes the category of unstable modules over the Steenrod algebra.

The Lannes linearization principle [19]

$$\operatorname{Hom}_{\mathcal{U}}(K, H^*(V))' \simeq \mathbf{F}_p^{\operatorname{Hom}_{\mathcal{K}}(K, H^*(V))}\,,$$

where $'$ denotes the (continuous) vector space dual, now immediately leads to an analogous theorem concerning unstable algebras.

Theorem 1.5. [21] *For an augmented unstable algebra K the following conditions are equivalent:*

1. *$\operatorname{Hom}_{\mathcal{K}}(K, H^*(V)) = 0$ for all elementary abelian p-groups V.*

2. *K is F-equivalent to the trivial unstable algebra $\mathbf{F}_p$.*
3. *$\bar{K}$ is nilpotent as an unstable module over the Steenrod algebra.*
4. *$\bar{K}$ is a nil ideal in K, i.e. it consists of nilpotent elements.*

Here the bar denotes the augmentation ideal. The equivalence of 2., 3., and 4. are direct consequences of the definitions. Theorem 1.1 now follows from the growth formula and the preceding general theorems.

It is worth noting that in the preceding theorems one has to consider elementary abelian groups V of an arbitrary size. Restriction to e.g. $V = \mathbf{Z}/p$ would not be enough, which is seen by for example setting $K = H^*(K(\mathbf{Z}, 3))$. This makes the property 'nilpotent' a bit less well behaved than for example the property 'locally finite'.

To prove Theorem 1.2 we examine work of Henn, Lannes and Schwartz [16] on the structure of unstable algebras and look at it in terms of growth properties. This leads to a characterization of the transcendence degree of an unstable algebra K in terms of the growth of the characteristic numbers $\gamma_v(K) := \log_p |\operatorname{Hom}_{\mathcal{K}}(K, H^*(V))|$ in $v := \operatorname{rk}_p V$, under mild restrictions on K. More precisely we prove:

Theorem 1.6. *Let K be an unstable algebra with $1 \neq 0$. Assume that K has transcendence degree $d(K)$ and that $\operatorname{Hom}_{\mathcal{K}}(K, H^*(V))$ is finite for all V. If $d(K)$ is finite then $\gamma_v(K) \sim d(K)v$. If $d(K)$ is infinite then $\gamma_v(K)$ grows faster than linearly in v.*

Theorem 1.6 is a powerful tool for calculating the transcendence degree of unstable algebras. To demonstrate this we rederive Quillen's theorem in the finite group case, by proving that the Krull dimension of $H^*(G)$ is equal to the p-rank $\operatorname{rk}_p G$ for every finite group G.

We call the numbers $\gamma_v(K)$ (which are not necessarily integers) 'unstable Betti numbers' since they show some similar properties of the traditional Betti numbers of a graded algebra, $\beta_n(K) = \dim_{\mathbf{F}_p} K^n$. One should think of the $\log_p$ as a replacement of $\dim_{\mathbf{F}_p}$ necessary since $\operatorname{Hom}_{\mathcal{K}}(K, H^*(\mathbf{F}_p^{\oplus n}))$ in general is not an $\mathbf{F}_p$-vector space. Theorem 1.6 shows that an analog of the well known formula relating the growth of the Betti numbers and the transcendence degree (Krull dimension) of a *noetherian* connected graded algebra holds for these new numbers, now with much weaker restrictions on the unstable algebra.

Applying Theorem 1.6 to the obtained growth formula for $\log_p |[BV, E]|$, where E is a finite Postnikov system, and doing a bit of work now leads to Theorem 1.2.

We also include a section where we see that the above results imply that the n-connected cover of a finite complex always has infinite Lusternik-Schnirelmann (LS) category, generalizing earlier partial results of McGibbon and Møller [23].

Finally we discuss and conjecture generalizations of the above results to polyGEMs, correcting a small mistake put forth in [12].

In the proofs of the theorems we several times need a small but useful fact about nilpotent actions. Since we have been unable to find this fact stated in the literature, and believe that it ought to be better known, we give a proof in a short appendix.

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2. Notation

By a space we, for simplicity, mean an object in the pointed homotopy category Ho_* of CW-complexes. The bracket $[-,-]$ means a *free* homotopy class of maps (Theorem 1.3 is the main reason why this is convenient). We will also need to refer to pointed homotopy classes of pointed maps which we denote by $[-,-]_{pt}$.

We define the p-rank, rk_p, of a group G as the maximal rank of an elementary abelian subgroup V contained in G. At all times we employ the convention that $v = \mathrm{rk}_p V$, for the elementary abelian group V in question.

When talking about the asymptotic behavior of some sequence of numbers, we mean by the symbol $\lesssim$ that for all $\epsilon > 0$ there exists an N such that (left-hand side)$_n \leq (1+\epsilon)$(right-hand side)$_n$ for all $n \geq N$. We define $\gtrsim$ and $\sim$ analogously.

In our notation involving unstable modules over the Steenrod algebra, we follow the standard notation used in e.g. Schwartz [30]. We quickly review the basic definitions:

Definition 2.1. *An* unstable module M *is a graded module over the Steenrod algebra* $\mathcal{A}$ *satisfying the following instability conditions:*

- *If* $p = 2$ *then* $\mathrm{Sq}^i x = 0$ *for* $i > |x|$.
- *If* $p > 2$ *then* $\beta^e P^i x = 0$ *for* $e + 2i > |x|$.

Let $\mathcal{U}$ *denote the category whose objects are unstable modules and whose morphisms are (degree* 0*)* $\mathcal{A}$*-module maps. Note that this is an abelian category.*

Definition 2.2. *An* unstable algebra K *is an unstable module equipped with two maps* $\eta : \mathbf{F}_p \to K$ *and* $\mu : K \otimes K \to K$ *making* K *into a commutative unital* $\mathbf{F}_p$*-algebra such that*

- μ *is* $\mathcal{A}$*-linear (i.e. the Cartan formula holds).*
- $Sq^{|x|}x = x^2$ *for any* $x \in K$ *if* $p = 2$*;* $P^{|x|/2}x = x^p$ *for any* $x \in K$ *of even degree if* $p > 2$.

Let $\mathcal{K}$ *denote the category whose objects are unstable algebras, and whose morphisms are those (degree* 0*)* $\mathbf{F}_p$*-algebra maps which are* $\mathcal{A}$*-linear.*

Furthermore, throughout this paper, for convenience we make the standing assumption that all unstable algebras have $1 \neq 0$, i.e. we exclude the unstable algebra 0.

Definition 2.3. *We say that an unstable module is* nilpotent *if the following holds:*

- *If* $p = 2$ *then for every* $x \in M$ *there exists an integer* N *such that* $\mathrm{Sq}^{2^{N-1}|x|}\,\mathrm{Sq}^{2^{N-2}|x|}\cdots Sq^{|x|}x = 0.$

- *If $p > 2$ then for every $x \in M$ where $|x|$ is even there exists an integer N such that $P^{p^{N-1}|x|/2} P^{p^{N-2}|x|/2} \dots P^{|x|/2} x = 0$.*

Let $\mathcal{N}il$ denote the full subcategory of $\mathcal{U}$ whose objects are nilpotent modules.

Note that this definition extends the usual definition if I is an ideal in an unstable algebra, in the sense that we have that I lies in $\mathcal{N}il$ iff it is nil as an ideal, that is if all its elements are nilpotent. Beware however, for a non-noetherian algebra a nil ideal I need not be nilpotent (i.e. there exists n such that $I^n = 0$), which makes the terminology slightly ambiguous. Note also that an unstable algebra with $1 \neq 0$ of course cannot itself be nilpotent.

We need a last definition:

Definition 2.4. *A morphism of unstable algebras $\varphi : K \to K'$ is said to be an* F-monomorphism *if the kernel of φ in the (abelian) category of unstable modules is nilpotent as an unstable module. We define* F-epimorphism *and* F-isomorphism *analogously.*

This definition coincides with Quillen's original definition of an F-isomorphism, and is also the same as what is sometimes referred to as a (purely) inseparable isogeny.

