KØBENHAVNS UNIVERSITET

Geometric Satake Equivalence

Daniel Pham Nguyen

Vejleder: Jesper Grodal

Department of Mathematical Sciences. 4. juni 2026

Abstract

In [MV18], Mirković and Vilonen proved a landmark result known as the *geometric Satake equivalence*. If G is a complex connected reductive group and Λ is a field of characteristic 0 (or, more generally, a Noetherian ring of finite global dimension), then the category $\mathrm{Rep}_\Lambda(\check{G}_\Lambda)$ of finitely generated representations of the Langlands dual group $\check{G}_\Lambda$ is canonically equivalent, as a symmetric monoidal category, to the Satake category Sat_G of L^+G-equivariant perverse sheaves on the affine Grassmannian Gr_G, equipped with the convolution product. This project is devoted to a detailed exposition of the proof of this equivalence.

Contents

1 Introduction

This bachelor project is devoted to a detailed exposition of the proof of the geometric Satake equivalence. At its core, this equivalence provides a canonical construction of the representations of the Langlands dual group of a reductive group, remedying the classical reconstruction theorem which is non-canonical. Most of the exposition is inspired/following [BR18] and [Zhu16].

Let G be a connected, reductive group scheme of finite type over $\mathbb{C}$. The main geometric object is the affine Grassmannian, Gr_G, defined as the fpqc-quotient LG/L^+G where the positive loop group L^+G acts on LG by right multiplication. It also acts by left multiplication, allowing us to define the category Sat_G of L^+G-equivariant perverse sheaves on (the underlying analytic topology of) Gr_G. The main occupation of the proof is the verify that Sat_G is a neutral Tannakian cateogory (see Appendix A.0.1), which includes promoting Sat_G to a symmetric monoidal category, which, arguably is the hardest part of the proof. Once this has been established, the Tannakian reconstruction theorem (Appendix A.0.2) due to Deligne and Milne in [DM18] supplies an affine group scheme and an equivalence of symmetric monoidal categories between Sat_G and the representations of $\check{G}_\Lambda$.

More precisely,

Theorem 1.0.1 (Geometric Satake equivalence)**.** Let G be a connected, reductive group scheme of finite type over $\mathbb{C}$, and let Λ be a field. The Satake category Sat_G is abelian and Λ-linear (and if $\mathrm{char}\,\Lambda = 0$, then Sat_G is semisimple). There is a "convolution product" $\star$ which promotes Sat_G to a symmetric monoidal category $\mathrm{Sat}_G^\star$ that is rigid, and there is a canonical equivalence $\mathrm{End}_{\mathrm{Sat}_G}(1) \simeq \Lambda$. The functor $H^* = \bigoplus_{k\in\mathbb{Z}} \mathsf{H}^k(\mathrm{Gr}_G, -) : \mathrm{Sat}_G \to \mathrm{Mod}_\Lambda^{\mathrm{fd}}$ is (strong) symmetric monoidal, exact, and faithful, witnessing Sat_G as a neutral Tannakian category.

Consequently, by Tannakian reconstruction (see Theorem A.0.2), there is an affine group scheme $\check{G}_\Lambda \simeq \underline{\mathrm{Aut}}^\otimes(H^*)$ and an equivalence $\mathsf{ST} : \mathrm{Sat}_G \to \mathrm{Rep}_\Lambda(\check{G}_\Lambda)$ of symmetric monoidal categories such that the diagram

$$\begin{tikzcd}
\mathrm{Sat}_G^\star \arrow[rr, "\mathsf{ST}"] \arrow[dr, "H^*"'] & & \mathrm{Rep}_\Lambda(\check{G}_\Lambda)^\otimes \arrow[dl, "\text{Forget}"] \\
& \mathrm{Mod}_\Lambda^\otimes &
\end{tikzcd}$$

commutes. Furthermore, $\check{G}_\Lambda$ is a connected, split reductive group of finite type over Λ. There is a canonical $\mathbb{X}_*(T)$-grading on H^* which exhibits a maximal torus $\check{T} \subset \check{G}_\Lambda$, with respect to which the root datum of $\check{G}_\Lambda$ is the dual root datum of G. In particular, $\check{G}_\Lambda$ is the Langlands dual group of G over Λ.

While our primary focus will be constructible Betti sheaves with coefficients in a field Λ (of characteristic 0), other sheaf-theoretic variants exist:

- Over the base field $\mathbb{C}$, the coefficients can be any commutative Noetherian ring of finite global dimension[1] (like $\mathbb{Z}$), provided we restrict the Satake category to sheaves with flat or torsion-free stalks and costalks, and target the category of finitely generated projective Λ-modules.
- If we change the sheaf theory to étale sheaves, we may even work over an algebraically closed field k of positive characteristic, with coefficient ring $\mathbb{Z}_l$ or $\mathbb{F}_l$ ($l \neq \mathrm{char}\,k$).

Because of this, I have chosen to do the exposition as algebraically as I could, even though there might be short cuts to some results/proofs using properties of the underlying analytic topology. I wont mention these two variants in the main body; I just including them here for completeness.

1.1 Quick review

This project is structured as follows. In Section 2, we introduce the primary geometric object of study: the affine Grassmannian Gr_G associated with a linear algebraic group G over k. We show that Gr_G is a strict ind-scheme of ind-finite type, and specifically ind-projective when G is reductive. We conclude the section by equipping Gr_G with a natural stratification by L^+G-orbits.

[1]So we have a reasonable six-functor formalism of constructible sheaves on the underlying analytic topology, see [Ach21].

In Section 3, we define the Satake category Sat_G as the category of L^+G-equivariant perverse sheaves on Gr_G, or equivalent, the category of perverse sheaves on the Hecke stack Hck_G. We establish that Sat_G is semisimple when $\mathrm{char}\,\Lambda = 0$, a property that ensures the reconstructed group scheme $\check{G}_\Lambda$ is reductive. Furthermore, we equip Sat_G with a monoidal structure $\star$ called *convolution* and a fiber functor $H^* : \mathrm{Sat}_G \to \mathrm{Mod}_\Lambda$, paving the way for Tannakian reconstruction.

In Section 4 we construct relative versions of the affine Grassmannian and the Hecke stack over a reduced, connected complex curve X. Here, the classical condition of universal local acyclicity (ULA) is reinterpreted categorically as *suaveness* within the constructible six-functor formalism. This relative framework allows us to identify the convolution product on Sat_G with a "fusion product" on the relative Satake category Sat_X. Using an Eckmann-Hilton style argument, we subsequently prove the symmetric monoidality of both the convolution product and the fiber functor.

Finally, in Section 5, we employ the weight functors of Mirković and Vilonen to construct a closed immersion $\check{T}_\Lambda \to \check{G}_\Lambda$, identifying the source as a split maximal torus of the target. This explicit construction verifies that the root datum of the reconstructed group matches that of the Langlands dual.

In Appendix A we state the Tannakian reconstruction theorem and related results that we will use. This is taken from [DM18].

In Appendix B we recall some standard notions from the theory of reductive groups. This is taken from [Spr98].

In Appendix C we construct the *convolution product* in a general ∞-categorical setting. The main statement is that if $\mathcal{X}$ is an ∞-topos and $A \in \mathrm{Grp}(\mathcal{X})$ is a group object, and $M \in \mathrm{Mon}(\mathcal{X})$ is a monoid object (or group object) with compatible A-actions form both left and right, then the double quotient quotient $A\backslash M/A$ becomes a monoid/group object in the ∞-category of correspondences $\mathrm{Corr}(\mathcal{X})$. Consequently, any if $D : \mathrm{Corr}(\mathcal{X})^\otimes \to \mathrm{Cat}^\times$ is a six-functor formalism, then $D(A\backslash M/A)$ is a monoidal ∞-category whose tensor product is different from the "ordinary" tensor product, and (at least) as "coherent" as the multiplication on M is.

1.2 Notation and conventions

Throughout the text, we use standard notation from the theory of reductive groups, which is briefly reviewed in Appendix B. We also use theory of G-torsors (principal G-bundles), following [NSS14]. In particular, $\mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{Aff}_k)$ will denote the ∞-topos of An-valued sheaves in fpqc-topology, however, all stacks that we consider will (at worst) 1-truncated. The sheaf theoretic aspects are sourced from [Ach21], but we have chosen to write in the modern language of stable ∞-categories opposed to triangulated categories.

2 The affine Grassmannian

The goal of this section is to introduce the affine Grassmannian Gr_G associated to a linear algebraic group G. In Section 2.2 We show that Gr_G is represented by a strict ind-scheme that is ind-of finite type, and is ind-projective when G is reductive (Theorem 2.2.3). In Section 2.3, we equip Gr_G with a stratification into L^+G-orbits under the left action of the positive loop group L^+G, indexed by the dominant cocharacters $X_*(T)^+$ (Theorem 2.3.1). However, before constructing the affine Grassmannian, we discuss properties of the loop group LG and the positive loop group L^+G in Section 2.1.

2.1 Formal geometry and loop spaces

To define the affine Grassmannian, we have to introduce the loop group and positive loop group LG and L^+G whose functor of points are $LG(R) = G(R((t)))$ and $L^+G([[t]])$.

Definition 2.1.1. Let k be a field. We define *the n^{th} infinitesimal disk*, *the disk*, and *the punctured disk* by

$$D_n \simeq \mathrm{Spec}(k[t]/t^n), \quad D \simeq \mathrm{Spec}(k[[t]]), \quad \text{and} \quad D^\circ \simeq \mathrm{Spec}(k((t)))$$

respectively. If R is a k-algebra, we define their relative versions over R as

$$D_{n,R} \simeq \mathrm{Spec}(R[t]/t^n), \quad D_R \simeq \mathrm{Spec}(R[[t]]), \quad \text{and} \quad D_R^\circ \simeq \mathrm{Spec}(R((t))).$$

Let $\pi_{D_n} : D_n \to \mathrm{Spec}(k)$ be the canonical structural morphism. The composite functor

$$\mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{Aff}_k) \xrightarrow{\pi_{D_n}^*} \mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{Aff}_k)_{/D_n} \xrightarrow{\pi_{D_n *}} \mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{Aff}_k)$$

is called the n^{th} *loop functor* and is denoted L^n. The natural quotients $k[t]/t^n \to k[t]/t^{n-1}$ induce maps $L^n \to L^{n-1}$, and we define the *positive loop functor* as the limit $L^+ \simeq \lim_{n\in\mathbb{N}^{\mathrm{op}}} L^n$.

Furthermore, we define the *loop functor* $L : \mathrm{PShv}(\mathrm{Aff}_k) \to \mathrm{PShv}(\mathrm{Aff}_k)$ globally by precomposition with the Laurent series functor $R \mapsto R((t))$.

Remark 2.1.2.
1. If X is an affine scheme of finite type, we will see in Proposition 2.1.4 that LX is representable by a strict ind-scheme, which ensures it is an fpqc sheaf. Consequently, L preserves finite limits of such X, so if G is a linear algebraic group, then LG is a group object.

2. By definition, $L^nX(R) = X(R[t]/t^n)$ and $L^+X(R) = X(\lim_{n\in\mathbb{N}^{\mathrm{op}}} R[t]/t^n) = X(R[[t]])$. Also, L^n preserves limits by construction, and so does L^+.

3. Note that $D \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R) \neq D_R$ nor is $D^\circ \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R) \neq D_R^\circ$.

Example 2.1.3. If $X = \mathbb{A}^1$, then $L^+X(R) = R[[t]]$ and $LX(R) = R((t))$. Therefore $L^+X \simeq \mathrm{Spec}\, k[r_0, r_1, \dots,]$ and $LX = \mathrm{colim}_{i\in\mathbb{N}} \mathrm{Spec}\, k[r_{-i}, r_{-i+1}, \dots,]$.

The next proposition concerns the geometry of (positive) loop spaces.

Proposition 2.1.4. Let k be a field.

(1) If X is a scheme of finite type over k, then L^nX is represented by a scheme of finite type over k, and $L^+X \simeq \lim_{n\in\mathbb{N}^{\mathrm{op}}} L^nX$ is represented by a scheme. If X is affine, then so is L^+X. If $Z \to X$ is an open (resp. closed) immersion, then so is $L^+Z \to L^+X$.

(2) If X is smooth over k, then both L^+X and LX are formally smooth. Also, L^+X is reduced.

(3) If X is an affine scheme of finite type over k, then LX is represented by a strict ind-scheme that is ind-of finite type. If $Z \to X$ is a closed immersion, then so is $LZ \to LX$.

(4) If X is an affine scheme of finite type over k, then $L^+X \to LX$ is a closed immersion.

Proof. In our applications, X will always be affine, so suppose $X \simeq \mathrm{Spec}(k[x_1, \dots, x_N]/(f_1, \dots, f_r))$ is affine. For any k-algebra R, an R-point of L^nX is a k-algebra homomorphism from $k[x_1, \dots, x_N]/(f_1, \dots, f_r)$ to $R[t]/(t^n)$. This is determined by assigning $x_i \mapsto \sum_{l=0}^{n-1} a_{i,l}t^l$ such that each f_j vanishes in $R[t]/(t^n)$.

Expanding f_j and setting the coefficient of each t^k to zero gives a finite system of polynomial equations in the variables $a_{i,l}$. Thus, $L^n X$ is represented by

$$L^n X = \operatorname{Spec}\left(\frac{k[\{a_{i,l}\}_{i,l}]}{(\text{coefficients of } t^k \text{ in } f_j)}\right)$$

which is clearly an affine scheme of finite type over k. Since $L^+X \simeq \lim_{n\in\mathbb{N}^{\mathrm{op}}} L^n X$ and an inverse limit of affine schemes is an affine scheme, L^+X is affine.

If $Z \to X$ is a closed immersion, Z is defined by additional equations. The functor $L^n Z$ is cut out from $L^n X$ by setting the coefficients of these new equations to zero, meaning $L^n Z \to L^n X$ is a closed immersion. The claim for L^+ follows by taking limits.

If $Z \simeq U \to X$ is an open immersion, say a principal open $U = D(f)$, then a map $\operatorname{Spec}(R[t]/(t^n)) \to X$ factors through U if and only if the pullback of f is invertible in $R[t]/(t^n)$. I.e. if and only if its constant term is invertible in R. Thus, $L^n U$ is the open subscheme of $L^n X$ defined by the non-vanishing of the constant term of f, making $L^n U \to L^n X$ an open immersion. Taking the limit, this preserves the open immersion for L^+.

For (2), we first show formal smoothness. We must show that for any k-algebra B and any nilpotent ideal $I \subset B$, the maps $X(B[[t]]) \to X((B/I)[[t]])$ and $X(B((t))) \to X((B/I)((t)))$ are surjective. Note that the kernel of the projection $B[[t]] \to (B/I)[[t]]$ is the ideal $I[[t]]$. If $I^N = 0$ in B, then $(I[[t]])^N = I^N[[t]] = 0$, meaning $I[[t]]$ is a nilpotent ideal in $B[[t]]$. Similarly, the kernel $I((t))$ of $B((t)) \to (B/I)((t))$ is a nilpotent ideal in $B((t))$. Now the lifts exist by smoothness of X. This proves both L^+X and LX are formally smooth.

To see that L^+X is reduced, note that the exact same formal smoothness argument applies to $L^n X$ (where the nilpotent kernel is $I[t]/(t^n)$). Since $L^n X$ is formally smooth and of finite type over k, it is smooth. Any smooth scheme over a field is regular, and therefore reduced. Because X is smooth, the transition maps $L^{n+1}X \to L^n X$ are smooth, which makes the corresponding maps of coordinate rings $\mathcal{O}(L^n X) \to \mathcal{O}(L^{n+1}X)$ injective. Since $\mathcal{O}(L^+X) = \operatorname{colim}_{n\in\mathbb{N}} \mathcal{O}(L^n X)$ is a direct limit of reduced rings with injective transition maps, it contains no nilpotents. Thus, L^+X is reduced.

For (3), let $X = \operatorname{Spec}(k[x_1, \dots, x_N]/(f_j))$. Define the subfunctor $L^{(m)}X \subset LX$ of those maps $\operatorname{Spec}(R((t))) \to X$ that take $x_i \mapsto \sum_{l=-m}^{\infty} a_{i,l} t^l$ (i.e. pole of order at most m). Substituting these into the relations $f_j = 0$ yields a system of equations on the coefficients $a_{i,l}$, so $L^{(m)}X$ is represented by the affine scheme $\operatorname{Spec}(k[\{a_{i,l}\}]/(\text{relations}))$. There are canonical closed immersions $L^{(m)}X \to L^{(m+1)}X$ taking $a_{i,-m-1}$ to 0 (but allowing a pole of order $m+1$). It follows that $LX = \operatorname{colim}_{n\in\mathbb{N}} L^{(m)}X$, making LX a strict ind-scheme, ind-of finite type.

If $Z \to X$ is a closed immersion, then $L^{(m)}Z$ is a closed subscheme of $L^{(m)}X$ by the exact same logic as in part (1). Since a colimit of closed immersions of ind-schemes is a closed immersion, $LZ \to LX$ is a closed immersion.

Finally, (4) follows from (3) by noting that $L^+X \simeq L^{(0)}X$. □

Proposition 2.1.5. Let k be a field, and let G be an affine group scheme over k.

(1) The stack $B(L^+G)$ of L^+G-torsors represents the functor

$$\operatorname{Tors}_G(D_-) : \operatorname{Alg}_k \to \operatorname{Grpd} \subset \operatorname{An},$$

which assigns to a k-algebra R the category of G-torsors on the disk D_R, $\operatorname{Tors}_G(D_R)$.

(2) If $G \to H$ is a closed immersion of affine group schemes over k, then the canonical map $L^+H/L^+G \to L^+(H/G)$ is an isomorphism. Furthermore, the canonical map $LH/LG \to L(H/G)$ is a monomorphism.

(3) Let $K_n = \ker(L^+G \to L^n G)$. The *congruence* subgroup $L^{>0}G = K_1$ is a pro-algebraic unipotent group. Specifically, $L^{>0}G \simeq \varprojlim_{n\in\mathbb{N}^{\mathrm{op}}} K_1/K_n$, where each quotient K_n/K_{n+1} is represented by an affine space.

Proof. For (1), recall that an L^+G-torsor P over $\operatorname{Spec}(R)$ is an fpqc sheaf on Aff_R for which there exists a faithfully flat cover $R \to R'$ on which P trivializes. The descent datum for this torsor is classified by an element

$$g \in L^+G(R' \otimes_R R') \simeq G((R' \otimes_R R')[[t]]) \simeq G(R'[[t]] \otimes_{R[[t]]} R'[[t]])$$

Because $R \to R'$ is an fpqc cover, the induced map $R[[t]] \to R'[[t]]$ is also faithfully flat, hence an fpqc cover. Therefore, this exact same 1-cocycle g gives the descent datum for gluing the trivial G-torsor over $D_{R'}$ into a G-torsor over D_R. Since G is an affine group scheme, G-torsors on affine schemes are themselves affine. Fpqc descent for affine schemes is effective, meaning this descent datum uniquely determines a globally defined G-torsor on D_R.

For (2), Let $X = H/G$. Since G is a closed subgroup of an affine group scheme, the quotient X exists as a scheme, and the projection $H \to X$ is an fpqc G-torsor. Let $D_R \to X$ be an R-point of L^+X, and let P be the G-torsor defined by $D_R \times_X H$. By part (1), P corresponds to an L^+G-torsor on $\mathrm{Spec}(R)$. Let $f : R \to R'$ be an fpqc-cover on which this torsor becomes trivial. A trivialization of $f^*(\mathcal{P})$ on $D_{R'}$ is the same as a lift of $D_{R'} \to X$ to a map $D_{R'} \to H$. Since every point of L^+X lifts locally to L^+H, the map $L^+H/L^+G \to L^+X$ is an epimorphism. It is canonically a monomorphism because the fibers are exactly the L^+G-orbits. Thus, it is an isomorphism.

For the final assertion, we show that $LH(R)/LG(R) \to L(H/G)(R)$ is injective. Suppose $a, b \in LH(R)$ map to the same element in $LX(R)$. By definition, a and b are maps $D_R^\circ \to H$ whose compositions with $H \to X$ coincide. This induces a map $D_R^\circ \to H \times_X H$. Because $H \to X$ is a principal G-bundle, we have an isomorphism $H \times_X H \xrightarrow{\sim} H \times G$ given by $(h, h\cdot g) \leftrightarrow (h,g)$. Therefore, our map to the fiber product corresponds to the map $D_R^\circ \to H$ (which is a) and a uniquely determined map $D_R^\circ \to G$. This second map gives an element $g \in G(D_R^\circ) = LG(R)$ such that $b = a\cdot g$. This means a and b belong to the same $LG(R)$-orbit, proving injectivity.

(3) will be omitted. □

2.2 The affine Grassmannian

From now on, let G be a smooth affine group scheme of finite type over k. Note that L^+G is a subgroup of LG and thus acts by left and right multiplication, denoted $\alpha_L : L^+G \times LG \to LG$ and $\alpha_R : LG \times L^+G \to LG$. The natural inclusion $L^+G \to LG$ induces a push-forward map of stacks $B(L^+G) \to B(LG)$ (extension of the structure group). Furthermore, by Proposition 2.1.4.(4), $L^+G \to LG$ is a closed immersion.

Definition 2.2.1. The *Hecke stack* and the *affine Grassmannian* are the fpqc-sheaves Hck_G and Gr_G defined by Cartesian squares in $\mathrm{Shv}(\mathrm{Aff}_k^{\mathrm{fpqc}})$

$$\begin{tikzcd}
\mathrm{Gr}_G \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & \mathrm{Hck}_G \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & B(L^+G) \arrow[d] \\
* \arrow[r] & B(L^+G) \arrow[r] & B(LG)
\end{tikzcd}$$

where $* \simeq \mathrm{Spec}(k)$ picks out the trivial L^+G-bundle. Equivalently, $\mathrm{Gr}_G \simeq LG/L^+G$ and $\mathrm{Hck}_G \simeq L^+G\backslash LG/L^+G$.

Remark 2.2.2. 1. As $B(L^+G)$ and $B(LG)$ are 1-truncated (values in Grpd), so are Gr_G and Hck_G, but it turns out that Gr_G is 0-truncated. Indeed, according to Proposition 2.1.5.(1) the R-points of Gr_G are apriori the groupoid with objects

$$\mathrm{Gr}_G(R) = \{(\mathcal{E}, \beta) \mid \mathcal{E} \in \mathrm{Tors}_G(D_R), \beta : \mathcal{E}_{|D_R^\circ} \simeq \mathcal{E}^0\},$$

where a map between pairs $(\mathcal{E}, \beta)$ and $(\mathcal{E}', \beta')$ is a map $\phi : \mathcal{E} \to \mathcal{E}'$ such that $\beta'\phi_{|D_R^\circ} = \beta$; the latter uniquely determines ϕ, as $D_R^\circ \subset D_R$ is dense.

The Hecke stack is however genuinely 1-truncated. Its R-points are the groupoid with objects

$$\mathrm{Hck}_G(R) = \{(\mathcal{E}, \mathcal{E}', \beta \mid \mathcal{E}, \mathcal{E}' \in \mathrm{Tors}_G(D_R), \beta : \mathcal{E}_{|D_R^\circ} \simeq \mathcal{E}'_{|D_R^\circ}\}$$

2. By definition, Gr_G is the fpqc sheafification of the presheaf $R \mapsto LG(R)/L^+G(R)$. If R is a ring for which every principal G-bundle on D_R is trivial, then the sheafification evaluated at R coincides exactly with the "naive" quotient:

$$\mathrm{Gr}_G(R) = LG(R)/L^+G(R) = G(R((t)))/G(R[[t]]).$$

This is the case if R is an algebraically closed field (e.g. $\mathbb{C}$ or $\bar{\mathbb{F}}_p$). Over such rings, points of the Affine Grassmannian are literally just the cosets.

3. Since G is a smooth affine group scheme, the loop group LG and positive loop group L^+G are formally smooth. Consequently, the quotient Gr_G inherits this property and is a formally smooth (apriori) ind-scheme, see Proposition 2.2.5.

4. If $G = \mathrm{GL}_n$, then $B(L^+G)$ classifies rank n vector bundles on the disk D_R, which are *lattices* in $R((t))^n$. Explicitly, a lattice is a pair (V, β) where:

   (a) V is a locally free $R[[t]]$-module of rank n (the vector bundle on D_R).

   (b) $\beta : V \otimes_{R[[t]]} R((t)) \xrightarrow{\sim} R((t))^n$ is an $R((t))$-linear isomorphism (the trivialization over D_R°).

   Via the isomorphism β, we can identify V with an $R[[t]]$-submodule of the standard loop vector space $R((t))^n$. The *standard lattice* is defined as the canonical submodule $\Lambda_R = R[[t]]^n \subset R((t))^n$.

Theorem 2.2.3. The sheaf Gr_G is represented by a strict ind-scheme, ind-of finite type over k. If G is reductive, then Gr_G is ind-projective.

If $X \subset \mathrm{Gr}_G$ is quasi-compact subscheme (e.g. any finite-dimensional closed subvariety), then the L^+G-action on C factors through a finite-dimensional algebraic group quotient $L^N G \cong L^+G/K_N$ for some integer $N > 0$, where $K_N = \ker(L^+G \to L^N G)$.

Remark 2.2.4. The first assertion in Theorem 2.2.3 asserts that Gr_G is a "nice" geometric object (it is formally smooth), but one must be careful with its scheme-theoretic structure: it is not necessarily reduced. In fact, by a theorem of Beilinson and Drinfeld, LG (or equivalently Gr_G) is reduced if and only if $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m) = 0$ if and only if G is semi-simple, see [BD94, Theorem 4.5.1.]. However, the sheaf-theory that we will consider only depends on the underlying (analytic) topology, so we may as well work with $(\mathrm{Gr}_G)_{\mathrm{red}}$.

The second assertion will be relevant when we discuss L^+G-equivariant perverse sheaves on Gr_G.

The rest of this subsection is dedicated to the proof of Theorem 2.2.3. The general case will be reduced to $G = \mathrm{GL}_n$, and in that case, it follows from explicit computations with lattices.

Proposition 2.2.5. The stack $\mathrm{Gr}_{\mathrm{GL}_n}$ is represented by a strict ind-scheme that is ind-of finite type over k and ind-projective.

First, we need some lemmas on lattices.

Lemma 2.2.6. If V is a finitely generated $R[[t]]$-submodule of $R((t))^n$, then the canonical map $V \otimes_{R[[t]]} R((t)) \to R((t))^n$ is an equivalence if and only if there exists an integer $N > 0$ such that $t^N \Lambda_R \subset V \subset t^{-N}\Lambda_R$.

Proof. Suppose the map is an equivalence. Because V is finitely generated over $R[[t]]$, let $v_1, \dots, v_k \in R((t))^n$ be its generators. Since each v_i is a vector of Laurent series, there exists an integer $N_1 > 0$ such that $t^{N_1} v_i \in R[[t]]^n = \Lambda_R$ for all i. Thus $t^{N_1} V \subset \Lambda_R$, meaning $V \subset t^{-N_1}\Lambda_R$. Since V spans $R((t))^n$ after inverting t, the standard basis vectors $e_1, \dots, e_n$ of Λ_R can be written as finite sums $e_j = \sum c_{ji} v_i$ with $c_{ji} \in R((t))$. Pick N_2 large enough such that $t^{N_2} c_{ji} \in R[[t]]$. Thus $t^{N_2} e_j \in V$ for all j, implying $t^{N_2}\Lambda_R \subset V$. Taking $N = \max(N_1, N_2)$ yields the desired bound.

Conversely, applying $- \otimes_{R[[t]]} R((t))$ to $t^N \Lambda_R \subset V \subset t^{-N}\Lambda_R$ turns both bounding modules into $R((t))^n$, forcing the middle term to be an equivalence as well. □

Before the next lemma, we introduce some notation over k. Let $\Lambda_k = k[[t]]^n$ be the standard lattice over the base field, and define the finite-dimensional k-vector space $V_N = t^{-N}\Lambda_k / t^N \Lambda_k$. We consider the standard Grassmannian $\mathrm{Gr}(V_N)$ which is projective and of finite type over k.

For any k-algebra R, we have $V_N \otimes_k R \simeq t^{-N}\Lambda_R / t^N \Lambda_R =: M_N$. The R-points of the Grassmannian are exactly given by:

$$\mathrm{Gr}(V_N)(R) = \{U \subset M_N \mid \mathrm{coker}(U \subset M_N) \in \mathrm{Vect}(R)\}.$$

Let $0 \to \mathcal{U} \to \mathcal{O}_{\mathrm{Gr}} \otimes_k V_N \to \mathcal{Q} \to 0$ be the universal exact sequence of vector bundles on the scheme $\mathrm{Gr}(V_N)$. The k-linear map $t : V_N \to V_N$ induces a global vector bundle endomorphism $\mathcal{O}_{\mathrm{Gr}} \otimes_k V_N \xrightarrow{t} \mathcal{O}_{\mathrm{Gr}} \otimes_k V_N$, and we define $\mathrm{Gr}^{(N)}$ as the closed subscheme of $\mathrm{Gr}(V_N)$ given by the vanishing locus of the composite section

$$s : \mathcal{U} \to \mathcal{O}_{\mathrm{Gr}} \otimes_k V_N \xrightarrow{t} \mathcal{O}_{\mathrm{Gr}} \otimes_k V_N \twoheadrightarrow \mathcal{Q},$$

viewed as a global section of $\underline{\mathrm{Hom}}(\mathcal{U}, \mathcal{Q})$.

Lemma 2.2.7. $\mathrm{Gr}^{(N)}$ is a projective scheme of finite type over k, and its R-points are canonically given by

$$\mathrm{Gr}^{(N)}(R) = \{U \in \mathrm{Gr}(V_N)(R) \mid tU \subset U\}.$$

Proof. $\mathrm{Gr}^{(N)} \subset \mathrm{Gr}(V_N)$ is closed, so it inherits projectiveness and is of finite type over k. Recall that an R-point of $\mathrm{Gr}(V_N)$ corresponds to a quotient module $(V_N \otimes_k R)/U \in \mathrm{Vect}(R)$, which is equivalent to pulling back the universal exact sequence to $\mathrm{Spec}(R)$. R-point factors through $\mathrm{Gr}^{(N)}$ if and only if the pullback of the section s to $\mathrm{Spec}(R)$ is zero. That is, the R-points of $\mathrm{Gr}^{(N)}$ are the $U \in \mathrm{Gr}(V_N)(R)$ for which the composite

$$U \to V_N \otimes_k R \xrightarrow{t} V_N \otimes_k R \twoheadrightarrow (V_N \otimes_k R)/U$$

is the zero map. □

Proof of Proposition 2.2.5. We will show that the geometric scheme $\mathrm{Gr}^{(N)}$ represents the functor $F^{(N)}$: $\mathrm{Aff}_k^{\mathrm{op}} \simeq \mathrm{Alg}_k \to \mathrm{Set}$ that assigns to R the set of bounded lattices:

$$F^{(N)}(R) = \{V \subset R((t))^n \mid V \text{ is a lattice and } t^N\Lambda_R \subset V \subset t^{-N}\Lambda_R\}.$$

If V is such a lattice, then it is an $R[[t]]$-module, so stable under t ($tV \subset V$). Because $t^N\Lambda_R \subset V$, the quotient $U = V/t^N\Lambda_R$ is a well-defined R-submodule of $M_N = t^{-N}\Lambda_R/t^N\Lambda_R \simeq V_N \otimes_k R$.

By Lemma 2.2.8, V being a projective $R[[t]]$-module bounded inside $t^{-N}\Lambda_R$ is equivalent to the quotient $t^{-N}\Lambda_R/V$ being a finitely generated projective R-module. Since $M_N/U \simeq t^{-N}\Lambda_R/V$, we deduce that U defines a valid R-point $U \in \mathrm{Gr}(V_N)(R)$. Furthermore, the $R[[t]]$-module structure $tV \subset V$ pushes forward to the quotient, implying $tU \subset U$. By the preceding Lemma, U is an R-point of the closed subscheme $\mathrm{Gr}^{(N)}$.

Conversely, pulling back an R-point $U \in \mathrm{Gr}^{(N)}(R)$ uniquely reconstructs an $R[[t]]$-submodule $V \subset R((t))^n$ satisfying the required bounds, establishing a canonical isomorphism of functors $\mathrm{Gr}^{(N)} \simeq \mathcal{F}^{(N)}$.

Finally, there are canonical transition maps $\mathrm{Gr}^{(N)} \to \mathrm{Gr}^{(N+1)}$. Because projective schemes are proper over k, and any proper monomorphism of schemes is a closed immersion, these transition maps are closed immersions. By Lemma 2.2.6 and Item 4 we a presentation $\mathrm{Gr}_{\mathrm{GL}_n} = \mathrm{colim}_N\, \mathrm{Gr}^{(N)}$. □

Lemma 2.2.8. Let R be a k-algebra, and let E be a finitely generated free $R[[t]]$-module (e.g., $E = t^{-N}R[[t]]^n$). Let $V \subset E$ be an $R[[t]]$-submodule containing $t^K E$ for some integer $K > 0$.

Then V is a finitely generated projective $R[[t]]$-module if and only if the quotient $Q = E/V$ is a finitely generated projective R-module.

