UNIVERSITY OF COPENHAGEN
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Bachelor project

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Cohomology of Groups of low Order

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Abstract

This bachelor thesis investigates group cohomology, focusing on computation with an algebraic approach. In the first section of this project, we introduce the theory of group cohomology, as well as our chosen machinery for computing group cohomology. In the second section, we perform concrete calculations of the cohomology of, for example, the cyclic groups $\mathbb{Z}/2^n$, the Klein four group $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ and the dihedral group D_8 of order 8. We limit our calculations to cohomology with coefficients in $\mathbb{F}_2$. Initially, we perform calculations using explicit resolutions. Thereafter, we move on to the more powerful machinery provided by the Lyndon-Hochschild-Serre spectral sequence.

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1 Homological algebra

1.1 Basic definitions

This subsection serves to lay down some notational conventions and recall some basic homological algebra. For a more detailed exposition, see e.g. Rotman (2009) or Weibel (1994).

Let R be a ring. A *graded R-module* is a sequence $C = (C_n)_{n\in\mathbb{Z}}$ of R-modules. We say that $x \in C_n$ has *degree* n and write $\deg(x) = |x| = n$. A *map of degree* p from C to another graded R-module C' is a sequence $f = (f_n : C_n \to C'_{n+p})_{n\in\mathbb{Z}}$ or R-module homomorphisms. A *chain complex* (C, d) over R is a graded R-module C together with a map d of degree -1 satisfying $d \circ d = 0$. We call d the *differential*. The *homology* of a chain complex (C, d) is $H_n(C) = Z_n(C)/B_n(0)$ where $Z_n(C) = \ker d_n$ and $B_n(C) = \operatorname{im} d_{n+1}$ which we call the *cycles* and *boundaries*, respectively. A *cochain complex* (C, d) over R is a graded R-module $C = (C^n)_{n\in\mathbb{Z}}$ together with a map $d = (d^n : C^n \to C^{n+1})$ of degree 1 satisfying $d \circ d = 0$. We still call d the *differential*, but note that we use superscript instead of subscript. The *cohomology* of a cochain complex (C, d) is $H^n(C) = Z^n(C)/B^n(0)$ where $Z^n(C) = \ker d_n$ and $B^n(C) = \operatorname{im} d_{n-1}$ which we call the *cocycles* and *coboundaries*, respectively. If (C, d) and (C', d') are cochain complexes over R, then a *chain map* $f : C \to C'$ is a graded module homomorphism over R of degree 0, satisfying $d'f = fd$. Note that a cochain map $f : C \to C'$ induces a map on cohomology $H^*(f) : H^*(C) \to H^*(C')$ and similarly for chain maps and homology (boundaries are sent to boundaries, since $d'f = fd$).

A *cochain homotopy* or *homotopy* h from a chain map $f : C \to C'$ to a chain map $g : C \to C'$ is a graded module homomorphism $h : C \to C'$ of degree -1 satisfying $d'h + hd = f - g$. We can visualize the maps by the following (noncommutative) diagram.

$$
\begin{tikzcd}
\cdots \arrow[r] & C_{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "f_{n-1}-g_{n-1}"] & C_n \arrow[r, "d^n"] \arrow[d, "f_n-g_n"] \arrow[dl, "h_n"'] & C_{n+1} \arrow[r] \arrow[d, "f_{n+1}-g_{n+1}"] \arrow[dl, "h_{n+1}"'] & \cdots \\
\cdots \arrow[r] & C'_{n-1} \arrow[r, "(d')^{n-1}"] & C'_n \arrow[r, "(d')^n"] & C'_{n+1} \arrow[r] & \cdots
\end{tikzcd}
$$

If such an homotopy exists, we say that f is *homotopic* to g and write $f \simeq g$. Note that if $f \simeq g$ then $H^*(f) = H^*(g)$ (this follows from a diagram chase). The same could be said for *chain homotopies* and homology.

A (co)chain map $f : C \to C'$ is a *homotopy equivalence* if there is a (co)chain map $f' : C' \to C$ satisfying $f'f \simeq \mathrm{id}_C$ and $ff' \simeq \mathrm{id}_{C'}$. A (co)chain complex C is *contractible* if it is homotopy equivalent to the zero complex. In other words, if $\mathrm{id}_C \simeq 0$. Such a homotopy from id_C to 0 is called a *contracting homotopy*. A (co)chain complex is *exact* if its (co)homology is zero. Note in particular, that contractible complexes are exact.

For the rest of this project, we will work over a base ring k. So we use abbreviations $\mathrm{Hom}(M, N) = \mathrm{Hom}_k(M, N)$ and $M \otimes N = M \otimes_k N$. Let G be a group. By a *G-module*, we mean a kG-module. Note that a G-module can be viewed as a k-module together with a G-action. The Hom-functor in the category of G-modules, Hom_{kG}, will be abbreviated Hom_G. We can think of a map $f\colon M \to N$ of G-modules as a map of k-modules which further satisfies $gf(m) = f(gm)$ for all $g \in G$ and $m \in M$. For G-modules M and N, we consider $\mathrm{Hom}(M, N)$ and $M \otimes N$ as G-modules by assigning them the actions

$$(g.f)(m) = gf(g^{-1}m),$$
$$g.(m \otimes n) = gm \otimes gn,$$

for $f \in \mathrm{Hom}(M, N)$, $m \in M$, $n \in N$ and $g \in G$. We will consider k as a G-module, by assigning it the trivial G-action.

There exists a projective resolution

$$\cdots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0$$

of k over kG with augmentation $\varepsilon\colon P_0 \to k$. We abbreviate this chain by P. This makes

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\varepsilon} k \to 0$$

into an exact sequence, which we will abbreviate $\varepsilon\colon P \to k$ and refer to as the *augmented chain complex* associated to the resolution. In both chains we consider P_i as laying in degree i. A maybe surprising fact about projective resolutions is that they are, unique up to (canonical) homotopy equivalence in the following sense.

Proposition 1.1.1. *Let P and P' be projective resolutions of k. There is an augmentation-preserving map $f\colon P \to P'$, unique up to homotopy, and f is a homotopy equivalence.*

Proof. See Brown (1982, Chapter I, Theorem 7.5). □

In this project, we will in practice only construct *free* resolutions, i.e. where each P_i is free. To use free resolutions, we simply note the following.

Proposition 1.1.2. *Free modules are projective.*

Proof. One characterization of projective modules is, that a module P is projective if and only if there exist another module Q such that $P \oplus Q$ is a free module. □

It is sometimes useful to look at the so-called *standard resolution* of k over kG. To construct it, we let P_i be the free k-module with generating set G^{i+1} and with G-action $g.(g_0, \ldots, g_i) =$

$(gg_0, \ldots, gg_i)$. The differentials are given by

$$\partial_i = \sum_{j=0}^{i} (-1)^j d_j,$$

where $d_j(g_0, \ldots, g_i) = (g_0, \ldots, \hat{g}_j, \ldots, g_i)$, and the augmentation is given by $\varepsilon(g_0) = 1$.

Proposition 1.1.3. *The standard resolution of k over kG defined above is a resolution.*

Proof. We see that P is a chain complex, since

$$\begin{aligned}
\partial_i \circ \partial_{i+1} &= \sum_{\ell=0}^{i} \sum_{j=0}^{i+1} (-1)^{j+\ell} d_\ell \circ d_j \\
&= \sum_{0 \le j \le \ell \le i} (-1)^{j+\ell} d_\ell \circ d_j + \sum_{0 \le \ell < j \le i+1} (-1)^{j+\ell} d_\ell \circ d_j \\
&= \sum_{0 \le j \le \ell \le i} (-1)^{j+\ell} d_j \circ d_{\ell+1} + \sum_{0 \le \ell < j \le i+1} (-1)^{j+\ell} d_\ell \circ d_j \\
&= \sum_{0 \le j < \ell \le i+1} (-1)^{j+\ell-1} d_j \circ d_\ell + \sum_{0 \le j < \ell \le i+1} (-1)^{\ell+j} d_j \circ d_\ell \\
&= 0.
\end{aligned}$$

In the third equality, we used that

$$(d_\ell \circ d_j)(g_0, \ldots, g_{i+1}) = \begin{cases} (g_0, \ldots, \hat{g}_j, \ldots, g_{\widehat{\ell+1}}, \ldots, g_i) & \text{if } j \le \ell \\ (g_0, \ldots, \hat{g}_\ell, \ldots, \hat{g}_j, \ldots, g_i) & \text{if } j > \ell. \end{cases}$$

In the fourth equality we used the substitution $\ell := \ell - 1$ in the first sum and $(\ell, j) := (j, \ell)$ in the second sum. Furthermore, the whole of $\varepsilon \colon P \to k$ is a chain, since also

$$(\varepsilon \circ \partial_1)(g_0, g_1) = \varepsilon(g_1 - g_0) = \varepsilon(g_1) - \varepsilon(g_0) = 1 - 1 = 0.$$

We almost already knew all this from algebraic topology, since these differentials are essentially the same as the differentials in simplicial homology.