We note that everything in this paper only depends on the space E up to $\mathbf{F}_p$-equivalence (E here, as everywhere, connected nilpotent with finite π_1), and that Bousfield-Kan $\mathbf{F}_p$-completion [5] of E coincides with Bousfield $\mathbf{F}_p$-localization [3]. As noted by Miller [26] $\mathrm{map}_*(BV, E) \simeq \mathrm{map}_*(BV, \hat{E}_p)$, and of course $H^*(E; \mathbf{F}_p) \simeq H^*(\hat{E}_p; \mathbf{F}_p)$ – Likewise we have that $[BV, E] \simeq [BV, \hat{E}_p]$ [5, 3]. Thus we could reformulate everything by writing: "Let E be $\mathbf{F}_p$-equivalent to..." – For the sake of clarity we refrain from doing so and instead leave reformulations like that to the reader.

3. Growth Properties

In this section we establish the growth formula for $\log_p |[BV, E]|$ mentioned in the introduction. To do this we first need to prove a couple of lemmas. The first lemma in its present form is due to Lannes and Schwartz [22].

Lemma 3.1. *Let G be an abelian group, and assume that $t = \mathrm{rk}_p({}_pG)$ and $s = \mathrm{rk}_p(G \otimes \mathbf{Z}/p)$ are finite. Furthermore let V be an elementary abelian p-group of rank v. Then the Poincaré series of $H^*(V; G)$, $P_G(x) = \sum_{k>0} \dim_{\mathbf{F}_p}(H^k(V; G) \otimes \mathbf{Z}/p) x^k$ is given by*

$$P_G(x) = \frac{t + sx}{1 + x}((1 - x)^{-v} - 1).$$

Especially

$$\mathrm{rk}_p(H^1(V; G)) \sim tv$$

for $v \to \infty$ and

$$\mathrm{rk}_p(H^k(V; G)) \sim \begin{cases} \frac{tv^k}{k!} & \text{if } t > 0 \\ \frac{sv^{k-1}}{(k-1)!} & \text{if } t = 0 \end{cases}$$

for $v \to \infty$, $k \geq 2$.

Proof. Note that induction on the rank v of V, using the cohomology of $H^*(\mathbf{Z}/p;\mathbf{Z})$ together with the Künneth formula, gives us that $pH^k(V;\mathbf{Z}) = 0$ for all $k > 0$ and all V.

The universal coefficient theorem now gives us the following exact sequence

$$0 \to H^k(V;\mathbf{Z}) \otimes G \to H^k(V;G) \to \mathrm{Tor}(H^{k+1}(V;\mathbf{Z}), G) \to 0,$$

i.e. for $k \geq 1$

$$0 \to H^k(V;\mathbf{Z}) \otimes (G \otimes \mathbf{Z}/p) \to H^k(V;G) \to H^{k+1}(V;\mathbf{Z}) \otimes ({}_pG) \to 0.$$

This shows that $xP_G(x) = (t+sx)P_{\mathbf{Z}}(x)$. Setting $G = \mathbf{Z}/p$ we obtain

$$P_{\mathbf{Z}}(x) = \frac{x}{1+x} P_{\mathbf{Z}/p}(x) = \frac{x}{1+x}((1-x)^{-v} - 1)$$

and hence

$$P_G(x) = \frac{t+sx}{1+x}((1-x)^{-v} - 1).$$

From this formula the growth formulas are immediate. □

Lemma 3.2. *Let X be an arbitrary space and let E be a connected finite Postnikov system. Then*

$$|[X,E]_{pt}| \leq \prod_{i>0} |H^i(X, \pi_i(E))|$$

when E is simple.

If more generally E is nilpotent, then there exists, for each i, a filtration $0 = F_{i,0} \lhd \cdots \lhd F_{i,t_i} = \pi_i(E)$ such that $\pi_1(E)$ acts trivially on $F_{i,j}/F_{i,j-1}$ and

$$|[X,E]_{pt}| \leq \prod_{i>0} \prod_{j=1}^{t_i} |H^i(X; F_{i,j}/F_{i,j-1})| \ .$$

Proof. The proof is by induction on the number of nontrivial homotopy groups. Assume that the top homotopy group sits in dimension n. Consider first the case where E is simple. We have a fibration sequence, which is principal since E is simple:

$$K(\pi_n(E), n) \to E \to P_{n-1}E,$$

where P_kE denotes the kth Postnikov stage of E. Since the fibration is principal we have an action $E \times K(\pi_n(E), n) \to E$, and thus an induced action $* : [X,E]_{pt} \times [X, K(\pi_n(E), n)]_{pt} \to [X,E]_{pt}$. This action has the property that in the exact sequence

$$[X, K(\pi_n(E), n)]_{pt} \xrightarrow{\iota} [X,E]_{pt} \xrightarrow{\kappa} [X, P_{n-1}E]_{pt}$$

we have that $\kappa(f) = \kappa(g)$ iff there exists $h \in [X, K(\pi_n(E), n)]$ such that $f * h = g$. This implies that $|[X,E]_{pt}| \leq |[X, P_{n-1}E]_{pt}||H^n(X, \pi_n(E))|$, so by induction we get that $|[X,E]_{pt}| \leq \prod_{i>0} |H^i(X, \pi_i(E))|$ as wanted.

Now assume that E is only nilpotent. The fibration $K(\pi_n(E), n) \to E \to P_{n-1}E$ might no longer be principal, but it does have a principal refinement corresponding to a filtration $0 = F_{i,0} \lhd \cdots \lhd F_{i,t_i} = \pi_i(E)$ of $\pi_i(E)$ (cf. [17]). Using induction as before finishes the proof in this case too. □

We are now ready to prove the key growth theorem:

Theorem 3.3. *Let E be a connected nilpotent finite Postnikov system with finite $\pi_1(E)$. Assume that $H^*(E)$ is of finite type and that $\bar{H}^*(E) \neq 0$. Let n denote dimension of the highest homotopy group $\pi_n(E)$ which is not uniquely p-divisible and set $k = n$ if $\pi_n(E)$ has p-torsion, $k = n-1$ if not. Then*

$$cv^k \lesssim \log_p |[BV, E]| \lesssim Cv^k ,$$

for $v \to \infty$, where c, C are positive constants.

Remark 3.4. The condition that $H^*(X)$ is of finite type is equivalent to the condition that $H_*(X; \mathbf{F}_p)$ is of finite type. By the universal coefficient theorem this is again equivalent to requiring ${}_pH_n(X; \mathbf{Z})$ and $H_n(X; \mathbf{Z}) \otimes \mathbf{Z}/p$ to be finite for all n. If X is connected nilpotent with finite π_1, this is equivalent to asking that ${}_p\pi_n(X)$ and $\pi_n(X) \otimes \mathbf{Z}/p$ are finite for all $n \geq 2$, e.g. by the Whitehead theorem modulo Serre classes. Note also that for a connected nilpotent space, X, the condition $\bar{H}^*(X) \neq 0$ is equivalent to $\pi_*(X)$ not being uniquely p-divisible (by the above or [5]). Also note that the k which appears in the theorem also could be defined as the dimension of the highest non-trivial mod p homotopy group minus 1.

Proof of Theorem 3.3. By replacing E by the $\mathbf{F}_p$-equivalent space P_nE, we can assume that n is the dimension of the top non-trivial homotopy group. Since E is connected and $\pi_1(E)$ is finite we might as well show the theorem for $[BV, E]_{pt}$, which is what we do. We have a principal fibration sequence $\Omega P_{n-1}E \to K(\pi_n(E), n) \to E$ so we get an exact sequence of pointed sets

$$[BV, \Omega P_{n-1}E]_{pt} \to [BV, K(\pi_n(E), n)]_{pt} \to [BV, E]_{pt}.$$

We can also write this as

$$[BV, \Omega_0 P_{n-1}E]_{pt} \to H^n(V; \pi_n(E)) \to [BV, E]_{pt} , \tag{3.1}$$

where $[BV, \Omega_0 P_{n-1}E]_{pt}$ acts on $H^n(V; \pi_n(E))$ as described in the previous proof and $\Omega_0 P_{n-1}E$ denotes the zero component of $\Omega P_{n-1}E$.