Proof. First, assume $Q = E/V$ is a finitely generated projective R-module. $R = R[[t]]/(t)$ admits the length-one free resolution $0 \to R[[t]] \xrightarrow{t} R[[t]] \to R \to 0$ as $R[[t]]$-module, which implies that R has projective dimension 1 over $R[[t]]$. Because Q is a projective R-module, it inherits the bound: $\mathrm{proj.\,dim}_{R[[t]]}(Q) \leq 1$. The middle term in the exact sequence

$$0 \to V \to E \to Q \to 0$$

is free, and thus the kernel V must be projective over $R[[t]]$.

Conversely, assume V is a finitely generated projective $R[[t]]$-module. Because projective modules are flat, and $R[[t]]$ is flat over R, both E and V are flat R-modules. The quotient $Q = E/V$ is finitely presented over R, and for finitely presented modules, flatness is equivalent to projectivity. By the Local Criterion of Flatness, Q is flat over R if and only if $\mathrm{Tor}_1^R(Q, \kappa) = 0$ for every residue field $\kappa = R/\mathfrak{m}$. Tensoring the exact sequence above with κ yields

$$0 \to \mathrm{Tor}_1^R(Q, \kappa) \to V \otimes_R \kappa \xrightarrow{\phi} E \otimes_R \kappa.$$

The modules $V \otimes_R \kappa$ and $E \otimes_R \kappa$ are free modules over the principal ideal domain $\kappa[[t]]$. Because V bounds a power of t in E, the cokernel of ϕ is t-torsion, which over a PID forces the map ϕ to be injective. Thus $\mathrm{Tor}_1^R(Q, \kappa) = 0$, proving Q is flat and therefore projective over R. □

Theorem 2.2.3 will follow from the following lemmas:

Lemma 2.2.9. There is a closed immersion $G \to \mathrm{GL}_n$ of group schemes over k such that the quotient GL_n / G is quasi-affine. Furthermore, if G is reductive, then the closed immersion $G \to \mathrm{GL}_n$ can be chosen so GL_n / G is affine.

Lemma 2.2.10. If $\rho : G \to \mathrm{GL}_n$ is a closed immersion of $k[[t]]$-group schemes such that GL_n / G is quasi-affine, then the induced map $\mathrm{Gr}_G \to \mathrm{Gr}_{\mathrm{GL}_n}$ is a locally closed embedding. Furthermore, if GL_n / G is affine, then $\mathrm{Gr}_G \to \mathrm{Gr}_{\mathrm{GL}_n}$ is a closed embedding.

References for Lemma 2.2.9. see [PR08, Proposition 1.3] and [Alp18, Corollary 9.7.7]. □

Proof of Lemma 2.2.10. Write H for GL_n. It suffices to show that the left vertical map in diagram below to the left is (locally) closed:

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & \mathrm{Gr}_G \arrow[d] \\ \operatorname{Spec} R \arrow[r] & \mathrm{Gr}_H \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & \operatorname{Spec} R \times_{B(L^+H)} B(L^+G) \arrow[d, "g"] \\ \operatorname{Spec} R \arrow[r] & \operatorname{Spec} R \times_{B(LH)} B(LG) \end{tikzcd}$$

By definition of Gr, the diagram to the right above is a pullback square as well. It suffices to verify that g is (locally) closed. Pick an fpqc-cover $\operatorname{Spec} R' \to \operatorname{Spec} R$ such that $\operatorname{Spec} R' \to \operatorname{Spec} R \to B(L^+H)$ factors through $*$ (see [NSS14, Proposition 3.12]). It follows that the diagram below is cartesian, and the horizontal maps are fpqc-covers.

$$\begin{tikzcd} \operatorname{Spec} R' \times L^+H/L^+G \arrow[r] \arrow[d, "g'"'] \arrow[dr, phantom, "\text{cart}"] & \operatorname{Spec} R \times_{B(L^+H)} B(L^+G) \arrow[d, "g"] \\ \operatorname{Spec} R' \times LH/LG \arrow[r] & \operatorname{Spec} R \times_{B(LH)} B(LG) \end{tikzcd}$$

Since the property of being (locally) closed is local on target in the fpqc-topology ([Sta26, Tag 02YJ]), we may equivalently verify that g' is (locally) closed. g' is induced by the map $L^+H/L^+G \to LH/LG$, and according Proposition 2.1.4.(4), Proposition 2.1.5.(2), and the assumption, there is a diagram like this:

$$\begin{tikzcd} L^+H/L^+G \arrow[r] \arrow[d, "\simeq"'] & LH/LG \arrow[d, hook] \\ L^+(H/G) \arrow[r, hook] & L(H/G) \end{tikzcd}$$

Consequently, the top horizontal map is a closed immersion. □

Proof of Theorem 2.2.3. Take a closed immersion $G \to \mathrm{GL}_n$ such that $\mathrm{Gr}_G \to \mathrm{Gr}_{\mathrm{GL}_n}$ is a closed immersion (Lemma 2.2.9 and Lemma 2.2.10). The first assertion follows from Proposition 2.2.5.

For the second assertion, let $X \subset \mathrm{Gr}_G$ be a quasi-compact subscheme, and let X_{GL_n} denote its image in $\mathrm{Gr}_{\mathrm{GL}_n}$ under the closed embedding, which is also quasi-compact $\mathrm{Gr}_{\mathrm{GL}_n}$, and thus factors through a finite stage $X_{\mathrm{GL}_n} \subset \mathrm{Gr}^{(M)}$ (here we use the presentation from the proof of Proposition 2.2.5, $\mathrm{Gr}_{\mathrm{GL}_n} = \operatorname{colim}_M \mathrm{Gr}^{(M)}$, where the M-th stage $\mathrm{Gr}^{(M)}$ parameterizes lattices V bounded by $t^M \Lambda \subset V \subset t^{-M}\Lambda$).

This implies that for every point $x \in X$, its corresponding lattice $V_x \subset k((t))^n$ satisfies $t^M\Lambda \subset V_x \subset t^{-M}\Lambda$. We claim that the subgroup

$$K_{2M}^{\mathrm{GL}_n} = \{g(t) \in L^+ \mathrm{GL}_n \mid g(t) \equiv I \pmod{t^{2M}}\} = \ker(L^+ \mathrm{GL}_n \to L^{2M} \mathrm{GL}_n)$$

stabilizes every such lattice V_x. If $f(t) = I + t^{2M} A(t) \in K_{2M}^{\mathrm{GL}_n}$ (where $A(t)$ is a matrix with entries in $k[[t]]$) and $v \in V_x$, then

$$f(t) \cdot v = (I + t^{2M} A(t))v = v + t^{2M} A(t)v.$$

Since $v \in V_x \subset t^{-M}\Lambda$, we have $A(t)v \in t^{-M}\Lambda$, so $t^{2M}A(t)v \in t^M\Lambda \subset V_x$. Thus $f(t) \cdot v \in V_x$, which means $f(t) \cdot V_x \subseteq V_x$. Since the inverse of $f(t)$ also lies in $K_{2M}^{\mathrm{GL}_n}$, we have $f(t)^{-1} \cdot V_x \subseteq V_x$, yielding $f(t) \cdot V_x = V_x$.

Since this holds for every $x \in X$, the subgroup $K_{2M}^{\mathrm{GL}_n}$ acts trivially on the entirety of X_{GL_n}. The action of L^+G on X restricts from the action of $L^+\mathrm{GL}_n$ on X_{GL_n}. Therefore, every element of $K_{2M}^G = L^+G \cap K_{2M}^{\mathrm{GL}_n}$ must fix every point of X, showing that the global L^+G-action on X factors uniformly through the finite-dimensional algebraic group quotient $L^+G/K_{2M}^G \cong L^{2M}G$. □

2.3 The Schubert cells Gr_λ

Next, we aim to construct a good stratification of Gr_G indexed by the partial odering of the dominant cocharacters, $(X_*(T)^+, \leq)$. If $\lambda \in X_*(T)$ is a cocharacter, let t^λ denote the point $* \xrightarrow{t} L\mathbb{G}_m \xrightarrow{\lambda} LG$ and, abusively, also the point $* \xrightarrow{t^\lambda} LG \to \mathrm{Gr}_G$. Note that L^+G acts on Gr_G by left multiplication. The orbit of L_λ under this action is denoted Gr_G^λ and is called a *Schubert cell.*

We write $\mathcal{O}$ and $\mathcal{K}$ for $k[[t]]$ and $k((t))$. Throughout, we fix a split maximal torus $T \subset G$ and a Borel subgroup $B \supset T$, which determines a system of positive roots and identifies the quotient $X_*(T)/W$ with the set of dominant cocharacters $X_*(T)^+$ (see Appendix B).

Theorem 2.3.1. (1) There is a set-theoretic decomposition called the *Cartan decomposition*

$$\mathrm{Gr}_G = \bigsqcup_{\lambda \in X_*(T)^+} G(\mathcal{O}) t^\lambda G(\mathcal{O}) = \bigsqcup_{\lambda \in X_*(T)^+} \mathrm{Gr}_\lambda .$$

(2) Gr_λ is a smooth connected quasi-projective variety of dimension $\langle 2\rho, \lambda \rangle$, and $\overline{\mathrm{Gr}_\lambda}$ is an irreducible projective variety (which is not neccesarily smooth). Furthermore,

$$\overline{\mathrm{Gr}_\lambda} = \bigcup_{\eta \in X_*(T)^+, \eta \leq \lambda} \mathrm{Gr}_\eta = \mathrm{Gr}_{\leq \lambda} \quad \text{and} \quad \mathrm{Gr}_G \simeq \operatorname*{colim}_{\lambda \in X_*(T)^+} \overline{\mathrm{Gr}_\lambda}.$$

(3) The orbit Gr_λ is an affine bundle over the partial flag variety G/P_λ^-, where P_λ^- is the parabolic[2] subgroup generated by the opposite Borel B^- and the root subgroups U_α for α such that $\langle \alpha, \lambda \rangle = 0$. In particular, Gr_λ is simply connected.

(4) The connected components of Gr_G are canonically indexed by $X_*(T)/W \simeq X_*(T)^+$, and Gr^λ and Gr^η lie in the same connected component if and only if $\lambda - \eta \in Q^\vee$ if and only if $\dim \mathrm{Gr}_\lambda \equiv \dim \mathrm{Gr}_\mu \pmod 2$

Proof. For the full proof of (1), see [Tit79, §3.3.3]. However, in the case of $G = \mathrm{GL}_n$, this is fundamentally a reformulation of the Smith Normal Form theorem for the PID $\mathcal{O} = k[[t]]$.

Let $g \in GL_n(\mathcal{K})$. By the Smith Normal Form theorem, there exist $A, B \in \mathrm{GL}_n(\mathcal{O})$ and a diagonal matrix $D \in GL_n(\mathcal{K})$ such that $g = ADB$, with $D = \mathrm{diag}(d_1, \dots, d_n)$ satisfying $d_1 \mid d_2 \mid \dots \mid d_n$ in $\mathcal{O}$. The only elements of $\mathcal{K}$ (up to multiplication by units in $\mathcal{O}^\times$) are powers of the uniformizer t, so we can write $D = \mathrm{diag}(t^{m_1}, t^{m_2}, \dots, t^{m_n})$ for some integers $m_i \in \mathbb{Z}$. The divisibility condition implies $m_1 \leq m_2 \leq \dots \leq m_n$. Because permutation matrices belong to $GL_n(\mathcal{O})$, we can absorb left and right multiplications by permutation matrices into A and B to reorder the diagonal entries in decreasing order, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$.

Consequently, $g \in GL_n(\mathcal{O}) t^\lambda GL_n(\mathcal{O})$ for $\lambda = (\lambda_1, \dots, \lambda_n) \in X_*(T)^+$. The uniqueness of the invariant factors in the Smith Normal Form ensures that these double cosets are disjoint, completing the proof.

For (2), the assertions about Gr_λ and $\overline{\mathrm{Gr}_\lambda}$ is a consequence of the L^+G-action on the point t^μ factoring through a finite quotient (Theorem 2.2.3). Note that the stabilizer of t^λ is $L^+G \cap t^\lambda L^+G t^{-\lambda}$, so $\mathrm{Gr}_\lambda \simeq L^+G/L^+G \cap t^\lambda L^+G t^{-\lambda}$. Therefore, $\dim(\mathrm{Gr}_\lambda)$ is given by the dimension of the quotient of their Lie algebras. $\mathrm{Lie}(L^+G) \simeq \mathfrak{g}[[t]]$ decomposes into its Cartan root spaces $\mathfrak{g}[[t]] = \mathfrak{t}[[t]] \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha[[t]]$. The adjoint action of t^λ on the Cartan subalgebra is trivial, contributing 0 to the codimension. On a root space $\mathfrak{g}_\alpha$, the adjoint action scales by $t^{\langle \alpha, \lambda \rangle}$. Thus, the contribution to the codimension from α is the dimension of

$$\frac{\mathfrak{g}_\alpha[[t]]}{\mathfrak{g}_\alpha[[t]] \cap t^{\langle \alpha, \lambda \rangle} \mathfrak{g}_\alpha[[t]]}.$$

[2] Quick definition: A closed subgroup $P \subset G$ is *parabolic* if it contains a Borel

If $\langle \alpha, \lambda \rangle \le 0$, this intersection is $\mathfrak{g}_\alpha[[t]]$, yielding a codimension of 0. If $\langle \alpha, \lambda \rangle > 0$, the quotient has dimension $\langle \alpha, \lambda \rangle$ with basis $\{x_\alpha, tx_\alpha, \dots, t^{\langle \alpha,\lambda\rangle -1}x_\alpha\}$. Since $\lambda \in X_*(T)^+$ is dominant, $\langle \alpha, \lambda \rangle > 0$ precisely for the positive roots Φ^+, so

$$\dim(\mathrm{Gr}_\lambda) = \sum_{\alpha\in\Phi^+} \langle \alpha, \lambda \rangle = \langle 2\rho, \lambda \rangle.$$

To show that $\bigcup_{\mu\le\lambda} \mathrm{Gr}_\mu \subset \overline{\mathrm{Gr}_\lambda}$, it suffices to show that $t^\mu \in \overline{\mathrm{Gr}_\lambda}$ for every $\mu \le \lambda$. By induction on the dominant order, we may assume $\lambda = \mu + \alpha^\vee$ for a positive root α. Let $x_\alpha : \mathbb{G}_a \to U_\alpha \subset G$ be the positive root map. We claim that the curve

$$\mathbb{A}^1 \simeq \mathbb{G}_a \xrightarrow{c\mapsto ct^{-1}} L\mathbb{G}_a \xrightarrow{x_\alpha} LG \xrightarrow{\cdot t^\mu} LG \longrightarrow \mathrm{Gr}_G$$

is contained in Gr_λ but at $c = 0$, where it hits t^μ. The cocharacter μ corresponds to $\mathrm{diag}(t^k, t^{-k})$ in the associated SL_2, where $k = \langle \alpha, \mu \rangle \ge 0$, so the matrix representation of $C(c)$ is

$$\begin{pmatrix} 1 & ct^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^k & 0 \\ 0 & t^{-k} \end{pmatrix} = \begin{pmatrix} t^k & ct^{-k-1} \\ 0 & t^{-k} \end{pmatrix}.$$

Since c is a unit, the ideal generated by the entries is strictly (t^{-k-1}). Since the determinant is 1, the invariant factors are exactly t^{-k-1} and t^{k+1}, which corresponds to the cocharacter $\mu + \alpha^\vee = \lambda$. Thus, $C(c) \in Gr_\lambda$ for all $c \ne 0$. At $c = 0$, $C(0) = t^\mu$. Since $\mathbb{A}^1$ is irreducible, the limit point $C(0)$ must lie in the topological closure of the generic image, proving $t^\mu \in \overline{Gr_\lambda}$. See [Zhu16, Proposition 2.1.5(1)] for the other inclusion.

For (3), first note that there is a short exact sequence in which $\mathrm{ev} : L^+G \to G = L^1G, t \mapsto 0$ admits a section,

$$1 \longrightarrow K_1 \longrightarrow L^+G \underset{\mathrm{ev}}{\overset{\longleftarrow}{\longrightarrow}} G \longrightarrow 1$$

so $L^+G = K_1 \rtimes G$. Let $Q_\lambda = L^+G \cap t^\lambda L^+G t^{-\lambda}$ be the stabilizer of t^λ. Because L^+G splits, the stabilizer also splits $Q_\lambda = (Q_\lambda \cap K_1) \rtimes \mathrm{ev}(Q_\lambda)$. We determine the image $\mathrm{ev}(Q_\lambda) \subset G$. An element $g \in G \subset L^+G$ lies in $\mathrm{ev}(Q_\lambda)$ if and only if $t^{-\lambda} g t^\lambda \in L^+G$. We test this condition on the root subgroups $U_\alpha \subset G$. For $u \in U_\alpha$, conjugation by $t^{-\lambda}$ scales the element by $t^{-\langle\alpha,\lambda\rangle}$. For this to remain in L^+G, we require $\langle \alpha, \lambda \rangle \le 0$. Because λ is a dominant cocharacter, $\langle \alpha, \lambda \rangle \ge 0$ for all positive roots. Thus, for a positive root α, the condition is satisfied if and only if $\langle \alpha, \lambda \rangle = 0$. For a negative root, the condition is trivially satisfied.

Therefore, $\mathrm{ev}(Q_\lambda)$ contains the full maximal torus T, all negative root subgroups, and exactly the positive root subgroups orthogonal to λ, making it the opposite parabolic subgroup P_λ^-.

We can now express the Schubert cell as a quotient of semidirect products,

$$\mathrm{Gr}_\lambda \simeq \frac{L^+G}{Q_\lambda} = \frac{K_1 \rtimes G}{(Q_\lambda \cap K_1) \rtimes P_\lambda^-} \simeq G \times_{P_\lambda^-} \left(\frac{K_1}{Q_\lambda \cap K_1}\right)$$

which identifies Gr_λ is an associated bundle over G/P_λ^-. Let $V = K_1/(Q_\lambda \cap K_1)$. Since the action of L^+G on Gr_λ factors through a finite-dimensional quotient L^nG for $n \gg 1$, V is determined by a finite-dimensional quotient of unipotent algebraic groups, which in characteristic 0, is isomorphic to $\mathbb{A}^N$ (explicitly, $N = \dim(\mathrm{Gr}_\lambda) - \dim(G/P_\lambda^-) = \langle 2\rho, \lambda \rangle - |\Phi^+ \setminus \Phi_\lambda^+|$). Because the projection map $G \to G/P_\lambda^-$ is Zariski-locally trivial, the associated bundle $G \times_{P_\lambda^-} \mathbb{A}^N \to G/P_\lambda^-$ is a Zariski-locally trivial fibration with fiber $\mathbb{A}^N$, proving that Gr_λ is an affine bundle over G/P_λ^-.

Finally, G/P_λ^- decomposes into affine cells $\mathbb{A}^k$ (see [Spr98, Proposition 8.5.1]). Since the lowest-dimensional cell (excluding the point $\mathbb{A}^0$) is $\mathbb{A}^1$, there are no cells in real dimension 2, making G/P_λ^- simply-connected. Since $\pi_1(G/P_\lambda) = 0$, we conclude that Gr_λ is simply connected (by the long-exact sequence).

For (4), partition the set of dominant cocharacters $X_*(T)^+$ into equivalence classes modulo $Q^\vee$. For each class $[\lambda]$, we set

$$\mathrm{Gr}_{[\lambda]} = \bigcup_{\substack{\eta\in X_*(T)^+ \\ \eta\equiv\lambda \pmod{Q^\vee}}} \mathrm{Gr}_\eta\,.$$

From (1) and (2), it follows that these are open, closed, connected, and partition Gr_G, so they form the connected components.

Recall that the Weyl group W acts $X_*(T)$ by $s_\alpha(\mu) = \mu - \langle \alpha, \mu \rangle \alpha^\vee$ (s_α a standard reflection), so the induced action on $X_*(T)/Q^\vee$ is trivial. But every cocharacter in $X_*(T)$ is conjugate under W to exactly one dominant cocharacter in $X_*(T)^+$. Since the Weyl group preserves the equivalence classes modulo $Q^\vee$, the inclusion $X_*(T)^+ \to X_*(T)$ induces a canonical bijection

$$X_*(T)^+/Q^\vee \xrightarrow{\simeq} X_*(T)/Q^\vee \simeq \pi_1(G)$$

See [Zhu16, Proposition 2.1.5, Theorem 1.3.11], [BR18, Proposition 3.2], and [BD94] for other treatments. □

Remark 2.3.2. The decomposition $\overline{\mathrm{Gr}_\lambda} = \bigcup_{\mu \le \lambda} \mathrm{Gr}_\lambda$ in L^+G-orbits is a L^+G-equivariantly good stratification in the sense of [Ach21, Definition 2.3.1, 2.3.20, 6.5.2], which basically means that if L is a local system on Gr_μ, then its $*$-pushforward is constructible with respect to the stratification. Indeed, we must verify that the L^+G-action on $\overline{\mathrm{Gr}_\lambda}$ has finitely many orbits, and factors through a finite stage. This is the content of Theorem 2.2.3 and Theorem 2.3.1, as there are only finitely many dominant cocharacters μ with $\mu \le \lambda$.

3 The Satake Category Sat_G

In this section we will introduce the Satake category. As explained in the introduction, we work with constructible Betti sheaves (i.e. sheaves on the underlying analytic topology), and so from now on k will be $\mathbb{C}$. in Section 3.1, we briefly recall the notion of constructible sheaves, perverse sheaves, and intersection cohomology, originally introduced in [BBD] (although for the exposition, we follow [Ach21]). In Section 3.2 we define the Satake category Sat_G and show it is semisimple and identify its simple objects. In Section 3.3, we equip Sat_G with a monoidal structure called *convolution*. We identify the tensor-unit and explain why Sat_G is rigid.

3.1 The perverse t-structure

Throughout this text, we assume the existence of the standard six-functor formalism for the bounded constructible derived category $D^b_c(X,\Lambda)$ on the category of complex algebraic varieties. Everything in this chapter can be found in [Ach21, Chapter 1-6].

Recall 3.1.1. Let X be a complex algebraic variety. The *perverse t-structure* on $D^b_c(X,\Lambda)$ is defined by the two strictly full subcategories $({}^pD^{\leq 0}(X,\Lambda), {}^pD^{\geq 0}(X,\Lambda))$ spanned by objects A satisfying the following support conditions for some (and hence any sufficiently fine) algebraic stratification $(X_s)_{s\in S}$:

$$\mathcal{H}^j(i_s^* A) = 0 \text{ for all } j > -\dim_{\mathbb{C}} X_s, \quad \text{resp.} \quad \mathcal{H}^j(i_s^! A) = 0 \text{ for all } j < -\dim_{\mathbb{C}} X_s,$$

where $i_s : X_s \hookrightarrow X$ denotes the locally closed inclusion of a stratum. The category of *perverse sheaves* is the heart of this t-structure $\mathrm{Perv}(X,\Lambda) = {}^pD^{\leq 0}(X,\Lambda) \cap {}^pD^{\geq 0}(X,\Lambda)$.

Definition 3.1.2. Let G be a smooth affine algebraic group of dimension d acting on a complex algebraic variety X, and let $q : X \to X/G$ be the canonical smooth projection to the quotient stack.

The G-equivariant constructible derived category is defined as $D^b_G(X,\Lambda) \coloneqq D^b_c(X/G,\Lambda)$. We define the *equivariant perverse t-structure* on this category by declaring sheaf A to be in ${}^pD^{\leq 0}$ (resp. ${}^pD^{\geq 0}$) if and only if its shifted pullback to X satisfies

$$q^*\mathcal{A}[d] \in {}^pD^{\leq 0}(X,\Lambda) \quad (\text{resp. } q^*\mathcal{A}[d] \in {}^pD^{\geq 0}(X,\Lambda)).$$

The category of G-equivariant perverse sheaves, denoted $\mathrm{Perv}_G(X,\Lambda) = \mathrm{Perv}(X/G,\Lambda)$, is the heart of this t-structure.

Explicitly, by unravelling the definitoins, an object in $\mathrm{Perv}_G(X,\Lambda)$ is equivalently described by the datum of a pair (A,ι), where $A \in \mathrm{Perv}(X,\Lambda)$ is a perverse sheaf on X, and $\iota : a^*A \xrightarrow{\sim} p^*A$ is an isomorphism of perverse sheaves on $G \times X$ satisfying the "cocycle condition" on $G \times G \times X$

$$(m \times \mathrm{id})^*(\iota) = (\mathrm{id} \times p)^*(\iota) \circ (\mathrm{id} \times a)^*(\iota),$$

where the maps $a, p : G \times X \to X$ denote the action and projection maps, respectively, and $m : G \times G \to G$ is the group multiplication. Indeed, this comes from the simplicial diagram

$$\begin{tikzcd} G \times G \times X \arrow[r, shift left=2, "\mathrm{id} \times a"] \arrow[r, "m\times\mathrm{id}" description] \arrow[r, shift right=2, "\mathrm{id} \times p"'] & G \times X \arrow[r, shift left, "a"] \arrow[r, shift right, "p"'] & X. \end{tikzcd}$$

Note that because a and p are both smooth morphisms of relative dimension d, their shifted pullbacks $a^*[d]$ and $p^*[d]$ are t-exact functors, ensuring that the pullbacks in the definition of ι remain perverse (see [Ach21, Theorem 3.6.6] or [BBD, Proposition 4.2.5]).

Recall 3.1.3. Let $h : U \hookrightarrow X$ be a locally closed embedding of varieties. There is a fully faithful functor $h_{!*} : \mathrm{Perv}(U,\Lambda) \to \mathrm{Perv}(X,\Lambda)$ called the *intermediate extension*, given by:

$$h_{!*}(A) = \mathrm{Im}({}^p\mathsf{H}^0(h_!A) \to {}^p\mathsf{H}^0(h_*A)).$$

Let $i : Z = \overline{U} \smallsetminus U \hookrightarrow X$ be the inclusion of the boundary. A perverse sheaf $A \in \mathrm{Perv}(X,\Lambda)$ lies in the essential image of $h_{!*}$ if and only if:

$$\mathrm{Supp}(A) = \overline{U}, \quad i^!(A) \in {}^pD^{\geq 1}_c(Z,\Lambda), \quad \text{and} \quad i^*(A) \in {}^pD^{\leq -1}_c(Z,\Lambda).$$

If U is smooth and irreducible, and $\mathcal{L}$ is a local system on U, we define the *Intersection Cohomology sheaf* (IC-sheaf) as the intermediate extension of the shifted local system:

$$\mathbf{IC}(\overline{U}, \mathcal{L}) = h_{!*}(\mathcal{L}[\dim_{\mathbb{C}} U]).$$

When $\mathcal{L} = \underline{\Lambda}$ is the constant sheaf, we write $\mathbf{IC}(\overline{U}, \Lambda)$.

The importance of Remark 2.3.2 and the reason for the assumption of finitely many orbits is due to the following result.

Proposition 3.1.4. Let G be a connected linear algebraic group acting on a complex variety X, and assume that there are only finitely many orbits. For each orbit $\mathcal{O} \subset X$, choose a basepoint $x \in \mathcal{O}$ and let $\mathrm{Stab}(x)$ denote its stabilizer subgroup. The simple objects of $\mathrm{Perv}_G(X, \Lambda)$ correspond bijectively to pairs $(\mathcal{O}, \tau)$, where τ is an irreducible finite-dimensional representation of $\pi_0(\mathrm{Stab}(x)) = \mathrm{Stab}(x)/\mathrm{Stab}(x)^\circ$. Explicitly, the simple perverse sheaf associated to the pair $(\mathcal{O}, \tau)$ is given by the intermediate extension of the corresponding equivariant local system:

$$\mathbf{IC}(\overline{\mathcal{O}}, \tau) \coloneqq \iota_{\mathcal{O}*}(j_{\mathcal{O}})_{!*} L_\tau[\dim_{\mathbb{C}} \mathcal{O}],$$

where L_τ is the irreducible G-equivariant local system on $\mathcal{O}$ determined by the representation τ, $j_{\mathcal{O}} : \mathcal{O} \hookrightarrow \overline{\mathcal{O}}$ is the open immersion into the orbit closure, and $\iota_{\mathcal{O}} : \overline{\mathcal{O}} \hookrightarrow X$ is the closed immersion into X.

Proof. See [Cla08, Proposition B.13] □

We bootstrap these notions to the setting of strict ind-schemes that are ind-of finite type with an action of a pro-algebraic group that factors through a finite stage, and with finitely many orbits.

Definition 3.1.5. Let $X = \mathrm{colim}_i X_i$ be a strict ind-scheme of ind-finite type, where the transition maps $\iota_i : X_i \to X_{i+1}$ are closed immersions. Let $G = \lim_k G_k$ be a pro-algebraic group acting on X.

Assume that for every finite stage X_i, the action of G factors through a smooth, finite-dimensional quotient group $G \to G_k$. Because the kernel of this projection is pro-unipotent, the pullback functor along the quotient map induces an equivalence of equivariant categories. We may therefore define

$$\mathrm{Perv}_G(X_i, \Lambda) \coloneqq \mathrm{Perv}_{G_k}(X_i, \Lambda).$$

The category of G-equivariant perverse sheaves on the entire ind-scheme X is then defined as the colimit over the closed immersions:

$$\mathrm{Perv}_G(X, \Lambda) \coloneqq \operatorname*{colim}_i \mathrm{Perv}_G(X_i, \Lambda)$$

where the transition functors are the proper pushforwards $\iota_{i*} : \mathrm{Perv}_G(X_i, \Lambda) \to \mathrm{Perv}_G(X_{i+1}, \Lambda)$.

3.2 Semisimplicity of Sat_G

In view of the discussion above, there is a well-defined abelian, Λ-linear category

$$\mathrm{Perv}(\mathrm{Hck}_G, \Lambda) = \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G, \Lambda) = \operatorname*{colim}_{\lambda \in X_*(T)^+} \mathrm{Perv}_{L^+G}(\overline{\mathrm{Gr}_\lambda}, \Lambda)$$

which we denote by Sat_G and call the *(absolute) Satake category*. The goal of this section is to equip Sat_G with the necessary structure for it to promote to a neutral Tannakian category. We will construct an exact, Λ-linear, faithful functor $F : \mathrm{Sat}_G \to \mathrm{Mod}_\Lambda$ and a monoidal structure, $\star$, called *convolution* on Sat_G. When Λ is a field of characteristic 0, Sat_G turns out to be a semisimple category.

Note that every object $A \in \mathrm{Sat}_G$ is supported on a finite stage $\overline{\mathrm{Gr}_\lambda}$ in the sense that A lies in the essential image of the pushforward $\iota_{\lambda*} : \mathrm{Perv}_{L^+G}(\overline{\mathrm{Gr}_\lambda}, \Lambda) \to \mathrm{Sat}_G$.

Notation 3.2.1. Theorem 2.3.1 gives an explicit representation of $\mathrm{Gr}_G \simeq \mathrm{colim}_{\lambda \in X_*(T)^+} \overline{\mathrm{Gr}_\lambda}$ as an ind-scheme of ind-finite type and ind-projective. We fix the following notation:

- $\iota_{\lambda,\mu} : \overline{\mathrm{Gr}_\lambda} \hookrightarrow \overline{\mathrm{Gr}_\mu}$ denotes the closed immersions in the colimit system.
- $\iota_\lambda : \overline{\mathrm{Gr}_\lambda} \hookrightarrow \mathrm{Gr}_G$ denotes the canonical global inclusion.

- $j_\lambda : \mathrm{Gr}_\lambda \hookrightarrow \overline{\mathrm{Gr}_\lambda}$ denotes the open embedding of the stratum.
- $i_\lambda : \partial\,\mathrm{Gr}_\lambda = \bigcup_{\mu<\lambda} \mathrm{Gr}_\mu \hookrightarrow \overline{\mathrm{Gr}_\lambda}$ denotes the closed embedding of the boundary.

Lemma 3.2.2. The simple objects of Sat_G are exactly the IC-sheaves

$$\mathbf{IC}_\lambda \coloneqq \iota_{\lambda*}\,\mathbf{IC}(\mathrm{Gr}_\lambda, \Lambda) = \iota_{\lambda*}(j_\lambda)_{!*}\underline{\Lambda}[\dim \mathrm{Gr}_\lambda]$$

for $\lambda \in X_*(T)^+$.

Proof. This follows from Proposition 3.1.4, noting that the stabilizers $L^+G \cap t^\lambda L^+Gt^{-\lambda}$ are connected. □

Before proving semisimplicity, we recall a fundamental property regarding the stalks of the intersection cohomology complexes on the affine Grassmannian (for references, see [BR18, Lemma 4.5]):

Lemma 3.2.3 (Parity Lemma)**.** The non-zero cohomology sheaves of $\mathbf{IC}_\lambda$ can only occur in degrees $k \equiv \dim \mathrm{Gr}_\lambda \pmod 2$.

Theorem 3.2.4. The abelian category Sat_G is semisimple.

Proof. Since Sat_G is Noetherian and the simple objects are exactly the $\mathbf{IC}_\lambda$, it suffices to show that $\mathrm{Ext}^1(\mathbf{IC}_\lambda, \mathbf{IC}_\mu) = \mathrm{Hom}(\mathbf{IC}_\lambda, \mathbf{IC}_\mu[1]) = 0$ for all $\lambda, \mu \in X_*(T)^+$. If Gr_λ and Gr_μ are not in the same connected component, then the claim is evidently true. Thus, we may assume that $\dim(\mathrm{Gr}_\lambda) \equiv \dim(\mathrm{Gr}_\mu) \pmod 2$. We distinguish three cases: (i) $\lambda = \mu$, (ii) $\lambda \nleq \mu$ and $\lambda \ngeq \mu$, and (iii) $\lambda < \mu$ or $\lambda > \mu$.