Now to show, that $\varepsilon \colon P \to k$ is an exact sequence. It is sufficient to construct a contracting homotopy h from the identity on the augmented chain complex to the zero chain map (Rotman, 2009, p. 337). Choose $h_i(g_0, \ldots, g_i) = (1, g_0, \ldots, g_i)$ for $i \ge 0$ and $h_{-1}(a) = a1 \in P_0$ for all $a \in k$.

Now, since $h_i \circ d_j = d_{j+1} \circ h_i$, we get for $i > 0$ that

$$\begin{aligned}\partial_{i+1} \circ h_i + h_{i-1} \circ \partial_i &= \sum_{j=0}^{i+1} (-1)^j d_j \circ h_i + \sum_{j=0}^{i} (-1)^j d_{j+1} \circ h_i \\ &= \sum_{j=0}^{i+1} (-1)^j d_j \circ h_i + \sum_{j=1}^{i+1} (-1)^{j-1} d_j \circ h_i \\ &= d_0 h_i \\ &= \mathrm{id}_{P_i}.\end{aligned}$$

Also,

$$\begin{aligned}(\partial_1 \circ h_0 + h_{-1} \circ \varepsilon)(g_0) &= \partial_1(1, g_0) + h_{-1}(1) \\ &= g_0 - 1 + 1 \\ &= \mathrm{id}_{P_0}(g_0),\end{aligned}$$

and

$$\begin{aligned}(\varepsilon \circ h_{-1})(a) &= \varepsilon(a1) \\ &= a \\ &= \mathrm{id}_k(a).\end{aligned}$$

So the identity map on the augmented chain map is homotopic to the zero chain map, and they thus induce the same maps on homology. Since the identity chain map induces an isomorphism on homology, and the zero chain map induces the zero map on homology, the homology of the augmented chain map must be trivial. In other words $\varepsilon\colon P \to k$ is exact. □

1.2 Group cohomology

Let G be a group, P a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ and M a G-module. The *cohomology of G with coefficients in M* is given by

$$H^*(G; M) = H^*(\mathrm{Hom}_G(P, M)).$$

Note that this definition is independent of the choice of projective resolution, since projective resolutions are unique up to homotopy equivalence.

Luckily, group cohomology is a (contravariant) functor in the first variable. Let $\phi\colon G \to G'$ be a group homomorphism and M a G-module. We can consider M as a G'-module through ϕ, by letting $g'.m = \phi(g').m$ for all $g' \in G'$ and $m \in M$. It can then be shown, that ϕ induces a map

$\phi^*\colon H^*(G';M) \to H^*(G;M)$ in a functorial manner (Brown, 1982, Section III.8).

Group cohomology is also a functor in the second variable, but we will not use this fact.

While this definition of group cohomology is very algebraic, it is also possible to use a more topological definition. We say that a topological space X is a $K(G,1)$ if $\pi_1(X) = G$ and $\pi_n(X) = 0$ for all $n > 1$. Such an X exists for any group G and is unique up to homotopy equivalence (Hatcher, 2001, Example 1B.7 and Theorem 1B.8).

Theorem 1.2.1. *Let G be a group and M a G-module with trivial action. Then*

$$H^*(G;M) \cong H^*(K(G,1);M)$$

as k-modules.

Proof. See Benson (1991, Theorem 2.2.3). □

Let us now present an interesting application of the functoriality of group cohomology. Let $H \trianglelefteq G$ be a normal subgroup of G. For each $g \in G$ we have a map $c_g\colon H \to H$ with $c_g(h) = ghg^{-1} \in gHg^{-1} = H$. This map induces a map $c_g^*\colon H^*(H;M) \to H^*(H;M)$. Note that c_g^* is the identity, for all $g \in G$, if $H \leq Z(G)$, where $Z(G)$ denotes the center of G. This gives us the following Lemma.

Proposition 1.2.2. *Let G be a group, $H \trianglelefteq G$ a normal subgroup and M a G-module. The conjugation action of G on H induces an action of G/H on $H^*(H;M)$.*

(a) This action is trivial if $H \leq Z(G)$. In particular, it is trivial if G is abelian.

(b) If P is a projective resolution of k over kG, then the action on cohomology is induced by the action of G on $\operatorname{Hom}_G(P,M)$ given by

$$(gf)(x) = gf(g^{-1}x),$$

for each $g \in G$, $f \in \operatorname{Hom}_G(P,M)$ and $x \in P$.

Proof. See Brown (1982, p. 80). □

We end this subsection with another way to interpret H^0 and H^1.

Proposition 1.2.3. *Let G be a group and M a G-module. Then*

$$H^0(G;M) \cong \{m \in M \mid \forall g \in G : g.m = m\}$$

as k-modules.

Proof. Let $\varepsilon\colon P \to k$ be the standard resolution of k over kG. Then $\operatorname{Hom}_G(P, M)$ is the (co)chain

$$\operatorname{Hom}_G(kG, M) \xrightarrow{\delta_0} \operatorname{Hom}_G(k[G \oplus G], M) \xrightarrow{\delta_1} \cdots .$$

So $H^0(G; M) \cong \ker \delta_0$. Let $f \in \operatorname{Hom}_G(kG, M)$. Note that for $g_0, g_1 \in G$,

$$g.\delta_0(f)(g_0, g_1) = g.f(\partial_1(g_0, g_1)) = f(g.\partial_1(g_0, g_1)) = f(\partial_1(gg_0, gg_1)) = \delta_0(f)(gg_0, gg_1).$$

We get the chain of implications

$$\begin{aligned}
\delta_0(f) = 0 &\implies \forall g_0, g_1 \in G : \delta_0(f)(g_0, g_1) = 0 \\
&\implies \forall g \in G : \delta_0(f)(1, g) = 0 \\
&\implies \forall g_0, g_1 \in G : g_0.\delta_0(f)(1, g_0^{-1}g_1) = 0 \\
&\implies \forall g_0, g_1 \in G : \delta_0(f)(g_0, g_1) = 0 \\
&\implies \delta_0(f) = 0,
\end{aligned}$$

so $\delta_0(f) = 0$ if and only if $\delta_0(f)(1, g) = 0$ for all $g \in G$. Since for any $g \in G$,

$$\delta_0(f)(1, g) = f(\partial_1(1, g)) = f(g - 1) = gf(1) - f(1),$$

this tells us that $f \in H^0(G; M) \iff \forall g \in G : g.f(1) = f(1)$. Since f is determined completely by its value on 1, we are done. ☐

Proposition 1.2.4. *Let G be a group and M a G-module with trivial action. Then*

$$H^1(G; M) \cong \operatorname{Hom}_{\text{Groups}}(G, M)$$

as abelian groups.

Proof. Note that M is abelian as a group, so the right-hand side is an abelian group. See Weibel (1994, Theorem 6.4.6) for a proof. ☐

1.3 The cup product

It is possible to endow $H^*(G; k)$ with a multiplicative structure which turns it into a commutative graded ring. This means that we will get an associative product

$$H^r(G; k) \otimes H^s(G; k) \to H^{r+s}(G; k)$$

with $\alpha\beta = (-1)^{\deg(\alpha)\deg(\beta)}\beta\alpha$. Let 1 be the identity element of $k = H^0(G; k)$. Then 1 will be the identity element with respect to the product.

