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Advances in Mathematics 200 (2006) 525–538

ADVANCES IN Mathematics

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Propagating sharp group homology decompositions

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Received 9 March 2004; accepted 21 January 2005

Communicated by D.J. Benson
Available online 20 April 2005

Abstract

A collection $\mathcal{C}$ of subgroups of a finite group G can give rise to three different standard formulas for the cohomology of G in terms of either the subgroups in $\mathcal{C}$ or their centralizers or their normalizers. We give a short but systematic study of the relationship among such formulas for nine standard collections $\mathcal{C}$ of p-subgroups, obtaining some new formulas in the process. To do this, we exhibit some sufficient conditions on the poset $\mathcal{C}$ which imply comparison results.

MSC: primary: 20J06; secondary: 20J05; 55R35; 55P91

Keywords: Cohomology decomposition; Induction theory; Subgroup complex

1. Introduction and statement of the two theorems

Induction theory, for representations of a group G over a commutative ring R, is about calculating the value $\mathfrak{F}(G)$ of a functor $\mathfrak{F}$ from the opposite orbit category $\mathbf{O}(G)^{\mathrm{op}}$ of G to R-modules, in terms of the values $\mathfrak{F}(H)$ for various subgroups H of G. From "topological induction theory", which investigates homology decompositions, one gets, for a fixed collection $\mathcal{C}$ of subgroups of G, three different potential induction formulas:

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[1] Partially supported by NSF Grant DMS-0104318.
[2] Partially supported by NSA Grant MDA 904-01-1-0045.

0001-8708/$ - see front matter
doi:10.1016/j.aim.2005.01.006

by considering the subgroups H themselves, their normalizers, or their centralizers. Furthermore, restricting attention now to collections of (non-trivial) p-subgroups of a finite group G, for a fixed prime p, there are at least nine different collections commonly studied, which thus give us some 27 potential induction formulas.

The goal of this paper is to give a brief but systematic study of the interrelationship among these 27 different potential formulas. Namely, we investigate when the validity of one formula implies the validity of other formulas, for geometric reasons, reasons which do not depend on the specific functor $\mathfrak{F}$, or only depend on $\mathfrak{F}$ in a mild way—this is what we mean by "propagation" in the title of the paper. We proceed by studying homotopy properties of certain associated poset spaces, extending an approach of Dwyer [Dwy[97,Dwy98]; see also [Gro02]. Our proofs basically consist of three short lemmas in Section 2, which each state assumptions on the poset $\mathcal{C}$ which guarantee comparison results among the formulas. As a consequence we are able to settle, either in the positive or the negative, which of these 27 formulas give rise to "sharp homology decompositions" [Dwy98]—recovering many previous results, as well as closing the gaps in the existing literature.

For some collection of subgroups $\mathcal{C}$ (always assumed closed under G-conjugation), let $\mathbf{O}_{\mathcal{C}}$ denote the full subcategory of the orbit category $\mathbf{O}(G)$ with objects G/H, where we take only those H which are members of $\mathcal{C}$. The most naïve approach to induction theory is to ask if the functor

$$F^{\beta} : \mathbf{O}_{\mathcal{C}}^{\mathrm{op}} \to R\text{-}\mathrm{mod} \ \text{ given by } \ H \mapsto \mathfrak{F}(H)$$

satisfies that

$$\mathfrak{F}(G) \xrightarrow{\cong} \lim\nolimits^0_{\mathbf{O}_{\mathcal{C}}} F^{\beta},$$

i.e., that $\mathfrak{F}$ is *(subgroup) $\mathcal{C}$-computable* in the language of [Dre75, p. 293]. A more ambitious demand is to require that also

$$\lim\nolimits^i_{\mathbf{O}_{\mathcal{C}}} F^{\beta} = 0 \ \text{ for } i > 0,$$

where $\lim^i$ denotes the ith higher derived functor of the inverse limit functor; in which case $\mathfrak{F}$ is called *subgroup $\mathcal{C}$-acyclic* [Gro02, Definition 8.5]. See also [Gro02] for motivation.

One can ask the same for two corresponding functors obtained via normalizers and centralizers:

$$\begin{aligned} F^{\delta} : ((\mathrm{sd}\,\mathcal{C})/G)^{\mathrm{op}} &\to R\text{-}\mathrm{mod} \ \text{ given by} \\ (H_0 < \cdots < H_k) &\mapsto \mathfrak{F}\big(N_G(H_0) \cap \cdots \cap N_G(H_k)\big) \end{aligned}$$

$$\text{and } \ F^{\alpha} : \mathbf{A}_{\mathcal{C}} \to R\text{-}\mathrm{mod} \ \text{ given by } \ H \mapsto \mathfrak{F}\big(C_G(H)\big).$$

Here $(\mathrm{sd}\,\mathcal{C})/G$ is the orbit simplex category, with objects the G-conjugacy classes of strict chains of subgroups in $\mathcal{C}$, and morphisms the refinement of chains; and $\mathbf{A}_{\mathcal{C}}$ is the conjugacy category, with objects the subgroups in $\mathcal{C}$ and morphisms the homomorphisms between subgroups induced by conjugation in G.

Independently of whether a collection $\mathcal{C}$ gives induction formulas in the above senses of computability or acyclicity, it is of interest to study the relationship among the limits $\lim^*_{\mathbf{O}_{\mathcal{C}}} F^{\beta}$, $\lim^*_{(\mathrm{sd}\,\mathcal{C})/G} F^{\delta}$, and $\lim^*_{\mathbf{A}_{\mathcal{C}}} F^{\alpha}$, as well as to examine how these limits change when the collection $\mathcal{C}$ is varied. Aspects of this question have already been much studied because of applications in homotopy theory, where these higher limits often play an important role; see e.g., [DH01,Gro02], and their references.