For (i), applying $\mathrm{Hom}(\mathbf{IC}_\lambda, -)$ to the recollement sequence $i_{\lambda*}i_\lambda^!\,\mathbf{IC}_\lambda \to \mathbf{IC}_\lambda \to j_{\lambda*}j_\lambda^*\,\mathbf{IC}_\lambda$ coming from the open-closed decomposition of $\overline{\mathrm{Gr}_\lambda}$, we obtain an exact sequence:

$$\mathrm{Hom}(i_\lambda^*\,\mathbf{IC}_\lambda, i_\lambda^!\,\mathbf{IC}_\lambda[1]) \longrightarrow \mathrm{Hom}(\mathbf{IC}_\lambda, \mathbf{IC}_\lambda[1]) \longrightarrow \mathrm{Hom}(j_\lambda^*\,\mathbf{IC}_\lambda, j_\lambda^*\,\mathbf{IC}_\lambda[1])$$

The first term vanishes because $i_\lambda^*\,\mathbf{IC}_\lambda \in {}^pD^{\le -1}$ and $i_\lambda^!\,\mathbf{IC}_\lambda \in {}^pD^{\ge 1}$ (by Recall 3.1.3). The right term vanishes because Gr_λ is an affine bundle, which yields the identification:

$$\mathrm{Hom}(j_\lambda^*\,\mathbf{IC}_\lambda, j_\lambda^*\,\mathbf{IC}_\lambda[1]) \simeq \mathrm{Hom}(1, 1[1]) \simeq H^1(\mathrm{Gr}_\lambda, \Lambda) \simeq 0.$$

Thus, the middle term is 0.

For (ii), consider the diagrams

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\text{cart}"] & \overline{\mathrm{Gr}_\lambda} \arrow[d, "\iota_\lambda"] \\ \overline{\mathrm{Gr}_\mu} \arrow[r, "\iota_\mu"'] & \mathrm{Gr}_G \end{tikzcd}, \qquad \begin{tikzcd} Z \arrow[r, "\tilde f"] \arrow[dr, "f"'] & \partial\,\mathrm{Gr}_\lambda \arrow[d, "i_\lambda"] \\ & \overline{\mathrm{Gr}_\lambda} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Z \arrow[r, "\tilde g"] \arrow[dr, "g"'] & \partial\,\mathrm{Gr}_\mu \arrow[d, "i_\mu"] \\ & \overline{\mathrm{Gr}_\mu} \end{tikzcd}$$

where the left-most diagram defines $Z = \overline{\mathrm{Gr}_\lambda} \cap \overline{\mathrm{Gr}_\mu}$. The other two factorizations follow from the incomparability assumption that $\mathrm{Gr}_\lambda \notin \overline{\mathrm{Gr}_\mu}$ and $\mathrm{Gr}_\mu \notin \overline{\mathrm{Gr}_\lambda}$. As in (i), $i_\lambda^*\,\mathbf{IC}_\lambda \in {}^pD_c^b(\partial\,\mathrm{Gr}_\lambda)^{\le -1}$. Since $\tilde f$ is a closed immersion, $\tilde f^*$ is right t-exact, so $f^*\,\mathbf{IC}_\lambda \simeq \tilde f^* i_\lambda^*\,\mathbf{IC}_\lambda \in {}^pD_c^b(Z)^{\le -1}$. Similarly, $i_\mu^!\,\mathbf{IC}_\mu \in {}^pD_c^b(\partial\,\mathrm{Gr}_\mu)^{\ge 1}$ and $\tilde g^!$ is left t-exact, so $g^!\,\mathbf{IC}_\mu \simeq \tilde g^! i_\mu^!\,\mathbf{IC}_\mu \in {}^pD_c^b(Z)^{\ge 1}$. By proper base-change, we conclude:

$$\mathrm{Hom}(\iota_{\lambda*}\,\mathbf{IC}_\lambda, \iota_{\mu*}\,\mathbf{IC}_\mu[1]) \simeq \mathrm{Hom}(f^*\,\mathbf{IC}_\lambda, g^!\,\mathbf{IC}_\mu[1]) \simeq 0.$$

For (iii), since Verdier duality $\mathbb{D}$ is an anti-autoequivalence of $D_c^b(\mathrm{Gr}_G, \Lambda)$ that preserves IC-sheaves, we may assume without loss of generality that $\mu < \lambda$, meaning $\mathrm{Gr}_\mu \subset \overline{\mathrm{Gr}_\lambda}$. Let h_μ denote the composite $\mathrm{Gr}_\mu \overset{j_\mu}{\hookrightarrow} \overline{\mathrm{Gr}_\mu} \overset{\iota_\mu}{\hookrightarrow} \mathrm{Gr}_G$. Note that $h_{\mu*}$ is left t-exact and $h_\mu^*\,\mathbf{IC}_\mu \simeq \underline{\Lambda}[\dim \mathrm{Gr}_\mu]$. Consider the fiber sequence

$$\mathbf{IC}_\mu \longrightarrow h_{\mu*}h_\mu^*\,\mathbf{IC}_\mu \simeq h_{\mu*}\underline{\Lambda}[\dim \mathrm{Gr}_\mu] \longrightarrow C$$

which induces an exact sequence

$$\mathrm{Hom}(\mathbf{IC}_\lambda, C) \longrightarrow \mathrm{Hom}(\mathbf{IC}_\lambda, \mathbf{IC}_\mu[1]) \longrightarrow \mathrm{Hom}(\mathbf{IC}_\lambda, h_{\mu*}(1)[\dim \mathrm{Gr}_\mu + 1])$$

We will show that the left and right terms of this exact sequence vanish.

Since $h_{\mu *}$ is left t-exact, $h_{\mu *}\underline{\Lambda}[\dim \mathrm{Gr}_\mu] \in {}^p D_c^{\geq 0}(\mathrm{Gr}_G, \Lambda)$. Since $\mathbf{IC}_\mu$ is perverse, the canonical map $\mathbf{IC}_\mu \to {}^p\mathsf{H}^0(h_{\mu *}\underline{\Lambda}[\dim \mathrm{Gr}_\mu])$ is injective. It follows that the cone $C \in {}^p D_c^{\geq 0}(\mathrm{Gr}_G, \Lambda)$. By applying h_μ^* to the fiber sequence above, we see that $h_\mu^* C \simeq 0$, so C is supported strictly on the boundary $\partial \mathrm{Gr}_\mu = \overline{\mathrm{Gr}_\mu} \smallsetminus \mathrm{Gr}_\mu$, which is contained within the boundary $\partial \mathrm{Gr}_\lambda = \overline{\mathrm{Gr}_\lambda} \smallsetminus \mathrm{Gr}_\lambda$ by the assumption $\mu < \lambda$. Let $i : \partial \mathrm{Gr}_\lambda \hookrightarrow \mathrm{Gr}_G$ be the closed inclusion. From the recollement sequence applied to C, it follows that

$$\mathrm{Hom}(\mathbf{IC}_\lambda, C) \simeq \mathrm{Hom}(i^* \mathbf{IC}_\lambda, i^! C) \simeq 0$$

because $i^* \mathbf{IC}_\lambda \in {}^p D_c^{\leq -1}$, and by the left t-exactness of $i^!$, $i^! C \in {}^p D_c^{\geq 0}$.

Finally, we evaluate the rightmost term by adjunction:

$$\mathrm{Hom}(\mathbf{IC}_\lambda, h_{\mu *}\underline{\Lambda}[\dim \mathrm{Gr}_\mu + 1]) \simeq \mathrm{Hom}_{\mathrm{Gr}_\mu}(h_\mu^* \mathbf{IC}_\lambda, \underline{\Lambda}[\dim \mathrm{Gr}_\mu + 1]).$$

First, note that in the standard t-structure, $\underline{\Lambda}[\dim \mathrm{Gr}_\mu + 1] \in D^{\geq -\dim \mathrm{Gr}_\mu - 1}$. Second, by the definition of the perverse t-structure, the standard cohomology sheaves $\mathsf{H}^k(h_\mu^* \mathbf{IC}_\lambda)$ must vanish for all $k \geq -\dim \mathrm{Gr}_\mu$, so the domain lives in $D^{\leq -\dim \mathrm{Gr}_\mu - 1}$. However, recall that $\dim \mathrm{Gr}_\lambda \equiv \dim \mathrm{Gr}_\mu \pmod 2$. Thus, by Lemma 3.2.3, the top potentially non-zero standard cohomology group $\mathsf{H}^{-\dim \mathrm{Gr}_\mu - 1}(h_\mu^* \mathbf{IC}_\lambda)$ must actually vanish. It follows that $h_\mu^* \mathbf{IC}_\lambda \in D^{\leq -\dim \mathrm{Gr}_\mu - 2}$, which means there can be no non-zero maps between these complexes. □

Remark 3.2.5. There is an alternative, highly conceptual proof of Theorem 3.2.4 using the theory of "parity sheaves" (see [Ach21, Section 7.5, and Proposition 9.3.10]).

To make Sat_G a neutral Tannakian category, we have to exhibit a fiber functor. The functor that we will consider is the *total cohomology functor* $H^* : \mathrm{Sat}_G \to \mathrm{Mod}_\Lambda$ defined by $H^*(A) = \bigoplus_{k \in \mathbb{Z}} \mathsf{H}^k(\mathrm{Gr}_G, A)$.

3.3 Convolution: Monoidal structure on Sat_G

Construction 3.3.1. If H is a group scheme acting on two schemes X and Y from the right and left, respectively, we may form the *twisted product* $X \times^H Y$ by taking the colimit over the two-sided simplicial bar construction $B_\bullet(X, H, Y)$ in the ∞-topos $\mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{Aff}_{\mathbb{C}})$.

Because X, H, and Y are ordinary schemes, they are 0-truncated objects in the ∞-topos. Equivalently, the group multiplication and actions are strictly associative, and all higher coherence homotopies are trivial, so the diagram below actually reduces to just the three first terms (degeneracy maps are left out on purpose):

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} X \times H^2 \times Y \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} X \times H \times Y \substack{\longrightarrow\\\longrightarrow} X \times Y$$

The generating face maps are given explicitly on points by the action and the strict group multiplication:

$$(x, g, y) \mapsto \begin{cases} (xg, y) \\ (x, gy) \end{cases} \qquad \text{and} \qquad (x, g, h, y) \mapsto \begin{cases} (xg, h, y) \\ (x, gh, y) \\ (x, g, hy). \end{cases}$$

Example 3.3.2. (1) L^+G still acts on Y from both left and right, and the multiplication on LG is compatible with the both actions, and therefore induce a map $LG \times^{L^+G} LG \to LG$, and further a map $L^+G\backslash Y/L^+G \to L^+G\backslash LG/L^+G = \mathrm{Hck}_G$.

$$\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G \simeq \operatorname*{colim}_{\lambda, \mu \in X_*(T)^+} \overline{\mathrm{Gr}_\lambda} \tilde{\times} \overline{\mathrm{Gr}_\mu},$$

The pro-group L^+G still acts on $\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G$ form the left, and there is a well-defined category of bounded constructible L^+G-equivariant (perverse) sheaves on $\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G$. Quotienting out by the action of L^+G, we obtain the *convolution Hecke stack*,

$$\mathrm{Hck}_{G,2}^{\mathrm{conv}} = \mathrm{Hck}_G \tilde{\times} \mathrm{Hck}_G \simeq L^+G\backslash LG \times^{L^+G} LG/L^+G,$$

whose sheaves are the L^+G-equivariant sheaves on $\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G$.

Example 3.3.3. (1) L^+G acts on LG from both left and right (by multiplication), and we may form the twisted product $LG \times^{L^+G} LG$. Since the multiplication $m : LG \times LG \to LG$ is L^+G-equivariant with respect to the diagonal L^+G-action, it descends to a map of twisted products:$LG \times^{L^+G} LG \to LG$. Taking the quotient by the remaining left/right L^+G-actions, this induces the multiplication map for the *convolution Hecke stack*

$$\mathrm{Hck}_{G,2}^{\mathrm{conv}} := L^+G \backslash (LG \times^{L^+G} LG)/L^+G \simeq B(L^+G) \times_{B(LG)} B(L^+G) \times_{B(LG)} B(L^+G) \to \mathrm{Hck}_G$$

(see [NSS14, Lemma 4.5] for the equivalence).

(2) Analogously, we define the *convolution affine Grassmannian* as the twisted product:

$$\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G := LG \times^{L^+G} \mathrm{Gr}_G .$$

As an ind-scheme, this satisfies the factorization

$$\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G \simeq \operatorname*{colim}_{\lambda,\mu \in X_*(T)^+} (\overline{\mathrm{Gr}_\lambda} \tilde{\times} \overline{\mathrm{Gr}_\mu}),$$

where each $\overline{\mathrm{Gr}_\lambda} \tilde{\times} \overline{\mathrm{Gr}_\mu} := LG \times^{L^+G} \overline{\mathrm{Gr}_\mu}$ is a well-defined ind-projective ind-scheme. The group L^+G acts on the left of $\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G$ via the identification $\mathrm{Gr}_G \simeq LG/L^+G$. The convolution Hecke stack $\mathrm{Hck}_{G,2}^{\mathrm{conv}}$ is exactly the quotient $L^+G \backslash (\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G)$, and its derived category $D^b_c(\mathrm{Hck}_{G,2}^{\mathrm{conv}}, \Lambda)$ is equivalent to the category of L^+G-equivariant bounded constructible complexes on $\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G$.

Construction 3.3.4 (Convolution)**.** By considering the diagram

$$\begin{tikzcd} & \mathrm{Hck}_{G,2}^{\mathrm{conv}} \arrow[dl, "p"'] \arrow[dr, "m"] & \\ \mathrm{Hck}_G \times \mathrm{Hck}_G & & L^+G\backslash LG/L^+G \simeq \mathrm{Hck}_G, \end{tikzcd}$$

we can define the *convolution* of two sheaves $A, B \in D^b_c(\mathrm{Hck}_G)$ as follows:

$$A \star B \simeq m_! p^*(A \boxtimes B).$$

Note that m is ind-proper, as $\mathrm{Hck}_{G,2}^{\mathrm{conv}}$ is ind-projective and LG is separated (recall that every map between a projective S-scheme and a separated S-scheme is proper). In particular, $m_! \simeq m_*$, canonically.

Theorem 3.3.5. (1) The convolution product promotes $D^b_c(\mathrm{Hck}_G, \Lambda)$ to a monoidal category, $D^b_c(\mathrm{Hck}_G, \Lambda)^\star$.
(2) The tensor-unit is the **IC**-sheaf $\mathbf{IC}_0$ corresponding to the trivial coroot, and $\mathrm{End}(\mathbf{IC}_0) \simeq \Lambda$.
(3) $\star$ preserves perversity, so Sat_G also promotes to a monoidal category $\mathrm{Sat}_G^\star$.
(4) The monoidal category $\mathrm{Sat}_G^\star$ promotes to a symmetric monoidal category.

Comments on the proof. (1) is a consequence of a greater theme in category-theory, as explained and proven in Section C. The idea is that Hck_G becomes a group-object in the category of correspondences $\mathrm{Corr}(\mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{Aff}_k))$. This promotes the convolution product $\star$ to a (fully coherent) monoidal structure on $D^b_c(\mathrm{Hck}_G)$. $\star$ and Hck_G inherits the associativity of LG, and whatever $\mathbb{E}_k$-structure it may admit, but it doesn't automatically gain any (higher) commutativity. So (3) is really a consequence of geometry, rather than category theory.

For (2), we claim that there is a diagram like this

$$\begin{tikzcd} \mathrm{Hck}_G \arrow[r, "\iota'"'] \arrow[d, "\mathrm{id}"'] \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, phantom, "\mathrm{cart}"] & \mathrm{Hck}_{G,2}^{\mathrm{conv}} \arrow[r, "m"'] \arrow[d, "p"] & \mathrm{Hck}_G \\ \mathrm{Hck}_G \times \{t^0\} \arrow[r, "\mathrm{id} \times \iota_0"'] & \mathrm{Hck}_G \times \mathrm{Hck}_G & \end{tikzcd}$$

Given this, we compute

$$A \star \mathbf{IC}_0 \simeq m_! p^*(A \boxtimes \iota_{0!}(1)) \simeq m_! p^*(\mathrm{id} \times \iota_0)_!(A \boxtimes 1) \simeq m_! \iota'_!(A) \simeq A$$

Since $\iota_0 : \{t^0\} \to \mathrm{Gr}_G$ is a closed immersion, ι_{0*} is fully faithful, so

$$\mathrm{Hom}(\mathbf{IC}_0, \mathbf{IC}_0) \simeq \mathrm{Hom}(\Lambda, \Lambda) = \Lambda$$

(3) is called a "miraculous theorem" in [BD94] and is originally due to Luztig (see 5.3.6 of *loc. cit.*). We will postpone the proof to Section 4.

(4) is the main content of Section 4. In [Noc25], it is proven that $D_c^b(\mathrm{Hck}_G)^\star$ promotes to an $\mathbb{E}_3$-structure. In *loc. cit.*, Nocera equips $D_c^b(\mathrm{Hck}_G)^\star$ with a compatible $\mathbb{E}_2$-structure (the number two comes from $\mathbb{C} \simeq \mathbb{R}^2$), and then he applies Lurie's work on $\mathbb{E}_k$-algebras via the topological Ran space ([Lur17, Chapter 5]). It is expected to be no more than $\mathbb{E}_3 \dots$ But luckily for us, Sat_G is an (ordinary) 1-category, so we can just construct the symmetric constraint "by hand." $\square$

4 Fusion: Symmetric monoidality of $\star$ and H^*

In this section we will sketch a proof of symmetric monoidality of the convolution product. Whenever we have an absolute mathematical object, we should always consider a relative/in-families version. This is exactly what Beilinson and Drinfeld did; they considered a relative version of the affine Grassmannian and the Hecke stack over a reduced connected complex curve X.

4.1 The relative loop group

We imitate the construction of the loop functors from Section 2.1 by replacing the absolute disks over $\mathrm{Spec}(k)$ with relative disks parameterized by a smooth, reduced curve X over $\mathbb{C}$.

Construction 4.1.1 (Relative loop functors)**.** Let I be a finite set. Consider the graph of the i^{th} projection

$$\begin{tikzcd} X^I \arrow[r] \arrow[d, "\Gamma_i"'] \arrow[dr, phantom, "\text{cart}"] & X \arrow[d, "\Delta"] \\ X \times X^I \arrow[r, "(\mathrm{id},\mathrm{pr}_i)"'] & X \times X \end{tikzcd}$$

which defines a Cartier divisor $\Gamma_i = \Gamma_i(X^I) \subset X \times X^I$. Taking the union yields a divisor $\Gamma_I = \bigcup_{i\in I}\Gamma_i \subset X \times X^I$.

Rather than forming a formal scheme, we consider the finite truncations. Let $\Gamma_{I,n}$ denote the n^{th} infinitesimal neighborhood of Γ_I in $X \times X^I$ (defined by the ideal sheaf $\mathcal{I}^n_{\Gamma_I}$). These subschemes parameterize $|I|$ infinitesimal disks varying along the curve X. The canonical projections induce a diagram:

$$X \xleftarrow{p_n} \Gamma_{I,n} \xrightarrow{q_n} X^I,$$

and we define the *relative n^{th} loop functor* $L^n_{X^I}(-)$ as the composite

$$\mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{Aff}_{\mathbb{C}})_{/X} \xrightarrow{p_n^*} \mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{Aff}_{\mathbb{C}})_{/\Gamma_{I,n}} \xrightarrow{q_{n*}} \mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{Aff}_{\mathbb{C}})_{/X^I}$$

The closed immersions $\Gamma_{I,n} \to \Gamma_{I,n+1}$ induce transition maps $L^{n+1}_{X^I} \to L^n_{X^I}$, allowing us to define the *relative positive loop functor* as the limit:

$$L^+_{X^I} := \lim_{n\in\mathbb{N}^{\mathrm{op}}} L^n_{X^I}.$$

To define the full loop functor, we construct the *relative disk* D_{X^I} and the *relative punctured disk* $D^\circ_{X^I}$ over X^I by

$$D_{X^I} \simeq \underline{\mathrm{Spec}}_{X^I}\left(\lim_{n\in\mathbb{N}^{\mathrm{op}}}(q_n)_*\mathcal{O}_{\Gamma_{I,n}}\right) \quad \text{and} \quad D^\circ_{X^I} \simeq D_{X^I} \smallsetminus \Gamma_I.$$

More generally, for any subset $I_k \subset I$, we define the *partially punctured relative disk*

$$D^{(I_k)}_{X^I} := D_{X^I} \smallsetminus \Gamma_{I_k}$$

which informally parameterizes $|I|$ disks, but where only the chosen $|I_k|$ centers are punctured.

Finally, we define the corresponding *relative loop functors* L_{X^I} and $L^{(I_k)}_{X^I}$ directly via their relative functor of points. For any target space $Z \in \mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{Aff}_k)_{/X}$ and any test point $x : \mathrm{Spec}(R) \to X^I$, we define their evaluations over x as:

$$(L_{X^I}Z)(x) := \mathrm{Hom}_{/X}(D^\circ_x, Z) \quad \text{and} \quad (L^{(I_k)}_{X^I}Z)(x) := \mathrm{Hom}_{/X}(D^{(I_k)}_x, Z)$$

where the disks are base-changed along x.

Remark 4.1.2. (1) Let $x : \mathrm{Spec}(R) \to X^I$ be an R-point. We write the base-changes along x as:

$$D_{n,x} = \Gamma_{I,n} \times_{X^I} \mathrm{Spec}\, R, \quad D_x = D_{X^I} \times_{X^I} \mathrm{Spec}\, R, \quad \text{and} \quad D_x^\circ = D_{X^I}^\circ \times_{X^I} \mathrm{Spec}\, R.$$

If $|I| = 1$, choosing a local coordinate t at the point $x \in X$ yields non-canonical isomorphisms,

$$D_{n,x} \simeq \mathrm{Spec}(R[t]/t^n), \quad D_x \simeq \mathrm{Spec}(R[[t]]), \quad \text{and} \quad D_x^\circ \simeq \mathrm{Spec}(R((t))).$$

Furthermore, there is a canonical equivalence $\Gamma_x \simeq \mathrm{Spec}(R)$, as the central point has already been chosen by x.

(2) For any object $Z \in \mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{Aff}_k)_{/X}$, the R-points of $L^+_{X^I} Z$ and $L_{X^I} Z$ are given by

$$(L^+_{X^I} Z)(R) = \{(x, \beta) \mid x \in X^I(R),\ \beta \in \mathrm{Hom}_{/X}(D_x, Z)\},$$

$$(L_{X^I} Z)(R) = \{(x, \beta) \mid x \in X^I(R),\ \beta \in \mathrm{Hom}_{/X}(D_x^\circ, Z)\}.$$

In our applications, Z will be a constant affine group scheme G ($Z \simeq G \times X \to X$). In this case, maps over X are equivalent to absolute maps into G, yielding

$$(L^+_{X^I} G)(R) = \{(x, \beta) \mid x \in X^I(R),\ \beta \in G(D_x)\}$$

$$(L_{X^I} G)(R) = \{(x, \beta) \mid x \in X^I(R),\ \beta \in G(D_x^\circ)\}.$$

Remark 4.1.3. (1) Let $x : S \simeq \mathrm{Spec}(R) \to X^I$ be an R-point. For any subset $I_k \subset I$, we write the base-changes along x as:

$$D_{n,x} = \Gamma_{I,n} \times_{X^I} S, \quad D_x = D_{X^I} \times_{X^I} S, \quad D_x^\circ = D_{X^I}^\circ \times_{X^I} S, \quad \text{and} \quad D_x^{(I_k)} = D_{X^I}^{(I_k)} \times_{X^I} S.$$

If $|I| = 1$, choosing a local coordinate t at the point $x \in X$ yields non-canonical isomorphisms,

$$D_{n,x} \simeq \mathrm{Spec}(R[t]/t^n), \quad D_x \simeq \mathrm{Spec}(R[[t]]), \quad \text{and} \quad D_x^\circ \simeq \mathrm{Spec}(R((t))),$$

and a canonical isomorphism $\Gamma_x \simeq \mathrm{Spec}(R)$, as the central point has already been chosen by x. Geometrically, if x is an I-tuple of points, $D_x^{(I_k)}$ represents the union of formal disks around the coordinates of x, punctured precisely at the coordinates belonging to the subset I_k.

(2) For any object $Z \in \mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{Aff}_k)_{/X}$, the R-points of the relative loop functors are given by

$$(L^+_{X^I} Z)(R) = \{(x, \beta) \mid x \in X^I(R),\ \beta \in \mathrm{Hom}_{/X}(D_x, Z)\},$$

$$(L^{(I_k)}_{X^I} Z)(R) = \{(x, \beta) \mid x \in X^I(R),\ \beta \in \mathrm{Hom}_{/X}(D_x^{(I_k)}, Z)\},$$

$$(L_{X^I} Z)(R) = \{(x, \beta) \mid x \in X^I(R),\ \beta \in \mathrm{Hom}_{/X}(D_x^\circ, Z)\}.$$

In our applications, Z will be a constant affine group scheme G (i.e. $Z \simeq G \times X \to X$). In this case, maps over X are equivalent to absolute maps into G, yielding

$$(L^+_{X^I} G)(R) = \{(x, \beta) \mid x \in X^I(R),\ \beta \in G(D_x)\}$$

$$(L^{(I_k)}_{X^I} G)(R) = \{(x, \beta) \mid x \in X^I(R),\ \beta \in G(D_x^{(I_k)})\}$$

$$(L_{X^I} G)(R) = \{(x, \beta) \mid x \in X^I(R),\ \beta \in G(D_x^\circ)\}.$$

(3) As in the absolute case, both $L^n_{X^I}$ and $L^+_{X^I}$ preserve all limits. In view of Theorem 4.2.2, L_{X^I} preserves finite limits over "mild" diagrams. In particular, they all take a linear algebraic group to a group object.

Theorem 4.1.4. If G is a constant group-scheme over X^I, then

(1) $L_{X^I} G$ is represented by an ind-scheme over X^I, and
(2) $L^+_{X^I} G$ is represented by a scheme affine over X^I,

Theorem 4.1.5. If G be an affine group scheme of finite type over $\mathbb{C}$ viewed as constant over X^I, then

(1) $L^+_{X^I}G$ is represented by a scheme that is affine over X^I.
(2) For any subset $I_k \subset I$, $L^{(I_k)}_{X^I}G$ is represented by strict ind-schemes that are ind-affine over X^I.

Proof. See [Zhu16, Proposition 3.1.6]. However, the proofs a similar to those in Section 2.1. ☐

Remark 4.1.6. Note if $I_k = I$, then $L^{(I_k)}_{X^I}G \simeq L_{X^I}G$. Also, the closed immersions

$$D^\circ_{X^I} \longrightarrow D^{(I_k)}_{X^I} \longrightarrow D_{X^I}$$

induce

$$L^+_{X^I}G \longrightarrow L^{(I_k)}_{X^I}G \longrightarrow L_{X^I}G.$$

Geometrically, this just says that a loop with no poles can be viewed as a loop with poles along Γ_{I_k}, which in turn can be viewed as a loop with poles along the entire divisor Γ_I. These are precisely the transition maps that induce the morphisms of classifying stacks $B(L^+_{X^I}G) \to B(L^{(I_k)}_{X^I}G)$ required to form the factorizable convolution stack.

4.2 The relative affine Grassmannian and Hecke stack

In this subsection we carry out the relative constructions of the affine Grassmannian and the Hecke stack.

Definition 4.2.1. If $G \to X$ is a group-object in $\mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{Aff}_{\mathbb{C}})_{/X}$, define the *Beilinson-Drinfeld affine Grassmannian* (or *relative affine Grassmannian*) Gr_{X^I} and the *Beilinson-Drinfeld Hecke stack* (or *relative Hecke stack*) Hck_{X^I} by the diagram

$$\begin{tikzcd} \mathrm{Gr}_{X^I} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & \mathrm{Hck}_{X^I} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & B(L^+_{X^I}G) \arrow[d] \\ * \arrow[r] & B(L^+_{X^I}G) \arrow[r] & B(L_{X^I}G) \end{tikzcd}$$

over X^I. Equivalently,

$$\mathrm{Gr}_{X^I} \simeq L_{X^I}G/L^+_{X^I}G \quad \text{and} \quad \mathrm{Hck}_{X^I} \simeq L^+_{X^I}G\backslash L_{X^I}G/L^+_{X^I}G.$$

Denote the structure maps by

$$\begin{tikzcd} \mathrm{Gr}_{X^I} \arrow[rr] \arrow[dr, "q_I"'] & & \mathrm{Hck}_{X^I} \arrow[dl, "q'_I"] \\ & X^I & \end{tikzcd}$$

As in the absolute case, we have a representability result:

Theorem 4.2.2. If G is a constant group-scheme over X^I, then Gr_{X^I} is represented by an ind-scheme, ind-of finite type over X^I. If G is reductive, then it is ind-projective.

Proof. See [Zhu16, Theorem 3.1.3] ☐

Construction 4.2.3 (Convolution Hecke Stacks)**.** For a fixed finite set I, we may form the *relative convolution Hecke stack* (analogous to the absolute case in Construction 3.3.4):

$$\begin{aligned} \mathrm{Hck}^{\mathrm{conv}}_{X^I,2} = \mathrm{Hck}_{X^I} \tilde{\times} \mathrm{Hck}_{X^I} &\simeq L^+_{X^I}G\backslash L_{X^I}G \times^{L^+_{X^I}G} L_{X^I}G/L^+_{X^I}G \\ &\simeq B(L^+_{X^I}G) \times_{B(L_{X^I}G)} B(L^+_{X^I}G) \times_{B(L_{X^I}G)} B(L^+_{X^I}G). \end{aligned}$$

This stack is used to define the convolution product on the derived category $D^b_c(\mathrm{Hck}_{X^I})$.

To apply the "factorization property" of Hck_{X^I} (see Theorem 4.2.5), we must generalize this to allow multiple finite sets to interact. Let $I_1, \dots, I_n$ be finite sets and let $I = I_1 \sqcup \dots \sqcup I_n \xrightarrow{\phi} \langle n \rangle^\circ$ be the obvious partition. Using the partially punctured loop functors $L^{(I_k)}_{X^I} G$ from Construction 4.1.1, we define the *factorizable convolution Hecke stack*:

$$\mathrm{Hck}^{\mathrm{conv}}_{I_1,\dots,I_n} \simeq B(L^+_{X^I} G) \times_{B(L^{(I_1)}_{X^I} G)} \cdots \times_{B(L^{(I_n)}_{X^I} G)} B(L^+_{X^I} G).$$

Note there are canonical maps

$$\prod_{k=1}^n \mathrm{Hck}_{X^{I_k}} \xleftarrow{\pi_\phi} \mathrm{Hck}^{\mathrm{conv}}_{I_1,\dots,I_n} \xrightarrow{m_\phi} \mathrm{Hck}_{X^I}$$

where π_ϕ is induced by the restrictions $L^+_{X^I} G \to L^+_{X^{I_k}} G$ and $L^{(I_k)}_{X^I} G \to L_{X^{I_k}} G$. The morphism m_ϕ is induced by the maps from Remark 4.1.6.