First, we define the "cross product". Let G and G' be groups. Let $X \to k$ be a projective resolution over kG and $Y \to k$ a projective resolution over kG'. We define the *cross* product[1]

$$\mathrm{Hom}_G(X;k) \otimes \mathrm{Hom}_{G'}(Y;k) \to \mathrm{Hom}_{G\times G'}(X \otimes Y;k)$$

by

$$(f \times g)(x \otimes y) = f(x) \otimes g(y),$$

where we identify $k \otimes k$ with k. This induces a map on cohomology which together with the Künneth map gives a homomorphism

$$H^*(G;k) \otimes H^*(G';k) \to H^*(G \times G';k)$$

which preserves total degree. We denote the image of $\alpha \otimes \beta$ under this map by $\alpha \times \beta$.

Now, define the diagonal homomorphism $\Delta\colon G \to G \times G$ given by $\Delta(x) = x \times x$. By functoriality we get an induced homomorphism

$$\Delta^*\colon H^*(G \times G;k) \to H^*(G;k).$$

Finally, define the *cup product* of $\alpha \in H^r(G;k)$ and $\beta \in H^s(G;k)$ as

$$\alpha\beta = \alpha \cup \beta = \Delta^*(\alpha \times \beta) \in H^{r+s}(G;k).$$

It can be shown, that the cup product respects the properties written in the start of this subsection (Evens, 1991, Section 3.1).

1.4 The Universal Coefficients Theorem

A basic result from homological algebra, the Universal Coefficients Theorem, can be written nicely in terms of group cohomology. We here present a useful special case.

Corollary 1.4.1. *Let k be a field, G a group and M a kG-module with trivial action. Then*

$$H^*(G;M) \cong H^*(G;k) \otimes_k M.$$

Proof. See Evens (1991, p. 30). □

[1] We haven't introduced the tensor product of chain complexes, but we work with one in the proof of Proposition 2.2.3.

1.5 Spectral sequences

The Lyndon-Hochschild-Serre spectral sequence will help us in cohomology calculations, by approximating the cohomology of a group through the cohomology of a normal subgroup and the cohomology of its quotient group. In this subsection we introduce the concept of a bigraded cohomological spectral sequence. Then in subsection 1.6 we specialize to the spectral sequence of a double complex. Finally, in subsection 1.7 we specialize to the LHS spectral sequence. For further technical details about spectral sequences, see e.g. Benson (1991, Chapter 3).

By a *spectral sequence* we will mean a sequence $\{E_r, d_r\}_{r\geq 0}$ of bigraded G-modules $E_r = \bigoplus_{p,q\in\mathbb{Z}^2} E_r^{pq}$ and endomorphisms $d_r = \{d_r^{pq} \colon E_r^{pq} \to E_r^{(p+r)(q-r+1)}\}$ satisfying $d_r \circ d_r = 0$ and $E_{r+1}^{pq} \cong H^{pq}(E_r, d_r) := \ker(d_r^{pq})/\operatorname{im}(d_r^{(p-r)(q+r-1)})$. We call E_r the *E_r-page*, since we can think of a spectral sequence as a book, where flipping to the next page corresponds to taking homology of the page. We call the d_rs *differentials*.

We will only work with spectral sequences, where E_r vanishes outside the first quadrant, i.e. where $E_r^{pq} = 0$ if $p < 0$ or $q < 0$. For a fixed position (p, q), both the differential d_r^{pq} starting at E_r^{pq} and the differential $d_r^{(p-r)(q+r-1)}$ ending at E_r^{pq} will be trivial for sufficiently large r, since they will end or start outside the first quadrant. If $d_r^{(p-r)(q+r-1)} = d_r^{pq} = 0$ for some r, then $E_{r+1}^{pq} = H^{pq}(E_r, d_r) = E_r^{pq}$, so $E_{r'}^{pq} = E_r^{pq}$ for all $r' > r$, and we will denote this stabilized value $E_\infty^{pq} = E_r^{pq}$. Note that this doesn't guarantee the existence of a global value r such that $E_r^{pq} = E_\infty^{pq}$ for all positions (p, q).

1.6 The spectral sequence of a double complex

Let G be a group. Let $E_0 = (E_0^{pq}, d_0, d_1)$ denote a *double cochain complex* of G-modules, where d_1 and d_0 are maps of bidegree $(1, 0)$ and $(0, 1)$, respectively. This means that we require $d_1^2 = d_0^2 = d_1 d_0 + d_0 d_1 = 0$. We will restrict to the case where E_0 vanishes outside the first quadrant. We can visualize E_0 by the following (anticommutative) diagram.

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
E_0^{0,2} \arrow[u, "d_0"] \arrow[r, "d_1"] & E_0^{1,2} \arrow[u, "d_0"] \arrow[r, "d_1"] & E_0^{2,2} \arrow[u, "d_0"] \arrow[r, "d_1"] & \cdots \\
E_0^{0,1} \arrow[u, "d_0"] \arrow[r, "d_1"] & E_0^{1,1} \arrow[u, "d_0"] \arrow[r, "d_1"] & E_0^{2,1} \arrow[u, "d_0"] \arrow[r, "d_1"] & \cdots \\
E_0^{0,0} \arrow[u, "d_0"] \arrow[r, "d_1"] & E_0^{1,0} \arrow[u, "d_0"] \arrow[r, "d_1"] & E_0^{2,0} \arrow[u, "d_0"] \arrow[r, "d_1"] & \cdots
\end{tikzcd}$$

We can take cohomology of this complex with respect to d_0, getting

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
H^{02}(E_0, d_0) & \xrightarrow{(d_1)^*} & H^{12}(E_0, d_0) & \xrightarrow{(d_1)^*} & H^{22}(E_0, d_0) & \xrightarrow{(d_1)^*} & \cdots \\
H^{01}(E_0, d_0) & \xrightarrow{(d_1)^*} & H^{11}(E_0, d_0) & \xrightarrow{(d_1)^*} & H^{21}(E_0, d_0) & \xrightarrow{(d_1)^*} & \cdots \\
H^{00}(E_0, d_0) & \xrightarrow{(d_1)^*} & H^{10}(E_0, d_0) & \xrightarrow{(d_1)^*} & H^{20}(E_0, d_0) & \xrightarrow{(d_1)^*} & \cdots
\end{array}$$

We will denote the qth row by $H^q(E_0, d_0)$. Note that each row is a chain, since $(d_1)^* \circ (d_1)^* = (d_1 \circ d_1)^* = (0)^* = 0$. We could then take cohomology of each row with respect to d_1, i.e. $H^*(H^q(E_0, d_0), d_1)$.

To turn E_0 into a single chain complex, we define the *total complex* $T^n = \mathrm{Tot}(E_0)^n = \bigoplus_{p+q=n} E_0^{pq}$ and let $d = d_1 + d_0$. By the above required relations on d_1 and d_0, we have $d^2 = d_1^2 + d_1 d_0 + d_0 d_1 + d_0^2 = 0$. Note that the summands of T^n lie on an "antidiagonal" line in the above diagram.

To *filter* the complex T, we let

$$F^p T^n = \bigoplus_{p' \geq p} E_0^{p'(n-p')}.$$

Then $F^0 T = T$ and $F^p T^n = 0$ for $p > n$. Now $F^p T^{p+q} / F^{p+1} T^{p+q} \cong E_0^{pq}$. Each $F^p T$ is of course again a chain complex with maps induced by restricting d. We define

$$F^p H^{p+q}(T) = \mathrm{im}(H^{p+q}(F^p T) \to H^{p+q}(T)),$$

i.e. the image of the map on cohomology induced by the inclusion of chains $F^p T \to T$.