Using the definitions it is possible (see e.g., [Dwy97, §1.2; Gro02, 2.8] or [DH01, §10]) to reformulate these higher limits in terms of Bredon cohomology, which makes dealing with them more amenable to homotopy theory. More precisely, let $E\mathbf{O}_{\mathcal{C}}$ be the poset category with objects $(G/H, gH)$, where $H \in \mathcal{C}$, and morphisms from $(G/H, x)$ to $(G/H', x')$ given by the G-maps $G/H \to G/H'$ sending x to x'; and let $E\mathbf{A}_{\mathcal{C}}$ be the category with objects the monomorphisms $i : H \to G$, and morphisms from i to i' given by the homomorphisms $\varphi : H \to H'$ such that $i = \varphi i'$. These spaces admit natural G-actions. With this notation we have the following models for the higher limits:

$$\lim_{\mathbf{O}_{\mathcal{C}}}{}^* F^{\beta} = H^*_G(|E\mathbf{O}_{\mathcal{C}}|; \mathfrak{F}), \quad \lim_{(\mathrm{sd}\,\mathcal{C})/G}{}^* F^{\delta} = H^*_G(|\mathcal{C}|; \mathfrak{F}), \text{ and } \lim_{\mathbf{A}_{\mathcal{C}}}{}^* F^{\alpha} = H^*_G(|E\mathbf{A}_{\mathcal{C}}|; \mathfrak{F}).$$

Here $|\cdot|$ denotes the nerve and $H^*_G(X; \mathfrak{F})$ the denotes Bredon cohomology of a space X with values in the generic coefficient system $\mathfrak{F}$—in other words, $H^*_G(X; \mathfrak{F})$ is the homology of the canonical cochain complex which in degree n is given by $\prod_{[\sigma]\in X_n/G} \mathfrak{F}(G_\sigma)$, where G_σ is the stabilizer of the n-simplex σ. (Different, smaller, models are found in [Gro02] under assumptions on $\mathcal{C}$.)

Note that we have natural comparison functors

$$E\mathbf{O}_{\mathcal{C}} \to \mathcal{C} \leftarrow E\mathbf{A}_{\mathcal{C}}$$

given by $(G/Q, x) \mapsto G_x$ and $(i : Q \to G) \mapsto i(Q)$. These functors are in fact equivalences of categories, with inverses being given by $Q \mapsto (G/Q, eQ)$ and $Q \mapsto (incl : Q \to G)$, and so the three categories have homotopy equivalent nerves. However, these inverses generally do not respect the G-action; and the nerves are in general not G-homotopy equivalent.

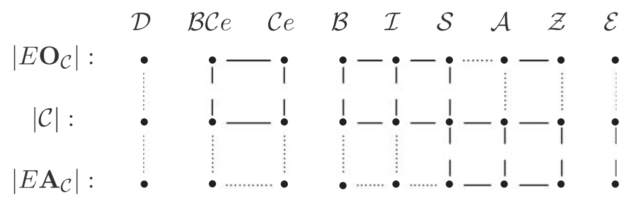
More generally, for a subcollection $\mathcal{C}'$ of $\mathcal{C}$ we can consider the diagram of G-spaces and G-maps

$$\begin{tikzcd}
|E\mathbf{O}_{\mathcal{C}'}| \arrow[r] \arrow[d] & |\mathcal{C}'| \arrow[d] & |E\mathbf{A}_{\mathcal{C}'}| \arrow[l] \arrow[d] \\
|E\mathbf{O}_{\mathcal{C}}| \arrow[r] & |\mathcal{C}| & |E\mathbf{A}_{\mathcal{C}}| \arrow[l]
\end{tikzcd}$$

If a map in this diagram is a G-homotopy equivalence, then it induces an isomorphism between the corresponding higher limits, by applying $H^*_G(-;\mathfrak{F})$. In some applications, $\mathfrak{F}$ has the further structure of a cohomological Mackey functor [Yos83]—that is, a Mackey functor with the standard property [AM94, II.5.3] of group cohomology functors $H^n(-;M)$ that restriction to a subgroup H followed by transfer back to G is just multiplication by the index $|G:H|$. Then if $|G:H|$ is also assumed invertible in R, we only need an H-equivalence, to get an induced isomorphism above. In particular when $R = \mathbf{Z}_{(p)}$ it is enough to have an S-equivalence, for S a Sylow p-subgroup of G.

Our main Theorem 1.1 below concerns 9 particular collections $\mathcal{C}$ of p-subgroups, whose definitions we review after the statement of the theorem. (These collections are interesting because of their known acyclicity and "sharpness" properties—see Theorem 1.2.) The theorem says exactly when the maps in the diagram of the previous paragraph for these $\mathcal{C}$ are either G-equivalences, S-equivalences for S a Sylow p-subgroup of G, or just ordinary homotopy equivalences.

Theorem 1.1. *Fix an arbitrary prime p, and let G denote a finite group, with Sylow p-subgroup S. Then we have the following table*:



Here a node denotes the space indicated by its row, for the collection $\mathcal{C}$ indicated by its column. A dotted line denotes a homotopy equivalence; a dashed line denotes an S-homotopy equivalence; and a solid line denotes a G-homotopy equivalence.

These results are best-possible in the sense that if a certain line is not present, then (for any prime p) there exists a finite group G for which that kind of equivalence does not hold.

We will later in Remark 3.1 comment on the history behind various parts of this result—in particular, the horizontal lines in the middle row correspond to classical equivalence theorems in the literature.

Notation for collections of non-trivial p-subgroups: The collection $\mathcal{S} = \mathcal{S}_p(G)$ is the collection of all non-trivial p-subgroups [Bro75]. Next $\mathcal{B} = \mathcal{B}_p(G)$ is the subcollection of $\mathcal{S}$ given by all non-identity *p-radical subgroups* [Bou84] i.e., all non-trivial p-subgroups Q such that $O_p\big(N_G(Q)/Q\big) = 1$. Also $\mathcal{C}e = \mathcal{C}e_p(G)$ is the subcollection of $\mathcal{S}$ of *p-centric subgroups* [Dwy97], i.e., p-subgroups Q such that $Z(Q)$ is a Sylow p-subgroup in $C_G(Q)$. Then we let $\mathcal{BC}e = \mathcal{B} \cap \mathcal{C}e$ denote the collection of non-trivial p-centric and p-radical subgroups. Further, $\mathcal{D} = \mathcal{D}_p(G)$ is the subcollection of $\mathcal{BC}e$ given by *principal* p-radical subgroups [Gro02], i.e., the subgroups Q in $\mathcal{C}e$ such that $O_p\big(N_G(Q)/QC_G(Q)\big) = 1$. We let $\mathcal{I}$ denote the subcollection of $\mathcal{S}$ given by the *Sylow intersections*: all non-trivial subgroups which are intersections of a set of Sylow p-subgroups in G. Next $\mathcal{A} = \mathcal{A}_p(G)$ is the subcollection of $\mathcal{S}$ of all nontrivial *elementary abelian p-subgroups* [Qui78]. Then $\mathcal{Z} = \mathcal{Z}_p(G)$ is the subcollection of $\mathcal{A}$ in [Ben98, Section 6.6], given by subgroups V such that $\Omega_1 O_p Z\big(C_G(V)\big) = V$ (the former expression denotes the elements of order dividing p in the center of the centralizer of V). Finally $\mathcal{E} = \mathcal{E}_p(G)$ [Ben94, Section 3] is the smallest subcollection of $\mathcal{A}$ which contains the conjugates of the subgroups of order p in the center of a Sylow p-subgroup of G, and is closed under taking products of commuting members.