Now by Lemma 3.2 and 3.1 we get:

$$\begin{aligned} \log_p |[BV, \Omega_0 P_{n-1}E]_{pt}| &\leq \sum_{i=1}^{n-2} \log_p |H^i(V; \pi_{i+1}(E))| \\ &= \sum_{i=1}^{n-2} \mathrm{rk}_p\, H^i(V; \pi_{i+1}(E)) \\ &\lesssim Kv^{n-2} \end{aligned}$$

for $v \to \infty$, since $\Omega_0 P_{n-1}E$ is an H-space and hence simple. Since $\log_p |H^n(V; \pi_n(E))| \sim cv^k$, where k is as defined in the theorem, (3.1) shows us:

$$\begin{aligned} \log_p |H^n(V; \pi_n(E))| &\sim \log_p |H^n(V; \pi_n(E))| - \log_p |[BV, \Omega_0 P_{n-1}E]|_{pt} \\ &\lesssim \log_p |[BV, E]|_{pt} . \end{aligned}$$

We want to get the other inequality by appealing to Lemma 3.2. In the simple case it is immediate that

$$\log_p |[BV, E]_{pt}| \leq \sum_{i=1}^{n} \log_p |H^i(V; \pi_i(E))| \lesssim Cv^k .$$

If $\pi_n(E)$ does contain p-torsion we can take $C = c$, whereas C in the case where $\pi_n(E)$ does not contain p-torsion is given in terms of $\pi_n(E)$ and $\pi_{n-1}(E)$.

In the non-simple case we need to worry a little bit about actions. We now have that

$$\log_p |[BV, E]_{pt}| \overset{<}{\sim} \sum_{i=1}^{n} \sum_{j=1}^{t_i} \log_p |H^i(V; F_{i,j}/F_{i,j-1})| \,.$$

It is still clear that $\log_p |[BV, E]_{pt}| \leq Cv^n$ for some C. This takes care of the case where $\pi_n(E)$ does contain p-torsion. In the case where $\pi_n(E)$ does not contain p-torsion however we want the better estimate Cv^{n-1}. This is not obvious since the filtration quotients could have p-torsion, even though $\pi_n(E)$ did not. To resolve this problem we have to appeal to Proposition 7.1.

By replacing E by an $\mathbf{F}_p$-equivalent space, we can assume that $\pi_1(E)$ is a p-group, since every finite nilpotent group is a product of p-groups. By Proposition 7.1 $\pi_1(E)$ acts trivially on $\pi_n(E)$, so we can take the filtration of $\pi_n(E)$ to be the trivial filtration, and thus there is no p-torsion introduced in dimension n. This shows that, in this case, we can conclude that $\log_p |[BV, E]_{pt}| \overset{<}{\sim} Cv^{n-1}$. □

Remark 3.5. It follows from the proof of the theorem above that one can actually get concrete estimates for c and C. This does have some interest, especially in the case $k = 1$ where c and C actually turn out to give lower and upper bounds on the transcendence degree of $H^*(E)$.

The first main theorem now follows from the preceding growth theorem.

Theorem 3.6. *Let E be a connected nilpotent finite Postnikov system with finite $\pi_1(E)$. Assume that $H^*(E)$ is of finite type and that $\bar{H}^*(E) \neq 0$. Then $\bar{H}^*(E)$ contains an element of infinite height.*

Proof. Theorem 3.3 shows the asymptotic growth of $\log_p |[BV, E]|$—in particular it shows that $[BV, E]$ is nontrivial for some V. But by Theorem 1.3 and 1.5 this now implies that $\bar{H}^*(E) \notin \mathcal{N}il$, so $\bar{H}^*(E)$ has to contain an element of infinite height. □

4. Applications to n-connected covers

We now turn to investigating the consequences for n-connected covers of finite complexes:

Corollary 4.1. *Let X be a finite complex, 1-connected with finite $\pi_2(X)$. Assume furthermore that $\bar{H}^*(P_n X) \neq 0$. Then $\bar{H}^*(X\langle n\rangle)$ contains an element of infinite height.*

Remark 4.2. The assumption $\bar{H}^*(P_n X) \neq 0$ could be formulated in a number of alternative ways. Since X is 1-connected the assumption is equivalent to the natural map $H^*(X) \to H^*(X\langle n\rangle)$ not being an isomorphism by a Serre spectral sequence argument. Also, since X is of finite type, $\bar{H}^*(P_n X) \neq 0$ iff $\bar{H}^*(P_n X; \mathbf{Z})$ is not entirely q-torsion for primes $q \neq p$, which, by for example the Whitehead theorem modulo Serre classes, is equivalent to $\pi_*(P_n X)_{(p)} \neq 0$.

Proof of Corollary 4.1. We have an exact sequence

$$[BV, \Omega X] \to [BV, \Omega P_n X] \to [BV, X\langle n\rangle] \to [BV, X] \,.$$

By Miller's theorem $[BV, \Omega X] = [BV, X] = 0$, so $[BV, \Omega P_n X] = [BV, X\langle n\rangle]$. Since $\bar{H}^*(P_n X) \neq 0$ we have that $\bar{H}^*(\Omega P_n X) \neq 0$. Now observe that $\Omega P_n X$ satisfies the assumptions of Theorem 3.3, so in particular we get that $[BV, X\langle n\rangle] =$

$[BV, \Omega P_n X] \neq 0$ for some V. By Theorem 1.3 and 1.5 this is equivalent to $\bar{H}^*(X\langle n\rangle)$ containing an element of infinite height. □

Remark 4.3. The relatively strong assumption on the connectivity of X cannot be weakened, as is shown by the Hopf fibration $S^1 \to S^3 \to S^2$.

Corollary 4.4. *Let X be a finite complex, 1-connected with finite $\pi_2(X)$. Assume that $X \neq X\langle n\rangle$. Then there exists a prime q such that $\bar{H}^*(X\langle n\rangle; \mathbf{F}_q)$ contains an element of infinite height.*

Proof. If $X \neq X\langle n\rangle$, we have to have that $\bar{H}^*(P_n X; \mathbf{F}_q) \neq 0$ for some prime q. The statement is now obvious from Corollary 4.1. □

Corollary 4.5. *Let X be a finite complex, 1-connected with finite $\pi_2(X)$. Assume that $X \neq X\langle n\rangle$. Then $X\langle n\rangle$ has infinite LS category.*

Proof. By Corollary 4.4 there exists a prime q such that $\bar{H}^*(X\langle n\rangle; \mathbf{F}_q)$ has infinite cup-length, so especially $X\langle n\rangle$ has infinite LS category. □

These corollaries generalize earlier partial results of Møller and McGibbon [23]. As they point out, it can be interesting to note the radical difference between these results and the results obtained in the rational case. Here the rational LS category of $X\langle n\rangle$ is always less than or equal to the rational LS category of X by the mapping theorem of Félix and Halperin [13]. But, on the other hand, in the rational setting we have no Serre's theorem either—indeed the rational cohomology of $K(\mathbf{Z}, 3)$ does not contain an element of infinite height.

5. The transcendence degree of $H^*(E)$

In this section we give a complete classification of the Postnikov systems whose cohomology is of finite transcendence degree. We start by using work of Henn, Lannes and Schwartz [16] to relate the growth of $\gamma_v(K) := \log_p |\mathrm{Hom}_{\mathcal{K}}(K, H^*(V))|$ in v to the transcendence degree of K, under mild restrictions on K. Combining these results with our growth formula for $[BV, E] = \mathrm{Hom}_{\mathcal{K}}(H^*(E), H^*(V))$ now leads to a classification theorem for finite Postnikov systems of finite transcendence degree.

In order to state and prove our results we need to review some work of Henn, Lannes and Schwartz. For the ease of the reader we follow their notation; we refer the reader to [16] for more details.