Theorem 1.6.1. *Let (E_0^{pq}, d_0, d_1) be a double complex and T its total complex. There is a spectral sequence with*

$$\begin{aligned}
E_1^{pq} &= H^{pq}(E_0, d_0), \\
E_2^{pq} &= H^p(H^q(E_0, d_0), d_1), \\
E_\infty^{pq} &= F^p H^{p+q}(T) / F^{p+1} H^{p+q}(T).
\end{aligned}$$

Proof. See Benson (1991, Theorem 3.4.2). ∎

The shorthand for this theorem is

$$H^p(H^q(E_0, d_0), d_1) \Rightarrow H^{p+q}(\mathrm{Tot}(E_0), d_1 + d_0).$$

1.7 The LHS spectral sequence

In this subsection, we construct the Lyndon-Hochschild-Serre spectral sequence of a group extension $0 \to H \to G \to G/H \to 0$. Let G be a group, $H \triangleleft G$ a normal subgroup and M a kG-module. Let $X \to k$ be a projective resolution over kG and let $Y \to k$ be a projective resolution over $k(G/H)$. We can consider $X \to k$ also as a projective resolution over kH, through the inclusion map. Recall that G acts on $\mathrm{Hom}_H(X, M)$ by $(gf)(x) = gf(g^{-1}x)$, so H acts trivially since $(hf)(x) = hf(h^{-1}x) = hh^{-1}f(x) = f(x)$. Therefore, we can consider $\mathrm{Hom}_H(X, M)$ as a G/H-module. We thus have a double complex

$$A^{pq} = \mathrm{Hom}_{G/H}(Y_p, \mathrm{Hom}_H(X_q, M)),$$

with

$$\begin{aligned}(d_0)^{pq} &= (-1)^p \mathrm{Hom}_{G/H}(\mathrm{id}, \mathrm{Hom}_H((d_X)_q, \mathrm{id})),\\ (d_1)^{pq} &= \mathrm{Hom}_{G/H}((d_Y)_p, \mathrm{Hom}_H(\mathrm{id}, \mathrm{id})).\end{aligned}$$

As required, we get

$$
\begin{aligned}
(d_0)^{p(q+1)} \circ (d_0)^{pq} &= (-1)^{p+p}\mathrm{Hom}_{G/H}(\mathrm{id}, \mathrm{Hom}_H((d_X)_{q+1}, \mathrm{id})) \circ \mathrm{Hom}_{G/H}(\mathrm{id}, \mathrm{Hom}_H((d_X)_q, \mathrm{id}))\\
&= \mathrm{Hom}_{G/H}(\mathrm{id}, \mathrm{Hom}_H((d_X)_{q+1}, \mathrm{id}) \circ \mathrm{Hom}_H((d_X)_q, \mathrm{id}))\\
&= \mathrm{Hom}_{G/H}(\mathrm{id}, \mathrm{Hom}_H((d_X)_q \circ (d_X)_{q+1}, \mathrm{id}))\\
&= \mathrm{Hom}_{G/H}(\mathrm{id}, \mathrm{Hom}_H(0, \mathrm{id}))\\
&= 0,\\
(d_1)^{(p+1)q} \circ (d_1)^{pq} &= \mathrm{Hom}_{G/H}((d_Y)_{p+1}, \mathrm{Hom}_H(\mathrm{id}, \mathrm{id})) \circ \mathrm{Hom}_{G/H}((d_Y)_p, \mathrm{Hom}_H(\mathrm{id}, \mathrm{id}))\\
&= \mathrm{Hom}_{G/H}((d_Y)_p \circ (d_Y)_{p+1}, \mathrm{Hom}_H(\mathrm{id}, \mathrm{id}))\\
&= \mathrm{Hom}_{G/H}(0, \mathrm{Hom}_H(\mathrm{id}, \mathrm{id}))\\
&= 0,\\
(d_0)^{(p+1)q}(d_1)^{pq} &= (-1)^{p+1}\mathrm{Hom}_{G/H}(\mathrm{id}, \mathrm{Hom}_H((d_X)_q, \mathrm{id})) \circ \mathrm{Hom}_{G/H}((d_Y)_p, \mathrm{Hom}_H(\mathrm{id}, \mathrm{id}))\\
&= (-1)^{p+1}\mathrm{Hom}_{G/H}((d_Y)_p \circ \mathrm{id}, \mathrm{Hom}_H((d_X)_q, \mathrm{id}) \circ \mathrm{Hom}_H(\mathrm{id}, \mathrm{id}))\\
&= (-1)^{p+1}\mathrm{Hom}_{G/H}(\mathrm{id} \circ (d_Y)_p, \mathrm{Hom}_H(\mathrm{id}, \mathrm{id}) \circ \mathrm{Hom}_H((d_X)_q, \mathrm{id}))\\
&= -\mathrm{Hom}_{G/H}((d_Y)_p, \mathrm{Hom}_H(\mathrm{id}, \mathrm{id})) \circ (-1)^p\mathrm{Hom}_{G/H}(\mathrm{id}, \mathrm{Hom}_H((d_X)_q, \mathrm{id}))\\
&= -(d_1)^{p(q+1)}(d_0)^{pq}.
\end{aligned}
$$

The double complex can be visualized by the following anticommutative diagram.

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \\
{\scriptstyle d_0}\uparrow & & {\scriptstyle d_0}\uparrow & & \\
\mathrm{Hom}_{G/H}(Y_0, \mathrm{Hom}_H(X_2, M)) & \xrightarrow{d_1} & \mathrm{Hom}_{G/H}(Y_1, \mathrm{Hom}_H(X_2, M)) & \xrightarrow{d_1} & \cdots\\
{\scriptstyle d_0}\uparrow & & {\scriptstyle d_0}\uparrow & & \\
\mathrm{Hom}_{G/H}(Y_0, \mathrm{Hom}_H(X_1, M)) & \xrightarrow{d_1} & \mathrm{Hom}_{G/H}(Y_1, \mathrm{Hom}_H(X_1, M)) & \xrightarrow{d_1} & \cdots\\
{\scriptstyle d_0}\uparrow & & {\scriptstyle d_0}\uparrow & & \\
\mathrm{Hom}_{G/H}(Y_0, \mathrm{Hom}_H(X_0, M)) & \xrightarrow{d_1} & \mathrm{Hom}_{G/H}(Y_1, \mathrm{Hom}_H(X_0, M)) & \xrightarrow{d_1} & \cdots
\end{array}
$$

By Theorem 1.6.1 we get the following theorem.

Theorem 1.7.1. *Let G be a group, $H \triangleleft G$ a normal subgroup and M a G-module. Then we have a spectral sequence with*

$$
\begin{aligned}
E_1^{pq} &= \mathrm{Hom}_{kG/H}(Y_p; H^q(H; M)),\\
E_2^{pq} &= H^p(G/H; H^q(H; M)),\\
E_\infty^{pq} &= F^pH^{p+q}(G; M)/F^{p+1}H^{p+q}(G; M),
\end{aligned}
$$

where for each $n \geq 0$,

$$H^n(G;M) = F^0H^n(G;M) \supseteq F^1H^n(G;M) \supseteq \cdots \supseteq F^nH^n(G;M) \supset F^{n+1}H^n(G;M) = 0$$

is some filtration of $H^n(G;M)$.

Proof. This follows from Theorem 1.6.1 and $\mathrm{Hom}_{kG/H}(Y_p, -)$ commuting with cohomology. See Benson (1991, p. 113) for a more detailed argument. □

The shorthand for this theorem is

$$H^p(G/H; H^q(H;M)) \Rightarrow H^{p+q}(G;M).$$

In the concrete computations we will perform, the following result will be useful.

Proposition 1.7.2. *Let* k *be a field,* G *be a group and* M *a* kG*-module. Then*

$$H^n(G;M) \cong E_\infty^{0n} \oplus E_\infty^{1(n-1)} \oplus \cdots \oplus E_\infty^{n0}$$

for all $n \geq 0$.

Proof. Since k is a field, every kG-module is a vector space. There is a filtration $H^n(G;M) \cong F_0 \supseteq F_1 \supseteq \cdots \supseteq F_n \supset F_{n+1} = 0$ with $F_i/F_{i+1} \cong E_\infty^{i(n-i)}$ and, in particular, $F_n \cong E_\infty^{n0}$. We therefore have

$$H^n(G;M) \cong F_0/F_1 \oplus F_1/F_2 \oplus \cdots \oplus F_{n-1}/F_n \oplus F_n,$$

which gives us what we want. □

When performing calculations, we will start with determining the E_2-page, skipping the E_0- and E_1-pages. One of the first steps is to determine the first row and column on the E_2-page. In our calculations, the rest of the E_2-page will follow, since we know a graded multiplicative structure of the page. We collect the following small results for easy reference.

Proposition 1.7.3. *Let* G *be a group,* $H \trianglelefteq G$ *a normal subgroup and* M *a* G*-module. Then*

$$E_2^{0*} \cong H^0(G/H; H^*(H;M)).$$

Proof. Clear from the above. □

Corollary 1.7.4. *Let* G *be a group,* $H \trianglelefteq G$ *a normal subgroup and* M *a* G*-module. If* $H \leq Z(G)$, *(or in particular if* G *is abelian) then*

$$E_2^{0*} \cong H^*(H;M).$$

Proof. Combine Proposition 1.7.3, Proposition 1.2.2 and Proposition 1.2.3. □

Proposition 1.7.5. *Let G be a group, $H \trianglelefteq G$ a normal subgroup and M a G-module. Then*

$$E_2^{*0} \cong H^*(G/H; H^0(H; M)).$$

Proof. Clear from the above. □

Corollary 1.7.6. *Let G be a group, $H \trianglelefteq G$ a normal subgroup and M a G-module. If the action of H on M is trivial, then*

$$E_2^{*0} \cong H^*(G/H; M).$$

In particular, the identity holds if the action of G on M is trivial.