We recall that a collection $\mathcal{C}$ is said to give rise to a *sharp subgroup (co)homology decomposition*, or to be *subgroup sharp* for short, if $H^n(-;\mathbf{F}_p)$ is subgroup $\mathcal{C}$-acyclic for all $n \geqslant 0$, i.e., if $\lim^i_{G/Q \in \mathbf{O}_\mathcal{C}} H^n(Q;\mathbf{F}_p) = 0$ for $i > 0$ and $H^n(G;\mathbf{F}_p) \xrightarrow{\cong} \lim^0_{G/Q \in \mathbf{O}_\mathcal{C}} H^n(Q;\mathbf{F}_p)$ for all $n \geqslant 0$. *Normalizer* and *centralizer sharpness* are defined similarly; we refer to e.g. [Dwy98,DH01, §8], and [Gro02, §9] for background and motivation.

Combining Theorem 1.1 with known results, we are able to complete the following table, showing which types of sharpness hold for each of the collections studied above—obtaining alternative proofs of sharpness for many of them in the process. Here the row names are abbreviations for subgroup, normalizer, and centralizer sharpness; a "*y*" in the table indicates that sharpness holds for all finite groups G and all primes p; while "*n*" indicates that for any prime p, sharpness fails for some G. (If a positive result was previously known, we have given a reference to the place where it first seemed to be stated in the literature.)

Theorem 1.2.

	$\mathcal{D}$	$\mathcal{B}\mathcal{C}e$	$\mathcal{C}e$	$\mathcal{B}$	$\mathcal{I}$	$\mathcal{S}$	$\mathcal{A}$	$\mathcal{Z}$	$\mathcal{E}$
s:	y [Gro02]	y [Dwy97]	y [Dwy97]	y [Dwy97]	y [Not01]	y [Dwy97]	n	n	n
n:	y [Gro02]	y [Gro02]	y [Gro02]	y [Web91]	y [Not01]	y [Web91]	y [Web91]	y [Ben98]	y
c:	n	n	n	n	n	y [Dwy97]	y [JM92]	y	y [JM92], [Ben94]

In particular, observe that the new positive results here are that the collection $\mathcal{Z}$ is centralizer sharp, and that the collection $\mathcal{E}$ is normalizer sharp. These are of some interest, since both $\mathcal{Z}$ and $\mathcal{E}$ are subcollections—often proper—of the more common collection $\mathcal{A}$ of all non-trivial elementary abelian p-subgroups; for example, the normalizer sharpness of $\mathcal{E}$ is applied to a number of sporadic simple groups at $p = 2$ in [BS].

A smaller version of the table in 1.2 was given by Dwyer at his talk at the 1996 AMS Summer Research Institute on Representations and Cohomology; and the present note arose as an attempt to go back and complete and extend that table.

2. Three lemmas and a pre-lemma

We start with some recollections and notation. Recall that a map is a G-homotopy equivalence if and only if the induced map on H-fixed points is an (ordinary) homotopy equivalence for all subgroups $H \leqslant G$ (see [Ben98, 6.4.2]). We say that a poset $\mathcal{X}$ is contractible if its nerve $|\mathcal{X}|$ is contractible. For a poset $\mathcal{X}$ and $x \in \mathcal{X}$, $\mathcal{X}_{\leqslant x}$ is the subposet of elements less than or equal to x, with $\mathcal{X}_{\geqslant x}$, $\mathcal{X}_{<x}$, and $\mathcal{X}_{>x}$ defined analogously. Further define $\mathbf{star}_{\mathcal{X}}(x) = \{y \in \mathcal{X} | x \leqslant y \text{ or } y \leqslant x\}$ and $\mathbf{link}_{\mathcal{X}}(x) = \{y \in \mathcal{X} | x < y \text{ or } y < x\}$; the nerves of these posets are of course the star and link of the vertex x in the nerve $|\mathcal{X}|$ of $\mathcal{X}$. Note that

$$\mathbf{link}_{\mathcal{X}}(x) = \mathcal{X}_{<x} \star \mathcal{X}_{>x} \text{ and } \mathbf{star}_{\mathcal{X}}(x) = \mathcal{X}_{<x} \star x \star \mathcal{X}_{>x}, \tag{$\star$}$$

where $\star$ denotes the join of posets, obtained by taking the disjoint union as sets and imposing the additional order relation that all elements in left poset are smaller than the ones in the right poset. By Quillen [Qui78, Proposition 1.9], on nerves the join of posets produces the join of spaces, which we also denote by $\star$. (For example we see using remarks above that we have a G_x-homeomorphism $|\mathbf{star}_{\mathcal{X}}(x)| \approx |x| \star |\mathbf{link}_{\mathcal{X}}(x)|$.)

We of course have that

$$\mathcal{C}^H = \{Q \in \mathcal{C} \mid H \leqslant N_G(Q)\}.$$

The elementary but fundamental observation which is needed for the lemmas of this section is that the functors $E\mathbf{O}_{\mathcal{C}} \to \mathcal{C} \leftarrow E\mathbf{A}$ described in the introduction induce equivalences of categories

$$E\mathbf{O}_{\mathcal{C}}^H \longrightarrow \mathcal{C}_{\geqslant H} \qquad \text{and} \tag{$\dagger$}$$

$$\mathcal{C}_{\leqslant C_G(H)} \longleftarrow E\mathbf{A}_{\mathcal{C}}^H$$

which are natural in the variable $\mathcal{C}$. This can be used to examine the more precise failure of the functors $E\mathbf{O}_{\mathcal{C}} \to \mathcal{C} \leftarrow E\mathbf{A}_{\mathcal{C}}$ to induce G-homotopy equivalences.

Recollection 2.1. The most fundamental fact in the homotopy theory of categories is the observation that a natural transformation between two functors induces a homotopy between their nerves (see e.g., [Qui78, 1.3]). In particular if F is an endomorphism of a poset $\mathcal{X}$ that is order-related to the identity functor, i.e., if either $F \leqslant \mathrm{Id}_{\mathcal{X}}$ (that is, $F(x) \leqslant x$ for all $x \in \mathcal{X}$) or $F \geqslant \mathrm{Id}_{\mathcal{X}}$, then we can construct a natural transformation from F to $\mathrm{Id}_{\mathcal{X}}$ or vice versa. Hence F induces a homotopy deformation retraction of the nerve of $\mathcal{X}$ onto the nerve of any poset $\mathcal{X}'$ such that $F(\mathcal{X}) \subseteq \mathcal{X}' \subseteq \mathcal{X}$. If $\mathcal{X}'$ is a G-subposet of a G-poset $\mathcal{X}$ and F is G-equivariant, then the retraction will be a G-homotopy deformation retraction, and $|\mathcal{X}'|$ is G-homotopy equivalent to $|\mathcal{X}|$. In particular a poset with a unique largest or smallest element x is contractible, since we can take F to be the endomorphism sending everything to x.

