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The Atiyah-Segal Completion Theorem

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Resumé

Atiyah-Segals fuldstændiggørelsessætning er et vigtigt bindeled mellem repræsentationsteori og algebraisk topologi. Sætningen relaterer fuldstændiggørelse forstået algebraisk og geometrisk, og giver en sammenligning af ækvivarient K-teori for et rum med en virkning af en kompakt liegruppe med almindelig K-teori for borelkonstruktionen for rummet.

I dette speciale vil vi gennemgå de nødvendige forudsætninger for at forstå sætningen og dens bevis, og vi vil præsentere et moderne bevis. Teorien vil blive belyst gennem eksempler.

Abstract

The Atiyah-Segal completion theorem is an important link between representation theory and algebraic topology. The theorem relates the algebraic and the geometric notions of completion, and gives a comparison between the equivariant K-theory of a space with an action of a compact Lie group with the ordinary K-theory of the Borel construction of the space.

In this thesis we will cover the prerequisites necessary in order to understand the theorem and its proof, and we will present a modern proof. The theory will be illustrated by means of examples.

Preface

This thesis is submitted for the degree of cand.scient. in Mathematics at The University of Copenhagen.

As a service to the reader there's an index to key concepts and notation before the reference list.

Acknowledgements I am very grateful to all my friends for their support. I would also like to thank my advisor, Jesper Grodal, for helpful discussions, support and encouragement along the way. Thank you all!

An apology I am sorry that I didn't have time to polish this thesis properly, but of course, I have only myself to blame for that. Therefore, a lot is left out that I wanted to say, and many misprints will surely be present, that I'd rather were not. Ah, well!

Copenhagen, July 2008

Ulrik Torben Buchholtz

Addendum I have made some corrections to the presentation that hopefully clears out most errors in the original. Most importantly, I have corrected the treatment of Example 8.1. I hope these changes may benefit future readers!

Palo Alto, August 2010

Contents

1 Introduction

This thesis is about various aspects of the Atiyah-Segal completion theorem and its generalizations.

1.1 Organization This thesis is organized as follows:
Chapter 2 is a cursory introduction to Lie groups and their representation theory;
Chapter 3 gives an overview of prerequisites from equivariant homotopy theory;
Chapter 4 introduces complex equivariant K-theory;
Chapter 5 discusses the technicalities of progroups;
Chapter 6 does the same for algebraic machinery of completion;
Chapter 7 finally covers the completion theorem in its various guises;
Chapter 8 presents some examples.
The reader is assumed to be familiar with ordinary (non-equivariant) homotopy theory, and we'll develop the equivariant story from that.

1.2 Category theory We will extensively employ the language and results of category theory; a basic reference is Mac Lane [15]. Here we work with a naive notion of categories, but see the note by Feferman [8] for a discussion of the foundations.

1.3 Notation and terminology By a *space* we shall always mean a compactly generated weak Hausdorff space.

We will use the following notation for standard categories:

Set	sets and functions,
Grp	groups and group homomorphisms,
Ab	abelian groups and homomorphisms,
Rng	commutative unital rings and unital ring homomorphisms,
R**Mod**	left R-modules and homomorphism for a commutative unital ring R.
Top	spaces and continuous functions (also called maps),
$\mathbf{Top}_*$	based spaces and based continuous functions (also called based maps),

2 Groups and representations

2.1 Group objects A *group object* in a category $\mathbf{C}$ with finite products is an object G of $\mathbf{C}$ together with structure morphisms

- $m\colon G \times G \to G$ (multiplication)
- $e\colon 1 \to G$ (identity)
- $i\colon G \to G$ (inverse)

such that the following axioms modeled on the usual group axioms are satisfied:

- m is associative: $m \circ (m \times 1_G) = m \circ (1_G \times m)$ as morphims $G \times G \times G \to G$.
- e is a two-sided unit: $m \circ (e \times 1_G) = 1_G = m \circ (1_G \times e)$ as morphism $G \to G$.
- i provides two-sided inverses: .

The group objects of $\mathbf{C}$ form a subcategory in which we can do group theory.

2.2 Actions A left G-object in $\mathbf{C}$ is an object X of $\mathbf{C}$ with an action morphism $a\colon G \times X \to X$ such that

- the action commutes with the multiplication: $a \circ (m \times 1_X) = a \circ (1_G \times a)$ as morphisms $G \times G \times X \to X$.
- the identity fixes X: $a \circ (e \times 1_X) = 1_X$.

A left G-morphism from (X_1, a_1) to (X_2, a_2) is a morphism $f\colon X_1 \to X_2$ in $\mathbf{C}$ with $f \circ a_1 = a_2 \circ (1_G \times f)$ as morphisms $G \times X_1 \to X_2$. The left G-objects with left G-morphisms thus form a subcategory of $\mathbf{C}$.

We define right G-objects and right G-morphisms analogously.

2.3 Topological groups We will in the sequel concern ourselves with topological groups. A topological group is a group object in the category of spaces, **Top**. In the sequel, the word *subgroup* will mean a *closed* subgroup of a topological group.

If G is a topological group, then the left G-objects in **Top**form a category of spaces with a continuous left G-action, G**Top**, whose objects we call G-spaces. By a G-equivariant map (or G-map for short) we mean a left G-morphism of G-spaces.

2.4 Lie groups A Lie group is a group object in the category of finite dimensional smooth manifolds. That is, it is a smooth manifold equipped with a group structure, and whose structure maps are smooth. Lie groups are of course special cases of topological groups.

2.5 Compenent group Given any topological group G, we denote by G_0 the set of components of G. By continuity of the group maps, the group structure on G induces a group structure on G_0. We call G_0 the *component group* of G.

2.6 Descending chain condition Let us here note a fairly obvious, but quite important, properly of compact Lie groups, namely, that they satisfy the descending chain condition. That is, given a compact Lie group G, every descending sequence of (closed) subgroups is eventually stationary. This follows, since a subgroup will either be of smaller dimension, or else have a smaller component group. Thus, any strictly descending chain of subgroups will terminate after at most ds terms, where d is the dimension of G and s is the number of subgroups of the component group (which is finite since G is compact).

A partially ordered set satisfies the descending chain condition if and only if it is well-founded,[1] so the importance of the descending chain condition is that it allows us to do proofs by well-founded induction over subgroups.

2.7 Representations As usual, we will confine ourselves to the case of *complex* representations. A basic reference is Adams' book [2].

Fix a topological group G. A *representation* of G (or a G-vector space, or a G-module) is a finite dimensional vector space V together with a continuous homomorphism $\rho\colon G \to \operatorname{Aut} V$. If we choose an n-element basis for V, then we can regard ρ as taking values in $\mathrm{GL}(n, \mathbb{C})$.

A G-homomorphism of G-vector spaces V and W is a linear map $f\colon V \to W$ such that $f(gv) = gf(v)$ for all $g \in G$ and $v \in V$.

Given G-vector spaces V and W we can form new G-vector spaces: the *direct product* $V \oplus W$ and the *tensor product* $V \otimes W$ by setting

$$g(v, w) = (gv, gw) \quad \text{and} \quad g(v \otimes w) = gv \otimes gw \quad \text{for } g \in G,\ v \in V \text{ and } w \in W.$$

We can also form the vector space $\mathrm{Hom}_{\mathbb{C}}(V, W)$ and equip it with a G-action such that

$$(gf)(v) = g(h(g^{-1}v)) \quad \text{for } g \in G,\ f \in \mathrm{Hom}_{\mathbb{C}}(V, W) \text{ and } v \in V.$$

Note that the G-homomorphisms from V to W are exactly the fixed points of this action.

Having defined the Hom-representation, we can also define the *dual* representation, V^*, of a representation V by setting $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, and the canonical isomorphism, $\mathrm{Hom}_{\mathbb{C}}(V, W) \cong V^* \otimes W$, still holds equivariantly.

2.8 Compact groups Now we will restrict ourselves further to the case of complex groups, which are very nicely behaved with respect to representation theory. This is because we can *integrate* over complex groups [2, p. 33]. Namely, there is a positive linear functional,

$$\int_G -\colon C(G, \mathbb{R}) \to \mathbb{R},$$

that is invariant under left and right translation,

$$\int_{y \in G} f(xy) = \int_{y \in G} f(y) = \int_{y \in G} f(yx) \quad \text{for } x \in G \text{ and } f \in C(G, \mathbb{R}),$$

and normed such that the constant function 1 has integral 1. This integral/measure was first constructed by Haar [10] in 1933.

We'll mention two important results of using integration for a compact topological group G.

[1] In full generality this statement requires the Axiom of Choice, but we will not be needing the full force here.