Remark 4.2.4. The R-points of Gr_{X^I}, Hck_{X^I}, and $\mathrm{Hck}^{\mathrm{conv}}_{I_1,\dots I_n}$ admit "local" descriptions (note that only $\mathrm{Gr}_{X^I}(R)$ is a genuine set, while the others are groupoids):

$$\mathrm{Gr}_{X^I}(R) \simeq \{(x, \mathcal{E}, \beta) \mid x \in X^I(R), \mathcal{E} \in \mathrm{Tors}_G(D_x), \beta : \mathcal{E}_{|D^\circ_x} \simeq \mathcal{E}^0_{|D^\circ_x}\},$$
$$\mathrm{Hck}_{X^I}(R) \simeq \{(x, \mathcal{E}, \mathcal{E}', \beta) \mid x \in X^I(R), \mathcal{E}, \mathcal{E}' \in \mathrm{Tors}_G(D_x), \beta : \mathcal{E}_{|D^\circ_x} \simeq \mathcal{E}'_{|D^\circ_x}\},$$
$$\mathrm{Hck}^{\mathrm{conv}}_{X^I,2}(R) \simeq \left\{ (x, \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \beta_1, \beta_2) \;\middle|\; \begin{array}{l} x \in X^I(R) \\ \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2 \in \mathrm{Tors}_G(D_x) \\ \beta_1 : \mathcal{E}_{0|D^\circ_x} \simeq \mathcal{E}_{1|D^\circ_x}, \quad \beta_2 : \mathcal{E}_{1|D^\circ_x} \simeq \mathcal{E}_{2|D^\circ_x} \end{array} \right\},$$
$$\mathrm{Hck}^{\mathrm{conv}}_{I,I}(R) \simeq \left\{ (x, y, \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \beta_1, \beta_2) \;\middle|\; \begin{array}{l} (x,y) \in X^I(R) \times X^I(R) \\ \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2 \in \mathrm{Tors}_G(D_{x,y}) \\ \beta_1 : \mathcal{E}_{0|D^{(I_1)}_{x,y}} \simeq \mathcal{E}_{1|D^{(I_1)}_{x,y}}, \quad \beta_2 : \mathcal{E}_{1|D^{(I_2)}_{x,y}} \simeq \mathcal{E}_{2|D^{(I_2)}_{x,y}} \end{array} \right\},$$

and more generally,

$$\mathrm{Hck}^{\mathrm{conv}}_{I_1,\dots I_n}(R) \simeq \left\{ (x, \mathcal{E}_0, (\mathcal{E}_k, \beta_k)^n_{k=1}) \;\middle|\; \begin{array}{l} x = (x_k)^n_{k=1} \in \prod^n_{k=1} X^{I_k}(R) \simeq X^I(R) \\ \mathcal{E}_0, \mathcal{E}_k \in \mathrm{Tors}_G(D_x), \beta_k : \mathcal{E}_{k-1|D^{(I_k)}_x} \simeq \mathcal{E}_{k|D^{(I_k)}_x} \end{array} \right\}.$$

The difference between $\mathrm{Hck}^{\mathrm{conv}}_{X^I,2}$ and $\mathrm{Hck}^{\mathrm{conv}}_{I,I}$ will be explored in Lemma 4.4.2. There is a "global" interpretation of the R-points of Gr_{X^I}, due to the Beauville-Laszlo theorem:

$$\mathrm{Gr}_{X^I}(R) = \{(x, \mathcal{E}, \beta) \mid x \in X^I(R), \mathcal{E} \in \mathrm{Tors}_G(X_R), \beta : \mathcal{E}_{|X_R \setminus \Gamma_x} \simeq \mathcal{E}^0_{|X_R \setminus \Gamma_x}\}$$

For references, see [Zhu16, Section 1.4.], and for the original reference, see *Un lemme de descente* ([BL95]).

The true power of the Beilinson-Drinfeld affine Grassmannian is the following "factorization property." For a surjective map $\phi : J \to I$, denote by $X^{(\phi)} \subset X^J$ the open subset of tuples $x = (x_j)_{j \in J} \in X^J$ such that if $\phi(j) \neq \phi(j')$, then $\Gamma_{x_j} \cap \Gamma_{x_{j'}} = \varnothing$.

Theorem 4.2.5 (Factorization property)**.** The following assertions also hold if Gr_{X^I} is replaced by Hck_{X^I}.

(1) For every finite set I, there is a unit section $e_I : X^I \to \mathrm{Gr}_{X^I}$ classifying the trivial G-torsor.
(2) For every surjection $\phi : J \to I$, the commutative square

$$\begin{tikzcd} \mathrm{Gr}_{X^I} \arrow[r, "\Delta(\phi)"] \arrow[d, "q_I"'] \arrow[dr, phantom, "\text{cart}"] & \mathrm{Gr}_{X^J} \arrow[d, "q_J"] \\ X^I \arrow[r, "-\circ\phi"'] & X^J \end{tikzcd}$$

is cartesian. The map takes a tuple of points $x : I \to X$ to the tuple $J \xrightarrow{\phi} I \xrightarrow{x} X$.

(3) Let $\phi : J \to I$ be a surjection, $J_i = \phi^{-1}(i)$, and let $X^{(\phi)} \subset X^J$ denote the open subset of those $\{x_j\}_{j\in J}$ such that $\Gamma_{x_j} \cap \Gamma_{x_{j'}} = \varnothing$ if $\phi(j) \neq \phi(j')$. There is a canonical isomorphism c_ϕ

$$\mathrm{Gr}_{X^J|X^{(\phi)}} \xrightarrow{c_\phi} (\textstyle\prod_{i\in I} \mathrm{Gr}_{X^{J_i}})_{|X^{(\phi)}}$$

If $K \xrightarrow{\phi} J \xrightarrow{\psi} I$ are surjections and $\phi_i : K_i \to J_i$ are the maps over $i \in I$, then

(a) $\Delta(\phi) \circ e_I = e_{J|\Delta(\phi)}$ and $c_\phi(e_{J|X^{(\phi)}}) = (e_{J_i})_{i\in I|X^{(\phi)}}$.

(b) $\Delta(\phi)\Delta(\psi) = \Delta(\psi\phi)$,

(c) the following diagram commutes

$$\begin{tikzcd}
\mathrm{Gr}_{X^J|X^{(\psi)}} \arrow[r, "\Delta(\phi)_{|X^{(\psi)}}"] \arrow[d, "c_\psi"'] & \mathrm{Gr}_{X^K|X^{(\psi\phi)}} \arrow[d, "c_{\psi\phi}"] \\
(\prod_{i\in I} \mathrm{Gr}_{X^{J_i}})_{|X^{(\psi)}} \arrow[r, "(\Delta(\phi_i))_{i\in I|X^{(\phi)}}"'] & (\prod_{i\in I} \mathrm{Gr}_{X^{K_i}})_{X^{(\psi\phi)}}
\end{tikzcd}$$

(d) $c_\phi = (c_{\phi_i})_{i\in I} \circ (c_{\psi\phi|X^{(\phi)}})$.

Proof. We prove the statements for Gr_{X^I}; however, the proof remains identical for Hck_{X^I}. For (1), note that the map $e_I : X^I \to \mathrm{Gr}_{X^I}$ is the map induced by the unit section $X^I \to B(L^+_{X^I}G)$.

For (2), since ϕ is surjective, the set of points in the tuples $x \circ \phi$ and x coincide, whence $\bigcup_{i\in I} \Gamma_{x_i} = \bigcup_{j\in J} \Gamma_{x_{\phi(j)}}$. Consequently,

$$L_{X^J}G \times_{X^J} X^I \simeq L_{X^I}G \quad \text{and} \quad L^+_{X^J}G \times_{X^J} X^I \simeq L^+_{X^I}G$$

and thus

$$B(L_{X^J}G) \times_{X^J} X^I \simeq B(L_{X^I}G) \quad \text{and} \quad B(L^+_{X^J}G) \times_{X^J} X^I \simeq B(L^+_{X^I}G)$$

by universality of colimits. The desired cartesian square follows.

For (3), recall that $L^+_{X^J}G(R) = \{(x,\beta) \mid x \in X^J(R), \beta \in G(D_x)\}$. We group the relative divisors into pieces corresponding to these fibers, where the last union is disjoint, as we are working over $X^{(\phi)}$:

$$\bigcup_{j\in J} \Gamma_{x_j} = \bigcup_{i\in I} \left(\bigcup_{j\in J_i} \Gamma_{x_j} \right) = \bigsqcup_{i\in I} \Gamma_{x_{J_i}}$$

The relative disk over a disjoint union of closed subschemes is canonically isomorphic to the disjoint union of their individual relative disks, yielding $D_x \simeq \coprod_{i\in I} D_{x_{J_i}}$ over $X^{(\phi)}$. Similarly, $D^\circ_x \simeq \coprod_{i\in I} D^\circ_{x_{J_i}}$. By the moduli-description, it follows that

$$L_{X^J}G_{|X^{(\phi)}} \simeq \prod_{i\in I} L_{X^{J_i}}G_{|X^{(\phi)}} \quad \text{and} \quad L^+_{X^J}G_{|X^{(\phi)}} \simeq \prod_{i\in I} L^+_{X^{J_i}}G_{|X^{(\phi)}}$$

Again, by universality of colimits, we get a canonical equivalence

$$\mathrm{Gr}_{X^J|X^{(\phi)}} \xrightarrow{c_\phi} (\textstyle\prod_{i\in I} \mathrm{Gr}_{X^{J_i}})_{|X^{(\phi)}}$$

(a) and (b) follow from the diagram

$$\begin{tikzcd}
X^I \arrow[r, "-\circ\psi"] \arrow[d, "e_I"'] \arrow[dr, phantom, "\text{cart}"] & X^J \arrow[d, "e_J"] & \\
\mathrm{Gr}_{X^I} \arrow[r, "\Delta(\psi)"] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & \mathrm{Gr}_{X^J} \arrow[r, "\Delta(\phi)"] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & \mathrm{Gr}_{X^K} \arrow[d] \\
X^I \arrow[r, "-\circ\psi"'] & X^J \arrow[r, "-\circ\phi"'] & X^K
\end{tikzcd}$$

(c) and (d) will be checked on the level of loop groups.

For (c), let $(x,\beta) \in L^+_{X^J}G(R)$ be a point over $X^{(\psi)}$, where $x : J \to X(R)$ and $\beta : D_x \to G$. Because $x \in X^{(\psi)}$, the relative disk splits as $D_x = \coprod_{i\in I} D_{x_{J_i}}$. Going right-down: The diagonal map $\Delta(\phi)$ pulls the points back to K, yielding $(x\circ\phi,\beta) \in L^+_{X^K}G(R)$, and $c_{\psi\phi}$ splits this element over the fibers of $\psi\phi : K \to I$, i.e. $K_i = \phi^{-1}(J_i)$, yielding the tuple $(x\circ\phi,(\beta_i)_{i\in I})$, where β_i is the restriction of β to $D_{(x\circ\phi)_{K_i}}$. Going down-left: c_ψ splits (x,β) into $(x,(\tilde\beta_i)_{i\in I})$, where $\tilde\beta_i$ is the restriction of β to $D_{x_{J_i}}$, and then $(\Delta(\phi_i))_{i\in I}$ pulls back the base points of each component along $\phi_i : K_i \to J_i$, yielding $(x\circ\phi,(\tilde\beta_i)_{i\in I})$. Because $K_i = \phi^{-1}(J_i)$, the subschemes of the curve defined by the points in $(x\circ\phi)_{K_i}$ and in x_{J_i}, respectively, are equal. So $D_{(x\circ\phi)_{K_i}} = D_{x_{J_i}}$, and thus $\beta_i = \tilde\beta_i$ as desired.

For (d), note that the equality $c_\phi = (c_{\phi_i})_{i\in I} \circ (c_{\psi\phi|X^{(\phi)}})$ represents the fact that splitting a relative disk is an associative operation. Let $(x,\beta) \in L^+_{X^K}G(R)$ over $X^{(\phi)}$, meaning the relative disk D_x completely separates into disjoint components $D_{x_{K_j}}$ for each $j \in J$. The left-hand side, c_ϕ, splits the group element $\beta : D_x \to G$ into a J-indexed tuple of maps $(\beta_j : D_{x_{K_j}} \to G)_{j\in J}$. The right-hand side performs this splitting in two stages. First, $c_{\psi\phi}$ groups the domain K by the fibers of $\psi\phi$ (which are the sets K_i), splitting β into an I-indexed tuple $(\beta_i : D_{x_{K_i}} \to G)_{i\in I}$. Second, the map $(c_{\phi_i})_{i\in I}$ takes each component β_i and splits it further along the fibers of $\phi_i : K_i \to J_i$, yielding $((\beta_{i,j})_{j\in J_i})_{i\in I}$. Because the set K partitions identically in both cases ($K = \coprod_{j\in J} K_j = \coprod_{i\in I}\coprod_{j\in J_i} K_j$), we have

$$\coprod_{j\in J} D_{x_{K_j}} = \coprod_{i\in I}\left(\coprod_{j\in J_i} D_{x_{K_j}}\right)$$

and thus $\prod_{j\in J} G(D_{x_{K_j}}) \simeq \prod_{i\in I}\prod_{j\in J_i} G(D_{x_{K_j}})$. Thus, the one-step splitting and the two-step splitting produce the exact same collection of local maps, proving the equality. □

Example 4.2.6. Lets unravel Theorem 4.2.5 for small $|I|$ and $|J|$. For $\{1,2\} \to \{1\}$, which induces the diagonal map $\Delta : X \to X^2$, (2) gives a canonical isomorphism $\Delta : \mathrm{Gr}_X \simeq \mathrm{Gr}_{X^2|\Delta}$.

(2) applied to the flip map $\sigma : \{1,2\} \to \{1,2\}$ gives a canonical isomorphism $\Delta(\sigma) : \mathrm{Gr}_{X^2} \to \mathrm{Gr}_{X^2}$ over $X^2 \to X^2$. (3) applied to $\mathrm{id} : \{1,2\} \to \{1,2\}$ gives a canonical isomorphism $c : \mathrm{Gr}_{X^2|X^2\setminus\Delta} \simeq (\mathrm{Gr}_X \times \mathrm{Gr}_X)_{|X^2\setminus\Delta}$. (c) applied to $\{1,2\} \xrightarrow{\sigma} \{1,2\} \xrightarrow{\mathrm{id}} \{1,2\}$ and (d) to $\{1,2\} \xrightarrow{\mathrm{id}} \{1,2\} \xrightarrow{\sigma} \{1,2\}$ says that the isomorphism c is compatible with $\Delta(\sigma)$ and the flip-map on $\mathrm{Gr}_X \times \mathrm{Gr}_X$. This is actually enough to construct the commutativity-constraint on Sat_G.

Remark 4.2.7. Theorem 4.2.5 can be neatly organized into a statement over the Ran-space. See [Zhu16, Section 3.3] for references.

The convolution Hecke stack interacts well with the splitting isomorphism c_ϕ, as captured in the next theorem:

Theorem 4.2.8. With the notation from Construction 4.2.3 and Theorem 4.2.5, the restrictions

$$\left(\textstyle\prod_{k=1}^n \mathrm{Hck}_{I_k}\right)_{|X^{(\phi)}} \xleftarrow{\pi_\phi} (\mathrm{Hck}^{\mathrm{conv}}_{I_1,\dots,I_n})_{|X^{(\phi)}} \xrightarrow{m_\phi} \mathrm{Hck}_{X^I|X^{(\phi)}}$$

are isomorphisms and there is a factorization $\pi_\phi = c_\phi \circ m_\phi$ over the disjoint locus $X^{(\phi)}$.

Proof. As in the proof of Theorem 4.2.5, over the disjoint locus $X^{(\phi)}$, the relative divisors corresponding to different I_j are strictly disjoint, so $D_x \simeq \coprod_{j=1}^n D_{x_{I_j}}$. When we remove the divisor $\Gamma_{x_{I_k}}$ to form the partially punctured disk, we are only puncturing the k-th component of this disjoint union; the other components remain, meaning $D_x^{(I_k)} \simeq D^\circ_{x_{I_k}} \sqcup \coprod_{j\neq k} D_{x_{I_j}}$. Consequently, over $X^{(\phi)}$, we have

$$L^{(I_k)}_{X^I}G_{|X^{(\phi)}} \simeq L_{X^{I_k}}G \times \prod_{j\neq k} L^+_{X^{I_j}}G$$

and thus by universality of colimits,

$$B(L^+_{X^I}G) \simeq \prod_{j=1}^n B(L^+_{X^{I_j}}G) \quad\text{and}\quad B(L^{(I_k)}_{X^I}G) \simeq B(L_{X^{I_k}}G) \times \prod_{j\neq k} B(L^+_{X^{I_j}}G).$$

Write $B_j^+ = B(L^+_{X^{I_j}}G)$ and $B_j = B(L_{X^{I_j}}G)$, and

$$P_m = B(L^+_{X^I}G) \times_{B(L^{(I_1)}_{X^I}G)} \cdots \times_{B(L^{(I_m)}_{X^I}G)} B(L^+_{X^I}G)$$

Note that $P_n = \mathrm{Hck}^{\mathrm{conv}}_{I_1,\dots,I_n}$. From above, we have $B(L^+_{X^I}G) \simeq \prod_{j=1}^n B_j^+$ and $B(L^{(I_k)}_{X^I}G) \simeq B_k \times \prod_{j\neq k} B_j^+$. We claim that over $X^{(\phi)}$,

$$P_m \simeq \left(\prod_{k=1}^m \mathrm{Hck}_{X^{I_k}}\right) \times \left(\prod_{j=m+1}^n B_j^+\right)$$

We proceed inductively. If $m = 0$, the empty fiber product is simply $P_0 = B(L^+_{X^I}G) \simeq \prod_{j=1}^n B_j^+$. Next, compute pointwise that

$$\begin{aligned} P_m \simeq P_{m-1} \times_{B(L^{(I_m)}_{X^I}G)} B(L^+_{X^I}G) &\simeq \left(\prod_{k=1}^{m-1} \mathrm{Hck}_{X^{I_k}} \times \prod_{j=m}^n B_j^+\right) \times_{\left(B_m \times \prod_{j\neq m} B_j^+\right)} \left(\prod_{j=1}^n B_j^+\right) \\ &\simeq \left(\prod_{k=1}^m \mathrm{Hck}_{X^{I_k}}\right) \times \left(\prod_{j=m+1}^n B_j^+\right) \end{aligned}$$

which gives the desired equivalence for $m = n$.

By unravelling the definitions, the factorization $\pi_\phi = c_\phi \circ m_\phi$ over $X^{(\phi)}$ follows. Consequently, m_ϕ is an equivalence over $X^{(\phi)}$ as well. □

4.3 Suaveness: The relative Satake categories Sat_{X^I}

In this section we will introduce the *relative Satake categories* Sat_{X^I}. The LG_{X^I}-equivariant perverse sheaves will not suffice; we have to impose a "Universal locally acyclic" condition (ULA), which geometrically is reminiscent to the requirement of a "flat family," but for constructible sheaves. Recent work due to Liu-Zheng, and Heyer-Mann ([HM24]) interprets the ULA-condition in purely categorical terms, which we adapt here.

Recall 4.3.1. To every three-functor formalism $D : \mathrm{Corr}(\mathcal{C}, \mathcal{C}_0) \to \mathrm{Cat}_\infty$ and object $S \in \mathcal{C}$ there is a $(\infty, 2)$-*category of kernels* denoted $K_{D,S}$ that can informally be described as follows:

- The object of $K_{D,S}$ are morphisms $X \to S$ in $\mathcal{C}_0$.
- The category of morphisms between $Y, X \in K_{D,S}$ is given by

$$\mathrm{Fun}_{K_{D,S}}(Y, X) \simeq D(X \times_S Y).$$

- If $M : Y \to X$ and $N : Z \to Y$ are morphisms, then the composite $M \circ N : Z \to X$ is given by

$$M \circ N \simeq \pi_{XZ!}(\pi^*_{XY} M \otimes \pi^*_{YZ} N) \in D(X \times_S Z).$$

Generally, given a map f in an 2-category, we can ask for the existence of a map g and 2-morphisms $\eta : \mathrm{id} \to gf$ and $\epsilon fg \to \mathrm{id}$ satisfying the usual triangle-identies for an adjunction[3]. If such datum exists, f is said to be a *left adjoint morphism.*

Let $f : Y \to X$ be a map in $\mathcal{C}_0$. An object $P \in D(Y) \simeq D(X \times_X Y) \simeq \mathrm{Fun}_{K_{D,X}}(Y, X)$ viewed as map $P : Y \to X$ is f-*suave* (or *ULA wrt.* f) if P is a left adjoint morphism. The full subcategory of f-suave objects is denoted $\mathrm{Suave}_f(Y) \subset D(Y)$.

The following two results are taken from [Zhu16, Appendix A.2.]. Part 6 is the "key" property of suaveness.

Lemma 4.3.2. (1) $A \in D(S)$ is id_S-suave if and only if A is dualizable.

(2) If $f : Y \to X$ is a proper map over S, and $A \in D(Y)$ is $(Y \to S)$-suave, then $f_*(A)$ is $(X \to S)$-suave.

[3] To be precise, we would need to specify a 2-simplex witnessing the commutativity.

(3) If $Y \to X$ is a smooth map over S, then $A \in D(X)$ is $(X \to S)$-suave if and only if $f^*(A)$ is $(Y \to S)$-suave.
(4) If $f_i : X_i \to S$ are maps and $A_i \in D(X_i)$ are f_i-suave for $i = 1, 2$, then $A_1 \boxtimes_S A_2$ is $f_1 \times_S f_2$-suave.
(5) If $f_i : X_i \to S_i$ are maps and $A_i \in D(X_i)$ are f_i-suave for $i = 1, 2$, then $A_1 \boxtimes A_2$ is $(f_1 \times f_2)$-suave.
(6) Let $f : X \to S$ be suave (i.e. $1 \in D(X)$ is f-suave) and let $g : S' \to S$ be a !-able map. $f' : X' \simeq X \times_S S' \to S'$ and $g' : X' \to X$ denotes the base-change maps. If $A \in D(X)$ is f-suave, then the canonical map

$$g'^*(A) \otimes f'^* g^!(-) \xrightarrow{\text{unit}} g'^! g'_!(g'^*(A) \otimes f'^* g^!(-)) \underset{\simeq}{\xrightarrow{\text{proj.}}} g'^!(A \otimes g'_! f'^* g^!(-))$$
$$\underset{\simeq}{\xrightarrow{\text{base.}}} g'^!(A \otimes f^* g_! g^!(-)) \xrightarrow{\text{counit}} g'^!(A \otimes f^*(-))$$

is an equivalence.

Proof. (4) and (5) follow from the fact that a functor of $(\infty, 2)$-categories preserves left adjoint morphisms.
(1), (2), (3), and (6) are all found in [HM24]. Specifically, see Example 4.4.3, Lemma 4.5.16, Lemma 4.5.13 and Remark 4.5.15.(ii) of *loc. cit.* □

Proposition 4.3.3. Let X is a complex algebraic variety. If $A \in D(X, \Lambda)$ is dualizable then it is locally perfect (i.e., there is a cover $\{U_i \to X\}$ such that $A_{|U_i}$ lies in $\langle 1 \rangle \subset D(U_i, \Lambda)$[4]). If $A \in D(X, \Lambda)$ is locally perfect, then the following conditions hold:

(i) There is a cover $\{U_i \to X\}$ such that for all i, $A_{|U_i} \in D(U_i, \Lambda)$ is bounded.

(ii) For all $k \in \mathbb{Z}$, $\mathcal{H}^k(A)$ is a local system.

(iii) For every point $i_x : * \to X$, the stalk $i_x^*(A) \in D(*, \Lambda) \simeq \mathrm{Mod}_\Lambda$ is perfect (i.e. lies in $\langle 1 \rangle$).

Proof. We first prove that if A is locally perfect, then it satisfies (i), (ii), and (iii). By definition, there is an open cover $\{U_i \to X\}$ such that $A_{|U_i}$ lies in $\langle 1 \rangle \subset D(U_i, \Lambda)$. For (i), the constant sheaf $1 \simeq \underline{\Lambda}$ is concentrated in degree 0, hence bounded. The operations of shifting, forming fiber sequences, and taking retracts preserve boundedness. Therefore, every object in $\langle 1 \rangle$ is bounded. Since $A_{|U_i} \in \langle 1 \rangle$, it is bounded.

For (ii), note that the cohomology sheaves of $\underline{\Lambda}$ are trivial except in degree 0, where it is the constant local system Λ. Taking shifts and retracts preserves local systems. Because fiber sequences induce long exact sequences in cohomology, and the kernel/cokernel of morphisms of local systems are local systems, the property of having local systems as cohomology sheaves is preserved. Thus, all objects in $\langle 1 \rangle$ have locally constant cohomology sheaves. Since this holds on every U_i, $\mathcal{H}^k(A)$ is a local system on X for all k.

For (iii), consider $U_i \subset X$ with $x \in U_i$. The pullback functor i_x^* is exact and symmetric monoidal, so it maps the thick subcategory generated by $\langle \underline{\Lambda}_{U_i} \rangle$ into $\langle \Lambda \rangle \subset D(*, \Lambda) \simeq \mathrm{Mod}_\Lambda$, which is exactly the perfect complexes (finite complexes of finitely generated projective modules).

Assume $A \in D(X, \Lambda)$ is dualizable. Locally on a sufficiently small a contractible open set U, the category $D(U, \Lambda)$ is compactly generated by the constant sheaf $\underline{\Lambda}_U$. Consequently, $A_{|U} \simeq \operatorname{colim} P_i$, where each $P_i \in \langle \underline{\Lambda}_U \rangle$. Becuase $A_{|U}$ is dualizable, $\underline{\mathrm{Hom}}(A_{|U}, M) \simeq A_{|U}^\vee \otimes M$ for any $M \in D(U, \Lambda)$, so

$$\mathrm{Map}(A_{|U}, M) = H^0(p_* \underline{\mathrm{Hom}}(A_{|U}, M)) \simeq H^0(p_*(A_{|U}^\vee \otimes M)).$$

As U is compact, p_* preserves filtered colimits, $A_{|U}^\vee \otimes -$ preserves all colimits, and H^0 preserves filtered colimits, making $A_{|U}$ compact. Thus

$$\mathrm{Map}(A_{|U}, A_{|U}) \simeq \mathrm{Map}(A_{|U}, \operatorname{colim} P_i) \simeq \operatorname{colim} \mathrm{Map}(A_{|U}, P_i)$$

so $\mathrm{id}_{A_{|U}}$ factors through a finite stage, say $g \circ f = id_{A_{|U}}$ for $f : A_{|U} \to P_k$ and $g : P_k \to A_{|U}$. It follows that $A_{|U}$ is a retract of a perfect complex, making it perfect. □

[4]Here $\langle 1 \rangle$ denotes the *thick subcategory* of 1, which is defined as the smallest subcategory containing 1 and closed under shifts, finite colimits, and retracts.

Remark 4.3.4. The converse to the above statements turn out to be true, but we will only need the forward direction.

Theorem 4.3.5. Let $f : X \to S$ and $S \to *$ be smooth maps of schemes of finite type over $\mathbb{C}$ and let $\iota : S' \to S$ be a closed immersion. $f' : Z \simeq X \times_S S' \to S'$ and $i : Z \to X$ denotes the base-change maps. Let $j : U \to X$ be the open complement to Z. Assume that S' is of pure codimension 1 in S, and that $j : U \to X$ is an affine morphism (e.g. an effective Cartier divisor). If $A \in D^b_c(X, \Lambda)$ is f-suave and $j^*(A) \in D^b_c(U, \Lambda)$ is perverse, then there is an isomorphism

$$i^!(A)[1] \simeq i^*(A)[-1]$$

and $i^!(A)[1] \simeq i^*(A)[-1]$ is perverse and f'-suave. Also, A is perverse and $A \simeq j_{!*}(j^*(A))$.

Proof. First, note that suaveness is (unconditionally) preserved by base-change (and shifts), see [HM24, Lemma 4.4.8.(i)].

Now, since $\operatorname{codim}(D, S) = 1$, plugging the tensor-unit into Lemma 4.3.2.(6) results the identification

$$i^!(A) \xleftarrow{\simeq} i^*(A) \otimes f'^* g^!(1) \xrightarrow{\simeq} i^*(A) \otimes f'^*(1[-2]) \simeq i^*(A)[-2]$$

so $i^!(A)[1] \simeq i^*(A)[-1]$. Since $j : U \to X$ is affine, $j_!$ is left t-exact (for the perverse t-structure), and $i^!$ is (unconditionally) left t-exact, so $i^! j_! j^*(A)$ is concentrated in non-negative perverse degrees (see [Ach21, Proposition 3.1.4 and Theorem 3.5.8]). Applying $i^!$ to the standard fiber sequence $j_! j^*(A) \to A \to i_* i^*(A)$ gives a fiber sequence

$$i^! j_! j^*(A) \longrightarrow i^*(A)[-2] \simeq i^!(A) \longrightarrow i^*(A)$$

which implies that ${}^p\mathcal{H}^k(i^*(A)) \simeq {}^p\mathcal{H}^{k-2}(i^*(A))$ for $k \le -2$. But since $i^*(A)$ is a bounded, it follows that ${}^p\mathcal{H}^k(i^*(A)) \simeq 0$ for $k \le -2$. Hence $i^*(A) \in {}^pD^{\ge -1}(Z, \Lambda)$.

Similarly, the functor j_* is right t-exact and so is i^*, so $i^* j_* j^!(A)$ is concentrated in non-positive perverse degrees. Applying i^* to the standard fiber sequence $i_* i^!(A) \to A \to j_* j^*(A)$ gives a fiber sequence

$$i^*(A)[-2] \simeq i^!(A) \longrightarrow i^*(A) \longrightarrow i^* j_* j^!(A)$$

which, as before, implies that ${}^p\mathcal{H}^{k-2}(i^*(A)) \simeq {}^p\mathcal{H}^k(i^*(A))$ for $k \ge -2$, so $i^*(A) \in {}^pD^{\le 1}(Z, \Lambda)$.

Consequently, $K \simeq i^!(A)[1] \simeq i^*(A)[-1]$ is perverse. If $U \subset X$ is dense, then Lemma 3.3.4. of *loc. cit.* implies that A is perverse and $A \simeq j_{!*}(j^*(A))$. If $U \subset X$ is not dense, let $W \subset Z$ be the complement of $\overline{U} \backslash U \subset Z$ in Z, and note that $W \subset X$ is open, so $i^*(A)_{|W} \simeq A_{|W} \simeq i^!(A)_{|W}$, so $A_{|W} \simeq 0$, and the same lemma applies. □

Definition 4.3.6. The *relative Satake category* Sat_{X^I} is the full subcategory of $\mathrm{Perv}_{L^+G_{X^I}}(\mathrm{Gr}_{X^I}, \Lambda) = \mathrm{Perv}(\mathrm{Hck}_{X^I}, \Lambda)$ spanned by objects that are q'_I-suave.

Remark 4.3.7.
1. Since $L^+_{X^I} G \to X^I$ is smooth, Sat_{X^I} is equivalently the category of $L^+_{X^I} G$-equivariant perverse on Gr_{X^I} that are q_I-suave.
2. If $\phi : J \to I$ is a surjection, then by factoring ϕ as surjections $J = K_0 \to K_1 \cdots \to K_n = I$ where $n = |J| - |I|$ and $|K_i| + 1 = |K_{i+1}|$, there is map

$$\Delta(\phi)^\bullet \simeq \Delta(\phi)^*[|I| - |J|] \simeq \Delta(\phi)^![|J| - |I|] : \mathrm{Sat}_{X^J} \to \mathrm{Sat}_{X^I}$$

comming from repeated applications of Theorem 4.3.5.

4.4 Fusion and Convolution.

In this section we explain how to embed Sat_G into Sat_X, and how to make the convolution product on Sat_X (and therefore also on Sat_G) symmetric monoidal. Informally, the idea is to embed the curve X into $X \times X$ via the diagonal and then let the torsors “fuse” together when two points collide.

Notation 4.4.1. To formalize the collision of points, we fix the following notation for the remainder of this section. Let I be a finite set.

- Let I_1 and I_2 denote two identical, disjoint copies of I, and set $J = I_1 \sqcup I_2$.
- Let $\Delta : I \sqcup I \to I$ be the map induced by the identity id_I on both factors. This induces the diagonal embedding $\Delta : X^I \to X^J \simeq X^I \times X^I$.
- Let $\sigma : J \to J$ be the map that swaps the two factors I_1 and I_2. This induces the flip automorphism $\sigma : X^J \to X^J$ defined by $(x,y) \mapsto (y,x)$, which clearly satisfies $\sigma \circ \Delta = \Delta$.
- Let $U \subset X^J$ denote the open complement to the diagonal $\Delta(X^I)$. We denote the corresponding open immersion by $j : U \hookrightarrow X^J$. By construction, U is stable under the flip automorphism, so $\sigma(U) = U$.

Lemma 4.4.2. The outer rectangle in the diagram below is cartesian

$$\begin{tikzcd}
\mathrm{Hck}^{\mathrm{conv}}_{X^I,2} \arrow[r, "m"] \arrow[d] \arrow[dr, phantom, "\mathrm{cart}"] & \mathrm{Hck}_{X^I} \arrow[r] \arrow[d, "\Delta"] \arrow[dr, phantom, "\mathrm{cart}"] & X^I \arrow[d] \\
\mathrm{Hck}^{\mathrm{conv}}_{I,I} \arrow[r, "m_\Delta"'] & \mathrm{Hck}_{X^J} \arrow[r] & X^J \simeq X^I \times X^I
\end{tikzcd}$$

Consequently, the multiplication map for the group object Hck_{X^I} in $\mathrm{Corr}(\mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{Aff}_{\mathbb{C}})_{/X^I})$ factors as

$$\left[\begin{tikzcd}[column sep=small]
& \mathrm{Hck}^{\mathrm{conv}}_{X^I,2} \arrow[dl, "p"'] \arrow[dr, "m"] & \\
\mathrm{Hck}_{X^I} \times \mathrm{Hck}_{X^I} & & \mathrm{Hck}_{X^I}
\end{tikzcd}\right] \simeq \left[\begin{tikzcd}[column sep=small]
& \mathrm{Hck}_{X^I} \arrow[dl, "\Delta"'] \arrow[dr, "\mathrm{id}"] & \\
\mathrm{Hck}_{X^J} & & \mathrm{Hck}_{X^I}
\end{tikzcd}\right] \circ \left[\begin{tikzcd}[column sep=small]
& \mathrm{Hck}^{\mathrm{conv}}_{I,I} \arrow[dl, "\pi_\Delta"'] \arrow[dr, "m_\Delta"] & \\
\mathrm{Hck}_{X^I} \times \mathrm{Hck}_{X^I} & & \mathrm{Hck}_{X^J}
\end{tikzcd}\right]$$

and thus there is a canonical equivalence

$$A \star B \simeq m_! p^*(A \boxtimes B) \simeq \Delta^* m_{\Delta !} \pi_\Delta^*(A \boxtimes B).$$

Proof. Note that by definition of the partially punctured loops, pulling back along the diagonal $\Delta : X^I \to X^J$ identifies the formal disks:

$$L^+_{X^J} G \times_{X^J} X^I \simeq L^+_{X^I} G, \quad L^{(I_1)}_{X^J} G \times_{X^J} X^I \simeq L_{X^I} G, \quad \text{and} \quad L^{(I_2)}_{X^J} G \times_{X^J} X^I \simeq L_{X^I} G.$$

Since colimits are universal and limits commute with limits, applying this to the fiber product definition of the factorizable Hecke stack yields $\mathrm{Hck}^{\mathrm{conv}}_{I,I} \times_{X^J} X^I \simeq \mathrm{Hck}^{\mathrm{conv}}_{X^I,2}$, proving the cartesian square. The functorial equivalence for $A \star B$ follows immediately from base change. ☐

Warning 4.4.3. The formula above does unfortunately not preserve perversity, but only after shifting by an appropriate amount. The functor Δ^* should be shifted by $|I| - 2|I| = -|I|$, and the functor π_Δ^* should be shifted at each of its finite stages.