While the above technique is obviously useful to get results about $\mathcal{C}$ from functors F on $\mathcal{C}$, it can also be propagated to uses on $E\mathbf{O}_{\mathcal{C}}$ and $E\mathbf{A}_{\mathcal{C}}$ via (†).

Lemma 2.2. *Let $\mathcal{C}$ be a collection of subgroups of a discrete group G. Suppose that F is a G-equivariant poset endomorphism of $\mathcal{C}$ satisfying either $F \geqslant \mathrm{Id}_{\mathcal{C}}$ or $F \leqslant \mathrm{Id}_{\mathcal{C}}$. Set $\mathcal{C}' = F(\mathcal{C})$.*

(1) *Assume that the case $F \geqslant \mathrm{Id}_{\mathcal{C}}$ of the hypothesis holds.*
Then the inclusion $E\mathbf{O}_{\mathcal{C}'} \to E\mathbf{O}_{\mathcal{C}}$ induces a G-homotopy equivalence on nerves.
(2) *Assume for all $P \in \mathcal{C}$ that $C_G(P) \leqslant C_G\big(F(P)\big)$.*
Then the inclusion $E\mathbf{A}_{\mathcal{C}'} \to E\mathbf{A}_{\mathcal{C}}$ induces a G-homotopy equivalence on nerves.
(*Notice that the further hypothesis holds in the case $F \leqslant \mathrm{Id}_{\mathcal{C}}$ of the hypothesis.*)
(3) *The inclusion $\mathcal{C}' \to \mathcal{C}$ induces a G-homotopy equivalence on nerves.*

Proof. First note that (3) follows directly from 2.1.

To see (1), we want to see that $E\mathbf{O}^H_{\mathcal{C}'} \to E\mathbf{O}^H_{\mathcal{C}}$ induces a homotopy equivalence on nerves, for all subgroups $H \leqslant G$. For this, observe by (†) that this map identifies with the inclusion $\mathcal{C}'_{\geqslant H} \to \mathcal{C}_{\geqslant H}$, up to equivalence of categories. The assumption $F \geqslant \mathrm{Id}_{\mathcal{C}}$ guarantees that F takes $\mathcal{C}_{\geqslant H}$ into $\mathcal{C}_{\geqslant H}$, and so induces a homotopy deformation retraction of $|\mathcal{C}_{\geqslant H}|$ onto $|\mathcal{C}'_{\geqslant H}|$ by 2.1.

Finally, (2) follows by a parallel argument, since now the assumptions on F guarantee that F induces a homotopy deformation retraction of the nerve of $\mathcal{C}_{\leqslant C_G(H)}$ onto the nerve of $\mathcal{C}'_{\leqslant C_G(H)}$: since if $Q \leqslant C_G(H)$, then $H \leqslant C_G(Q) \leqslant C_G(F(Q))$, so that $F(Q) \leqslant C_G(H)$. $\square$

Next, we work towards a lemma that says when we can remove subgroups, one conjugacy class at a time, from a collection, while preserving G-homotopy type—again we propagate standard methods from $\mathcal{C}$ to $E\mathbf{O}_{\mathcal{C}}$ and $E\mathbf{A}_{\mathcal{C}}$. First a recollection and a pre-lemma.

Recollection 2.3. Let $\mathcal{X}$ be a G-poset, and $\mathcal{X}'$ the subposet of $\mathcal{X}$ obtained by removing from $\mathcal{X}$ the G-conjugates of an element x. Then we have the following (homotopy) pushout square of G-spaces:

$$\begin{array}{ccc} G \times_{G_x} |\mathbf{link}_{\mathcal{X}}(x)| & \longrightarrow & |\mathcal{X}'| \\ \downarrow & & \downarrow \\ G \times_{G_x} |\mathbf{star}_{\mathcal{X}}(x)| & \longrightarrow & |\mathcal{X}| \end{array}$$

To see that we have a pushout square of spaces, just observe that every simplex in $|\mathcal{X}|$ which is not in $|\mathcal{X}'|$ lies in the G-orbit of $|\,\mathbf{star}_{\mathcal{X}}(x)|$, and the intersection of this G-orbit with $|\mathcal{X}'|$ equals the G-orbit of $|\,\mathbf{link}_{\mathcal{X}}(x)|$. Since the left-hand vertical map is an inclusion of G-spaces, we have a homotopy pushout diagram of G-spaces. (See e.g., [DS95,Dwy98,DH01] for basics on pushouts and homotopy pushouts.) Note also that $|\,\mathbf{star}_{\mathcal{X}}(x)|$ is G_x-contractible, e.g., by 2.1: since in view of ($\star$), the mapping taking elements of $\mathcal{X}_{\leqslant x}$ to x extends to a poset endomorphism F of $\mathbf{star}_{\mathcal{X}}(x)$ with image $\mathcal{X}_{\geqslant x}$, which satisfies $F \geqslant \mathrm{Id}_{\mathbf{star}_{\mathcal{X}}(x)}$; and then we can send $\mathcal{X}_{\geqslant x}$ to its unique smallest element x. Thus if we can show that $|\,\mathbf{link}_{\mathcal{X}}(x)|$ is also G_x-contractible, then the left-hand vertical map is a G-homotopy equivalence; and it follows that right-hand vertical map is a G-homotopy equivalence from $|\mathcal{X}'|$ to $|\mathcal{X}|$, since we have a homotopy pushout of G-spaces.

