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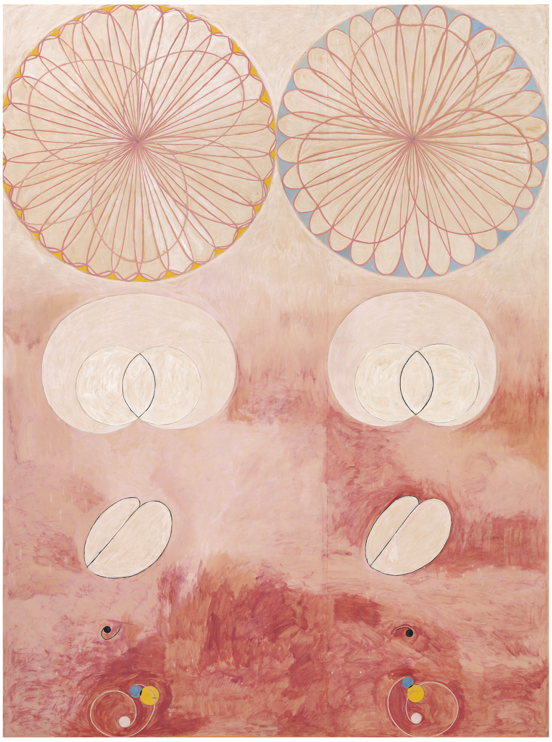

PhD thesis

Sheaves and moduli spaces of manifolds

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Cover image: *The Ten Largest, No. 9 (Old Age)*, Hilma af Klint, 1907. Public domain.

With solidarity and love for Palestine.

Summary

This thesis examines two topics in the intersection between topology and geometry: sheaves and moduli spaces of manifolds. These themes are tied together by the fourth and last article contained in the synopsis below. In this article, we use the six-functor formalism for sheaves on locally compact Hausdorff spaces (as developed by Verdier, Lurie, and Volpe) to produce relations in the cohomology of moduli spaces of manifolds with boundary. The key ingredient in this work is a sheaf-theoretic enhancement of the family Stokes theorem, which we hope may be of independent interest.

Of the remaining three articles, the first gives a new proof of Harer's celebrated stability theorem for moduli spaces of two-dimensional manifolds (also known as surfaces) with the optimal slope. This is joint work with Max Vistrup and Nathalie Wahl.

The second and third article are concerned with sheaf categories. Specifically, the second article classifies the compact objects in the category of sheaves on a locally compact Hausdorff space with values in an arbitrary presentable stable ∞-category, extending results of Neeman. Interestingly, it follows from our classification that the subcategory of compact sheaves on a nice compact space only depend on the homotopy type of the space. In the third article, we pursue further the problem of extracting information about a topological space from its category of sheaves. In this direction, we show that the homeomorphism type of a locally finite one-dimensional CW complex can be recovered from its category of sheaves by examining the Serre functor on this category.

Sammenfatning

Denne afhandling undersøger to emner i krydsfeltet mellem topologi og geometri: knipper og modulirum af mangfoldigheder. Bindeleddet mellem disse temaer er den fjerde og sidste artikel, der indgår i synopsen nedenfor. I denne artikel bruger vi seksfunktorformalismen for knipper på lokalkompakte Hausdorffrum (udviklet af Verdier, Lurie og Volpe) til at fremstille relationer i kohomologien af modulirum af mangfoldigheder med rand. Hovedingrediensen i dette arbejde er en knippeteoretisk forbedring af Stokes' sætning for fiberbundter, som vi håber kan have uafhængig interesse.

Blandt de resterende tre artikler giver den første et nyt bevis for Harers berømte stabilitetssætning for modulirum af todimensionelle mangfoldigheder (det vil sige flader) med den optimale hældning. Dette projekt er udført i samarbejde med Max Vistrup og Nathalie Wahl.

Den anden og tredje artikel omhandler knippekategorier. Nærmere bestemt klassificerer den anden artikel de kompakte objekter i kategorien af knipper på et lokalkompakt Hausdorffrum med værdier i en vilkårlig præsentabel stabil ∞-kategori og udvider derved Neemans resultater. Det følger interessant nok af vores klassifikation, at delkategorien af kompakte knipper på et pænt kompakt rum kun afhænger af rummets homotopitype. I den tredje artikel forfølger vi yderligere problemet om at udlede information om et topologisk rum fra dets kategori af knipper. I denne retning viser vi, at homeomorfitypen af et lokalt endeligt endimensionelt CW-kompleks kan rekonstrueres ud fra dets kategori af knipper ved at undersøge Serre-funktoren på denne kategori.

Acknowledgements

For the past seven-or-so years, I have benefitted enormously from the kindness and sagacity of my advisors, Jesper Grodal and Nathalie Wahl. It is my pleasure to thank you here, not only for teaching me how to be a mathematician, but also for heading/parenting what is probably the liveliest and warmest topology center in the world. I could truly not have asked for better supervisors (that's why I stuck around for so long)!

Aside from my math parents, I also wish to warmly thank my math uncle, Lars Hesselholt. Your generous instruction and encouragement has had a huge influence not just on the work contained in this thesis, but also on various ongoing projects that will perhaps occupy me for years to come.

I am deeply grateful to Oscar Randal-Williams for hosting and mentoring me during my stay in Cambridge. A large portion of this thesis grew out of the many fruitful conversations I had with you.

I heartily thank Johannes Ebert, Henrik Holm, and Achim Krause for agreeing to be on my committee.

I have already mentioned the Copenhagen topology group. It breaks my heart to say goodbye and thanks to this wonderful collection of people (including its non-topologist hangarounds): Alexis Aumonier, Bai Qingyuan, Nena Batenburg, Tim With Berland, Robert Burklund, Shachar Carmelli, Jonathan Clivio, Adriano Córdova, Marie-Camille Delarue, Desirée Gijón Gómez, Kaif Hilman, Branko Juran, Dani Kaufman, Priya Kaveri, Marius Kjærsgaard, Ishan Levy, Erik Lindell, Liu Jinyang, Fadi Mezher, Thomas Jan Mikhail, Isaac Moselle, Marius Verner Bach Nielsen, Azélie Picot, Maxime Ramzi, Cecilie Olesen Recke, Florian Riedel, Robert Szafarczyk, Jan Steinebrunner, Vignesh Subramanian, Jan Tapdrup, Philippe Vollmuth, and Adela Zhang. Thanks also to Lukas Junge and Boris Bolvig Kjær for volunteering to help with my grading duties during the weeks leading up to my submission.

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Oscar Bendix Jin-Harr
August 2025

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Introduction

I veed vel nok det hedder: løst Knippe kan let glippe!
— Saxo Grammaticus, *Gesta Danorum*[1]

One of the most fruitful ideas in geometry is that we should not only work with one thing at a time, but also with the collection of all things of a given kind treated as a single whole. Frequently, this collection—known as a *moduli space*—has a rich geometric, topological or algebraic structure in its own right, containing subtle information about the original objects of interest.

An early and influential manifestation of this idea is the moduli space of surfaces, first introduced by Riemann [Rie57]. In this moduli space, points correspond to surfaces and nearby points represent surfaces that differ by a small deformation, see Figure 1 or the painting by Hilma af Klint on the cover. The study of Riemann's moduli space remains a central and active area of research, connecting geometry, physics, and number theory. In this

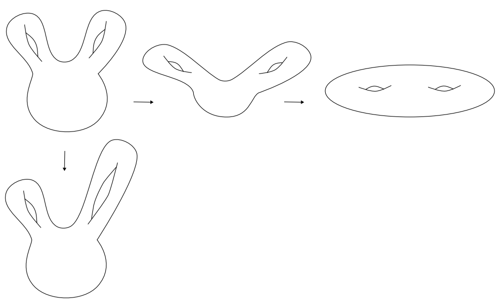

Figure 1: Nearby points in the moduli space of surfaces

thesis, we focus on one particular aspect of moduli spaces of surfaces and higher-dimensional manifolds, namely their *cohomology*.

Like moduli spaces, the notion of cohomology first emerged in the second half of the 19th century. Cohomology, even more than moduli spaces, has proved to be such a powerful idea that it now pervades mathematics and mathematical physics.

[1]translated from the original Latin by N. F. S. Grundtvig

It is particularly illuminating to view cohomology through the lens of *sheaf theory*. From this perspective, cohomology arises from the subtleties of assembling global information from local information. Furthermore, sheaf theory comes with a so-called *formalism of six operations*, which is an exceptionally powerful and user-friendly interface for studying and calculating cohomology. Cohomology with or without sheaves plays a key role in this thesis.

In the first article of this thesis, we give a new proof of Harer's stability theorem for the moduli space of surfaces. Usually, homological stability results are proved by placing the groups of interest (implicitly or explicitly) in a monoidal category with a braiding. The monoidal category that appears in our work does not admit a braiding. It does however come with a Yang–Baxter operator on the stabilizing object, and we observe that this structure is sufficient to run the stability machine of Randal-Williams–Wahl and Krannich. The upshot is a very simple proof of Harer's stability theorem with the optimal slope. This is joint work with Max Vistrup and Nathalie Wahl, and has appeared in *Higher Structures* 8(1), 193–223.

In article two, we classify the compact objects in the category of sheaves on a locally compact Hausdorff space valued in a presentable stable ∞-category. The classification shows that there are few compact objects; in particular, we recover Neeman's result that there are no non-trivial compact objects in the derived category of sheaves of abelian groups on a non-compact connected manifold. It follows from our classification that the category of derived sheaves on a locally compact Hausdorff space is compactly generated if and only if the space is totally disconnected. This paper was first published in *Proceedings of the American Mathematical Society*, vol. 153(1), American Mathematical Society, Providence, RI, 2025.

In the third article, we consider the problem of extracting information about a space from its derived category of sheaves. Analogous problems have received a lot of attention in algebraic geometry, and have given rise to the field of *non-commutative geometry* (à la Kontsevich). The maximal amount of information that one could hope to extract from the category of sheaves on a space is its homeomorphism type. This is trivially possible if the space is discrete, or in other words a zero-dimensional CW complex. We show that it is also possible for one-dimensional CW complexes. The proof is inspired by Bondal and Orlov's proof of their non-commutative reconstruction theorem for Fano and anti-Fano varieties.

In the fourth article, we apply sheaf-theoretic methods to study the unstable cohomology of moduli spaces of manifolds with boundary. The strategy is closely inspired by work of Randal-Williams on moduli spaces of closed manifolds. The key ingredient in our proof (which distinguishes it from the closed manifold case) is a sheaf-theoretic enhancement of the family Stokes theorem. We hope that this theorem will be of interest beyond our applications to the cohomology of moduli spaces.

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1 Disordered arcs and Harer stability

DISORDERED ARCS AND HARER STABILITY

OSCAR HARR, MAX VISTRUP, AND NATHALIE WAHL

Abstract. We give a new proof of homological stability with the best known isomorphism range for mapping class groups of surfaces with respect to genus. The proof uses the framework of Randal-Williams–Wahl and Krannich applied to disk stabilization in the category of bidecorated surfaces, using the Euler characteristic instead of the genus as a grading. The monoidal category of bidecorated surfaces does not admit a braiding, distinguishing it from previously known settings for homological stability. Nevertheless, we find that it admits a suitable Yang–Baxter element, which we show is sufficient structure for homological stability arguments.

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1. Introduction

Let $S^s_{g,r}$ be a surface of genus g with r boundary components and s punctures. The mapping class group $\Gamma(S^s_{g,r}) := \pi_0\mathrm{Homeo}(S^s_{g,r} \text{ rel } \partial S)$ of S satisfies *homological stability*: the homology group $H_i(\Gamma(S^s_{g,r});\mathbb{Z})$

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is independent of g and r when g is large relative to i. This stability result was originally proved by Harer in [Har85], and later improved by Ivanov, Boldsen and Randal-Williams [Iva89, Bol12, RW16], see also [Har93, Wah13, HV17, GKRW19]. We recast the result here as a stability theorem in the category of *bidecorated surfaces*, and give a new proof of the best know stability range using the most straightforward inductive argument originally designed by Quillen, and formalized in [RWW17, Kra19]. Our proof at the same time illustrates how little is needed to run the stability machines of these two papers.

Our main stability result is the following, recovering precisely the ranges of [Bol12, Thm 1] and [RW16, Thm 7.1 (i),(ii)]:

Theorem A. Let $S^s_{g,b}$ be a surface of genus $g \geq 0$, with $r \geq 1$ marked boundary components and $s \geq 0$ punctures, and let $\Gamma(S^s_{g,r}) = \pi_0\mathrm{Homeo}(S^s_{g,r} \text{ rel } \partial S)$ denote its mapping class group. The map

$$H_i(\Gamma(S^s_{g,r});\mathbb{Z}) \longrightarrow H_i(\Gamma(S^s_{g,r+1});\mathbb{Z})$$

induced by gluing a pair or pants along one boundary component is always injective, and an isomorphism when $i \leq \frac{2g}{3}$, and the map

$$H_i(\Gamma(S^s_{g,r+1});\mathbb{Z}) \longrightarrow H_i(\Gamma(S^s_{g+1,r});\mathbb{Z})$$

induced by gluing a pair of pants along two boundary components is an epimorphism when $i \leq \frac{2g+1}{3}$ and an isomorphism when $i \leq \frac{2g-2}{3}$.

Combining the two maps in the theorem gives a genus stabilization that is known to be close to optimal by a computation of Morita [Mor03] and low dimensional computations, see Remarks 2.5 and 4.11. While we do not know whether the two ranges in the above statement can be individually improved, it is remarkable that three rather different proofs (those of Boldsen [Bol12], Randal-Williams [RW16], and ours) end up with the exact same ranges.

A particular feature of our proof is that the two maps occurring in the theorem will be for us "the same map", namely a disk stabilization in the category $\mathbf{M}_2$ of *bidecorated surfaces*. A bidecorated surface is a surface S with two marked intervals I_0, I_1 in its boundary. The two intervals may lie on the same or on different boundary components. Morphisms in $\mathbf{M}_2$ are mapping classes, i.e. isotopy classes of homeomorphisms, and $\mathbf{M}_2$ admits a monoidal structure # defined by identifying the marked intervals in pairs.

Our main example of a bidecorated surface will be the bidecorated disk D. As shown in Lemma 3.1, taking sums of the disk with itself in $\mathbf{M}_2$ produces surfaces of any genus: $D^{\#2g+1}$ is a surface $S_{g,1}$ of genus g with a single boundary component, while $D^{\#2g+2}$ is a surface $S_{g,2}$ of genus g with two boundary components, each containing a marked interval. To obtain any surface $S^s_{g,r}$ with $r \geq 1$, we will consider the object $S \# D^{\#2g}$ in $\mathbf{M}_2$, for $S = S^s_{0,r}$ a genus 0 surface with r boundary components and s punctures. Now the maps in Theorem A are precisely the disk stabilization maps in $\mathbf{M}_2$:

$$\mathrm{Aut}_{\mathbf{M}_2}(S \# D^{\#2g}) \xrightarrow{\#D} \mathrm{Aut}_{\mathbf{M}_2}(S \# D^{\#2g+1}) \xrightarrow{\#D} \mathrm{Aut}_{\mathbf{M}_2}(S \# D^{\#2g+2})$$

for these particular choices of surfaces.

Theorem A is thus the statement that disk stabilization $\#D$ in $\mathbf{M}_2$ induces isomorphisms on the homology of these automorphism groups in a range.

We show in the present paper that this result can be obtained as a direct application of the main result of [Kra19], from which an additional stability statement with twisted coefficients automatically follows. We start by stating this additional result.

Twisted coefficients. Fix $r \geq 1$ and $s \geq 0$. In our setting, a coefficient system F for the mapping class group $\Gamma(S^s_{g,r})$ is a collection of $\mathbb{Z}[\Gamma(S^s_{g,r})]$-modules F_{2g} and $\mathbb{Z}[\Gamma(S^s_{g,r+1})]$-modules F_{2g+1} for each $g \geq 0$, together with maps

$$F_n \longrightarrow F_{n+1}$$

equivariant with respect to the disk stabilization and satisfying that a certain Dehn twist acts trivially on the image of F_n in F_{n+2} under double stabilization (see Definition 4.6). Given a coefficient system, one can define a notion of degree; a constant coefficient systems has degree 0 and for example the coefficient system $F_{2g+i} = H_1(S^s_{g,r+i};\mathbb{Z})^{\otimes k}$, $i \in \{0,1\}$, has degree k (see Example 4.7).

We obtain the following twisted stability result:

Theorem B. Let $\Gamma(S^s_{g,r})$ be as in Theorem A, and F be a coefficient system of degree k. The stabilization map

$$H_i(\Gamma(S^s_{g,r}); F_{2g}) \longrightarrow H_i(\Gamma(S^s_{g,r+1}); F_{2g+1})$$

is an epimorphism for $i \leq \frac{2g-3k-2}{3}$ and an isomorphism for $i \leq \frac{2g-3k-5}{3}$, and the map

$$H_i(\Gamma(S^s_{g,r+1}); F_{2g+1}) \longrightarrow H_i(\Gamma(S^s_{g+1,r}); F_{2g+2})$$

is an epimorphism for $i \leq \frac{2g-3k-1}{3}$ and an isomorphism for $i \leq \frac{2g-3k-4}{3}$. In these bounds, $3k$ can be replaced by k if F is in addition split in the sense of Definition 4.6.

Stability theorems for mapping class groups with twisted coefficients can be found in the work of Ivanov, Boldsen, Randal-Williams–Wahl, and Galatius–Kupers–Randal-Williams [Bol12, Iva93, RWW17, GKRW19]. The results are not easy to compare as the types of coefficient system that are permitted depend on the paper, but some classical examples such as the one described above fit all frameworks (see Remarks 4.8 and 4.11 for more details).

Braided action and Yang–Baxter operators. We want to obtain Theorems A and B as consequences of Theorems A and C of [Kra19]. For this, we first have to show that disk stabilization in the monoidal category $(\mathbf{M}_2, \#)$ comes from an action of a braided monoidal groupoid.

Let $\mathbf{B}$ denote the groupoid of braid groups, with object the natural numbers and the braid group B_n as automorphisms of n. We will construct an action of $\mathbf{B}$ on $\mathbf{M}_2$ using an appropriate *Yang–Baxter operator* in $\mathbf{M}_2$: The sum of bidecorated disks $D \# D$ in $\mathbf{M}_2$ is a cylinder, whose mapping class group is an infinite cyclic group generated by the Dehn twist T along the core circle of the cylinder. It turns out that this morphism $T \in \mathrm{Aut}_{\mathbf{M}_2}(D \# D)$ is a Yang–Baxter operator in $\mathbf{M}_2$, in the sense that it satisfies the equation

$$(T \# 1)(1 \# T)(T \# 1) = (1 \# T)(T \# 1)(1 \# T)$$

in $\mathrm{Aut}_{\mathbf{M}_2}(D^{\#3})$. The same holds for the inverse twist T^{-1}, that will turn out more convenient for us. As explained in Section 5.1, we get an associated strong monoidal functor $\mathbf{B} \longrightarrow \mathbf{M}_2$ taking the object n to $D^{\#n}$. The corresponding homomorphism $B_n \longrightarrow \mathrm{Aut}_{\mathbf{M}_2}(D^{\#n})$ can be identified with the geometric embedding in the sense of [Waj99], associated to the chain of curves $a_1, \ldots, a_{n-1}$ in

$$D^{\#n} = D \# D \# \cdots \# D,$$

where the ith curve a_i is the core circle in the ith cylinder $D \# D$ in the above sum, see Lemma 3.5 and Example 5.3.

The strong monoidal functor $\mathbf{B} \longrightarrow \mathbf{M}_2$ from above endows $\mathbf{M}_2$ with the structure of an E_1-module over the braid groupoid $\mathbf{B}$, and since the latter is braided monoidal, we can apply the results of [Kra19] to study disk stabilization in $\mathbf{M}_2$.

Remark 1.1. Homological stability frameworks such as [RWW17, Kra19, GKRW19] require an E_2-algebra, or the weaker structure of E_1-module over an E_2-algebra, as input. This is a priori a lot of data, and it may be that the most natural choice in a given context simply does not admit an E_2-structure. This turns out to be the case for the monoidal category of bidecorated surfaces $\mathbf{M}_2$: In the context of categories, E_2-structures are given by braided monoidal structures and we show in Section 5.3 that even the full monoidal subcategory of $\mathbf{M}_2$ generated by our stabilizing object, the disk D, does not admit a braiding. This distinguishes our situation from most previous examples of homological stability.

On the other hand, it does not take much to equip a given monoidal category $\mathscr{X}$ with the structure of an E_1-module over a braided monoidal category. In fact, as shown in Section 5.1, any Yang–Baxter operator in $\mathscr{X}$ determines a strong monoidal functor $\mathbf{B} \longrightarrow \mathscr{X}$ from the braid groupoid $\mathbf{B}$, and thus endows $\mathscr{X}$ with the structure of an E_1-module over $\mathbf{B}$. This perspective also makes sense if $\mathscr{X}$ itself acts on a category $\mathscr{M}$, and one is interested in the stabilization

$$\mathscr{M} \xrightarrow{\oplus X} \mathscr{M} \xrightarrow{\oplus X} \cdots$$

induced by acting with an object X of $\mathscr{X}$ admitting a Yang-Baxter operator $\tau \in \mathrm{Aut}_{\mathscr{X}}(X \oplus X)$. The category $\mathscr{M}$ becomes this way likewise a module over $\mathbf{B}$, where the object n of $\mathbf{B}$ acts on $A \in \mathscr{M}$ via $A \oplus n = A \oplus X^{\oplus n}$.

Disordered arcs. Given a category $\mathscr{M}$ as above, with the structure of an E_1-module over a monoidal category $\mathscr{X}$ with a distinguished Yang–Baxter operator (X, τ), such that acting by X satisfies a certain injectivity property (see Proposition 3.4), the main result of [Kra19] implies that homological stability for stabilization with X is controlled by the connectivity of certain *complexes of destabilizations*. In the category of bidecorated surfaces $\mathbf{M}_2$, stabilizing with the bidecorated disk D corresponds homotopically to attaching an arc, and we show in Proposition 4.4 that the relevant complex of destabilizations for stabilizing a surface S with a disk n times identifies with the "disordered arc complex"[1] associated to the surface $S \# D^{\#n}$. This

[1]We called those *disordered* arcs because it is the opposite ordering convention than the one used in the "ordered arc complex" of [RW16].

is a simplicial complex whose vertices are isotopy classes of non-separating arcs in the surface with endpoints $b_0 = I_0(1/2)$ and $b_1 = I_1(1/2)$, and where a collection of isotopy classes forms a simplex if the classes can be represented by arcs that are disjoint away from the endpoints, are jointly non-separating, and such that the arcs have the same ordering at I_0 and I_1.

Writing $\mathscr{D}^\nu(S_{g,r}, b_0, b_1)$ for the disordered arc complex of a surface $S_{g,r}$ with marked points b_0 and b_1 in $\nu = 1$ or $\nu = 2$ boundary components, the main ingredient of our proof of homological stability is the following connectivity result:

Theorem C. (Theorem 2.4) The disordered arc complex $\mathscr{D}^\nu(S_{g,r}, b_0, b_1)$ is $\left(\frac{2g+\nu-5}{3}\right)$-connected.

Remark 1.2. It is conjectured in [RWW17, Conj C] that the complex of destabilizations is highly connected if and only if stability holds with all appropriate twisted coefficients. The slope 2/3 bounds in Theorems A and B is precisely dictated by the same slope 2/3 in Theorem C in the connectivity of the arc complex, which is the complex of destabilizations in that case. This connectivity bound is best possible among linear bounds as a better bound would prove an incorrect stability statement, see Remark 2.5.

Organization of the paper. In Section 2 we prove the high connectivity of the disordered arc complex. In Section 3 we define the monoidal category of bidecorated surfaces $(\mathbf{M}_2, \#)$, as well as the action of the braid groupoid $\mathbf{B}$ on this category. In Section 4, we show Theorems A and B by showing that the disordered arc complex agrees with the complex of destabilizations, and applying the main result of [Kra19]. Finally, in Section 5 we explain the relationship between homological stability and Yang–Baxter operators, and show the non-braidedness of the category of bidecorated surfaces.

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2. High connectivity of the disordered arc complex

In this section, we prove that the disordered arc complex is highly connected. It will be defined as a subcomplex of the following simplicial complex of non-separating arcs:

Definition 2.1. Let S be an orientable surface[2] with nonempty boundary, and let b_0, b_1 be distinct points in ∂S. The *complex of non-separating arcs* $\mathscr{B}(S, b_0, b_1)$ is the simplicial complex whose p-simplices are collections of $p+1$ distinct isotopy classes of arcs between b_0, b_1 that admit representatives $a_0, \ldots, a_p$ such that

(a) $a_i \cap a_j = \{b_0, b_1\}$ for each $i \neq j$ and

[2] By *surface* we mean a topological 2-manifold S which is compact except for a finite number of punctures, i.e. there is a compact topological 2-manifold $\overline{S}$ and an embedding $i\colon S \longhookrightarrow \overline{S}$ so that $\overline{S} \setminus i(S)$ is a (possibly empty) finite union of points.

(b) $S - (a_0 \cup \cdots \cup a_p)$ is connected.

For convenience, we will add a superscript $\mathscr{B}^\nu(S, b_0, b_1)$ to the notation of the complex, with $\nu = 1$ indicating that b_0, b_1 lie on the same boundary component and $\nu = 2$ indicating that they do not.

Note that the orientation of the surface defines orderings of the arcs $a_0, \ldots, a_p$ representing a simplex at both b_0 and b_1.

Definition 2.2. Let (S, b_0, b_1) be as before. The *disordered arc complex* is the subcomplex $\mathscr{D}^\nu(S_{g,r}, b_0, b_1) \subseteq \mathscr{B}^\nu(S, b_0, b_1)$ consisting of those simplices σ that admit arc representatives $a_0, \ldots, a_p$, again subject to (a), (b), satisfying in addition

(c) the ordering of the arcs at b_0 agrees with the ordering of the arcs at b_1.

The name "disordered" was chosen to contrast with the pre-existing *ordered arc complex* used by Ivanov [Iva89] in the case $\nu = 1$ and Randall-Williams [RW16] in their proofs of homological stability for the mapping class group of surfaces; the "ordered" version is also a subcomplex of the $\mathscr{B}^\nu(S, b_0, b_1)$, but with the requirement that the order of the arcs at b_1 is reversed compared to the order at b_0. Fixing an ordering condition has the effect that the action of the mapping class group is transitive on the set of p-simplices for every p, see [Har85, Lem 3.2]. The ordered and disordered arc complexes represent the two extremes of how fast the genus of the surface decreases when cutting along larger and larger simplices: for the ordered arc complex, the genus goes down as fast as possible, essentially every time one removes an arc, while for the disordered arc complex, the genus goes down as slow as possible, only every other time:

Proposition 2.3. *For a p–simplex $\sigma = \langle a_0, \ldots, a_p\rangle \in \mathscr{D}^\nu(S_{g,r}, b_0, b_1)$, the surface $S_{g,r} \setminus \sigma$ obtained by removing a tubular neighborhood of a_i for each i has genus g' with r' boundary components for*

$$g' = g - \left\lfloor \frac{p + 3 - \nu}{2} \right\rfloor \quad \text{and} \quad r' = \begin{cases} r + (-1)^\nu, & \text{if } p \text{ is even,} \\ r & \text{else.} \end{cases}$$

Proof. The number of boundary components r' can be obtained by a direct inductive computation, with the genus g' then deduced using the Euler characteristic. The computation is a special case of [Bol12, Prop 2.11], applied to the case where the permutation α is the inversion $[p\,(p-1)\ldots 0]$, once one computes that the genus $S(\alpha)$ of a neighborhood of the arcs is $\lfloor \frac{p+2-\nu}{2} \rfloor$, e.g. using Corollary 2.15 of the same paper. □

The complex $\mathscr{B}^\nu(S, b_0, b_1)$ is known to be $(2g + \nu - 3)$-connected. (This was first stated in [Har85]; see [Wah08, Thm 3.2] or [Wah13, Thm 4.8] for a complete proof.) We will here use this fact to deduce that $\mathscr{D}^\nu(S_{g,r}, b_0, b_1)$ is also highly-connected. While the ordered arc complex is $(g - 2)$-connected [RW16, Thm A.1], the following result shows that the disordered arc complex is only slope $\frac{2}{3}$ connected with respect to the genus, despite being $\sim 2g$-dimensional.

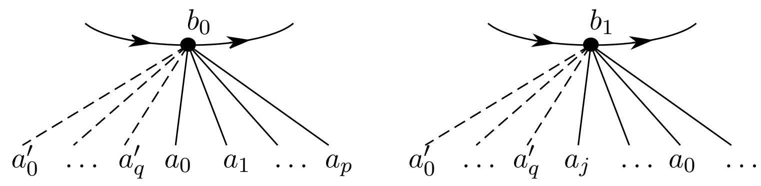

FIGURE 1. Maximal regular bad simplex $\{a_0, \dots, a_p\}$ and simplex $\{a_0', \dots, a_q'\}$ in its link.