Proof. Combine Proposition 1.7.5 and Proposition 1.2.3. □

We finish this subsection with a coverage of the interplay between the cup product in group cohomology and the multiplicative structure of the E_∞-page in the LHS spectral sequence.

Theorem 1.7.7. *Let G be a group, $H \trianglelefteq G$ a normal subgroup and M a G-module.*

(a) There is a graded product structure on the E_2-page which matches the tensor product structure we will see in computations. (This statement is of course very informal.)

(b) The product on the E_r-page for $r \geq 2$ induces a graded product structure on the E_{r+1}-page.

(c) The cup product in $H^*(G; M)$ restricts to maps $F^pH^m(G; M) \times F^sH^n(G; M) \to F^{p+s}H^{m+n}(G; M)$, which induce maps

$$\begin{aligned} F^pH^m(G; M)/F^{p+1}H^m(G; M) &\times F^sH^n(G; M)/F^{s+1}H^n(G; M) \\ &\to F^{p+s}H^{m+n}(G; M)/F^{p+s+1}H^{m+n}(G; M). \end{aligned}$$

These quotient maps induce maps $E_\infty^{p(m-p)} \times E_\infty^{s(n-s)} \to E_\infty^{(p+s)(m+n-p-s)}$, or in other words

$$E_\infty^{pq} \times E_\infty^{p'q'} \to E_\infty^{(p+p')(q+q')}.$$

And this product structure matches the one induced by the one originating from the E_2-page.

Proof. See Benson (1991, Section 3.9). □

The E_∞-page thus tells us the cup product structure of $H^*(G; M)$ up to a filtration.

2 Group cohomology calculations

We will calculate the cohomology of select finite groups with coefficients in $\mathbb{F}_2$. Where $\mathbb{F}_2$ is viewed as a G-module by assigning it the trivial action.Note that we see $\mathbb{F}_2$ as a G-module by assigning it the trivial action. First, we employ the algebraic topological approach using Eilenberg-MacLane spaces. Then, in subsequent subsections, we take an approach based more in homological algebra.

In this section, we work over the base field $k = \mathbb{F}_2$ and coefficient ring $M = \mathbb{F}_2$.

2.1 Calculations using Eilenberg-MacLane spaces

Proposition 2.1.1.

$$H^*(\mathbb{Z}/2; \mathbb{F}_2) \cong \mathbb{F}_2[x].$$

Proof. This follows from $\mathbb{RP}^\infty$ being a $K(\mathbb{Z}/2, 1)$ (Hatcher, 2001, Example 1B.3) and the singular cohomology of $\mathbb{RP}^\infty$ being $\mathbb{F}_2[x]$ where the polynomial ring indicates the cup product structure (Hatcher, 2001, Theorem 3.19). $\square$

2.2 Calculations using explicit resolutions

We now turn to more purely algebraic approaches. Sometimes it is possible to calculate the cohomology of a group G, by constructing an explicit resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. In this subsection, we don't make claims about the cup product structure of the graded cohomology rings we calculate. Nonetheless the results of this subsection will be useful, for our upcoming calculations using spectral sequences, where we do make claims about the cup product structure.

Proposition 2.2.1. *Let n be an even positive integer. Then for all $m \in \mathbb{Z}_{\geq 0}$, we have*

$$H^m(\mathbb{Z}/n; \mathbb{F}_2) \cong \mathbb{F}_2.$$

Proof. Let $C_n = \langle t \mid t^n = 1 \rangle$. We have the following free resolution of k over kC_n.

$$\cdots \xrightarrow{t-1} kC_n \xrightarrow{N} kC_n \xrightarrow{t-1} kC_n \xrightarrow{\varepsilon} k \to 0,$$

where $N = 1 + t + t^2 + \cdots + t^{n-1}$. It is clearly a chain, since $Nt = N$, so $N(t-1) = 0$. We will show exactness by constructing a contracting homotopy h. Let $h_{-1}(1) = 1$. For $i \geq 0$ and $0 \leq k < n$, let

$$h_i(t^k) = \begin{cases} \sum_{j=0}^{k-1} t^j & \text{if } i \text{ even and } k > 0 \\ 1 & \text{if } i \text{ odd and } k = n-1 \\ 0 & \text{else.} \end{cases}$$

Now for even $i > 0$ we have

$$\begin{aligned}((t-1)\circ h_i + h_{i-1}\circ N)(1) &= (t-1)0 + h_{i-1}(N) \\ &= 1.\end{aligned}$$

For even $i > 0$ and $0 < k < n$ we have

$$\begin{aligned}((t-1)\circ h_i + h_{i-1}\circ N)(t^k) &= (t-1)\sum_{j=0}^{k-1} t^j + h_{i-1}(Nt^k) \\ &= t^k - 1 + h_{i-1}(N) \\ &= t^k - 1 + 1 \\ &= t^k.\end{aligned}$$

For odd $i > 0$ and $0 \le k < n-1$ we have

$$\begin{aligned}(N\circ h_i + h_{i-1}\circ(t-1))(t^k) &= Nh_i(t^k) + h_{i-1}(t^{k+1} - t^k) \\ &= N0 + h_{i-1}(t^{k+1}) - h_{i-1}(t^k) \\ &= t^k.\end{aligned}$$

For odd $i > 0$ we have

$$\begin{aligned}(N\circ h_i + h_{i-1}\circ(t-1))(t^{n-1}) &= Nh_i(t^{n-1}) + h_{i-1}(1 - t^{n-1}) \\ &= N1 + h_{i-1}(1) - h_{i-1}(t^{n-1}) \\ &= N + 0 - \sum_{j=0}^{n-2} t^j \\ &= t^{n-1}.\end{aligned}$$

Furthermore,

$$\begin{aligned}((t-1)\circ h_0 + h_{-1}\circ\varepsilon)(1) &= (t-1)0 + h_{-1}(1) \\ &= 1,\end{aligned}$$

and for $0 < k < n$,

$$\begin{aligned}((t-1) \circ h_0 + h_{-1} \circ \varepsilon)(t^k) &= (t-1) \sum_{j=0}^{k-1} t^j + h_{-1}(1) \\ &= t^k - 1 + 1 \\ &= 1.\end{aligned}$$

Finally, $(\varepsilon \circ h_{-1})(1) = 1$. This shows, that h is a contracting homotopy from the augmented chain associated to our claimed resolution to the zero chain. Thus the augmented chain is exact and the claimed resolution is actually a resolution.

Note that $\mathrm{Hom}_{C_n}(kC_n, \mathbb{F}_2) = \mathbb{F}_2$ since any $f\colon kC_n \to \mathbb{F}_2$ must have $f(t^k) = f(t^k)^n = f(t^{tn}) = f(1)$. Because of this, we furthermore have $f \circ t^k = f$ for all k. Therefore, $f \circ (t-1) = f - f = 0$ and $f \circ N = nf = 0$ since n is even. When we apply $\mathrm{Hom}_{C_n}(-, \mathbb{F}_2)$ to the resolution, we thus get

$$\mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \xrightarrow{0} \cdots .$$

The cohomology of this chain is $\mathbb{F}_2$ in each degree. And the cohomology of this chain is exactly the cohomology of $C_n \cong \mathbb{Z}/n$ with coefficients in $\mathbb{F}_2$. $\square$

Proposition 2.2.2. *Let n be an odd positive integer. Then*

$$H^*(\mathbb{Z}/n; \mathbb{F}_2) \cong \mathbb{F}_2.$$

Proof. We can reuse the resolution from the the proof of Proposition 2.2.1. We still have $\mathrm{Hom}_{C_n}(kC_n, \mathbb{F}_2) = \mathbb{F}_2$. For any $f \in \mathrm{Hom}_{C_n}(kC_n, \mathbb{F}_2)$ we also still have $f \circ (t-1) = 0$. But we no longer have $f \circ N = 0$, since n is not even. Instead we have $f \circ N = f$. So when we apply $\mathrm{Hom}_{C_n}(-, \mathbb{F}_2)$ to the resolution, we get

$$\mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \xrightarrow{\mathrm{id}} \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \xrightarrow{\mathrm{id}} \cdots .$$