2.9 Proposition [2, 3.16] *Given any G-vector space V, there is a Hermitian inner product on V that is invariant under G.*

Proof. If $(-,-)$ is any inner product on V, define a new inner product by setting

$$\langle v, w\rangle = \int_{g\in G} (g^{-1}v, g^{-1}w).$$

This is easily seen to be invariant under G. □

2.10 Proposition [2, 3.18] *Any short exact sequence of G-vector spaces split.*

Proof. Given a short exact sequence of G-vector spaces

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\mu"] & N \arrow[r, "\nu"] & P \arrow[l, dashed, bend left, "\sigma"] \arrow[r] & 0 \end{tikzcd}$$

where σ is a nonequivariant splitting, we define an equivariant splitting τ by setting

$$\tau(p) = \int_{g\in G} g\sigma(g^{-1}p).$$ □

2.11 Reducibility We say that a nonzero G-space V is *reducible* if there is some proper subspace of V that is fixed by G. Otherwise V is called *irreducible*. A fundamental tool is the following classic:

2.12 Lemma (Schur) [2, 3.22] *Let G be a topological group.*
- *(i) If $f\colon V \to W$ is a G-homomorphism of irreducible G-vector spaces V and W, then f is either zero or an isomorphism.*
- *(ii) If $f\colon V \to V$ in a G-endomorphism of an irreducible G-vector space V, then f is multiplication by some $\lambda \in \mathbb{C}$.*

Proof. (i) The kernel and the image of f are G-subspaces, and so either 0 or everything, and the result follows.
(ii) Let λ be an eigenvalue for f, then $f - \lambda$ can't be an isomorphism, and so must be 0 by (i). □

2.13 The Grothendieck construction [13, p. 39] Given an additively written commutative monoid M, we define a commutative group $K(M)$ as follows. Let $\mathbb{Z}^{(M)}$ denote the free abelian group generated by the elements of M, and let $K(M)$ be the quotient of $\mathbb{Z}^{(M)}$ module the subgroup generated by elements of the form

$$\delta_{x+y} - \delta_x - \delta_y$$

for elements x and y in M, and where δ_x denotes the generator of $\mathbb{Z}^M$ corresponding to $x \in M$.

There is a canonical monoid homomorphism $\gamma\colon M \to K(M)$ defined by taking $x \in M$ to the class $[x] \in K(M)$ represented by $\delta_x \in \mathbb{Z}^{(M)}$. Evidently, γ is injective if and only if the cancellation law holds in M.

We call $K(M)$ the *Grothendieck group* corresponding to M, and the construction is sometimes called the *Grothendieck construction*. We have the following universal property. If $f\colon M \to A$ is a monoid homomorphism of M to an abelian group A, then there is a unique group homomorphism $f_*\colon K(M) \to A$ such that $f = f_*\gamma$.

If in addition to being a commutative monoid, M is a *semiring* (that is, M has a multiplication that distributes over addition), then $K(M)$ becomes a ring, which is commutative if the multiplication on M is.

2.14 The representation ring For a compact topological group G we let $R(G)$ be the free abelian group generated by the equivalence classes of irreducible representations of G. By induction using Proposition 2.10, every representation of G is the direct sum of irreducible representations, and by Schur's Lemma, such a decomposition is unique. Thus, the equivalence classes of representations of G are in 1-1 correspondence with the elements of $R(G)$ with nonnegative coefficients, and we see that we can also obtain $R(G)$ as the Grothendieck construction on the monoid of isomorphism classes of representations with direct sum.

Using the tensor product of representations, we can make $R(G)$ into a commutative ring, called the *representation ring* of G. The unit of $R(G)$ is the trivial representation.

There is an *augmentation* homomorphism $\varepsilon(G)\colon R(G) \to \mathbb{Z}$ defined by sending an irreducible representation to its dimension. The kernel of ε is denoted $I(G)$ – this is the *augmentation ideal*.

Segal published a papar [22] describing the representation ring of a compact Lie group. He attributes the following key result to Atiyah:

2.15 Theorem [22, Proposition 3.2] *If H is a subgroup of G, then the restriction $R(G) \to R(H)$ makes $R(H)$ a finitely generated module over $R(G)$. In particular, $R(G)$ is a finitely generated Noetherian ring for all G.*

2.16 Supports Given any subgroup $H \subset G$ we can consider the restriction homomorphism $r^G_H\colon R(G) \to R(H)$ and the induced map of prime ideal spectra $\operatorname{Spec} R(H) \to \operatorname{Spec} R(G)$. Fixing a prime $\mathfrak{p} \in \operatorname{Spec} R(G)$ we then consider the set of subgroups H such that $\mathfrak{p}$ comes from an element of $\operatorname{Spec} R(H)$. This set has minimal elements since the set of subgroups of G is well-founded, and any minimal element is called the *support* of $\mathfrak{p}$. This terminology is justified by the following result:

2.17 Proposition *The support of a prime $\mathfrak{p}$ of $R(G)$ is determined up to conjugation in G, and all supports are cyclic subgroups.*

This is a part of (i) in Proposition 3.7 in Segal's paper. It is proved by considering the various fibers of the canonical map $\operatorname{Spec} R(G) \to \operatorname{Spec} \mathbb{Z}$ (and thus corresponding to various primes (including 0) of $\mathbb{Z}$).

2.18 Lemma [1, Lemma 3.5] *If $S \subset H$ is a support of a prime ideal $\mathfrak{q} \subset R(H)$, then S is a support of $\mathfrak{p} = (r^G_H)^{-1}(\mathfrak{q}) \subset R(G)$.*

Proof. Since $r^G_S = r^H_S r^G_H$, we conclude that $\mathfrak{p}$ comes from $R(S)$. Since S is minimal for $\mathfrak{q}$, it is also minimal for $\mathfrak{p}$, and we conclude that S is a support of $\mathfrak{p}$. □

The important point of this lemma is that if a prime $\mathfrak{p}$ of $R(G)$ comes from $R(H)$, then $\mathfrak{p}$ has a support that is a subgroup of H.

3 Equivariant homotopy theory

In this chapter we will briefly outline the machinery of equivariant homotopy theory based on the introduction given by May [18, Chapter 1]. See also the papers by Matumoto [17] and Waner [24].

Fix a topological group G. We have a category $G\mathbf{Top}$ of G-spaces and G-maps. We define a G-homotopy of G-maps $X \to Y$ to be a G-map $X \times I \to Y$, where the interval I has the trivial G-action. We can then go on to develop equivariant homotopy theory analogously to the development of ordinary homotopy theory of spaces. For every subgroup $H \subset G$ we have a functor of fixed points $(-)^H \colon G\mathbf{Top} \to \mathbf{Top}$ sending a G-space X to the space

$$X^H = \{\, x \in X \mid \forall g \in H \colon gx = x \,\}.$$

We say that a G-map $f \colon X \to Y$ is a weak equivalence in $G\mathbf{Top}$ if $f^H \colon X^H \to Y^H$ is a weak equivalence for all $H \subset G$.

A cofibration in $G\mathbf{Top}$ is defined by the homotopy extension property, just as for ordinary spaces. That is, the G-map $f \colon A \to X$ is a cofibration if for every solid arrow commutative diagram in $G\mathbf{Top}$

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "f"'] & & A \times I \arrow[dl] \arrow[dd, "f \times 1_I"] \\
& Y & \\
X \arrow[ur] \arrow[rr, "i_0"'] & & X \times I \arrow[ul, dashed]
\end{tikzcd}$$

we can find the dotted arrow making the entire diagram commute.

Analogously, a fibration in $G\mathbf{Top}$ is defined by the homotopy lifting property.

3.1 Adjunctions We have several useful adjunctions relating different functors occurring in equivariant topology. First, let us note that the familiar nonequivariant adjunction

$$\mathrm{Map}_G(X \times Y, Z) \cong \mathrm{Map}_G(X, \mathrm{Map}_G(Y, Z))$$

is a G-homeomorphism for G-spaces X, Y and Z.

Given a subset $H \subset G$, consider the forgetful functor $U \colon G\mathbf{Top} \to H\mathbf{Top}$ that restricts the action. This is right adjoint to the functor $G \times_H - \colon H\mathbf{Top} \to G\mathbf{Top}$ defined for $X \in H\mathbf{Top}$ by setting

$$G \times_H X = G \times X/\sim, \quad \text{where } (gh, x) \sim (g, hx) \text{ for } g \in G,\ h \in H \text{ and } x \in X.$$

We call $G \times_H X$ the *induced G-space*. Thus,

$$\hom_{G\mathbf{Top}}(G \times_H X, Y) \cong \hom_{H\mathbf{Top}}(X, UY)$$

for $X \in H\mathbf{Top}$ and $Y \in G\mathbf{Top}$.

The forgetful functor is left adjoint to functor $\mathrm{Map}_H(G,-)\colon H\mathbf{Top} \to G\mathbf{Top}$, where the left G-action on $\mathrm{Map}_H(G,Y)$ is given by setting $(gf)(g') = f(g'g)$ for $g, g' \in G$ and $f \in \mathrm{Map}_H(G,Y)$. Then we have the adjunction

$$\hom_{H\mathbf{Top}}(UX, Y) \cong \hom_{G\mathbf{Top}}(X, \mathrm{Map}_H(G,Y)).$$

We call $\mathrm{Map}_H(G,Y)$ the *coinduced G-space*.