Theorem 2.4. *The disordered arc complex $\mathscr{D}^\nu(S_{g,r}, b_0, b_1)$ is $\left(\frac{2g+\nu-5}{3}\right)$-connected.*

To prove the result, we use essentially the same argument as the one given in [RW16] in the ordered case.

Proof. Let $S = S_{g,r}$. In the case $g = 0$, the statement for $\mathscr{D}^1(S)$ is vacuous, and for $\mathscr{D}^2(S)$ it states that the complex is (-1)-connected, i.e. nonempty, which holds as any arc in the surface connecting b_0 and b_1 defines a vertex in $\mathscr{D}^2(S)$. We prove the remaining cases by induction on g.

Let $g > 0$. Suppose we are given $f : \partial D^{k+1} \longrightarrow \mathscr{D}^\nu(S, b_0, b_1)$ for some $k \leq (2g + \nu - 5)/3$. We wish to exhibit a nullhomotopy of this map. Since $(2g + \nu - 5)/3 \leq 2g + \nu - 3$, Theorem 3.2 in [Wah08] enables us to choose a map $\hat{f}$ such that the outer diagram

$$
\begin{tikzcd}
\partial D^{k+1} \arrow[r, "f"] \arrow[d, hook] & \mathscr{D}^\nu(S, b_0, b_1) \arrow[d, hook] \\
D^{k+1} \arrow[r, "\hat{f}"] \arrow[ur, dotted] & \mathscr{B}^\nu(S, b_0, b_1),
\end{tikzcd}
\tag{2.1}
$$

commutes. Using PL-approximation, we may assume that $\hat{f}$ and f are simplicial with respect to some PL-triangulation of D^{k+1}. We will repeatedly replace $\hat{f}$ until the dotted arrow exists, thereby giving the desired nullhomotopy.

Write $<_0$ and $<_1$ for the anti-clockwise orderings at b_0 and b_1. We call a p-simplex σ in D^{k+1} *regular bad* if $\hat{f}(\sigma) = \langle a_0, \cdots, a_{p'} \rangle$, indexed in such a way that $a_0 <_0 \cdots <_0 a_{p'}$ are anticlockwise at b_0, and there is $j > 0$ with $a_j <_1 a_0$ at b_1. Here $p' \leq p$ is the dimension of the image simplex $\hat{f}(\sigma)$, and $p' \geq 1$ if σ is regular bad. This condition is "dense" in the sense that any simplex σ in D^{k+1} with image not included in $\mathscr{D}^\nu(S, b_0, b_1)$ must contain a regular bad simplex as a face. Thus it suffices to give a procedure for exchanging $\hat{f}$ with a map having strictly fewer regular bad simplices, while maintaining commutativity of the outer diagram (2.1).

Let σ be a regular bad simplex of D^{k+1} of maximal dimension p and consider its link $\operatorname{Lk}\sigma \subset D^{k+1}$. Maximality implies that $\hat{f}|_{\operatorname{Lk}\sigma}$ factors as

$$\hat{f}|_{\operatorname{Lk}\sigma} \colon \operatorname{Lk}\sigma \longrightarrow \mathscr{D}^\mu(S \setminus \hat{f}(\sigma), b_0', b_1') \longrightarrow \mathscr{D}^\nu(S, b_0, b_1) \subseteq \mathscr{B}^\nu(S, b_0, b_1),$$

where $S \setminus \hat{f}(\sigma)$ is the closure of the surface obtained from S by cutting out the collection of arcs $\hat{f}(\sigma)$, and b_0' and b_1' in $S \setminus \hat{f}(\sigma)$ are the first copies of b_0 and b_1 in the cut surfaces as depicted in Figure 1. Indeed, suppose

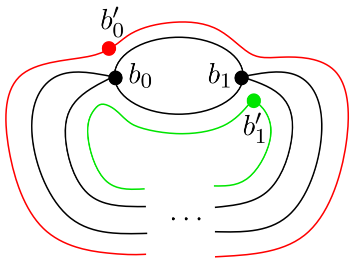

FIGURE 2. Regular bad 1-simplex with $\nu = 1$.

that $\tau \in \operatorname{Lk}\sigma$ and write $\hat{f}(\tau) = \langle a_0', \cdots, a_q' \rangle$. If $a_0 \leq_0 a_i'$ at b_0 for any a_i', then the simplex $\sigma * \langle a_i' \rangle$ is regular bad of a larger dimension, contradicting maximality. So we must have $a_i' <_0 a_0$ for each i, i.e. the arcs of τ are at b_0' in the cut surface. Now we must also have that each $a_i' <_1 a_0$ as otherwise $\sigma * \langle a_i' \rangle$ would again be regular bad. Finally, maximality of σ would also be contradicted if the orderings of the arcs $a_0', \cdots, a_q'$ does not agree at b_0 and b_1 as, if $a_i' <_0 a_j'$ with $a_j' <_1 a_i'$ for some i, j, then $\sigma * \langle a_i', a_j' \rangle$ would again be regular bad, of larger dimension. Thus $\hat{f}(\tau)$ must be disordered and, after cutting the surface at the arcs of σ, can be viewed as a simplex of $\mathscr{D}^\nu(S \setminus \hat{f}(\sigma), b_0', b_1')$.

The link $\operatorname{Lk}(\sigma)$ is a simplicial sphere $S^{k-p} \subset D^{k+1}$. We want to show that the map $\hat{f}|_{\operatorname{Lk}(\sigma)}$ extends to a simplicial map

$$F : D^{k-p+1} \longrightarrow \mathscr{D}^\mu(S - \hat{f}(\sigma), b_0', b_1') \longrightarrow \mathscr{D}^\nu(S, b_0, b_1) \subseteq \mathscr{B}(S, b_0, b_1) \tag{2.2}$$

for D^{k-p+1} a disk with some PL-structure extending that of $\operatorname{Lk}(\sigma)$. This will follow if we can show that the complex $\mathscr{D}^\mu(S \setminus \hat{f}(\sigma), b_0', b_1')$ is at least $(k-p)$-connected. Note that necessarily have $g(S \setminus \hat{f}(\sigma)) < g$ as $f(\sigma)$ is a non-separating p'-simplex with $p' \geq 1$. Hence we can use our induction hypothesis. We consider the cases $\nu = 1$ and $\nu = 2$ separately.

Case 1: $\nu = 1$. We have that $g(S \setminus \hat{f}(\sigma)) \geq g - p' - 1 \geq g - p - 1$, as removing $p' + 1$ arcs reduces the genus by at most $p' + 1 \leq p + 1$. Hence by induction we have that $\mathscr{D}^\mu(S \setminus \hat{f}(\sigma), b_0', b_1')$ is at least $(\frac{2(g-p-1)-4}{3})$-connected, using also that $\mu \geq 1$. If $p \geq 2$, we have

$$k - p \leq \frac{2g-4}{3} - p = \frac{2g-3p-4}{3} \leq \frac{2(g-p-1)-4}{3}.$$

For $p = p' = 1$, note that b_0', b_1' necessarily lie in different boundary components, so that $\mu = 2$ in that case. (See Figure 2.) Hence in that case $\mathscr{D}^\mu(S \setminus \hat{f}(\sigma), b_0', b_1')$ is $(\frac{2(g-2)-3}{3})$-connected, and

$$k - 1 \leq \frac{2g-4}{3} - 1 = \frac{2g-7}{3} = \frac{2(g-2)-3}{3}.$$

so we get the desired extension in both subcases.

Case 2: $\nu = 2$. The fact that b_0, b_1 lie in different components implies that

$$g(S \setminus \hat{f}(\sigma)) \geq g - p' \geq g - p$$

as cutting along the first arc has no effect on the genus. Hence induction here gives that $\mathscr{D}^\mu(S \setminus \hat{f}(\sigma), b_0', b_1')$ is at least $(\frac{2(g-p)-4}{3})$-connected. Now for all $p \geq 1$,

$$k - p \leq \frac{2g-4}{3} - p = \frac{2g-3p-4}{3} \leq \frac{2(g-p)-4}{3}$$

yielding the desired connectivity.

We will use the map F of (2.2) to modify $\hat{f}$ in the star $\mathrm{St}(\sigma)$. For this purpose, note that as simplicial subcomplexes of D^{k+1},

$$\mathrm{St}(\sigma) = \sigma * \mathrm{Lk}(\sigma),$$
$$\partial\mathrm{St}(\sigma) = \partial\sigma * \mathrm{Lk}(\sigma).$$

In particular, we get an identification $\partial(\partial\sigma * D^{k-p+1}) \cong \partial\mathrm{St}(\sigma)$ for D^{k-p+1} the simplicial disk that is the source of the map F above.

We replace $\hat{f}\,|_{\,\mathrm{St}(\sigma)}$ by the unique simplicial map

$$\hat{f} * F : \partial\sigma * D^{k-p+1} \longrightarrow \mathscr{B}^\nu(S, b_0, b_1).$$

It remains to show that this has improved the situation. Indeed, suppose that $\tau = \tau_0 * \tau_1$ is a regular bad simplex in $\partial\sigma * D^{k-p+1}$. By construction, τ_1 has image in $\mathscr{D}^\mu(S \setminus \hat{f}(\sigma), b_0', b_1') \subset \mathscr{D}^\nu(S, b_0, b_1)$, so the ordering of the arcs of τ at b_0 and b_1 starts with the arcs of τ_1, all in anti-clockwise order. Hence, if τ is regular bad, we must have $\tau = \tau_0$ is a strict face of σ. In particular, no new regular bad simplices have been added. As the simplex σ has been removed, we have thus reduced the total number of regular bad simplices in the disk. Repeating this procedure, we will after finitely many stages remove every regular bad simplex, thus making the dashed arrow exist, which proves the result. □

Remark 2.5. The connectivity estimate above can be shown to be optimal in certain low-genus examples, corresponding to known computations of the unstable homology of mapping class groups. Indeed, $\mathscr{D}^2(S_{1,r})$ is disconnected. To see this, consider the spectral sequence associated to the action of the mapping class group $\Gamma(S_{1,r})$ on the simplicial complex $\mathscr{D}^2(S_{1,r})$. This is the spectral sequence arising from the vertical filtration of the double complex $\mathbb{Z}\mathscr{D}^2(S_{1,r})_\bullet \otimes_{\Gamma(S_{1,r})} F_\bullet$, where $F_\bullet \longrightarrow \mathbb{Z}$ is a free resolution of the trivial $\Gamma(S_{1,r})$-module. By a standard argument using Shapiro's lemma (see e.g. [HW10, Thm 5.1] or [HV17, Sec 1]), one finds that the first page of this spectral sequence is given by

$$E^1_{p,q} \cong \begin{cases} \widetilde{H}_q(\Gamma(S_{1,r})) & \text{if } p = -1, \\ \widetilde{H}_q(\Gamma(S_{1,r-1})) & \text{if } p = 0, \\ \widetilde{H}_q(\Gamma(S_{0,r})) & \text{if } p = 1, \\ \widetilde{H}_q(\Gamma(S_{0,r-1})) & \text{if } p = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Assume for contradiction that $\mathscr{D}^2(S_{1,r})$ is connected. Then an analysis of the horizontal filtration of the double complex $\mathbb{Z}\mathscr{D}^2(S_{1,r})_\bullet \otimes_{\Gamma(S_{1,r})} F_\bullet$ shows that $E^\infty_{p,q} = 0$ for $p+q \leq 0$, so the differential $d^1 \colon H_1(\Gamma(S_{1,r-1})) \longrightarrow H_1(\Gamma(S_{1,r}))$ must be surjective. This contradicts the fact that $H_1(\Gamma(S_{1,s})) \cong \mathbb{Z}^s$ for $s \geq 1$ (see [Kor02, Thm 5.1]). Hence it is not true that $\mathscr{D}^\nu$ is $\left(\frac{2g+\nu-4}{3}\right)$-connected when $\nu = 2$.

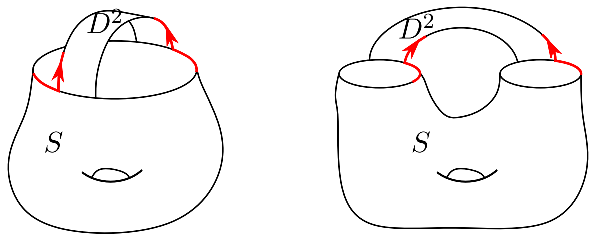

FIGURE 3. Gluing a disk $X_1 = D^2$ to a bidecorated surface S

Similarly, one finds that $H_1(\mathscr{D}^1(S_{3,r})) \neq 0$ by considering the spectral sequence associated to the action of $\Gamma(S_{3,r})$ on $\mathscr{D}^1(S_{2,r+1})$ and noting that the differential $d^1\colon H_1(\Gamma(S_{2,r+1})) \longrightarrow H_1(\Gamma(S_{3,r}))$ cannot be injective since the source identifies with $\mathbb{Z}/10\mathbb{Z}$ and the target is zero (see [Kor02, Thm 5.1]). Thus $\mathscr{D}^\nu$ fails to be $\left(\frac{2g+\nu-4}{3}\right)$-connected when $\nu = 1$ also.

Note that these low dimensional computations also show that the first and last ranges in Theorem A cannot be improved by a constant.

3. The monoidal category of bidecorated surfaces

In this section, we describe a monoidal groupoid $(\mathbf{M}_2, \#)$ of surfaces decorated by two intervals in their boundary, where the monoidal structure glues the intervals in pairs. We show that this groupoid is a module over the braided monoidal groupoid $\mathbf{B}$ of braid groups, giving, on classifying spaces, the structure of an E_1-module over an E_2-algebra in the sense of [Kra19].

3.1. Bidecorated surfaces and the monoidal structure.

The groupoid $\mathbf{M}_2$ has objects *bidecorated surfaces*, that are, informally, surfaces with two intervals marked in their boundary. To give a precise definition of the objects that is convenient for the monoidal structure, we start by constructing a special sequence of bidecorated surfaces X_n, built out of disks, and defined inductively.

Let $X_1 = D^2 \subset \mathbb{C}$ denote the unit disk in the complex plane, and define the embeddings $\iota_1^0, \iota_1^1\colon I \longrightarrow X_1$ by

$$\iota_1^0(t) = e^{i(\pi/4+t\pi/2)} \quad \text{and} \quad \iota_1^1(t) = e^{i(5\pi/4+t\pi/2)}.$$

We denote by $\overline{\iota_1^i}\colon I \longrightarrow X_1$ the reversed map $t \longmapsto \iota_1^i(1-t)$ for $i = 0, 1$.

Recursively, suppose we have defined $(X_m, \iota_m^0, \iota_m^1)$ for some $m \geq 1$. We construct X_{m+1} from X_m by gluing an additional disk along two half intervals, with new markings $\iota_{m+1}^0, \iota_{m+1}^1$ coming from the first half of the markings of X_m and the second half of the markings of the attached disk:

$$X_{m+1} := \frac{X_m \sqcup X_1}{\iota_m^i(t) \sim \overline{\iota_1^i}(t),\ t \in [1/2, 1],} \quad \text{with} \quad \iota_{m+1}^i(t) = \begin{cases} \iota_m^i(t), & \text{if } t \leq 1/2, \\ \iota_1^i(t), & \text{else.} \end{cases}$$

for $i = 0, 1$. Note that the marked intervals in the boundary of X_m might live in different boundary components (in fact this will happen every other time). Figure 3 shows what happens when a disk is glued to a surface in the above described manner, in each of these two possible cases.

Lemma 3.1. *Let $m \geq 1$. Then $X_m \cong S_{g,r}$ is a surface of genus g with r boundary components, where*

$$(g, r) = \begin{cases} (\frac{m}{2} - 1, 2), & \text{if } m \text{ is even,} \\ (\frac{m-1}{2}, 1), & \text{if } m \text{ is odd.} \end{cases}$$

Proof. Note first that X_m is a connected surface for each m, since X_1 is a disk and X_m is obtained from X_1 by successively adding disks (or strips), attached along two disjoint intervals in the boundary. For the same reason, we get that the Euler characteristic of X_m is

$$\chi(X_{m+1}) = \chi(X_m) - 1 = \cdots = 1 - m.$$

By the classification of surfaces, we are left to compute the number of boundary components of X_m. For this, observing Figure 3, we notice that if we glue a disk along two intervals of S that lie in the same boundary component, the new marked intervals given by the above procedure will give new intervals in different boundary components and vice versa, and no boundary component without marked intervals are ever created. It follows that the number of boundary components of X_m alternates between 1 and 2. The result follows. □

We are now ready to define the objects of the groupoid $\mathbf{M}_2$. We will use the boundary of the above defined surfaces X_m to parametrize the boundary components of the surfaces that contain the marked intervals, to allow us to work with parametrized boundary components instead of parametrized arcs, in order to simplify some definitions.

Definition 3.2. A *bidecorated surface* is a tuple (S, m, φ) where S is a surface, $m \geq 1$ is an integer, and

$$\varphi \colon \partial X_m \sqcup (\sqcup_k S^1) \xrightarrow{\sim} \partial S$$

is a homeomorphism, giving a parametrization of the boundary of S. We think of (S, m, φ) as a surface with two parametrized arcs

$$I_0 := \varphi \circ \iota_m^0 \quad \text{and} \quad I_1 := \varphi \circ \iota_m^1$$

in its boundary, and k additional parametrized boundaries. The surface S may also have punctures.

The monoidal groupoid $(\mathbf{M}_2, \#, U)$ has objects the bidecorated surfaces together with a formal unit U. There are no morphisms between two bidecorated surfaces (S, m, φ) and (S', m', φ') unless S and S' are homeomorphic and $m = m'$, in which case we define the set of morphisms to be all the mapping classes of homeomorphisms that preserve the boundary parametrizations

$$\begin{aligned} \mathrm{Hom}_{\mathbf{M}_2}((S, m, \varphi), (S', m, \varphi')) &:= \pi_0 \mathrm{Homeo}_\partial(S, S') \\ &= \pi_0\{f \in \mathrm{Homeo}(S, S') \mid f \circ \varphi = \varphi'\}, \end{aligned}$$

where $\mathrm{Homeo}(S, S')$ is endowed with the compact-open topology, and $\mathrm{Homeo}_\partial(S, S')$ with the subspace topology. The only morphism involving the unit U is the identity id_U.

Remark 3.3. Our definition of the morphisms in the category $\mathbf{M}_2$ is such that punctures in a surfaces S can be permuted by automorphisms of S in $\mathbf{M}_2$. Our argument works just as well with labeled punctures, that are not permutable by homeomorphims, or both labeled and unlabeled punctures, just like we could also have additional boundary components that are only marked up to a permutation. The only changes this would cause to the argument would be that it would make the notations and conventions more cumbersome.

The monoidal structure # is defined as follows. The object U is by definition a unit for #. For the remaining objects, the monoidal product # is defined by

$$(S, m, \varphi) \,\#\, (S', m', \varphi') := \left(\frac{S \sqcup S'}{I_i(t) \sim \overline{I'_i}(t), t \in [1/2, 1],}, m + m', \varphi \,\#\, \varphi' \right),$$

where $i = 0, 1$, and where

$$\varphi \,\#\, \varphi' \colon \partial X_{m+m'} \sqcup (\sqcup_{k+k'} S^1) \hookrightarrow \partial(S \,\#\, S'),$$

is obtained using the canonical identification $\partial X_{m+m'} \cong (\partial X_n \backslash \iota_m(\frac{1}{2}, 1)) \cup (\partial X_{m'} \backslash \iota_{m'}(0, \frac{1}{2}))$. On morphisms, the monoidal product is given by juxtaposition.

The monoidal category $\mathbf{M}_2$ has the following *injectivity property* with respect to gluing a disk, that will be useful in the proof of our stability result.

Proposition 3.4. *For any object $S = (S, m, \varphi)$ of $\mathbf{M}_2$, and any $p \geq 0$, the map*

$$\mathrm{Aut}_{\mathbf{M}_2}(S) \xrightarrow{\#D^{\#p+1}} \mathrm{Aut}_{\mathbf{M}_2}(S \,\#\, D^{\#p+1})$$

is injective, where $D = (X_1, 1, \mathrm{id})$ is our chosen disk.

Proof. Recall that the underlying surface of $D^{\#p+1}$ is the surface X_{p+1} defined above. Picking a smooth representative of the underlying surface of $S \,\#\, X_{p+1}$, with S a smooth subsurface in its interior, we can model the map in the statement using the description of the mapping class group of surfaces in terms of isotopy classes of diffeomorphisms rather than homeomorphisms. (See e.g., [Bol09, Thm 1.2] for a detailed account of the classical isomorphism $\pi_0\mathrm{Homeo}_\partial(S) \cong \pi_0\mathrm{Diff}_\partial(S)$ when S is compact.) Now the result follows by essentially the same argument as the case of attaching of surface along a single arc instead of two, as treated in [RWW17, Prop 5.18], using the fibration

$$\begin{aligned}\mathrm{Diff}(S \,\#\, X_{p+1} \text{ rel } \partial S \cup X_{p+1}) &\longrightarrow \mathrm{Diff}(S \,\#\, X_{p+1} \text{ rel } \partial(S \,\#\, X_{p+1})) \\ &\longrightarrow \mathrm{Emb}((X_{p+1}, I_0|_{[\frac{1}{2},1]} \cup I_1|_{[\frac{1}{2},1]}), (S \,\#\, X_{p+1}, I_0|_{[\frac{1}{2},1]} \cup I_1|_{[\frac{1}{2},1]}))\end{aligned}$$

where the fiber identifies with $\mathrm{Diff}(S \text{ rel } \partial_0 S)$ and where we note that $I_0|_{[\frac{1}{2},1]} \cup I_1|_{[\frac{1}{2},1]} = \partial X_{p+1} \cap \partial(S \,\#\, X_{p+1})$. Injectivity of the first map on π_0 follows if we can show that the base is simply-connected. In fact the base can be shown inductively to have contractible components, using that X_{p+1} is built inductively by attaching disks along two intervals, or homotopically attaching arcs, and using the contractibility of the components of embeddings of arcs in a surface, as proved in [Gra73, Thm 5]. $\square$

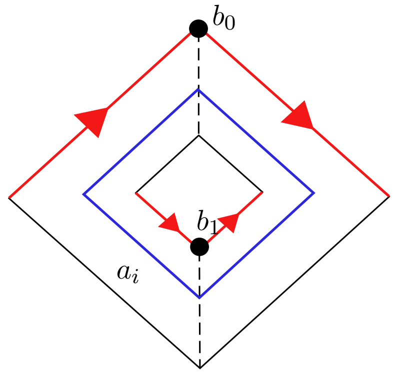

FIGURE 4. The curve a_i in $D_i \# D_{i+1}$

3.2. **Braided action.** We want to apply the homological stability machine of [Kra19] to stabilization in $\mathbf{M}_2$ with the bidecorated disk

$$D := (X_1, 1, \mathrm{id}).$$

For this, we need that the classifying space of $\mathbf{M}_2$ is an E_1–module over an E_2–algebra. This will follow if we can show on the categorical level that $\mathbf{M}_2$ admits an appropriate action of a braided monoidal groupoid. We will build such an action in this section, using as braided monoidal groupoid the groupoid of braid groups. In constrast with most classical examples of homological stability, we will show in Section 5.3 that this action of the braid groupoid does not come from a braided structure on $\mathbf{M}_2$, or the full monoidal subcategory generated by D. It is instead constructed using a *Yang–Baxter element* in $\mathbf{M}_2$, associated to a braid subgroup of the mapping class group of X_m, that we will describe now.

Write

$$D^{\# m} = D_1 \# \dots \# (D_i \# D_{i+1}) \# \dots \# D_m,$$

where we use subscript to enumerate the disks, and where the underlying surface is X_m. We let a_i denote the isotopy class of a curve in the interior $D_i \# D_{i+1} \cong S^1 \times I$ that is parallel to its boundary components, as shown in Figure 4.

Lemma 3.5. *The curves $a_1, \dots, a_{m-1}$ form a* chain *in $D^{\# m}$, i.e. a_i and a_{i+1} have intersection number 1 for each i, and $a_i \cap a_j = \emptyset$ if $|i-j| > 1$.*

Proof. The curve a_i lives in the disks D_i and D_{i+1}, so it can only intersect a_{i-1} and a_{i+1} non-trivially, and hence it suffices to consider the subsurface of $D^{\# m}$ corresponding to $D_i \# D_{i+1} \# D_{i+2}$. Here the claim can be checked by hand, see Figure 5. □

Let $T_i \in \mathrm{Aut}_{\mathbf{M}_2}(D^{\# m})$ denote the Dehn twist along the curve a_i in $D^{\# m}$. A classical fact states that the Dehn twists along a chain of embedded curves satisfy the braid relations (see e.g. [FM11, 3.9 and 3.11]):

$$\begin{aligned} T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \text{for all } i, \\ T_i T_j &= T_j T_i && \text{if } |i-j| > 1, \end{aligned} \tag{3.1}$$

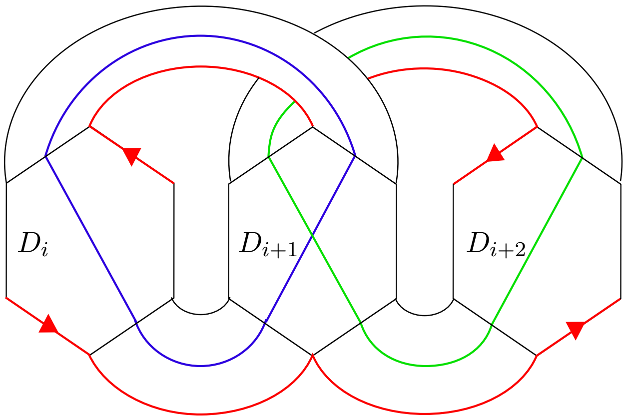

FIGURE 5. Intersection of a_i (blue) and a_{i+1} (green) in the underlying surface of $D_i \,\#\, D_{i+1} \,\#\, D_{i+2}$.

Note that the same relations are satisfied by the inverse twists T_i^{-1}, that will turn out to be more convenient for us. Also, adding a disk to the right or left of $D^{\#m}$ gives the relations

$$T_i \,\#\, \mathrm{id}_D = T_i \quad \text{and} \quad \mathrm{id}_D \,\#\, T_i = T_{i+1}$$

in $\mathrm{Aut}_{\mathbf{M}_2}(D^{\#m+1})$. In particular (3.1) includes the relation

$$(T_1^{-1} \,\#\, \mathrm{id}_D)(\mathrm{id}_D \,\#\, T_1^{-1})(T_1^{-1} \,\#\, \mathrm{id}_D) = (\mathrm{id}_D \,\#\, T_1^{-1})(T_1^{-1} \,\#\, \mathrm{id}_D)(\mathrm{id}_D \,\#\, T_1^{-1})$$

in $\mathrm{Aut}_{\mathbf{M}_2}(D^{\#3})$, so in other words, the inverse Dehn twist $T_1^{-1} \in \mathrm{Aut}_{\mathbf{M}_2}(D\#D)$ is a Yang–Baxter operator in the sense of Section 5.1.

Recall from the introduction that $\mathbf{B}$ denotes the groupoid of braid groups, with objects the natural numbers $\{0, 1, 2, \dots\}$, automorphisms of n the braid group B_n, and no other non-trivial morphisms. In Section 5.1 we show that, being a Yang-Baxter operator, T_1^{-1} yields a strong monoidal functor

$$\Phi = \Phi_{D,T_1^{-1}} \colon (\mathbf{B}, \oplus) \longrightarrow (\mathbf{M}_2, \#),$$

uniquely determined up to monoidal natural isomorphism by the fact that $\Phi(1) = D$ and, for the standard generator $\sigma_1 \in B_2 = \mathrm{Aut}_{\mathbf{B}}(1)$, $\Phi(\sigma_1) = T_1^{-1}$.

Such a functor Φ endows $\mathbf{M}_2$ with the structure of an E_1-module over $\mathbf{B}$ via the associated functor

$$\alpha = (- \,\#\, \Phi(-))\colon\ \mathbf{M}_2 \times \mathbf{B} \longrightarrow \mathbf{M}_2,$$

given on objects by $\alpha(S, n) = S \,\#\, \Phi(n) = S \,\#\, D^{\#n}$, and likewise for morphisms. On classifying spaces, this yields exactly the kind of input needed in Krannich's homological stability framework, see [Kra19, Lem 7.2].

Remark 3.6. For each m, the restriction of the functor $\Phi : \mathbf{B} \longrightarrow \mathbf{M}_2$ to $B_m = \mathrm{Aut}_{\mathbf{B}}(m)$ maps the standard generator σ_i to the inverse Dehn twist $T_i^{-1} \in \mathrm{Aut}_{\mathbf{M}_2}(D^{\#m}) = \pi_0\mathrm{Homeo}_\partial(X_m)$. By Birman–Hilden theory [BH72, BH73] the homomorphisms $\Phi|_{B_m} \colon B_m \longrightarrow \mathrm{Aut}_{\mathbf{M}_2}(D^{\#m})$ are actually injective.