So here the cohomology is $\mathbb{F}_2$ in degree 0 and trivial in all other degrees. $\square$

Proposition 2.2.3. *For all $m \in \mathbb{Z}_{\geq 0}$, we have*

$$H^m(\mathbb{Z}/2 \oplus \mathbb{Z}/2; \mathbb{F}_2) \cong F_2^{m+1}.$$

Proof. Let $G = C_2 \oplus C_2$. We have already established that

$$\cdots \xrightarrow{t-1} kC_2 \xrightarrow{N} kC_2 \xrightarrow{t-1} kC_2 \xrightarrow{\varepsilon} k \to 0,$$

is a projective resolution of k over kC_2. Denote this resolution by $\varepsilon\colon P \to k$. Then by Brown (1982, Chapter V, Proposition 1.1), $\varepsilon\otimes\varepsilon\colon P\otimes P\to k$ is a projective resolution of k over $k[C_2\times C_2]=kG$. We can visualize the tensor product $P\otimes P$ as follows.

$$
\begin{tikzcd}
\vdots \arrow[d, "(t-1)\otimes\mathrm{id}" description] & & & \\
kC_2\otimes kC_2 \arrow[d, "(t+1)\otimes\mathrm{id}" description] & \iddots \arrow[l, "-\mathrm{id}\otimes(t-1)"'] \arrow[d, "(t+1)\otimes\mathrm{id}" description] & & \\
kC_2\otimes kC_2 \arrow[d, "(t-1)\otimes\mathrm{id}" description] & kC_2\otimes kC_2 \arrow[l, "\mathrm{id}\otimes(t-1)"'] \arrow[d, "(t-1)\otimes\mathrm{id}" description] & \iddots \arrow[l, "\mathrm{id}\otimes(t+1)"'] \arrow[d, "(t-1)\otimes\mathrm{id}" description] & \\
kC_2\otimes kC_2 & kC_2\otimes kC_2 \arrow[l, "-\mathrm{id}\otimes(t-1)"'] & kC_2\otimes kC_2 \arrow[l, "-\mathrm{id}\otimes(t+1)"'] & \cdots \arrow[l, "-\mathrm{id}\otimes(t-1)"']
\end{tikzcd}
$$

Here

$$(P\otimes P)_n=\bigoplus_{j+k=n}P_j\otimes P_k=(kC_2\otimes kC_2)^{n+1}$$

is the sum along each antidiagonal. If d is the differential of P, then

$$
\begin{aligned}
&(d\otimes d)_n(x_n\otimes y_0,\dots,x_0\otimes y_n)\\
&\qquad=(d_nx_n\otimes y_0+(-1)^{n-1}x_{n-1}\otimes d_1y_1,\dots,d_1x_1\otimes y_{n-1}+x_0\otimes d_ny_n)).
\end{aligned}
$$

To get cohomology, we first want to take $\mathrm{Hom}_G(-,\mathbb{F}_2)$ of the chain

$$\cdots\to(kC_2\otimes kC_2)^3\to(kC_2\otimes kC_2)^2\to kC_2\otimes kC_2\overset{\varepsilon\otimes\varepsilon}{\to}k\to 0.$$

Note that if $f\in\mathrm{Hom}_G((kC_2\otimes kC_3)^{n+1},\mathbb{F}_2)$, then

$$
\begin{aligned}
f(0,\dots,0,1\otimes 1,0,\dots,0)&=(t^\alpha,t^\beta).f(0,\dots,0,1\otimes 1,0,\dots,0)\\
&=f(0,\dots,0,t^\alpha\otimes t^\beta,0,\dots,0),
\end{aligned}
$$

for all $\alpha, \beta \in \{0,1\}$. Therefore $\mathrm{Hom}_G((P \otimes P)_n, \mathbb{F}_2) = \mathbb{F}_2^{n+1}$. Since e.g.

$$\begin{aligned} f(0,\dots,0,((t+1)\otimes \mathrm{id})(t^\alpha \otimes t^\beta),0,\dots,0) &= f(0,\dots,0,t^{\alpha+1}\otimes t^\beta + t^\alpha \otimes t^\beta,0,\dots,0) \\ &= f(0,\dots,0,t^{\alpha+1}\otimes t^\beta,0,\dots,0) \\ &\quad + f(0,\dots,0,t^\alpha \otimes t^\beta,0,\dots,0) \\ &= 0, \end{aligned}$$

we get $(d \otimes d)_n = 0$ for all n. So the chain $\mathrm{Hom}_G(P \otimes P, \mathbb{F}_2)$ looks like

$$\mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2^2 \xrightarrow{0} \mathbb{F}_2^3 \xrightarrow{0} \cdots .$$

We conclude $H^n(\mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{F}_2) \cong H^n(\mathrm{Hom}_G(P \otimes P, \mathbb{F}_2)) \cong \mathbb{F}_2^{n+1}$. □

2.3 Calculations using the LHS spectral sequence

In this section we present the major results of the project. Using the LHS spectral sequence will allow us to examine cup product structure of our group cohomologies.

We will denote by $\bigwedge(x)$ the exterior algebra $\bigwedge_{\mathbb{F}_p}(x) = \mathbb{F}_p[x]/x^2$.

Proposition 2.3.1. *We have*

$$H^*(\mathbb{Z}/4; \mathbb{F}_2) \cong \mathbb{F}_2[z] \otimes \bigwedge(y),$$

where $|z| = 2$, $|y| = 1$.

Proof. Let $G = \mathbb{Z}/4$ and $H = 2\mathbb{Z}/4$. Then $H \triangleleft G$ with $G/H \cong \mathbb{Z}/2$. Recall that the action induced by conjugation of G/H on $H^*(H; M)$ is trivial by Proposition 1.2.2, since G is abelian. Using the LHS spectral sequence, we have

$$\begin{aligned} E_2^{**} &\cong H^*(G/H; H^*(H; M)) \\ &\cong H^*(G/H; k) \otimes H^*(H; M) \\ &\cong H^*(\mathbb{Z}/2; \mathbb{F}_2) \otimes H^*(\mathbb{Z}/2; \mathbb{F}_2) \\ &\cong \mathbb{F}_2[y] \otimes \mathbb{F}_2[x], \end{aligned}$$

where the second equality follows by Corollary 1.4.1 and the last by Proposition 2.1.1. By Corollary 1.7.4 we get

$$E_2^{0*} \cong H^*(H; M) \cong \mathbb{F}_2[x].$$

By Corollary 1.7.6 we get

$$E_2^{*0} \cong H^*(G/H; M) \cong \mathbb{F}_2[y].$$

The E_2-page is drawn below, where e.g. xy^2 represents $\mathbb{F}_2xy^2$.

$$\begin{tikzcd}
2 & x^2 & x^2y & x^2y^2 \\
1 & x \arrow[drr] & xy & xy^2 \\
0 & 1 & y & y^2 \\
 & 0 & 1 & 2
\end{tikzcd}$$

E_2-page.

$$\begin{array}{c|cccc}
3 & 0 & 0 & 0 & 0 \\
2 & x^2 & x^2y & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & y & 0 & 0 \\
\hline
 & 0 & 1 & 2 & 3
\end{array}$$

E_3-page.

We have drawn the differential d_2^{01}, which is either trivial or the identity. Note that $E_2^{10} \cong E_\infty^{10}$, since the differential into and out of y are trivial on all pages. If d_2^{01} is trivial, then $E_2^{01} \cong E_\infty^{01}$, since all differentials in and out of E_r^{01} are trivial on higher pages. This implies that $H^1(\mathbb{Z}/2; \mathbb{F}_2) \cong E_\infty^{01} \oplus E_\infty^{10}$ is 2-dimensional as an $\mathbb{F}_2$-vector space. But this contradicts Proposition 2.1.1. So d_2^{01} is the identity, i.e. $d_2(x) = y^2$.