Definition 5.1. *For any unstable algebra K define its transcendence degree $d(K)$ as the maximal number of algebraically independent homogeneous elements in K.*

Remark 5.2. If K is a connected graded noetherian algebra, the maximal number of algebraically independent elements, $d(K)$, is finite. In this case the number $d(K)$ coincides with the Krull dimension of K, and we can furthermore choose $d(K)$ algebraically independent elements such that K will be finite over the algebra spanned by those elements. If K is an integral domain, $d(K)$ is equal to the classical transcendence degree of the field of fractions of K. A nice and graded proof of these standard facts can be found in [1].

In the following we need to refer to elementary abelian groups of different rank. Therefore we sometimes equip the elementary abelian group V with a subscript which in those cases indicate the rank of V.

Definition 5.3. *Let $\mathcal{E}$ denote the category of elementary abelian p-groups, i.e. of finite dimensional $\mathbf{F}_p$-vector spaces. Let $\mathcal{PS}$ denote the category of profinite sets and let $\mathcal{G}$ be the category of functors $\mathcal{E} \to (\mathcal{PS})^{op}$. Note that we can view $\mathcal{G}^{op}$ as the category of contravariant functors $\mathcal{E} \to \mathcal{PS}$.*

One should realize that objects in $\mathcal{G}$ contain a rich structure. This stems from the fact that not only do we associate to each vector space a profinite set, but we do this in a natural way, which, loosely speaking, ties the profinite sets together.

By inverting all F-isomorphisms in $\mathcal{K}$ we obtain a quotient category of $\mathcal{K}$ which will be denoted by $\mathcal{K}/\mathcal{N}il$. Let $g : \mathcal{K} \to \mathcal{G}$ be given by $g(K)(V) = \operatorname{Hom}_{\mathcal{K}}(K, H^*(V))$ for all V. Here we equip $\operatorname{Hom}_{\mathcal{K}}(K, H^*(V))$ with the profinite topology induced by writing

$$\operatorname{Hom}_{\mathcal{K}}(K, H^*(V)) = \operatorname{Hom}_{\mathcal{K}}(\operatorname{colim}_\alpha K_\alpha, H^*(V)) = \lim_{\alpha} \operatorname{Hom}_{\mathcal{K}}(K_\alpha, H^*(V))\,,$$

where α runs over the finitely generated $\mathcal{A}$-subalgebras K_α of K.

By Theorem 1.4 and the Lannes linearization principle we have that a morphism between unstable algebras $K \to K'$ is an F-monomorphism (resp. F-epi) iff $g(K')(V) \to g(K)(V)$ is a surjective (resp. injective) map of sets for all V. This shows that g induces a a faithful functor $\mathcal{K}/\mathcal{N}il \to \mathcal{G}$ (likewise denoted g). In [16] Henn, Lannes and Schwartz actually identify the image of g—we shall however not need this.

Definition 5.4. *Let $\mathcal{PS} - \operatorname{End} V_d$ denote the category whose objects are profinite sets equipped with a continuous right action of the monoid $\operatorname{End} V_d$ and whose morphisms are maps of profinite sets respecting the $\operatorname{End} V_d$-action.*

To each $G \in \mathcal{G}^{op}$ and each vector space V_d we can associate a profinite right $\operatorname{End} V_d$-set $G(V_d)$. Namely define the right action of $\operatorname{End} V_d$ on $G(V_d)$ by to each $(s, \varphi) \in G(V_d) \times \operatorname{End} V_d$ associating $s\varphi = G(\varphi)s$. This gives us a 'restriction functor'

$$e_d : \mathcal{G}^{op} \to \mathcal{PS} - \operatorname{End} V_d\,.$$

Likewise we can define an 'induction functor'

$$i_d : \mathcal{PS} - \operatorname{End} V_d \to \mathcal{G}^{op} : S \mapsto (W \mapsto S \times_{\operatorname{End} V_d} \operatorname{Hom}(W, V_d))\,,$$

where $S \times_{\operatorname{End} V_d} \operatorname{Hom}(W, V_d)$ denotes the coequalizer in the category of profinite sets of the action of $\operatorname{End} V_d$ on S and $\operatorname{Hom}(W, V_d)$ respectively.

Now note that the canonical map $G(V_d) \times \operatorname{Hom}(W, V_d) \to G(W)$ induces a well defined map on the coequalizer $G(V_d) \times_{\operatorname{End} V_d} \operatorname{Hom}(W, V_d) \to G(W)$. Therefore, to each map $S \to G(V_d)$ of profinite $\operatorname{End} V_d$-sets and each W, we can associate a map

$$S \times_{\operatorname{End} V_d} \operatorname{Hom}(W, V_d) \to G(V_d) \times_{\operatorname{End} V_d} \operatorname{Hom}(W, V_d) \to G(W)\,.$$

Likewise a morphism in $\mathcal{G}^{op}$, $i_d(S) \to G$ of course induces a map $S \to G(V_d)$ of profinite $\operatorname{End} V_d$-sets. Since these operations are inverses of each other we get that $\operatorname{Hom}_{\mathcal{G}^{op}}(i_d(S), G) = \operatorname{Hom}_{\mathcal{PS} - \operatorname{End} V_d}(S, e_d(G))$, naturally in $G \in \mathcal{G}^{op}$, $S \in \mathcal{PS} - \operatorname{End} V_d$, so i_d is left adjoint of e_d. It is immediate that we have an equivalence $1_{\mathcal{PS} - \operatorname{End} V_d} \xrightarrow{\simeq} e_d \circ i_d$, given by the unit of the adjunction, so we get an embedding of $\mathcal{PS} - \operatorname{End} V_d$ as a subcategory of $\mathcal{G}^{op}$. Define $s_d = e_d^{op} \circ g$, where $e_d^{op} : \mathcal{G} \to (\mathcal{PS} - \operatorname{End} V_d)^{op}$ is the opposite functor of e_d.

In [16] Henn, Lannes and Schwartz prove the following key results about the structure of $\mathcal{G}^{op}$.

Proposition 5.5. [16] *The morphism $(i_d \circ e_d)(G) \to G$ given by the counit of the adjunction is a monomorphism in $\mathcal{G}^{op}$ for all $G \in \mathcal{G}^{op}$, i.e. we have that $(i_d \circ e_d)(G)(V) \to G(V)$ is an injective map of profinite sets for all V.*

This gives us a filtration of $\mathcal{G}$ which turns out to coincide with the filtration of $\mathcal{K}/\mathcal{N}il$ by transcendence degree.

Proposition 5.6. [16] *Let $G \in \mathcal{G}^{op}$. Define the transcendence degree of G as*

$$d(G) := \min\{d \mid ((i_d \circ e_d)G)(W) \to G(W) \text{ is bijective for all } W\},$$

where we take $d(G) = \infty$ if none such d exists. For an element $G \in \mathcal{G}$ we define the transcendence degree by viewing it as lying in $\mathcal{G}^{op}$. With these definitions we have that $d(K) = d(g(K))$.

We shall need some alternative ways of expressing the transcendence degree of an object in $\mathcal{G}^{op}$. These can be found implicit in [16]. Since they are important in their own right we find it useful to state them explicitly—we include proofs for the convenience of the reader. First a useful definition:

Definition 5.7. *Let S be an $\operatorname{End} V_d$-set, and let $s \in S$. Define the rank of $s \in S$ as*

$$\operatorname{rk} s = \min\{\operatorname{rk}\varphi \mid \text{ where } s = t\varphi \text{ for some } t \in S, \varphi \in \operatorname{End} V_d\}.$$

We say that s is regular if $\operatorname{rk} s = d$, i.e. if $s = t\varphi$ implies that φ is a regular (invertible) matrix.