4. Homological stability

Generalizing the main result of [RWW17], Krannich associates to an E_1-module $\mathscr{M}$ over an E_2-algebra $\mathscr{X}$ with a chosen stabilizing object $X \in \mathscr{X}$, a *space of destabilizations* at every $A \in \mathscr{M}$, whose high connectivity implies homological stability at A when stabilizing by X. We are interested in the case where $\mathscr{M} = B\mathbf{M}_2$ is the classifying space of $\mathbf{M}_2$ and $\mathscr{X} = B\mathbf{B}$, acting on $B\mathbf{M}_2$ via the map $\alpha\colon \mathbf{M}_2 \times \mathbf{B} \longrightarrow \mathbf{M}_2$ defined in Section 3.2. We will pick $A = S \in \mathbf{M}_2$ to be some surface, with $X = 1 \in \mathbf{B}$ modelling stabilization with the disk as $\alpha(-, X) = - \# D$ is the sum with the bidecorated disk $D = (X_1, 1, \mathrm{id})$ of Section 3.1.

Generally, the space of destabilizations is a semi-simplicial space, but in settings such as ours, it is actually levelwise homotopy discrete. Indeed, by [Kra19, Lem 7.6]), when the structure of E_1-module over an E_2-algebra is induced by an action of a braided monoidal category on a groupoid, and under the injectivity condition given in Proposition 3.4, the space of destabilizations is equivalent to the following semi-simplicial set, defined just as in [RWW17] in the case of a braided monoidal groupoid acting on itself.

Definition 4.1. ([Kra19, Def 7.5]) Let $(\mathscr{M}, \oplus)$ be a right module over a braided monoidal groupoid $(\mathscr{X}, \oplus, b)$, where we denote also by $\oplus$ the module action. Let A and X be objects of $\mathscr{M}$ and $\mathscr{X}$ respectively. The *space of destabilizations* $W_n(A, X)_\bullet$ is the semi-simplicial set with set of p-simplices

$$W_n(A, X)_p = \{(B, f) \mid B \in \mathrm{Ob}(\mathscr{M}) \text{ and } f\colon B \oplus X^{\oplus p+1} \longrightarrow A \oplus X^{\oplus n} \text{ in } \mathscr{M}\}/\sim$$

where $(B, f) \sim (B', f')$ if there exists an isomorphism $g\colon B \longrightarrow B'$ in $\mathscr{C}$ satisfying that $f = f' \circ (g \oplus \mathrm{id}_{X^{\oplus p+1}})$. The face map $d_i\colon W_n(A, X)_p \longrightarrow W_n(A, X)_{p-1}$ is defined by $d_i[B, f] = [B \oplus X, d_i f]$ for

$$d_i f\colon B \oplus X \oplus X^p \xrightarrow{\mathrm{id}_B \oplus b^{-1}_{X^{\oplus i}, X} \oplus \mathrm{id}_{X^{\oplus p-i}}} B \oplus X^{\oplus i} \oplus X \oplus X^{\oplus p-i} \xrightarrow{f} A \oplus X^{\oplus n},$$

for $b^{-1}_{X^{\oplus i}, X}\colon X \oplus X^{\oplus i} \longrightarrow X^{\oplus i} \oplus X$ coming from the braiding in $\mathscr{X}$.

4.1. Disk destabilizations and disordered arcs.

Given a bidecorated orientable[3] surface $S = (S, m, \varphi)$, with I_0, I_1 compatibly oriented, let $\mathscr{D}(S) = \mathscr{D}^\nu(S, b_0, b_1)$ denote the disordered arc complex of S as in Section 2, where

$$b_0 = I_0(1/2) \quad \text{and} \quad b_1 = I_1(1/2)$$

are the midpoints of the marked intervals, and $\nu = 1$ if I_0 and I_1 lie on the same boundary component and $\nu = 2$ otherwise. The vertices of a simplex in $\mathscr{D}(S)$ are canonically ordered by the anti-clockwise ordering at b_0 (or equivalently at b_1). Hence we can associate to this simplicial complex a semi-simplicial set that we denote $\mathscr{D}(S)_\bullet$, with same set of p-simplices and whose ith face map is given by forgetting the $(i+1)$st arc with respect to that ordering. As $\mathscr{D}(S)$ and $\mathscr{D}(S)_\bullet$ have homeomorphic realizations, they have the same connectivity.

[3]The definition of the disordered arc complex naturally extend to non-orientable bidecorated surfaces, ordering the arcs according to the orientations of I_0 and I_1, but we will only consider orientable surfaces here

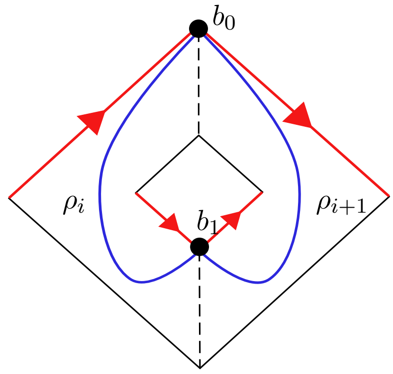

FIGURE 6. Ordering of the arcs ρ_i at their endpoints

Write $W_n(S,D)_\bullet$ for the space of destabilization of Definition 4.1 associated to the module $\mathscr{M} = \mathbf{M}_2$ over the E_2–algebra $\mathscr{X} = \mathbf{B}$ acting on $\mathbf{M}_2$ as above, with $X = 1 \in \mathbf{B}$, and $A = S = (S^s_{g,r}, m, \varphi)$ some bidecorated orientable surface of small genus $g \geq 0$, with r boundary components and s punctures. The space $W_n(S,D)_\bullet$ is then the space of destabilizations of the stabilization map

$$\mathrm{Aut}_{\mathbf{M}_2}(S \,\#\, D^{\#n-1}) \xrightarrow{\#D} \mathrm{Aut}_{\mathbf{M}_2}(S \,\#\, D^{\#n})$$

that attaches an additional disk to the surface along the two marked intervals.

We want to identify $W_n(S,D)_\bullet$ with $\mathscr{D}(S \,\#\, D^{\#n})_\bullet$. For this, we start by constructing a particular disordered collection of arcs in $D^{\#n}$. Write again

$$D^{\#n} = D_1 \,\#\cdots\#\, D_i \,\#\cdots\#\, D_n,$$

and let ρ_i denote the unique isotopy class of arc in the ith disk D_i going from $b_0 = I_0(1/2)$ to $b_1 = I_1(1/2)$.

Lemma 4.2. *The arcs $\rho_1, \ldots, \rho_m$ are ordered anti-clockwise at both b_0 and b_1.*

Proof. It suffices to show that ρ_i and ρ_{i+1} are ordered anti-clockwise at b_0 and b_1 for each i. Thus we need only consider what happens in the subsurface $D_i \,\#\, D_{i+1}$. The gluing being defined in exactly the same way at I_0 and I_1, the arcs are ordered in the same way at both endpoints, and the particular choice of gluing gives the anti-clockwise ordering, see Figure 6. □

Recall from Section 3.2 the Dehn twist T_i along the curve a_i in $D_i \,\#\, D_{i+1}$. The union of the arcs ρ_i in $D^{\#m}$ define a deformation retract of the surface, as each disk D_i retracts onto the corresponding arc ρ_i, and we can understand the action of the twists T_i on the surface by considering their action on the arcs ρ_i. The action is given by the following result, that will be needed to compare the face maps in the semi-simplicial sets $W_n(S,D)_\bullet$ with $\mathscr{D}(S \,\#\, D^{\#n})_\bullet$.

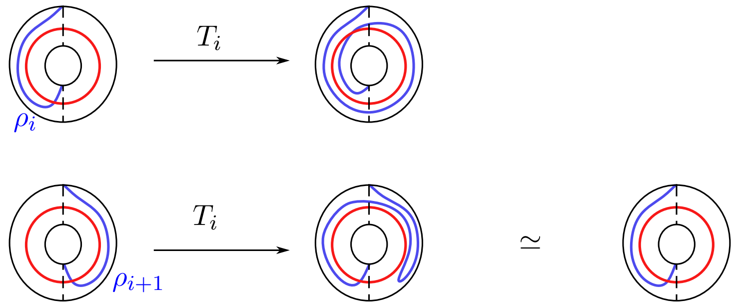

FIGURE 7. The action of the Dehn twist T_i on the arcs ρ_i (top) and ρ_{i+1} (bottom)

Lemma 4.3. *The action of the Dehn twist T_i along the curve a_i on the homotopy classes of the arcs ρ_i, relative to their endpoints, is*

$$T_i(\rho_j) = \begin{cases} \rho_i\overline{\rho_{i+1}}\rho_i & \text{if } j = i, \\ \rho_i & \text{if } j = i+1, \\ \rho_j & \text{else.} \end{cases}$$

Equivalently,

$$T_i^{-1}(\rho_i) = \rho_{i+1} \quad \text{and} \quad T_i^{-1}(\rho_{i+1}) = \rho_{i+1}\overline{\rho_i}\rho_{i+1}$$

and T_i^{-1} leaves the other ρ_j invariant.

Proof. The Dehn twist T_i can only affect ρ_i and ρ_{i+1} as the curve a_i only intersects these two arcs, from which the last case in the statement follows. The computation for the arcs ρ_i and ρ_{i+1} is local to $D_i \# D_{i+1}$, where, as shown in Figure 7, we have $T_i(\rho_i) \simeq \rho_i\overline{\rho_{i+1}}\rho_i$, giving the first case in the statement, and $T_i(\rho_{i+1}) \simeq \rho_i$, giving the second case. $\square$

Proposition 4.4. *Let $S = (S, m, \varphi)$ be an object of $\mathbf{M}_2$. There is an isomorphism of semi-simplicial sets*

$$W_n(S, D)_\bullet \cong \mathscr{D}^\nu(S \# D^{\# n})_\bullet$$

where the marked points b_0 and b_1 are the midpoints of the intervals I_0 and I_1 in $S \# D^{\# n}$ and with $\nu = \operatorname{parity}(m+n)$, that is $\nu = 1$ if I_0 and I_1 lie in the same boundary component of $S \# D^{\# n}$ and $\nu = 2$ otherwise.

Proof. We first show that both $W_n(S, D)_p$ and $\mathscr{D}^\nu(S \# D^{\# n})_p$ are isomorphic to $\operatorname{Aut}_{\mathbf{M}_2}(S \# D^{\# n})/\operatorname{Aut}_{\mathbf{M}_2}(S \# D^{\# n-p-1})$ for every $p \geq 0$. This holds by definition for the first semi-simplicial set. For $\mathscr{D}^\nu(S \# D^{\# n})_p$, it will follow from two facts: (1) the natural action of

$$\operatorname{Aut}_{\mathbf{M}_2}(S \# D^{\# n}) = \pi_0 \operatorname{Homeo}_\partial(S \# D^{\# n})$$

on this set of p-simplices is transitive, and (2) the stabilizer of a p-simplex is isomorphic to $\operatorname{Aut}_{\mathbf{M}_2}(S \# D^{\# n-p-1})$. The first fact follows because the homeomorphism type of the complement $S \setminus \sigma$ of a collection of non-separating arcs $\sigma = \langle a_0, \dots, a_p \rangle$ is determined by the orderings of the arcs at the endpoints as this determines the number of boundary components of the complement (see [Har85, Lem 3.2]), and the second from the fact that this complement is precisely diffeomorphic to $S \# D^{\# n-p-1}$ for any p-simplex in the disordered

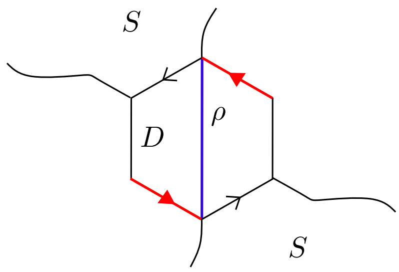

FIGURE 8. Cutting along the core of a disk

arc complex. Indeed, this diffeomorphism type does not depend on the simplex by transitivity of the action, so it is enough to check the claim for any chosen simplex. Let

$$\sigma_p = \langle \rho_{n-p}, \dots, \rho_n \rangle$$

be the collection of arcs in $S \,\#\, D^{\#n}$ consisting of the cores ρ_i of the last $p+1$ disks. Recall from Lemma 4.2 that this is a disordered simplex, once we note additionally that the arcs are also non-separating. Now Figure 8 shows that the operation of cutting along the core ρ of a disk exactly undoes the gluing operation, which proves the claim in that case.

Note that the actions on both sets of simplices are given by post-composition with mapping classes, where we think here of an arc as an isotopy class of embedding. There is then a unique equivariant isomorphism $\varphi_p\colon W_n(S,D)_p \xrightarrow{\cong} \mathscr{D}^\nu(S \,\#\, D^{\#n})_p$ taking the p-simplex

$$f_p = (S \,\#\, D^{\#n-p-1},\ \mathrm{id}_{S\#D^{\#n}})$$

of $W_n(S,D)$ to the p-simplex $\sigma_p = \langle \rho_{n-p}, \dots, \rho_n \rangle$ of the target already considered above.

We are left to check that the face maps d_i correspond to each other under the isomorphisms φ_p. Because the face maps are equivariant with respect to the $\mathrm{Aut}_{\mathbf{M}_2}(S \,\#\, D^{\#n})$-action in both cases, and the actions are transitive, it is enough to check that the face maps agree for the simplices f_p and $\sigma_p = \varphi_p(f_p)$. By definition,

$$d_i f_p = ((S \,\#\, D^{\#n-p-1}) \,\#\, D,\ \mathrm{id}_{S\#D^{\#n-p-1}} \#b^{-1}_{D^{\#i},D} \,\#\, \mathrm{id}_{D^{\#p-i}})$$

while

$$d_i\sigma_p = \langle \rho_{n-p}, \dots, \widehat{\rho_{n-p+i}}, \dots, \rho_n \rangle$$

is the simplex obtained by forgetting the $(i+1)$st arc. In particular, we immediately have that $d_0(f_p) = f_{p-1}$ and $d_0(\sigma_p) = \sigma_{p-1} = \varphi_{p-1}(f_{p-1})$ giving that the face maps agree in that case.

For the remaining face maps, note that

$$\mathrm{id}_{S\#D^{\#n-p-1}} \#b^{-1}_{D^{\#i},D} \oplus \mathrm{id}_{D^{\#p-i}} = T_{n-p+i-1} \circ \dots \circ T_{n-p}\colon S \,\#\, D^{\#n} \longrightarrow S \,\#\, D^{\#n}$$

as composition of Dehn twists T_i of Section 3.2. We need to compute the image of $\rho_{n-p+1}, \dots, \rho_n$ under this map. By Lemma 4.3, we have that for

$1 \le j \le i$,

$$\begin{aligned} T_{n-p+i-1} \circ \cdots \circ T_{n-p}(\rho_{n-p+j}) &= T_{n-p+i-1} \circ \cdots \circ T_{n-p+j-1}(\rho_{n-p+j}) \\ &= T_{n-p+i-1} \circ \cdots \circ T_{n-p+j}(\rho_{n-p+j-1}) \\ &= \rho_{n-p+j-1} \end{aligned}$$

while for $i+1 \le j \le p$,

$$T_{n-p+i-1} \circ \cdots \circ T_{n-p}(\rho_{n-p+j}) = \rho_{n-p+j}.$$

Hence $d_i(f_p)$ takes the arcs $\rho_{n-p+1}, \ldots, \rho_n$ to the arcs

$$\rho_{n-p}, \ldots, \rho_{n-p+i-1}, \rho_{n-p+i+1}, \ldots, \rho_n,$$

i.e. precisely to the arcs of $d_i(\sigma_p)$. So we indeed have that $\varphi_{p-1}(d_i(f_p)) = d_i(\varphi_{p-1}(f_p))$, which finishes the proof. □

4.2. **Coefficient systems.** Having identified the space of destabilizations with the semi-simplicial set of disordered arcs in Proposition 4.4, we can now input the connectivity computation of the disordered arc complex of Section 2 into the general stability theorem of [Kra19]. To state the resulting stability theorem in full generality, we need to introduce the notions of (split) finite degree coefficient systems. We follow [Kra19, Sec 4], which generalizes [RWW17, 4.1-4] that unify the earlier definitions of Dwyer for the general linear groups [Dwy80] and Ivanov for the mapping class groups [Iva93]. (The papers [Kra19, RWW17] consider in addition abelian coefficient systems, but these are not relevant here, because the abelianization of the mapping class group of surfaces of large enough genus is trivial by a theorem of Mumford–Birman–Powell, see Lemma 1.1 in [Har83].)

Fix a bidecorated surface $S = (S, m, \varphi)$, and let D be the bidecorated disk as above. Definition 4.1 of [Kra19] becomes in our case:

Definition 4.5. A *coefficient system* for the groups $\mathrm{Aut}_{\mathbf{M}_2}(S \# D^{\# n})$ with respect to the stabilization by D is a collection of $\mathbb{Z}[\mathrm{Aut}(S \# D^{\# n})]$-modules M_n for $n \ge 0$, together with maps $s_n \colon M_n \longrightarrow M_{n+1}$ that are equivariant with respect to the stabilization map $\mathrm{Aut}(S \# D^{\# n}) \xrightarrow{\# D} \mathrm{Aut}(S \# D^{\# n+1})$, satisfying the following condition:

(4.1)
$$T_{n+1} \in \mathrm{Aut}(S \# D^{\# n+2}) \text{ acts trivially on the image of } M_n \xrightarrow{s_{n+1} \circ s_n} M_{n+2}$$

for T_{n+1} the Dehn twist of Section 3.2 with support the last two disks in $S \# D^{\# n+2}$.

We will encode the data of a coefficient system as a pair (F, σ^F) with

$$F \colon \mathbf{M}_2|_{S,D} \longrightarrow \mathrm{Mod}_{\mathbb{Z}}$$

a functor from the full subcategory of $\mathbf{M}_2$ on the objects $S \# D^{\# n}$ for $n \ge 0$ to abelian groups, where $M_n = F(S \# D^{\# n})$ with its $\mathrm{Aut}(S \# D^{\# n})$-action induced by F, and

$$\sigma^F \colon F(-) \longrightarrow F(- \# D)$$

is a natural transformation encoding the suspension maps s_n, where we assume that $F(\mathrm{id} \# T)$ acts trivially on the image of $(\sigma^F)^2 \colon F(-) \longrightarrow F(- \# D^{\# 2})$ for T the Dehn twist supported on the added disks $D^{\# 2}$.

Given a coefficient system F, we define its *suspension* $\Sigma F\colon \mathbf{M}_2|_{S,D} \longrightarrow \mathrm{Mod}_{\mathbb{Z}}$ by $\Sigma F(-) = F(- \# D)$ with

$$\sigma^{\Sigma F}\colon \Sigma F(-) = F(- \# D) \xrightarrow{\sigma^F} F(- \# D^{\#2}) \xrightarrow{\mathrm{id}\,\# T} F(- \# D^{\#2}) = \Sigma F(- \# D),$$

where one checks that the triviality condition 4.1 is satisfied with this choice of structure map $\sigma^{\Sigma F}$. (See [Kra19, Def 4.4].)

The structure map σ^F induces a natural transformation $F \longrightarrow \Sigma F$, called the *suspension map.* We define the *kernel* $\ker F$ and *cokernel* $\mathrm{coker} F$ to be the kernel and cokernel functors of that natural transformation. We call F *split* if the suspension map is split injective in the category of coefficient systems.

Definition 4.6. [RWW17, Def 4.10] A coefficient system F is

(a) of *(split) degree* -1 *at* N if $F(S \# (D^{\# n})) = 0$ for all $n \geq N$;
(b) of *degree* $k \geq 0$ *at* N if $\ker(F)$ has degree -1 at N and $\mathrm{coker}(F)$ has degree $(k-1)$ at $(N-1)$;
(c) of *split degree* $k \geq 0$ *at* N if F is split and $\mathrm{coker}(F)$ is of split degree $(k-1)$ at $(N-1)$.

Example 4.7.

(a) A coefficient system F is of degree 0 at 0 if and only if σ^F is a natural isomorphism. This is in particular the case for constant coefficient systems.
(b) The functor $F_k\colon \mathbf{M}_2 \longrightarrow \mathrm{Mod}_{\mathbb{Z}}$ defined by
$$F_k(S) = H_1(S, \mathbb{Z})^{\otimes k}$$
is a split coefficient system of degree k at 0. (This is essentially a result of Ivanov [Iva93, Sec 2.8], who considers a version of the composite stabilization $\# D^{\# 2}$. See also [Bol12, Ex 4.3] for the case $k = 1$, and [Sou20, Lem 2.9] that proves this in a very general set-up, though in the case of a braided groupoid acting over itself only.)
(c) Given a k-connected space X, the coefficient system $F_n^k\colon \mathbf{M}_2 \longrightarrow \mathrm{Mod}_{\mathbb{Z}}$ defined by
$$F_n^k(S) = H_n(\mathrm{Map}(S/\partial S), X),$$
which appears in the work of Cohen–Madsen [CM09], is a coefficient system of degree $\lfloor n/k \rfloor$ (see [Bol12, Ex 4.3]).

Remark 4.8. Although the above examples all makes sense in the different set-ups considered in the literature, one should keep in mind that there are variations in what precisely a finite degree coefficient system for the mapping class groups of surfaces means in e.g. the papers [Iva93, CM09, Bol12, RWW17] and [Kra19]. This is due to two facts: first, the definition of the coefficient system depends on the category of surfaces considered and on the stabilization map(s) one works with, and second, the triviality condition (4.1) arising from Krannich's framework is actually weaker than the one used in earlier frameworks, see e.g. [Kra19, Rem 7.9].

In addition, the paper [GKRW19] uses a homological condition instead of a finite degree condition (see 5.5.1 in that paper). The relationship between that condition and finite degree conditions is discussed in [GKRW18, Rem 19.11].

4.3. **The stability theorem.** We are now ready to state our main theorem:

Theorem 4.9. *Let $S = (S, m, \varphi)$ be an object of $\mathbf{M}_2$ with m odd, i.e. such that I_0, I_1 are in the same boundary component. Let $F\colon \mathbf{M}_2|_{S,D} \longrightarrow \mathrm{Mod}_{\mathbb{Z}}$ be a coefficient system and write $F_n = F(S \# D^{\# n})$. The map*

$$H_i(\mathrm{Aut}_{\mathbf{M}_2}(S \# D^{\# n}); F_n) \longrightarrow H_i(\mathrm{Aut}_{\mathbf{M}_2}(S \# D^{\# n+1}); F_{n+1})$$

is

(a) an epimorphism for $i \leq \frac{n}{3}$ and an isomorphism for $i \leq \frac{n-3}{3}$ if F is constant.
(b) an epimorphism for $i \leq \frac{n-3k-2}{3}$ and an isomorphism for $i \leq \frac{n-3k-5}{3}$ if F has degree k at $N \geq 0$ and $n > N$.
(c) an epimorphism for $i \leq \frac{n-k-2}{3}$ and an isomorphism for $i \leq \frac{n-k-5}{3}$ if F has split degree k at $N \geq 0$ and $n > N$.

Remark 4.10. We have stated the theorem in the case of an initial surface S with I_0 and I_1 in the same boundary component for simplicity. The case of a surface S' where the two intervals lie in different components is actually also included in the statement, by writing $S' = S \# D$ for S of the previous type, or considering $S' \# D$ if S' does not admit such a decomposition. Indeed, as we have already seen in Section 3 (see Figure 3), gluing in a disk exactly changes whether I_0 and I_1 are in the same boundary or not.

We will first show that the above results implies the two main theorems stated in the introduction.

Proof of Theorems A and B from Theorem 4.9. Let $S^s_{0,r}$ be a surface of genus 0 with $r \geq 1$ boundary components and s punctures, and consider the associated object $S = (S^s_{0,r}, 1, \varphi)$ of $\mathbf{M}_2$, with two marked intervals in the first boundary component. Then $S \# D^{\# 2g}$ has the form $(S^s_{g,r}, 1 + 2g, \varphi)$ while $S \# D^{\# 2g+1}$ has the form $(S^s_{g,r+1}, 2 + 2g, \varphi)$. Moreover, the maps

$$S \# D^{\# 2g} \xrightarrow{\# D} S \# D^{\# 2g+1} \xrightarrow{\# D} S \# D^{\# 2g+2}$$

precisely induce on automorphism groups in $\mathbf{M}_2$ the two maps appearing in Theorems A and B.

The fact that the first map is always injective in homology follows from the fact that postcomposing the map $S^s_{g,r} \longrightarrow S^s_{g,r+1}$, defined by the sum $\# D$, with the map $S^s_{g,r+1} \longrightarrow S^s_{g,r+1} \cup_{S^1} D^2 \simeq S^s_{g,r}$ filling in one of the newly created boundary component, is homotopic to the identity. Now Theorem 4.9(a) gives that the map

$$H_i(\mathrm{Aut}_{\mathbf{M}_2}(S \# D^{\# 2g})) \xrightarrow{\# D} H_i(\mathrm{Aut}_{\mathbf{M}_2}(S \# D^{\# 2g+1}))$$

is surjective for $i \leq \frac{2g}{3}$ in homology with constant coefficients. Given that the map is always injective, we get an isomorphism in that same range, proving the first part of Theorem A. Applying (b) and (c) instead gives Theorem B for the first map.

For the second map, we now apply Theorem 4.9 in the case $n = 2g+1$, but in that case, there is no additional argument for injectivity, so the bounds translate directly to surjectivity and isomorphism bounds. $\square$

Proof of Theorem 4.9. Proposition 4.4 together with Theorem 2.4 give that $W_n(S,D)_\bullet$ is $\left(\frac{2g+\nu-5}{3}\right)$-connected, for g the genus of $S \# D^{\# n}$ and $\nu = 1$ if I_0 and I_1 are in the same boundary component of $S \# D^{\# n}$, which is the case precisely when n is even, and $\nu = 2$ otherwise. The surface $S \# D^{\# n}$ has genus greater than or equal to the genus of $D \# D^{\# n}$, that is $\frac{n}{2}$ if n is even and $\frac{n-1}{2}$ if n is odd (see Lemma 3.1). Hence $2g + \nu \geq n+1$ is both cases, and $W_n(S,D)_\bullet$ is at least $\left(\frac{n-4}{3}\right)$-connected.

Now $W_n(S,D)_\bullet$ is the semi-simplicial set denoted $W^{\mathrm{RW}}(S \# D^{\# n})_\bullet$ in [Kra19] (see Definition 7.5 in that paper). By Lemma 7.6 in the same paper, using Proposition 3.4, this semi-simplicial set has the same connectivity as the semi-simplicial space $W(S \# D^{\# n})_\bullet$ of [Kra19], which by Remark 2.7 of that paper determines the connectivity assumption of Theorem A in that paper: the canonical resolution of the assumption of the theorem is m-connected, if and only if the space $W(S \# D^{\# n})_\bullet$ is $(m-1)$-connected. Given that $W(S \# D^{\# n})_\bullet$ is $\left(\frac{n-4}{3}\right)$–connected, we have that the canonical resolution of is $\left(\frac{n-4+3}{3}\right)$–connected. Hence we can apply [Kra19, Thm A] with $k = 3$ and grading $g_{\mathbf{M}_2} : \mathbf{M}_2|_{S,D} \longrightarrow \mathbb{N}$ given by $g_{\mathbf{M}_2}(S \# D^{\# n}) = n-2$; see also [Kra19, Rem 2.24], where we can take $m = 4$. The theorem, with the improvement given by (i) in the remark, then gives that

$$H_i(\mathrm{Aut}_{\mathbf{M}_2}(S \# D^{\# n}); \mathbb{Z}) \longrightarrow H_i(\mathrm{Aut}_{\mathbf{M}_2}(S \# D^{\# n+1}); \mathbb{Z})$$

is an isomorphism for $i \leq \frac{n-3}{3}$ and an epimorphism for $i \leq \frac{n}{3}$, giving the stated result in the case of constant coefficients. For a coefficient system F of degree k at N, [Kra19, Thm C] gives that

$$H_i(\mathrm{Aut}_{\mathbf{M}_2}(S \# D^{\# n}); F_n) \longrightarrow H_i(\mathrm{Aut}_{\mathbf{M}_2}(S \# D^{\# n+1}); F_{n+1})$$

is an isomorphism for $i \leq \frac{n-3k-5}{3}$ and an epimorphism for $i \leq \frac{n-3k-2}{3}$ for $n > N$, improved to an isomorphism for $i \leq \frac{n-k-5}{3}$ and an epimorphism for $i \leq \frac{n-k-2}{3}$ if F is split. □

Remark 4.11 (Optimality of the stability bounds)**.** Combining the two maps in Theorem A, we obtain that the genus stabilization

$$H_i(\Gamma(S^s_{g,r}); \mathbb{Z}) \longrightarrow H_i(\Gamma(S^s_{g+1,r}); \mathbb{Z})$$

is an epimorphism when $i \leq \frac{2g}{3}$ and an isomorphism when $i \leq \frac{2g-2}{3}$. The slope $\frac{2}{3}$ is known to be optimal by a computation of Morita [Mor03], with optimal isomorphism range since for instance $H_1(\Gamma(S_{2,r}); \mathbb{Z}) \longrightarrow H_1(\Gamma(S_{3,r}); \mathbb{Z})$ is not injective as the source is isomorphic to $\mathbb{Z}/12$ and the target is trivial, see e.g. [Kor02, Theorem 5.1]. Our combined genus epimorphism range, on the other hand, falls short of the range $i \leq \frac{2g+1}{3}$, as given in [GKRW19], a range that is optimal by Morita's computation (see Theorem B (i) of [GKRW19]).