Lemma 4.4.4. The convolution product $- \star - : D^b_c(\mathrm{Hck}_{X^I}) \times D^b_c(\mathrm{Hck}_{X^I}) \to D^b_c(\mathrm{Hck}_{X^I})$ restricts to $\mathrm{Sat}_{X^I} \times \mathrm{Sat}_{X^I} \to \mathrm{Sat}_{X^I}$.

Proof. With the notation from Lemma 4.4.2, it suffices to show that $- \boxtimes -$, π_Δ^*, $m_{\Delta !}$, and Δ^* preserves suaveness and perversity (up to shifts). $- \boxtimes -$ preserves suaveness by Lemma 4.3.2.(4) and perversity by [Ach21, Lemma 3.2.1].

☐

Construction 4.4.5 (The symmetric monoidal structure on Sat_{X^I})**.** The goal is to construct a natural isomorphism of functors $\beta_{A,B} : A \star B \xrightarrow{\sim} B \star A$ that satisfies the symmetry and hexagon axioms. The only new data going into the construction is the canonical symmetric monoidal constraint for the exterior tensor product

$$A \boxtimes B \xrightarrow{\quad \tau \quad} \Delta(\sigma)^*(B \boxtimes A)$$

where σ is the flip automorphism defined above.

By abuse of notation, we will write Δ for the map $\Delta(\Delta)$ from Theorem 4.2.5. Over the disjoint locus U, the factorization property (Theorem 4.2.5) and the convolution equivalence (Theorem 4.2.8) allow us to identify:

$$(m_{\Delta !}\pi_{\Delta}^{*}(A \boxtimes B))_{|U} \simeq (m_{\Delta !}m_{\Delta}^{*}c_{\Delta}^{*}(A \boxtimes B))_{|U} \simeq (c_{\Delta}^{*}(A \boxtimes B))_{|U}.$$

The right-hand side is called the *fusion product*. This computation asserts that, away from the diagonal, the convolution product is canonically isomorphic to the usual exterior product.

By Theorem 4.3.5, there is a canonical equivalence

$$A \star B = \Delta^{*}m_{\Delta !}\pi_{\Delta}^{*}(A \boxtimes B) \simeq \Delta^{*}j_{!*}((c_{\Delta}^{*}(A \boxtimes B))_{|U}).$$

Both $j_{!*}$ and c_{Δ}^{*} respect the flip map $\Delta(\sigma)^{*}$. Therefore, applying the exterior symmetry τ induces the desired natural equivalence $\beta_{A,B}$:

$$A \star B = \Delta^{*}(j_{!*}c_{\Delta}^{*}(A \boxtimes B)_{|U}) \xrightarrow{\beta_{A,B}} \Delta^{*}(j_{!*}c_{\Delta}^{*}\Delta(\sigma)^{*}(B \boxtimes A)_{|U}) \simeq \Delta^{*}\Delta(\sigma)^{*}(j_{!*}c_{\Delta}^{*}(B \boxtimes A)_{|U}) \simeq B \star A$$

where the final identification uses the fact that $\sigma \circ \Delta = \Delta$. By the fully faithfulness of $j_{!*}$, the symmetry condition $\beta_{B,A} \circ \beta_{A,B} = \mathrm{id}_{A \star B}$ and the hexagon axiom can be checked on the open subset U, where they are trivially satisfied by the symmetry of the exterior product.

4.5 The fully faithful functor $\mathrm{Sat}_G \to \mathrm{Sat}_{\mathbb{A}^1}$

In this subsection, we assume that the base curve is the affine line, $X = \mathbb{A}^1$. Under this assumption, the geometry of the relative Hecke stack simplifies drastically. Because the affine line is an algebraic group under addition, it acts on itself by translation. This allows us to canonically identify the formal neighborhood of any point $x \in \mathbb{A}^1$ with the standard formal disk at the origin.

Lemma 4.5.1 (Global Splitting)**.** Let $X = \mathbb{A}^1$. There are canonical global isomorphisms of ind-group schemes and stacks over $\mathbb{A}^1$:

$$L_{\mathbb{A}^1}G \simeq \mathbb{A}^1 \times LG, \quad L^{+}_{\mathbb{A}^1}G \simeq \mathbb{A}^1 \times L^{+}G, \quad \text{and} \quad \mathrm{Hck}_{\mathbb{A}^1} \simeq \mathbb{A}^1 \times \mathrm{Hck}_G.$$

Proof. Let $\mathbb{A}^1 \simeq \mathrm{Spec}(\mathbb{C}[z])$. For any R-point $x \in \mathbb{A}^1(R)$, translation by x provides a canonical isomorphism of formal disks $\hat{\Gamma}_x \xrightarrow{\sim} \hat{D}_R$, $\quad z \mapsto t + x$. An R-point of $L_{\mathbb{A}^1}G$ is a pair (x, β) where $x \in \mathbb{A}^1(R)$ and $\beta \in G(\hat{\Gamma}_x^{\circ})$. Precomposing β with the translation isomorphism maps it canonically to an element of $G(D_R^{\circ}) = LG(R)$. This yields the equivalence of ind-group schemes $\Phi : \mathbb{A}^1 \times LG \simeq L_{\mathbb{A}^1}G$. The exact same translation logic applies to the positive loop group, yielding $\Phi : \mathbb{A}^1 \times L^{+}G \simeq L^{+}_{\mathbb{A}^1}G$. To show the last equivalence, we must show that Φ and Φ^{+} intertwines the actions in the sense that the diagram

$$\begin{tikzcd}
\mathbb{A}^1 \times (L^{+}G \times LG \times L^{+}G) \arrow[r, "\mathrm{id}_{\mathbb{A}^1} \times (\alpha_l, \alpha_r)"] \arrow[d, "(\Phi^{+}, \Phi, \Phi^{+})"'] & \mathbb{A}^1 \times LG \arrow[d, "\Phi"] \\
L^{+}_{\mathbb{A}^1}G \times_{\mathbb{A}^1} L_{\mathbb{A}^1}G \times_{\mathbb{A}^1} L^{+}_{\mathbb{A}^1}G \arrow[r, "(\alpha_l, \alpha_r)"'] & L_{\mathbb{A}^1}G
\end{tikzcd}$$

commutes. Using the formulas $\Phi^{+}(x, g(t)) = (x, g(z - x))$ and $\Phi(x, h(t)) = (x, h(z - x))$, we can trace the R-point $(x, g_1(t), h(t), g_2(t))$ right-down and down-right to get $(x, g_1(z - x)h(z - x)g_2(z - x))$. □

Let $\mathrm{pr}_2 : \mathrm{Hck}_X \simeq \mathbb{A}^1 \times \mathrm{Hck}_G \to \mathrm{Hck}_G$ denote the projection.

Theorem 4.5.2. The functor $(-)_{\mathbb{A}^1} : \mathrm{Sat}_G \longrightarrow \mathrm{Sat}_{\mathbb{A}^1}, A \mapsto \mathrm{pr}_2^{*} A[1]$ is fully faithful.

Proof. First, we claim that the canonical map $1 \to \mathrm{pr}_{2*}\,\mathrm{pr}_2^{*}(1) \simeq \mathrm{pr}_{2*}(1)$ is an equivalence. We check on stalks: Let $i_x : * \to \mathrm{Hck}_G$ be the inclusion. By smooth base-change, the map above becomes $1 \to i_x^{*}\,\mathrm{pr}_{2*}(1) \simeq p_{*}(1)$ where $p : \mathbb{A}^1 \to *$ is the canonical map. This computes the (singular) cohomology of (the underlying analytic topology of) $\mathbb{A}^1$, which is just 1. Now, if $A \in \mathrm{Sat}_G$, then the projection formula gives

$$\mathrm{pr}_{2*}\,\mathrm{pr}_2^{*}(A) \simeq \mathrm{pr}_{2*}\,\mathrm{pr}_2^{*}(A \otimes 1) \simeq A \otimes \mathrm{pr}_{2*}(1) \simeq A,$$

showing pr_{2*} is fully faithful. The shift by 1 is to there to preserve perversity. □

4.6 Symmetric monoidality of the fiber functor

Let $\mathrm{Loc}^{\mathrm{gr}}(X^I)$ denote the category of graded local systems on X^I[5]. The symmetric monoidal structure is by Koszul sign-rule.

Theorem 4.6.1. There is a functor

$$\mathcal{H}^* : \mathrm{Sat}_{X^I} \longrightarrow \mathrm{Loc}^{\mathrm{gr}}(X^I), \quad \mathcal{A} \mapsto \bigoplus_{i\in\mathbb{Z}} \mathcal{H}^i(q_{I*}(A))$$

which is (strong) symmetric monoidal. I.e., there is a commutative diagram

$$\begin{tikzcd} \mathcal{H}^*(A \star B) \arrow[r, "\simeq", "\beta_{A,B}"'] \arrow[d, "\simeq"'] & \mathcal{H}^*(B \star A) \arrow[d, "\simeq"] \\ \mathcal{H}^*(A) \otimes \mathcal{H}^*(B) \arrow[r, "\beta^{\mathrm{gr}}"'] & \mathcal{H}^*(B) \otimes \mathcal{H}^*(A) \end{tikzcd}$$

Furthermore, for every point $i_x : \mathrm{Spec}(\mathbb{C}) \to \mathbb{A}^1$, the diagram

$$\begin{tikzcd} \mathrm{Sat}_G \arrow[r, "(-)_{\mathbb{A}^1}"] \arrow[d, "H^*"'] & \mathrm{Sat}_{\mathbb{A}^1} \arrow[d, "\mathcal{H}^*"] \\ \mathrm{Mod}^{\mathrm{gr}}_\Lambda & \mathrm{Loc}^{\mathrm{gr}}(\mathbb{A}^1, \Lambda) \arrow[l, "i_x^*"] \end{tikzcd}$$

commutes. In particular, the fiber functor H^* promotes to a (strong) symmetric monoidal functor.

Proof. Since A q_I-suave, it follows that $q_I(A)$ is id_{X^I}-suave, so all $\mathcal{H}^i(A)$ are local systems, making the functor $\mathcal{H}^*$ well-defined (see Lemma 4.3.2.(3) and Proposition 4.3.3).

For strong monoidality, we use the fusion formula together with the Künneth formula. Recall that there are cartesian squares

$$\begin{tikzcd} \mathrm{Hck}_{X^J|U} \arrow[r, "j'"] \arrow[d, "q'_J"'] \arrow[dr, phantom, "\mathrm{cart}"] & \mathrm{Hck}_{X^J} \arrow[d, "q_J"] \\ U \arrow[r, "j"'] & X^I \times X^I \end{tikzcd} \qquad \begin{tikzcd} (\mathrm{Hck}_{X^I} \times \mathrm{Hck}_{X^I})_{|U} \arrow[r, "\tilde{j}"] \arrow[d, "(q_I\times q_I)'"'] \arrow[dr, phantom, "\mathrm{cart}"] & \mathrm{Hck}_{X^I} \times \mathrm{Hck}_{X^I} \arrow[d, "q_I\times q_I"] \\ U \arrow[r, "j"'] & X^I \times X^I \end{tikzcd}$$

and that the isomorphism c_Δ satisfies $(q_I \times q_I)' c_\Delta = q_J$ (here $J = I \sqcup I$). Thus

$$\begin{aligned} j^* q_{J*} j_{!*}(c_\Delta^*(A \boxtimes B)_{|U}) &\simeq q'_{J*} j'^* j_{!*}(c_\Delta^*(A \boxtimes B)_{|U}) \\ &\simeq (q_I \times q_I)'_* c_{\Delta*} c_\Delta^*(A \boxtimes B)_{|U} \\ &\simeq (q_I \times q_I)'_* \tilde{j}^*(A \boxtimes B) \\ &\simeq j^*(q_I \times q_I)_*(A \boxtimes B) \\ &\simeq j^*(q_I(A) \boxtimes q_I(B)), \end{aligned}$$

so $q_{J*} j_{!*}(c_\Delta^*(A \boxtimes B)_{|U}) \simeq q_I(A) \boxtimes q_I(B)$. Consequently,

$$\begin{aligned} q_{I*}(A \star B) \simeq q_{I*}\Delta^* j_{!*}(c_\Delta^*(A \boxtimes B)_{|U}) &\simeq \Delta^* q_{J*} j_{!*}(c_\Delta^*(A \boxtimes B)_{|U}) \\ &\simeq \Delta^*(q_I(A) \boxtimes q_I(B)) \\ &\simeq q_I(A) \otimes q_I(B) \end{aligned}$$

[5]The objects are local systems L together with a direct sum decomposition $L \simeq \bigoplus_{k\mathbb{Z}} L_{X^I}$ for local systems L_{X^I}. A map is a map of local systems that respect the grading.

so taking $\oplus_{i\in\mathbb{Z}}\mathcal{H}^i(-)[i]$ gives the desired result. Similarly, applying Δ^* and $\oplus_{i\in\mathbb{Z}}\mathcal{H}^i(-)[i]$ to the diagram

$$\begin{tikzcd} q_{J*}j_{!*}(c_\Delta^*(A\boxtimes B)_{|U}) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & q_{J*}j_{!*}(c_\Delta^*\Delta(\sigma)^*(B\boxtimes A)_{|U}) \arrow[r, "\simeq"] & \Delta(\sigma)^*q_{J*}j_{!*}(c_\Delta^*(B\boxtimes A)_{|U}) \arrow[d, "\simeq"] \\ q_I(A)\boxtimes q_I(B) \arrow[rr, "\simeq"'] & & \sigma^*(q_I(B)\boxtimes q_I(A)) \end{tikzcd}$$

gives the symmetric constraint.

The last diagram follows from base-change.

□

Remark 4.6.2. In Theorem 4.6.1, we established that the graded fiber functor $\mathcal{H}^*:\mathrm{Sat}_G\to\mathrm{Loc}^{\mathrm{gr}}(X^I)$ (and correspondingly the functor to $\mathrm{Mod}_\Lambda^{\mathrm{gr}}$) is symmetric monoidal. However, the standard symmetric monoidal structure on the category of graded modules $\mathrm{Mod}_\Lambda^{gr}$ comes with a Koszul sign rule for its commutativity constraint

$$\beta_{\mathrm{gr}}(v\otimes w)=(-1)^{\deg(v)\deg(w)}(w\otimes v).$$

To apply Tannakian reconstruction (Theorem A.0.2), our target category must be the ordinary, ungraded modules Mod_Λ equipped with the standard symmetry $v\otimes w\mapsto w\otimes v$. The forgetful functor $\mathrm{Mod}_\Lambda^{\mathrm{gr}}\to\mathrm{Mod}_\Lambda$ is unfortunately *not* symmetric monoidal with respect to the Koszul constraint.

To resolve this, we modify the geometric commutativity constraint $\beta_{A,B}$ on Sat_G by a sign. The parity of a simple object $\mathbf{IC}_\lambda\in\mathrm{Sat}_G$ is determined by the complex dimension of its support: $p(\mathbf{IC}_\lambda)\equiv\dim(\mathrm{Gr}_\lambda)\equiv\langle 2\rho,\lambda\rangle\pmod 2)$., the modified commutativity constraint as

$$c_{\mathbf{IC}_\lambda,\mathbf{IC}_{\lambda'}}=(-1)^{\langle 2\rho,\lambda\rangle\langle 2\rho,\lambda'\rangle}\beta_{\mathbf{IC}_\lambda,\mathbf{IC}_{\lambda'}}.$$

for simple objects $\mathbf{IC}_\lambda$ and $\mathbf{IC}_{\lambda'}$. For arbitrary objects $A,B\in\mathrm{Sat}_G$, we write $A\simeq\bigoplus\mathbf{IC}_\lambda^{\oplus m_\lambda}$ and $B\simeq\bigoplus\mathbf{IC}_{\lambda'}^{\oplus m_{\lambda'}}$ by Theorem 3.2.4, and extend the definition of the constraint $c_{A,B}$ component-wise.

With this modified constraint $c_{A,B}$ on Sat_G, the fiber functor $H^*:\mathrm{Sat}_G\to\mathrm{Mod}_\Lambda$ is strong symmetric monoidal functor, summarized by the following commutative diagram:

$$\begin{tikzcd} H^*(A\star B) \arrow[r, "H^*(c_{A,B})"] \arrow[d, "\simeq"'] & H^*(B\star A) \arrow[d] \\ H^*(A)\otimes H^*(B) \arrow[r] & H^*(B)\otimes H^*(A) \end{tikzcd}$$

5 Weight functors and the root datum of $\check{G}_\Lambda$

In this section, we will verify that H^* is actually a fiber functor and that Sat_G is rigid, and thus proving the first two paragraphs of Theorem 1.0.1. In Section 5.1, we identify Sat_T with the the category of (finitely generated) $X_*(T)$-graded Λ-modules $\mathrm{Mod}_\Lambda^{X_*(T)}$[6] and show that the fiber functor H^* factors through this category. In Section 5.2 we use this identification to construct a map $\mathrm{Sat}_G \to \mathrm{Mod}_\Lambda^{X_*(T)} \simeq \mathrm{Sat}_T$, which, once we have shown rigidity and that H^* is a fiber functor, will give a map $\check{T}_\lambda \to \check{G}_\Lambda$ by functoriality of Tannakian reconstruction (see Theorem A.0.3). The argument relies on the theory of "weight functors", attributed to Mirkovic-Vilonen (see [MV18] for the original paper). Finally, in Section 5.4 we prove the assertions in the last paragraph of Theorem 1.0.1 by that the map $\check{T}_\Lambda \to \check{G}_\Lambda$ is a closed immersion and constructing a canonical Borel subgroup $\check{B}_\Lambda \subset \check{G}_\Lambda$ containing $\check{T}_\Lambda$ using Theorem B.2.2.

5.1 Identification of Sat_T

We explain how to identify Sat_T with $\mathrm{Mod}_\Lambda^{X_*(T)}$. Once we have Theorem 5.3.1, this identification can be used to exhibit a closed immersion $\check{T}_\Lambda \to \check{G}_\Lambda$.

Example 5.1.1 (The Geometric Satake Equivalence for a Torus)**.** Assume that $G = T$ is a torus. Because T is abelian, the loop group LT and the positive loop group L^+T are also abelian. The Cartan decomposition, Gr_T is a disjoint union of points $\{t^\lambda\}_{\lambda \in X_*(T)}$. Thus the Hecke stack becomes a disjoint union of classifying stacks

$$\mathrm{Hck}_T \simeq \coprod_{\lambda \in X_*(T)} L^+T \backslash *_\lambda$$

where $*_\lambda$ denotes the point t^λ. Consequently, a perverse sheaf on Hck_T is uniquely determined by a set of sheaves on each $L^+T\backslash *_\lambda$.

In general, for any smooth algebraic group H, there are canonical equivalences

$$\mathrm{Perv}(*/H, \Lambda) \simeq \mathrm{Loc}(*/H, \Lambda)[-\dim H] \simeq \mathrm{Rep}_\Lambda(\pi_1(*/H)) \simeq \mathrm{Rep}_\Lambda(\pi_0(H)).$$[7]

Because $H = L^+T$ is connected, the latter becomes Mod_Λ. Taking the coproduct over all $\lambda \in X_*(T)$, we obtain an equivalence between the Satake category and $X_*(T)$-graded Λ-modules $\mathrm{Sat}_T \simeq \mathrm{Mod}_\Lambda^{X_*(T)}$. We claim that this equivalence is symmetric monoidal. Recall the convolution diagram:

$$\mathrm{Hck}_T \times \mathrm{Hck}_T \xleftarrow{p} L^+T\backslash(LT \times^{L^+T} LT)/L^+T \xrightarrow{m} \mathrm{Hck}_T$$

The target of p decomposes as

$$\mathrm{Hck}_T \times \mathrm{Hck}_T \simeq \coprod_{\mu,\nu \in X_*(T)} (L^+T \times L^+T)\backslash *_{(\mu,\nu)},$$

and the source decomposes as

$$L^+T\backslash(LT \times^{L^+T} LT)/L^+T \simeq \coprod_{\mu,\nu \in X_*(T)} \Delta(L^+T)\backslash *_{(\mu,\nu)} .$$

This holds since T is abelian, so the orbits of $LT \times LT$ under the action of $L^+T \times L^+T \times L^+T$ are strictly indexed by pairs $(\mu,\nu) \in X_*(T) \times X_*(T)$, and the stabilizer of the point (t^μ, t^ν) is exactly the diagonal subgroup $\Delta(L^+T)$. The fiber $p_{\mu,\nu}$ becomes the canonical map of classifying stacks induced by the subgroup inclusion $\Delta(L^+T) \hookrightarrow L^+T \times L^+T$. Because both $\Delta(L^+T)$ and $L^+T \times L^+T$ are connected, their component groups are trivial, and their perverse sheaf categories are canonically identified with Mod_Λ. Under this

[6] Recall that if G is a group, then a category of G-graded modules Mod_Λ^G where an object is a Λ-module M together with an isomorphism $M \simeq \bigoplus_{g\in G} M_g$ and a map $f : M \to N$ is a map of modules that respects the decomposition, i.e. $f(M_g) \subset N_g$. The group-structure on G induces a monoidal structure $\otimes$ on Mod_Λ^G with $(M \otimes N)_g = \bigoplus_{xy=g}(M_x \otimes_\Lambda N_y)$, which is symmetric monoidal if G is abelian. There is an (obvious) forget functor $\mathrm{Mod}_\Lambda^G \to \mathrm{Mod}_\Lambda$.

[7] This is true since the point $*$ admit no non-trivial stratifications; it is certainly not true for general classifying stacks X/H.

identification, the pullback $p^*_{\mu,\nu}$ becomes the identity functor. Hence p^* is an equivalence, allowing us to comput $A \star B$ as $m_!(A \boxtimes B)$.

The multiplication map $m([g_1, g_2]) = [g_1 g_2]$ becomes addition in the cocharacter lattice, $m(\mu,\nu) = \mu + \nu$, so taking stalks yields

$$(A \star B)_\lambda \simeq m_!(A \boxtimes B)_\lambda \simeq \bigoplus_{\mu+\nu=\lambda} (A \boxtimes B)_{(\mu,\nu)} = \bigoplus_{\mu+\nu=\lambda} (A_\mu \otimes B_\nu),$$

which is exactly the λ^{th} graded piece of the tensor product of two $X_*(T)$-graded Λ-modules. Preservation of the symmetric monoidal constraint follows similarly, noting that $m \circ \sigma = m$ (where σ swaps components), which is a direct consequence of T being abelian.

5.2 The Semi-Infinite Orbits S_μ and T_μ

In view of Example 5.1.1 and Theorem A.0.3, we aim to construct a symmetric monoidal functor $F : \mathrm{Sat}_G \to \mathrm{Mod}_\Lambda^{X_*(T)}$. To do so, we must introduce some further geometry on the affine Grassmannian.

The action of L^+G on Gr_G may be restricted along the composite

$$\mathbb{G}_m \xrightarrow{2\rho^\vee} T \longrightarrow G \longrightarrow L^+G,$$

making Gr_G a $\mathbb{G}_m$-space.

Recall 5.2.1 ($\mathbb{G}_m$-spaces)**.** First, some general theory. Let $\mathcal{X}$ be an ∞-topos. If $f : S \to T$ is a map in $\mathcal{X}$, there are adjunctions $f_! \dashv f^* \dashv f_*$

$$\begin{tikzcd} \mathcal{X}_{/S} \arrow[r, shift left=2, "f_!"] \arrow[r, shift right=2, "f_*"'] & \mathcal{X}_{/T} \arrow[l, "f^*" description] \end{tikzcd} \qquad\qquad \begin{array}{l} f_!(X \to S) \simeq (X \to S \to T) \\ f^*(Y \to T) \simeq (Y \times_T S \to S) \end{array}$$

Let $G \in \mathrm{Grp}(\mathcal{X})$ be a group-object. The maps to the left induce the adjunctions to the right

$$* \xrightarrow{\ s\ } BG \simeq */G \xrightarrow{\ p\ } * \qquad \begin{tikzcd} \mathcal{X} \arrow[r, shift left=2, "s_!"] \arrow[r, shift right=2, "s_*"'] & \mathcal{X}_{/BG} \arrow[l, "s^*" description] \arrow[r, shift left=2, "p_!"] \arrow[r, shift right=2, "p_*"'] & \mathcal{X} \arrow[l, "p^*" description] \end{tikzcd}$$

Objects in $\mathcal{X}_{/BG}$ are *G-actions*, and if $\tilde{X} \in \mathcal{X}_{/BG}$, then $X \simeq s^*\tilde{X}$ is the *underlying object* of the G-action $\tilde{X}$. The essential image of $s_!$ (resp. s_*) is the category of *free G-actions* (resp. *cofree G-actions*). The essential image of p^* is the category of *trivial G-actions*, while $p_!(\tilde{X})$ and $p_*(\tilde{X})$ compute the *orbits* and *fixed points*, respectively. For $X \simeq s^*\tilde{X}$ and $Y \simeq s^*\tilde{Y}$, we define the internal hom of G-actions as:

$$\underline{\mathrm{Hom}}^G(X, Y) = p_* \underline{\mathrm{Hom}}(\tilde{X}, \tilde{Y}).$$

We will apply this to $\mathcal{X} \simeq \mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{Aff}_k)$ and $G = \mathbb{G}_m$. The group $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by dilation and inverse dilation, yielding $\mathbb{G}_m$-spaces $\mathbb{A}^1_+$ and $\mathbb{A}^1_-$. If Z is a $\mathbb{G}_m$-space (the underlying object of a $\mathbb{G}_m$-action), we define the attractor, fixed points, and repeller as

$$Z^+ \simeq \underline{\mathrm{Hom}}^{\mathbb{G}_m}(\mathbb{A}^1_+, Z), \quad Z^0 \simeq \underline{\mathrm{Hom}}^{\mathbb{G}_m}(*, Z), \quad \text{and} \quad Z^- \simeq \underline{\mathrm{Hom}}^{\mathbb{G}_m}(\mathbb{A}^1_-, Z),$$

respectively (where s^* has been suppressed in the notation). The maps $* \overset{0}{\to} \mathbb{A}^1 \to *$ and $\mathbb{A}^1 \overset{s \mapsto s}{\leftarrow} \mathbb{G}_m \overset{s \mapsto s^{-1}}{\to} \mathbb{A}^1$ induce maps $Z^0 \overset{q^\pm}{\leftarrow} Z^\pm \overset{q^\pm}{\leftarrow} Z^0$ and $Z^+ \overset{p^+}{\to} Z \overset{p^-}{\leftarrow} Z^-$. The maps i^+ and i^- induce a map $Z^0 \to Z^+ \times_Z Z^-$ which is is not necessarily an isomorphism, however it is both open and closed.

Back to the setting at hand. We consider LG as a $\mathbb{G}_m$-space via the adjoint action

$$\mathrm{Ad}_{2\rho^\vee(s)}(g) = s^{2\rho^\vee} g s^{-2\rho^\vee} \in LG, \quad s \in \mathbb{G}_m.$$

Recall that for a root subgroup element $x_\alpha(f(t))$, the adjoint action acts via weights:

$$\mathrm{Ad}_{2\rho^\vee(s)}(x_\alpha(f(t))) = x_\alpha(s^{\langle \alpha, 2\rho^\vee \rangle} f(t)).$$

Thus, the attractor, fixed points, and repeller for the $\mathbb{G}_m$-action correspond exactly to the spaces where:

$$\langle \alpha, 2\rho^\vee \rangle > 0, \quad \langle \alpha, 2\rho^\vee \rangle = 0, \quad \langle \alpha, 2\rho^\vee \rangle < 0,$$

which are equivalently LB, LT, and LB^-. Similarly, by restricting the adjoint action to $L^+G \subset LG$, we obtain analogous isomorphisms $(L^+G)^+ \simeq L^+B, (L^+G)^0 \simeq L^+T$, and $(L^+G)^- \simeq L^+B^-$ for the positive loop group.

Under this identification, the maps $p^\pm, q^\pm, i^\pm$ from the general theory of $\mathbb{G}_m$-actions do as expected; $p^+ : (LG)^+ \to LG$ is the inclusion $LB \hookrightarrow LG$, the fixed-point inclusion $i^+ : (LG)^0 \to (LG)^+$ is the standard inclusion of the maximal torus $LT \hookrightarrow LB$, and $q^+ : (LG)^+ \to (LG)^0$ corresponds to killing the strictly positive weights, recovering the canonical projection $LB \to LB/LU \simeq LT$.

Note that the $\mathbb{G}_m$-action on LG and L^+G descend to the $\mathbb{G}_m$-action by left multiplication on Gr_G. Miraculously, the attractor, fixed points, and repellers behave well under this quotient:

Theorem 5.2.2. The canonical maps

$$\begin{aligned} X_*(T) \simeq \mathrm{Gr}_T \simeq LT/L^+T \simeq (LG)^0/(L^+G)^0 &\longrightarrow (\mathrm{Gr}_G)^0, \\ \mathrm{Gr}_B \simeq LB/L^+B \simeq (LG)^+/(L^+G)^+ &\longrightarrow (\mathrm{Gr}_G)^+, \\ \mathrm{Gr}_{B^-} \simeq LB^-/L^+B^- \simeq (LG)^-/(L^+G)^- &\longrightarrow (\mathrm{Gr}_G)^-, \end{aligned}$$

$$\begin{tikzcd} \mathrm{Gr}_T \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & \mathrm{Gr}_{B^-} \arrow[d] \\ \mathrm{Gr}_B \arrow[r] & \mathrm{Gr}_G \end{tikzcd}$$

are equivalences, and the diagram to the right is cartesian.

Proof. Let W be the $\mathbb{G}_m$-spaces $\mathbb{A}^1_+$, $\mathbb{A}^1_-$, or $*$. There is a canonical map

$$\underline{\mathrm{Hom}}^{\mathbb{G}_m}(W, LG)/\underline{\mathrm{Hom}}^{\mathbb{G}_m}(W, L^+G) \to \underline{\mathrm{Hom}}^{\mathbb{G}_m}(W, LG/L^+G) \simeq \underline{\mathrm{Hom}}^{\mathbb{G}_m}(W, \mathrm{Gr}_G),$$

which is a monomorphism. Thus it suffices to show it is an effective epimorphism, or equivalently, that $\underline{\mathrm{Hom}}^{\mathbb{G}_m}(W, LG) \to \underline{\mathrm{Hom}}^{\mathbb{G}_m}(W, \mathrm{Gr}_G)$ is an effective epimorphism.