From the Leibniz identity we get by induction that $d_2(x^n) = nx^{n-1}y^2$ which is trivial for even n and the identity for odd n. Using the Leibniz identity once more, together with the fact that $d_2(y^m) = 0$ for all m, we get that $d_2(x^ny^m) = d_2(x^n)y^m + x^nd_2(y^m) = d_2(x^n)y^m$ is trivial for even n and the identity for odd n.

For any $m > 0$ and odd $n > 0$ we thus get $E_3^{mn} = 0$, since d_2^{mn} is injective. For any $m \geq 2$ and even $n > 0$ we get $E_3^{mn} = 0$, since $d_2^{(m-2)(n-1)}$ is surjective. For any $0 < m < 2$ and even $n > 0$ we get $E_3^{mn} \cong E_2^{mn}$, since $d_2^{mn} = 0$ and $d_2^{(m-2)(n-1)} = 0$ since $m - 2 < 0$. The E_3-page is drawn above.

On this E_3-page all differentials are trivial, and thus $E_\infty^{**} \cong E_3^{**}$. Since each diagonal on the page has exactly one non-trivial entry, (or using Proposition 1.7.2) we regain that $H^n(G; M) \cong \mathbb{F}_2$ for all $n > 0$. And $H^{2n}(G; M)$ is generated by x^{2n} and H^{2n+1} is generated by $x^{2n}y$. With respect to the product structure on the $E_3 \cong E_\infty$-page, we have

$$E_\infty \cong \mathbb{F}_2[x^2] \otimes \bigwedge(y).$$

This matches the cup product structure up to a filtration. But since only one filtration quotient is non-trivial in each degree, this product matches the cup product directly. So

$$H^*(\mathbb{Z}/2; \mathbb{F}_2) \cong \mathbb{F}_2[x^2] \otimes \bigwedge(y).$$

□

Proposition 2.3.2. *For positive integers $n \geq 2$ we have*

$$H^*(\mathbb{Z}/2^n; \mathbb{F}_2) \cong \mathbb{F}_2[z] \otimes \bigwedge(y),$$

where $|z| = 2$, $|y| = 1$.

Proof. We will perform induction over n. The base case $n = 2$ is just Proposition 2.3.1. Let $n > 2$ be given and assume

$$H^*(\mathbb{Z}/2^{n-1}; \mathbb{F}_2) \cong \mathbb{F}_2[z] \otimes \bigwedge(y),$$

where $|z| = 2$ and $|y| = 1$. Let $G = \mathbb{Z}/2^n$ and $H = 2^{n-1}\mathbb{Z}/2^n$. Then $H \cong \mathbb{Z}/2$ and $G/H \cong \mathbb{Z}/2^{n-1}$. Let us construct the LHS spectral sequence for $H \to G \to G/H$. By the same argument as in the proof of Proposition 2.3.1, we get

$$\begin{aligned}
E_2^{**} &\cong H^*(G/H; H^*(H; M)) \\
&\cong H^*(G/H; k) \otimes H^*(H; M) \\
&\cong H^*(\mathbb{Z}/2^{n-1}; \mathbb{F}_2) \otimes H^*(\mathbb{Z}/2; \mathbb{F}_2) \\
&\cong \mathbb{F}_2[z] \otimes \bigwedge(y) \otimes \mathbb{F}_2[x], \\
E_2^{0*} &\cong H^*(H; M) \\
&\cong \mathbb{F}_2[x], \\
E_2^{*0} &\cong H^*(G/H; M) \\
&\cong \mathbb{F}_2[z] \otimes \bigwedge(y),
\end{aligned}$$

where $|x| = |y| = 1$ and $|z| = 2$. The E_2-page is drawn below.

$$\begin{array}{c|ccccc}
2 & x^2 & x^2y & x^2z & x^2yz & x^2z^2 \\
1 & x & xy & xz & xyz & xz^2 \\
0 & 1 & y & z & yz & z^2 \\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}$$

E_2-page.

$$\begin{array}{c|cccc}
3 & 0 & 0 & 0 & 0 \\
2 & x^2 & x^2y & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & y & 0 & 0 \\
\hline
 & 0 & 1 & 2 & 3
\end{array}$$

E_3-page.

Just like in the proof of Proposition 2.3.1, we know that $d_2^{01} = 0$ would imply that $H^1(\mathbb{Z}/2^n)$ is 2-dimensional as an $\mathbb{F}_2$-vector space. But that would contradict by Proposition 2.2.1, so d_2^{01} is the identity. In other words, $d_2(x) = z$. Also, $d_2(y) = d_2(z) = 0$, since d_2^{10} and d_2^{20} exit the first quadrant. By the Leibniz identity, we get

$$\begin{aligned} d_2(x^\alpha y^\beta z^\gamma) &= d_2(x^\alpha) y^\beta z^\gamma + x^\alpha d_2(y^\beta z^\gamma) \\ &= \alpha x^{\alpha-1} y^\beta z^{\gamma+1} \\ &= \begin{cases} x^{\alpha-1} y^\beta z^{\gamma+1} & \text{if } \alpha \text{ odd} \\ 0 & \text{else.} \end{cases} \end{aligned}$$

This is the same situation as in Proposition 2.3.1, so we get the E_3-page drawn above. This is the same E_3-page as in the proof of Proposition 2.3.1, so we again get

$$H^*(\mathbb{Z}/2^n; \mathbb{F}_2) \cong \mathbb{F}_2[z] \otimes \bigwedge(y).$$

□

Proposition 2.3.3. *We have*

$$H^*(\mathbb{Z}/2 \oplus \mathbb{Z}/2; \mathbb{F}_2) \cong \mathbb{F}_2[x, y].$$

Proof. Let $G = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $H = \mathbb{Z}/2 \oplus 0$. Then $H \triangleleft G$ with $G/H \cong \mathbb{Z}/2$ and $H \cong \mathbb{Z}/2$. Recall that the action induced by conjugation of G/H on $H^*(H; M)$ is trivial by Proposition 1.2.2, since G is abelian. Using the LHS spectral sequence, we have

$$\begin{aligned} E_2^{**} &\cong H^*(G/H; H^*(H; M)) \\ &\cong H^*(G/H; k) \otimes H^*(H; M) \\ &\cong H^*(\mathbb{Z}/2; \mathbb{F}_2) \otimes H^*(\mathbb{Z}/2; \mathbb{F}_2) \\ &\cong \mathbb{F}_2[y] \otimes \mathbb{F}_2[x], \end{aligned}$$

where the second equality follows by Corollary 1.4.1 and the last by Proposition 2.1.1. By Corollary 1.7.4 we get

$$E_2^{0*} \cong H^*(H; M) \cong \mathbb{F}_2[x].$$

By Corollary 1.7.6 we get

$$E_2^{*0} \cong H^*(G/H; M) \cong \mathbb{F}_2[y].$$

The E_2-page is drawn below.

$$\begin{tikzcd}
2 & x^2 & x^2y & x^2y^2 \\
1 & x \arrow[drr] & xy & xy^2 \\
0 & 1 & y & y^2 \\
 & 0 & 1 & 2
\end{tikzcd}$$

E_2-page.

Note that d_2^{01} is either trivial or the identity map. If it were the identity, then x would vanish on higher pages of the spectral sequence. Which would imply, that $H^1(G;M)$ is 1-dimensional as an F_2-vector space. But we know from Proposition 2.2.3 that $H^1(G;M)$ is 2-dimensional. So $d_2(x) = 0$. Since d_2^{10} exits the first quadrant, we also know $d_2(y) = 0$. Thus $d_2 = 0$ and so $E_\infty \cong E_2 \cong \mathbb{F}_2[x,y]$.