Also important is the fact that the fixed point functor $(-)^H\colon G\mathbf{Top} \to \mathbf{Top}$ is right adjoint to the functor $G/H \times -\colon \mathbf{Top} \to G\mathbf{Top}$, that is,

$$\hom_{G\mathbf{Top}}(G/H \times X, Y) \cong \hom_{\mathbf{Top}}(X, Y^H) \quad \text{for } X \in \mathbf{Top} \text{ and } Y \in G\mathbf{Top}.$$

3.2 The based story As for ordinary spaces, there's a seperate story for the based case. We take the category $G\mathbf{Top}_*$ to be the category of based G-spaces *where the basepoint is fixed by the action.* There is a functor $(-)_+\colon G\mathbf{Top} \to G\mathbf{Top}_*$ adjoining a disjoint basepoint to a G-space. The categorical coproduct is the wedge sum, and the smash product, defined as for ordinary spaces, is a symmetric monoidal product. We have adjunctions similar to the ones in the unbased case.

A based homotopy of based G-maps is given by a based G-map $X \wedge I_+ \to Y$. The construction of the homotopy category follows the usual route.

3.3 G-CW-complexes A short introduction is given by May [18, Section I.3]. We would like to have a family of spaces in $G\mathbf{Top}$ that are as nice as CW-complexes are in $\mathbf{Top}$. CW-complexes are constructed by attaching cells modeled on spheres. In the equivariant world, we must attach spaces with a nice G-action. It turns out that the right notion of an *equivariant n-cell* is a space of the form $(G/H) \times D^n$ for subgroups $H \subset G$ attached via G-maps out of $(G/H) \times S^{n-1}$. Here G acts trivially on D^n.

Of course, by adjunction, an attaching G-map $G/H \times S^{n-1} \to X$ is equivalent to a map $S^{n-1} \to X^H$.

Attachment makes sense since $G\mathbf{Top}$ has all small colimits (as well as limits). We can now define G-CW-complexes analogously to ordinary CW-complexes.

3.4 Definition A G-CW-decomposition of a pair (X, A) is a filtration $A = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X$ such that:

1. $X = \bigcup_n X_n$.
2. For each $n \geq 0$ the space X_n is obtained from X_{n-1} by attaching G-equivariant n-cells.
3. The topology on X is the topology of the union.

Let $G\mathbf{CW}$ denote the category of (absolute) G-CW-complexes, and let $G\mathbf{CW}_*$ denote the category of G-CW-complexes relative the the fixed basepoint $*$.

An important fact is that when G is compact Lie group, then all smooth G-manifolds are G-CW-complexes [16, 24].

There is an equivariant version of the homotopy extension and lifting property for G-CW-complexes [24, Theorem 3.3]. Using this we can prove the equivariant version of the Whitehead Theorem, and there is a good notion G-CW-approximation, unique up to G-homotopy equivalence.

Given a subgroup $H \subset G$ we have a restriction functor $G\mathbf{Top} \to H\mathbf{Top}$ given by considering a G-space as an H-space. This functor restricts to a functor $G\mathbf{CW} \to H\mathbf{CW}$.

4 Equivariant K-Theory

4.1 Convention In this chapter we shall introduce the theory of complex equivariant K-theory. In most of the following G will denote a fixed topological group, and we will concern ourselves exclusively with *complex* vector spaces, and thus our bundles will be complex vector bundles and so on. Much of the theory can of course also be developed also for real vector bundles.

4.2 G-vector bundles We will now go on to develop the theory of G-vector bundles. There are a lot of references for this material, I've used mainly Atiyah's book [4], Hatcher's online manuscript-in-progress [11], May's concise book [19], and, where the equivariant case differs from the nonequivariant case, Segal's paper on equivariant K-theory [21].

A *G-vector bundle* over a G-space X is a G-space E with a G-map $p\colon E \to X$ (called the *projection*) such that:

(i) The map $p\colon E \to X$ is the projection of an ordinary (nonequivariant) complex vector bundle on X.

(ii) For each $g \in G$ and $x \in X$ the map of fibers $g\colon E_x \to E_{gx}$ is linear.

We call the space E the *total space*, and the space X the *base space*.

We define a *section* of a G-vector bundle $p\colon E \to X$ to be a (nonequivariant) map $s\colon X \to E$ such that $ps = 1_X$ (as in the diagram on the right). The section form a vector space ΓE, and we denote by $\Gamma^G E$ the subspace of *equivariant sections*, that is, those that happen to be G-maps.

$$\begin{tikzcd} E \arrow[d, "p"', shift right] \\ X \arrow[u, dashed, "s"', shift right] \end{tikzcd}$$

Given G-vector bundles E and F over X we can form the direct sum $E \oplus F$ and the tensor product $E \otimes F$, which are again G-vector bundles over X with $(E \oplus F)_x = E_x \oplus E_x$ and $(E \otimes F)_x = E_x \otimes F_x$. We can also form the bundle $\operatorname{Hom}(E, F)$ whose fiber over x is the vector space $\operatorname{Hom}_{\mathbb{C}}(E_x, F_x)$.

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & F \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

The G-vector bundles over X form a category when we define a *G-homomorphism* from $p\colon E \to X$ to $q\colon F \to X$ to be a G-map $\varphi\colon E \to F$ such that $q\varphi = p$ and such that restricted to each fiber, φ is a linear map $\varphi_x\colon E_x \to F_x$. The G-homomorphisms from E to F form a vector space isomorphic to $\Gamma^G \operatorname{Hom}(E, F)$.

Note that a G-vector bundle over the point is just the same as a G-vector space defined as in Section 2.7. Given any G-vector space V and a G-space X we can form the G-vector bundle $X \times V \to X$. Any G-vector bundle isomorphic to a bundle of this form will be called *trivial*.

4.3 Restriction and pullback of bundles Given a G-subspace $Y \subset X$ and a G-vector bundle $p\colon E \to X$ we can take the *restriction* of E to Y. This is just the restriction $p\colon p^{-1}(Y) \to Y$, and we denote it $E|Y$.

More generally, given any G-map $f\colon Y \to X$, we can define the *pullback* of E along f to be $f^*(p)\colon f^*(E) \to Y$, where we take the pullback in the category of G-spaces

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d, "f^*(p)"'] & E \arrow[d, "p"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

so f^*E consists of pairs $(y, e) \in Y \times E$ such that $f(y) = p(e)$. For $y \in Y$, the fiber over y is then identified with the fiber $E_{f(y)}$ of E over $f(y)$, and it thus acquires a natural vector space structure.

Given two composable maps $f\colon Y \to X$ and $g\colon Z \to Y$ there is a natural isomorphism $g^*f^*(E) \cong (fg)^*E$.

From now on, we'll assume that G is a *compact* group.

4.4 Lemma [21, Proposition 1.1] *If E is a G-vector bundle over a compact G-space X, and A is a closed G-subspace of X, then every equivariant section of $E|A$ extends to an equivariant section of E.*

Proof. Let $s\colon A \to E$ be a section of $E|A$. Use the Tietze Extension Theorem to extend s to all of X. Use the Haar integral to average the extension over all X so as to obtain an equivariant extension. □

Using this on the bundle $\mathrm{Hom}(E, F)$, we can extend an isomorphism $E|A \cong F|A$ to an isomorphism over a G-neighborhood U of A. Using this, we get:

4.5 Lemma [21, Proposition 1.3] *Let X by a compact G-space, Y a G-space, and f_t a G-homotopy of maps $X \to Y$. If E is a G-vector bundle over Y, then f_0^*E and f_1^*E are isomorphic G-bundles over X.*

4.6 Generalized cohomology theories Let us now recall the axioms for a generalized (unreduced) cohomology theory, but phrase the axioms so that they apply equivariantly (the ordinary case emerges when we take G to be the trivial group).

Let $G\mathbf{CW}^{(2)}$ denote the category of all G-CW-pairs (X, A) and G-maps of pairs $(X, A) \to (Y, B)$. We have a restriction functor $R\colon G\mathbf{CW}^{(2)} \to G\mathbf{CW}^{(2)}$ that takes a pair (X, A) to $(A, \emptyset)$ and a map f to the restriction $f|A$.

On $G\mathbf{CW}^{(2)}$ we have a good notion of relative equivariant homotopies, and we let $hG\mathbf{CW}^{(2)}$ denote the corresponding homotopy category. Of course, R descends to a functor on the homotopy category, which we also denote R.