Proposition 5.8. *We have the following formula for the transcendence degree of $G \in \mathcal{G}^{op}$:*

$$\begin{aligned} d(G) &:= \min\{d \mid ((i_d \circ e_d)G)(W) \to G(W) \text{ is bijective for all } W\} \\ &= \max\{\operatorname{rk} s \mid s \in G(V_d) \text{ for some } d\} \\ &= \max\{d \mid G(V_d) \text{ contains a regular element}\}. \end{aligned}$$

Proof. We start by proving that

$$\max\{d \mid G(V_d) \text{ contains a reg. elt.}\} = \max\{\operatorname{rk} s \mid s \in G(V_d) \text{ for some } d\}. \tag{5.1}$$

First note that '$\leq$' is obvious. To prove '$\geq$' let $s \in G(W)$ and suppose that $\operatorname{rk} s = d$. We can thus choose $\pi \in \operatorname{End} W$, $t \in G(W)$ such that $s = t\pi$ and $\operatorname{rk}\pi = d$. By changing t we can assume that π is a projection. Let $\rho : W \to \pi W$ and $i : \pi W \to W$ be the canonical projection and inclusion associated to π. Now set $s' = G(i)s \in G(\pi W)$ and note that $G(\rho)s' = G(\rho)G(i)s = G(\rho)G(i)G(\pi)t = G(\pi i \rho)t = s$. We claim that s' is regular. Suppose we have $\varphi \in \operatorname{End}(\pi W), u \in G(\pi W)$ such that $s' = u\varphi$. Then we have

$$s = G(\rho)s' = G(\rho)G(\varphi)u = G(\rho)G(\varphi)G(i)G(\rho)u = G(i\varphi\rho)G(\rho)u = (G(\rho)u)(i\varphi\rho)$$

so φ has to be regular, since $\operatorname{rk} s = d$. This shows the wanted inequality.

We now prove that the number given by (5.1) actually coincides with the transcendence degree. We first see that it is less than or equal to the transcendence degree. Suppose therefore that we have d such that $G(V_d) \times_{\operatorname{End} V_d} \operatorname{Hom}(W, V_d) \to G(W)$ is bijective for all W, and let $s \in G(W)$ be arbitrary. Choose $t \in G(V_d)$ and $\varphi \in \operatorname{Hom}(W, V_d)$ such that $G(\varphi)t = s$. Since the rank of $\varphi : W \to V_d$ must be less than or equal to d, we can choose a projection $\pi \in \operatorname{End} W$ of rank less than or equal to d such that $\varphi\pi = \varphi$. But this gives us that $s = G(\varphi)t = G(\pi)G(\varphi)t = (G(\varphi)t)\pi$, which shows that $\operatorname{rk} s \leq d$, as wanted.

We finish the proof by showing $d(G) \leq \max\{d | G(V_d) \text{ contains a regular element}\}$. Let d be the maximal number such that $G(V_d)$ contains a regular element (if there exist regular elements in infinitely many dimension we are done). We want to prove that $G(V_d) \times_{\operatorname{End} V_d} \operatorname{Hom}(W, V_d) \to G(W)$ is surjective for all W. Let $s \in G(W)$ be arbitrary and set $n = \operatorname{rk} s$. Note that $n \leq d$ by (5.1). Choose an element $\pi \in \operatorname{End} W$ of rank n and $t \in G(W)$ such that $s = t\pi$. By changing the choice of t we can assume that π is a projection. Letting $\rho : W \to \pi W$ and $i : \pi W \to W$ denote the canonical projection and inclusion we get that $s = G(\pi)t = G(\pi i \rho)t = G(\rho)(G(\pi i)t)$. Therefore s is in the image of $G(V_n) \times_{\operatorname{End} V_n} \operatorname{Hom}(W, V_n)$ which implies that s is in the image of $G(V_d) \times_{\operatorname{End} V_d} \operatorname{Hom}(W, V_d)$, since $n \leq d$. This completes the proof, since $G(V_d) \times_{\operatorname{End} V_d} \operatorname{Hom}(W, V_d) \to G(W)$ is injective by Proposition 5.5. □

Example 5.9. Let X_n be the $\operatorname{End} V_n$-set consisting of two elements x_1 and x_0, with the $\operatorname{End} V_n$-action given as follows:

$$x_0\alpha = x_0 \text{ for all } \alpha \in \operatorname{End} V_n$$

$$x_1\alpha = \begin{cases} x_1 & \text{for } \alpha \in \operatorname{Aut} V_n \\ x_0 & \text{for } \alpha \notin \operatorname{Aut} V_n\,. \end{cases}$$

We have that x_1 has rank n and x_0 has rank 0. The $\operatorname{End} V_n$-set X_n is thus the smallest $\operatorname{End} V_n$-set containing a regular element. It has the universal property that any $\operatorname{End} V_n$-set containing a regular element surjects onto X_n as an $\operatorname{End} V_n$-set.

Just knowing the growth properties of the 'unstable Betti numbers' $\gamma_v(K) = \log_p |g(K)(V)|$ in v is actually enough to determine the transcendence degree of K, under mild restrictions on K.

Theorem 5.10. *Let K be an unstable algebra and assume that* $\operatorname{Hom}_{\mathcal{K}}(K, H^*(V))$ *is finite for all V. If $d(K)$ is finite then $\gamma_v(K) \sim d(K)v$. If $d(K)$ is infinite then $\gamma_v(K)$ grows faster than linearly in v.*

Remark 5.11. The assumption that $\operatorname{Hom}_{\mathcal{K}}(K, H^*(V))$ is finite for all V is a technical assumption which (by Theorem 3.3) is satisfied in all the applications we have in mind. Also it is clear that if for instance K is finitely generated as an $\mathcal{A}$-algebra, then $\operatorname{Hom}_{\mathcal{K}}(K, H^*(V))$ is likewise finite. Furthermore if K is $\mathcal{N}il$-closed (cf. [6, 16]) and of finite type then $\operatorname{Hom}_{\mathcal{K}}(K, H^*(V))$ is finite [16]. Note however that $H^*(\prod_{i=1}^{\infty} K(\mathbf{Z}/p, i))$ serves as an example of an unstable algebra of finite type where $\operatorname{Hom}_{\mathcal{K}}(K, H^*(V))$ is not finite.

Proof of Theorem 5.10. We want to establish the general growth formulas by establishing them for the 'largest' and the smallest finite $\operatorname{End} V_n$-set containing a regular element. Let X_n be the $\operatorname{End} V_n$-set defined in Example 5.9. For the elements in $i_n(X_n)(V) = X_n \times_{\operatorname{End} V_n} \operatorname{Hom}(V, V_n)$ we have the following relations:

$$\begin{aligned} (x_0, \varphi) &\sim (x_0, 0) \text{ for all } \varphi \in \operatorname{Hom}(V, V_n) \\ (x_1, \varphi) &\sim (x_0, 0) \text{ if } \operatorname{rk}\varphi < n \\ (x_1, \varphi) &\sim (x_1, \psi) \text{ if } \operatorname{rk}\varphi = \operatorname{rk}\psi = n \text{ and } \ker\varphi = \ker\psi\,. \end{aligned}$$

This shows that $|i_n(X_n)(V)| = \#(\text{ n dimensional subspaces in V }) + 1$

For $v > n$, the number of n dimensional subspaces of V is as follows:

$$\begin{aligned}\#(\text{ n dimensional subspaces in V }) &= \frac{(p^v-1)\cdots(p^v-p^{n-1})}{(p^n-1)\cdots(p^n-p^{n-1})} \\ &\sim Cp^{nv}\end{aligned}$$

for $v \to \infty$.

Let T be a finite set and consider the $\operatorname{End} V_n$-set $T \times \operatorname{End} V_n$. We have that $i_n(T \times \operatorname{End} V_n)(V) = T \times \operatorname{Hom}(V, V_n)$ so

$$|i_n(T \times \operatorname{End} V_n)(V)| = |T|p^{nv}\,.$$

From the above we conclude that

$$\log_p |i_n(X_n)(V)| \sim \log_p |i_n(T \times \operatorname{End} V_n)(V)| \sim nv\,.$$

Now let K be an unstable algebra and assume that $s_n(K)$ contains a regular element. Since $s_n(K)$ is finite we can find a surjection of $\operatorname{End} V_n$-sets $T \times \operatorname{End} V_n \to s_n(K)$ for some finite set T. Also, since $s_n(K)$ contains a regular element, we can find a surjection $s_n(K) \to X_n$ of $\operatorname{End} V_n$-sets. Now this means that, for all V, we have surjections of sets

$$i_n(T \times \operatorname{End} V_n)(V) \to i_n(s_n(K))(V) \to i_n(X_n)(V)\,,$$

so $\log_p |i_n^{op}(s_n(K))(V)|$ behaves asymptotically as nv.