Pre-Lemma 2.4. *Let G be a discrete group. Suppose $\mathcal{C}$ is a collection of subgroups, and let $\mathcal{C}'$ be the subcollection obtained by removing the G-conjugates of some subgroup P.*

(1) *Assume for all subgroups $H \leqslant P$ that the poset $\mathbf{link}_{\mathcal{C}}(P)_{\geqslant H}$ is contractible. Then the inclusion $|E\mathbf{O}_{\mathcal{C}'}| \to |E\mathbf{O}_{\mathcal{C}}|$ is a G-homotopy equivalence.*
(2) *Assume for all $H \leqslant C_G(P)$ that the poset $\mathbf{link}_{\mathcal{C}}(P)_{\leqslant C_G(H)}$ is contractible. Then the inclusion $|E\mathbf{A}_{\mathcal{C}'}| \to |E\mathbf{A}_{\mathcal{C}}|$ is a G-homotopy equivalence.*
(3) *Assume for all $H \leqslant N_G(P)$ that $\mathbf{link}_{\mathcal{C}}(P)^H$ is contractible. Then the inclusion $|\mathcal{C}'| \to |\mathcal{C}|$ is a G-homotopy equivalence.*

Proof. To establish (1), we examine the pushout square in 2.3, with $\mathcal{X} = E\mathbf{O}_{\mathcal{C}}$ and $x = (G/P, eP)$, so that $\mathcal{X}' = E\mathbf{O}_{\mathcal{C}'}$, and x has stabilizer $G_x = P$. We saw in 2.3 that we get the needed G-homotopy equivalence if we can show that $|\,\mathbf{link}_{E\mathbf{O}_{\mathcal{C}}}(G/P, eP)|$ is P-contractible. But for any $H \leqslant P$, by (†), $\mathbf{link}_{E\mathbf{O}_{\mathcal{C}}}(G/P, eP)^H$ is equivalent to $\mathbf{link}_{\mathcal{C}}(P)_{\geqslant H}$, the contractibility of which is exactly the hypothesis of (1).

The proof of (2) proceeds via a similar pushout square using the category $E\mathbf{A}_{\mathcal{C}}$: here the object x defined by $i : Q \to G$ with $i(Q) = P \in \mathcal{C}$ has stabilizer $G_x = C_G(P)$ so we need $C_G(P)$-contractibility of $|\,\mathbf{link}_{E\mathbf{O}_{\mathcal{C}}}(i)|$. Now (†) reduces us to verifying that for all $H \leqslant C_G(P)$, $\mathbf{link}_{\mathcal{C}}(P)_{\leqslant C_G(H)}$ is contractible, which again is just the assumption.

Finally (3) is again similar, since the stated assumption says exactly that $|\,\mathbf{link}_{\mathcal{C}}(P)|$ is $N_G(P)$-contractible, where $N_G(P)$ is the stabilizer of the object P of $\mathcal{C}$. □

Lemma 2.5. *Let G be a discrete group, and let $\mathcal{C}' \subset \mathcal{C}$ be collections such that $\mathcal{C} \setminus \mathcal{C}'$ contains finitely many G-conjugacy classes of subgroups.*

(1) *Assume for all $P \in \mathcal{C} \setminus \mathcal{C}'$ that $\mathcal{C}_{>P}$ is contractible. Then $|E\mathbf{O}_{\mathcal{C}'}| \to |E\mathbf{O}_{\mathcal{C}}|$ is a G-homotopy equivalence.*
(2) *Assume for all $P \in \mathcal{C} \setminus \mathcal{C}'$ that $\mathcal{C}_{<P}$ is contractible. Then $|E\mathbf{A}_{\mathcal{C}'}| \to |E\mathbf{A}_{\mathcal{C}}|$ is a G-homotopy equivalence.*
(3) *Assume either that $\mathcal{C}_{>P}$ is $N_G(P)$-contractible for all $P \in \mathcal{C} \setminus \mathcal{C}'$, or that $\mathcal{C}_{<P}$ is $N_G(P)$-contractible for all $P \in \mathcal{C} \setminus \mathcal{C}'$. Then $|\mathcal{C}'| \to |\mathcal{C}|$ is a G-homotopy equivalence.*

Proof. Assume the hypothesis of (1). We want to argue that we can successively remove the G-conjugacy classes of subgroups in $\mathcal{C} \setminus \mathcal{C}'$ in order of *increasing* size. Assume that P is a minimal subgroup in $\mathcal{C} \setminus \mathcal{C}'$, and let $\mathcal{C}''$ denote the poset obtained by removing the G-conjugates of P from $\mathcal{C}$. Then, for $H \leqslant P$, we have $\mathcal{C}_{>P} \subseteq \mathcal{C}_{\geqslant H}$; so using ($\star$) we see that $\mathbf{link}_{\mathcal{C}}(P)_{\geqslant H} = (\mathcal{C}_{<P})_{\geqslant H} \star \mathcal{C}_{>P}$. Since $\mathcal{C}_{>P}$ is assumed contractible, so is $\mathbf{link}_{\mathcal{C}}(P)_{\geqslant H}$. Hence Lemma 2.4(1) gives a G-homotopy equivalence $|E\mathbf{O}_{\mathcal{C}''}| \to |E\mathbf{O}_{\mathcal{C}}|$. Since for all $Q \in \mathcal{C}'' \setminus \mathcal{C}'$ we have $\mathcal{C}''_{>Q} = \mathcal{C}_{>Q}$ by minimal choice of P, we can continue by induction.

For (2) note that if $P \in \mathcal{C} \setminus \mathcal{C}'$, and $H \leqslant C_G(P)$, this time $\mathcal{C}_{<P} \subseteq \mathcal{C}_{\leqslant C_G(H)}$, so using ($\star$) we get $\mathbf{link}_{\mathcal{C}}(P)_{\leqslant C_G(H)} = \mathcal{C}_{<P} \star (\mathcal{C}_{>P})_{\leqslant C_G(H)}$; so that we may remove G-conjugates of P by Lemma 2.4(2) since $\mathcal{C}_{<P}$ is assumed contractible. By successively removing conjugacy classes of subgroups in $\mathcal{C} \setminus \mathcal{C}'$ in order of *decreasing* size in $\mathcal{C} \setminus \mathcal{C}'$, we conclude that $|E\mathbf{A}_{\mathcal{C}'}| \to |E\mathbf{A}_{\mathcal{C}}|$ is a G-homotopy equivalence.

Finally (3) is classical: First if $\mathcal{C}_{>P}$ or $\mathcal{C}_{<P}$ is $N_G(P)$-contractible, then so is $\mathbf{link}_{\mathcal{C}}(P)$. Thus we may apply Lemma 2.4(3) inductively—removing subgroups either bottom-up or top-down as above, depending on the assumptions on $\mathcal{C}$. $\square$

Remark 2.6. Note that Lemma 2.5 and its proof help explain why normalizer formulas tend to have the freedom of choice of $\mathcal{C}$ of *both* the subgroup and the centralizer formulas.

The next lemma will enable us to get the vertical lines in Theorem 1.1.