We now define an unreduced equivariant cohomology theory on $hG\mathbf{CW}^{(2)}$ to be a sequence of contravariant functors $h^n\colon (hG\mathbf{CW}^{(2)})^{\mathrm{op}} \to \mathbf{Ab}$ and natural transformations $\delta^n\colon h^n \circ R \to h^{n+1}$ for $n \in \mathbb{Z}$ satisfying:

1. Exactness: For every pair $(X, A) \in hG\mathbf{CW}^{(2)}$ the long sequence

$$\begin{tikzcd} \cdots \arrow[r] & h^{n-1}(X,\emptyset) \arrow[r] & h^{n-1}(A,\emptyset) \arrow[dll, "\delta^{n-1}"] \\ h^n(X,A) \arrow[r] & h^n(X,\emptyset) \arrow[r] & h^n(A,\emptyset) \arrow[dll, "\delta^{n}"] \\ h^{n+1}(X,A) \arrow[r] & h^{n+1}(X,\emptyset) \arrow[r] & \cdots \end{tikzcd}$$

   is exact where the short arrows are induced by the inclusions $(A, \emptyset) \hookrightarrow (X, \emptyset) \hookrightarrow (X, A)$.
2. Excision: For every G-CW-triad $(X; A, B)$ (A, B subcomplexes of X with $A \cup B = X$) the inclusion $(A, A \cup B) \hookrightarrow (X, B)$ induces an isomorphism on h^n for each $n \in \mathbb{Z}$.

Similarly, using the reduced suspension functor $\Sigma\colon hG\mathbf{CW}_* \to hG\mathbf{CW}_*$ on the homotopy category of the category of based G-CW-complexes, $hG\mathbf{CW}_*$, we define a reduced equivariant cohomology theory to be a sequence of contravariant functors $\tilde{h}^n\colon (hG\mathbf{CW}_*)^{\mathrm{op}} \to \mathbf{Ab}$ and natural equivalences $\sigma^n\colon \tilde{h}^n \circ \Sigma \to \tilde{h}^{n-1}$ for $n \in \mathbb{Z}$ satisfying:

1. Exactness: For every pointed pair (X, A, x_0) and $n \in \mathbb{Z}$ the sequence

$$\tilde{h}^n(X \cup CA, *) \to \tilde{h}^n(X, x_0) \to \tilde{h}^n(A, x_0)$$

   induced by the inclusions $(A, x_0) \hookrightarrow (X, x_0) \hookrightarrow (X \cup CA, *)$ is exact.
2. Wedge Axiom: For every set $\{ (X_\alpha, x_\alpha) \mid \alpha \in A \}$ of based G-CW-complexes and every $n \in \mathbb{Z}$ the inclusions $X_\alpha \hookrightarrow \bigvee_{\beta \in A} X_\beta$ together induce an isomorphism

$$\tilde{h}^n(\bigvee\nolimits_{\alpha \in A} X_\alpha) \xrightarrow{\cong} \bigoplus_{\alpha \in A} \tilde{h}^n(X_\alpha).$$

As usual, unreduced and reduces theories are in one-to-one correspondence with each other.

4.7 Equivariant K-theory Fix a compact G-space X. The isomorphism classes of G-vector bundles on X for a monoid, to which we can apply the Grothendieck Construction (Section 2.13). Thus we get an abelian group $K_G(X)$. The tensor product makes $K_G(X)$ into a commutative ring.

Since pullback preserve direct sums and tensor product, $K_G(-)$ becomes a contravariant functor from the category of compact G-spaces to the category of commutative rings.

Of course, when G is the trivial group, we recover ordinary K-theory. Given a subgroup $H \subset G$ we get a restriction $K_G(X) \to K_H(X)$.

4.8 Note Since a G-equivariant vector bundle over the point is the same as a complex representation of G, we have a canonical isomorphism $R(G) \cong K_G(*)$. Thus, $K_G(X)$ is always an algebra over $R(G)$.

By Lemma 4.5, $K_G(-)$ becomes a functor on the homotopy category.

The following property is fundamental:

4.9 Proposition [21, Proposition 2.1] *Let G be a compact Lie group. If G act freely on the G-space X, then there is a canonical ring isomorphism:*

$$K_G(X) \cong K(X/G)$$

Proof. Since G is a compact Lie group, given any G-vector bundle $p\colon E \to X$ we can form the nonequivariant vector bundle $E/G \to X/G$. This gives a ring homomorphism $K_G(X) \to K(X/G)$. The inverse is given by pulling back along the quotient map $X \to X/G$.[1] □

From now on, G will always be assumed to be a compact Lie group.

Of fundamental importance for the development of the theory is the following generalization of Proposition 2.10:

4.10 Proposition [21, Proposition 2.4] *Any G-vector bundle over X embeds into a trivial G-vector bundle.*

The proof relies on the Peter-Weyl Theorem.

For a based G-space X, we define reduced equivariant K-theory of X, $\tilde{K}_G(X)$, to be the kernel of the map $K_G(X) \to K_G(x_0)$ induced by the inclusion of the basepoint. If we define $\tilde{K}_G^{-q}(X) = \tilde{K}_G(\Sigma^q X)$ for $q \in \mathbb{N}$, then $K_G^{\leq 0}(X)$ becomes a graded ring. Then we have equivariant Bott periodicity:

4.11 Theorem [21, Proposition 3.5] $K_G^{-q}(X)$ *is naturally isomorphic to* $K_G^{-q-2}(X)$.

This allows us to define $K_G^n(X)$ for all $n \in \mathbb{Z}$, and then we can verify that equivariant K-theory becomes a generalized equivariant cohomology theory. The details of this are in Segal's paper [21].

4.12 The multiplicative structure Similarly to ordinary cohomology, K-theory is a multiplicative cohomology theory; we have already seen how to do the internal product in $K_G(X)$. Recall [23, Chapter 13] that a multiplicative cohomology theory is a generalized cohomology theory h^* equipped with a natural external product

$$h^p(X,A) \oplus h^q(Y,B) \xrightarrow{\times} h^{p+q}(X \times Y, X \times B \cup A \times Y).$$

For K-theory this is constructed in the following way: For vector bundles $\xi \to X$ and $\eta \to Y$ we can construct the tensor product bundle $\xi \otimes \eta \to X \times Y$. Composition with the map in K-theory induced by the diagonal $X \to X \times X$ gives us back the definition of the product in $K_G(X)$.

4.13 Thom isomorphism We are now ready to state the fundamental theorem of equivariant K-theory, which is actually a generalization of Bott periodicity.

4.14 Theorem *Let X be a compact G-space and V a G-vector space. Then there's an element $\lambda_V \in K_G(V)$ (the Bott class), multiplication by which induces an isomorphism*

$$K_G(X) \to K_G(V \times X),$$

where we use K-theory with compact support. For the proof we refer to Atiyah [5, Theorem 4.3].

[1] Another way to get the map $R(G) \to K(BG)$ is to map a representation $\rho\colon G \to U(n)$ to the composite $BG \to BU(n) \hookrightarrow \{n\} \times BU \hookrightarrow \mathbb{Z} \times BU$, and then use $[BG, \mathbb{Z} \times BU] \cong K(BG)$.

5 Progroups

We will have to works with progroups, because the cohomology theories we consider (primarily complex K-theory) are not well-behaved on infinite complexes. By letting the cohomology take values in progroups we can analyze the system of the cohomology groups on finite subcomplexes as a single entity.

Here we state the basic results about proobjects in an arbitrary category, and about progroups in particular. The basic references are Grothendieck [9] and the appendix of Artin-Mazur [3].

5.1 Representable functors Fix an arbitrary locally small category $\mathbf{C}$. We consider the functor category $\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$ of contravariant functors from $\mathbf{C}$ to the category of sets, in other words, the covariant functors from the dual of $\mathbf{C}$ to $\mathbf{Set}$.[1]

Any $X \in \mathbf{C}$ defines a functor $h_X \in \mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$ by setting

$$h_X(Y) = \hom_{\mathbf{C}}(Y, X)$$

for any $Y \in \mathbf{C}$. Since $\hom_{\mathbf{C}}$ is a bifunctor $\hom_{\mathbf{C}}\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Set}$ this defines a functor $h\colon \mathbf{C} \to \mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$, and the Yoneda Lemma [15, p. 61] tells us that h is fully faithful, *i.e.*, that the induced function

$$\hom_{\mathbf{C}}(X, X') \to \hom_{\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})}(h_X, h_{X'})$$

is bijective.

As a corollary of the Yoneda Lemma we note that if a functor $F \in \mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$ is isomorphic to a functor of the form h_X, then X is uniquely determined up to a unique isomorphism. We call such functors *representable*. Thus the functor h sets up an equivalence of categories between $\mathbf{C}$ and the full subcategory of $\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$ consisting of representable functors.