If K has transcendence degree $d < \infty$, then $g(K)$ also has transcendence degree d so $\log_p |g(K)(V)| = \log_p |(i_d^{op} \circ e_d^{op} \circ g)(K)(V)| = \log_p |i_d^{op}(s_d(K))(V)|$ which grows asymptotically as dv, since $s_d(K)$ contains a regular element by Proposition 5.8. If K has transcendence degree $d = \infty$ then by Proposition 5.5

$$\log_p |g(K)(V)| \geq \log_p |(i_n^{op} \circ e_n^{op} \circ g)(K)(V)| = \log_p |i_n^{op}(s_n(K))(V)|\,.$$

Since $s_n(K)$ contains a regular element for infinitely many n by Proposition 5.8, we get that $\log_p |g(K)(V)| \gtrsim nv$ for all n as wanted. $\square$

The unstable Betti numbers $\gamma_v(K)$ can be viewed as an unstable algebra alternative to the classical Betti numbers of a graded algebra. In the classical case we have a formula relating the growth of the Betti numbers and the transcendence degree of a *noetherian* connected graded $\mathbf{F}$-algebra given by

$$d(K) = \min\{k \in \mathbf{N}_0 | \text{ there exists a C such that } \dim_{\mathbf{F}} K^n \leq Cn^{k-1} \text{ for all } n\}\,.$$

Theorem 5.10 establishes a different but analogous formula for these new numbers, which has among its advantages that it holds with much weaker restrictions on the unstable algebra. Moreover, when applied to the cohomology of spaces, it is easy to see how these numbers behave with respect to fibrations of the underlying spaces. For instance we immediately get that the unstable Betti numbers are subadditive over principal fibrations, and hence that the transcendence degree is likewise so. More precisely:

Proposition 5.12. *Let B, F be connected nilpotent spaces with finite fundamental groups and mod p cohomology rings of finite type. Assume that we have a principal fibration $F \to E \to B$. Then $\gamma_v(H^*(E)) \leq \gamma_v(H^*(F)) + \gamma_v(H^*(B))$, and hence, if $\gamma_v(H^*(F))$ and $\gamma_v(H^*(B))$ are finite for all v, $d(H^*(E)) \leq d(H^*(F)) + d(H^*(B))$.*

Proof. Note that obviously E is also connected with finite π_1 and by [5, Prop. II.4.4] E is nilpotent as well. Since the fibration is principal $\pi_1(B)$ acts nilpotently on $H^*(F)$ (cf. [5, p. 62]), so the Serre spectral sequence guarantees that $H^*(E)$ is of finite type. The subadditivity of the unstable Betti numbers follows from Theorem 1.3 and the fact that, by the principal action, $|[BV, E]| \leq |[BV, F]||[BV, B]|$. The subadditivity of the transcendence degree now follows from Theorem 5.10. □

Remark 5.13. In [16] Henn, Lannes and Schwartz use Proposition 5.5 and 5.6 to derive a far reaching generalization of Quillen's theorem about the structure of the mod p cohomology ring of a finite group [28]. They prove that for any unstable algebra K of transcendence degree less than or equal to d there is an F-isomorphism

$$K \to \lim_{s_d(K)} H^*(V_d)\,.$$

Here we view $s_d(K)$ as a category by taking as objects the elements in $s_d(K)$, and as morphisms the maps induced on $s_d(K)$ from endomorphisms of V_d. (The theorem is strictly speaking only true as stated here when $s_d(K)$ is finite—when $s_d(K)$ is infinite one has only to consider the elements in $\lim_{s_d(K)} H^*(V_d)$ which are continuous as functions $s_d(K) \to H^*(V_d)$.)

The theorem however has the inherent weakness that it requires a priori knowledge of the transcendence degree of K. Theorem 5.10 can remedy this defect, since it gives an easily applicable way of calculating the transcendence of an unstable algebra K.

To illustrate the usefulness of this approach we rederive Quillen's theorem in the finite group case, by showing that the Krull dimension of $H^*(G)$ is equal to $\mathrm{rk}_p G$. By Cartan-Eilenberg [8], $H^*(G)$ is isomorphic to an inverse limit of the cohomology of its p-subgroups, so it is enough to show the claim for a finite p-group P. It was shown by Hurewicz that $[BV, BP] = \mathrm{Rep}(V, P) = \mathrm{Hom}(V, P)/(\text{conj. by } p \in P)$. It is furthermore an easy exercise to see that $\log_p |\mathrm{Rep}(V, P)|$ grows asymptotically as $\mathrm{rk}_p P\ v$ (cf. also Proposition 5.17 and its proof). Theorem 5.10 now implies that $d(H^*(P)) = \mathrm{rk}_p P$, and since we know that $H^*(P)$ is noetherian by the Evens-Venkov theorem (cf. [11]) we are done.

Tracing back what elements go into this proof one sees that one of the main ingredients is the use of Proposition 5.8, which in some sense can be seen as being the replacement of Serre's theorem about cohomological detection of elementary abelian groups.

Theorem 5.10 now enables us to prove the second main theorem.

Theorem 5.14. *Let E be a connected nilpotent finite Postnikov system with finite $\pi_1(E)$. Assume that $H^*(E)$ is of finite type. Then $H^*(E)$ has finite transcendence degree iff E is $\mathbf{F}_p$-equivalent to a space E' fitting into a principal fibration sequence of the form:*

$$\mathbf{CP}^\infty \times \cdots \times \mathbf{CP}^\infty \to E' \to K(P, 1)\,,$$

where P is a finite p-group.

Proof. By Theorem 3.3 and 5.10 all spaces E' as in Theorem 5.14 have cohomology of finite transcendence degree. To prove the converse, assume that $H^*(E)$ has finite transcendence degree. We see from Theorem 3.3 that $\mathrm{Hom}_{\mathcal{K}}(H^*(E), H^*(V))$ is finite for all V. Theorem 3.3 and 5.10 now imply that E has to be $\mathbf{F}_p$-equivalent to P_2E, and furthermore that $\pi_2(E)$ cannot contain p-torsion. Since $\pi_1(P_2E)$ is finite nilpotent, and thus a product of p-groups, we can by passing to a covering replace P_2E with an $\mathbf{F}_p$-equivalent space whose fundamental group is a finite p-group. These observations show that, without restriction, we can assume that E has nontrivial homotopy groups only in dimension 1 and 2, $\pi_2(E)$ being without p-torsion and $\pi_1(E)$ being a finite p-group. But now Proposition 7.1 tells us that the action of $\pi_1(E)$ on $\pi_2(E)$ has to be trivial. This means that the Postnikov fibration $K(\pi_2(E), 2) \to E \to P_1E$ is principal (cf. [17]). We thus have a fibration sequence

$$E \to P_1E \xrightarrow{k} K(\pi_2(E), 3)\,.$$

Upon passing to $\mathbf{F}_p$-completion we obtain a fibration sequence (cf. [5])

$$\hat{E}_p \to P_1E \xrightarrow{k} K(\mathrm{Ext}(\mathbf{Z}/p^\infty, \pi_2(E)), 3)\,.$$