Let $W = *$. Let $x \in (\mathrm{Gr}_G)^0(R)$ be an R-point. We must lift x to an element of LT, possibly on an fpqc-cover. By definition, there is a $\mathbb{G}_m$-equivariant map $x : \mathrm{Spec}(R) \to \mathrm{Gr}_G$, where the action on $\mathrm{Spec}(R)$ is trivial. Pulling back $LG \to \mathrm{Gr}_G$ along x gives a $\mathbb{G}_m$-equivariant principal L^+G-bundle P over $\mathrm{Spec}(R)$. Pick a fpqc-cover $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ and a trivialization $\phi\colon P_{R'} \xrightarrow{\sim} L^+G_{R'}$. The $\mathbb{G}_m$-action on P induce an action a on $LG_{R'}$ given by $a_c(g) = \phi(\mathrm{Ad}_{2\rho^\vee(c)}\,\phi^{-1}(g))$. Using equivariance, one computes the formula $a_c(g) = a_c(1)\,\mathrm{Ad}_{2\rho^\vee(c)}(g)$, so if $h(c) = a_c(1)2\rho^\vee(c)$, then $h(cd) = h(c)h(d)$ (used that $a\mathbb{G}_m \to \mathrm{Aut}(L^+G_{R'})$ is a group map). Consequently, we have a cocharacter $h : \mathbb{G}_m \to L^+G_{R'}$. But since $L^+G = G \ltimes L^{>0}G$, and $L^{>0}G$ is pro-unipotent (so admits no non-trivial cocharacters), h factors through $G_{R'}$ (up to conjugation). In a reductive group (over a commutative ring), any cocharacter can be conjugated étale locally, and therefore fpqc-locally, into a fixed maximal torus. Pick such an fpqc-cover $\mathrm{Spec}(R'') \to \mathrm{Spec}(R')$ and $k \in L^+G(R'')$ such that $k^{-1}h(c)k = \lambda(c)$ lies in $X_*(T_{R''})$. The trivialization ϕ pulls back to a trivialization $\phi : P_{R''} \xrightarrow{\sim} (L^+G)_{R''}$, and we define a new trivialization $\phi' = k^{-1}\cdot\phi$. Again, there is an induced action $a'_c(g) = \phi'(\mathrm{Ad}_{2\rho^\vee(c)}\,\phi'^{-1}(g))$, and it satisfies

$$a'_c(1) = k^{-1}\cdot a_c(k) = k^{-1}a_c(1)\,\mathrm{Ad}_{2\rho^\vee(c)}(g) = \lambda(c)2\rho^\vee(c)^{-1}.$$

Consider the lift $y = \phi'^{-1}(1) \in P_{R''}$ of x. Since x is a fixed point, there is some $h \in L^+G_{R''}$ s.t. $\mathrm{Ad}_{2\rho^\vee(c)}(y) = y\cdot h$. But then

$$h = \phi'(y)h = \phi'(yh) = \phi'(\mathrm{Ad}_{2\rho^\vee(c)}(y)) = a'_c(1),$$

so

$$\mathrm{Ad}_{2\rho^\vee(c)}(y) = y\cdot a'_c(1) = y\lambda(c)2\rho^\vee(c)^{-1},$$

which gives $y^{-1}\,2\rho^\vee(c)\,y = \lambda(c)$. Now, as $2\rho^\vee$ is regular and λ lands in T, we have

$$LT = C_{LG}(2\rho^\vee) \subset yC_{LG}(\lambda)y^{-1} = yLTy^{-1},$$

and thus by maximality of T, $LT = yLTy^{-1}$. Hence $y \in N_{LG}(T) = N_G(T) \ltimes LT = N_G(T) \ltimes (X_*(T) \times L^+T)$. Write $y = w\cdot t^\mu \cdot \tau$ where $w \in N_G(T)$, $t^\mu \in LT$, and $\tau \in L^+T \subset L^+G$. Finally, using that $wt^\mu = t^{w(\mu)}w$ where $w(\mu) : \mathbb{G}_m \to G$ is the cocharacter defined by $w\mu(-)w^{-1}$, we see that $x = [y] = [t^{w(\mu)}]$, lifting x into LT.

Let $W = \mathbb{A}^1_+ = \mathbb{A}^1$. Let $\mathrm{Spec}(R) \to (\mathrm{Gr}_G^+)$ be an R-point, which we need to lift, possibly after an fpqc-cover, to an R-point of $(LG)^+ = LB$. The given R-point corresponds to a $\mathbb{G}_m$-equivariant map $\mathbb{A}^1_R \to \mathrm{Gr}_G$ which we pull back along the fpqc-cover $LG \to \mathrm{Gr}_G$ to obtain a principal L^+G-bundle $P \to \mathbb{A}^1_R$. Because the $\mathbb{G}_m$-action on $\mathbb{A}^1_R$ fixes the origin, the restriction $\mathrm{Spec}(R) \overset{0}{\to} \mathbb{A}^1_R \to \mathrm{Gr}_G$ factors through $(\mathrm{Gr}_G)^0 \simeq \mathrm{Gr}_T$, so if P_0 is the restriction of P to the origin, then the diagram below supplies an isomorphism $P_0 \simeq L^+G$

$$\begin{tikzcd}
P_0 \simeq (L^+G)_R \arrow[r] \arrow[d] & L^+G \arrow[r] \arrow[d] & LG \arrow[d] \\
\mathrm{Spec}(R) \arrow[r] & \mathrm{Spec}(k) \arrow[r, "t^\mu"'] & Gr_G
\end{tikzcd}$$

for some $\mu \in X_*(T)$. The $\mathbb{G}_m$-action on P is by conjugation, so to make the isomorphism $\mathbb{G}_m$-equivariant, we compose it with the isomorphism $t^{-\mu} \cdot : L^+G \to L^+G$ (also use that $\mathcal{T}$ is abelian). Lemma 5.2.3 extends this $\mathbb{G}_m$-equivariant isomorphism to a global one: $P \simeq (L^+G) \times_{\mathrm{Spec}(R)} \mathbb{A}^1_R$. Composing with the zero-section, we get a map $\mathbb{A}^1_R \to P \to LG$ that sits over $f : \mathbb{A}^1_R \to \mathrm{Gr}_G$, as desired. □

Lemma 5.2.3 (Contraction lemma)**.** Let $R \in \mathrm{CAlg(Ab)}$ be a commutative ring and let $H \in \mathrm{Grp}(\mathrm{Aff}_{/R})$ be an affine group scheme over $\mathrm{Spec}(R)$ equipped with a $\mathbb{G}_m$-action. If P is a $\mathbb{G}_m$-equivariant H-torsor over $\mathbb{A}^1_R$ (equipped with the $\mathbb{G}_m$-action by dilation), then P is the pullback of its restriction to 0 along the map $\mathbb{A}^1_R \to \mathrm{Spec}(R)$. In other words, there are cartesian squares

$$\begin{tikzcd}
P \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & P_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & P \arrow[d] \\
\mathbb{A}^1_R \arrow[r] & \mathrm{Spec}(R) \arrow[r, "0"'] & \mathbb{A}^1_R
\end{tikzcd}$$

Proof. See □

Now for the construction of S_μ and T_μ for a cocharacter μ. Pulling back the cartesian square from Theorem 5.2.2 along $* \overset{t^\mu}{\to} \mathrm{Gr}_T$ defines the *positive-* and *negative semi-infinite orbits* S_μ and T_μ.

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & * \arrow[r, "t^\mu"'] \arrow[d, "t^\mu"] & T_\mu \arrow[r] \arrow[d] & * \arrow[d, "t^\mu"] \\
* \arrow[r, "t^\mu"'] \arrow[d, "t^\mu"] & \mathrm{Gr}_T \arrow[r, "i^-"'] \arrow[d, "i^+"] & \mathrm{Gr}_{B^-} \arrow[r, "q^-"'] \arrow[d, "p^-"] & \mathrm{Gr}_T \\
S_\mu \arrow[r] \arrow[d] & \mathrm{Gr}_B \arrow[r, "p^+"'] \arrow[d, "q^+"] & \mathrm{Gr}_G & \\
* \arrow[r, "t^\mu"'] & \mathrm{Gr}_T & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\{t^\mu\} \arrow[r] \arrow[d] & T_\mu \arrow[d] \\
S_\mu \arrow[r] & \mathrm{Gr}_G \\
\varnothing \arrow[r] \arrow[d] & S_\nu \arrow[d] \\
S_\mu \arrow[r] & \mathrm{Gr}_G
\end{tikzcd}$$

By construction, we have the cartesian squares to the right ($\mu \neq \nu$), and since Gr_T is a discrete set of isolated points, S_μ and T_μ are the connected components of Gr_B and Gr_{B^-}, respectively, making the composites $S_\mu \to \mathrm{Gr}_B \overset{p^+}{\to} \mathrm{Gr}_G$ and $T_\mu \to \mathrm{Gr}_{B^-} \overset{p^-}{\to} \mathrm{Gr}_G$, which we will denote by s_μ and t_μ, locally closed (see [DG14, Theorem 1.6.8]). Also, the underlying points of S_μ and T_μ are

$$\{x \in \mathrm{Gr}_G \mid \lim_{c \to 0} 2\rho^\vee(c) \cdot x = t^\mu\} \quad \text{and} \quad \{x \in \mathrm{Gr}_G \mid \lim_{c \to \infty} 2\rho^\vee(c) \cdot x = t^\mu\}.$$

The *contraction principle* ([DG14, Propositoin 3.2.2]) states that the maps canoncial maps

$$q^-_* \xrightarrow{\sim} q^-_* i^-_* i^{-*} \simeq i^{-*}, \qquad i^{+!} \xrightarrow{\sim} i^{+!} q^{+!} q^+_! \simeq q^+_!, \qquad \mathsf{H}^*(T_\mu, -) \xrightarrow{\sim} (t^\mu)^*, \qquad (t^\mu)^! \xrightarrow{\sim} \mathsf{H}^*_{\mathsf{c}}(S_\mu, -)$$

are equivalences. Braden's theorem (the main result of [DG14], originally proven in [Bra03]) states that the *hyperbolic localization map*

$$i^{-*}p^{-!} \xrightarrow{\text{unit}} i^{-*}p^{-!}p^{+}_{*}p^{+*} \xrightarrow[\sim]{\text{base.}} i^{-*}i^{-}_{*}i^{+!}p^{+*} \xrightarrow{\text{counit}} i^{+!}p^{+*}$$

is an equivalence. (Note that the composite above is a specialization of the hyperbolic localization map in [DG14] to the case where $j : Z^0 \to Z^+ \times_Z Z^-$ is an equivalence). Using that $* \xrightarrow{t^\mu} \mathrm{Gr}_T$ is a connected component, so an open immersion, yielding $(t^\mu)^* \simeq (t^\mu)^!$. Writing the hyperbolic localization map as $q^-_* p^{-!} \simeq q^+_! p^{+*}$ and pulling back along $t^\mu : * \to \mathrm{Gr}_T$ using !-pullback on the left and $*$-pullback on the right and using base-change yields an equivalence

$$\mathsf{H}^*(\mathrm{Gr}_G, t^!_\mu(-)) \simeq \mathsf{H}^*_{\mathsf{c}}(\mathrm{Gr}_G, s^*_\mu(-))$$

This functor is the so-called *weight functor with weight* μ of Mirković and Vilonen which we will consider in the next subsection. We end of this section with two results on the geometry of S_μ, T_μ, and $\overline{\mathrm{Gr}_\lambda}$, however, we will not provide proofs.

Proposition 5.2.4. S_μ and T_μ are the orbits of t^μ under the action of LU and LU^-, respectively. Furthermore, $\overline{S_\mu} = \bigcup_{\nu \le \mu} S_\nu$ and $\overline{T_\mu} = \bigcup_{\nu \ge \mu} T_\nu$.

Proof. See [MV18, Proposition 3.1]. □

Theorem 5.2.5. If λ and μ are cocharacters, and λ is dominant, then

$$\overline{\mathrm{Gr}_\lambda} \cap S_\mu \ne \varnothing \iff \mu \le \lambda \iff \overline{\mathrm{Gr}_\lambda} \cap T_\mu \ne \varnothing$$

If this is the case, then $\overline{\mathrm{Gr}_\lambda} \cap S_\mu$ has pure dimension $\langle \rho, \lambda + \mu \rangle$, and $\mathrm{Gr}_\lambda \cap S_\mu \subset \overline{\mathrm{Gr}_\lambda} \cap S_\mu$ is open and dense. In particular, there is a canonical bijection $\pi_0(\overline{\mathrm{Gr}_\lambda} \cap S_\mu) \simeq \pi_0(\mathrm{Gr}_\lambda \cap S_\mu)$.

Similarly, $\overline{\mathrm{Gr}_\lambda} \cap T_\mu$ has pure dimension $\langle \rho, \lambda - \mu \rangle$, and $\mathrm{Gr}_\lambda \cap T_\mu \subset \overline{\mathrm{Gr}_\lambda} \cap S_\mu$ is open and dense. In particular, there is a canonical bijection $\pi_0(\overline{\mathrm{Gr}_\lambda} \cap S_\mu) \simeq \pi_0(\mathrm{Gr}_\lambda \cap S_\mu)$.

Proof sketch. Evidently, $t^\mu \in S_\mu$, so if $\mu \le \lambda$, then $t^\mu \in \overline{\mathrm{Gr}_\lambda} \cap S_\mu$. Assume that $x \in \overline{\mathrm{Gr}_\lambda} \cap S_\mu$. By definition, $\lim_{c \to 0} 2\rho^\vee(c) \cdot x$ exists and is equal to t^μ. Since $\overline{\mathrm{Gr}_\lambda}$ is closed, $\lim_{c\to 0} 2\rho^\vee(c) \cdot x = t^\mu$ must be in $\overline{\mathrm{Gr}_\lambda}$. The proof is similar for T_μ.

I won't provide a full proof of the dimension statement, however, I will include the main claims. This follows [Zhu16, Theorem 5.3.9]. See [BR18, Theorem 5.2] for more details. The first claim is that $\overline{S_\mu} \smallsetminus S_\mu$ is a hyperplane section of $\overline{S_\mu}$ (under an embedding of Gr_G into projective space). Using this, one sees that if $\alpha^\vee$ is a simple coroot, then $\dim(\overline{\mathrm{Gr}_\lambda \cap S_{\lambda - \alpha^\vee}}) = \dim(\overline{\mathrm{Gr}_\lambda} \cap S_\lambda)$, which allows for induction. The base case is

$$\overline{\mathrm{Gr}_\lambda} \cap S_\mu \simeq \mathrm{Gr}_\lambda \cap S_\mu \simeq \begin{cases} t^{w_0(\lambda)} & \mu = w_0(\lambda) \quad (w_0 \in W \text{ is the longest element}) \\ \mathbb{A}^{\langle 2\rho, \lambda \rangle} & \mu = \lambda \end{cases}$$

□

5.3 The weight functors of Mirković and Vilonen

The goal of this subsection is partly to promote the functor $H^* : \mathrm{Sat}_G \to \mathrm{Mod}_\Lambda$ to a fiber functor, witnessing Sat_G as a neutral Tannakian category, and partly to construct a symmetric monoidal functor $F = \bigoplus_{\mu \in X_*(T)} F_\mu : \mathrm{Sat}_G \to \mathrm{Mod}_\Lambda^{X_*(T)}$ that H^* factors through and will be used in Section 5.4 to construct a split maximal torus of $\check{G}_\Lambda$. The construction of F is due to Mirković and Vilonen.

Theorem 5.3.1 (Weight Decomposition and the Fiber Functor)**.** If $A \in \mathrm{Sat}_G$ and k is an integer, then there are canonical maps

$$\bigoplus_{\substack{\mu \in X_*(T) \\ \langle 2\rho, \mu \rangle = k}} \mathsf{H}^k(\mathrm{Gr}_G, t^!_\mu A) \longrightarrow \mathsf{H}^k(\mathrm{Gr}_G, A) \quad \text{and} \quad \mathsf{H}^k(\mathrm{Gr}_G, A) \longrightarrow \bigoplus_{\substack{\mu \in X_*(T) \\ \langle 2\rho, \mu \rangle = k}} \mathsf{H}^k_{\mathsf{c}}(\mathrm{Gr}_G, s^*_\mu A)$$

which are isomoprhisms. Consequently, the total cohomology functor

$$H^* = \bigoplus_{k\in\mathbb{Z}} \mathsf{H}^k(\mathrm{Gr}_G, -) : \mathrm{Sat}_G \longrightarrow \mathrm{Mod}_\Lambda$$

factors through the category of $X_*(T)$-graded Λ-modules $F = \bigoplus_{\mu\in X_*(T)} F_\mu : \mathrm{Sat}_G \to \mathrm{Mod}_\Lambda^{X_*(T)}$ via the canonical isomorphisms

$$\bigoplus_{\mu\in X_*(T)} \mathsf{H}^{\langle 2\rho,\mu\rangle}(\mathrm{Gr}_G, t_\mu^! A) \simeq \bigoplus_{k\in\mathbb{Z}} \mathsf{H}^k(\mathrm{Gr}_G, A) \simeq \bigoplus_{\mu\in X_*(T)} \mathsf{H}_{\mathsf{c}}^{\langle 2\rho,\mu\rangle}(\mathrm{Gr}_G, s_\mu^* A).$$

Furthermore, H^* is exact and (strong) symmetric monoidal.

First, we establish a vanishing lemma:

Lemma 5.3.2. Let $A \in \mathrm{Sat}_G$ be a L^+G-equivariant perverse sheaf. If λ and μ are cocharacters and λ is dominant, then

$$\mathsf{H}_{\mathsf{c}}^k(S_\mu \cap \mathrm{Gr}_\lambda, A) = 0 \text{ for } k > \langle 2\rho,\mu\rangle \qquad \mathsf{H}^k_{T_\mu\cap\mathrm{Gr}_\lambda}(\mathrm{Gr}_G, A) = 0 \text{ for } k < \langle 2\rho,\mu\rangle.$$

and thus

$$\mathsf{H}_{\mathsf{c}}^k(\mathrm{Gr}_G, s_\mu^* A) = 0 \text{ for } k > \langle 2\rho,\mu\rangle \qquad \mathsf{H}^k(\mathrm{Gr}_G, t_\mu^! A) = 0 \text{ for } k < \langle 2\rho,\mu\rangle.$$

Consequently, $F_\mu \simeq \mathsf{H}_{\mathsf{c}}^{\langle 2\rho,\mu\rangle}(\mathrm{Gr}_G, s_\mu^* A) \simeq \mathsf{H}^{\langle 2\rho,\mu\rangle}(\mathrm{Gr}_G, t_\mu^! A)$ is an exact functor.

Proof. As A is perverse, $i_\lambda^* A \in D^{\le -2\langle\rho,\lambda\rangle}(\mathrm{Gr}_\lambda, \Lambda)$. If $j : S_\mu \cap \mathrm{Gr}_\lambda \hookrightarrow \mathrm{Gr}_\lambda$ denotes the inclusion, then

$$A_{|S_\mu\cap\mathrm{Gr}_\lambda} \simeq j^* i_\lambda^* A \in D^{\le -2\langle\rho,\lambda\rangle}(S_\mu \cap \mathrm{Gr}_\lambda, \Lambda),$$

As $\dim_{\mathbb{C}}(X_{\lambda,\mu}) = \langle\rho, \mu+\lambda\rangle$, its real dimension is $2\langle\rho,\mu+\lambda\rangle$. The compactly supported cohomology of a space of real dimension d with coefficients in a sheaf shifted to degree $\le m$ vanishes in degrees strictly greater than $d+m$, whence the claim.

As a topological space, S_μ is locally compact and admits a locally finite filtration $F_p S_\mu = \bigcup_{\langle 2\rho,\lambda\rangle\le p}(S_\mu \cap \mathrm{Gr}_\lambda)$. Note that $Z_p = F_p S_\mu \smallsetminus F_{p-1} S_\mu = \dot\bigcup_{\langle 2\rho,\lambda\rangle = p}(S_\mu\cap\mathrm{Gr}_\lambda)$ is a disjoint union. Let $h_p : Z_p \hookrightarrow S_\nu$ be the inclusion. The E_1 page of the spectral sequence from Theorem 5.3.3 applied to the complex $s_\nu^* A$ satisfies

$$E_1^{p,q} = \mathsf{H}_{\mathsf{c}}^{p+q}(Z_p, h_p^* s_\nu^* A) = 0 \quad \text{for } p+q > 2\langle\rho,\nu\rangle.$$

But as the abutment $\mathsf{H}_{\mathsf{c}}^k(S_\nu, s_\nu^* A)$ admits a filtration whose associated graded pieces are subquotients of the E_1 page, it follows that $\mathsf{H}_{\mathsf{c}}^k(S_\nu, s_\nu^* A) = 0$ for $k > 2\langle\rho,\nu\rangle$.

The claims for T_μ follow from similar arguments, using the fact that if $f : Z \to X$ is smooth of relative dimension d and $A \in D^{\ge m}(X,\Lambda)$, then $\mathsf{H}^k_Z(X,A) = \mathsf{H}^k(X, f^! A) = \mathsf{H}^k(X, f^* A[2d]) = 0$ for all $k < \dim X + 2d$.

Exactness follows from vanishing of most terms in the long exact sequence associated to short exact sequence $0 \to A \to B \to C \to 0$ in Sat_G upon applying F_μ. □

Lemma 5.3.3. Let X be a locally compact topological space equipped with a locally finite filtration by closed subsets:

$$\varnothing = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X$$

such that $X = \bigcup X_p$. Let $h_p : Z_p = X_p \smallsetminus X_{p-1} \hookrightarrow X$ be the natural locally closed inclusions.

For any complex of sheaves $A \in D^+(X,\Lambda)$, there exist spectral sequences converging to the compactly supported cohomology and global cohomology of X, with the E_1 pages given respectively by:

$$E_1^{p,q} = \mathsf{H}_{\mathsf{c}}^{p+q}(Z_p, h_p^* A) \Longrightarrow \mathsf{H}_{\mathsf{c}}^{p+q}(X,A) \quad \text{and} \quad E_1^{p,q} = \mathsf{H}^{p+q}(Z_p, h_p^! A) \Longrightarrow \mathsf{H}^{p+q}(X,A)$$

Proof of Theorem 5.3.1. For each integer $n \in \mathbb{Z}$, we claim that

$$U_n = \bigcup_{\langle 2\rho,\mu\rangle\le n} T_\mu$$

is open in Gr_G. Indeed, consider the complement $\mathrm{Gr}_G \setminus U_n = \bigcup_{\langle 2\rho,\mu\rangle > n} T_\mu = \bigcup_{\langle 2\rho,\mu\rangle > n} \overline{T_\mu}$, which intersected with $\overline{\mathrm{Gr}_\lambda}$ is

$$\bigcup_{\langle 2\rho,\mu\rangle > n} \overline{\mathrm{Gr}_\lambda} \cap \overline{T_\mu}.$$

But this is a finite union, for if $\overline{\mathrm{Gr}_\lambda} \cap \overline{T_\mu} \neq \varnothing$, then $\mu \leq \lambda$ by Proposition 5.2.4 and Theorem 5.2.5, and there are only finitely many such cocharacters satisfying $\langle 2\rho, \mu\rangle > n$ for a fixed n.

This gives a filtration of open sets $\varnothing \subset \cdots \subset U_{n-1} \subset U_n \subset \cdots \subset \mathrm{Gr}_G$ with $Z_n = U_n \setminus U_{n-1} = \dot\bigcup_{\langle 2\rho,\mu\rangle = n} T_\mu$. Let $i_n : Z_n \to U_n$ and $j_n : U_{n-1} \to U_n$ be the inclusions. From the fiber sequence $i_{n*}i_n^! A \to A \to j_{n*}j_n^* A$, there is a long exact sequence

$$\cdots \to \mathsf{H}^k_{Z_n}(U_n, A) \xrightarrow{i_\mu} \mathsf{H}^k(U_n, A) \longrightarrow \mathsf{H}^k(U_{n-1}, A) \longrightarrow \mathsf{H}^{k+1}_{Z_n}(U_n, A) \to \cdots$$

By excision and Lemma 5.3.2, we have

$$\mathsf{H}^k_{Z_n}(U_n, A) \simeq \bigoplus_{\langle 2\rho,\mu\rangle = n} \mathsf{H}^k(\mathrm{Gr}_G, t_\mu^! A) = \begin{cases} \bigoplus_{\langle 2\rho,\mu\rangle = k} \mathsf{H}^k(\mathrm{Gr}_G, t_\mu^! A) & n = \langle 2\rho, \mu\rangle = k, \\ 0 & \text{else.} \end{cases}$$

Using this, we find

- if $k > n$, then $\mathsf{H}^k(U_n, A) \simeq \mathsf{H}^k(U_{n-1}, A)$, and since U_n is empty for sufficiently small n, we must have $\mathsf{H}^k(U_{k-1}, A) \simeq 0$.
- If $k = n$, then $\mathsf{H}^k_{Z_n}(U_n, A)$ is non-zero, so U_n is non-empty. The number $\langle 2\rho, \mu\rangle$ has constant parity on each connected component of Gr_G (see Theorem 2.3.1), so existence of strata at degree k guarantees there are no strata at degree $k-1$ on those same components. Thus, $Z_{k-1} = \varnothing$, meaning $U_{k-1} = U_{k-2}$. Hence $\mathsf{H}^{k-1}(U_{k-1}, A) \simeq \mathsf{H}^{k-1}(U_{k-2}, A) \simeq 0$. The long exact sequence gives a canonical isomorphism

$$\bigoplus_{\langle 2\rho,\mu\rangle = k} \mathsf{H}^k(\mathrm{Gr}_G, t_\mu^! A) \xrightarrow{\sim} \mathsf{H}^k(U_k, A).$$

- If $k < n$ The local cohomology terms again vanish, yielding isomorphisms $\mathsf{H}^k(U_n, A) \simeq \mathsf{H}^k(U_{n-1}, A)$.

Taking the limit as $n \to \infty$, the global cohomology stabilizes after step k. Thus, $\mathsf{H}^k(\mathrm{Gr}_G, A) \simeq \mathsf{H}^k(U_k, A)$, completing the proof of the direct sum decomposition.

(The other isomorphism follows from similar considerations. Because the closures of S_μ drift downwards, the sets $V_n = \bigcup_{\langle 2\rho,\mu\rangle \leq n} S_\mu$ form a growing tower of closed subsets. Applying the long exact sequence for compactly supported cohomology yields the coimage decomposition).

Exactness follows from Lemma 5.3.2.

The idea for symmetric monoidality is to extend the $2\rho^\vee(\mathbb{G}_m)$-action to the relative affine Grassmannian, which we will not do here. See [BR18, Proposition 8.1] or [MV18, Proposition 6.4] for the original account. □

Corollary 5.3.4. If λ and μ are cocharacters, and λ is dominant, then

$$\dim F_\mu(\mathbf{IC}_\lambda) = |\pi_0(S_\mu \cap \mathrm{Gr}_\lambda)|.$$

In particular, H^* is faithful, proving that H^* is a fiber functor.

Proof. By the proof of Theorem 3.2.4 and Lemma 3.2.3, we have $\mathbf{IC}_{\lambda|\mathrm{Gr}_\eta} \in D^{-\langle 2\rho,\eta\rangle - 2}(\mathrm{Gr}_\eta, \Lambda)$ for $\eta < \lambda$. Compactly supported cohomology of a space of real dimension d with coefficients in $D^{\leq m}$ vanishes in degrees strictly greater than $d + m$, so for $\eta < \lambda$, $\mathsf{H}^k_c(\mathrm{Gr}_\eta \cap S_\mu, \mathbf{IC}_\lambda)$ vanishes for $k > \langle 2\rho, \mu\rangle - 2$.

Using the long exact sequence associated to the open-closed decomposition $\mathrm{Gr}_\lambda \cap S_\mu \to \overline{\mathrm{Gr}_\lambda} \cap S_\mu \leftarrow \sqcup_{\eta<\lambda} \mathrm{Gr}_\eta \cap S_\mu$ gives an isomorphism

$$0 \to \mathsf{H}^{\langle 2\rho,\mu\rangle}_c(\mathrm{Gr}_G^\lambda \cap S_\mu, \mathbf{IC}_\lambda) \xrightarrow{\sim} \mathsf{H}^{\langle 2\rho,\mu\rangle}_c(\overline{\mathrm{Gr}_G^\lambda} \cap S_\mu, \mathbf{IC}_\lambda) \to 0$$

Since $\mathbf{IC}_\lambda$ is supported on $\overline{\mathrm{Gr}_\lambda}$ and $\mathbf{IC}_{\lambda|\,\mathrm{Gr}_\lambda} \simeq \Lambda[\dim \mathrm{Gr}_\lambda] \simeq \Lambda[\langle 2\rho, \lambda\rangle]$, it follows that

$$F_\mu(\mathbf{IC}_\lambda) \simeq \mathsf{H}_c^{\langle 2\rho,\mu\rangle}(\overline{\mathrm{Gr}_\lambda} \cap S_\mu, \mathbf{IC}_\lambda) \simeq \mathsf{H}_c^{\langle 2\rho,\mu\rangle}(\mathrm{Gr}_\lambda \cap S_\mu, \Lambda[\langle 2\rho, \lambda\rangle]) = \mathsf{H}_c^{\langle 2\rho,\lambda+\mu\rangle}(\mathrm{Gr}_\lambda \cap S_\mu, \Lambda)$$

Note that the right-hand side is the compactly supported cohomology in top-degree, which we know is of dimension $|\pi_0(S_\mu \cap \mathrm{Gr}_\lambda)|$ (e.g. by Poincare-duality).

Faithfulness is equivalent to conservativity as H^* is an exact functor of abelian categories. Conservativity follows from semisimplicity and the dimension formula just established. □

Remark 5.3.5. The functors F_μ do not depend on the choice of T and B. See [BR18, Section 5.5] or [MV18, Theorem 3.6].

Corollary 5.3.6. The category Sat_G is rigid.

Proof. According to [DM18, Proposition 1.20], it suffices to verify that if $H^*(A)$ is of dimension 1, then there is an object A^{-1} such that $A \star A^{-1} \simeq 1 \simeq \mathbf{IC}_0$. For such A, Theorem 5.3.1 and Corollary 5.3.4 show that there is only one non-zero weight in the decomposition of $H^*(A)$, so $A \simeq \mathbf{IC}_\lambda$ for some central cocharacter λ. Because λ is central, its Schubert cell is a single point $\mathrm{Gr}_\lambda = \{t^\lambda\}$.

We claim the inverse is $A^{-1} = \mathbf{IC}_{-\lambda}$. Because t^λ commutes with L^+G, the fiber of the projection p over the point $(t^\lambda, t^{-\lambda})$ is a trivial L^+G-torsor, collapsing to a single point. This yields the following Cartesian square:

$$\begin{tikzcd} * \arrow[r] \arrow[d, "\mathrm{id}"'] \arrow[rr, bend left, "t^0"] \arrow[dr, phantom, "\mathrm{cart}"] & \mathrm{Hck}_{G,2}^{\mathrm{conv}} \arrow[r, "m"] \arrow[d, "p"] & \mathrm{Hck}_G \\ * \simeq \{t^\lambda\} \times \{t^{-\lambda}\} \arrow[r, "\iota_\lambda \times \iota_{-\lambda}"'] & \mathrm{Hck}_G \times \mathrm{Hck}_G & \end{tikzcd}$$

which we use to compute

$$\mathbf{IC}_\lambda \star \mathbf{IC}_{-\lambda} \simeq m_! p^*(\mathbf{IC}_\lambda \boxtimes \mathbf{IC}_{-\lambda}) \simeq m_! p^*(\iota_\lambda \times \iota_{-\lambda})(1 \boxtimes 1) \simeq \iota_{0!}(1) \simeq \mathbf{IC}_0\,.$$

□

Finally, Tannakian reconstruction supplies the affine group scheme $\check{G}_\Lambda$ and the diagram from Theorem 1.0.1.

5.4 The root datum of $\check{G}_\Lambda$

In this subsection we will prove the last paragraph of Theorem 1.0.1, by constructing a canonical split maximal torus $\check{T}_\Lambda \to \check{G}_\Lambda$ and Borel $\check{B}_\Lambda \subset \check{G}_\Lambda$ containing $\check{T}_\Lambda$ (see Appendix Section B). But first, we show that $\check{G}_\Lambda$ is reductive.

By construction, we only have access to $\check{G}_\Lambda$ through its representations $\mathrm{Rep}_\Lambda(\check{G}_\Lambda) \simeq \mathrm{Sat}_G$, but we are in luck due to Lemma 5.4.1.

Lemma 5.4.1. Let Λ be a field.

(1) If H is an affine group scheme over Λ, then G is of finite over Λ if and only if there exist an object $X \in \mathrm{Rep}_\Lambda(H)$ such that X generates $\mathrm{Rep}_\Lambda(H)$ under direct sums, tensor products, duals, and subquotients.

(2) Let H be a affine group scheme of finite type over Λ. If H is not connected, then there exists a non-trivial object $X \in \mathrm{Rep}_\Lambda(H)$ such that the full subcategory of $\mathrm{Rep}_\Lambda(H)$ spanned by subquotients of $X^{\oplus n}$ for $n \geq 0$ is stable under $\otimes$.

(3) Let H be a affine group scheme of finite type over Λ that is connected. Assume that $\mathrm{char}\,\Lambda = 0$. If $\mathrm{Rep}_{\overline{\Lambda}}(H \times_{\mathrm{Spec}(\Lambda)} \mathrm{Spec}(\overline{\Lambda}))$ is semisimple, then H is reductive.

Proof. See [BR18, Proposition 2.11]. □

Proposition 5.4.2. The affine group scheme $\check{G}_\Lambda$ is reductive.

Proof. We verify (1)-(3) in Lemma 5.4.1 for $\mathrm{Rep}_\Lambda(\check{G}_\Lambda) \simeq \mathrm{Sat}_G$. For (1), let $\lambda_1, \dots \lambda_n$ be generators for the monoid $X_*(T)^+$ and let $X = \mathbf{IC}_{\lambda_1} \star \cdots \star \mathbf{IC}_{\lambda_n}$. Fix $\lambda = \sum_{i=1}^n k_i \lambda_i$ for $k_i \geq 0$ and let

$$X = \mathbf{IC}_{\lambda_1}^{\star k_1} \star \cdots \star \mathbf{IC}_{\lambda_n}^{\star k_n}.$$

Note that $\mathrm{Supp}(X) \subset \overline{\mathrm{Gr}_\lambda}$. As Sat_G is semisimple, we may write $X \simeq \bigoplus_{\mu \leq \lambda} \mathbf{IC}_\mu^{\oplus m_\mu}$ for some $m_\mu \geq 0$. But $C_{|\mathrm{Gr}_\lambda} \simeq 1[\dim \mathrm{gr}_\lambda]$, so $m_\lambda = 1 > 0$, witnessing $\mathbf{IC}_\lambda$ as a direct summand of C. We finish by semisimplicity of Sat_G.