Our results for twisted coefficients are most easily compared with those of Boldsen [Bol12, Thm 3], whose coefficient systems are coefficient systems of finite split degree in our sense, though with a stricter triviality condition upon double stabilization. For these coefficient systems, he obtains slightly better ranges, with improvement $+{}^2\!/_3$ for the first map and $+{}^5\!/_3$ for the second. The papers [RWW17, GKRW19] only consider genus stability. In

[RWW17], the stability slope obtained is only $\frac{1}{2}$, while in [GKRW19, Sec 5.5.1], the finite degree condition is replaced by a more general homological condition that applies to some finite coefficient systems [GKRW18, Sec 19.2]. In the particular case of the kth tensor power of the first homology of the surface, they do however only get the epimorphism range $i \leq \frac{2g-2k+1}{3}$ and isomorphism range $i \leq \frac{2g-2k-2}{3}$, see Example 5.22 in that paper.

5. Braiding and homological stability for groups

In order to use the framework of Krannich [Kra19] to prove homological stability for a sequence of groups, one needs the structure of an "E_1-module over an E_2-algebra". We give in Proposition 5.1 below a simple way to construct such a module structure, in terms of Yang–Baxter operators. Compared to earlier approaches to homological stability such as [RWW17], which Krannich's work generalizes, this has the advantage of being very lightweight. Instead of having to provide the structure of a braiding on the monoidal category whose automorphism groups one is interested in, it suffices to provide a single morphism satisfying a simple equation.

Our main example of a Yang–Baxter operator is the inverse Dehn twist $T_1^{-1} \in \mathrm{Aut}_{\mathbf{M}_2}(D \# D)$, defined in Section 3.2 and used to prove our main result. In Section 5.3, we show that this Yang–Baxter operator is not part of a braided monoidal structure on the category $\mathbf{M}_2$, but gives instead a twisted version of such a structure.

5.1. **Yang–Baxter operators and braid groupoid actions.** Let $\mathscr{X} = (\mathscr{X}, \oplus, \mathbb{1})$ be a monoidal category. A *Yang–Baxter operator* in $\mathscr{X}$ is a pair (X, τ) consisting of an object $X \in \mathscr{X}$ and a morphism $\tau \in \mathrm{Aut}_{\mathscr{C}}(X \oplus X)$, satisfying the Yang–Baxter equation

$$(\tau \oplus 1)(1 \oplus \tau)(\tau \oplus 1) = (1 \oplus \tau)(\tau \oplus 1)(1 \oplus \tau) \in \mathrm{Aut}_{\mathscr{C}}(X \oplus X \oplus X),$$

where we suppress associators from the notation.

Yang–Baxter operators are closely related to the braid groupoid: Recall from Section 3.2 the braid groupoid $\mathbf{B}$, with objects the natural numbers and only non-trivial morphisms $\mathrm{Aut}_{\mathbf{B}}(n) = B_n$. A variant of the coherence theorem for braided monoidal categories says that the category of strong monoidal functors from the braid groupoid into $\mathscr{X}$ is equivalent to a naturally defined category of Yang–Baxter operators in $\mathscr{X}$ [JS93, Prop 2.2].[4] To a Yang–Baxter operator (X, τ) in $\mathscr{X}$, this equivalence associates the strong monoidal functor $\Phi_{X,\tau} \colon \mathbf{B} \longrightarrow \mathscr{X}$ given by $\Phi_{X,\tau}(n) = X^{\oplus n}$ on objects, and on morphisms by letting

$$\Phi_{X,\tau} \colon B_n \longrightarrow \mathrm{Aut}_{\mathscr{X}}(X^{\oplus n})$$

send the ith standard generator σ_i to $\mathrm{id}_{X^{\oplus i-1}} \oplus \tau \oplus \mathrm{id}_{X^{\oplus n-i-1}}$, where the required maps $\Phi_{X,\tau}(m) \oplus \Phi_{X,\tau}(n) \longrightarrow \Phi_{X,\tau}(m+n)$ are given by the monoidal structure of $\mathscr{X}$.

Suppose now that the monoidal category $\mathscr{X}$ acts on a category $\mathscr{M}$ via a functor $\mathscr{M} \times \mathscr{X} \longrightarrow \mathscr{M}$, which we also denote by $\oplus$, compatible with

[4]In other words, the pair consisting of the braid groupoid $\mathbf{B}$ and the Yang–Baxter operator $\sigma_1 \in \mathrm{Aut}_{\mathbf{B}}(2)$, is the initial monoidal category with a distinguished Yang–Baxter element.

the monoidal sum in $\mathscr{X}$. The following result shows that the choice of a Yang–Baxter operator defines an action of the braid groupoid $\mathbf{B}$ on $\mathscr{M}$, and hence is appropriate data to apply the stability framework of [Kra19]:

Proposition 5.1. *Let $(\mathscr{X}, \oplus, \mathbb{1})$ be a monoidal category with $\tau \in \mathrm{Aut}_{\mathscr{X}}(X \oplus X)$ a Yang–Baxter operator in $\mathscr{X}$. Suppose $\mathscr{X}$ acts on a category $\mathscr{M}$. Then there is an action of the braid groupoid*

$$\alpha_\tau\colon \mathscr{M} \times \mathbf{B} \longrightarrow \mathscr{M}$$

given on objects by $\alpha_\tau(A, n) = A \oplus X^{\oplus n}$ and determined on morphisms by

$$\alpha_\tau(f, \sigma_i) = f \oplus id_{X^{\oplus i-1}} \oplus \tau \oplus id_{X^{\oplus n-i-1}},$$

for σ_i the ith elementary braid in B_n. Furthermore, taking classifying spaces this endows $B\mathscr{M}$ with the structure of an E_1-module over the E_2-algebra $B\mathbf{B}$.

Note that if we are interested in homological stability for stabilization by X for the automorphism groups $G_n := \mathrm{Aut}_{\mathscr{M}}(A \oplus X^{\oplus n})$ for some object A of $\mathscr{M}$, only the full subcategory $\mathscr{M}_{A,X} \subseteq \mathscr{M}$ spanned by objects of the form $A \oplus X^{\oplus n}$, is relevant. So for stability purposes, it is enough to consider the subfunctor

$$\alpha_\tau\colon \mathscr{M}_{A,X} \times \mathbf{B} \longrightarrow \mathscr{M}_{A,X}.$$

In fact, to make sure that the structure of E_1–module over the E_2–algebra $B\mathbf{B}$ is graded, one can even replace the category $\mathscr{M}_{A,X}$ by a category with objects the natural numbers and setting $\mathrm{Aut}(n) = \mathrm{Aut}_{\mathscr{M}}(A \oplus X^{\oplus n})$, avoiding any potential issue coming from unwanted equalities $A \oplus X^{\oplus n} = A \oplus X^{\oplus m}$ for $m \neq n$.

Proof. The functor $\alpha_\tau\colon \mathscr{M} \times \mathbf{B} \longrightarrow \mathscr{M}$ is defined as the composite functor

$$\alpha(-,-) = (-) \oplus \Phi_{X,\tau}(-),$$

for $\Phi_{X,\tau}\colon \mathbf{B} \longrightarrow \mathscr{X}$ as above. The result follows from [Kra19, Lem 7.2] because α makes $\mathscr{M}$ into a module over $\mathbf{B}$ and $\mathbf{B}$ is braided monoidal. $\square$

Example 5.2. If $\mathscr{X} = (\mathscr{X}, \oplus, \mathbb{1})$ admits a braiding b, then $\tau = b_{X,X} \in \mathrm{Aut}_{\mathscr{X}}(X \oplus X)$ is a Yang–Baxter operator for any object X. For $\mathscr{X}$ a groupoid acting on itself or $\mathscr{X}$ acting on a category $\mathscr{M}$, this recovers the basic set-up for homological stability of the paper [RWW17], or Section 7 of [Kra19].

Example 5.3 (Mapping class groups of surfaces)**.** As explained above, a Yang–Baxter operator $\tau \in \mathrm{Aut}_{\mathscr{X}}(X \oplus X)$ gives in particular a collection of homomorphisms $\Phi_{X,\tau}\colon B_n \longrightarrow \mathrm{Aut}_{\mathscr{X}}(X^{\oplus n})$ from the braid groups to the automorphism group of n copies of X. There are two standard ways to embed braid groups in mapping class groups of surfaces, and we explain here how they both come from Yang–Baxter elements in appropriate categories of surfaces.

(a) Let $\mathbf{M}_2$ be the category of bidecorated surfaces of Section 3. As explained in Section 3.2, the Dehn twist $T \in \mathrm{Aut}_{\mathbf{M}_2}(D \# D) \cong \pi_0 \mathrm{Homeo}_\partial(S^1 \times I) \cong \mathbb{Z}$, or its inverse T^{-1}, is a Yang–Baxter operator. The associated map $\Phi_{D,T}\colon B_n \longrightarrow \mathrm{Aut}_{\mathbf{M}_2}(D^{\# n})$ is the embedding of braid group in the mapping class groups of $S_{g,1}$ (when

$n = 2g+1$) and of $S_{g,2}$ (when $n = 2g+2$) associated to Dehn twists along the chain of embedded curves in the surfaces described in Lemma 3.5. This embedding goes back at least to the work of Birman and Hilden [BH72, BH73].

(b) Let $\mathbf{M}_1$ denote instead the category of surfaces decorated by a single interval, with monoidal structure $\oplus$ defined just as in the case of $\mathbf{M}_1$ but gluing only along one interval. Then $\mathbf{M}_1$ in braided monoidal, see [RWW17, Sec 5.6.1]. Hence by Example 5.2, for any object X of $\mathbf{M}_1$, we have a Yang–Baxter element $\tau_X \in \mathrm{Aut}_{\mathbf{M}_1}(X \oplus X)$. For $X = S_{1,2}$, this can be used to prove genus stabilization (albeit with the suboptimal slope $1/2$), and in the case $X = S^1 \times I$ marked by an interval in one of its boundary components, we have that $X^{\oplus n}$ has underlying surface an n-legged pair of pants $D^2 \backslash (\sqcup_n \mathring{D}^2)$ and the associated morphism

$$\Phi_{X,\tau_X} \colon B_n \longrightarrow \mathrm{Aut}_{\mathbf{M}_1}(X^{\oplus n}) = \pi_0 \,\mathrm{Homeo}_\partial(D^2 \backslash (\sqcup_n \mathring{D}^2))$$

is the standard embedding of the braid group as the subgroup of the mapping class group of the multi-legged pants that does not twist the legs, see e.g. [RWW17, Sec 5.6.1].

We will show in Proposition 5.7 below that the Yang–Baxter operator T of the first example, in the category $\mathbf{M}_2$, does not come from a braiding in $\mathbf{M}_2$.

5.2. **Homological stability from Yang–Baxter elements.** Suppose we are given the data of a monoidal category $(\mathscr{X}, \oplus, \mathbb{1})$ acting on a category $\mathscr{M}$, along with a choice of stabilizing object $X \in \mathscr{X}$ and Yang–Baxter operator $\tau \in \mathrm{Aut}_{\mathscr{X}}(X \oplus X)$. Proposition 5.1 above allows to apply [Kra19, Thm A], which in this case says that for any $A \in \mathscr{M}$, there is a sequence of simplicial spaces $W_n(X, A)_\bullet$, for $n \geq 0$, so that if $W_n(X, A)$ is highly-connected for large n, then the sequence

$$\mathrm{Aut}_{\mathscr{M}}(A) \xrightarrow{-\oplus X} \mathrm{Aut}_{\mathscr{M}}(A \oplus X) \xrightarrow{-\oplus X} \mathrm{Aut}_{\mathscr{M}}(A \oplus X \oplus X) \xrightarrow{-\oplus X} \cdots$$

satisfies homological stability. Theorem B of the same paper gives in addition a stability statement with twisted coefficients. Under an injectivity assumption of the form of Proposition 3.4, this simplicial space is homotopy discrete, and modeled by the space of destabilizations as described in Definition 4.1.

Remark 5.4. The fact that (X, τ) is a Yang–Baxter operator is precisely what is needed for the collection of sets $W_n(A, X)_p$ and maps $d_i \colon W_n(A, X)_p \longrightarrow W_n(A, X)_{p-1}$, defined as in Definition 4.1, to assemble into a semi-simplicial set; indeed, the Yang–Baxter equation implies the necessary simplicial identities.

For a fixed monoidal category $\mathscr{X} = (\mathscr{X}, \oplus, \mathbb{1})$ acting on a category $\mathscr{M}$, and a stabilizing object $X \in \mathscr{X}$, the choice of Yang–Baxter element will not affect the stabilizing map, but it will affect the spaces $W_n(X, A)_\bullet$. The identity map $1 \in \mathrm{Aut}_{\mathscr{X}}(X \oplus X)$ is a trivial choice of Yang–Baxter operator. But, as is to be expected, this trivial twist is not useful for proving stability:

Proposition 5.5. *Let $\mathscr{X}$, $\mathscr{M}$, A and X be as above. If we choose the Yang–Baxter operator $\tau \in \mathrm{Aut}_{\mathscr{X}}(X \oplus X)$ to be the identity element, then the semi-simplicial set $W_n(A, X)_\bullet$ is connected if and only if the map*

$$G_{n-1} = Aut_{\mathscr{M}}(A \oplus X^{\oplus n-1}) \xrightarrow{-\oplus X} Aut_{\mathscr{M}}(A \oplus X^{\oplus n}) = G_n$$

is an isomorphism.

Proof. If τ is the identity element, all face maps d_i are equal to the canonical map $G_n/G_{n-p-1} \longrightarrow G_n/G_{n-p}$. In particular, the vertices of any p-simplex are all equal, so the semi-simplicial set $W_n(A, X)_\bullet$ is isomorphic to a disjoint union of semi-simplicial sets, one for each 0-simplex. The result follows from the fact that the set of 0-simplices is precisely the quotient G_n/G_{n-1}. $\square$

In fact, Barucco proved in his master thesis a result that translates to the following stronger statement (stated in the thesis in the context of a groupoid acting on itself, i.e. $\mathscr{M} = \mathscr{X}$):

Lemma 5.6. [Bar17, Lem 3.1] *The space $W_n(A, X)$ is connected if and only if $1^{\oplus n-2} \oplus \tau$ and $G_{n-1} \oplus 1$ together generate $G_n = \mathrm{Aut}(A \oplus X^{\oplus n})$.*

The connectivity of the semi-simplicial set $W_n(A, X)$ (or of the associated simplical complex defined in [RWW17, Def 2.8]) can be thought of as a measure a form of *higher generation* of the group G_n by the cosets of the subgroups G_{n-p} for $p \geq 1$ and braid subgroups generated by the chosen Yang–Baxter element t, in a way similar to the notion of higher generation for a family of subgroups of a group defined in [AH93, 2.1].

5.3. **Braidings and bidecorated surfaces.** We show in this section that the Yang–Baxter operator T on the bidecorated disk D in the groupoid $\mathbf{M}_2$ does not come from a braiding on the subcategory of $\mathbf{M}_2$ generated by the disk. In fact, we will show that this subcategory does not admit a braiding.

Let $D = (D^2, 1, \mathrm{id})$ be the standard bidecorated disk of Section 3, where we recall that $X_1 = D^2$. We define a "rotated" bidecorated disk $\overline{D} = (D^2, 1, r_\pi)$, where r_π is the rotation of $\partial X_1 = \partial D^2$ by π radians, which has the effect of interchanging the intervals I_0 and I_1. Rotating all of D^2 by π then induces a morphism $\iota\colon D \longrightarrow \overline{D}$ in $\mathbf{M}_2$, and likewise morphisms

$$\iota^{\#m}\colon D^{\#m} \longrightarrow \overline{D}^{\#m}$$

for every $m \geq 1$, each which we will by abuse of notation also denote by ι. The morphism ι can be identified with the hyperelliptic involution of the underlying surface depicted in Figure 9 for the two cases $m = 2g$ and $m = 2g+1$, where in the latter case the boundary components are exchanged by ι. The morphism ι induces an identification

$$\begin{aligned} \mathrm{Aut}_{\mathbf{M}_2}(D^{\#m}) &\xrightarrow{\cong} \mathrm{Aut}_{\mathbf{M}_2}(\overline{D}^{\#m}) \\ f &\longmapsto \iota \circ f \circ \iota^{-1} \end{aligned}$$

In order to precisely state the failure of T to extend to a braiding, we will also need the identification

$$\begin{aligned} I\colon \mathrm{Aut}_{\mathbf{M}_2}(D^{\#m}) &\xrightarrow{\cong} \mathrm{Aut}_{\mathbf{M}_2}(\overline{D}^{\#m}) \\ f &\longmapsto f \end{aligned}$$

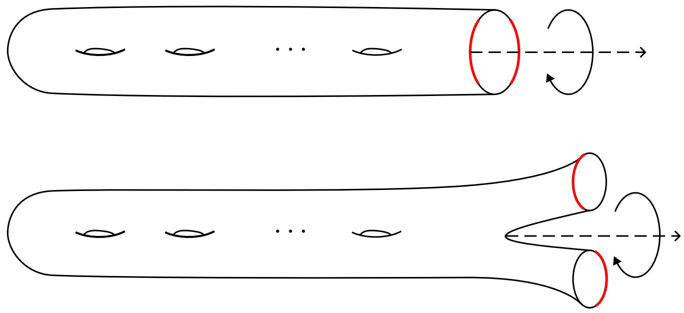

FIGURE 9. The hyperelliptic involutions ι of $S_{g,1}$ and $S_{g,2}$

that comes from the fact that an element $f \in \mathrm{Aut}_{\mathbf{M}_2}(D^{\#m})$ is just a mapping class for the underlying surface of $D^{\#m}$, which is the same as the underlying surface of $\overline{D}^{\#m}$, so f can just as well be viewed as an element of $\mathrm{Aut}_{\mathbf{M}_2}(\bar{D}^{\#m})$. In contrast with the identification induced by ι, the second identification is "external", in the sense that it does not come from a morphism in $\mathbf{M}_2$.

Viewing ι as a diffeomorphism of the underlying surface X_m of $D^{\#m}$ that does not fix the boundary, and specifically exchanges the marked points $b_0 = I_0(1/2)$ and $b_1 = I_1(1/2)$, we see that it takes the isotopy class of arc ρ_i of Section 4.1 to the reversed arc $\overline{\rho_i}$. We will use in the proof of the following result that the homotopy classes ρ_i generate the fundamental groupoid of X_m based at the points b_0, b_1.[5] The mapping class ι is in fact completely determined by the fact that $\iota(\rho_i) = \overline{\rho_i}$.

Proposition 5.7. *Let $\mathbf{D} \subset \mathbf{M}_2$ denote the full monoidal subcategory generated by D.*

(i) The monoidal category $\mathbf{D}$ does not admit a braiding. In particular, the monoidal functor

$$\Phi\colon (\mathbf{B}, \oplus) \longrightarrow (\mathbf{D}, \#) \subset (\mathbf{M}_2, \#)$$

does not come from a braiding on $\mathbf{D}$.

(ii) Let $f \in \mathrm{Aut}_{\mathbf{M}_2}(D^{\#m})$ and $g \in \mathrm{Aut}_{\mathbf{M}_2}(D^{\#n})$, and put $\beta_{m,n} = \Phi(b_{m,n})$, where the block braid $b_{m,n}$ is the braid which passes the last n strands over the first m strands. Then

$$\beta_{m,n} \circ (f \# g) \circ \beta_{n,m}^{-1} = \begin{cases} g \# (\iota^{-1} \circ f \circ \iota) & \text{if } n \text{ is odd,} \\ g \# f & \text{else,} \end{cases}$$

for $\iota\colon D^{\#m} \longrightarrow \overline{D}^{\#m}$ the involution defined above, and where f in the rightmost expression is the map f considered as an element of $\mathrm{Aut}_{\mathbf{M}_2}(\overline{D}^{\#m})$ via the isomorphism I defined above.

Proof. We start by proving (ii). It is enough to check the statement when f and g are Dehn twists, as those generate the mapping class groups. Note that if c is a curve in the underlying surface X_{m+n} of $D^{\#m+n}$, and T_c denotes

[5]As a full subgroupoid of the ordinary fundamental groupoid of X_m, this groupoid is the one spanned by the objects corresponding to the points $b_0, b_1 \in X_m$.

the Dehn twist along c, then conjugating T_c by a diffeomorphism φ of the surface gives

$$\varphi \circ T_c \circ \varphi^{-1} = T_{\varphi(c)}.$$

Recall further that the isotopy class of a Dehn twist T_c depends only on the free homotopy class of the curve c. We are therefore to compute the images of curves in $D^{\#m}$ and $D^{\#n}$ under the map $\beta_{m,n}$, as free homotopy classes. A curve c can be written, up to free homotopy, as a concatenation of the arcs ρ_i and their inverses $\overline{\rho}_i$, as the homotopy classes of these arcs generate the fundamental groupoid of the surface X_{m+n} based at b_0, b_1. In particular, write

$$c \simeq \rho_{i_1} * \overline{\rho}_{i_2} * \rho_{i_3} \cdots * \overline{\rho}_{i_k}. \tag{5.1}$$

The mapping class $\beta_{m,n}$ can be written as the composition

$$\beta_{m,n} = (T_n \circ \cdots \circ T_{m+n-1}) \circ \cdots \circ (T_2 \circ \cdots \circ T_{m+1}) \circ (T_1 \circ \cdots \circ T_m)$$

and hence we can compute the image of each ρ_i using Lemma 4.3. For $r > 0$, denote by $T_{i,i+r}$ the composition of Dehn twists $T_i \circ T_{i+1} \circ \cdots \circ T_{i+r}$. Note first that

$$T_{i,j}(\rho_{j+1}) \simeq T_{i,j-1}(\rho_j) \simeq \cdots \simeq \rho_i.$$

From this, it follows that for $i \geq 1$,

$$\begin{aligned}
\beta_{m,n}(\rho_{m+i}) &\simeq (T_{n,m+n-1}) \circ \cdots \circ (T_{1,m})(\rho_{m+i}) \\
&\simeq (T_{n,m+n-1}) \circ \cdots \circ (T_{i,m+i-1})(\rho_{m+i}) \\
&\simeq (T_{n,m+n-1}) \circ \cdots \circ (T_{i+1,m+i})(\rho_i) \\
&\simeq \rho_i.
\end{aligned}$$

On the other hand, for $i \leq k \leq j$, we have

$$T_{i,j}(\rho_k) \simeq T_{i,k}(\rho_k) \simeq T_{i,k-1}(\rho_k * \overline{\rho}_{k+1} * \rho_k) \simeq \rho_i * \overline{\rho}_{k+1} * \rho_i,$$

from which we can deduce that for $i \leq m$,

$$\begin{aligned}
\beta_{m,n}(\rho_i) &\simeq (T_{n,m+n-1}) \circ \cdots \circ (T_{1,m})(\rho_i) \\
&\simeq (T_{n,m+n-1}) \circ \cdots \circ (T_{2,m+1})(\rho_1 * \overline{\rho}_{i+1} * \rho_1) \\
&\simeq (T_{n,m+n-1}) \circ \cdots \circ (T_{3,m+2})(\rho_1 * \overline{\rho}_2 * \rho_{i+2} * \overline{\rho}_2 * \rho_1) \\
&\simeq \cdots \\
&\simeq \rho_1 * \iota(\rho_2) * \cdots * \iota^{n-1}(\rho_n) * \iota^n(\rho_{i+n}) * \iota^{n-1}(\rho_n) * \cdots * \iota(\rho_2) * \rho_1
\end{aligned}$$

since $\iota^j(\rho_i)$ is ρ_i when j is even and $\overline{\rho}_i$ when j is odd.

If the curve c lies in the last n disks $D^{\#n}$ inside $D^{\#m+n}$, it can be written as a product (5.1) with each $i_j > m$. Then the above computation gives that

$$\beta_{m,n}(c) \simeq \rho_{i_1-m} * \overline{\rho}_{i_2-m} * \rho_{i_3-m} * \cdots * \overline{\rho}_{i_k-m},$$

that is, c is mapped to the corresponding curve in the *first* n disks $D^{\#n}$ inside $D^{\#n+m} = D^{\#m+n}$.

If the curve c instead lies in the first m disks $D^{\#m}$ inside $D^{\#m+n}$, it can be written as a product (5.1) with each $i_j \leq m$. Then the above computation gives that

$$\begin{aligned}\beta_{m,n}(c) &\simeq \rho_1 * \iota(\rho_2) * \cdots * \iota^{n-1}(\rho_n) * \iota^n(\rho_{i_1+n}) * \iota^{n+1}(\rho_{i_2+n}) * \cdots * \iota^{n+1}(\rho_{i_k+n}) \\ &\qquad * \iota^n(\rho_n) * \cdots * \iota^2(\rho_2) * \iota(\rho_1) \\ &\simeq \iota^n(\rho_{i_1+n}) * \iota^{n+1}(\rho_{i_2+n}) * \iota^n(\rho_{i_3+n}) \cdots * \iota^{n+1}(\rho_{i_k+n}) \\ &\simeq \iota^n(\rho_{i_1+n} * \overline{\rho}_{i_2+n} * \rho_{i_3+n} \cdots * \overline{\rho}_{i_k+n})\end{aligned}$$

Hence c is mapped to the curve $\iota^n(c)$ in the last m disks $D^{\#m}$ inside $D^{\#n+m} = D^{\#m+n}$, from which the statement follows.

We are left to prove (i). To see that the images $\beta_{m,n}$ of block braids under β do not define a braiding in $\mathbf{D}$, using (ii) it is enough to find a curve c in $D^{\#m}$ for some m so that $\iota(c) \not\simeq c$, and such curves are plentiful. The same argument shows that the inverses $\beta_{m,n}^{-1}$ likewise do not define a braiding.

Now suppose that $\widetilde{\beta}$ is a braiding on $\mathbf{D}$. The braiding is determined by $\widetilde{\beta}_{1,1} \in \mathrm{Aut}_{\mathbf{M}_2}(D^{\#2}) \cong \mathbb{Z}$, a group generated by the Dehn twist T_1. We have excluded the possibilities $\widetilde{\beta}_{1,1} = T_1^{\pm 1}$, and $\widetilde{\beta}_{1,1} = \mathrm{id}$ is similarly ruled out using now the fact that curves are not moved at all by the identity. So assume that $\widetilde{\beta}_{1,1} = T_1^k$, with $|k| > 1$. Then $\widetilde{\beta}_{2,1} = T_1^k T_2^k$ would have to satisfy $T_1^k T_2^k(a_1) = a_2$ in order for naturality to hold, where T_i is the Dehn twist along the curve a_i as in Section 3.2. Applying Proposition 3.2 in [FM11] twice, we get that the intersection number $i(a_2, T_2^k(a_1)) = i(T_1^k(T_2^k(a_1)), T_2^k(a_1)) = |k| i(a_1, T_2^k(a_1)^2 = |k|^2 i(a_1, a_2)^4 = |k|^2$. On the other hand, using Proposition 3.4 in [FM11] we obtain

$$|k|^2 = i(a_2, T_2^k(a_1)) = |i(T_2^k(a_1), a_2) - |k| i(a_1, a_1) i(a_1, a_2)| \leq i(a_1, a_2) = 1,$$

where we have also used that $i(a_1, a_1) = 0$. This contradicts our assumption of $\widetilde{\beta}_{1,1}$. □

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2 Compact sheaves on a locally compact space

COMPACT SHEAVES ON A LOCALLY COMPACT SPACE

OSCAR BENDIX HARR

Abstract. Let X be a hypercomplete locally compact Hausdorff space and let $\mathcal{C}$ be a compactly generated stable ∞-category. We describe the compact objects in the ∞-category of $\mathcal{C}$-valued sheaves $\mathrm{Shv}(X, \mathcal{C})$. When X is a non-compact connected manifold and $\mathcal{C}$ is the unbounded derived ∞-category of a ring, our result recovers a result of Neeman. Furthermore, if $\mathcal{C}$ is a nontrivial compactly generated stable ∞-category, we show that $\mathrm{Shv}(X, \mathcal{C})$ is compactly generated if and only if X is totally disconnected.

The aim of this paper is to clarify and expand on a point made by Neeman [Nee01]. Let M be a non-compact connected manifold, and let $\mathrm{Shv}(M, \mathcal{D}(\mathbb{Z}))$ denote the unbounded derived ∞-category of sheaves of abelian groups on M. Neeman shows that the only compact object in $\mathrm{Shv}(M, \mathcal{D}(\mathbb{Z}))$ is the zero sheaf. This implies that $\mathrm{Shv}(M, \mathcal{D}(\mathbb{Z}))$ is very far from compactly generated. Nevertheless, it follows from Lurie's covariant Verdier duality theorem [Lur17, Thm 5.5.5.1] that $\mathrm{Shv}(M, \mathcal{D}(\mathbb{Z}))$ satisfies a related smallness condition: it is *dualizable* in the symmetric monoidal ∞-category $\mathcal{P}\mathrm{r}^{\otimes}_{\mathrm{stab}}$ of stable presentable ∞-categories and left adjoints, which holds more generally if M is replaced with any locally compact Hausdorff space X. Although every compactly generated presentable stable ∞-category is dualizable [Lur18, Prop D.7.2.3], Neeman's example thus shows that the converse is false. The existence of this large and interesting class of stable presentable ∞-categories that are dualizable but not compactly generated forms part of the motivation behind Efimov's continuous extensions of localizing invariants [Efi24], see also [Efi22, Hoy18].