Lemma 2.7. *Let $\mathcal{C}$ be a collection of p-subgroups in a finite group G, with $S \in \mathrm{Syl_p(G)}$.*

(1) *Suppose that $\mathcal{C}$ is closed under passage to p-overgroups.*
Then the canonical functor $E\mathbf{O}_{\mathcal{C}} \to \mathcal{C}$ induces an S-homotopy equivalence on nerves.
(2) *Suppose that $\mathcal{C}$ is closed under passage to non-trivial subgroups.*
Then the canonical functor $E\mathbf{A}_{\mathcal{C}} \to \mathcal{C}$ induces an S-homotopy equivalence on nerves.

Proof. Let H denote an arbitrary subgroup of S. By (†) the map $E\mathbf{O}^H_{\mathcal{C}} \to \mathcal{C}^H$ identifies with the inclusion $\mathcal{C}_{\geqslant H} \to \mathcal{C}^H$. However if Q and H are p-groups and $H \leqslant N_G(Q)$, then QH is a p-group also normalized by H, so that $QH \in \mathcal{C}^H$ using the closure hypothesis of (1); then $Q \mapsto QH$ defines a poset endomorphism on $\mathcal{C}^H$ with image in $\mathcal{C}_{\geqslant H}$. Now 2.1 shows that $|\mathcal{C}_{\geqslant H}| \to |\mathcal{C}^H|$ is a homotopy equivalence. Then (1) follows.

For (2), note that by (†) the map $E\mathbf{A}^H_{\mathcal{C}} \to \mathcal{C}^H$ identifies with the inclusion $\mathcal{C}_{\leqslant C_G(H)} \to \mathcal{C}^H$. By elementary group theory, if Q is non-trivial and $Q \leqslant N_G(H)$, then $C_Q(H)$ is non-trivial as well, since both H and Q are p-groups. Then $C_Q(H) \in \mathcal{C}^H$ using the closure hypothesis in (2). Hence $Q \mapsto C_Q(H)$ gives a poset endomorphism on $\mathcal{C}^H$ with image in $\mathcal{C}_{\leqslant C_G(H)}$. By 2.1, this induces a deformation retraction of $|\mathcal{C}^H|$ onto $|\mathcal{C}_{\leqslant C_G(H)}|$. So $|E\mathbf{A}_{\mathcal{C}}| \to |\mathcal{C}|$ is an S-homotopy equivalence as wanted. $\square$

3. Proofs of the two theorems

Proof of Theorem 1.1. We first establish the horizontal lines. Since vertical dotted lines always exist (by (†) for $H = 1$) it is enough to establish the solid horizontal lines.

Consider the lines between columns $\mathcal{B}$ and $\mathcal{S} = \mathcal{S}_p(G)$, and between columns $\mathcal{BC}e$ and $\mathcal{C}e$. Note that the p-centric condition is closed under p-overgroups, so for $\mathcal{X} = \mathcal{S}$ or $\mathcal{C}e$, we have $\mathcal{X}_{>P} = \mathcal{S}_p(G)_{>P}$ for any $P \in \mathcal{X}$. Observe that by elementary group theory, if $Q > P$ then $N_Q(P) > P$; i.e., $N_Q(P) \in \mathcal{S}_p(G)_{>P}$. If $P \notin \mathcal{B}$, then $O_p\big(N_G(P)\big) \in \mathcal{S}_p(G)_{>P}$. Hence the standard inequalities

$$Q \geqslant N_Q(P) \leqslant N_Q(P) O_p\big(N_G(P)\big) \geqslant O_p\big(N_G(P)\big)$$

describe a zig-zag of $N_G(P)$-equivariant functors which by 2.1 show that $\mathcal{X}_{>P}$ is $N_G(P)$-contractible to the point $O_p\big(N_G(P)\big)$. The two solid lines between the respective columns now follow, using (1) and (3) of Lemma 2.5.

Next consider the lines between columns $\mathcal{S}$ and $\mathcal{A}$: For $P \in \mathcal{S}$, we denote by $\Phi(P)$ the Frattini subgroup of P, the smallest normal subgroup of P such that $P/\Phi(P)$ is elementary abelian. By elementary group theory (see [Gor68, Theorem 5.1.1]) $\Phi(P) < P$, and if $Q < P$ then also $\Phi(P)Q < P$. Furthermore, if $P \notin \mathcal{A}$ then $\Phi(P) \neq 1$. Hence the standard inequalities $Q \leqslant \Phi(P)Q \geqslant \Phi(P)$ show that $\mathcal{S}_p(G)_{<P}$ is $N_G(P)$-contractible using 2.1. The two solid lines between columns $\mathcal{S}$ and $\mathcal{A}$ now follow, using (2) and (3) of Lemma 2.5.

We turn to arguments via the functor method of Lemma 2.2.

Consider the lines between columns $\mathcal{I}$ and $\mathcal{S}$: For $P \in \mathcal{S}$, define

$$F(P) = \bigcap_{P \leqslant S \in Syl_p(G)} S;$$

and observe that $P \leqslant F(P)$, so that $F \geqslant \mathrm{Id}_{\mathcal{S}}$. Likewise if $P \leqslant Q \in \mathcal{S}$, the Sylow groups above Q are also above P, so that $F(P) \leqslant F(Q)$. So the two solid lines between columns $\mathcal{I}$ and $\mathcal{S}$ now follow using (1) and (3) of Lemma 2.2. These lines, together with the earlier solid lines between $\mathcal{B}$ and $\mathcal{S}$, now by composition imply the lines between $\mathcal{B}$ and $\mathcal{I}$.

Consider the lines between columns $\mathcal{A}$ and $\mathcal{Z}$: For $P \in \mathcal{A}$, define $F(P) = \Omega_1 O_p Z\big(C_G(P)\big)$. We see that $P \leqslant F(P)$, and also that $C_G(P) \leqslant C_G\big(F(P)\big)$; while for $P \leqslant Q \in \mathcal{A}$, we have $Q \leqslant C_G(Q) \leqslant C_G(P)$, so that $F(P)$ centralizes $C_G(Q)$ and in particular lies in $C_G(Q)$, and hence $F(P) \leqslant F(Q)$. Thus again F defines an G-equivariant poset endomorphism on $\mathcal{A}$ with $F \geqslant \mathrm{Id}_{\mathcal{A}}$. We may let F^∞ denote the repeated iteration of F; for any finite group G, a finite number of iterations suffices. Then F^∞ is idempotent, with image $\mathcal{Z}$. The three horizontal solid lines between columns $\mathcal{A}$ and $\mathcal{Z}$ now follow using (1)–(3) of Lemma 2.2. This completes the proof of the horizontal solid lines between columns, and hence as we mentioned of the dotted horizontal lines as well.