5.2 Proposition *Suppose I is a small category and $F\colon I \to \mathbf{C}$ is an I-shaped diagram in $\mathbf{C}$ such that the limit $\varprojlim F$ exists. Then $h(\varprojlim F)$ is (canonically) isomorphic to $\varprojlim h \circ F$ (this limit always exists and can be computed pointwise in $\mathbf{Set}$).*

Proof. By the universal property of $\varprojlim F \circ h$ there is a unique morphism $g\colon h_{\varprojlim F} \to \varprojlim F \circ h$. This is given on an object $X \in \mathbf{C}$ by the function

$$g(X)\colon \hom_{\mathbf{C}}(X, \varprojlim F) \to \varprojlim \hom_{\mathbf{C}}(X, F(-))$$

and this is bijective by the universal property of $\varprojlim F$.[2] □

[1] This category will in general have hom-*classes* instead of hom-sets. See Section 1.2 for notes on foundational issues

[2] This is discussed in further detail in Mac Lane [15, p. 116].

5.3 Algebraic objects As a consequence of this proposition we see that to give a object $X \in \mathbf{C}$ some algebraic structure based on one or more compositions (for instance, to make X a group-object or a ring-object in $\mathbf{C}$) is the same as to give a lifting of the functor $h_X \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ to the corresponding algebraic category (for instance, $\mathbf{Grp}$ or $\mathbf{Rng}$) compatible with the forgetful functor from that category to $\mathbf{Set}$. This is because, by the proposition, giving a composition $X \times X \to X$ is equivalent to giving a composition $h_{X\times X} = h_X \times h_X \to h_X$, and checking the relevant algebraic axioms amount to the same in the two instances.

5.4 Corepresentable functors What we have said for representable functors can of course be dualized by considering the dual of $\mathbf{C}$, $\mathbf{C}^{\mathrm{op}}$. In this case we have a contravariant functor $k \colon \mathbf{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathbf{C}, \mathbf{Set})$, where for $X \in \mathbf{C}$ the functor k_X is the covariant functor

$$k_X \colon \mathbf{C} \to \mathbf{Set}, \qquad Y \mapsto k_X(Y) = \hom_{\mathbf{C}^{\mathrm{op}}}(Y, X) = \hom_{\mathbf{C}}(X, Y).$$

The analogue of Proposition 5.2 says that k carries colimits in $\mathbf{C}$ to limits in $\mathrm{Fun}(\mathbf{C}, \mathbf{Set})$ such that

$$k_{\varinjlim F}(X) = \hom_{\mathbf{C}}(\varinjlim F, X) \cong \varprojlim \hom_{\mathbf{C}}(F(-), X)$$

when the colimit of $F \colon I \to \mathbf{C}$ exists.

We will have use for the following proposition:

5.5 Proposition *Suppose I is a small category and $F \colon I \to \mathbf{C}$ is I-shaped diagram in $\mathbf{C}$. Then we have a natural bijection*

$$\hom_{\mathrm{Fun}(\mathbf{C},\mathbf{Set})}(k_X, \varinjlim k \circ F) \cong \varinjlim \hom_{\mathbf{C}}(F(-), X)$$

for any $X \in \mathbf{C}$.

Proof. To ease notation, we'll drop the subscript on homsets for this proof. First note that for any $Y \in \mathbf{C}$ we have

$$(\varinjlim k \circ F)(Y) = \varinjlim \hom(F(-), Y)$$

since limits and colimits in the functor category are computed pointwise. Thus we get a natural function

$$\Phi \colon \hom(k_X, \varinjlim k \circ F) \to \varinjlim \hom(F(-), X), \qquad \varphi \mapsto \varphi_X(1_X).$$

Conversely, a natural transformation $\varphi \in \hom(k_X, \varinjlim k \circ F)$ is determined by its value $\varphi_X(1_X)$, since for any $g \colon X \to Y$ we have a naturality square

$$\begin{tikzcd}
\hom(X,X) \arrow[r, "\varphi_X"] \arrow[d, "g_*"'] & \varinjlim \hom(F(-), X) \arrow[d, "g_*"] \\
\hom(X,Y) \arrow[r, "\varphi_Y"'] & \varinjlim \hom(F(-), Y)
\end{tikzcd}$$

and therefore $\varphi_Y(g) = g_* \varphi_X(1_X)$. $\square$

5.6 Filtered categories Let us here recall the theory of filtered limits and colimits [15, p. 211]. First we must define filtered categories.

We say a category I is *filtered* if I is not empty, has coequalizers, and satisfies that for any two objects $i, j \in I$ there is a $k \in I$ with morphisms $i \to k$ and $j \to k$.

Note that the product of two filtered categories again is filtered.

5.7 Filtered limits We define a filtered colimit to be the colimit of a functor from a filtered category, while a filtered limit is the limit of a functor defined on a cofiltered category (*i.e.*, a category whose dual is filtered).

Sometimes we use the term *projective limit* instead of filtered limit.

5.8 Proobjects From now on, we will use the notation $\mathcal{X} = (X_i)_{i\in I}$ to indicate an I-shaped diagram in $\mathbf{C}$ with I a small cofiltered category. We call such diagrams *projective systems* in $\mathbf{C}$.

A projective system $\mathcal{X} = (X_i)_{i\in I}$ defines a covariant functor $k_{\mathcal{X}}\colon \mathbf{C} \to \mathbf{Set}$ by setting

$$k_{\mathcal{X}} = \varinjlim_i k_{X_i}$$

Such functors are called *prorepresentable.* For two projective systems $\mathcal{X} = (X_i)_{i\in I}$ and $\mathcal{Y} = (Y_j)_{j\in J}$ we have a sequence of canonical bijections

$$\begin{aligned} \hom_{\mathrm{Fun}(\mathbf{C},\mathbf{Set})}(k_{\mathcal{Y}}, k_{\mathcal{X}}) &\cong \varprojlim_j \hom_{\mathrm{Fun}(\mathbf{C},\mathbf{Set})}(k_{Y_j}, k_{\mathcal{X}}) \\ &\cong \varprojlim_j \varinjlim_i \hom_{\mathbf{C}}(X_i, Y_j) \end{aligned} \tag{5.1}$$

where the second is Proposition 5.5. This inspires us to define the category $\mathbf{Pro}(\mathbf{C})$ of *proobjects* in $\mathbf{C}$, whose objects are the projective systems and whose homsets are

$$\hom_{\mathbf{Pro}(\mathbf{C})}(\mathcal{X}, \mathcal{Y}) = \varprojlim_j \varinjlim_i \hom_{\mathbf{C}}(X_i, Y_j)$$

for proobjects $\mathcal{X} = (X_i)_{i\in I}$ and $\mathcal{Y} = (Y_j)_{j\in J}$. Futhermore, (5.1) shows that we can consider k as a contravariant functor

$$k\colon \mathbf{Pro}(\mathbf{C})^{\mathrm{op}} \to \mathrm{Fun}(\mathbf{C}, \mathbf{Set})$$

that establises an equivalence of categories between the dual of $\mathbf{Pro}(\mathbf{C})$ and the full subcategory of prorepresentable functors.

An object $X \in \mathbf{C}$ defines a canonical proobject, also denoted X, defined on the category $*$ with a single object and only the identity morphism on that object. Thus $\mathbf{C}$ is equivalent to a full subcategory of $\mathbf{Pro}(\mathbf{C})$.

Using these equivalences we see that for a projective system $\mathcal{X} = (X_i)_{i\in I}$ we have an isomorphism

$$\mathcal{X} \cong \varprojlim_i X_i$$

in $\mathbf{Pro}(\mathbf{C})$ (and with the limit taken in $\mathbf{Pro}(\mathbf{C})$) coming from the defined equality

$$k_{\mathcal{X}} = \varinjlim_i k_{X_i}$$

in $\mathrm{Fun}(\mathbf{C}, \mathbf{Set})$.

5.9 Warning! If a projective system $\mathcal{X} = (X_i)_{i\in I}$ has a projective limit $\varprojlim X_i$ in $\mathbf{C}$ then this is generally *not* isomorphic to the projective limit $\mathcal{X}$ in $\mathbf{Pro}(\mathbf{C})$.

5.10 Proposition [3, Proposition A.4.4]

The category $\mathbf{Pro}(\mathbf{C})$ *has all small projective limits.*

Proof. Let $(\mathcal{X}^j)_{j\in J}$ be a projective system in $\mathbf{Pro}(\mathbf{C})$ with J a small cofiltering category, where the $\mathcal{X}^j$ are themselves projective systems $\mathcal{X}^j = (X_i^j)_{i\in I_j}$ in $\mathbf{C}$. Define a new small category K with objects

$$(j,i) \in \bigcup_{j\in J} I_j$$

and hom-sets

$$\hom_K\Big((j,i),(j',i')\Big) = \Big\{ (j \xrightarrow{\alpha} j', X_i^j \xrightarrow{f} X_{i'}^{j'}) \Big\}$$

where f represents the i'-th factor of $\alpha^*\colon \mathcal{X}^j \to \mathcal{X}^{j'}$. Then K is a cofiltered category, and we have an obvious projective system

$$\mathcal{Y} = (X_i^j)_{(j,i)\in K}$$

with a bijection

$$\varprojlim_{j\in J} \hom_{\mathbf{Pro}(\mathbf{C})}(\mathcal{Z},\mathcal{X}^j) \cong \varprojlim_{(j,i)\in K} \hom_{\mathbf{Pro}(\mathbf{C})}(\mathcal{Z},\mathcal{X}_i^j) = \hom_{\mathbf{Pro}(\mathbf{C})}(\mathcal{Z},\mathcal{Y}).$$

Thus, $\mathcal{Y}$ is the limit of the system $(\mathcal{X}^j)_{j\in J}$ in $\mathbf{Pro}(\mathbf{C})$. □

5.11 Abelian categories If $\mathbf{C}$ is an additive (Abelian) category, then $\mathbf{Pro}(\mathbf{C})$ is additive (Abelian). [3, Proposition A.4.5]

6 Algebraic completion

6.1 Change of rings Consider a homomorphism of commutative rings $\varphi\colon A \to B$. Any B-module then becomes an A-module through φ and we get a functor $B\mathbf{Mod} \to A\mathbf{Mod}$ called *restriction of scalars*. This functor is obviously exact.