Using Proposition 1.7.2, we conclude that

$$H^n(\mathbb{Z}/2 \oplus \mathbb{Z}/2; \mathbb{F}_2) \cong \mathbb{F}_2[x^n, x^{n-1}y, \dots, y^n].$$

It remains to be shown, that the product structure of the E_∞-page lifts nicely to $H^*(\mathbb{Z}/2 \oplus \mathbb{Z}/2; \mathbb{F}_2)$. $\square$

Proposition 2.3.4. *We have*

$$H^*(D_8; \mathbb{F}_2) \cong \mathbb{F}_2[z^2] \otimes \mathbb{F}_2[x,y]/(x^2 + xy).$$

Proof. Let us write $G = D_8 = \langle \sigma, \tau \mid \sigma^4 = \tau^2 = (\sigma\tau)^2 = 1 \rangle$ and $H = Z(D_8) = \langle \sigma^2 \rangle$, where $Z(D_8) = \{z \in D_8 \mid \forall g \in D_8 : gz = zg\}$ denotes the center of D_8. Then $H \cong \mathbb{Z}/2$ and $G/H \cong Z/2 \oplus Z/2$. Since H is the center of G, the G/H-action on $H^*(H;M)$ is trivial by Proposition 1.2.2. We get an associated LHS spectral sequence with

$$\begin{aligned}
E_2 &\cong H^*(G/H; H^*(H;M)) \\
&\cong H^*(G/H; k) \otimes H^*(H;M) \\
&\cong H^*(\mathbb{Z}/2 \oplus \mathbb{Z}/2; \mathbb{F}_2) \otimes H^*(\mathbb{Z}/2; \mathbb{F}_2) \\
&\cong \mathbb{F}[x,y] \otimes \mathbb{F}_2[z],
\end{aligned}$$

where the second equality follows by Corollary 1.4.1 and the last by Proposition 2.3.3 and Proposition 2.1.1. By Corollary 1.7.4 and Proposition 1.2.2 and we have

$$E_2^{0*} \cong H^*(H;M) \cong \mathbb{F}_2[z]$$

and by Corollary 1.7.6 we have

$$E^{*0} \cong H^*(G/H; M) \cong F_2[x, y].$$

Part of the E_2-page is drawn below.

$$\begin{tikzcd}
2 & z^2 & & & \\
1 & z \arrow[drr] & & & \\
0 & 1 & x,y & x^2,y^2,xy & \\
 & 0 & 1 & 2 & 3
\end{tikzcd}$$

E_2-page.

We want to determine d_2^{01}. We know that it is on the form $d_2^{01}(z) = ax^2 + by^2 + cxy$ for $a, b, c \in \mathbb{F}_2$. By Proposition 1.2.4, we have $H^1(G/H; \mathbb{F}_2) \cong \mathrm{Hom}_{\mathrm{Groups}}(D_8/\langle\sigma^2\rangle, \mathbb{F}_2)$ which is represented by x and y. Without loss of generality, we can assume $x(\sigma) = y(\tau) = 1$ and $x(\tau) = y(\sigma) = 0$, since x and y then form a basis of $H^1(G/H)$.

Let us restrict the spectral sequence to the group extension $\langle\sigma^2\rangle \to \langle\sigma\rangle \to \langle\sigma\rangle/\langle\sigma^2\rangle$, which is isomorphic to $2\mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2$. This restriction corresponds to setting $y = 0$, since we've removed τ. So in the restricted spectral sequence, $d_2^{01} = ax^2$. We know the restricted spectral sequence from Proposition 2.3.1, where d_2^{01} was non-zero. We claim that we must therefore have $a = 1$. This follows from the fact that the map of group extensions

$$\begin{tikzcd}
\langle\sigma^2\rangle \arrow[r] \arrow[d, equal] & \langle\sigma\rangle \arrow[r] \arrow[d, hook] & \langle\sigma\rangle/\langle\sigma^2\rangle \arrow[d, hook] \\
H \arrow[r] & G \arrow[r] & G/H
\end{tikzcd}$$

induces a map of spectral sequences, commuting with differentials. This is known as the naturality of the Serre spectral sequence.

Let us now instead restrict to $\langle\sigma^2\rangle \to \langle\sigma^2, \tau\rangle \to \langle\sigma^2, \tau\rangle/\langle\sigma^2\rangle$, which is isomorphic to $\mathbb{Z}/2 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z}/2$. This corresponds to setting $x = 0$, since we've removed all powers of σ from the factor group. So in the restricted spectral sequence, $d_2^{01} = by^2$. We know the restricted spectral sequence from Proposition 2.3.3, where d_2^{01} was trivial. So $b = 0$.

Let us finally restrict to $\langle\sigma^2\rangle \to \langle\sigma^2, \sigma\tau\rangle \to \langle\sigma^2, \sigma\tau\rangle/\langle\sigma^2\rangle$, which is isomorphic to $\mathbb{Z}/2 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z}/2$. This corresponds to setting $x = y$, since the factor group is generated by $\sigma\tau$. So in the restricted spectral sequence, $d_2^{01} = (a + b + c)x^2$. We again know the restricted spectral sequence from Proposition 2.3.3, where d_2^{01} was trivial. So $a + b + c = 0$, i.e. $c = 1$.

We conclude that $d_2(z) = x^2 + xy$ in the original spectral sequence. Using the Leibniz identity, we get $d_2(z^2) = d_2(z)z + zd_2(z) = 0$, and can further conclude by induction, that

$$\begin{aligned} d_2(z^\gamma) &= d_2(z)z^{\gamma-1} + zd_2(z^{\gamma-1}) \\ &= \begin{cases} x^2 z^{\gamma-1} + xyz^{\gamma-1} & \text{if } \gamma \text{ odd} \\ 0 & \text{else.} \end{cases} \end{aligned}$$

Since d_2^{10} exits the first quadrant, it is trivial, so we get

$$\begin{aligned} d_2(x^\alpha y^\beta z^\gamma) &= d_2(x^\alpha y^\beta)z^\gamma + x^\alpha y^\beta d_2(z^\gamma) \\ &= x^\alpha y^\beta d_2(z^\gamma) \\ &= \begin{cases} x^{\alpha+2} y^\beta z^{\gamma-1} + x^{\alpha+1} y^{\beta+1} z^{\gamma-1} & \text{if } \gamma \text{ odd} \\ 0 & \text{else.} \end{cases} \end{aligned}$$

This tells us, that $\ker d_2^{pq} = 0$ for odd q, i.e. $E_3^{pq} = 0$ for odd q. Since $\operatorname{im} d_2^{01}$ is generated by $x^2 + xy$, we get

$$E_3^{20} \cong \frac{\mathbb{F}_2[x^2, y^2, xy]}{x^2 + xy} \cong \mathbb{F}_2[x^2, y^2]$$

and similarly, by observing $\operatorname{im} d_2^{(p-2)(q-1)}$ for $p \geq 2$ and even $q \geq 1$, we get $E_3^{pq} = \mathbb{F}_2[x^p, y^p]$. So we get $E_3 = \mathbb{F}_2[z^2] \otimes \mathbb{F}_2[x, y]/(x^2 + xy)$ which can be visualized as follows.

$$\begin{array}{c|ccccc} 3 & 0 & 0 & 0 & 0 & \\ 2 & z^2 & xz^2, yz^2 & x^2z^2, y^2z^2 & x^3z^2, y^3z^2 & \\ 1 & 0 & 0 & 0 & 0 & \\ 0 & 1 & x, y & x^2, y^2 & x^3, y^3 & \\ \hline & 0 & 1 & 2 & 3 & 4 \end{array}$$

E_3-page.

We know d_3^{02} is of the form $d_3(z^2) = ax^3 + by^3$ for some $a, b \in \mathbb{F}$. We can once again restrict the spectral sequence to two the group extensions mentioned above. In each case, we know from previous proofs that d_3 is trivial in the restriction. Since one of the restrictions corresponded to setting $x = 0$ and another to setting $y = 0$, we get $b = 0$ and $a = 0$, respectively, in the restrictions, and thus also in the original spectral sequence. We also have $d_3(x) = d_3(y) = 0$, since d_3^{10} exits the first quadrant. So we have $d_3 = 0$, and therefore $E_\infty \cong E_3 \cong \mathbb{F}_2[z^2] \otimes \mathbb{F}_2[x, y]/(x^2 + xy)$.

Using Proposition 1.7.2, we conclude that

$$H^n(G;M) \cong \mathbb{F}_2[x^n, y^n, x^n z^2, y^n z^2, \ldots, x^n z^n, y^n z^n]$$

for even n, and

$$H^n(G;M) \cong \mathbb{F}_2[x^n, y^n, x^n z^2, y^n z^2, \ldots, x^n z^{n-1}, y^n z^{n-1}]$$

for odd n.

The multiplicative structure $\mathbb{F}_2[z^2] \otimes \mathbb{F}_2[x,y]/(x^2+xy)$ of the $E_3 \cong E_\infty$-page in the proof above corresponds to the cup product structure of $H^*(D_8;\mathbb{F}_2)$ up to a filtration. It remains to be shown, how this structure lifts to $H^*(D_8;\mathbb{F}_2)$. $\square$

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