Let X be a locally compact Hausdorff space and let $\mathcal{C}$ be a compactly generated stable ∞-category (e.g. the unbounded derived ∞-category of a ring or the ∞-category of spectra). This paper is concerned with the following two questions about the ∞-category of $\mathcal{C}$-valued sheaves on X:

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(1) How rare is it for $\mathrm{Shv}(X, \mathcal{C})$ to be compactly generated?
(2) How far is $\mathrm{Shv}(X, \mathcal{C})$ from being compactly generated in general?

With a relatively mild completeness assumption on X (see Section 1), we answer question (2) by showing that a $\mathcal{C}$-valued sheaf $\mathscr{F}$ on X is compact as an object of $\mathrm{Shv}(X, \mathcal{C})$ if and only if it has compact support, compact stalks, and is locally constant (Theorem 2.6).[1] Thus if X is for instance a CW complex, the subcategory of compact objects $\mathrm{Shv}(X, \mathcal{C})^{\omega}$ remembers only the *homotopy type* of the compact path components of X. Therefore, it is impossible to reconstruct the entire sheaf ∞-category $\mathrm{Shv}(X, \mathcal{C})$ from this information.

In his 2022 ICM talk, Efimov mentions that the ∞-category of $\mathcal{D}(R)$-valued sheaves on a locally compact Hausdorff space X 'is almost never compactly generated (unless X is totally disconnected, like a Cantor set)' [Efi22, slide 13]. Modulo the same completeness assumption mentioned above, as a corollary to our description of the compact objects of $\mathrm{Shv}(X, \mathcal{C})$, we verify that the only locally compact Hausdorff spaces X with $\mathrm{Shv}(X, \mathcal{C})$ compactly generated, for some nontrivial $\mathcal{C}$, are the totally disconnected ones (Proposition 3.1). This answers question (1).

Notation and conventions. Throughout this paper, we use the theory of higher categories and higher algebra, an extensive textbook account of which can be found in the work of Lurie [Lur09, Lur17, Lur18]. We will also make frequent use of the six-functor formalism for derived sheaves on locally compact Hausdorff spaces, as described classically by [Ver65, KS90] and with general coefficients by [Vol23].

For convenience, we assume the existence of an uncountable Grothendieck universe $\mathcal{U}$ of *small* sets and further Grothendieck universes $\mathcal{U}'$ and $\mathcal{U}''$ of *large* and *very large* sets respectively, such that $\mathcal{U} \in \mathcal{U}' \in \mathcal{U}''$. 'Topological space' always implicitly refers to a small topological space, and similarly with 'spectrum'. On the other hand, '∞-category' refers to a large ∞-category unless otherwise stated. We let $\widehat{\mathcal{C}\mathrm{at}}_{\infty}$ denote the very large ∞-category of (large) ∞-categories.

Because we are dealing with sheaves on topological spaces, thinks it is best to make a clear distinction between actual topological spaces on the one hand, and on the other hand the objects of the ∞-category $\mathcal{S}$ of 'spaces' in the sense of Lurie. Following the convention suggested in [CS23], we will refer to the latter as *anima*.

Recall that an object C in an ∞-category $\mathcal{C}$ is said to be *compact* if the presheaf of large anima $D \mapsto \mathrm{Map}_{\mathcal{C}}(C, D)$ preserves small filtered colimits. We let $\mathcal{C}^{\omega} \subseteq \mathcal{C}$ denote the subcategory spanned by the compact objects.

1. $\mathcal{C}$-hypercomplete spaces

Given a ∞-category $\mathcal{C}$ and a topological space X, we let $\mathrm{Shv}(X, \mathcal{C})$ denote the ∞-category of $\mathcal{C}$-valued sheaves on X in the sense of Lurie [Lur09]. That

[1] Since posting this paper on the arXiv, we became aware that Scholze has indicated a proof of this statement for $\mathcal{C} = D(\mathbb{Z})$ in his notes on six-functor formalisms [Sch, Prop 7.11]. The approach taken there, which uses descent to deduce the general statement from the case where X is a profinite set, is different from the one we take.

is, $\mathrm{Shv}(X, \mathcal{C})$ is the full subcategory of the presheaf ∞-category

$$\mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, \mathcal{C})$$

consisting of presheaves $\mathscr{F}$ satisfying the *sheaf condition*: for any open set $U \subseteq X$ and any open cover $\{U_i \to U\}_{i \in I}$, the canonical map

$$\mathscr{F}(U) \to \lim_V \mathscr{F}(V)$$

is an equivalence, where V ranges over open sets $V \subseteq U_i \subseteq X$, $i \in I$, considered as a poset under inclusion. When $\mathcal{C} = \mathcal{S}$ is the ∞-category of anima, we will abbreviate $\mathrm{Shv}(X) = \mathrm{Shv}(X, \mathcal{S})$.

Remark 1.1. When $\mathcal{C} = \mathcal{D}(R)$ is the unbounded derived ∞-category of a ring, the ∞-category $\mathrm{Shv}(X, \mathcal{D}(R))$ is related to, but generally not the same as, the derived ∞-category $\mathcal{D}(\mathrm{Shv}(X, R))$ of the ordinary category of sheaves of R-modules on X, which is the object studied (via its homotopy category) by Neeman [Nee01]. However, they do coincide under the completeness assumption that we will impose on X below, see [Sch, Prop 7.1]. Since this completeness assumption is verified when X is a topological manifold, our results include those of Neeman.

We are interested in topological spaces satisfying the following condition:

Definition 1.2. Let $\mathcal{C}$ be a presentable ∞-category. A topological space X is *$\mathcal{C}$-hypercomplete* if the stalk functors $x^*\colon \mathrm{Shv}(X, \mathcal{C}) \to \mathcal{C}$ are jointly conservative for x ranging over the points of X.

The reason for our choice of terminology is that X is $\mathcal{S}$-hypercomplete if and only if the 0-localic ∞-topos $\mathrm{Shv}(X)$ has enough points, which is equivalent to $\mathrm{Shv}(X)$ being hypercomplete as an ∞-topos by Claim (6) in [Lur09, § 6.5.4]. (This is *not* true for arbitrary ∞-topoi, i.e. there are hypercomplete ∞-topoi that do not have enough points.) This subtlety, whereby a morphism of sheaves may fail to be an equivalence even though it is so on all stalks, does not occur for non-derived sheaves In fact, if $\mathcal{C}$ is an n-category for $n < \infty$, i.e. has $(n-1)$-truncated mapping spaces, then every topological space X is $\mathcal{C}$-hypercomplete, see e.g. [Hai22a, Rem 1.8].

We refer to [Lur09, § 6.5.4] for a discussion of why it is often preferable to work with non-hypercomplete sheaves, rather than, say, imposing hypercompleteness by replacing $\mathrm{Shv}(X)$ with its hypercompletion $\mathrm{Shv}(X)^\wedge$.

The following observation will provide us with a source of $\mathcal{C}$-hypercomplete spaces. First, recall that a presentable ∞-category is *compactly assembled* if, when viewed as an object of the ∞-category of presentable ∞-categories and left adjoints, it is a retract of a compactly generated ∞-category.

Proposition 1.3. *Let X be an $\mathcal{S}$-hypercomplete topological space. Then X is also $\mathcal{C}$-hypercomplete for any compactly assembled ∞-category $\mathcal{C}$.*

Proof. Given $x \in X$, let us write $x^*_{\mathcal{C}}\colon \mathrm{Shv}(X, \mathcal{C}) \to \mathcal{C}$ to distinguish the $\mathcal{C}$-valued stalk functor from the $\mathcal{S}$-valued stalk functor $x^*\colon \mathrm{Shv}(X) = \mathrm{Shv}(X, \mathcal{S}) \to$

$\mathcal{S}$. For each x there is a commutative diagram

$$\begin{tikzcd} \mathrm{Shv}(X) \otimes \mathcal{C} \arrow[r, "x^* \otimes \mathcal{C}"] \arrow[d, "\simeq"] & \mathcal{S} \otimes \mathcal{C} \arrow[d, "\simeq"] \\ \mathrm{Shv}(X, \mathcal{C}) \arrow[r, "x^*_{\mathcal{C}}"] & \mathcal{C} \end{tikzcd}$$

where the vertical maps are the equivalences of [Lur18, Rem 1.3.1.6], see also [Jan22, Lem B.3]. By assumption the maps x^* are jointly conservative as x varies over X, and it follows from [Hai22b, Lem 2.12] that the functors $x^* \otimes \mathcal{C}$ are also jointly conservative. But then the $\mathcal{C}$-valued stalk functors $x^*_{\mathcal{C}}$ must also be jointly conservative. $\square$

The literature describes several classes of topological spaces that are $\mathcal{S}$-hypercomplete. Here is a list of some classes of topological spaces that have this property:

- paracompact spaces that are locally of covering dimension $\leq n$ for some fixed n [Lur09, Cor 7.2.1.12],
- arbitrary CW complexes [Hoy16],
- finite-dimensional Heyting spaces [Lur09, Rem 7.2.4.18], and
- Alexandrov spaces associated to posets with binary joins [Aok23, Exmp A.12].[2]
- spectral spaces (recalled in Subsection 3.2 below) of finite Krull dimension [CM21, Thm 3.12].

2. When is a sheaf compact?

Let $\mathcal{C}$ be a compactly generated stable ∞-category, e.g. the unbounded derived ∞-category $\mathcal{D}(R)$ of a ring R or the ∞-category of spectra Sp.

Definition 2.1. Given a sheaf $\mathscr{F} \in \mathrm{Shv}(X, \mathcal{C})$, the *support* of $\mathscr{F}$ is the subspace

$$\operatorname{supp} \mathscr{F} = \{x \in X \mid \mathscr{F}_x \not\simeq 0\} \subseteq X.$$

As in [Nee01], our study of the compact objects of $\mathrm{Shv}(X, \mathcal{C})$ proceeds from an analysis of their supports. By slightly adapting the proof of [Nee01, Lem 1.4], we get the following description of the support of a compact sheaf:

Lemma 2.2. *Let X be a $\mathcal{C}$-hypercomplete locally compact Hausdorff space and let $\mathscr{F} \in \mathrm{Shv}(X, \mathcal{C})^{\omega}$. Then the support* $\operatorname{supp} \mathscr{F}$ *is compact.*

Proof. We first show that $\operatorname{supp} \mathscr{F}$ is contained in a compact subset of X. Consider the canonical map

$$\operatorname{colim}_U (j_U)_! j_U^* \mathscr{F} \to \mathscr{F}, \tag{2.1}$$

where the colimit ranges over the poset of relatively compact open sets ordered by the rule $U \leq V$ if $\overline{U} \subseteq V$, and for each such U we have denoted by $j_U : U \hookrightarrow X$ the inclusion. Since X is locally compact Hausdorff, the relatively compact open subsets of X form a basis for its topology. Hence the map (2.1) is an equivalence of sheaves. Let $\phi \colon \mathscr{F} \xrightarrow{\sim} \operatorname{colim}_U (j_U)_! (j_U)^* \mathscr{F}$

[2] Contrary to what was stated in an earlier version of this article, not all Alexandrov spaces are $\mathcal{S}$-hypercomplete, see [Aok23, Exmp A.13].

be some choice of inverse. Any finite union of relatively compact open sets is again relatively compact open, so the poset of relatively compact open sets is filtered. Hence compactness of $\mathscr{F}$ implies that ϕ factors through $(j_U)_! j_U^* \mathscr{F}$ for some relatively compact open U, and it follows that $\operatorname{supp} \mathscr{F}$ is contained in a compact subset $\overline{U} \subseteq X$, as claimed.

By the above, it remains only to be seen that $\operatorname{supp} \mathscr{F}$ is closed, or equivalently that its complement $X \setminus \operatorname{supp} \mathscr{F}$ is open. Suppose $x \in X \setminus \operatorname{supp} \mathscr{F}$. Then we have a recollement fiber sequence

$$j_! j^* \mathscr{F} \to \mathscr{F} \to i_* i^* \mathscr{F},$$

where $j\colon X \setminus \{x\} \hookrightarrow X$ and $i\colon \{x\} \hookrightarrow X$ are the inclusions, and since $x \notin \operatorname{supp} \mathscr{F}$ we have $j_! j^* \mathscr{F} \simeq \mathscr{F}$. Since $j_!$ is a fully faithful left adjoint, it reflects compact objects, and we conclude that $j^* \mathscr{F}$ is again compact. But then $j^* \mathscr{F}$ is supported on a compact subset of $X \setminus \{x\}$ by the above, which must be closed as a subset of X, and hence x lies in the interior of $X \setminus \operatorname{supp} \mathscr{F}$ as desired. □

Notation 2.3. Let X be a topological space. Given $E \in \mathcal{C}$, we denote by E_X the constant sheaf on X with value E.

Lemma 2.4. *If $f\colon X \to Y$ is a proper map of locally compact Hausdorff spaces, then the pullback functor f^* preserves compact objects. In particular, if X is a compact Hausdorff space and $E \in \mathcal{C}^\omega$, then $E_X \in \operatorname{Shv}(X, \mathcal{C})^\omega$.*

Proof. Since f is proper, the pullback f^* is left adjoint to $f_* \simeq f_!$, which is itself left adjoint to $f^!$. Hence f_* preserves colimits, and it follows that its left adjoint f^* preserves compact objects. The statement about constant sheaves follows by taking f to be the projection from X to a point. □

Remark 2.5. As pointed out by the anonymous referee, the previous lemma is also true without our standing assumption that $\mathcal{C}$ is stable. Indeed, the proof only used that f_* preserves filtered colimits, and the fact that f is proper means that this holds with coefficients in any (not necessarily stable) compactly generated ∞-category, see [Lur09, Rem 7.3.1.5, Thm 7.3.1.16] and [Hai22b, Cor 3.11].

Our main result is the following description of the compact objects in $\operatorname{Shv}(X, \mathcal{C})$:

Theorem 2.6. *Let X be a $\mathcal{C}$-hypercomplete locally compact Hausdorff space. A sheaf $\mathscr{F} \in \operatorname{Shv}(X, \mathcal{C})$ is compact if and only if*

(i) $\operatorname{supp} \mathscr{F}$ *is compact;*
(ii) $\mathscr{F}$ *is locally constant; and*
(iii) $\mathscr{F}_x \in \mathcal{C}^\omega$ *for each* $x \in X$.

In particular, note that conditions (i) and (ii) together imply that if $\mathscr{F}$ is compact, then the support of $\mathscr{F}$ must be a compact open subset of X. Indeed, every locally constant sheaf $\mathscr{F} \in \operatorname{Shv}(X, \mathcal{C})$ has open support. To see this, suppose $x \in \operatorname{supp} \mathscr{F}$. Since $\mathscr{F}$ is locally constant, there is some open neighborhood U of x in which $\mathscr{F}$ is constant. It follows in particular that every $y \in U$ will have $\mathscr{F}_y \simeq \mathscr{F}_x$, which is nonzero by assumption; hence $U \subseteq \operatorname{supp} \mathscr{F}$, showing that $\operatorname{supp} \mathscr{F}$ is open.

Proof. 'Necessity.' Suppose we are given $\mathscr{F} \in \mathrm{Shv}(X, \mathcal{C})^\omega$. Lemma 2.2 shows that $\mathscr{F}$ satisfies (i). For each $x \in X$ the inclusion $i_x \colon \{x\} \hookrightarrow X$ is proper and the stalk $\mathscr{F}_x$ is the same as the pullback $i_x^* \mathscr{F}$, hence Lemma 2.4 shows that $\mathscr{F}$ satisfies (iii). It remains only to be seen that $\mathscr{F}$ is locally constant. Fix a point $x \in X$, and let i_x again denote the inclusion of this point into X. Let $E = i_x^* \mathscr{F}$ denote the stalk of $\mathscr{F}$ at x. By [Lur09, Cor 7.1.5.6], there is an equivalence $E \simeq \mathrm{colim}_U \mathscr{F}(U)$, where U ranges over the poset of open neighborhoods of x. As E is compact, this equivalence must factor through the canonical map $\mathscr{F}(V) \to \mathrm{colim}_U \mathscr{F}(V)$ for some V, exhibiting E as a retract of $\mathscr{F}(V)$. Pick a relatively compact open neighborhood $W \ni x$ with $\overline{W} \subseteq V$, and let $i \colon \overline{W} \hookrightarrow X$ denote the inclusion. As the canonical map $\mathscr{F}(V) \to E$ factors through the restriction $\mathscr{F}(V) \to (i^*\mathscr{F})(\overline{W}) \to \mathscr{F}(W)$, the map $(i^*\mathscr{F})(\overline{W}) \to E$ also admits a section $E \to (i^*\mathscr{F})(\overline{W})$. Viewing the latter as a morphism from the constant presheaf with value E to $i^*\mathscr{F}$, we get an induced map $\sigma \colon E_{\overline{W}} \to i^*\mathscr{F}$ of sheaves over $\overline{W}$ which by construction induces an equivalence of stalks at x. Here both $E_{\overline{W}}$ and $i^*\mathscr{F}$ are compact, so the cofiber $\mathscr{Q} = \mathrm{cofib}(\sigma)$ is again compact. But then $\mathrm{supp}\,\mathscr{Q}$ is compact, so $W' = W \setminus \mathrm{supp}\,\mathscr{Q}$ is open and $\mathscr{Q}_x \simeq 0$ so $x \in W'$ (see Figurefig:espace-etale). Furthermore, σ restricts to an equivalence of sheaves on W' by construction,

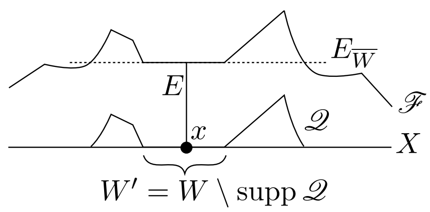

FIGURE 1. 'Espace étale' visualization of the fiber sequence $E_{\overline{W}} \to \mathscr{F} \to \mathscr{Q}$

so $\mathscr{F}|_{W'}$ is equivalent to the constant sheaf on W' with value E, as desired.

'Sufficiency.' Let $i \colon \mathrm{supp}\,\mathscr{F} \hookrightarrow X$ denote the inclusion. Recall from the discussion following the statement of the theorem that since $\mathscr{F}$ is locally constant, its support $\mathrm{supp}\,\mathscr{F}$ is open. Thus i is both proper and an open immersion, and we therefore have that the functors $i_* \simeq i_!$ and $i^* \simeq i^!$ both preserve compact objects. By replacing X with $\mathrm{supp}\,\mathscr{F}$, we may therefore assume that X is compact. Pick a finite collection of closed subsets $Z_i \subseteq X$, $i = 1, \dots, n$, such that $\mathscr{F}$ is constant in a neighborhood of each Z_i and such that X is covered by the interiors Z_i°. Descent (Corollary A.3) implies that the canonical functor

$$\mathrm{Shv}(X, \mathcal{C}) \to \lim_{\Delta_{\leq n}} \left(\mathrm{Shv}\big(\textstyle\bigcap_1^n Z_i, \mathcal{C}\big) \overset{\longleftarrow}{\underset{\longleftarrow}{\vdots}} \cdots \overset{\leftarrow}{\underset{\leftarrow}{\rightleftarrows}} \prod_{i,j} \mathrm{Shv}(Z_i \cap Z_j, \mathcal{C}) \rightleftarrows \prod_i \mathrm{Shv}(Z_i, \mathcal{C}) \right)$$

is an equivalence. Write $I = \{1, \dots, n\}$ for short and put $Z_J = \bigcap_{j \in J} Z_j$ for each $J \subseteq I$. The canonical projection from $\mathrm{Shv}(X, \mathcal{C})$ to $\mathrm{Shv}(Z_J, \mathcal{C})$ is the restriction map. By construction, we have that for each $J \subseteq I$, the restriction $\mathscr{F}|_{Z_J}$ is constant with value a compact object, and hence compact as an object of $\mathrm{Shv}(Z_J, \mathcal{C})$ by the preceding lemma. According to [Lur09, Lem 6.3.3.6], the identity functor $\mathrm{id} \colon \mathrm{Shv}(X, \mathcal{C}) \to \mathrm{Shv}(X, \mathcal{C})$ is the colimit of a diagram $\Delta_{\leq n} \to \mathrm{Fun}(\mathrm{Shv}(X, \mathcal{C}), \mathrm{Shv}(X, \mathcal{C}))$ which sends the object $[k] \in \Delta_{\leq n}$ to the composition

$$‘\,\mathrm{Shv}(X,\mathcal{C}) \longrightarrow \prod_{\substack{J\subseteq I,\\ |J|=k}} \mathrm{Shv}(Z_J,\mathcal{C}) \;\simeq\; \mathrm{Shv}\left(\coprod_{\substack{J\subseteq I,\\ |J|=k}} Z_J, \mathcal{C}\right) \xrightarrow{(i_k)_*} \mathrm{Shv}(X,\mathcal{C}),$$

and so for any filtered system $\{\mathscr{G}_\alpha\}_{\alpha\in A}$, we find

$$\begin{aligned}
\mathrm{Map}(\mathscr{F}, \mathrm{colim}_A\, \mathscr{G}_\alpha) &\simeq \lim\nolimits_{[k]\in\Delta_{\leq n}} \mathrm{Map}(\mathscr{F}, (i_k)_* i_k^* \,\mathrm{colim}\, \mathscr{G}_\alpha) \\
&\simeq \lim\nolimits_{[k]\in\Delta_{\leq n}} \mathrm{Map}(i_k^*\mathscr{F}, \mathrm{colim}_A\, i_k^*\mathscr{G}_\alpha) \\
&\simeq \lim\nolimits_{[k]\in\Delta_{\leq n}} \mathrm{colim}_A\, \mathrm{Map}(i_k^*\mathscr{F}, i_k^*\mathscr{G}_\alpha) \\
&\simeq \mathrm{colim}_A \lim\nolimits_{[k]\in\Delta_{\leq n}} \mathrm{Map}(i_k^*\mathscr{F}, i_k^*\mathscr{G}_\alpha) \\
&\simeq \mathrm{colim}_A\, \mathrm{Map}(\mathscr{F}, \mathscr{G}_\alpha),
\end{aligned}$$

where the third equivalence uses that the restriction $i_k^*\mathscr{F}$ is compact[3] and the second-last equivalence uses that filtered colimits commute are left exact in $\mathcal{S}$. $\square$

As an immediate corollary, we have:

Corollary 2.7. *Let X be a $\mathcal{C}$-hypercomplete locally compact Hausdorff space whose quasicomponents are all non-compact. Then $\mathscr{F} \in \mathrm{Shv}(X, \mathcal{C})^\omega$ if and only if $\mathscr{F} \simeq 0$.*

Note that this recovers Neeman's result when X is a connected non-compact manifold and $\mathcal{C}$ is the ∞-category $\mathcal{D}(\mathbb{Z})$.

As a further corollary to our theorem, we will describe the dualizable objects in the ∞-category of sheaves on a locally compact Hausdorff space. Suppose that $\mathcal{C}$ has the structure of a presentably monoidal ∞-category $\mathcal{C}^\otimes \in \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{P}\mathrm{r}^\otimes_{\mathrm{stab}})$, meaning roughly that $\mathcal{C}$ has a coherently associative and unital tensor product $\otimes$ that commutes with colimits in each variable. We let $\mathbf{1} \in \mathcal{C}$ denote the unit with respect to $\otimes$. Recall that an object $D \in \mathcal{C}$ is said to be *right dualizable* if there exists an object $D^\vee \in \mathcal{C}$ and a morphism $e\colon D^\vee \otimes D \to \mathbf{1}$ such that for all $E, F \in \mathcal{C}$, the map
(2.2)
$$\mathrm{Map}_{\mathcal{C}}(E, F\otimes D^\vee) \xrightarrow{-\otimes D} \mathrm{Map}_{\mathcal{C}}(E\otimes D, F\otimes D^\vee \otimes D) \xrightarrow{(F\otimes e)\circ} \mathrm{Map}_{\mathcal{C}}(E\otimes D, F)$$

is an equivalence. Right dualizability is an algebraic smallness condition, just as compactness is a purely categorical smallness condition. Indeed, if the unit $\mathbf{1}$ is compact as an object of $\mathcal{C}$, then by a well-known observation every right dualizable object of $\mathcal{C}$ is compact. To see this, suppose $D \in \mathcal{C}$ is right dualizable and $I \to \mathcal{C}$, $i \mapsto E_i$, is a filtered diagram of objects in $\mathcal{C}$.

[3]Indeed, we have already observed that $\mathscr{F}|_{Z_J}$ is compact for each J, and hence the associated object $i_k^*\mathscr{F}$ in the product $\Pi_J\, \mathrm{Shv}(Z_J, \mathcal{C})$is also compact according to [Lur09, Lem 5.3.4.10].

Then we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(D, \mathrm{colim}_I E_i) \arrow[r] \arrow[d, "\simeq"] & \mathrm{colim}_I \mathrm{Map}_{\mathcal{C}}(D, E_i) \arrow[d, "\simeq"] \\
\mathrm{Map}_{\mathcal{C}}(\mathbf{1} \otimes D, \mathrm{colim}_I E_i) \arrow[r] \arrow[d, "\simeq"] & \mathrm{colim}_I \mathrm{Map}_{\mathcal{C}}(D, E_i) \arrow[d, "\simeq"] \\
\mathrm{Map}_{\mathcal{C}}(\mathbf{1}, (\mathrm{colim}_I E_i) \otimes D^\vee) \arrow[r] \arrow[d, "\simeq"] & \mathrm{colim}_I \mathrm{Map}_{\mathcal{C}}(\mathbf{1}, E_i \otimes D^\vee) \\
\mathrm{Map}_{\mathcal{C}}(\mathbf{1}, \mathrm{colim}_I (E_i \otimes D^\vee)) \arrow[ur, "\simeq"] &
\end{tikzcd}$$

where the vertical maps in the top square are induced by the unit equivalences $D \simeq \mathbf{1} \otimes D$, the vertical maps in the second square are the equivalences of the form (2.2) coming from the assumption that D is dualizable, and the lower triangle shows that the lowest straight horizontal arrow factors as post-composition with the canonical equivalence

$$(\mathrm{colim}_I E_i) \otimes D^\vee \simeq \mathrm{colim}_I (E_i \otimes D^\vee),$$

where we use that $\otimes$ preserves colimits, followed by the canonical map

$$\mathrm{Map}_{\mathcal{C}}(\mathbf{1}, \mathrm{colim}_I (E_i \otimes D^\vee)) \to \mathrm{colim}_I \mathrm{Map}_{\mathcal{C}}(\mathbf{1}, E_i \otimes D^\vee),$$

which we know to be an equivalence by our assumption that $\mathbf{1}$ is compact.

Given a presentably monoidal stable ∞-category $\mathcal{C}^\otimes$ as above and a topological space X, we can equip also the ∞-category of $\mathcal{C}$-valued sheaves $\mathrm{Shv}(X, \mathcal{C})$ with the structure of a presentably monoidal ∞-category, which is roughly given by defining the tensor product of $\mathscr{F}, \mathscr{G} \in \mathrm{Shv}(X, \mathcal{C})$ to be the sheafification of the presheaf

$$U \mapsto \mathscr{F}(U) \otimes \mathscr{G}(U).$$

(For a precise definition, see e.g. the discussion following [Vol23, Thm 1.3].) The unit with respect to this tensor product is the constant sheaf $\mathbf{1}_X$ at the unit $\mathbf{1} \in \mathcal{C}$, and for each continuous map $f \colon Y \to X$, the pullback functor $f^* \colon \mathrm{Shv}(X, \mathcal{C}) \to \mathrm{Shv}(Y, \mathcal{C})$ can be canonically endowed with the structure of a monoidal functor. In a similar vein to the question answered by Theorem 2.6, one could ask for a classification of the dualizable objects of $\mathrm{Shv}(X, \mathcal{C})^\otimes$ with respect to the monoidal structure defined above, when X is a $\mathcal{C}$-hypercomplete locally compact Hausdorff space. For an $\mathbb{E}_\infty$-ring R and $\mathcal{C}^\otimes = \mathrm{Mod}_R^\otimes$ the associated category of module spectra, this question has been answered in great generality by Martini and Wolf [MW22]; they characterize the dualizable sheaves of Mod_R-modules on an arbitrary ∞-topos, and in particular they do not require a hypercompleteness assumption. We extend their characterization to other coefficient categories, but assume hypercompleteness in order to invoke Theorem 2.6:

Corollary 2.8 (cf. [MW22, Thm E]). *Let $\mathcal{C}^\otimes$ be a presentably monoidal stable ∞-category, whose underlying ∞-category is compactly generated and such that the unit $\mathbf{1} \in \mathcal{C}$ is compact. Let X be a $\mathcal{C}$-hypercomplete locally compact Hausdorff space. With respect to the induced symmetric monoidal structure on $\mathrm{Shv}(X, \mathcal{C})$, a sheaf $\mathscr{F} \in \mathrm{Shv}(X, \mathcal{C})$ is dualizable if and only if*

(i) $\mathscr{F}$ *is locally constant, and*
(ii) $\mathscr{F}_x$ *is dualizable for each* $x \in X$.