For (2), let $X \in \mathrm{Sat}_G$ be a non-trivial object. There is some non-zero dominant cocharacter λ such that $\mathbf{IC}_\lambda$ appears as a direct summand of X. Hence $\mathbf{IC}_{n\lambda}$ appears as a direct summand of $X^{\star n}$ for every $n \geq 0$. But the objects $\mathbf{IC}_{n\lambda}$, $n \geq 0$, are pairwise non-isomorphic (they have distinct supports), contradicting stability under $\star$. This shows that $\check{G}_\Lambda$ is connected.

For (3), note that $\check{G}_{\overline{\Lambda}} \simeq \check{G}_\Lambda \times_{\mathrm{Spec}(\Lambda)} \mathrm{Spec}(\overline{\Lambda})$, and that $\mathrm{Rep}_\Lambda(\check{G}_{\overline{\Lambda}}) \simeq \mathrm{Sat}_G$ semisimple. □

Theorem 5.4.3. The reductive group $\check{G}_\Lambda$ contains a canonical split maximal torus $\check{T}_\Lambda$ and a canonical Borel subgroup $\check{B}_\Lambda$. Furthermore, the root datum of $\check{G}_\Lambda$ with respect to these subgroups is exactly the root datum of the Langlands dual of G.

Proof. First, we consctruct a map $\varphi : \check{T}_\Lambda \to \check{G}_\Lambda$. From Example 5.1.1 and Theorem 5.3.1 there is a commuting triangle to the left below, which induces the triangle on the right by Tannakian reconstruction (Theorem A.0.3):

$$\begin{tikzcd} \mathrm{Sat}_G \arrow[rr, "F"] \arrow[dr, "H^*"'] & & \mathrm{Mod}_\Lambda^{X_*(T)} \simeq \mathrm{Sat}_T \arrow[dl, "\mathrm{Forget}"] \\ & \mathrm{Mod}_\Lambda & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Rep}_\Lambda(\check{G}_\Lambda) \arrow[rr, "\varphi^*"] \arrow[dr, "\mathrm{Forget}"'] & & \mathrm{Rep}_\Lambda(\check{T}_\Lambda) \arrow[dl, "\mathrm{Forget}"] \\ & \mathrm{Mod}_\Lambda & \end{tikzcd}$$

for some map $\varphi : \check{T}_\Lambda \to \check{G}_\Lambda$. Note that by constrution, the characters and cocharacters of T and $\check{T}_\Lambda$ are swapped:

$$(X^*(T), X_*(T)) = (X_*(\check{T}_\Lambda), X^*(\check{T}_\Lambda)).$$

Second, we show that $\varphi : \check{T}_\Lambda \to \check{G}_\Lambda$ is a closed immersion and that it witnesses the source as a split maximal torus of the target. According to [DM18, Proposition 2.21(b)], the map $\varphi : \check{T}_\Lambda \to \check{G}_\Lambda$ is a closed immersion if and only if for every object $V \in \mathrm{Rep}_\Lambda(\check{T}_\Lambda) \simeq \mathrm{Mod}_\Lambda^{X_*(T)}$ there is some $A \in \mathrm{Rep}_\Lambda(\check{G}_\Lambda) \simeq \mathrm{Sat}_G$ such that V is a subquotient of $\phi^*(A) \simeq \bigoplus_{\mu \in X_*(T)} F'_\mu(A)$. Since $\mathrm{Sat}_T \simeq \mathrm{Mod}_\Lambda^{X_*(T)}$ is semisimple, it suffices to show this for the 1-dimensional graded vector space $V = \Lambda_{(\mu)}$ supported in degree μ. Write $\mu = \lambda - \lambda'$ for $\lambda, \lambda' \in X_*(T)$ and let $A = \mathbf{IC}_\lambda \star \mathbf{IC}_{\lambda'}^\vee$, where $\mathbf{IC}_{\lambda'}^\vee$ is the dual to $\mathbf{IC}_{\lambda'}$ (recall that Sat_G is rigid). We have

$$F_\mu(A) \simeq \bigoplus_{\nu + \nu' = \mu} F_\nu(\mathbf{IC}_\lambda) \otimes_\Lambda F_{\nu'}(\mathbf{IC}_{\lambda'}^\vee),$$

and from Corollary 5.3.4 and (the proof sketch of) Theorem 5.2.5, $F_\lambda(\mathbf{IC}_\lambda) = 1 = F_{-\lambda'}(\mathbf{IC}_{\lambda'}^\vee)$, which yields a 1-dimensional subspace $F_\lambda(\mathbf{IC}_\lambda) \otimes_\Lambda F_{\lambda'}(\mathbf{IC}_{\lambda'}^\vee) \subset F'_\mu(A)$.

For maximality, it suffices to show that the induced map on Grothendieck rings is injective. Let $V_\lambda = \mathsf{ST}(\mathbf{IC}_\lambda)$ be the irreducible representations of $\check{G}_\Lambda$. Because $F_\mu(\mathrm{IC}_\lambda) = 0$ for $\mu \nleq \lambda$, and $\dim F_\lambda(\mathrm{IC}_\lambda) = 1$, the formal character $\varphi^*([V_\lambda]) \in K_0(\mathrm{Rep}_\Lambda(\check{T}_\Lambda)) \simeq \mathbb{Z}[X_*(T)]$ is of the form $e^\lambda + \sum_{\mu < \lambda} m_\mu e^\mu$. Since the leading coefficients are different for varying λ, images of the basis elements of $K_0(\mathrm{Rep}_\Lambda(\check{G}_\Lambda))$ are linearly independent, making the induced map of Grothendieck rings injective.

Next, we will construct the Borel $\check{B}_\Lambda \subset \check{G}_\Lambda$ containing $\check{T}_\Lambda$. For each dominant coweight $\lambda \in X_*(T)^+$, define the 1-dimensional line $\ell_\lambda = F_\lambda(\mathrm{IC}_\lambda) \subset V_\lambda$. The convolution $\mathrm{IC}_\lambda \star \mathrm{IC}_{\lambda'}$ decomposes as $\mathrm{IC}_{\lambda+\lambda'} \oplus X$, where the remainder X is supported on $\overline{\mathrm{Gr}_{\lambda+\lambda'}} \setminus \mathrm{Gr}_{\lambda+\lambda'}$. Applying the monoidal functor $F = \bigoplus_\mu F_\mu$ to the canonical projection $p : \mathrm{IC}_\lambda \star \mathrm{IC}_{\lambda'} \to \mathrm{IC}_{\lambda+\lambda'}$ and restricting to degree $\lambda + \lambda'$ yields a map

$$F_{\lambda+\lambda'}(p) : \ell_\lambda \otimes \ell_{\lambda'} \longrightarrow \ell_{\lambda+\lambda'},$$

which we claim is an isomosphim. Recall that the weight functor $F_{\lambda+\lambda'}$ evaluates compactly supported cohomology on the semi-infinite orbit $S_{\lambda+\lambda'}$. But as $S_{\lambda+\lambda'} \cap (\overline{\mathrm{Gr}_{\lambda+\lambda'}} \smallsetminus \mathrm{Gr}_{\lambda+\lambda'}) \simeq \varnothing$, we see that $X_{|S_{\lambda+\lambda'}} \simeq 0$. Consequently, p restricts to an isomorphism on $S_{\lambda+\lambda'} \cap \mathrm{Gr}_{\lambda+\lambda'}$, which implies $F_{\lambda+\lambda'}(p)$ is an isomorphism. This defines a point in the flag-variety $\mathcal{B} = \mathcal{B}(\check{G}_\Lambda) \subset \prod_{\lambda \in X^*(\check{T}_\Lambda)} \mathbb{P}(V_\lambda)$ (Theorem B.2.2), and therefore a Borel subgroup $\check{B}_\Lambda \subset \check{G}_\Lambda$, and the monoid of dominant characters for that $\check{B}_\Lambda$ cut out is exactly $X^*(\check{T}_\Lambda) = X_*(T)^+$.

Finally, let $\tilde{Q}^+$ be the positive root semigroup of $\check{G}_\Lambda$. A weight $\lambda - \mu$ appears in V_λ if and only if $\mu \in \tilde{Q}^+$. So if $\mu \in \tilde{Q}^+$, then $F_{\lambda-\mu}(\mathrm{IC}_\lambda) \neq 0$, which is equivalent to $S_{\lambda-\mu} \cap \overline{\mathrm{Gr}_\lambda} \neq \varnothing$, or equivalently, $\lambda - \mu \leq \lambda$, making μ a positive coroot of G. Thus, $\tilde{Q}^+$ coincides exactly with the positive coroot semigroup of G, meaning the simple roots of $\check{G}_\Lambda$ are exactly the simple coroots of G. By Lemma B.1.1, this completes the proof that $\check{G}_\Lambda$ has the Langlands dual root datum. □

Remark 5.4.4. The grading on H^* corresponds to the one parameter subgroup

$$2\rho : \mathbb{G}_m \to \check{T}_\Lambda \to \check{G}_\Lambda,$$

where $2\rho \in X_*(\check{T}_\Lambda) = X^*(T)$ is the sum of the positive coroots of $\check{G}_\Lambda$. In particular, $\check{T}_\Lambda$ is the centralizer of $2\rho(\mathbb{G}_m) \subset \check{G}_\Lambda$ and $\check{B}_\Lambda$ is the unique Borel that contains $\check{T}_\Lambda$ and such that 2ρ is dominant with respect to $\check{B}_\Lambda$. Consequently, the pair $(\check{T}_\Lambda, \check{B}_\Lambda)$ is independent on the choice of $T \subset B \subset G$.

A Tannakian Reconstruction

In this section we collect the necessary results from the theory of Tannakian reconstruction, which gives a criterion for when a "nice" abelian category $\mathcal{C}$ is equivalent the category of finite-dimensional representation of a group scheme. For references, see [DM18].

Definition A.0.1 (Neutral Tannakian Category)**.** Let Λ be a field. A *neutral Tannakian category* over Λ is a (ordinary) category $\mathcal{C}$ equipped with a Λ-linear structure (enriched in Mod_Λ) and a symmetric monoidal structure satisfying the following properties:

1. $\mathcal{C}$ is an abelian category.
2. The symmetric monoidal structure is rigid (every object admits a dual).
3. The tensor product $\otimes$ is Λ-bilinear.
4. $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) \simeq \Lambda$.
5. There exists a faithful, exact, (strong) symmetric monoidal functor $\omega : \mathcal{C} \to \mathrm{Mod}_\Lambda^{\mathrm{fd}}$, called a *fiber functor*.

Theorem A.0.2 (Tannakian Reconstruction)**.** Let $\mathcal{C}$ be a neutral Tannakian category over a field Λ, and let $\omega : \mathcal{C} \to \mathrm{Mod}_\Lambda^{\mathrm{fd}}$ be a fiber functor for $\mathcal{C}$. The group functor $\underline{\mathrm{Aut}}^{\otimes}(\omega) : \mathrm{Aff}_\Lambda^{\mathrm{op}} \to \mathrm{Grp}$, which assigns to an affine scheme $S \simeq \mathrm{Spec}(R)$ the group of monoidal natural automorphisms of the base-changed functor $\omega_R \coloneqq \omega \otimes_\Lambda R$, is representable by an affine group scheme G over Λ. Furthermore, ω induces an equivalence of rigid symmetric monoidal categories $\mathcal{C} \xrightarrow{\sim} \mathrm{Rep}_\Lambda(G)$ such that the diagram

$$\begin{tikzcd}
\mathcal{C}^{\otimes} \arrow[rr, "\sim"] \arrow[dr, "\omega"'] & & \mathrm{Rep}_\Lambda(G)^{\otimes} \arrow[dl, "\mathrm{Forget}"] \\
& \mathrm{Vect}_\Lambda^{\otimes} &
\end{tikzcd}$$

commutes.

Proof idea. The proof proceeds by constructing the underlying algebra of G as the direct sum $\bigoplus_{X \in \mathcal{C}} \omega(X)^\vee \otimes_\Lambda \omega(X)$ quotiented out by the relations generated by $\omega(f)^\vee(\phi) \otimes v \sim \phi \otimes \omega(f)(v)$ for all maps $f : X \to Y$. The symmetric monoidal structure of $\mathcal{C}$ induces a comultiplication, while the rigidity supplies the antipode map, to make it a Hopf algebra. With G constructed, the canonical functor $\mathcal{C} \to \mathrm{Rep}_\Lambda(G)$ is defined by mapping an object X to the vector space $\omega(X)$ on which G acts as follows: Using the moduli-description of G, any element $g \in G$ provides a canonical automorphism g_X of $\omega(X)$. Exactness and faithfulness of ω is used to show that this functor is an equivalence. $\square$

Theorem A.0.3 (Functoriality of Tannakian Reconstruction)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be neutral Tannakian categories over a field Λ, equipped with fiber functors $\omega : \mathcal{C} \to \mathrm{Mod}_\Lambda$ and $\omega' : \mathcal{D} \to \mathrm{Mod}_\Lambda$. Let $G = \mathrm{Aut}^{\otimes}(\omega)$ and $H = \mathrm{Aut}^{\otimes}(\omega')$ be the respective reconstructed affine group schemes, so that $\mathcal{C} \simeq \mathrm{Rep}_\Lambda(G)$ and $\mathcal{D} \simeq \mathrm{Rep}_\Lambda(H)$.

If $F : \mathcal{C} \to \mathcal{D}$ is an exact symmetric monoidal functor there is a monoidal natural isomorphism $\eta : \omega \xrightarrow{\sim} \omega' \circ F$, then F canonically induces a homomorphism of affine group schemes $\varphi : H \longrightarrow G$ such that under the Tannakian equivalences, F is canonically isomorphic to the pullback functor $\varphi^* : \mathrm{Rep}_\Lambda(G) \to \mathrm{Rep}_\Lambda(H)$. This yields the following commutative diagram of categories:

$$\begin{tikzcd}
\mathcal{C} \simeq \mathrm{Rep}_\Lambda(G) \arrow[rr, "F \simeq \varphi^*"] \arrow[dr, "\omega"'] & & \mathcal{D} \simeq \mathrm{Rep}_\Lambda(H) \arrow[dl, "\omega'"] \\
& \mathrm{Mod}_\Lambda &
\end{tikzcd}$$

B Miscellanea: Linear algebraic groups

In this appendix we recall some classical notions from the theory of linear algebraic groups. Everthing in here can be found in [Spr98].

B.1 Root datum and the choice of torus and Borel

Let G be a split connected reductive algebraic group over a field.[8] The classification theorem says that G is uniquely determined up to isomorphism by its *root datum* $(X, \Phi, X^\vee, \Phi^\vee)$. In this section, we will recall how extracting this combinatorial data from G corresponds to choosing a split maximal torus[9] $T \subset G$, and a Borel[10] subgroup $B \subset G$ containing T.

- **The Torus T pins down the lattices and the roots:** By choosing a split maximal torus $T \subset G$, we get the *character* and *cocharacter lattices* $X = \mathbb{X}^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ and $X^\vee = \mathbb{X}_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$.

  There is a canonical pairing $\langle -, - \rangle : X \times X^\vee \to \mathbb{Z}$ that takes a pair (χ, λ) to the unique integer exponent defining the composition $\chi \circ \lambda : \mathbb{G}_m \to \mathbb{G}_m$.

  The group G acts on its Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ via the adjoint representation. Because T is split, restricting this action strictly to T yields a decomposition $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, where $\mathfrak{t} = \mathrm{Lie}(T)$ is the zero-weight space, and $\Phi \subset X$ is the finite set of non-zero characters that appear, called the *roots*. To every such root, structural theory assigns a unique *coroot* $\alpha^\vee \in X^\vee$ satisfying $\langle \alpha, \alpha^\vee \rangle = 2$.

  This defines the (unoriented) root datum $(X, \Phi, X^\vee, \Phi^\vee)$.

- **The Borel B pins down positive and simple roots:** Let $B \subset G$ be a Borel subgroup containing T. It decomposes as a semi-direct product $B = T \ltimes U$, where U is the unipotent radical. The roots that appear in the Lie algebra $\mathfrak{u} = \mathrm{Lie}(U)$ form exactly the set of *positive roots* $\Phi^+ \subset \Phi$. Inside Φ^+, there is a unique minimal subset $\Delta \subset \Phi^+$ of *simple roots*, defined such that every positive root is a non-negative integer linear combination of elements in Δ.

- **Further structure:** Once Φ^+ and Δ are locked in by the choice of B, we may define:
  - *The root and coroot lattices* $Q = \mathbb{Z}\Phi \subset X$ and $Q^\vee = \mathbb{Z}\Phi^\vee \subset X^\vee$.
  - *The positive semigroups* $Q^+ = \mathbb{Z}_{\geq 0}\Phi^+$ and $(Q^\vee)^+ = \mathbb{Z}_{\geq 0}(\Phi^\vee)^+$.
  - *The dominance ordering on X and $X^\vee$*: For weights $\lambda, \mu \in X$, we define $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$. For coweights $\lambda, \mu \in X^\vee$, we write $\lambda \geq \mu$ if $\lambda - \mu \in (Q^\vee)^+$.
  - *The dominant chambers X^+ and $(X^\vee)^+$*: A weight $\lambda \in X$ is *dominant* if $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Delta$. Dually, a coweight $\lambda \in X^\vee$ is *dominant* if $\langle \alpha, \lambda \rangle \geq 0$ for all $\alpha \in \Delta$.
  - *The opposite Borel B^-* is the unique Borel subgroup $B^- \subset G$ such that $B \cap B^- = T$. It decomposes as $B^- = T \ltimes U^-$, where the roots appearing in the Lie algebra of its unipotent radical U^- form exactly the set of *negative roots* $\Phi^- = -\Phi^+$.
  - *The Weyl vector* $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$. It satisfies $\langle \rho, \alpha^\vee \rangle = 1$ for all $\alpha \in \Delta$, making it strictly dominant.
  - *The Weyl group* $W = N_G(T)/T$. It acts faithfully on the lattices X and $X^\vee$. The action of $w = [\dot{w}] \in W$ on $\chi \in X$ and $\lambda \in X^\vee$ is given by post- and precompositition with conjugation by $\dot{w}$. For each root $\alpha \in \Phi$, there is a corresponding reflection $s_\alpha \in W$. The choice of B (and thus Δ) singles out a set of *simple reflections* $S = \{s_\alpha \mid \alpha \in \Delta\}$, which generate W and make (W, S) a Coxeter group.
  - *The longest element w_0 wrt. S* is the unique element $w_0 \in W$ of maximal length. It satisfies $w_0(\Phi^+) = \Phi^-$, and thus $w_0(\Delta) = -\Delta$.
  - The *Langlands* dual to G is the reductive group $\check{G}$ with root datum $(X^\vee, \Phi^\vee, X, \Phi)$.

    For the classical groups, the Langlands duals are given by the following table:

[8] A linear algebraic group G over a field is *reductive* if it is smooth, affine, connected, and its unipotent radical (the maximal connected normal unipotent subgroup) over the algebraic closure is trivial.

[9] A *torus* T is a linear algebraic group that is isomorphic over the algebraic closure to a finite product of multiplicative groups, $\mathbb{G}_m^r$. We say T is *split* if this isomorphism is already defined over the base field.

[10] A *Borel subgroup* $B \subset G$ is a maximal Zariski-closed, connected, solvable algebraic subgroup.

G	SL_n	GL_n	SO_{2n+1}	Sp_{2n}	SO_{2n}	T
$\check{G}$	PGL_n	GL_n	Sp_{2n}	SO_{2n+1}	SO_{2n}	$\check{T}$

Lemma B.1.1. A root datum $(X, \Phi, X^\vee, \Phi^\vee)$ is uniquely determined by its lattices X and $X^\vee$, its semigroup of dominant weights $X^+ \subset X$, and its set of simple roots $\Delta \subset \Phi$.

Example B.1.2 (The root datum of SL_n)**.** Let me spell out the terms above for the case of $G = \mathrm{SL}_n$ over a field k. Let $T \subset SL_n$ be the standard split maximal torus consisting of diagonal matrices with determinant 1, and let $B \subset SL_n$ be the standard Borel subgroup consisting of upper-triangular matrices with determinant 1.

Let $\epsilon_i : T \to \mathbb{G}_m$ be the homomorphism extracting the i-th diagonal entry. We have $\sum \epsilon_i = 0$. A cocharacter $\mathbb{G}_m \to T$ takes the form $s \mapsto \operatorname{diag}(s^{a_1}, \dots, s^{a_n})$. We have $\sum a_i = 0$. So if $\epsilon_i^\vee$ is the canonical dual basis, then the character and cocharacter lattices are

$$X \simeq \left(\bigoplus_{i=1}^n \mathbb{Z}\epsilon_i \right) \Big/ \left\langle \sum_{i=1}^n \epsilon_i \right\rangle \quad \text{and} \quad X^\vee = \left\{ \sum_{i=1}^n a_i \epsilon_i^\vee \in \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i^\vee \;\middle|\; \sum_{i=1}^n a_i = 0 \right\}$$

The canonical pairing is simply the standard evaluation: $\langle \epsilon_i, \epsilon_j^\vee \rangle = \delta_{ij}$.

The Lie algebra $\mathfrak{g} = \mathfrak{sl}_n$ consists of $n \times n$ matrices with trace zero. T acts on $\mathfrak{g}$ by conjugation. For a standard matrix unit $E_{i,j}$ (with a 1 in the (i,j) position and 0 elsewhere), the adjoint action of $t = \operatorname{diag}(t_1, \dots, t_n)$ $t \cdot E_{i,j} \cdot t^{-1} = t_i t_j^{-1} E_{i,j}$, which means the weight of the T-action on $E_{i,j}$ is $\epsilon_i - \epsilon_j$. Hence the roots and coroots are

$$\Phi = \{\epsilon_i - \epsilon_j \mid i \neq j\} \quad \text{and} \quad \Phi^\vee = \{\epsilon_i^\vee - \epsilon_j^\vee \mid i \neq j\},$$

which evidently satisfy $\langle \epsilon_i - \epsilon_j, \epsilon_i^\vee - \epsilon_j^\vee \rangle = 2$.

Our choice of B as the upper-triangular matrices restricts us to the matrix units $E_{i,j}$ where $i < j$. Therefore the positive roots, simple roots, and the Weyl vector become

$$\Phi^+ = \{\epsilon_i - \epsilon_j \mid i < j\}, \quad \Delta = \{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq n-1\}, \quad \text{and} \quad \rho = \frac{1}{2} \sum_{i<j} (\epsilon_i - \epsilon_j) = \sum_{i=1}^n \frac{n+1-2i}{2} \epsilon_i.$$

A weight $\lambda = \sum c_i \epsilon_i \in X$ is dominant if $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for all simple coroots $\alpha_i^\vee = \epsilon_i^\vee - \epsilon_{i+1}^\vee$, or equivalently, if $c_i - c_{i+1} \geq 0$. Therefore, X^+ consists of those $\lambda = \sum c_i \epsilon_i \in X$ with $c_1 \geq c_2 \geq \dots \geq c_n$.

B.2 Highest weight classification and the flag variety

Recall B.2.1 (Highest weight classification of irreducible representations)**.** Let G be a connected reductive algebraic group over an algebraically closed field of characteristic zero (e.g., $\mathbb{C}$). Fix a maximal torus T and a Borel subgroup B containing T, with unipotent radical U.

Let V be a representation of G. A non-zero vector $v \in V$ is called a *highest weight vector* of weight λ if

- $t \cdot v = \lambda(t) v$ for all $t \in T$ (it is a weight vector for T).
- $u \cdot v = v$ for all $u \in U$ (it is killed by the positive root spaces).

A representation V is a *highest weight representation* if it is generated by a single highest weight vector v_λ, meaning $V = \operatorname{span}(G \cdot v_\lambda)$.

For every finite-dimensional irreducible representation V of G, there exists a unique highest weight vector up to scalar multiplication. Its weight λ is necessarily dominant. This induces a bijection:

$$\left\{ \begin{matrix} \text{Dominant weights} \\ \lambda \in X^*(T)^+ \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Isomorphism classes of finite-dimensional} \\ \text{irreducible representations of } G \end{matrix} \right\}$$

The corresponding irreducible representation to λ is denoted V_λ. It decomposes into T-weight spaces $V_\lambda = \bigoplus_\mu V_{\lambda,\mu}$ which satisfies the following:

- If $V_{\lambda,\mu} \neq 0$, then $\mu \leq \lambda$.
- The highest weight space $\ell_\lambda = V_{\lambda,\lambda}$ is one-dimensional, meaning $\dim \ell_\lambda = 1$.

Recall that the Borel subgroups of G are all conjugate. The space $\mathcal{B}$ that parametrizes these subgroups is a projective variety known as the *flag variety* of G. For any fixed Borel subgroup B, this variety is canonically isomorphic to the quotient G/B. The following theorem demonstrates how embed this flag variety into a product of projective spaces, reminiscent of the classical Plücker embedding for the Grassmannian. The upshot is that to pick a Borel, it is enough to pick certain irreducible representations of G together with "tensor-compatible lines", which will be relevant in the proof of Theorem 5.4.3.

Theorem B.2.2 (The Plücker embedding and Borel characterization)**.** With the notation from Recall B.2.1, the canonical map

$$G/B \longrightarrow \textstyle\prod_{\lambda\in X^*(T)^+} \mathbb{P}(V_\lambda), \qquad gB \mapsto (g\cdot\ell_\lambda)_{\lambda\in X^*(T)^+}$$

exhibits the flag variety as a closed subscheme of the target. This subscheme is cut out by the multi-homogeneous ideal generated by the kernels of the dual projections

$$V_\lambda^\vee \otimes V_\mu^\vee \xrightarrow{p_{\lambda,\mu}^\vee} V_{\lambda+\mu}^\vee$$

for all $\lambda,\mu \in X^*(T)^+$. Consequently, there is a canonical isomorphism $G/B \simeq \operatorname{Proj}\big(\bigoplus_{\lambda\in X^*(T)^+} V_\lambda^\vee\big)$.

Conversely, if $M \subset X^*(T)$ is a monoid that generates the character lattice and V_λ is an irreducible representation for each $\lambda \in M$, then a family of lines $(\ell_\lambda)_{\lambda\in M}$ with $\ell_\lambda \subset V_\lambda$ corresponds to a unique Borel subgroup of G if and only if, for all $\lambda,\mu \in M$, the canonical projections

$$V_\lambda \otimes V_\mu^\vee \xrightarrow{p_{\lambda,\mu}} V_{\lambda+\mu}$$

restrict to isomorphisms $\ell_\lambda \otimes \ell_\mu \xrightarrow{\sim} \ell_{\lambda+\mu}$. In which case, the corresponding Borel subgroup is uniquely determined as the stabilizer $\{g \in G \mid g\cdot\ell_\lambda = \ell_\lambda \text{ for all } \lambda \in M\}$, and the monoid M corresponds to the dominant weights of G with respect to this Borel.

Proof of Theorem B.2.2. For the first assertion, see [Wan, Theorem 12].

For the converse, note that $x = (\ell_\lambda)_{\lambda\in M}$ is a point in $P_M = \prod_{\lambda\in M} \mathbb{P}(V_\lambda)$.

Recall that $V_\lambda \otimes V_\mu$ contains the irreducible component $V_{\lambda+\mu}$ if and only if λ and μ lie in the same dominant Weyl chamber. This is the case since, by assumption, the maps $p_{\lambda,\mu}^\vee$ are non-zero. So there is some Borel $B_0 \in \mathcal{B}$ s.t. M is in the corresponding dominant chamber $X^*(T)^+_{B_0}$.

Because M generates the full character lattice $X^*(T)$ as a group, it contains a weight η that is strictly dominant with respect to B_0, and thus the map $G/B_0 \to \mathbb{P}(V_\eta), gB_0 \mapsto g\cdot\ell_\eta$ is a closed immersion, or equivalently, the associated line bundle $\mathcal{L}(\eta)$ is very ample on $G/B_0 \simeq \mathcal{B}$. Thus $\operatorname{Proj}(\bigoplus_{n\geq 0} V_{n\eta}^\vee) \simeq \operatorname{Proj}(\bigoplus_{n\geq 0} H^0(X, \mathcal{L}(\eta)^{\otimes n}) \simeq G/B_0$. The maps

$$G/B_0 \simeq \operatorname{Proj}(\textstyle\bigoplus_{\lambda\in X^*(T)^+_{B_0}} V_\lambda^\vee) \longrightarrow \operatorname{Proj}(\bigoplus_{\lambda\in M} V_\lambda^\vee) \longrightarrow \operatorname{Proj}(\bigoplus_{n\geq 0} V_{n\eta}^\vee) \simeq G/B_0$$

imply that $\operatorname{Proj}(\bigoplus_{\lambda\in M} V_\lambda^\vee) \simeq \mathcal{B}$.

By the first part of the theorem, the ideal sheaf of $\prod_{\lambda\in M} \mathbb{P}(V_\lambda)$ generated by the kernels of the maps $p_{\lambda,\mu}^\vee$ for all $\lambda,\mu \in M$. Applying $p_{\lambda,\mu}^\vee$ on x restricts $p_{\lambda,\mu}$ to the specific lines $\ell_\lambda \otimes \ell_\mu$. Thus $x \in \mathcal{B}$ if and only if $p_{\lambda,\mu}$ restricts to an isomorphism $\ell_\lambda \otimes \ell_\mu \xrightarrow{\sim} \ell_{\lambda+\mu}$.

In that case, the associated Borel, B_x, is the stabilizer of x in $\mathcal{B} \simeq G/B_0$ under the G-action, namely

$$B_x = \{g \in G \mid g\cdot\ell_\lambda = \ell_\lambda \text{ for all } \lambda \in M\}.$$

By construction, B_x stabilizes the line $\ell_\lambda \subset V_\lambda$. By the highest weight classification of representations, ℓ_λ is therefore the unique highest weight line with respect to B_x. Since an irreducible representation V_λ exists if and only if its highest weight is dominant, the M is the monoid of dominant weights for the Borel B_x. □

C General Convolution

Let $\mathcal{X}$ be an ∞-topos, and let $H \in \mathrm{Grp}(\mathcal{X}) \subset \mathrm{Mon}(\mathcal{X}) \simeq \mathrm{Alg}(\mathcal{X}^\times)$ be a group-object and $X \in \mathrm{Alg}({}_H\mathrm{BMod}_H(\mathcal{X}^\times)^{\otimes_H})$ be a monoid-object in $\mathcal{X}$ with an A-A-equivariant multiplication for some action of A from both left and right. If $D : \mathrm{Corr}(\mathcal{X})^\otimes \to \mathrm{Cat}^\times$ is a six-functor formalism (or even a three-functor formalism), then the category $D(H\backslash X/H)$ admits a monoidal structure different from the usual tensor called the *convolution product*, denoted $\star$. The multiplication is explicitly given by "pull-push" along the span

$$\begin{tikzcd} & H\backslash(X \otimes_H X)/H \arrow[dl, "p"'] \arrow[dr, "m"] & \\ H\backslash X/H \times H\backslash X/H & & H\backslash X/H. \end{tikzcd}$$

That is,

$$\mathcal{F} \star \mathcal{G} \simeq m_! p^*(\mathcal{F} \boxtimes \mathcal{G})$$

In this section, we will make this into a fully coherent monoidal structure, promoting $D(H\backslash X/H)$ to a monoidal ∞-category $D(H\backslash X/H)^\star \in \mathrm{Mon} \simeq \mathrm{Alg}(\mathrm{Cat}^\times)$ of "H-H-equivariant sheaves on X."

An example of interest is the case $\mathcal{X} \simeq \mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{Aff}_k^{)}$ and the monoid-object is the loop group LG, and the group-object is positive loop group L^+G acting by multiplcation on both the left and right. This yields a convolution product on the double-fpqc-quotient $Hck_G = L^+G\backslash LG/L^+G$, or equivalently, on L^+G-equivariant constructible sheaves on the affine Grassmannian $Gr_G = LG/L^+G$, with an analogous version for the Beilinson-Drinfeld affine Grassmannian.

C.1 Double categories and the category of correspondences

the main technical tool is the fully faithful functor from *orthogonal factorization systems*[11] to *double categories*[12],

$$\mathrm{OFS} \xrightarrow{\mathrm{Fact}} \mathrm{DCat}$$

constructed and proven in [Jur25, Theorem A]. Explicitly, if $\mathcal{A}^\dagger \in \mathrm{OFS} \subset \mathrm{Fun}(\Lambda^2_2, \mathrm{Cat})$ is an orthogonal factorization system, then $\mathrm{Fact}(\mathcal{A}^\dagger)$ is the bisimplicial object given by

$$\mathrm{Fact}(\mathcal{A}^\dagger)([m],[n]) = \mathrm{Map}([m]\overline{\times}[n], \mathcal{A}^\dagger),$$

where $[m]\overline{\times}[n]$ denotes the orthogonal factorization system $([m] \times [n]^\simeq \to [m] \times [n] \leftarrow [m]^\simeq \times [n])$.

Theorem C.1.1 (Hauseng? Where is reference)**.** If $G : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ is a lax monoidal map of monoidal ∞-categories whose underlying functor $G : \mathcal{D} \to \mathcal{C}$ admits a a left adjoint $F : \mathcal{C} \to \mathcal{D}$, then F promotes to an oplax monoidal functor $F : \mathcal{C}_\otimes \to \mathcal{D}_\otimes$.