Now $\mathrm{Ext}((\mathbf{Z}/p^\infty, \pi_2(E))$ is a torsion free Ext $-p$ complete abelian group, and hence classified by the $\mathbf{F}_p$-dimension of $\mathrm{Ext}(\mathbf{Z}/p^\infty, \pi_2(E)) \otimes \mathbf{Z}/p$ (cf. [5, p. 181], [15]), which is finite since $H^*(E)$ is of finite type. In other words $\mathrm{Ext}(\mathbf{Z}/p^\infty, \pi_2(E)) \simeq \hat{\mathbf{Z}}_p \times \cdots \times \hat{\mathbf{Z}}_p$. Note that $H^3(\pi_1(E); \mathbf{Z}) = H^3(\pi_1(E); \hat{\mathbf{Z}}_p)$ by the universal coefficient theorem. We can thus let $k' : P_1E \to K(\mathbf{Z} \times \cdots \times \mathbf{Z}, 3)$ be the map corresponding to k under the equivalence $[P_1E, K(\mathrm{Ext}(\mathbf{Z}/p^\infty, \pi_2(E)), 3)] \simeq [P_1E, K(\mathbf{Z} \times \cdots \times \mathbf{Z}, 3)]$. Letting E' denote the homotopy fiber of k' we obtain a diagram

$$\begin{tikzcd}
E' \arrow[r] \arrow[d, "f"] & P_1E \arrow[r, "k'"] \arrow[d, equal] & K(\mathbf{Z} \times \cdots \times \mathbf{Z}, 3) \arrow[d, "\mathbf{F}_p\simeq"] \\
\hat{E}_p \arrow[r] & P_1E \arrow[r, "k"] & K(\mathrm{Ext}(\mathbf{Z}/p^\infty, \pi_2(E)), 3)
\end{tikzcd}$$

where f is any lifting which makes the diagram commute. From this diagram we see that $f : E' \to \hat{E}_p$ has to be an $\mathbf{F}_p$-equivalence as well. By construction E' is $\mathbf{F}_p$-equivalent to E and fits into a principal fibration sequence of the form stated in the theorem. □

Remark 5.15. Remember that a p-toral group G is a group which arises as a group extension $1 \to \mathbf{T}^n \to G \to P \to 1$, where $\mathbf{T}^n$ is an n-dimensional torus and P is a finite p-group. The spaces E' which appear in Theorem 5.14 are just those classifying spaces of p-toral groups which arise as central extensions. They are classified by n, P and their one extension class (= Postnikov invariant) $k \in [BP, K(\mathbf{Z} \times \cdots \times \mathbf{Z}, 3)] = H^3(P; \mathbf{Z} \times \cdots \times \mathbf{Z}) \simeq H^2(P; \mathbf{T}^n)$, where in the cohomology P acts trivially on the coefficients.

Remark 5.16. By Venkov's theorem [32], classifying spaces of p-toral groups have noetherian cohomology. Theorem 5.14 thus in particular shows that for the cohomology of a finite Postnikov system, being of finite transcendence degree is *equivalent* to being noetherian. This is indeed a very striking and unusual property which of course does not hold for spaces in general. The cohomology of the loop space of a (1-connected, say) finite complex for instance is *always* non-noetherian and has transcendence degree 0. Just knowing this intriguing property of finite Postnikov systems would actually be enough to rederive the above theorem using the original Betti numbers estimates of Serre—knowing this equivalence would secure that the wild growth of the Betti numbers could only be caused by an infinite number of polynomial generators in the ring.

We end this section by showing a proposition which precisely determines the transcendence degree of the cohomology the spaces E' of Theorem 5.14, and whose proof illustrates a calculation using Theorem 5.10. The proposition can also be obtained by using Quillen's theorem for compact Lie groups.

Proposition 5.17. *Let E be a space which fits into a fibration sequence of the form*

$$\underbrace{\mathbf{CP}^\infty \times \cdots \times \mathbf{CP}^\infty}_{n} \to E \to K(P,1) \xrightarrow{k} K(\mathbf{Z} \times \cdots \times \mathbf{Z}, 3)\,,$$

where P is a finite p-group. Then

$$d(H^*(E)) = n + \max\{\operatorname{rk}_p V | V \overset{\iota}{\hookrightarrow} P, \iota^*(k) = 0 \in H^3(V; \mathbf{Z} \times \cdots \times \mathbf{Z})\}\,.$$

Especially $n + \operatorname{rk}_p P \geq d(H^*(E)) \geq n+1$, when P is non-trivial.

Proof. Consider the sequence

$$[BV, E]_{pt} \to [BV, K(P,1)]_{pt} \xrightarrow{k_*} [BV, K(\mathbf{Z} \times \cdots \times \mathbf{Z}, 3)]_{pt}\,.$$

Let $\iota \in [BV_d, BP]_{pt} = \operatorname{Hom}(V_d, P)$ be an injection and assume that $\iota \in \ker(k_*)$. We have that ι naturally gives rise to $\sim Cp^{dv}$ elements in $\ker(k_*)$ as $v \to \infty$ coming from maps arising by precomposing ι with projections onto d dimensional subspaces of V. Since $\operatorname{Hom}(V_{\operatorname{rk}_p P}, P)$ is finite we obtain that

$$\log_p |\ker(k_*)| \sim \max\{\operatorname{rk}_p V | V \overset{\iota}{\hookrightarrow} P, \iota^*(k) = 0\} v\,.$$

But since it is easy to see that $[BV, K(\mathbf{Z} \times \cdots \times \mathbf{Z}, 2)]_{pt}$ acts freely on $[BV, E]_{pt}$ we get that

$$\log_p |[BV, E]_{pt}| \sim nv + \max\{\operatorname{rk}_p V | V \overset{\iota}{\hookrightarrow} P, \iota^*(k) = 0\} v\,,$$

so $d(H^*(E)) = n + \max\{\operatorname{rk}_p V | V \overset{\iota}{\hookrightarrow} P, \iota^*(k) = 0\}$ by Theorem 5.10. To get the last part of the proposition, note that $H^3(\mathbf{Z}/p; \mathbf{Z}) = 0$. $\square$

Example 5.18. Consider the space $E = \operatorname{Fib}(K(\mathbf{Z}/p \times \mathbf{Z}/p, 1) \xrightarrow{k} K(\mathbf{Z}, 3))$, where k is some nonzero element in $H^3(\mathbf{Z}/p \times \mathbf{Z}/p; \mathbf{Z}) \simeq \mathbf{Z}/p$. From the above proposition it follows that $d(H^*(E)) = 2$.

Remark 5.19. Note that the formula for the transcendence degree of $H^*(E)$ involves the behavior of a certain class in the *integral* cohomology of P when restricting to elementary abelian subgroups.

6. Generalizations to PolyGEMs

Recall that a GEM is a (possibly infinite) product of $K(G, n)$'s, where G is an abelian group. A space is a polyGEM if it belongs to the smallest full subcategory (polyGEMs) of spaces containing all GEMs and which is closed under taking extensions by fibrations, i.e. which satisfies that if $F \to E \to B$ is an arbitrary fibration then $B, F \in$ polyGEMs $\Rightarrow E \in$ polyGEMs. We say that a space is an oriented polyGEM if it belongs to the smallest full subcategory polyGEMs$^{\text{ori}}$ of spaces containing all GEMs and which is closed under taking extensions by principal fibrations, i.e. which satisfies that if $F \to E \to B$ is a principal fibration then $F, B \in$ polyGEMs$^{\text{ori}} \Rightarrow E \in$ polyGEMs$^{\text{ori}}$. Note that a nilpotent finite Postnikov system is an oriented polyGEM by [17].

In [12] Dror Farjoun conjectures that if X is a non-trivial p-complete polyGEM then $[B\mathbf{Z}/p, X]$ is always non-trivial. As stated the conjecture is false—$\hat{S}^1_p$ and

$K(\hat{\mathbf{Z}}_p, 3)$ serve as counterexamples. The problem with $\hat{S}^1_p$ is easy—it's a fundamental group problem. The problem with $K(\hat{\mathbf{Z}}_p, 3)$ is deeper, it demonstrates that it in general is not enough to look at maps from just a single $B\mathbf{Z}/p$ (it is however interesting to note that for any nilpotent, connected space X with finite π_1 and with $H^*(X)$ assumed to be *noetherian*, we have that $[BV, X] \neq 0 \Rightarrow [B\mathbf{Z}/p, X] \neq 0$ (cf. [16, Prop II 7.2])). Since the idea that polyGEMs should behave like finite Postnikov systems still seems very plausible, we dare to state the following more modest conjecture:

Conjecture 6.1. *Let E be a connected nilpotent polyGEM with finite $\pi_1(E)$. Assume that $H^*(E)$ is of finite type and that $\bar{H}^*(E) \neq 0$. Then $\bar{H}^*(E)$ contains an element of infinite height.*

We likewise believe that Theorem 5.14 should generalize:

Conjecture 6.2. *Let E be a connected nilpotent polyGEM with finite $\pi_1(E)$. Assume that $H^*(E)$ is of finite type. Then the transcendence degree of $H^*(E)$ is finite iff E is $\mathbf{F}_p$-equivalent to a space E' fitting into a principal fibration sequence of the form:*

$$\mathbf{CP}^\infty \times \cdots \times \mathbf{CP}^\infty \to E' \to K(P, 1)\,,$$

where P is a finite p-group.