Proof. 'Sufficiency.' Let $\mathscr{F}$ be a sheaf satisfying conditions (i) and (ii), and let $\mathcal{U}$ be an open cover of X such that $\mathscr{F}|_U$ is equivalent to a constant sheaf for each $U \in \mathcal{U}$. Čech descent implies that $\mathrm{Shv}(X, \mathcal{C})$ is equivalent to the limit $\lim_V \mathrm{Shv}(V, \mathcal{C})$, as V runs over the poset of open sets V such that $V \subseteq U$ for some $U \in \mathcal{U}$. For each of these V we have that $\mathscr{F}|_V$ is equivalent to a constant sheaf, which if $V \neq \varnothing$ will be of the form $\pi^*\mathscr{F}_x$, where $\pi\colon V \to *$ is the projection and $\mathscr{F}_x$ is the stalk at any $x \in V$. But π^* is monoidal and hence preserves right dualizable objects, whence by dualizability of $\mathscr{F}_x$ we know that $\mathscr{F}|_V$ is right dualizable too. It now follows from the descent property for dualizability [Lur17, Prop 4.6.1.11] that $\mathscr{F}$ is right dualizable as an object of $\mathrm{Shv}(X, \mathcal{C})$.

'Necessity.' Assume that $\mathscr{F}$ is dualizable, and let $x \in X$ be some point. The condition on the stalks is immediate, since pullback preserves dualizable sheaves. We must show that $\mathscr{F}$ is locally constant in a neighborhood of x. Pick a relatively compact open neighborhood $U \ni x$. Then $\mathscr{F}|_{\overline{U}}$ is again dualizable, and since the monoidal unit $R_{\overline{U}} = \pi^*R \in \mathrm{Shv}(\overline{U}, \mathcal{C})$ is compact, it follows that $\mathscr{F}|_{\overline{U}}$ is compact as an object of $\mathrm{Shv}(\overline{U}, \mathcal{C})$. But then the previous theorem implies that it must be locally constant on $\overline{U}$, and hence also on the subset U as desired. □

3. When is $\mathrm{Shv}(X, \mathcal{C})$ compactly generated?

In this section, we prove the following characterization of those locally compact Hausdorff spaces X that have $\mathrm{Shv}(X, \mathcal{C})$ compactly generated:

Proposition 3.1. *Let* $\mathcal{C}$ *be a non-trivial compactly generated stable* ∞*-category, and let* X *be a* $\mathcal{C}$*-hypercomplete locally compact Hausdorff space. Then* $\mathrm{Shv}(X, \mathcal{C})$ *is compactly generated if and only if* X *is totally disconnected.*

3.1. **Proof of the proposition.** The proof will use the following observation:[4]

Lemma 3.2. *Let* $\mathcal{C}$ *be a compactly generated stable* ∞*-category, and let* $\{C_i\}_{i \in I}$ *and* $\{D_i\}_{i \in I}$ *be filtered systems in* $\mathcal{C}$ *indexed over the same poset* I.

(1) Suppose that for each $i \in I$*, there is some* $j \geq i$ *so that the transition map* $C_i \to C_j$ *factors through the zero object* $*$*. Then* $\mathrm{colim}_I\, C_i \simeq *$*. If each* C_i *is compact, then the converse holds.*

[4] I am thankful to Maxime Ramzi for pointing out that an earlier incarnation of this lemma, which appeared in the first arXiv version of this paper, was incorrect. The following proof of the more restricted lemma was suggested to me by Jesper Grodal (and also by Ramzi when he pointed out the error). Fortunately, the arguments in this paper only ever required the current version of the lemma.

(2) *Suppose that for each comparable pair $i \leq j$ in I there are horizontal equivalences making*

$$\begin{tikzcd} C_i \arrow[r, dashed, "\simeq"] \arrow[d] & D_i \arrow[d] \\ C_j \arrow[r, dashed, "\simeq"] & D_j \end{tikzcd}$$

commute, where the vertical maps are the transition maps. If each C_i is compact, then $\operatorname{colim}_I C_i \simeq *$ if and only if $\operatorname{colim}_I D_i \simeq *$.

Proof. Note that (2) follows from (1), since the existence of such commutative squares implies that $\{C_i\}_I$ has the vanishing property for transition maps described in (1) if and only if $\{D_i\}_I$ has that property.

For the first claim in (1), it suffices to show that $\operatorname{Map}_{\mathcal{C}}(D, \operatorname{colim}_{i\in I} C_i)$ is contractible for each compact object $D \in \mathcal{C}^\omega$. For this, first observe that

$$\pi_0 \operatorname{Map}_{\mathcal{C}}(D, \operatorname{colim}_{i\in I} C_i) \cong \operatorname{colim}_{i\in I} \pi_0 \operatorname{Map}(D, C_i) \cong *,$$

since our assumption guarantees that any homotopy class $D \to C_i$ is identified $D \to * \to C_i$ after postcomposing with the transition map $C_i \to C_j$ for sufficiently large $j \geq i$, where we have also used that π_0 preserves filtered colimits. Applying the same argument for the compact object $\Sigma^n D$, $n \geq 1$, we find that

$$\pi_n \operatorname{Map}_{\mathcal{C}}(D, \operatorname{colim}_{i\in I} C_i) \cong \pi_0 \operatorname{Map}_{\mathcal{C}}(\Sigma^n D, \operatorname{colim}_{i\in I} C_i)$$

vanishes also.

Assume now that each C_i is compact and that $\operatorname{colim}_I C_i \simeq *$. Then

$$\operatorname{colim}_{j\in I} \operatorname{Map}_{\mathcal{C}}(C_i, C_j) \simeq \operatorname{Map}_{\mathcal{C}}(C_i, \operatorname{colim}_{j\in I} C_j) \simeq *,$$

and since π_0 commutes with filtered colimits of anima, it follows that for sufficiently large $j \geq i$ the transition map $C_i \to C_j$ is homotopic to $C_i \to * \to C_j$. □

Proof of Proposition 3.1. 'Sufficiency.' The ∞-category of sheaves of anima $\operatorname{Shv}(X)$ is compactly generated by [Lur09, Prop 6.5.4.4], and hence so is $\operatorname{Shv}(X, \mathcal{C}) \simeq \operatorname{Shv}(X) \otimes \mathcal{C}$ according to [Lur17, Lem 5.3.2.11].

'Necessity.' Let $x \in X$. We must show that if $y \in X$ lies in the same connected component as X, then $y = x$. For this, pick an object $C \not\simeq 0$ in $\mathcal{C}$ and let x_*C denote the skyscraper sheaf at x with value C. By assumption there is a filtered system $\{\mathscr{F}_i\}_{i\in I}$ of compact sheaves with $\operatorname{colim}_I \mathscr{F}_i \simeq x_*C$. For each i, the fact that $\mathscr{F}_i$ is locally constant and that x and y lie in the same connected component means there is a non-canonical equivalence of stalks $x^*\mathscr{F}_i \simeq y^*\mathscr{F}_i$. One should not expect to find a system of such non-canonical equivalences assembling into a natural transformation, essentially because the neighborhoods on which the $\mathscr{F}_i$ are constant could get smaller and smaller as i increases. Nevertheless, given a comparable pair $i \leq j$ in I, one can pick equivalences making the diagram

$$\begin{tikzcd} x^*\mathscr{F}_i \arrow[r, dashed, "\simeq"] \arrow[d] & y^*\mathscr{F}_i \arrow[d] \\ x^*\mathscr{F}_j \arrow[r, dashed, "\simeq"] & y^*\mathscr{F}_j \end{tikzcd} \tag{3.1}$$

where the vertical maps are the transition maps. To see this, simply note that the set of $z \in Z$ for which there is a commutative diagram

$$\begin{tikzcd} x^*\mathscr{F}_i \arrow[r, dashed, "\simeq"] \arrow[d] & z^*\mathscr{F}_i \arrow[d] \\ x^*\mathscr{F}_j \arrow[r, dashed, "\simeq"] & z^*\mathscr{F}_j \end{tikzcd}$$

is a clopen subset of X, since any point admits a neighborhood on which both $\mathscr{F}_i$ and $\mathscr{F}_j$ are constant. Since all of the $\mathscr{F}_i$ have compact stalks by Theorem 2.6, it follows from Lemma 3.2 that the stalk $(x_*C)_y \simeq \operatorname{colim}_I y^*\mathscr{F}_i$ is nonzero. But X is Hausdorff, so this implies that $y = x$ as desired. $\square$

Remark 3.3. Lemma 3.2 is also true if $\mathcal{C}$ is any ordinary pointed category, e.g. the category of abelian groups Ab. It is illuminating to consider why the lemma holds in this concrete setting. Given a filtered system of abelian groups $\{A_i\}_{i\in I}$, the associated colimit can be described as the quotient of $\bigoplus_I A_i$ by the subgroup consisting of elements $a - \varphi_{ij}(a)$ where $a \in A_i$ and $\varphi_{ij} \colon A_i \to A_j$ is the transition map for some $j \geq i$. Clearly $\operatorname{colim}_I A_i \cong 0$ is implied by the assumption that for every $i \in I$, there is $j \geq i$ with $\varphi_{ij} = 0$. For the partial converse, assume now that each A_i is a compact object of Ab, i.e. a finitely generated abelian group, and that $\operatorname{colim}_I A_i \cong 0$. Let $i \in I$ and pick a generating set $a_1, \ldots, a_n \in A_i$. Since $\operatorname{colim}_I A_j \cong 0$, there is $j_1, \ldots, j_n$ with $\varphi_{ij_s}(a_s) = 0$ for each s. Using that I is filtered, pick $j \in I$ so that $j \geq j_s$ for each s. Then $\varphi_{ij} = \varphi_{j_sj}\varphi_{ij_s}(a) = 0$ for each s, and hence $\varphi_{ij} = 0$.

3.2. **Hausdorff schemes.** A topological space is said to be *locally spectral* if it is homeomorphic to the underlying space of a scheme, and *spectral* if this scheme can be taken to be affine. A celebrated result of Hochster completely characterizes these spaces in terms of point-set topology, and allows us to give a conceptual rephrasing of Proposition 3.1.

Recall that a topological space X is *quasi-separated* if the compact open subsets of X are closed under finite intersections and *sober* if every nonempty irreducible closed subset $A \subseteq X$ contains a unique point $a \in A$ such that $A = \overline{\{a\}}$.

Theorem 3.4 (Hochster [Hoc69, Thms 6,9])**.** *A topological space X is spectral if and only if X is compact, quasi-separated, and sober, and has the property that its compact open subsets form a basis for its topology. Similarly, X is locally spectral if it has an open cover $X = \bigcup_{U\in\mathcal{U}} U$ where each $U \in \mathcal{U}$ is spectral.*

Note that if X is Hausdorff, then it is automatically sober and quasi-separated, so in this case X is spectral if and only if (i) it is compact and (ii) the compact open subsets of X form a basis. Similarly, X is locally spectral if and only if it satisfies (ii).

Unlike in point-set topology, compactly generated categories of sheaves are abundant in algebraic geometry. Intuitively, there are very few locally compact Hausdorff spaces which also appear in the category of schemes; in the latter, these are exactly the zero-dimensional schemes. The point of

Proposition 3.1 is that if X is locally compact Hausdorff and $\mathcal{C}$ is a nontrivial compactly generated stable ∞-category, then the ∞-category of sheaves $\mathrm{Shv}(X, \mathcal{C})$ will only very rarely be compactly generated. One way to emphasize this is to reinterpret our result as saying that this only occurs if X belongs to the small class of Hausdorff spaces which happen to also appear in the category of schemes:

Proposition 3.5. *Let $\mathcal{C}$ be a nontrivial compactly generated stable ∞-category, and let X be a $\mathcal{C}$-hypercomplete locally compact Hausdorff space Then* $\mathrm{Shv}(X, \mathcal{C})$ *is compactly generated if and only if X is the underlying space of a zero-dimensional scheme.*

Proof. By Hochster's theorem (see also the discussion following Theorem 3.4) and Proposition 3.1, it suffices to show that a locally compact Hausdorff space is totally disconnected if and only if it admits a basis of open sets.

We first show that if the compact open subsets of X form a basis, then X is totally disconnected. Note that every $x \in X$ has

$$\{x\} = \bigcap_{U \ni x} U, \tag{3.2}$$

with U ranging over compact open neighborhoods of x. Indeed if $y \neq x$, then since X is Hausdorff there is some open neighborhood $V \ni x$ with $y \notin V$, and since the compact open subsets of X form a basis, there is some compact open $U \ni x$ with $U \subseteq V$, and in particular $y \notin U$. Since each compact open neighborhood is clopen, the equality (3.2) shows that $\{x\}$ is a quasi-component in X, and hence that X is totally disconnected.

For the other direction, we must show that for every $x \in X$ and every open neighborhood $V \ni x$, there is a compact open W with $x \in W \subseteq V$. Since X is locally compact, we may assume that V is relatively compact. By assumption $\{x\} = \bigcap_{U \ni x} U$, with U ranging over clopen neighborhoods of x. Since each of these U is in particular closed, we have that each $U \cap \partial\overline{V}$ is compact. By the finite intersection property, it therefore follows from $\bigcap_{U \ni x} U \cap \partial\overline{V} = \varnothing$ that for small enough clopen $U \ni x$, $U \cap \partial\overline{V} = \varnothing$. Hence $U \cap \overline{V} = U \cap V$ is a compact open neighborhood of x contained in V, as desired. □

3.3. **When is** $\mathrm{Shv}(X)$ **compactly generated?** Proposition 3.1 says that the ∞-category of sheaves on X with coefficients in a stable ∞-category is rarely compactly generated when X is a locally compact Hausdorff space. If we had asked the same question 'without coefficients,' this would have been an easier observation:

Proposition 3.6. *Let X be a quasi-separated topological space. The ∞-topos* $\mathrm{Shv}(X)$ *of sheaves of anima on X is compactly generated if and only if the sobrification of X is the underlying space of a scheme.*

Proof. A topological space and its sobrification have the same frame of open sets. Thus if the sobrification of X is the underlying space of a scheme, it follows that the compact open subsets of X must form a basis for its topology. But then $\mathrm{Shv}(X)$ is compactly generated by [Lur09, Prop 6.5.4.4]. For the other direction, assume that $\mathrm{Shv}(X)$ is compactly generated. Then so is the frame $\mathcal{U} \simeq \tau_{\leq -1} \operatorname{Shv}(X)$ of open subsets of X by [Lur09, Cor 5.5.7.4]. But this means that X admits a basis of compact open sets, and hence the sobrification of X is the underlying space of a scheme according to Hochster's theorem. $\square$

Appendix A. Descent for maps with local sections

In this short appendix, we prove a descent lemma that was used in the proof of Theorem 2.6, which is an immediate generalization of [SD72, Cor 4.1.6].

Let $\mathcal{C}$ be a presentable ∞-category and let $f\colon X \to Y$ be a continuous map of topological spaces. Recall that the *Čech nerve* of f is the augmented simplicial topological space $X_\bullet$ with $X_{-1} = Y$ and p-simplices

$$X_p = \underbrace{X \times_Y \cdots \times_Y X}_{p \text{ times}}$$

for $p \geq 0$, with face maps given by projections and degeneracy maps given in the obvious way. More formally, if Δ_+ is the category of finite (possibly empty) ordinals and $\mathcal{T}\mathrm{op}$ is the category of topological spaces, then $X_\bullet\colon \Delta_+^{\mathrm{op}} \to \mathcal{T}\mathrm{op}$ is defined by right Kan extending $(f\colon X \to Y)\colon \Delta_{+,\leq 0}^{\mathrm{op}} \to \mathcal{T}\mathrm{op}$ along the inclusion functor $\Delta_{+,\leq 0}^{\mathrm{op}} \subset \Delta_+^{\mathrm{op}}$.

Letting $\operatorname{Shv}^*(-,\mathcal{C})$ denote the contravariant functor from $\mathcal{T}\mathrm{op}$ to $\widehat{\mathcal{C}\mathrm{at}}_\infty$ given informally by $X \mapsto \operatorname{Shv}(X,\mathcal{C})$ on objects and $f \mapsto f^*$ on morphisms, we then have the following useful definition:

Definition A.1. The map f is of $\mathcal{C}$-*descent type* if the canonical functor

$$\operatorname{Shv}(X,\mathcal{C}) \to \lim_\Delta \operatorname{Shv}^*(X_\bullet,\mathcal{C})$$

is an equivalence.

Let us say that f *admits local sections* if for every $x \in X$, there is an open set $U \ni x$ such that the restriction $f\colon f^{-1}(U) \to U$ admits a section.

Proposition A.2. *If f admits local sections, then f is of $\mathcal{C}$-descent type.*

Proof. By ordinary Čech descent (see e.g. [JT24, Cor B.5]), after possibly passing to an open cover of X, we may assume that f admits a section globally on X. Let $\varepsilon\colon Y \to X$ be a choice of such a section. The section ε allows us to endow the Čech nerve $X_\bullet$ with the structure of a split augmented simplicial space, by defining the extra degeneracies $h_i\colon X_p \to X_{p+1}$ by

$$h_i(x_0, \ldots, x_p) = (x_0, \ldots, x_{i-1}, \varepsilon(y), x_i, \ldots, x_p)$$

where $y = f(x_0) = \cdots = f(x_p)$. It then follows that the split coaugmented cosimplicial ∞-category $\operatorname{Shv}^*(X_\bullet,\mathcal{C})$ is a limit diagram by [Lur09, Lem 6.1.3.16] $\square$

Corollary A.3. *Let $\{A_i\}_{i\in I}$ be a collection of subsets of X such that $X = \bigcup_I A_i^\circ$, where A_i° is the interior of A_i. Then the canonical map $\coprod_I A_i \to X$ is of $\mathcal{C}$-descent type.*

Proof. The canonical map $\coprod_I A_i \to X$ admits a section on A_j° given by $A_j^\circ \hookrightarrow A_j \to \coprod_I A_i$, where the second map is the canonical injection. $\square$

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3 Reconstruction of a one-dimensional space from the derived category

RECONSTRUCTION OF A ONE-DIMENSIONAL SPACE FROM THE DERIVED CATEGORY

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Abstract. We prove that the homeomorphism type of a one-dimensional CW complex is uniquely determined by its derived category of sheaves. The proof is inspired by Bondal and Orlov's non-commutative reconstruction theorem for Fano and anti-Fano varieties. As in their proof, a key role is played by the Serre functor. We also discuss the general problem of finding Fourier–Mukai partners in topology.

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0. Introduction

Let $\mathcal{C} = \{X, Y, Z, \dots\}$ be a category of geometric objects which comes with some theory of sheaves encoded by a category-valued functor $X \mapsto D(X)$, such that the pushforward $f_* = D(f)$ has a left adjoint f^* for each morphism $f\colon X \to Y$ in $\mathcal{C}$. If $\mathcal{C}$ has a terminal object pt, then we can produce an invariant of our geometric objects for every $M \in D(\mathrm{pt})$ by sending an object $X \in \mathcal{C}$ to the *cohomology of X with coefficients in M*

$$X_* X^* M \in D(\mathrm{pt}), \tag{1}$$

where we abuse notation by writing $X\colon X \to \mathrm{pt}$ for the unique map to the terminal object. Many invariants in geometry and topology arise in such a way, or by slight variations on this theme.

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In general, we expect the invariants $X \mapsto X_*X^*M$ to lose information about X. For instance, if $\mathcal{C}$ is the category of CW complexes and $D = D\,\mathrm{Shv}(-;\mathrm{Ab})$ is the functor which sends a CW complex X to the category of derived sheaves of abelian groups on X, then the invariant $X \mapsto X_*X^*M = C^*(X;M)$ cannot tell the difference between homotopy equivalent CW complexes

Going one categorical level up, we can also consider the entire category $D(X)$ as an invariant of X. This may be a much stronger invariant than $X \mapsto X_*X^*M$, i.e. it may remember much more information about X. A maximal instance of this is given in algebraic geometry by the following celebrated theorem:

Theorem (Bondal–Orlov [BO01])**.** *Let X be a smooth irreducible projective variety such that either its canonical sheaf ω_X is ample or anti-ample, where the latter means that ω_X^{-1} is ample. If Y is a smooth algebraic variety with $D^{perf}(Y) \simeq D^{perf}(X)$ then Y is isomorphic to X.*

The theorem of Bondal and Orlov fits into a well-developed story in algebraic geometry, starting with the 1962 thesis of Gabriel [Gab62]. If we have access to more structure on the category $D^{\mathrm{perf}}(Y)$, such as the standard t-structure or the standard symmetric monoidal structure, then Y can be reconstructed under very mild assumptions by work of Gabriel and Rosenberg [Ros98] or Balmer [Bal02], respectively. Without this extra information, examples are known of non-isomorphic varieties X and Y (whose canonical sheaves are necessarily neither ample nor anti-ample) with $D^{\mathrm{perf}}(Y) \simeq D^{\mathrm{perf}}(X)$ [Muk81, Orl97] and are predicted to exist in abundance [BO95].

In topology, analogous questions have received little attention.[1] One notable exception is the recent result of Aoki showing that a sober topological space can be recovered from its derived category together with its pointwise symmetric monoidal structure [Aok24], giving a topological counterpart to Balmer's Tannakian reconstruction theorem in algebraic geometry.

The following question remains: how much does the derived category of sheaves on a CW complex remember about the topological space if we forget about monoidal structures? Trivially a zero-dimensional CW complex (i.e. a discrete space) can be reconstructed from its derived category. In this article we study the question in the simplest case where it is non-trivial, namely one-dimensional CW complexes. For these we prove the following topological pastiche of Bondal and Orlov's theorem:

Theorem A. *Let X be a one-dimensional locally finite CW complex, and let Y be an arbitrary CW complex. Let $\mathbf{k}$ be a field and let $\mathrm{Mod}_{\mathbf{k}}$ denote its derived category. If $\mathrm{Shv}(X;\mathrm{Mod}_{\mathbf{k}})$ and $\mathrm{Shv}(Y;\mathrm{Mod}_{\mathbf{k}})$ are equivalent, then Y is homeomorphic to X.*

Remark 0.1. As in the theorem of Bondal and Orlov, we do not require the equivalence $\mathrm{Shv}(X;\mathrm{Mod}_{\mathbf{k}}) \simeq \mathrm{Shv}(Y;\mathrm{Mod}_{\mathbf{k}})$ to be induced by a map $X \to Y$.

[1]One reason for this might be that the category of derived sheaves on a topological space (assumed to be locally compact and Hausdorff) is frighteningly large, in the sense that it is almost never compactly generated [Nee01, Har25, Efi25].

The statement is therefore not just that the functor $X \mapsto \mathrm{Shv}(X; \mathrm{Mod}_{\mathbf{k}})$ reflects equivalences whose source or target is a one-dimensional CW complex, but that it creates such equivalences.[2] In fact, the functor $X \mapsto \mathrm{Shv}(X; \mathrm{Mod}_{\mathbf{k}})$ is conservative for all sober topological spaces X, since it factors as

$$\{\text{sober topological spaces}\} \to \mathrm{CAlg}(\mathrm{Pr}^{L,\otimes}_{\mathrm{st}}) \to \mathrm{Pr}^{L}_{\mathrm{st}},$$

where the first functor is conservative by Aoki's theorem and the second functor is conservative by general nonsense, e.g. [Lur17, Lem 3.2.2.7]. As in algebraic geometry, the subtletly lies in the possible existence of exotic equivalences of derived categories, which are not induced by maps of the underlying geometric objects or by twisting with an invertible sheaf.

The proof of Theorem A is inspired by Bondal and Orlov's proof. As in their proof, we start by contemplating the inherent duality of the derived category of a locally compact Hausdorff space, as exhibited by Verdier [Ver65] and Lurie [Lur17]. Part of this duality is captured by a Serre functor. Bondal and Orlov use the Serre functor on the derived category of a variety to identify those sheaves which are skyscrapers at closed points. The Serre functor in topology is less generous, but it does allow us to identify singular points (§ 3.1). By removing the singular points (categorically, by forming a Verdier quotient), we are left with the derived category of a one-dimensional manifold. We identify the components of this manifold and their homeomorphism types (§ 3.2), and finally use the gluing functor of the Verdier sequence to reconstruct our space (§ 3.3).

0.1. **Acknowledgements.** I thank Shaul Barkan for stimulating conversations about the problem of finding non-homeomorphic CW complexes with equivalent derived categories.

1. Notation and terminology

In this article, the word "category" refers to an ∞-category. With this convention, a category in the classical sense is simply a category with the property that all its mapping spaces are homotopy discrete. We let $\mathrm{Pr}^{L}_{\mathrm{st}}$ denote the category of presentable stable categories. The Lurie tensor product endows this category with a symmetric monoidal structure $\mathrm{Pr}^{L,\otimes}_{\mathrm{st}}$. In general, a symmetric monoidal category is written $\mathcal{C}^{\otimes}$ where $\mathcal{C}$ is its underlying category and $\otimes$ is notation for the symmetric monoidal structure.

We let $\mathrm{Sp}^{\otimes}$ denote the category of spectra, viewed as a symmetric monoidal category under the smash product $\otimes$. Given a commutative algebra $R \in \mathrm{CAlg}(\mathrm{Sp})$, we let Mod_R denote the category of R-modules. If R is a classical ring, in the sense that R belongs to the heart $\mathrm{Sp}^{\heartsuit} = \mathrm{Ab}$ of the standard t-structure on spectra, then our convention means that Mod_R is the category which is classically known as the derived category of R, and denoted $D(R)$.

Given a topological space X, we abuse notation by also writing $X \colon X \to \mathrm{pt}$ for the projection from X to a point. Given a point $x \in X$, we similarly

[2]Recall that a functor $F \colon \mathcal{C} \to \mathcal{D}$ is said to *create* I-shaped limits if it preserves and reflects I-shaped limits, and for every universal cone $j \colon I^{\triangleleft} \to \mathcal{D}$ lifts to a universal cone $j' \colon I^{\triangleleft} \to \mathcal{C}$. Creating equivalences is the special case $I = \Delta^0$.

abuse notation by writing $x\colon \mathrm{pt} \to X$ for the map sending the unique point in pt to X. We use the six-functor formalism for sheaves on locally compact Hausdorff spaces. Given a stable category $\mathcal{C}$, this theory assigns to every map $f\colon X \to Y$ (of locally compact Hausdorff spaces) a pair of adjunctions

$$\begin{tikzcd} \mathrm{Shv}(X;\mathcal{C}) \arrow[r, bend right, "f_*"'] \arrow[r, phantom, "\perp"] & \mathrm{Shv}(Y;\mathcal{C}) \arrow[l, bend right, "f^*"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{Shv}(X;\mathcal{C}) \arrow[r, bend left, "f_!"] \arrow[r, phantom, "\perp"] & \mathrm{Shv}(Y;\mathcal{C}) \arrow[l, bend left, "f^!"] \end{tikzcd}$$

where f_* and f^* are the usual pushforward and pullback, and $f_!$ and $f^!$ are the so-called *exceptional pushforward and pullback*, respectively. These functors satisfy various properties that make them amenable to calculation. Classical references for this theory are [Ver65, KS90], and a good modern reference is [Vol23b]. If $\mathcal{C}$ is the underlying category of a presentably symmetric monoidal stable category $\mathcal{C}^{\otimes} \in \mathrm{CAlg}(\mathrm{Pr}^{L,\otimes}_{\mathrm{st}})$, we get two further kinds of functors in the form of the pointwise symmetric monoidal structure $\otimes$ on sheaves and its internal Hom. If $\mathbf{1} \in \mathcal{C}$ is the monoidal unit, we write

(constant sheaf) $$\mathbf{1}_X = X^*\mathbf{1} \in \mathrm{Shv}(X;\mathcal{C}),$$

which is the monoidal unit for the pointwise tensor product, and

(dualizing sheaf) $$\omega_X = X^!\mathbf{1} \in \mathrm{Shv}(X;\mathcal{C}).$$

2. Preliminaries

We start by fixing some basic notions that will be needed for the proof.

2.1. **One-dimensional spaces.** Bondal and Orlov reconstruct varieties by gradually reconstructing the graded coordinate rings of their canonical sheaves. Since a smooth projective variety with ample or anti-ample canonical sheaf is isomorphic to the projectivization of this coordinate ring [Sta25, Tag 01Q1], this suffices to prove their reconstruction theorem. Whereas Bondal and Orlov reconstruct varieties by extracting algebraic data from their derived categories, we will reconstruct spaces by extracting combinatorial data from their derived categories.