We can also go the other way an get a functor $A\mathbf{Mod} \to B\mathbf{Mod}$ called *extension of scalars*. This is obtained by associating to an A-module M the tensor product $B \otimes_A M$ with the obvious B-scalar multiplication. Extension is left-adjoint to restriction.

The extension functor is always right-exact and it is exact if and only if φ is a flat ring homomorphism.

The is a another functor $A\mathbf{Mod} \to B\mathbf{Mod}$ called *induction*. This takes an A-module M to the B-module $\mathrm{Hom}_A(B, M)$ with the scalar multiplication

$$B \times \mathrm{Hom}_A(B, M) \ni (b, f) \mapsto \Big(B \ni c \mapsto f(bc) \in M\Big) \in \mathrm{Hom}_A(B, M).$$

Induction is right-adjoint to restriction.

6.2 Adjointness and exactness Let (F, G) be a pair of adjoint functors of abelian categories. Then F is right-exact and G is left-exact.

Recall also, that any left adjoint functor is cocontinuous and any right adjoint functor is continuous.

6.3 Completion Let R be a commutative ring, and let $I \subset R$ be a proper ideal. Then we define the completion $R\hat{_I}$ to be the projective limit $R\hat{_I} = \varprojlim R/I^n$. There is another way to view this completion. Give R the I-adic topology in which a basis of the open neighborhoods of 0 is given by the sets I^n for $n \geq 1$, and the the other neighborhoods bases are obtained by translation. Then $R\hat{_I}$ can be identified with the completion of R with respect to the I-adic topology. The completion is a complete topological ring.

Completion can also be applied to modules $M \in R\mathbf{Mod}$ by taking the projective limit $M\hat{_I} = \varprojlim M/I^nM$. Completion is then a functor $R\mathbf{Mod} \to R\hat{_I}\mathbf{Mod}$.

If R is Noetherian, then $R\hat{_I}$ is flat over R. For finitely generated modules, we can also view completion as extension by scalars as in Section 6.1. The completion functor is then exact on sequences of finitely generated modules [7, Lemma 7.15].

6.4 Computation by power series If R is Noetherian, and $\mathfrak{a} = (a_1, \ldots, a_n)$ an ideal, then

$$R\hat{_\mathfrak{a}} \cong R[\![x_1, \ldots, x_n]\!]/(x_1 - a_1, \ldots, x_n - a_n).$$

This follows from the development in Chapter 7 in Eisenbud [7].

7 The completion theorem

7.1 Outline In this chapter we will discuss the various forms of the completion theorem, how they relate to each other and how to prove them. First, we will apply the machinery of proobjects to define progroup-valued K-theory. This will allow us to state our main theorem as it appeared in the article by Adams, Haeberly, Jackowski and May [1]. Then we can go on to the proof of the theorem, the bulk of which lies in the analysis of a special case. When we are done with the proof, we can then finish the chapter by discussing how to derive other versions of the theorem.

7.2 Progroup K-theory We define a sequence of functors for $\mathcal{K}_G^n(-)\colon G\mathbf{CW} \to \mathbf{Pro}(\mathbf{Grp})$ for $n \in \mathbb{Z}$ that takes a G-CW-complex X to the prosystem $\{K_G^n(X_\alpha)\}$, where X_α runs through the finite subcomplexes of X. This defines a progroup-valued multiplicative cohomology theory. We will also have use for corresponding reduced theory. It is clear that relevant axioms are satisfied, since term-wise exactness implies pro-exactness.

We will interpret $\mathcal{K}_G^*(-)$ as taking values in pro-$R(G)$-modules or pro-$R(G)$-algebras when convenient.

7.3 The general theorem We a now ready to deal with the generalized completion theorem as it appeared in the article by Adams, Haeberly, Jackowski and May [1].

Fix a compact Lie group G and a family of closed subgroups, $\mathcal{J}$, closed under subconjugacy (taking subgroups and conjugating). Given a closed subgroup H of G we have a restriction homomorphism $r_H^G\colon R(G) \to R(H)$. Let I_H^G denote the kernel. Thus, $I(G) = I_1^G$.

Finite products of the ideals I_H^G for $H \in \mathcal{J}$ determine a directed system of ideals in $R(G)$ with respect to which we can form a completion functor, $(-)\hat{}_{\mathcal{J}}$. Since $R(G)$ is Noetherian, completion is exact, so we get a completed multiplicative cohomology theory, $\mathcal{K}_G^*(-)\hat{}_{\mathcal{J}}$.

Explicitly, to a space $X \in G\mathbf{CW}$ consider the cofiltered category $\mathcal{F}(X)^{\mathrm{op}} \times \mathcal{I}$, where $\mathcal{F}(X)$ is the set of finite subcomplexes of X, and $\mathcal{I}$ is the set of ideals of $R(G)$ that are finite products of ideals I_H^G with $H \in \mathcal{J}$. The image of X is then the progroup determined by the projective system

$$\{K_G^n(X_\alpha)/JK_G^n(X_\alpha)\}_{(X_\alpha, J)\in\mathcal{F}(X)^{\mathrm{op}}\times\mathcal{I}}.$$

Now we can state our main theorem:

7.4 Theorem [1, Theorem 1.1] *If a G-map $f\colon X \to Y$ restricts to a homotopy equivalence $f^H\colon X^H \to Y^H$ for each $H \in \mathcal{J}$, then $\mathcal{K}_G^n(f)\hat{}_{\mathcal{J}}$ is an isomorphism of progroups.*

Note, that if $\mathcal{J}$ is a *finite* set, then completion with respect to $\mathcal{J}$ is the same as completion with respect to the ideal

$$I(\mathcal{J}) = \bigcap_{H \in \mathcal{J}} I_H^G.$$

In particular, if we take $\mathcal{J} = \{1\}$, then we recover the Atiyah-Segal completion theorem as stated in the original article [6].

7.5 Proof strategy We will prove Theorem 7.4 by proving the following, equivalent, theorem:

7.6 Theorem *If $X \in G\mathbf{CW}_*$ has X^H contractible for all $H \in \mathcal{J}$, then $\mathcal{K}_G^*(X)\hat{}_{\mathcal{J}} \cong 0$.*

Proof (that theorems 7.4 and 7.6 are equivalent). Assume have proved Theorem 7.4, and let $X \in G\mathbf{CW}_*$ satisfy the hypothesis of Theorem 7.6. Then the conclusion of Theorem 7.6 follows from consideration of the map $* \to X$.

Conversely, if $f\colon X \to Y$ satisfies the hypothesis of Theorem 7.4, then we can apply Theorem 7.6 to the cofiber sequence corresponding to f together with the Five-Lemma. □

We will now go on to the proof of Theorem 7.6. We start with some useful lemmas about representation spheres. If V is a representation on G, we can form the one-point compactification S^V, and this is the representation sphere.

7.7 Lemma *If $i\colon V \hookrightarrow W$ is inclusion of a subrepresentation such that the complement $W - V$ has a fixpoint by the subgroup $H \subset G$, then i is H-nullhomotopic.*

Proof. Suppose $w \in (W - V)^H$. We have an explicit H-nullhomotopy:

$$\begin{aligned} S^V \times I &\to S^W \\ (v,t) &\mapsto \begin{cases} v + \frac{1}{1-t} w, & \text{for } v \neq \infty \text{ and } t \neq 1, \\ \infty, & \text{otherwise.} \end{cases} \end{aligned}$$

□

7.8 Lemma *Let V be a subrepresentation of W. Then the map $i^*\colon \tilde{K}_G^*(S^W) \to \tilde{K}_G^*(S^V)$ induced by the inclusion $i\colon S^V \hookrightarrow S^W$ is given by multiplication with the Euler class of the complement, $\chi_{W-V} \in R(G)$.*

Proof. The Euler class $\chi_V \in K_G(*) \cong R(G)$ of a representation V is obtained from the Bott class $\lambda_V \in K_G(V)$ by applying the map induced from the inclusion $0 \hookrightarrow V$.