Theorem C.1.2 (Lurie? Where is the precise reference)**.** If $\mathcal{C}^\otimes$ is symmetric monoidal and $f : A \to B$ is a map in $\in \mathrm{Alg}(\mathcal{C}^\otimes)$, then there is an adjunction of ∞-categories

$$\begin{tikzcd} {}_A\mathrm{BMod}_A(\mathcal{C}) \arrow[r, shift left, "f^*"] & {}_B\mathrm{BMod}_B(\mathcal{C}) \arrow[l, shift left, "f_*"] \end{tikzcd}$$

where f_* is the identity on objects and $f^*(M) = B \otimes_A M \otimes_A B$. Furthermore, f_* promotes to a lax monoidal map, and (therefore) f^* promotes to an oplax monoidal map.

From these results, we find

[11] an ∞-category $\mathcal{C}$ and two subcategories $\mathcal{C}_{\mathrm{eg}}, \mathcal{C}_{\mathrm{in}} \subset \mathcal{C}$ containing every equivalence such that the map

$$(\mathcal{C}_{\mathrm{eg}})^\simeq \times_{\mathcal{C}^\simeq} (\mathcal{C}_{\mathrm{in}})^\simeq \to (\mathcal{C})^\simeq, \quad (f, g) \mapsto g \circ f$$

is an equivalence. Informally, this just means that every map in $\mathcal{C}$ factors uniquely as a map in $\mathcal{C}_{\mathrm{eg}}$ followed by a map in $\mathcal{C}_{\mathrm{in}}$.

[12] a bisimplicial object in An satisfying Segal condition in each coordiante. Informally, this is a grid where the first category lies horizontally, and the second lies vertically.

Corollary C.1.3. If $\mathcal{X}$ is an ∞-topos, and $A \in \mathrm{Mon}(\mathcal{X}) \simeq \mathrm{Alg}(\mathcal{X}^\times)$ is a monoid-object, then there is an oplax monoidal functor

$$ {}_A\mathrm{BMod}_A(\mathcal{X}^\times) \longrightarrow {}_1\mathrm{BMod}_1(\mathcal{X}^\times) \simeq \mathcal{X} $$

induced by the canonical map $A \to *$. It takes an object M to $* \otimes_A M \otimes_A *$.

Remark C.1.4. We denote $* \otimes_A M \otimes_A *$ by $A\backslash M/A$, and call it the *double quotient*. Explicitly, it is computed as the colimit over the simplicial diagram

$$ \cdots \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}} A^2 \times M \times A^2 \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} A \times M \times A \rightrightarrows M. $$

The monoidal structure on ${}_A\mathrm{BMod}_A(\mathcal{X}^\times)$ called the *twisted product* or the *diagonal action* and will be denoted by $- \times^A -$. Explicitly, $M \times^A N$ is computed as the colimit over the simplicial diagram

$$ \cdots \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}} M \times A^2 \times N \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} M \times A \times N \rightrightarrows M \times N $$

Lemma C.1.5. If $\mathcal{X}$ is an ∞-topos, and $A \in \mathrm{Grp}(\mathcal{X}) \subset \mathrm{Mon}(\mathcal{X})$ is a group-object, then there is a cartesian square

$$ \begin{tikzcd} A\backslash(M \times^A N)/A \arrow[r] \arrow[d] & (A\backslash M/A) \times (A\backslash N/A) \arrow[d, "(\alpha_R, \alpha_L)"] \\ */A \arrow[r, "(\mathrm{id}, I)"'] & (*/A) \times (A\backslash *) \end{tikzcd} $$

for every $M, N \in {}_A\mathrm{BMod}_A(\mathcal{X}^\times)$, where $I : */A \to A\backslash *$ denotes the equivalence provided by the inverse map $(-)^{-1} : A \xrightarrow{\sim} A$, and $\alpha_R : (A\backslash M)/A \to */A$ and $\alpha_L : A\backslash(N/A) \to A\backslash *$ are the maps classifying the right action on M and the left action on N, respectively.

Proof. This follows from [NSS14, Lemma 4.5] and the fact that the left and right action commute, so

$$ \frac{(M/A \times A\backslash N)}{A_{\mathrm{diag}}} \simeq A\Big\backslash \frac{M \times N}{A_{\mathrm{diag}}} \Big/ A \simeq A\backslash(M \times^A N)/A, $$

where A_{diag} denotes the diagonal action. □

We recall some notation from [HM24].

Notation C.1.6. (a) Let $\Sigma^m \subset [m] \times [m]^{\mathrm{op}}$ denote the poset of pairs (i, j) such that $0 \le i \le j \le m$, and let $\mathbb{A}^m$ denote the full subposet spanned on objects (i, j) with $|i - j| \le 1$.

(b) A functor $F : \Sigma^m \to \mathcal{A}$ is *cartesian* if f is a pointwise right Kan extension of its restriction to $\mathbb{A}^m$. Equivalently, if each square is a pullback in $\mathcal{A}$. Let $\mathrm{Fun}^{\mathrm{cart}}(\Sigma^m, \mathcal{A}) \subset \mathrm{Fun}(\Sigma^m, \mathcal{A})$ denote the full subcategory spanned by cartesian functors.
Note that the restriction $\mathrm{Fun}(\Sigma^{m,n}, \mathcal{A}) \to \mathrm{Fun}(\mathbb{A}^{m,n}, \mathcal{A})$ is fully faithful ([Lur09, Theorem 4.3.2.15]), and that it induces an equivalence $\mathrm{Map}^{\mathrm{cart}}(\Sigma^{m,n}, \mathcal{A}) = \mathrm{Fun}(\Sigma^{m,n}, \mathcal{A})^\simeq \to \mathrm{Map}(\mathbb{A}^{m,n}, \mathcal{A})$ of anima.

(c) A functor $f : [m]^{\mathrm{op}} \times [n] \to \mathcal{A}$ is *cartesian* if it is a right Kan extension of its restriction to the poset $\mathbb{L}^{m,n} \subset [m]^{\mathrm{op}} \times [n]$ spanned on vertices (i, n) and $(0, j)$ for $i \in [m]$ and $j \in [n]$.
Let $\mathrm{Fun}^{\mathrm{cart}}([m]^{\mathrm{op}} \times [n], \mathcal{A}) \subset \mathrm{Fun}([m]^{\mathrm{op}} \times [n], \mathcal{A})$ denote the full subcategory of cartesian functors.

(d) If $\mathcal{A}$ is an ∞-category, then there is a double-category $\mathrm{Sq}(\mathcal{A}) \in \mathrm{DCat}$ given by

$$ \mathrm{Sq}(\mathcal{A})([m], [n]) \simeq \mathrm{Map}([m] \times [n], \mathcal{A}) \simeq \mathrm{Fun}([m] \times [n], \mathcal{A})^\simeq $$

There is a double subcategory of $\mathrm{Sq}(\mathcal{A})^{1\,\mathrm{op}}$ given by

$$ \mathrm{Sq}^{\mathrm{cart}}(\mathcal{A})^{1\,\mathrm{op}}([m], [n]) \simeq \mathrm{Fun}^{\mathrm{cart}}([m]^{\mathrm{op}} \times [n], \mathcal{A})^\simeq. $$

Recall C.1.7. If $(\mathcal{C}, \mathcal{C}_0)$ is a geometric setup where $\mathcal{C}$ admits finite products, then the category $\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)$ promotes to a symmetric monoidal category

$$ \mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)^\otimes \simeq \mathrm{Corr}((\mathcal{C}^{\mathrm{op}})^{\sqcup, \mathrm{op}}, \mathcal{C}_0^-). $$

where $(\mathcal{C}^{op})^{\sqcup,op} \to \mathsf{Fin}_*$ is the cocartesian operad dual to $\mathcal{C}^\times$, and $\mathcal{C}_0^- = (\mathcal{C}_0^{op})^{\sqcup,op} \times_{\mathsf{Fin}_*^{op}} \mathsf{Fin}_*^{\simeq}$. The symmeric monoidal structure on $\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)$ is given by cartesian product in $\mathcal{C}$. For example, the cocartesian lift of the map $\alpha : \langle 2\rangle \to \langle 1\rangle$ given by $\alpha(1) = \alpha(2) = 1$ is the span

$$\begin{tikzcd} & X \times Y \arrow[dl] \arrow[dr, "\mathrm{id}"] & \\ X \oplus Y & & X \times Y \end{tikzcd}$$

C.2 Convolution

Theorem C.2.1. If $\mathcal{X}$ is an ∞-topos, and $A \in \mathrm{Grp}(\mathcal{X}) \subset \mathrm{Mon}(\mathcal{X}) \simeq \mathrm{Alg}(\mathcal{X}^\times)$ is a group-object, then the oplax monoidal functor from Corollary C.1.3

$$ {}_A\mathrm{BMod}_A(\mathcal{X}^\times) \xrightarrow{A\backslash -/A} \mathcal{X}, \qquad M \mapsto A\backslash M/A$$

induce a lax monoidal map

$$ {}_A\mathrm{BMod}_A(\mathcal{X}^\times)^{\otimes_A} \longrightarrow \mathrm{Corr}(\mathcal{X})^{\otimes}.$$

In particular, $A\backslash M/A$ is a monoid-object in $\mathrm{Corr}(\mathcal{X})$.

Lemma C.2.2. Let $F : \mathcal{O} \to \mathrm{Cat}_\infty$ be a functor with corresponding coCartesian fibration $p : \mathcal{C} \to \mathcal{O}$ and Cartesian fibration $F' : \mathcal{C}' \to \mathcal{O}^{op}$.

(1) There is a orthogonal factorization system

$$\mathcal{C}_{p\text{-coCart},\mathrm{fib}} = (\mathcal{C}_{p\text{-coCart}} \to \mathcal{C} \leftarrow \mathcal{C}_{\mathrm{fib}})$$

where $\mathcal{C}_{p\text{-cocart}} \subset \mathcal{C}$ is the wide subcategory spanned on p-cocartesian lifts, and $\mathcal{C}_{\mathrm{fib}} = \mathcal{C} \times_{\mathcal{O}} \mathrm{ob}(\mathcal{O})$ is the wide subcategory consisting of only maps in the same fiber.

(2) There is an equivalence of double-categories

$$\mathrm{Sq}(\mathcal{C})_{p\text{-coCart},\mathrm{fib}} \simeq \mathrm{Sq}(\mathcal{C}')^{1\,\mathrm{op}}_{q\text{-Cart},\mathrm{fib}}$$

that on $[1] \times [1]$-simplicies is given by

$$\begin{tikzcd} X \arrow[r] \arrow[d] & f_!X \arrow[d] & & X \arrow[d] & f^*X \arrow[l] \arrow[d] \\ Y \arrow[r] & f_!Y & \longleftrightarrow & Y & f^*Y \arrow[l] \end{tikzcd}$$

Proof. This is a consequence of Luries straightening-unstraightening. □

Lemma C.2.3. Let $(\mathcal{A}, \mathcal{A}_0)$ is a geometric setup.

(1) The category $\mathrm{Corr}(\mathcal{A}, \mathcal{A}_0)$ promotes to an orthogonal factorization system

$$\mathrm{Corr}(\mathcal{A}, \mathcal{A}_0)_{\mathrm{Leg},\mathrm{Reg}} = (\mathcal{A}^{op} \xrightarrow{\mathrm{Leg}} \mathrm{Corr}(\mathcal{A}, \mathcal{A}_0) \xleftarrow{\mathrm{Reg}} \mathcal{A}_0) \in \mathrm{OFS},$$

where Leg and Reg denote the left and right leg inclusions.

(2) There is an equivalence of double-categories

$$\mathrm{Fact}(\mathrm{Corr}(\mathcal{A}, \mathcal{A}_0)_{\mathrm{Leg},\mathrm{Reg}}) \simeq \mathrm{Sq}^{\mathrm{cart},\mathcal{A}_0}(\mathcal{A})^{1\,\mathrm{op}}$$

In particular,

$$\mathrm{Fact}(\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)^{\otimes}_{\mathrm{Leg},\mathrm{Reg}}) \simeq \mathrm{Sq}^{\mathrm{cart},\mathcal{C}_0^-}((\mathcal{C}^{op})^{\sqcup,op}) \simeq \mathrm{Sq}^{\mathrm{cart},\mathcal{C}_0^-}(\mathcal{C}_\times)$$

Proof. (1) is evident, as any span uniquely factors as

$$\left[\begin{tikzcd}[column sep=small] & Y \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Z \end{tikzcd}\right] \simeq \left[\begin{tikzcd}[column sep=small] & Y \arrow[dl, "\mathrm{id}"'] \arrow[dr, "g"] & \\ Y & & Z \end{tikzcd}\right] \circ \left[\begin{tikzcd}[column sep=small] & Y \arrow[dl, "f"'] \arrow[dr, "\mathrm{id}"] & \\ X & & Y \end{tikzcd}\right]$$

For (2), note that $\mathrm{Fact}(\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)^\dagger)([m],[n])$ sits in the cartesian square

$$\begin{tikzcd} \mathrm{Fact}(\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)^\dagger)([m],[n]) \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{cart}"] & \mathrm{Map}([m]\times[n]^\simeq,\mathcal{A}^{\mathrm{op}})\times\mathrm{Map}([m]^\simeq\times[n],\mathcal{A}_0) \arrow[d, "{(\mathrm{Leg},\mathrm{Reg})}"] \\ \mathrm{Map}([m]\times[n],\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)) \arrow[r] & \mathrm{Map}([m]\times[n]^\simeq,\mathrm{Corr}(\mathcal{A},\mathcal{A}_0))\times\mathrm{Map}([m]^\simeq\times[n],\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)). \end{tikzcd}$$

There are functorial maps

$$\begin{aligned} \mathrm{Sq}^{\mathrm{cart},\mathcal{A}_0}(\mathcal{A})^{1\,\mathrm{op}}([m],[n]) &\longrightarrow \mathrm{Map}(\mathrm{Corr}([m]^{\mathrm{op}}\times[n],[m]^\simeq\times[n]),\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)) \\ &\xrightarrow{(\mathrm{Leg}^{\mathrm{op}},\mathrm{Reg})} \mathrm{Map}([m]\times[n],\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)) \end{aligned}$$

and

$$\begin{aligned} \mathrm{Sq}^{\mathrm{cart},\mathcal{A}_0}(\mathcal{A})^{1\,\mathrm{op}}([m],[n]) &\subset \mathrm{Map}([m]^{\mathrm{op}}\times[n],\mathcal{A}) \\ &\longrightarrow \mathrm{Map}([m]\times[n]^\simeq,\mathcal{A}^{\mathrm{op}})\times\mathrm{Map}([m]^\simeq\times[n],\mathcal{A}_0), \end{aligned}$$

which induce a map of double-categories

$$\mathrm{Sq}^{\mathrm{cart},\mathcal{A}_0}(\mathcal{A})^{1\,\mathrm{op}} \longrightarrow \mathrm{Fact}(\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)^\dagger)$$

By the Segal-condition, it suffices to show that this map is an equivalence on the bisimplicies $[0]\times[0]$, $[0]\times[1]$, $[1]\times[0]$, and $[1]\times[1]$. The first three are follow more or less by defintion. For $[1]\times[1]$, note that there is a commuting diagram

$$\begin{tikzcd} \mathrm{Sq}^{\mathrm{cart},\mathcal{A}_0}(\mathcal{A})^{1\,\mathrm{op}}([1],[1]) \arrow[rr, "(!)"] \arrow[dr] & & \mathrm{Fact}(\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)^\dagger)([1],[1]) \arrow[dl] \\ & \mathrm{Map}(\Lambda^2_2,\mathcal{A}) & \end{tikzcd}$$

where the left map is induced by $\Lambda^2_2 = (0\to 2\leftarrow 1)\mapsto((0,0)\to(0,1)\leftarrow(1,1))$ (the bottom left corner), and the right map is induced by

$$\begin{tikzcd} \mathrm{Fact}(\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)^\dagger)([1],[1]) \arrow[r] \arrow[dd, dotted] & \mathrm{Map}([1]\times[1]^\simeq,\mathrm{Corr}(\mathcal{A},\mathcal{A}_0))\times\mathrm{Map}([1]^\simeq\times[1],\mathrm{Corr}(\mathcal{A},\mathcal{A}_0) \arrow[d] \\ & \mathrm{Map}([1]\times\{1\},\mathrm{Corr}(\mathcal{A},\mathcal{A}_0))\times\mathrm{Map}(\{0\}\times[1],\mathrm{Corr}(\mathcal{A},\mathcal{A}_0) \arrow[d] \\ \mathrm{Map}(\Lambda^2_2,\mathcal{A}) \arrow[r] & \mathrm{Map}([1],\mathcal{A})\times\mathrm{Map}([1],\mathcal{A}_0) \end{tikzcd}$$

The fiber over $(X\xrightarrow{f}Y\xleftarrow{g}Z):\Lambda^2_2\to\mathcal{A}$ in $\mathrm{Sq}^{\mathrm{cart},\mathcal{A}_0}(\mathcal{A})^{1\,\mathrm{op}}([1],[1])$ is contractible, as it parametrizes pullbacks of that cospan. Similarly, the fiber in $\mathrm{Fact}(\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)^\dagger)([1],[1])$ is the full sub-anima on quadruples of spans satisfying

$$\left[\begin{tikzcd}[column sep=small] & Z \arrow[dl, "g"'] \arrow[dr, "\mathrm{id}"] & \\ Y & & Z \end{tikzcd}\right] \circ \left[\begin{tikzcd}[column sep=small] & X \arrow[dl, "\mathrm{id}'"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}\right] \simeq \left[\begin{tikzcd}[column sep=small] & W \arrow[dl, "\mathrm{id}"'] \arrow[dr, "f'"] & \\ W & & Z \end{tikzcd}\right] \circ \left[\begin{tikzcd}[column sep=small] & W \arrow[dl, "g'"'] \arrow[dr, "\mathrm{id}"] & \\ X & & W \end{tikzcd}\right].$$

Consequently, the square

$$\begin{tikzcd} X & W \arrow[l, "f'"'] \arrow[d, "g'"] \\ Y \arrow[from=u, "f"'] & Z \arrow[l, "g"] \end{tikzcd}$$

is cartesian, so the fiber is contractible. Hence by the 2-out-of-3 property, the map (!) is an equivalence. ☐

Proposition C.2.4. Let $p : \mathcal{C}^{\otimes} \to \mathsf{Assoc}^{\otimes}$ be a monoidal operad, and let $(\mathcal{D}, \mathcal{D}_0)$ be a geometric setup such that $\mathcal{D}$ has finite finite limits. If $F : \mathcal{C}_{\otimes} \to \mathcal{D}_{\times} \simeq (\mathcal{D}^{\mathrm{op}})^{\sqcup,\mathrm{op}})$ is an oplax monoidal functor such that

(i) for every $f : X \simeq \bigoplus_{i=1}^{m} X_i \to Y \simeq \bigoplus_{i=1}^{m} Y_i$, $\alpha : \langle m \rangle \to \langle n \rangle$ in $\mathsf{Assoc}^{\otimes}$, and $i = 1, 2, \ldots, m$, the square

$$\begin{tikzcd} \prod_{\alpha^{-1}(i)} F(X_i) \arrow[d] & F(\bigotimes_{j\in\alpha^{-1}(i)} X_j) \arrow[l] \arrow[d] \\ \prod_{\alpha^{-1}(i)} F(Y_j) & F(\bigotimes_{j\in\alpha^{-1}(i)} Y_j) \arrow[l] \end{tikzcd}$$

is cartesian (in $\mathcal{C}$), and

(ii) $F : \mathcal{C} \to \mathcal{D}$ factors through $\mathcal{D}_0$,

then F promotes to a lax monoidal functor

$$\mathcal{C}^{\otimes} \longrightarrow \mathrm{Corr}(\mathcal{D}, \mathcal{D}_0)^{\otimes}$$

that takes

(1) an object $X \simeq \oplus_{i=1}^{m} X_i \in \mathcal{D}^{\otimes}$ over $\langle m \rangle \in \mathsf{Assoc}^{\otimes}$ to $F(X) \simeq \oplus_{i=1}^{m} F(X_i) \in (\mathcal{C}^{\mathrm{op}})^{\sqcup,\mathrm{op}}$,

(2) a morphism $X \simeq \oplus_{i=1}^{m} X_i \xrightarrow{f} Y \simeq \oplus_{i=1}^{n} Y_i$ over $\alpha : \langle m \rangle \to \langle n \rangle \in \mathsf{Assoc}^{\otimes}$ to the span

$$\begin{tikzcd} & \bigoplus_{i=1}^{n} F(\bigotimes_{j\in\alpha^{-1}(i)} X_j) \arrow[dl, "\text{oplax}"'] \arrow[dr, "F(f)"] & \\ \bigoplus_{i=1}^{m} F(X_i) & & \bigoplus_{i=1}^{n} F(Y_i) \end{tikzcd}$$

(3) and a 2-cell $X \xrightarrow{f} Y \xrightarrow{g} Z$ over $\langle m \rangle \xrightarrow{\alpha} \langle n \rangle \xrightarrow{\beta} \langle k \rangle$ to the span

$$\begin{tikzcd} & & \bigoplus_{i=1}^{k} F(\bigotimes_{j\in(\beta\alpha)^{-1}(i)} X_j) \arrow[dl, "\text{oplax}"'] \arrow[dr, "F(f)"] & & \\ & \bigoplus_{i=1}^{m} F(\bigotimes_{j\in\alpha^{-1}(i)} X_j) \arrow[dl, "\text{oplax}"'] \arrow[dr, "F(f)"'] & & \bigoplus_{i=1}^{k} F(\bigotimes_{j\in\beta^{-1}(i)} Y_j) \arrow[dl, "\text{oplax}"'] \arrow[dr, "F(g)"] & \\ \bigoplus_{i=1}^{n} F(X_i) & & \bigoplus_{i=1}^{m} F(Y_i) & & \bigoplus_{i=1}^{k} F(Z_i) \end{tikzcd}$$

Proof. In general, if $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and $q : \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ are $\mathcal{O}$-monoidal operads, and $F : \mathcal{C}_{\otimes} \to \mathcal{D}_{\otimes}$ is an oplax $\mathcal{O}$-monoidal functor (i.e., a functor of dual operads over $\mathcal{O}_{\otimes}$), then there is a map

$$\mathrm{Fact}(\mathcal{C}^{\otimes}_{p\text{-coCart,fib}}) \longrightarrow \mathrm{Sq}(\mathcal{C}^{\otimes})_{p\text{-coCart,fib}} \simeq \mathrm{Sq}(\mathcal{C}_{\otimes})^{1\,\mathrm{op}}_{q\text{-Cart,fib}} \xrightarrow{F\circ-} \mathrm{Sq}(\mathcal{D}_{\otimes})^{1\,\mathrm{op}}_{\text{general,fib}}$$

where $\mathrm{Sq}(\mathcal{D}_{\otimes})^{1\,\mathrm{op}}_{\text{general,fib}}$ denotes the category of squares in $\mathcal{D}_{\otimes}$ for which the vertical maps are in the same fiber.

In our setting, the two assumptions assert that this composite factors through $\mathrm{Sq}^{\mathrm{cart},\mathcal{D}_0^-}((\mathcal{D}^{\mathrm{op}})^{\sqcup,\mathrm{op}})$; indeed, we need F to take the (cartesian) square (in $\mathcal{C}_\otimes$) to the left to a cartesian square (in $\mathcal{D}_\times$) to the right:

$$\begin{tikzcd}
\bigoplus_{i=1}^m X_i \arrow[d] & \bigoplus_{i=1}^n(\bigotimes_{j\in\alpha^{-1}(i)} X_j) \arrow[l] \arrow[d] \\
\bigoplus_{i=1}^m Y_i & \bigoplus_{i=1}^n(\bigotimes_{j\in\alpha^{-1}(i)} Y_j) \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
\bigoplus_{i=1}^m F(X_i) \arrow[d] & \bigoplus_{i=1}^n F(\bigotimes_{j\in\alpha^{-1}(i)} X_j) \arrow[l] \arrow[d] \\
\bigoplus_{i=1}^m F(Y_i) & \bigoplus_{i=1}^n F(\bigotimes_{j\in\alpha^{-1}(i)} Y_j) \arrow[l]
\end{tikzcd}$$

Since $\mathcal{D}_\times$ is a (dual) operad, we can find cocartesian lifts factoring the square on the right as

$$\begin{tikzcd}
\bigoplus_{i=1}^m F(X_i) \arrow[d] & \bigoplus_{i=1}^n \prod_{j\in\alpha^{-1}(i)} F(X_j) \arrow[l] \arrow[d] & \bigoplus_{i=1}^n F(\bigotimes_{j\in\alpha^{-1}(i)} X_j) \arrow[l] \arrow[d] \\
\bigoplus_{i=1}^m F(Y_i) & \bigoplus_{i=1}^n \prod_{j\in\alpha^{-1}(i)} F(X_j) \arrow[l] & \bigoplus_{i=1}^n F(\bigotimes_{j\in\alpha^{-1}(i)} Y_j) \arrow[l]
\end{tikzcd}$$

The left square is cartesian, so it suffices to show that the right square is as well, but this is exactly assumption (i). So by Lemma C.2.3.(2) and fully faithfulness of $\mathrm{Fact} : \mathrm{OFS} \to \mathrm{DCat}$ [Jur25, Theorem A], there is a map of orthogonal factorization systems

$$\mathcal{C}^{\otimes}_{p\text{-coCart},\mathrm{fib}} \longrightarrow \mathrm{Corr}(\mathcal{D},\mathcal{D}_0)^{\otimes}_{\mathrm{Leg},\mathrm{Reg}}.$$

By construction, the underlying functor is a lax monoidal functor

$$\mathcal{C}^{\otimes} \longrightarrow \mathrm{Corr}(\mathcal{D},\mathcal{D}_0)^{\otimes}$$

which does exactly as claimed (1), (2), and (3). $\square$

Proof of Theorem C.2.1. It suffices to verify condition (i) of Proposition C.2.4 as (ii) is trivially satisfied in the case at hand. For simplicity, we will treat the case $|\alpha^{-1}(i)| = 2$. From Lemma C.1.5, the outer square and the right square are pullbacks, and therefore the left is as well:

$$\begin{tikzcd}
A\backslash(X_1 \times^A X_2)/A \arrow[r] \arrow[d] & A\backslash(Y_1 \times^A Y_2)/A \arrow[r] \arrow[d] & */A \arrow[d] \\
A\backslash X_1/A \times A\backslash X_2/A \arrow[r] & A\backslash Y_1/A \times A\backslash Y_2/A \arrow[r] & */A \times A\backslash *
\end{tikzcd}$$

Recall that the operadic structure on $\mathrm{LMod}_A(\mathcal{X}^\times)^{\otimes_A}$ is the diagonal action, so this proves (i). $\square$

Remark C.2.5. If $D : \mathrm{Corr}(\mathcal{C})^{\otimes} \to \mathrm{Cat}^{\times}$ is a six-functor formalism (a lax symmetric monoidal map), then Theorem C.2.1 implies that there are maps

$$\mathrm{Alg}({}_A\mathrm{BMod}_A(\mathcal{C}^\times)^{\otimes_A}) \longrightarrow \mathrm{Alg}(\mathrm{Corr}(\mathcal{C})^{\otimes}) \longrightarrow \mathrm{Alg}(\mathrm{Cat}^\times) \simeq \mathrm{Mon}, \quad X \mapsto D(A\backslash X/A)^{\star}$$

We think of on object $X \in \mathrm{Alg}({}_A\mathrm{BMod}_A(\mathcal{C}^\times)^{\otimes_A})$ as an "A-A-equivariant monoid in $\mathcal{C}$", and the monoidal category $D(A\backslash X/A)^{\star}$ as "A-A-equivariant sheaves on X". The monoidal structure $-\star-$ is called *convolution*. Explicitly, if $m : X \times^A X \to X$ is the multiplication on X, then the multiplication on $A\backslash X/A$ in $\mathrm{Corr}(\mathcal{C})$ is given by the span

$$\begin{tikzcd}
& A\backslash(X \otimes_A X)/A \arrow[dl, "{(p_1,p_2)}"'] \arrow[dr, "m"] & \\
A\backslash X/A \times A\backslash X/A & & A\backslash X/A,
\end{tikzcd}$$

and thus the monoidal structure on $D(X)$ is given by

$$D(A\backslash X/A) \times D(A\backslash X/A) \xrightarrow{-\boxtimes-} D(A\backslash X/A \times A\backslash X/A) \xrightarrow{(p_1,p_2)^*} D(A\backslash(X \otimes_A X)/A) \xrightarrow{m_!} D(X)$$

Example C.2.6. Let $\mathcal{C} \simeq \mathrm{Shv}(\mathrm{Aff}_k^{\mathrm{fpqc}})$, our monoid-object is LG which is acted upon by L^+G by left and right multiplication, and D is the constructible six-functor formalism on underling analytic topology with a given stratification (e.g. the one in L^+G-orbits), then we get a convolution product on the double-fpqc-quotient $\mathrm{Hck}_G = L^+G\backslash LG/L^+G$, or equivalently, L^+G-equivariant constructible sheaves on the affine grassmannian $\mathrm{Gr}_G = LG/L^+G$ with L^+G-orbit stratification.

We also get a version for the Beilinson-Drinfeld affine Grassmannian.

Remark C.2.7. The idea of embedding the problem into doublecategories was communicated to me by Robert Burklund.

References

[Ach21] Pramod N. Achar. *Perverse Sheaves and Applications to Representation Theory*, volume 258 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, Rhode Island, 2021.

[Alp18] Jarod Alper. Adequate moduli spaces and geometrically reductive group schemes, 2018. URL: https://arxiv.org/abs/1005.2398, arXiv:1005.2398.

[BBD] Alexander A. Beilinson, Joseph Bernstein, and Pierre Deligne. Faisceaux pervers. 100:5–171. URL: https://publications.ias.edu/sites/default/files/Faisceaux%20pervers.pdf.

[BD94] Alexander Beilinson and Vladimir Drinfeld. Quantization of hitchin's integrable system and hecke eigensheaves. Preprint, 1994. URL: https://math.uchicago.edu/~drinfeld/langlands/QuantizationHitchin.pdf.

[BL95] Arnaud Beauville and Yves Laszlo. Un lemme de descente. *C. R. Acad. Sci. Paris Sér. I Math.*, 320(3):335–340, 1995.

[BR18] Pierre Baumann and Simon Riche. Notes on the geometric satake equivalence, 2018. URL: https://arxiv.org/abs/1703.07288, arXiv:1703.07288.

[Bra03] Tom Braden. Hyperbolic localization of intersection cohomology, 2003. URL: https://arxiv.org/abs/math/0202251, arXiv:math/0202251.

[Cla08] Dustin Clausen. The springer correspondence, 2008. URL: https://www.math.harvard.edu/media/clausen.pdf.

[DG14] Vladimir Drinfeld and Dennis Gaitsgory. On a theorem of braden, 2014. URL: https://arxiv.org/abs/1308.3786, arXiv:1308.3786.

[DM18] Pierre Deligne and James S. Milne. Tannakian categories, 2018. Updated and corrected version of the article originally published in *Hodge Cycles, Motives, and Shimura Varieties*, Lecture Notes in Mathematics 900, Springer, 1982, pp. 101–228. URL: https://www.jmilne.org/math/xnotes/tc2018.pdf.

[HM24] Claudius Heyer and Lucas Mann. 6-functor formalisms and smooth representations, 2024. URL: https://arxiv.org/abs/2410.13038, arXiv:2410.13038.

[Jur25] Branko Juran. On orthogonal factorization systems and double categories, 2025. URL: https://arxiv.org/abs/2501.01363, arXiv:2501.01363.

[Lur09] Jacob Lurie. *Higher topos theory.* Princeton University Press, 2009.

[Lur17] Jacob Lurie. Higher algebra, 2017. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf.

[MV18] I. Mirkovic and K. Vilonen. Geometric langlands duality and representations of algebraic groups over commutative rings, 2018. URL: https://arxiv.org/abs/math/0401222, arXiv:math/0401222.

[Noc25] Guglielmo Nocera. A model for the e3 fusion-convolution product of constructible sheaves on the affine grassmannian, 2025. URL: https://arxiv.org/abs/2012.08504, arXiv:2012.08504.

[NSS14] Thomas Nikolaus, Urs Schreiber, and Danny Stevenson. Principal ∞-bundles: general theory. *Journal of Homotopy and Related Structures*, 10(4):749–801, June 2014. URL: http://dx.doi.org/10.1007/s40062-014-0083-6, doi:10.1007/s40062-014-0083-6.

[PR08] G. Pappas and M. Rapoport. Twisted loop groups and their affine flag varieties, 2008. URL: https://arxiv.org/abs/math/0607130, arXiv:math/0607130.

[Spr98] Tonny Albert Springer. *Linear Algebraic Groups*, volume 9 of *Progress in Mathematics*. Birkhäuser, 2 edition, 1998. doi:10.1007/978-0-8176-4840-4.

[Sta26] The Stacks project authors. The stacks project. https://stacks.math.columbia.edu, 2026.

[Tit79] J. Tits. Reductive groups over local fields. In *Automorphic forms, representations and L-functions*, volume XXXIII of *Proc. Sympos. Pure Math.*, pages 29–69, Providence, R.I., 1979. Amer. Math. Soc.

[Wan] Jonathan P. Wang. Functor of points description of the flag variety. Personal notes. URL: https://www.jonathanpwang.com/notes/Flag.Variety.pdf.

[Zhu16] Xinwen Zhu. An introduction to affine grassmannians and the geometric satake equivalence, 2016. URL: https://arxiv.org/abs/1603.05593, arXiv:1603.05593.