First, some pedantry. Recall that a CW complex is a Hausdorff space X together with a stratification $X_0 \subseteq X_1 \subseteq \cdots \subseteq X = \bigcup_k X_k$ by closed subspaces, such that for each component e^k_α (referred to as a *cell*) of the stratum $X_k \setminus X_{k-1}$ there exists a continuous surjection $\varphi_\alpha\colon D^k \to \overline{e}^k_\alpha$ which restricts to a homeomorphism $\operatorname{int} D^k \to e^k_\alpha$ and has $\varphi_\alpha(\partial D^k) \subseteq X_{k-1}$. The filtration is extra combinatorial structure on X, and are not at all uniquely determined by its homeomorphism type. This is also true in the one-dimensional case: there is no canonical CW complex structure on the circle or the real line (see Figure 1). It is therefore hopeless to reconstruct a CW complex from its

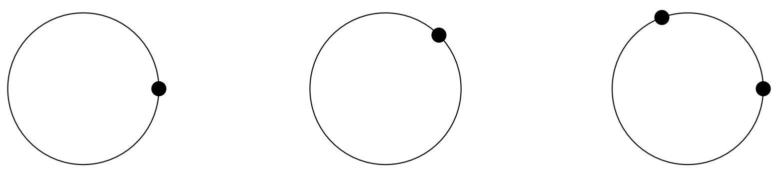

Figure 1. Three CW complex structures on a circle

derived category, which only depends on the underlying topological space.

Our reconstruction theorem concerns spaces that *admit* one-dimensional CW complex structures. In order to emphasize this distinction between structure and property, we will take the view that these spaces are equivalently one-dimensional *manifolds with singularities*, à la [Baa73]. With this as our starting point, we show that the line and the circle are the only obstructions for one-dimensional spaces to have canonical CW complex structures, and by modifying the definition of a CW complex slightly one can describe the homeomorphism type of a one-dimensional space in terms of well-defined combinatorial data. It is this combinatorial data which our reconstruction theorem extracts from the derived category.

Like ordinary manifolds, manifolds with singularities are defined in terms of local charts. We recall the local charts in the one-dimensional case here. For each $k \geq 1$, the *open corolla with k legs*, denoted Cor_k, is the union of the non-negative parts of all the coordinate axes in $\mathbb{R}^k$; that is,

$$\mathrm{Cor}_k = \bigcup_{1 \leq i \leq k} \mathbb{R}_{\geq 0} \cdot e_i \subseteq \mathbb{R}^k, \qquad \text{(open corolla)}$$

where $\{e_i\}_{1 \leq i \leq k}$ denotes the standard basis of $\mathbb{R}^k$. We will also put $\mathrm{Cor}_0 = \{0\}$.

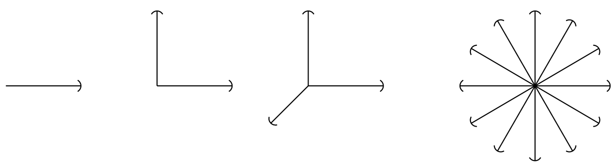

FIGURE 2. From left to right: Cor_1, Cor_2, Cor_3, and Cor_{12}

Definition 2.1. A *one-dimensional space* is a paracompact Hausdorff space X such that every $x \in X$ has an open neighborhood which is homeomorphic to the open corolla Cor_k, for some k which may depend on x.

Remark 2.2. By our definition, a zero-dimensional (i.e. discrete) space is a special case of a one-dimensional space.

Note that a one-dimensional space is the same as a topological space which admits the structure of a one-dimensional locally finite CW complex. A CW complex structure on a one-dimensional space is a representation of the space as a graph, determined by appropriately distributing vertices around the space. Although a one-dimensional space without this structure does not have vertices per se, it does retain the graph-theoretical notion of valence:

Definition 2.3. The *valence* of a point x in a one-dimensional space X, denoted $\nu(x)$, is zero if x is an isolated point of X and

$$\nu(x) = \operatorname{rank} H_1(X, X \setminus \{x\}; \mathbb{Z}) + 1$$

otherwise.

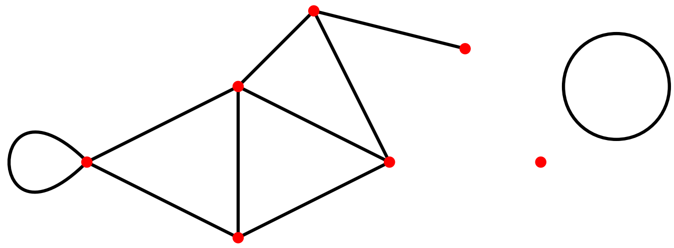

FIGURE 3. A one-dimensional space with its singular locus highlighted in red.

Alternatively, excision implies that the valence $k = \nu(x)$ of a point x in a one-dimensional space X is also the unique non-negative integer k such that there is a pointed open map $(\mathrm{Cor}_k, 0) \to (X, x)$ which is a homeomorphism onto its image.

The valence of a point classifies the type of singularity at that point. In particular, a point has valence two if and only if it has an open neighborhood homeomorphic to the real line; in other words, the points of valence two are exactly the non-singular points. This gives rise to a canonical two-term stratification of X, with open and closed stratum given as follows:

Definition 2.4. Let X be a one-dimensional space. The *singular locus* of X, denoted X_{sing}, is the closed subspace

$$X_{\mathrm{sing}} = \{x \in X \mid \nu(x) \neq 2\} \subseteq X.$$

The (open) complement of this subspace is called the *regular locus*, and is denoted X_{reg}; that is,

$$X_{\mathrm{reg}} = X \setminus X_{\mathrm{sing}} \subseteq X.$$

By the classification of one-dimensional manifolds, components E of the regular locus X_{reg} come in five flavors:

(A)	$E \cong \mathbb{R}$ and $\overline{E} \cong [0,1]$	(edge with two endpoints);
(B)	$E \cong \mathbb{R}$ and $\overline{E} \cong S^1$	(edge with one multiplicity-two endpoint);
(C)	$E \cong \mathbb{R}$ and $\overline{E} \cong [0,1)$	(edge with one multiplicity-one endpoint);
(D)	$E = \overline{E} \cong \mathbb{R}$	(edge without endpoints); and
(E)	$E = \overline{E} \cong S^1$	(isolated circle).

We relax the definition of a one-dimensional CW complex to allow all of these flavors of cells:

Definition 2.5. A *generalized one-dimensional CW complex* is a Hausdorff space X together with a discrete subset $X_0 \subseteq X$ such that every component $E_\alpha \subseteq X \setminus X_0$ admits a continuous surjection $\varphi_\alpha \colon A_\alpha \to \overline{E}_\alpha$ which restricts to a homeomorphism $\operatorname{int} A_\alpha \to E_\alpha$ and has $\varphi_\alpha(\partial A_\alpha) \subseteq X_0$, where $A_\alpha \in \{[0,1], [0,1), \mathbb{R}, S^1\}$. We refer to the choice of such an $X_0 \subseteq X$ as a *generalized CW structure on X*.

Like an (ordinary) one-dimensional CW complex (aka a graph), the underlying homeomorphism type of a generalized one-dimensional CW complex can be reconstructed from combinatorial data. Namely, if (X, X_0) is a generalized one-dimensional CW complex, we define an endpoint function

$$\begin{aligned} \pi_0(X \setminus X_0) &\xrightarrow{\ell} \mathcal{P}_{\leq 2}(X_0) \\ E_\alpha &\longmapsto \overline{E}_\alpha \cap X_0, \end{aligned} \tag{2}$$

where $\mathcal{P}_{\leq 2}(X_0)$ denotes the set of subsets of X_0 that are of cardinality less than or equal to two.

Lemma 2.6. *A generalized one-dimensional CW complex (X, X_0) is uniquely determined by the tuple $(X_0, \pi_0(X \setminus X_0), \ell, \pi_0^{[0,1)}, \pi_0^{\mathbb{R}})$, where ℓ is the endpoint function (2), and $\pi_0^{[0,1)}$ and $\pi_0^{\mathbb{R}} \subseteq \pi_0(X \setminus X_0)$ are the subsets consisting of components $E \subseteq X \setminus X_0$ with $\overline{E} \subseteq X$ homeomorphic to the half-open interval and the real line, respectively.*

Proof. Let $\widetilde{\mathcal{P}}_{\leq 2}(X_0)$ denote the set of totally ordered subsets of X_0 of cardinality less than or equal to two, and pick a lift $\widetilde{\ell}\colon \pi_0(X \setminus X_0) \to \widetilde{\mathcal{P}}_{\leq 2}(X_0)$. The topological space X fits into a pushout diagram

$$\begin{tikzcd} \bigsqcup_{E \in \pi_0(X \setminus X_0)} \partial A_E \arrow[r] \arrow[d, hook] & X_0 \arrow[d, hook] \\ \bigsqcup_{E \in \pi_0(X \setminus X_0)} A_E \arrow[r] & X, \end{tikzcd}$$

where

- $A_E = [0, 1]$ if $\widetilde{\ell}(E)$ has cardinality two, and $\{0, 1\} = \partial A_E \to X_0$ maps 0 to the minimal element of $\widetilde{\ell}(E)$ and 1 to the maximal element;
- $A_E = [0, 1]$ if $\widetilde{\ell}(E)$ has cardinality one and $E \notin \pi_0^{[0,1)}$, and $\{0, 1\} = \partial[0, 1] \to X_0$ maps both 0 and 1 to the unique element of $\ell(E)$;
- $A_E = [0, 1)$ if $\widetilde{\ell}(E)$ has cardinality one and $E \in \pi_0^{[0,1)}$, and $\{0\} = \partial[0, 1) \to X_0$ maps to the unique element of $\ell(E)$;
- $A_E = \mathbb{R}$ if $\widetilde{\ell}(E)$ is empty and $E \in \pi_0^{\mathbb{R}}$; and
- $A_E = S^1$ if $\widetilde{\ell}(E)$ is empty and $E \notin \pi_0^{\mathbb{R}}$.

Clearly the tuple $(X_0, \pi_0(X \setminus X_0), \ell, \pi_0^{[0,1)}, \pi_0^{\mathbb{R}})$ also determines the generalized CW structure on X. $\square$

Definition 2.7. Let X be a one-dimensional space. A generalized CW structure $X_0' \subseteq X$ is said to *refine* another generalized CW structure $X_0 \subseteq X$ if $X_0 \subseteq X_0'$. A generalized CW structure is *minimal* if it is minimal with respect to refinement.

The benefit of working with the combinatorics of generalized one-dimensional CW structures (as opposed to ordinary CW structures) is that they are canonical, and hence we can hope to reconstruct them from the derived category:

Proposition 2.8. *Every one-dimensional space admits a unique minimal generalized CW structure.*

Proof. Let X be a one-dimensional space. We put $X_0 = X_{\mathrm{sing}}$, and note that this is a generalized CW structure on X by design. It is also the unique minimal one; indeed, one can check locally on a corolla Cor_k that every generalized CW structure $X'_0 \subseteq \mathrm{Cor}_k$ must contain the point $0 \in \mathrm{Cor}_k$. □

2.2. **Serre functors.** As in [BO01], our proof starts by extracting information from the Serre functor of the derived category. From a modern perspective, Bondal and Orlov consider Serre functors via their action on the compact objects of compactly generated stable categories. This will not suffice for our purposes, since the derived category of a locally compact Hausdorff space is almost never compactly generated and often contains no non-trivial compact objects; indeed, specializing the results of [Har25] to one-dimensional spaces, we find that the derived category of a one-dimensional space is only compactly generated in the degenerate case where the space is discrete (see Remark 2.2).

In [Lur17], Lurie upgrades Verdier's duality theory for locally compact Hausdorff spaces [Ver65] by proving that the derived category of such a space is dualizable as an object of $\mathrm{Pr}^{L,\otimes}_{\mathrm{st}}$. This theorem implies that these derived categories have non-trivial Serre functors, even in the absence of non-trivial compact objects.

Let $R \in \mathrm{CAlg}(\mathrm{Sp})$, and let

$$\mathrm{LinCat}^{\otimes_R}_{\mathrm{st},R} = \mathrm{Mod}_{\mathrm{Mod}_R}(\mathrm{Pr}^L_{\mathrm{st}})^{\otimes_R}$$

denote the category of (presentable) R-linear stable categories equipped with the R-linear tensor product $\otimes_R = \otimes_{\mathrm{Mod}_R}$. We also denote by $\mathrm{Fun}^L_R(\mathcal{C}, \mathcal{D})$ the category of R-linear colimit-preserving functors from $\mathcal{C}$ to $\mathcal{D}$, which functions as an internal Hom with respect to $\otimes_R$. A dualizable R-linear stable category $\mathcal{C}$ has, by definition, an essentially unique duality datum

$$(\mathcal{C}^\vee, \mathrm{ev}\colon \mathcal{C}^\vee \otimes_R \mathcal{C} \to \mathrm{Mod}_R, \mathrm{coev}\colon \mathrm{Mod}_R \to \mathcal{C} \otimes_R \mathcal{C}^\vee) \tag{3}$$

witnessing $\mathcal{C}^\vee$ as a dual to $\mathcal{C}$. This means in particular that the functor

$$\mathcal{C} \otimes_R \mathcal{C}^\vee \to \mathrm{Fun}^L_R(\mathcal{C}, \mathcal{C}) \tag{4}$$

adjoint to $\mathcal{C} \otimes_R \mathcal{C}^\vee \otimes_R \mathcal{C} \xrightarrow{\mathcal{C}\otimes\mathrm{ev}} \mathcal{C}$ is an equivalence.

Example 2.9. Let X be a locally compact Hausdorff space. The sheaf category $\mathrm{Shv}(X; \mathrm{Mod}_R)$ is a dualizable R-linear stable category by [Lur17, Thm 5.5.5.1]. Recall that there is a canonical equivalence $\mathrm{Shv}(X; \mathrm{Mod}_R) \otimes_R \mathrm{Shv}(Y; \mathrm{Mod}_R) \simeq \mathrm{Shv}(X \times Y; \mathrm{Mod}_R)$, according to [Lur09, Prop 7.3.1.11] and [Lur17, Exmp 4.8.1.19]. Under this identification, an explicit duality datum witnessing $\mathrm{Shv}(X; \mathrm{Mod}_R)$ as its own dual is given by the functors

$$\mathrm{Shv}(X \times X; \mathrm{Mod}_R) \xrightarrow{\Delta^*} \mathrm{Shv}(X; \mathrm{Mod}_R) \xrightarrow{X_!} \mathrm{Mod}_R$$

and

$$\mathrm{Mod}_R \xrightarrow{X^*} \mathrm{Shv}(X; \mathrm{Mod}_R) \xrightarrow{\Delta_!} \mathrm{Shv}(X \times X; \mathrm{Mod}_R),$$

where $\Delta\colon X \to X \times X$ is the diagonal inclusion.

Since the evaluation ev and coevaluation coev of a duality datum (3) are both morphisms in $\mathrm{LinCat}_{\mathrm{st},R}$, they admit R-linear right adjoints. We will use the right adjoint of the evaluation functor to define the Serre functor (cf. [Lur18, Cons D.1.5.3]):

Construction 2.10. Let $\mathcal{C}$ be a dualizable stable R-linear category, and choose a duality datum $(\mathcal{C}^\vee, \mathrm{ev}, \mathrm{coev})$ for $\mathcal{C}$. The *Serre functor* is the value of the unit $R \in \mathrm{Mod}_R$ under the functor

$$\mathrm{Mod}_R \xrightarrow{\mathrm{ev}^R} \mathcal{C}^\vee \otimes_R \mathcal{C} \simeq \mathcal{C} \otimes_R \mathcal{C}^\vee \xrightarrow{\sim} \mathrm{Fun}^L_R(\mathcal{C}, \mathcal{C}),$$

where ev^R denotes the right adjoint of ev and the last map is the equivalence (4).

As explained in [Lur18, Rem 11.1.5.2], the construction above recovers Bondal and Orlov's definition of Serre functors in the compactly generated case.

Specializing to the case of sheaves on locally compact Hausdorff spaces, we get:

Proposition 2.11. *Let X be a locally compact Hausdorff space, and consider the dualizable stable R-linear category* $\mathrm{Shv}(X; \mathrm{Mod}_R)$*. The Serre functor is given by*

$$\omega_X \otimes -\colon\ \mathrm{Shv}(X; \mathrm{Mod}_R) \to \mathrm{Shv}(X; \mathrm{Mod}_R).$$

Proof. By Example 2.9, the Serre functor on $\mathrm{Shv}(X; \mathrm{Mod}_R)$ is the value of the unit $\mathbf{1} = R \in \mathrm{Mod}_R$ under the functor

$$\begin{aligned}\mathrm{Mod}_R \xrightarrow{X^!} \mathrm{Shv}(X; \mathrm{Mod}_R) \xrightarrow{\Delta_*} {} & \mathrm{Shv}(X \times X; \mathrm{Mod}_R) \\ & \simeq \mathrm{Shv}(X; \mathrm{Mod}_R) \otimes_R \mathrm{Shv}(X; \mathrm{Mod}_R).\end{aligned}$$

Here the last identification is given by the Fourier–Mukai functor

$$\begin{aligned}\mathrm{Shv}(X \times X; \mathrm{Mod}_R) &\longrightarrow \mathrm{Fun}^L_R(\mathrm{Shv}(X; \mathrm{Mod}_R), \mathrm{Shv}(X; \mathrm{Mod}_R)) \\ F &\longmapsto (p_2)_!(F \otimes p_1^*(-)),\end{aligned}$$

where $p_1, p_2\colon X \times X \to X$ are the projections onto the first and second coordinates, respectively. Hence the Serre functor is given by

$$\begin{aligned}(p_2)_!(\Delta_* X^! \mathbf{1} \otimes p_1^* -) &\simeq (p_2)_!(\Delta_! \omega_X \otimes p_1^* -) \\ &\simeq (p_2)_! \Delta_!(\omega_X \otimes \Delta^* p_1^* -) \\ &\simeq \omega_X,\end{aligned}$$

where the second equivalence is the projection formula and the last equivalence is the identity $p_1 \Delta = p_2 \Delta = \mathrm{id}$. $\square$

Remark 2.12. Bondal and Orlov observe that the Serre functor on the derived category of a smooth variety is given by tensoring with (a shift of) the canonical sheaf, see Eq. (7) in [BO01]. The previous proposition is a direct topological analog of this description.

Remark 2.13. The Serre functor on a dualizable stable R-linear category does not depend on the R-linear structure. This follows from [Lur17, Cor 4.6.5.14]. More broadly, Arinkin–Gaitsgory–Kazhdan–Raskin–Rozenblyum–Varshavsky have shown that if $\mathcal{A} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ is locally rigid, then an $\mathcal{A}$-linear stable category $\mathcal{M} \in \mathrm{Mod}_{\mathcal{A}}(\mathrm{Pr}^L_{\mathrm{st}})$ is dualizable if and only if it is so as an object of $\mathrm{Pr}^L_{\mathrm{st}}$, and duality data are related via the unit morphism $\mathrm{Sp} \to \mathcal{A}$, see [KNP24, Thm 4.3.1].

On well-behaved spaces, the dualizing sheaf can be described more explicitly in terms of singular homology. Let us write $C_*(X; R) \simeq R \otimes \Sigma^\infty_+ X$ for the homology of an ∞-groupoid X with coefficients in R.

Proposition 2.14. *Let X be a locally compact Hausdorff space with dualizing sheaf ω_X.*

(a) The stalk $\omega_{X,x} = x^*\omega_X$ at a point $x \in X$ is given by

$$\operatorname{cofib}\left(\Gamma_c(X, j_!\omega_U) \to \Gamma_c(X, \omega_X)\right),$$

where $U = X \setminus \{x\}$ and $j\colon U \hookrightarrow X$ denotes the inclusion. (Recall that $\Gamma_c(X, -) = X_!$.)

(b) If X is locally of singular shape in the sense of [Lur17, Defn A.4.15], *then $\omega_{X,x}$ is given by the local homology*

$$C_*(X, X \setminus \{x\}; R) = \operatorname{cofib}\left(C_*(X \setminus \{x\}; R) \to C_*(X; R)\right).$$

Proof. Part (a) is the content of Remark 4.6.19 in [KNP24]. The statement in (b) now follows from Lurie's monodromy theorem [Lur17, Thm A.1.15]. □

Remark 2.15. The purpose of the shape assumption in (a) is to ensure that sheaf (co)homology agrees with singular (co)homology. For this, a much milder assumption suffices. Namely, Petersen has shown that it is enough for the space to be locally connected with respect to the coefficient ring R assuming R is discrete [Pet22], and this has been extended to general coefficient rings by [Vol23a].

2.3. **Connected components of stable categories.** In the proof of our reconstruction theorem, we will need a way to extract the set of connected components of a space from its derived category.

Definition 2.16. Let $\mathcal{C}$ be a stable category.

(i) Say that $\mathcal{C}$ is *connected* if whenever $\mathcal{C} \simeq \mathcal{C}_1 \times \mathcal{C}_2$, then either $\mathcal{C}_1$ or $\mathcal{C}_2$ is the trivial stable category.

(ii) A *connected component* of $\mathcal{C}$ is a maximal connected stable subcategory of $\mathcal{C}$. The collection of connected components of $\mathcal{C}$ is denoted $\pi_0\mathcal{C}$.

Remark 2.17. Note that in many other contexts, a category $\mathcal{C}$ is said to be connected if the groupoid $\mathcal{C}[\mathrm{all}^{-1}]$ produced by inverting all morphisms in $\mathcal{C}$ is connected, or in other words if any pair of objects in $\mathcal{C}$ is connected by a zigzag of morphisms. If $\mathcal{C}$ is stable, this will always be true, since any pair of objects is connected by the zero morphism. The definition we are working with here is very different from this definition; indeed, it follows from the results below that there are many stable categories which are not connected in our sense.

Proposition 2.18. *Let $R \in \mathrm{CAlg}(\mathrm{Sp})$. The following are equivalent:*

(i) The ring $\pi_0 R$ has no nontrivial idempotents.

(ii) R is indecomposable as a R-module.

(iii) R is indecomposable as a R-algebra.

(iv) The category Mod_R is connected.

Proof. To see that (i) and (ii) are equivalent, observe that $\pi_0 R = \pi_0 \operatorname{Hom}_{\mathbb{S}}(\mathbb{S}, R) \simeq \pi_0 \operatorname{Hom}_R(R, R)$ and a nontrivial idempotent in the latter is exactly a nontrivial retract of R as a R-module.

In order to see that conditions (ii)-(iv) are equivalent, consider the functors

$$\begin{tikzcd} & R \arrow[rr, mapsto] & & \mathrm{Mod}_R^{\otimes} & \\ \mathrm{Mod}_R & \mathrm{CAlg}(\mathrm{Mod}_R) \arrow[l, "\mathrm{fgt}"'] \arrow[rr, "\theta"] & & \mathrm{CAlg}(\mathrm{LinCat}_{\mathrm{st},R}) \arrow[r, "\mathrm{fgt}"] & \mathrm{LinCat}_{\mathrm{st},R} \arrow[d, "\mathrm{fgt}"] \\ & & & & \mathrm{Pr}^L_{\mathrm{st}} \end{tikzcd}$$

It now suffices to observe that each of these functors preserve and create finite products. For θ this follows from [Lur17, Cor 4.8.5.22] and [Lur18, Lem D.3.5.5], and for the remaining functors it is general nonsense. □

Recall that a topological space X is said to be *R-hypercomplete* if the stalk functors $x^*\colon \operatorname{Shv}(X; \mathrm{Mod}_R) \to \mathrm{Mod}_R$ are jointly conservative as x varies over points in X [Har25]. For example, a space which admits the structure of a CW complex is R-hypercomplete for any choice of R [Hoy16].

Proposition 2.19. *Let $R \in \mathrm{CAlg}(\mathrm{Sp})$ be a ring such that $\pi_0 R$ has no nontrivial idempotents, and let X be a locally compact Hausdorff space.*

(a) The category $\operatorname{Shv}(X; \mathrm{Mod}_R)$ is connected if and only if X is connected.

(b) Assume that X is R-hypercomplete, and let $\pi_0 X$ denote the collection of connected components of X. There is a bijection

$$\pi_0 X \to \pi_0 \operatorname{Shv}(X; \mathrm{Mod}_R) \tag{5}$$

given by sending a connected component $Z \subseteq X$ to the essential image of

$$i_*\colon \operatorname{Shv}(Z; \mathrm{Mod}_R) \to \operatorname{Shv}(X; \mathrm{Mod}_R)$$

where $i\colon Z \hookrightarrow X$ is the inclusion.

Proof. We first prove (a). Assume first that X is not connected. Then we can pick a separation $X = U \cup V$, with U and V open, disjoint, and nonempty. It now follows from the sheaf condition that $\operatorname{Shv}(X; \mathrm{Mod}_R) \simeq \operatorname{Shv}(U; \mathrm{Mod}_R) \times \operatorname{Shv}(V; \mathrm{Mod}_R)$, where neither factor is equivalent to the trivial stable category. Thus $\operatorname{Shv}(X; \mathrm{Mod}_R)$ is not connected either. For the opposite implication, assume that X is connected, and suppose

$$\operatorname{Shv}(X; \mathrm{Mod}_R) \simeq \mathcal{C}_1 \times \mathcal{C}_2 \tag{6}$$

with projection functors $p_i\colon \operatorname{Shv}(X; \mathrm{Mod}_R) \to C_i$ for $i = 1, 2$. We must show that either $\mathcal{C}_1$ or $\mathcal{C}_2$ is trivial. Since the forgetful functor $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}}) \to \mathrm{Pr}^L_{\mathrm{st}}$ creates finite products, we can pick symmetric monoidal structures $\otimes_1$ and $\otimes_2$ on $\mathcal{C}_1$ and $\mathcal{C}_2$ in such a way that the projections promote to symmetric monoidal functors. It will therefore suffice to show that the unit $\mathbf{1}_X \in \operatorname{Shv}(X; \mathrm{Mod}_R)$ projects to zero in either $\mathcal{C}_1$ or $\mathcal{C}_2$.

From the decomposition (6), we get that

$$\mathbf{1}_X \simeq F_1 \oplus F_2, \tag{7}$$

where F_i projects to zero in $\mathcal{C}_{1-i}$. We claim that F_1 and F_2 are locally constant sheaves. Since X is locally compact, it is enough to check this upon restricting to a compact subset K of X. On K we again get $\mathbf{1}_K \simeq F_1|_K \oplus F_2|_K$. Since $\mathbf{1}_K$ is a compact object of $\mathrm{Shv}(K; \mathrm{Mod}_R)$ and compact objects are closed under retracts, we get that $F_1|_K$ and $F_2|_K$ are compact as well. It then follows from [Har25, Thm 2.6] that $F_1|_K$ and $F_2|_K$ are locally constant, and since K was arbitrary this proves that F_1 and F_2 are also locally constant. In particular, the sheaves F_1 and F_2 have open supports. Furthermore, since R is indecomposable as an R-module and $R \simeq x^*\mathbf{1}_X \simeq x^*F_1 \oplus x^*F_2$ for each $x \in X$, we find that the supports of F_1 and F_2 are disjoint. But $X = \operatorname{supp} \mathbf{1}_X = \operatorname{supp}(F_1) \cup \operatorname{supp}(F_2)$, so without loss of generality we can assume that $\operatorname{supp}(F_1) = \varnothing$. Since F_1 is locally constant, this implies that $F_1 \simeq 0$ as desired.

We now prove (b). We must first show that the essential image of

$$\mathrm{Shv}(Z; \mathrm{Mod}_R) \to \mathrm{Shv}(X; \mathrm{Mod}_R)$$

is a *maximal* connected subcategory of $\mathrm{Shv}(X; \mathrm{Mod}_R)$, or in other words that the map (5) is well-defined. Given a subspace $A \subseteq X$, we will abuse notation by identifying $\mathrm{Shv}(A; \mathrm{Mod}_R)$ with its essential image in $\mathrm{Shv}(X; \mathrm{Mod}_R)$ under (ordinary) pushforward. Suppose that $\mathrm{Shv}(Z; \mathrm{Mod}_R) \subseteq \mathcal{C} \subseteq \mathrm{Shv}(X; \mathrm{Mod}_R)$ where $\mathcal{C}$ is again connected. We must show that $\mathrm{Shv}(Z; \mathrm{Mod}_R) = \mathcal{C}$.

Because of our hypercompleteness assumption, the subcategory

$$\mathrm{Shv}(Z; \mathrm{Mod}_R) \subseteq \mathrm{Shv}(X; \mathrm{Mod}_R)$$

is equal to the full subcategory spanned by sheaves F such that $\operatorname{supp}(F) \subseteq Z$. Suppose $x \in X \setminus Z$. Since Z is a connected component of X, we can pick a separation $X = U_1 \cup U_2$ of X into disjoint nonempty open subsets such that $Z \subseteq U_1$ and $x \in U_2$. But then we get an equivalence

$$\mathrm{Shv}(X; \mathrm{Mod}_R) \xrightarrow{(i_1^*, i_2^*)} \mathrm{Shv}(U_1; \mathrm{Mod}_R) \times \mathrm{Shv}(U_2; \mathrm{Mod}_R),$$

where $i_r \colon U_r \hookrightarrow X$ denote the inclusions, for $r = 1, 2$. Here the projection i_1^* restricts non-trivially to $\mathcal{C}$ because the latter contains $\mathrm{Shv}(Z; \mathrm{Mod}_R)$. If the projection i_2^* also restricted non-trivially to $\mathcal{C}$, then this would induce a non-trivial decomposition, which cannot exist by connectedness of $\mathcal{C}$. Thus i_2^* must be zero when restricted to $\mathcal{C}$, and it follows in particular that x^* must be zero when restricted to $\mathcal{C}$, i.e. that any $F \in \mathcal{C}$ has $\operatorname{supp} F \subseteq Z$. Hence $\mathcal{C} = \mathrm{Shv}(Z; \mathrm{Mod}_R)$.