We turn to the vertical lines. We observed that we at least have dotted vertical lines in all columns. So it remains to establish the stronger dashed vertical lines: namely S-equivalences for S Sylow in G.

We have observed that $\mathcal{C}e$ is closed under p-overgroups, while the definition of $\mathcal{E}$ shows it is closed under non-trivial subgroups; and by its definition $\mathcal{S}$ is of course closed both above and below. Hence the dashed lines in columns $\mathcal{C}e$, $\mathcal{S}$, and $\mathcal{E}$ follow, respectively, from (1), (1) and (2), and (2) of Lemma 2.7. The remaining dashed lines now follow from the composition of the parallel vertical dashed lines in the above columns with adjacent horizontal solid lines already established. This completes the proof of all the vertical lines shown, and hence of all the lines in the table of Theorem 1.1.

It remains to show that there are no more lines than those stated. We provide counterexamples below, considering lines of each type in turn.

Nonexistence of further dotted lines: The horizontal and vertical lines established so far show that we have at most four homotopy types in the table, represented by $\mathcal{D}$, $\mathcal{C}e$, $\mathcal{S}$, and $\mathcal{E}$. To rule out further dotted lines, we will give counterexamples showing that these four homotopy types can be distinct.

First consider the group $G = ((\mathbf{Z}/p)^p \times \mathbf{Z}/q) \rtimes \mathbf{Z}/p$, for a prime q such that $p \mid q-1$, where $\mathbf{Z}/p$ acts on $(\mathbf{Z}/p)^p$ via permutation and on $\mathbf{Z}/q$ by multiplying by a pth root unity in $\mathbf{Z}/q$. This example shows that $\mathcal{D}$ has a homotopy type distinct from that of the others, since in that case $\mathcal{D}$ is non-contractible, while $\mathcal{C}e$, $\mathcal{S}$, and $\mathcal{E}$ are contractible.

Next consider $G = \mathbf{Z}/p \times (\mathbf{Z}/q \rtimes \mathbf{Z}/p)$, with q as above and the same action. Here $\mathcal{C}e$ is non-contractible, while $\mathcal{S}$ and $\mathcal{E}$ (which here equals $\mathcal{S}$) are contractible.

Finally we claim that $\mathcal{S}$ is connected while $\mathcal{E}$ is disconnected for $G = \mathrm{SL}_2(\mathbf{F}_{p^2}) \rtimes \mathbf{Z}/p$, where $\mathbf{Z}/p$ acts by field automorphisms: Since the inclusion $\mathrm{SL}_2(\mathbf{F}_{p^2}) < G$ gives an identification of $\mathcal{S}_p(\mathrm{SL}_2(\mathbf{F}_{p^2}))$ with $\mathcal{E}_p(G)$, we see that $\mathcal{E}$ is disconnected. To see that $\mathcal{S}$ is connected, first observe that the subgroups S and S' of strictly upper and lower triangular matrices in $\mathrm{SL}_2(\mathbf{F}_{p^2})$ are connected in $\mathcal{S}_p(G)$ (we have a zig-zag $S \leqslant SQ \geqslant Q \leqslant S'Q \geqslant S'$, for Q the order p subgroup of field automorphisms, since S and S' are normalized by Q); this in fact holds for any two Sylow p-subgroups in $\mathrm{SL}_2(\mathbf{F}_{p^2})$, since a Sylow p-subgroup is either equal to S or an S-conjugate of S'. The result now follows since every p-subgroup of G is contained in a Sylow p-subgroup, and every Sylow p-subgroup of G contains a Sylow p-subgroup of $\mathrm{SL}_2(\mathbf{F}_{p^2})$. (Interesting counterexamples also arise from sporadic simple groups for $p=2$; e.g., Co_3 in [Ben94], and other groups such as M_{12} in [BS].) This completes the proof that there can be no more dotted lines than the ones displayed.

Since we have now established the non-existence of further dotted lines, further dashed or solid lines could only arise within the four homotopy types already indicated. For ease of notation, for the rest of the proof let D denote the dihedral group of order 8 if $p=2$, and the extraspecial p-group p_+^{1+2} of order p^3 and exponent p, if p is odd.

Nonexistence of further dashed lines: We work first within the homotopy type of $\mathcal{C}e$: Taking $G = D$, so that $S = G$ and $C_G(S) = Z(G)$ is of order p, we see that $\mathcal{BC}e_{\leqslant C_G(S)}$ and $\mathcal{C}e_{\leqslant C_G(S)}$ are empty, while $\mathcal{C}e^S$ is not; so by (†) there are no corresponding dashed lines. Further taking H to be a maximal abelian subgroup in $G = S$, we see that $\mathcal{BC}e_{\leqslant C_G(H)}$ is empty while $\mathcal{C}e_{\leqslant C_G(H)}$ is not.

Now consider the homotopy type of $\mathcal{E}$; again taking $G = D = S$ shows via (†) that $\mathcal{E}_{\geqslant S}$ is empty but $\mathcal{E}^S$ is not.

Next consider the homotopy type of $\mathcal{S}$. Again taking $G = D = S$, and taking H to be a rank-2 elementary abelian p-subgroup shows that $\mathcal{B}_{\leqslant C_G(H)}$ and $\mathcal{I}_{\leqslant C_G(H)}$ are empty while $\mathcal{A}_{\geqslant H}$ and $\mathcal{A}^H$ are not. However $\mathcal{A}_{\geqslant S}$ is empty while $\mathcal{A}^S$ is not. Finally for $G = \mathrm{SL}_3(\mathbf{F}_p)$ and $S = D$, $\mathcal{B}_{\leqslant C_G(S)}$ is empty but $\mathcal{I}_{\leqslant C_G(S)}$ is not.

We are hence left with the homotopy type of $\mathcal{D}$. Note that for $G = D = S$, $\mathcal{D}_{\leqslant C_G(S)}$ is empty while $\mathcal{D}^G$ and $\mathcal{D}_{\geqslant G}$ are not. This reduces us to showing that there is no dashed line between $\mathcal{D}$ and $E\mathbf{O}_{\mathcal{D}}$, which requires a slightly more elaborate argument:

Fix a prime p; and take some $a \geqslant 1$ for $p > 3$, but $a \geqslant 3, 2$ when $p = 2, 3$—the reason for this choice of a will emerge later. Now choose a prime q with $p^a \mid q - 1$ but $p^{a+1} \nmid q - 1$. Let $G = \mathbf{Z}/q \rtimes \mathrm{GL}_p(\mathbf{F}_q)$, where $\mathrm{GL}(\mathbf{F}_q)$ acts on $\mathbf{Z}/q$ by letting the index p subgroup H generated by $\mathrm{SL}_p(\mathbf{F}_q)$ and the scalar matrices act trivially and the quotient act faithfully.