One piece of evidence for the first conjecture is a theorem of Félix, Halperin, Lemaire and Thomas [14] which (in the language of Dror Farjoun) states that a 1-connected oriented polyGEM E with $H^*(E)$ of finite type and $\bar{H}^*(E) \neq 0$ has infinite LS category.

We give another piece of evidence for the conjecture, whose proof makes use of the Neisendorfer localization functor [27], which we now define. By [3, 12] we can choose a map f such that $L_f = L_{H\mathbf{Z}/p}$, where $L_{H\mathbf{Z}/p}$ denotes localization with respect to mod p homology and L_f denotes localization with respect to the map f. Let g be the map $g : B\mathbf{Z}/p \to *$, and note that, by definition, $L_g = P_{B\mathbf{Z}/p}$, the $B\mathbf{Z}/p$-nullification functor (cf. [12]). We define the Neisendorfer localization functor as $L = L_{f\vee g}$, where $f \vee g$ just denotes the wedge of the two maps. Note that, by the universal properties of the functors, we have a natural transformation of coaugmented functors $L_{H\mathbf{Z}/p}P_{B\mathbf{Z}/p} \to L$. This induces a weak equivalence $L_{H\mathbf{Z}/p}P_{B\mathbf{Z}/p}X \xrightarrow{\sim} LX$ whenever $P_{B\mathbf{Z}/p}X$ is a nilpotent space, since in that case we know that $L_{H\mathbf{Z}/p}P_{B\mathbf{Z}/p}X$ is $B\mathbf{Z}/p$-null. (Our definition differs slightly from the one given by Neisendorfer, since we want to to ensure that L is an idempotent localization functor on all of Ho_*.)

We need the following definition.

Definition 6.3. *Let* $\mathrm{polyGEMs}^{\mathrm{ft}}$ *denote the smallest full subcategory of spaces which is closed under taking extensions by fibrations and which contains all connected GEMs with finite π_1.*

Remark 6.4. The class $\mathrm{polyGEMs}^{\mathrm{ft}}$ contains all connected finite Postnikov systems with finite solvable π_1, and is probably very close to being equal to all connected polyGEMs with finite π_1.

Remember that an unstable module M is called locally finite if for all $x \in M$ we have that $\mathcal{A}x$ is finite dimensional.

Proposition 6.5. *Let E be a nilpotent polyGEM which belongs to* $\mathrm{polyGEMs}^{\mathrm{ft}}$*. Assume that $H^*(E)$ is of finite type and that $\tilde{H}^*(E) \neq 0$. Then $H^*(E)$ is not locally finite.*

Before giving the proof we need a lemma:

Lemma 6.6. *We have that $LE = *$ for all $E \in \mathrm{polyGEMs}^{\mathrm{ft}}$, where L denotes the Neisendorfer localization functor.*

Proof of Lemma 6.6. Our main technical tool is a theorem of Dror Farjoun which states that if L_f is any localization functor with respect to some map f, and if $F \to E \to B$ is any fibration sequence, then $L_f F = *$ implies that $L_f E \xrightarrow{\simeq} L_f B$ (cf. [12]). Hence the class of spaces which are acyclic with respect to Neisendorfer localization, i.e. the spaces E which satisfies $LE = *$, is closed under taking extensions by fibrations. It is therefore enough to prove the claim for connected GEMs with finite π_1. Furthermore note that it is enough to prove the claim for 1-connected GEMs, since we can apply the theorem of Dror Farjoun to the fibration sequence $E\langle 1 \rangle \to E \to K(\pi_1(E), 1)$, where $LK(\pi_1(E), 1)$ is easily seen to be zero.

Now let E be a 1-connected GEM and write this $E = \Omega \tilde{E}$ where $\tilde{E}$ is 2-connected. By a theorem of Dror Farjoun [12, Prop 7.B.5] $P_{B\mathbf{Z}/p}$ and $P_{M(\mathbf{Z}/p,1)}$ coincide on GEMs (where $M(\mathbf{Z}/p, 1)$ denotes the mod p Moore space). This gives us

$$P_{B\mathbf{Z}/p} E = P_{M(\mathbf{Z}/p,1)} E = \Omega P_{M(\mathbf{Z}/p,2)} \tilde{E}\,,$$

where the last equality is by another theorem of Dror Farjoun [12, Prop 3.A.1]. Bousfield [4] has shown that

$$\pi_i(P_{M(\mathbf{Z}/p,2)} \tilde{E}) = \pi_i(\tilde{E}) \otimes \mathbf{Z}[\frac{1}{p}]\,,$$

where we use that $\tilde{E}$ is a 2-connected GEM. But this shows that $P_{B\mathbf{Z}/p} E$ is a connected nilpotent space with uniquely divisible homotopy groups, and hence $LE = L_{H\mathbf{Z}/p} P_{B\mathbf{Z}/p} E = *$. □

Proof of Proposition 6.5. By Lemma 6.6 we have that $LE = *$, so especially $\mathrm{map}_*(B\mathbf{Z}/p, E) \neq *$, since $\mathrm{map}_*(B\mathbf{Z}/p, E) = *$ would imply that $LE = \hat{E}_p \neq *$. But saying that $\mathrm{map}_*(B\mathbf{Z}/p, E) \neq *$ is equivalent to saying that $H^*(E)$ is not locally finite (cf. [26, 21]). □

Remark 6.7. Using recent results of Bousfield [2] it is in fact possible to replace the category $\mathrm{polyGEMs}^{\mathrm{ft}}$ in Proposition 6.5 with the larger and somewhat more natural category consisting of all connected polyGEMs with finite π_1.

Remark 6.8. Neisendorfer's theorem [27], stating that $LX\langle n \rangle = \hat{X}_p$ for a 1-connected finite complex X with finite π_2, follows immediately from Lemma 6.6, by applying L to the fibration sequence $\Omega P_n X \to X\langle n \rangle \to X$ and using Miller's theorem to conclude that $LX = \hat{X}_p$.

Remark 6.9. There is no direct relation between the LS category and the property locally finite. For instance $\Sigma K(\mathbf{Z}/2, 2)$ is an example of a space with non-locally finite cohomology but with LS category 1, whereas $\prod_{n \in \mathbf{N}} S^n$ is an example of a space of infinite LS category whose cohomology is locally finite.

7. Appendix: Nilpotent actions

In this short appendix we prove a proposition about nilpotent actions, which we have used several times in the paper. The author thanks W. Chachólski for pointing out a proof somewhat simpler than the author's original proof.

Proposition 7.1. *Let M be an abelian group which does not contain any p-torsion and let P be a p-group. Assume that P acts nilpotently on M. Then P acts trivially on M.*

Proof. Suppose that P acts nilpotently on M and let $0 = F_0 \subset \cdots \subset F_n = M$ be a filtration on M such that P acts trivially on the filtration quotients. We want to do an induction on the length of the filtration, the initial step being trivial. Let $g \in P$ and $x \in F_2$ be arbitrary. We can write $gx = x + x_1$ where $x_1 \in F_1$. By iterating this we get that $g^n x = x + nx_1$. Setting $n = |g|$ gives us $|g|x_1 = 0$. But since M does not contain p-torsion we have that multiplication by $|g|$ is injective on M so $x_1 = 0$. Since x and g were arbitrary we conclude that P acts trivially on F_2. Induction on the length of the filtration now finishes the proof. □

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