By construction of the Bott class [5], we have $\lambda_W = \lambda_{W-V}\lambda_V$. By Theorem 4.14, $\tilde{K}_G^*(S^V)$ is a free $\tilde{K}_G^*(S^0)$-module generated by the Bott class $\lambda_V \in \tilde{K}_G^0(S^V)$. We then have

$$i^*(x\lambda_W) = x\chi_{W-V}\lambda_V,$$

so i^* is indeed multiplication by χ_{W-V}. □

We now prove Theorem 7.6 by reducing to the following special case. Consider a countable set $\{V_i\}_{i \in I}$ of non-trivial representations of G such that $V_i^G = 0$ for all i, and for each proper subgroup H of G there is some i with $V_i^H \neq 0$. Then form the infinite dimensional representation U by taking a sum of the representations V_i, where each V_i occurs countably many times.

The finite dimensional subrepresentations of U form a directed system, and we take Y to be the colimit of the one-point compactifications of these.

7.9 Lemma *We have $Y^G = S^0$ and for all proper subgroups $H \subset G$, we have that Y^H is contractible.*

Proof. Since $V_i^G = 0$ for all $i \in I$, we get Y^G equals S^0 on the nose. Given a proper subgroup $H \subset G$, we can choose an ascending sequence of finite-dimensional subspaces $V_1 \subset V_2 \subset \cdots \subset U$, such that $(V_{n+1} - V_n)^H \neq 0$ and U is the union of the V_n. Then by Lemma 7.7, each inclusion $S^{V_n} \hookrightarrow S^{V_{n+1}}$ is H-nullhomotopic, and so $Y = \varinjlim S^{V_n}$ is H-contractible. Therefore, Y^H is contractible. $\square$

7.10 Lemma [1, Lemma 3.2] *If $G \notin \mathcal{J}$, then $\tilde{\mathcal{K}}_G^*(Y)\hat{}_{\mathcal{J}} \cong 0$.*

Proof. By the proof of Proposition 5.10, $\tilde{\mathcal{K}}_G^*(Y)\hat{}_{\mathcal{J}}$ is the limit in the category of proobjects of the $\tilde{\mathcal{K}}_G^*(Y)/J\tilde{\mathcal{K}}_G^*(Y)$ for J in $\mathcal{I}$. Thus we need only show that each of these is prozero. That is, we must show that for any finite module $V \in U$ there is a module $W \supset V$ such that the inclusion $i\colon S^V \hookrightarrow S^W$ induces the zero map

$$i^*\colon \tilde{\mathcal{K}}_G^*(S^W)/J\tilde{\mathcal{K}}_G^*(S^W) \to \tilde{\mathcal{K}}_G^*(S^V)/J\tilde{\mathcal{K}}_G^*(S^V).$$

Let $J = I_{H_1}^G \cdots I_{H_k}^G$ be given, and choose W such that $W - V$ is the sum of modules W_i with $W_i^{H_i} \neq 0$. Then by Lemma 7.8, $i^*\colon \tilde{\mathcal{K}}_G^*(S^W) \to \tilde{\mathcal{K}}_G^*(S^V)$ is multiplication by $\chi_{W_1} \cdots \chi_{W_k}$, and by Lemma 7.7, thus product lies in J. $\square$

7.11 Lemma *For any based G-space X,*

$$\tilde{\mathcal{K}}_G^*((G/H)_+ \wedge X)\hat{}_{\mathcal{J}} \cong \tilde{\mathcal{K}}_H^*(X)\hat{}_{\mathcal{J}|H}.$$

Proof. As pro-$R(G)$-modules,

$$\tilde{\mathcal{K}}_G^*((G/H)_+ \wedge X) \cong \tilde{\mathcal{K}}_H^*(X).$$

Thus we get the stated isomorphim from the following Lemma. $\square$

7.12 Lemma *The $\mathcal{J}$-adic and $(\mathcal{J}|H)$-adic topologies coincide on $R(H)$.*

Proof. Here we are considering $R(H)$ as a module over $R(G)$ via the restriction homomorphism.

Consider first a subgroup $L \in \mathcal{J}|H$. Then $r_H^G(I_L^G) \subset I_L^H$ since r_L^G is the composition

$$R(G) \xrightarrow{r_H^G} R(H) \xrightarrow{r_L^H} R(L).$$

So $\mathcal{J}$ induces on $R(H)$ a topology finer than the $(\mathcal{J}|H)$-adic topology.

Conversely, consider $K \in \mathcal{J}$ and the induced ideal $I = r_H^G(I_K^G)R(H)$. Since $R(H)$ is Noetherian, we have primary decompositions [13, Theorem 3.3; 14, Proposition III.2.1], so we can find contained in I a finite product of prime ideals of $R(H)$ each containing I. Note that any prime of $R(H)$ contains I_S^H, where S is a support of the prime. Thus, we need only show that for a prime $\mathfrak{q} \supset I$, the support S of $\mathfrak{q}$ lies in $\mathcal{J}$.

Let $\mathfrak{p} = (r_H^G)^{-1}(\mathfrak{q})$ and note that

$$\mathfrak{p} \supset (r_H^G)^{-1}(r_H^G(I_K^G)) \supset I_K^G.$$

Since $R(K)$ is finitely generated over $R(G)$ by Proposition 2.15, we have that $R(K)$ is an integral ringextension of $R(G)$, so we can "go up" [7, Proposition 4.15] and find

$\mathfrak{p}' \in \operatorname{Spec} R(K)$ mapping to $\mathfrak{p}$. Thus, $\mathfrak{p}$ has a support S' in K, and since any two supports are conjugate in G by Proposition 2.17 and $\mathcal{J}$ is closed under subconjugacy, we conclude that $S' \in \mathcal{J}$. □

Finally, at long last, we are ready for:

Proof (of Theorem 7.6). By the Equivariant Whitehead Theorem, X is is H-contractible for all $H \in \mathcal{J}$, so $\tilde{\mathcal{K}}^*_H(X)$ is prozero for $H \in \mathcal{J}$. If $G \in \mathcal{J}$, there is nothing to prove, so assume $G \notin \mathcal{J}$. The descending chain condition on the subgroups of G (Section 2.6) allows us to proceed by well-founded induction on the subgroup lattice. The base case $G = 1$ is trivial.

For the induction step, assume $\tilde{\mathcal{K}}^*_H(X)\hat{}_{\mathcal{J}|H}$ is prozero for all proper subgroups $H \subset G$. We have a cofiber sequence

$$S^0 \to Y \to Y/S^0$$

from which we obtain a new cofiber sequence

$$X \to X \wedge Y \to X \wedge (Y/S^0)$$

by smashing with X.

By the long-exact sequence for cofiber sequences together with the Five-Lemma, it suffices to show that $\tilde{\mathcal{K}}^*_G(X \wedge Y)\hat{}_{\mathcal{J}}$ and $\tilde{\mathcal{K}}^*_G(X \wedge (Y/S^0))\hat{}_{\mathcal{J}}$ are prozero. This follows from the following observations:

(i) $\mathcal{K}^*_G(W \wedge Y)\hat{}_{\mathcal{J}}$ is prozero for any $W \in G\mathbf{CW}_*$.

(ii) $\mathcal{K}^*_G(X \wedge Z)\hat{}_{\mathcal{J}}$ is prozero for any $Z \in G\mathbf{CW}_*$ with $Z^G = *$.

As for (i), by induction over cells we are reduced to showing this for cells $W = (G/H)_+ \wedge S^n$, and thus by the suspension isomorphism for $W = (G/H)_+$. For $H = G$ we can use Lemma 7.10, and for $H \neq G$ we can use lemmata 7.9 and 7.11.

As for (ii), we again reduce to $Z = (G/H)_+$, but here H must be a proper subgroup, so we are done by Lemma 7.11 and the induction hypothesis. □

This concludes the proof of the main theorem, Theorem 7.4.

7.13 The special case We will now dwelve a bit on the special case where $\mathcal{J} = \{1\}$.

7.14 A note on Lusternik-Schnirelmann category We shall need a result related to the notion of *Lusternik-Schnirelmann category* (or LS-category for short). The LS-category of a space X, denoted $\operatorname{cat} X$, is the minimum cardinality of an open covering of X by subsets that are contractible in X. Thus, a contractible space has LS-category 1, while suspensions (spheres in particular) have LS-category 2. It can be shown that LS-category is an invariant of homotopy type. [12].

Suppose given a path-connected based space X with LS-category at most n. Then we will show that the product of n elements of $\tilde{K}^*(X)$ is zero. In fact, this holds for any reduced multiplicate cohomology theory $\tilde{h}^*$, as defined in Section 4.12.