In this case, we claim that G has exactly two conjugacy classes of principal p-radical subgroups, represented by: a Sylow p-subgroup S of order p^{pa+1}; and another subgroup Q of order p^{a+2} given by the central product of the cyclic subgroup of order p^a of scalar matrices with an extraspecial group of order p^3 (of exponent p for odd p, but quaternion of order 8 when $p = 2$). To see this, note that it suffices to work in the quotient group $\mathrm{GL}_p(\mathbf{F}_q)$, where the statement follows for example using [AF90, 4A] for p odd, and the analogous result [An92, 2B] for $p = 2$. (The only possible values for the parameters given in those results are $V_0 = 0$, $s = 1$, $\alpha = 0$; and either $(e, \gamma) = (p, 0)$ corresponding to S, or $(e, \gamma) = (1, 1)$ corresponding to Q.) By construction Q is a subgroup of H, but since $pa > a + 2$ (by the choice of a) it is not a Sylow p-subgroup of H. We can hence pick a p-subgroup $P \leqslant N_H(Q)$ strictly containing Q. Then the poset $\mathcal{D}^P$ consists of the G-conjugates of Q and S normalized by P, which is easily seen to be connected: if Q' is G-conjugate to Q and normalized by P, then for S' a Sylow p-subgroup containing $Q'P$, Q' will be connected to Q via $Q' \leqslant S' \geqslant Q$; and likewise for a Sylow p-subgroup S'' normalized by P, we have $P \leqslant S''$ so $Q \leqslant S''$. On the other hand the poset $\mathcal{D}_{\geqslant P}$ is disconnected, since the fact that P is contained in H and the non-triviality of the action of $\mathrm{GL}_p(\mathbf{F}_q)$ on $\mathbf{Z}/q$ ensures that there is more than one G-conjugate of S containing P.

Nonexistence of further solid lines: We can continue to work within the indicated homotopy types. Since there is at least a dotted line in each row of these, and we have just seen in particular that dotted lines cannot be strengthened to dashed lines, it is now enough to see that there can be no new solid lines going between rows. For $G = \mathbf{Z}/p \rtimes \mathbf{Z}/q$, $q \mid p - 1$, for p odd or $G = \mathrm{Alt}_4$ for $p = 2$, and for each $\mathcal{C}$ in the table, $\mathcal{C}_{\leqslant C_G(G)}$ and $\mathcal{C}_{\geqslant G}$ are empty while $\mathcal{C}^G$ is not. For $G = \mathbf{Z}/p \times \mathbf{Z}/q$, q a prime different from p, and each $\mathcal{C}$ in the table, $\mathcal{C}_{\geqslant G}$ is empty, while $\mathcal{C}_{\leqslant C_G(G)}$ and $\mathcal{C}^G$ are not.

This finishes the elimination of any further lines, and hence the proof of Theorem 1.1. □

Remark 3.1. The G-homotopy equivalences along the normalizer row in Theorem 1.1 were at least known classically: The ordinary equivalence between $\mathcal{S}$ and $\mathcal{A}$ was observed by Quillen [Qui78, 2.1], and between $\mathcal{S}$ and $\mathcal{B}$ by Bouc [Bou84, Corollary p. 50]. The G-homotopy equivalence was first observed by Thévenaz-Webb [TW91]. The G-equivalence between $\mathcal{I}$ and $\mathcal{S}$ was probably first observed by Alperin sometime in the 1990s via a nerve-of-covering argument; see also [Not01]. That $\mathcal{A}$ and $\mathcal{Z}$ are G-homotopy equivalent was observed in [Ben98, Section 6.6] (though seemingly our argument differs from the one intended there). Finally, a number of the remaining horizontal equivalences can be found implicitly in [Dwy98].

Proof of Theorem 1.2. The table in the theorem gives references to the first publication known to us of any particular result. The two new positive results follow from the earlier positive results in the same columns via dashed lines in Theorem 1.1.

A counterexample to centralizer sharpness for $\mathcal{D}, \mathcal{B}\mathcal{C}e, \mathcal{C}e, \mathcal{B}$, and $\mathcal{I}$ is provided by $G = D = p_+^{1+2}$, the extraspecial group of order p^3 and exponent p for p odd, and $D = D_8$ for $p = 2$: This is easy for all the indicated collections $\mathcal{C}$ except $\mathcal{C}e$, since in those other cases $\mathcal{C}$ consists of just D, and the mod p cohomology of D is different from that of $Z(D) = \mathbf{Z}/p$. For $\mathcal{C} = \mathcal{C}e_p(D)$ we calculate directly that $H^*(D; \mathbf{F}_p) \to \lim^0_{P \in \mathbf{A}_\mathcal{C}} H^*(C_G(P); \mathbf{F}_p)$ is not an isomorphism, for instance by observing that the two sides do not have the same Krull dimension.

That each of $\mathcal{C} = \mathcal{A}$, $\mathcal{Z}$, and $\mathcal{E}$ is not subgroup sharp is likewise easy, since taking $G = \mathbf{Z}/p^2$, we observe that $H^*(\mathbf{Z}/p^2; \mathbf{F}_p) \to H^*(p\mathbf{Z}/p^2; \mathbf{F}_p)$ is not an isomorphism. $\square$

Remark 3.2. Note that by propagation in the graph of Theorem 1.1, the normalizer sharpness of $\mathcal{S}$ (for example) implies 10 other sharpness results. (See e.g., [Web91, §2.5; AM94, §V.3; Dwy98, 7.2], or [Gro02, 8.2, §9] for various proofs of the normalizer sharpness of $\mathcal{S}$.) Also note that Theorem 1.1 allows us for example to obtain normalizer sharpness of $\mathcal{C}e$ from subgroup sharpness of $\mathcal{C}e$, and vice versa. (Compare [Dwy98,SY97,Gro02].)

Remark 3.3. We have seen that $|E\mathbf{O}_\mathcal{D}|$ and $|\mathcal{D}|$ are in general not S-homotopy equivalent. However it is possible to analyze the map $E\mathbf{O}_\mathcal{D} \to \mathcal{D}$ to show that it induces an equivalence on Bredon cohomology with values in any cohomological Mackey functor $\mathfrak{F}$ with $|G:S|$ invertible in R and with the *additional* property that $\mathfrak{F}$ vanishes on p'-subgroups.

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