Let $U_1 \cup \cdots \cup U_n$ be an open covering of X by subsets contractibe in X. Picking a basepoint x_i in each U_i, for $1 \le i \le n$, the inclusion $U_i \hookrightarrow X$ induces the trivial map $\tilde{h}^*(X, x_i) \to h^*(U_i, x_i)$. By the long exact sequence of the triple (X, U_i, x_i), the map $h^*(X, U_i) \to \tilde{h}^*(X, x_i)$ is surjective. Of course, since X is path-connected we can identify each of the $\tilde{h}^*(X, x_i)$ with $\tilde{h}^*(X, *)$, so if we are given n elements $\alpha_1, \ldots, \alpha_n$ in $\tilde{h}^*(X, *)$, we can then find elements $\bar{\alpha}_i \in h^*(X, U_i)$ mapping to the α_i.

Consider now the commutative diagram

$$\begin{tikzcd} h^*(X,U_1)\oplus\cdots\oplus h^*(X,U_n) \arrow[r] \arrow[d, "\times"'] & \tilde h^*(X)\oplus\cdots\oplus\tilde h^*(X) \arrow[d, "\times"] \\ h^*(X^n,\mathcal{U}) \arrow[r] \arrow[d, "\Delta^*"'] & \tilde h^*(X^n) \arrow[d, "\Delta^*"] \\ h^*(X,X) \arrow[r] & \tilde h^*(X) \end{tikzcd}$$

where

$$\mathcal{U} = U_1\times X\times\cdots\times X\cup X\times U_2\times\cdots\times X\cup X\times X\times\cdots\times U_n.$$

Starting with the $\bar\alpha_i$'s, if we go right along the top and then down, we arrive in the bottom right at the product $\alpha_1\cdots\alpha_n\in\tilde h^*(X)$, which is then zero, since going down and right, we pass through $h^*(X,X)$, which is trivial.

7.15 Milnor's Construction Milnor showed [20] how to construct a functorial model for EG. Recall the definition of the join of two spaces X and Y: $X*Y = X\times Y\times I/\sim$, where

$$(x_0,y_0,t_0)\sim(x_1,y_1,t_1)\Leftrightarrow\Big((t_0=t_1=0\wedge y_0=y_1)\vee(t_0=t_1=1\wedge x_0=x_1)\Big).$$

We identify X and Y with the subspaces of $X*Y$ with $t=0$ and $t=1$ respectively. The join is then the union of linesegments from a point in X to a point in Y.

The join is commutative and associative up to homeomorphism. In fact, we can think of the join $X_0*\cdots*X_n$ of the spaces $X_0,\dots,X_n$ as the formal convex combinations $t_0x_0+\cdots+t_nx_n$ of points $x_i\in X_i$, $0\le i\le n$, where the point x_i is irrelevant when $t_i=0$.

When X and Y are CW-complexes, there is a natural CW-complex construction of $X*Y$, whose topology agrees with the quotient topology we used above in the cases, we're interested in.

We note that the join of a space X with a point is the cone CX, and the join with S^0 is the suspension SX. Thus, the join of two spheres is another sphere,

$$S^p*S^q\cong\overbrace{(S^0*\cdots*S^0)}^{p+1}*\overbrace{(S^0*\cdots*S^0)}^{q+1}\cong\overbrace{(S^0*\cdots*S^0)}^{p+q+2}\cong S^{p+q+1}.$$

This illustrates the crucial property of joins, namely that if $X_0,\dots X_n$ are spaces such that X_i is c_i-connected, then the join $X_0*\cdots*X_n$ is $(c_0+\cdots+c_n+2n)$-connected. [20, Lemma 2.3]

Given a topological group G, we can then define for $n\ge1$ the space E^nG as the join of n copies of G. The action of G by multiplication on the individual factors is then a free action, and since the connectivity increases, $EG=\varinjlim E^nG$ is contractible.

If we set $B^nG=E^nG/G$, then we have principal G-bundles $E^nG\to B^nG$. Letting $BG=\varinjlim B^nG$ we obtain the *universal principal G-bundle $EG\to BG$*, and we call BG the *classifying space* for G.

7.16 The original article Consider the map $\alpha_n\colon K_G^*(*) \to K_G^*(E^nG)$ induced by the map $E^nG \to *$. Using Proposition 4.9 and Note 4.8 we can identify it with the middle vertical map in the diagram below with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & I(G) \arrow[r] \arrow[d] & R(G) \arrow[r, "\varepsilon"] \arrow[d, "\alpha_n"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \tilde{K}^*(B^nG) \arrow[r] & K^*(B^nG) \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

We can write E^nG as a union of the n open subsets V_i, for $1 \leq i \leq n$, where the ith join coordinate is non-zero. Then V_i is G-homotopy equivalent to G, so $U_i = V_i/G$ is contractible. Thus, B^nG is the union of the n open contractible subsets U_i. By the result of Section 7.14, this means that the product of any n elements of $\tilde{K}^*(B^nG)$ is zero. Thus, the map α_n factors through the projection $R(G) \to R(G)/I(G)^n$.

Using this we can analyse the map $p^*\colon K_G^*(X) \to K_G^*(X \times E^nG)$ induced by the projection $p\colon X \times E^nG \to X$. Since the external product is natural, we have a commutative square

$$\begin{tikzcd}
K_G^*(X) \oplus K_G^*(*) \arrow[r, "\times"] \arrow[d, "1\oplus\alpha_n"'] & K_G^*(X) \arrow[d, "p^*"] \\
K_G^*(X) \oplus K_G^*(E^nG) \arrow[r, "\times"] & K_G^*(X \times E^nG)
\end{tikzcd}$$

from which we gather that p^* factors through the projection

$$K_G^*(X) \to K_G^*(X)/I(G)^n K_G^*(X).$$

It follows that the prohomomorphism $\mathcal{K}_G^*(X)\hat{}_{\mathcal{J}} \to \mathcal{K}_G^*(X \times EG)\hat{}_{\mathcal{J}}$ is represented by the maps

$$K_G^*(X)/I(G)^n K_G^*(X) \to K_G^*(X \times E^nG)$$

This is the prohomomorphism that Atiyah and Segal originally studied in their 1969 article [6]. From the completion theorem we can now conclude that since $\mathcal{K}_G^*(X \times EG)\hat{}_{\mathcal{J}}$ satisfies the Mittag-Leffler condition, then $\mathcal{K}_G^*(X)\hat{}_{\mathcal{J}}$ does so as well, so there is no $\lim^1$ term, and we get the well-known formulation of the Atiyah-Segal completion theorem:

7.17 Theorem *If X is a finite G-CW complex, then the projection $X \times EG \to X$ induces an isomorphism*

$$K_G^*(X)\hat{}_{I(G)} \to K_G^*(X \times EG).$$

In particular, for $X = *$ we get $K^0(BG) \cong R(G)\hat{}_{I(G)}$ and $K^1(BG) \cong 0$.

8 Examples

8.1 Cyclic groups For a cyclic group $G = C_n$, the representation ring is $\mathbb{Z}[x]/(x^n - 1)$, where we can take x to be the representation taking a generator of C_n to $\exp(2\pi i/n)$.

The base space of the universal principal C_n-bundle is the infinite lens space, L_n. By the completion theorem, we get an isomorphism

$$\begin{aligned} K(L_n) &\cong (\mathbb{Z}[x]/(x^n - 1))\hat{}_{(x-1)} \\ &\cong (\mathbb{Z}[t]/((t+1)^n - 1))\hat{}_{(t)} \\ &\cong \mathbb{Z}[\![t]\!]/((t+1)^n - 1), \end{aligned}$$

where we set $t = x - 1$.

Now consider the case where $n = p$, a prime. Let $R = \mathbb{Z}[t]/((t+1)^p - 1)$. Note that in the module tR, the ideals (t) and (p) generate the same topology. Indeed, $0 \equiv (t+1)^p - 1$ is congruent to $t^p \pmod{pt}$ and to $pt \pmod{t^2}$, so

$$(t)^p \subset (t)(p) \subset (t)^2,$$

from which we get

$$(t)(p)^{k-1} \subset (t)^k \qquad \text{and} \qquad (t)^{1+pk} \subset (t)(p)^k.$$

Thus, for reduced K-theory, we get $\tilde{K}(L_p) \cong (x-1)\mathbb{Z}\hat{}_{(p)}[x]/(x^p - 1)$. A similar conclusion holds for any p-group.

8.2 The circle Consider now the circle group, $G = S^1$. A model for BS^1 is $\mathbb{C}P^\infty$. The representation ring is the Laurant polynomial ring $\mathbb{Z}[x, x^{-1}]$, where we take x to be the canonical character

$$x\colon S^1 \hookrightarrow \mathbb{C}^\times.$$

The augmentation ideal is $(x - 1)$, and by changing variables such that $t = x - 1$, we see that

$$K(\mathbb{C}P^\infty) \cong \mathbb{Z}[t, (1+t)^{-1}][\![y]\!]/(y - t) \cong Z[\![t]\!]$$

since $(1 + t)$ is invertible in the power series ring.

Both of these examples can by the way also be done via the Atiyah-Hirzebruch spectral sequence, which collapses in these cases.

Index

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