It is also clear from our description of $\mathrm{Shv}(Z; \mathrm{Mod}_R) \subseteq \mathrm{Shv}(X; \mathrm{Mod}_R)$ that the map (5) is injective, so it only remains to be seen that it is surjective. For this, let $\mathcal{C} \subseteq \mathrm{Shv}(X; \mathrm{Mod}_R)$ be a maximal connected subcategory. By a similar argument to the one in the previous paragraph, one finds that

$$\operatorname{supp} \mathcal{C} = \{x \in X \mid x^*F \simeq 0 \text{ for some } F \in \mathcal{C}\}$$

must be connected, otherwise a choice of separation would give rise to a non-trivial decomposition of $\mathcal{C}$. But then $\operatorname{supp} \mathcal{C}$ it is contained in some connected component $Z \subseteq X$, and clearly $\mathcal{C} \subseteq \mathrm{Shv}(Z; \mathrm{Mod}_R)$, finishing the proof. $\square$

3. Proof of the reconstruction theorem

Throughout this section, fix a ring $R \in \mathrm{CAlg}(\mathrm{Sp})$ such that $\pi_0 R$ has no non-trivial idempotents.

3.1. **Reconstruction of the singular locus.** We first show how to reconstruct the singular locus of a one-dimensional space from its derived category. This is the step where we extract information from the Serre functor.

Given a one-dimensional space X, it follows from Proposition 2.14 that the stalk of ω_X at a point $x \in X$ of valence k is given by the local homology

$$C_*(X, X \setminus \{x\}; R) \simeq C_*(\mathrm{Cor}_k, \mathrm{Cor}_k \setminus \{0\}; R) \simeq \begin{cases} \Sigma R^{\oplus k-1}, & \text{if } k \geq 1, \\ R, & \text{if } k = 0. \end{cases} \tag{8}$$

where Cor_k is the k-legged corolla and the first equivalence is excision. Since the Serre functor S on $\mathrm{Shv}(X; \mathrm{Mod}_R)$ is given by tensoring with the dualizing sheaf according to Proposition 2.11, the calculation (8) suggests that S is sensitive to singularities.

Definition 3.1. Let X be a one-dimensional space, and let k be a non-negative integer different from two. A sheaf $V \in \mathrm{Shv}(X; \mathrm{Mod}_R)$ is *vertex-like of valence* k if it is indecomposable and satisfies

$$S(V) \simeq \begin{cases} V, & \text{if } k = 0, \\ \Sigma V^{\oplus k-1}, & \text{otherwise,} \end{cases}$$

where $S\colon \mathrm{Shv}(X; \mathrm{Mod}_R) \to \mathrm{Shv}(X; \mathrm{Mod}_R)$ denotes the Serre functor on the sheaf category $\mathrm{Shv}(X; \mathrm{Mod}_R)$.

Proposition 3.2. *Let X be a one-dimensional space, and let k be a non-negative integer different from two. The following are equivalent for $V \in \mathrm{Shv}(X; \mathrm{Mod}_R)$:*

(a) The sheaf V is vertex-like of valence k;
(b) There is a point $x \in X$ of valence k such that V is equivalent to the skyscraper sheaf $x_* I$ for some indecomposable R-module I.

Proof. It follows from (8) and indecomposability of the skyscraper sheaves $x_* I$ that (b) implies (a).

To see that (a) implies (b), suppose that $V \in \mathrm{Shv}(X; \mathrm{Mod}_R)$ is vertex-like of valence k. Let y be an element of the support $\operatorname{supp} V$. We set $l = \nu(y)$ to be the valence of y. Then the calculation (8) implies that

$$S(V)_y \simeq (V \otimes \omega_X)_y \simeq V_y \otimes \omega_{X,y} \simeq \begin{cases} V_y, & \text{if } l = 0, \\ \Sigma V_y^{\oplus l-1}, & \text{otherwise.} \end{cases}$$

On the other hand, we have by assumption that

$$S(V)_y \simeq \begin{cases} V_y, & \text{if } k = 0, \\ \left(\Sigma V^{\oplus k-1}\right)_y \simeq \Sigma V_y^{\oplus k-1}, & \text{otherwise.} \end{cases}$$

Since $V_y \not\simeq 0$ by assumption, this implies that $l = k$. Thus $\operatorname{supp} V \subseteq \{y \in X \mid \nu(y) = k\} \subseteq X_{\mathrm{sing}}$. In particular, $\operatorname{supp} V$ is discrete, so by the sheaf condition we must have

$$V \simeq \bigoplus_{y \in \operatorname{supp} V} y_* V_x.$$

Since V is indecomposable we find that $\operatorname{supp} V$ must consist of a single point x. The functor $x_*\colon \mathrm{Mod}_R \to \mathrm{Shv}(X;\mathrm{Mod}_R)$ is fully faithful, and its essential image consists of sheaves F with $\operatorname{supp} F \subseteq \{x\}$. Hence we must have $V \simeq x_* I$ for some indecomposable R-module I as desired. □

Corollary 3.3. *Let X be a one-dimensional space, and let $i\colon X_{\mathrm{sing}} \hookrightarrow X$ denote the inclusion of its singular locus. The essential image of*

$$i_*\colon \mathrm{Shv}(X_{\mathrm{sing}};\mathrm{Mod}_R) \to \mathrm{Shv}(X;\mathrm{Mod}_R)$$

is equal to the localizing subcategory generated by vertex-like sheaves.

Notation 3.4. Given a one-dimensional space X, we let $\mathrm{Shv}_{\mathrm{sing}}(X;\mathrm{Mod}_R)$ denote the essential image of the functor i_* in the previous proposition. We have shown that this subcategory of $\mathrm{Shv}(X;\mathrm{Mod}_R)$ only depends on $\mathrm{Shv}(X;\mathrm{Mod}_R)$ as a category.

Corollary 3.5. *Let X be a one-dimensional space. There is a bijection*

$$X_{\mathrm{sing}} \to \pi_0 \mathrm{Shv}_{\mathrm{sing}}(X;\mathrm{Mod}_R)$$

given by sending $x \in X_{\mathrm{sing}}$ to the full subcategory spanned by F with $\operatorname{supp} F \subseteq \{x\}$.

Remark 3.6. The closest analog to Proposition 3.2 in [BO01] is the characterization of the skyscrapers concentrated at closed points. We recall this here for contrast. Bondal and Orlov define a *point object* of a dualizable stable R-linear category $\mathcal{C}$ with Serre functor S to be an object $P \in \mathcal{C}$ such that $S(P) \simeq \Sigma^s P$ for some s, and such that the canonical map

$$\begin{aligned} R &\longrightarrow \mathrm{Hom}(P,P) \\ a &\longmapsto a \cdot - \end{aligned}$$

is an equivalence. If X is a smooth projective variety with ample or anti-ample canonical sheaf and $\mathcal{C}$ is its derived category, Bondal and Orlov show that the point objects of $\mathcal{C}$ are exactly the skyscraper sheaves of the form $x_* I$ for I indecomposable and x a closed point. If $\mathcal{C}$ is the derived category of a locally compact Hausdorff space, there are many point objects in $\mathcal{C}$ which do not correspond to closed points. For instance, any interval $[a,b] \subseteq \mathbb{R}$ gives rise to a point object $i_* \mathbf{1}_{[a,b]} \in \mathrm{Shv}(\mathbb{R};\mathrm{Mod}_R)$, where $i\colon [a,b] \to \mathbb{R}$ denotes the inclusion. We are not aware of a way to distinguish between these point objects and the ones arising from actual points. Bondal and Orlov's proof crucially uses that tensoring with an ample sheaf is highly nontrivial, and more specifically that any sheaf which is fixed by this action must be supported on a finite collection of points, and there is no obvious replacement for this in topology.

3.2. **Reconstruction of the regular locus.** Now that we have a handle on the singular locus, reconstructing the regular locus is an easy application of the theory of recollements and the results from § 2.3.

Proposition 3.7. *Let X be a one-dimensional space, and let $j\colon X_{\mathrm{reg}} \hookrightarrow X$ denote the inclusion of its regular locus. The essential image of*

$$j_*\colon \mathrm{Shv}(X_{\mathrm{reg}};\mathrm{Mod}_R) \to \mathrm{Shv}(X;\mathrm{Mod}_R)$$

is equal to the full subcategory $\mathrm{Shv}_{\mathrm{sing}}(X; \mathrm{Mod}_R)^{\perp} \subseteq \mathrm{Shv}(X; \mathrm{Mod}_R)$ *spanned by sheaves* F *such that* $\mathrm{Map}(V, F)$ *is contractible for each* $V \in \mathrm{Shv}_{\mathrm{sing}}(X; \mathrm{Mod}_R)$.

Proof. Recall that $\mathrm{Shv}_{\mathrm{sing}}(X; \mathrm{Mod}_R)$ is the essential image of the functor

$$i_*\colon \mathrm{Shv}(X_{\mathrm{sing}}; \mathrm{Mod}_R) \to \mathrm{Shv}(X; \mathrm{Mod}_R)$$

induced by the inclusion $i\colon X_{\mathrm{sing}} \hookrightarrow X$ of the singular locus. The functors i_* and j_* exhibit $\mathrm{Shv}(X; \mathrm{Mod}_R)$ as a recollement, and the characterization of the open part

$$\mathrm{img}\,(j_*\colon \mathrm{Shv}(X_{\mathrm{reg}}; \mathrm{Mod}_R) \to \mathrm{Shv}(X; \mathrm{Mod}_R))$$

as the right orthogonal complement of the closed part $\mathrm{Shv}_{\mathrm{sing}}(X; \mathrm{Mod}_R)$ is a standard fact about recollements, see [Lur17, Prop A.8.20]. □

Notation 3.8. Given a one-dimensional space X, we let $\mathrm{Shv}_{\mathrm{reg}}(X; \mathrm{Mod}_R)$ denote the essential image of the functor j_* in the previous proposition. Again we have shown that this subcategory only depends on the category $\mathrm{Shv}(X; \mathrm{Mod}_R)$.

Corollary 3.9. *Let* X *be a one-dimensional space. There is a bijection*

$$\pi_0 X_{\mathrm{reg}} \to \pi_0 \mathrm{Shv}_{\mathrm{reg}}(X; \mathrm{Mod}_R) \tag{9}$$

given by sending a component E *of* X_{reg} *to the full subcategory spanned by* F *with* $\mathrm{supp}\, F \subseteq E$.

We still need to be able to identify whether a given component $E \in \pi_0 X_{\mathrm{reg}}$ is homeomorphic to the circle or to the real line. Here we have several options, since there are several ways to tell the topological spaces S^1 and $\mathbb{R}$ apart. One way to tell them apart is that S^1 is compact, which $\mathbb{R}$ is not:

Proposition 3.10. *Let* X *be a one-dimensional space, and let* $\pi_0^{\mathbb{R}}(\mathrm{Shv}_{\mathrm{reg}}(X; \mathrm{Mod}_R))$ *denote the collection of connected components* $\mathcal{C} \subseteq \mathrm{Shv}_{\mathrm{reg}}(X; \mathrm{Mod}_R)$ *such that* $\mathcal{C}$ *contains a nontrivial compact object. The bijection* (9) *restricts to a bijection*

$$\pi_0^{\mathbb{R}} \to \pi_0^{\mathbb{R}} \mathrm{Shv}_{\mathrm{reg}}(X; \mathrm{Mod}_R),$$

where $\pi_0^{\mathbb{R}} \subseteq \pi_0 X_{\mathrm{reg}}$ *is the subset of components* E *such that* $E = \overline{E} \subseteq X$ *is homeomorphic to the real line.*

Proof. This follows immediately from the previous corollary together with [Har25, Thm 2.6]. □

3.3. **Gluing everything back together.** Given a one-dimensional space X, the results of the previous subsections show how to reconstruct full subcategories

$$\mathrm{Shv}_{\mathrm{sing}}(X; \mathrm{Mod}_R) \text{ and } \mathrm{Shv}_{\mathrm{reg}}(X; \mathrm{Mod}_R) \subseteq \mathrm{Shv}(X; \mathrm{Mod}_R),$$

such that if we denote the inclusions by i_* and j_* respectively, we have a recollement

$$\begin{tikzcd} \mathrm{Shv}_{\mathrm{sing}}(X; \mathrm{Mod}_R) \arrow[r, "i_*"] & \mathrm{Shv}(X; \mathrm{Mod}_R) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathrm{Shv}_{\mathrm{reg}}(X; \mathrm{Mod}_R). \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} \tag{10}$$

Furthermore, these subcategories categories were used to reconstruct the sets X_{sing}, $\pi_0 X_{\mathrm{reg}}$, and $\pi_0^{\mathbb{R}}$ (in the notation of § 2.1).

Note that from the recollement we also have access to the gluing functor

$$i^* j_* \colon \mathrm{Shv}_{\mathrm{reg}}(X; \mathrm{Mod}_R) \to \mathrm{Shv}_{\mathrm{sing}}(X; \mathrm{Mod}_R), \tag{11}$$

which can be thought of informally as sending a sheaf on X_{reg} to the "intersection" of its "closure" inside X with the singular locus X_{sing}. We use this functor to retrieve the final piece of data needed to recover the homeomorphism type of X, namely the endpoint function

$$\ell \colon \pi_0 X_{\mathrm{reg}} \to \mathcal{P}_{\leq 2}(X_{\mathrm{sing}})$$

constructed in § 2.1.

Proposition 3.11. *Let X be a one-dimensional space, let $E \in \pi_0 X_{\mathrm{reg}}$ be a component of the regular locus which is homeomorphic to the real line, and let $x \in X_{\mathrm{sing}}$. Let $\mathcal{C}_E \subseteq \mathrm{Shv}_{\mathrm{reg}}(X; \mathrm{Mod}_R)$ and $\mathcal{C}_x \subseteq \mathrm{Shv}_{\mathrm{sing}}(X; \mathrm{Mod}_R)$ denote the corresponding connected components. Then x belongs to $\ell(E)$ if and only*

$$\mathcal{C}_E \to \mathrm{Shv}_{\mathrm{reg}}(X; \mathrm{Mod}_R) \xrightarrow{i^* j_*} \mathrm{Shv}_{\mathrm{sing}}(X; \mathrm{Mod}_R) \to \mathcal{C}_x \tag{12}$$

is nonzero, where the functor on the right is left adjoint to the inclusion.

Proof. Equivalently, we must show that x belongs to $\ell(E)$ if and only if $x^* f_*$ is nonzero, where $f \colon E \hookrightarrow X$ is the inclusion. Recall that x belongs to $\ell(E)$ if and only if x is contained in $\overline{E}$, where $\overline{E}$ denotes the closure of E inside X. Let $g \colon E \hookrightarrow \overline{E}$ and $h \colon \overline{E} \hookrightarrow X$ denote the inclusions. Thus we must show that $x \in \overline{E}$ if and only if $x^* h_* g_*$ is nonzero. If $x \notin \overline{E}$, then $x^* h_* \simeq 0$ since h is a closed inclusion, so we can apply proper base change to the diagram

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \overline{E} \arrow[d, "h"] \\ \mathrm{pt} \arrow[r, "x"] & X. \end{tikzcd}$$

If on the other hand $x \in \overline{E}$, we can rewrite $x^* h_* g_* \simeq x^* g_*$. Assume first that the pair $(\overline{E}, x)$ is homeomorphic to $(\mathbb{R}_{\geq 0}, 0)$, so we can pick a cofinal sequence of contractible open neighborhoods $U_0 \supseteq U_1 \supseteq U_2 \supseteq \cdots \ni x$ in $\overline{E}$ such that $U_n \cap E$ is also contractible (and in particular nonempty) for each n. Then we have equivalences

$$\begin{tikzcd} (g_* \mathbf{1}_E)(U_0) \arrow[r] \arrow[d, "\simeq"] & (g_* \mathbf{1}_E)(U_1) \arrow[r] \arrow[d, "\simeq"] & (g_* \mathbf{1}_E)(U_2) \arrow[r] \arrow[d, "\simeq"] & \cdots \\ \mathbf{1}_E(U_0 \cap E) \arrow[r] \arrow[d, "\simeq"] & \mathbf{1}_E(U_1 \cap E) \arrow[r] \arrow[d, "\simeq"] & \mathbf{1}_E(U_2 \cap E) \arrow[r] \arrow[d, "\simeq"] & \cdots \\ R \arrow[r, "\mathrm{id}"] & R \arrow[r, "\mathrm{id}"] & R \arrow[r, "\mathrm{id}"] & \cdots \end{tikzcd}$$

showing that $x^* g_* \mathbf{1}_E$, which is computed as the colimit of the top row, is equivalent to R, and since $R \not\simeq 0$ this proves the claim. If on the other hand $(\overline{E}, x)$ is homeomorphic to $(S^1, 1)$. Then we can instead pick a cofinal sequence of open neighborhoods $U_0 \supseteq U_1 \supseteq U_2 \supseteq \cdots \ni x$ with $U_n \cap E \simeq$

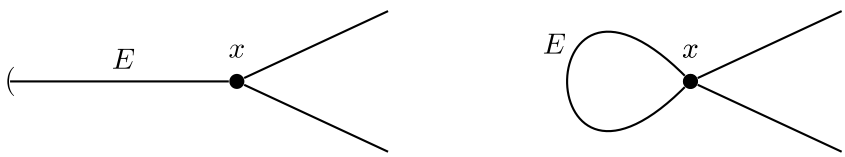

FIGURE 4. Two situations in which the edge E only has a single endpoint $x \in X_{\mathrm{sing}}$.

pt $\sqcup$ pt. We now get a diagram as above, but with the bottom row replaced by

$$R \oplus R \xrightarrow{\mathrm{id}} R \oplus R \xrightarrow{\mathrm{id}} R \oplus R \xrightarrow{\mathrm{id}} \cdots,$$

showing that $x^* g_* \mathbf{1}_E \simeq R \oplus R \not\simeq 0$. □

In the notation of § 2.1, it remains to determine the set $\pi_0^{[0,1)} \subseteq \pi_0 X_{\mathrm{reg}}$ of components whose closure in X is homeomorphic to the half-open interval. More specifically, it remains to separate these components from the only other components that have exactly one endpoint, namely those components $E \subseteq X_{\mathrm{reg}}$ characterized by $E \cong \mathbb{R}$ and $\overline{E} \cong S^1$, see Figure 4. In order to distinguish between these cases, we can again use that we know how to detect compactness:

Proposition 3.12. *Let X be a one-dimensional space, and let $E \in \pi_0 X_{\mathrm{reg}}$ be a component of its regular locus homeomorphic to the real line such that $\ell(E)$ only contains a single point x. Let $\mathcal{C}_E \subseteq \mathrm{Shv}_{\mathrm{reg}}(X; \mathrm{Mod}_R)$ and $\mathcal{C}_x \subseteq \mathrm{Shv}_{\mathrm{sing}}(X; \mathrm{Mod}_R)$ denote the corresponding connected components, and let $\mathcal{C}_{\overline{E}}$ denote the lax limit over the Δ^1-indexed diagram corresponding to this functor coming from the functor* (12). *Then $\overline{E} \cong S^1$ if and only if $\mathcal{C}_{\overline{E}}$ contains a nonzero compact object.*

Proof. By [Lur17, A.8.11], the category $\mathcal{C}_{\overline{E}}$ is equivalent to $\mathrm{Shv}(\overline{E}; \mathrm{Mod}_R)$, where either $\overline{E}$ is homeomorphic either to S^1 or $[0, 1)$. The result now follows from [Har25, Thm 2.6] and the fact that S^1 is compact which $[0, 1)$ is not. □

We now assemble all the results of this section into a proof of our main theorem:

Proof of Theorem A. Suppose $\Phi\colon \mathrm{Shv}(Y; \mathbf{k}) \to \mathrm{Shv}(X; \mathbf{k})$ is an equivalence of categories. Also, let S_Y and S_X denote the Serre functors on $\mathrm{Shv}(Y; \mathbf{k})$ and $\mathrm{Shv}(X; \mathbf{k})$, respectively. We start by showing that Y is one-dimensional. For this, we assume for contradiction that Y is not one-dimensional. Then we can find $F \in \mathrm{Shv}(Y; \mathbf{k})$ such that $S_Y(F) \simeq \Sigma^d F$ for some $d > 1$, but $\Sigma^k F \not\simeq F$ for any $0 \neq k \in \mathbb{Z}$ (e.g. take $F = y_* \mathbf{k}$ for y an interior point of a top-dimensional cell in Y). Then

$$S_X \Phi(F) \simeq \Phi(S_Y(F)) \simeq \Sigma^d \Phi(F),$$

and a stalkwise consideration shows that we must have $\operatorname{supp}\Phi(F) \subseteq X_{\mathrm{reg}}$. From this we get $S_X\Phi(F) \simeq \Sigma\Phi(F)$, whence $\Sigma^{d-1}\Phi(F) \simeq \Phi(F)$. But then $\Sigma^{d-1}F \simeq F$, contradiction.

Since X and Y are both one-dimensional spaces, it follows from the results of this section together with Lemma 2.6 that X and Y are homeomorphic. □

Remark 3.13. With a little extra work, one can probably prove a reconstruction result for one-dimensional spaces without the paracompactness assumption in Definition 2.1. By the classification of non-paracompact one-dimensional manifolds [Fro62], one needs to distinguish between components $E \subseteq X_{\mathrm{reg}}$ which are homeomorphic to the circle, the real line, the long line, and the open long ray.

We have already shown how to recognize circle components using compactness. Unlike the long line and the open long ray, the real line is second countable, which implies that $\mathrm{Shv}(\mathbb{R}; \mathrm{Mod}_R)$ has a countable set of $\aleph_1$-compact generators, see for instance [Cla25, Exmp 6.14]. On the other hand, if L is the long line or the open long ray, then $\mathrm{Shv}(L; \mathrm{Mod}_R)$ does not admit such a generating set. Indeed, assume for contradiction that such a generating set existed. We can pick a closed subspace $S \subseteq L$ which is both uncountable and discrete. Since the pullback functor $\mathrm{Shv}(L; \mathrm{Mod}_R) \to \mathrm{Shv}(S; \mathrm{Mod}_R)$ is a strongly continuous localization, we find that $\mathrm{Shv}(S; \mathrm{Mod}_R)$ must also admit a countable set of $\aleph_1$-compact generators.

We claim that a sheaf $F \in \mathrm{Shv}(S; \mathrm{Mod}_R)$ is $\aleph_1$-compact if and only $\operatorname{supp} F$ is countable and each stalk x^*F is $\aleph_1$-compact. Here the 'if' direction follows from the fact that $\aleph_1$-compact objects are closed under countable sum. As for 'only if', we can write

$$F \simeq \operatorname{colim}_{T \subseteq S} (i_T)_* F|_T,$$

where T ranges over the poset of countable subsets $T \subseteq S$. This poset if $\aleph_1$-filtered, so we find that F must be a retract of $(i_T)_*F|_T$ for some countable subset $T \subseteq S$, proving the support condition. The stalk condition follows from the fact that taking stalks is strongly continuous.

If $\{F_n\}_1^\infty$ were a countable set of $\aleph_1$-compact generators for $\mathrm{Shv}(S; \mathrm{Mod}_R)$, we would have that $S = \bigcup_1^\infty \operatorname{supp} F_n$ But we have shown that each $\operatorname{supp} F_n$ is countable, and a countable union of countable sets is again countable, contradiction.

4. Fourier–Mukai partners in topology and homotopy theory

Let $\mathcal{C} = \{X, Y, Z, \dots\}$ be a category of geometric objects equipped with a sheaf theory functor $X \mapsto D(X)$ as in the introduction.

Definition 4.1. Two objects X and $Y \in \mathcal{C}$ are said to be *Fourier–Mukai partners* if $D(X) \simeq D(Y)$.

The phenomenon of Fourier–Mukai partners is well-studied in algebraic geometry, but has not been considered in topology. The problem of finding Fourier–Mukai partners in topology is further related to an analogous question in homotopy theory:

Proposition 4.2. *Let X and Y be a compact topological spaces, both locally of singular shape, and let $R \in \mathrm{CAlg}(\mathrm{Sp})$. If X and Y are Fourier–Mukai partners in the sense that*

$$\mathrm{Shv}(X; \mathrm{Mod}_R) \simeq \mathrm{Shv}(Y; \mathrm{Mod}_R),$$

then their underlying homotopy types $|X|$ and $|Y|$ are Fourier–Mukai partners in the sense that

$$\mathrm{Fun}(|X|, \mathrm{Perf}_R) \simeq \mathrm{Fun}(|Y|, \mathrm{Perf}_R),$$

where $\mathrm{Perf}_R = \mathrm{Mod}_R^\omega$ is the category of perfect R-modules.

Proof. By [Har25, Thm 2.6], the full subcategory of compact objects

$$\mathrm{Shv}(X; \mathrm{Mod}_R)^\omega \subseteq \mathrm{Shv}(X; \mathrm{Mod}_R)$$

is spanned by sheaves F which are locally constant and have $x^*F \in \mathrm{Mod}_R^\omega = \mathrm{Perf}_R$ for each $x \in X$. Because of our shape assumption, Lurie's monodromy theorem [Lur17, Thm A.1.15] lets us identify this subcategory with $\mathrm{Fun}(|X|, \mathrm{Mod}_R^\omega)$. □

Remark 4.3. The topological Fourier–Mukai problem is interesting also for topological spaces that are homotopy equivalent. One may as well look for partners of this type, so as to remove the obstruction posed by the previous proposition. For instance, by analogy with Bondal and Orlov's conjecture that smooth projective varieties that are related by a flop (a kind of contractible surgery in algebraic geometry) are Fourier–Mukai partners, one might hope to prove that h-cobordant manifolds are derived equivalent.

Appendix A. Homology manifolds and the topological Serre functor

Recall that a Euclidean neighborhood retract X is said to be a *homology manifold of dimension d* if

$$H_i(X, X \setminus \{x\}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } i = d, \\ 0, & \text{else,} \end{cases}$$

for each $x \in X$. The Serre functor lets us rephrase this condition in a non-commutative way, i.e. referencing only the category $\mathrm{Shv}(X; \mathrm{Mod}_\mathbb{Z})$:

Proposition A.1. *Let X be a Euclidean neighborhood retract. Then X is a homology manifold if and only if the Serre functor $S\colon \mathrm{Shv}(X; \mathrm{Mod}_\mathbb{Z}) \to \mathrm{Shv}(X; \mathrm{Mod}_\mathbb{Z})$ is an equivalence.*

Proof. It is well-known that X is a homology manifold if and only if ω_X is invertible with respect to the pointwise tensor product, see [KNP24, Cor 4.6.20]. The result now follows immediately from the description of the Serre functor given in Proposition 2.11. □

Remark A.2. Kontsevich showed that the derived category $D(X)$ of a scheme X remembers whether X is smooth. Namely, for a scheme X (over a base field $\mathbf{k}$) subject to some mild conditions, we have that X is smooth if and only if the coevaluation functor $\mathrm{Mod}_\mathbf{k} \to D(X) \otimes_\mathbf{k} D(X)$ is strongly continuous [Kon05]. This is not a useful notion in topology: the derived category of a locally compact Hausdorff space X is smooth if and only if X is finite discrete [Har23]. The proposition above provides a non-commutative criterion for checking that a locally compact Hausdorff space is smooth, in a topologically reasonable sense.

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4 Stokes theorem for sheaves and moduli spaces of manifolds with a free boundary

STOKES THEOREM FOR SHEAVES AND MODULI SPACES OF MANIFOLDS WITH A FREE BOUNDARY

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ABSTRACT. We lift the family Stokes theorem to a formula in the derived category of the family's base space. This makes it possible to extract information about fiber integrals from information about the cohomology of the fiber. We give an application of this to the cohomology of moduli spaces with a free boundary, giving new relations among κ-classes and ψ-classes.

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0. INTRODUCTION

Let $p\colon E \to B$ be a smooth submersion, where B is a closed connected smooth manifold and E is a smooth manifold with boundary, and let $i\colon \partial E \hookrightarrow$

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E denote the inclusion of the boundary. An orientation of p is a smoothly varying choice of orientation of every fiber $E_b = p^{-1}(b)$, $b \in B$. Integrating differential forms against this fiberwise orientation defines *fiber integration* maps

$$(0.1) \qquad \int_p \colon \Omega^{*+d}_{\mathrm{dR}}(E, \partial E) \to \Omega^*_{\mathrm{dR}}(B) \quad \text{and} \quad \int_{p\,i} \colon \Omega^{*+d-1}_{\mathrm{dR}}(\partial E) \to \Omega^*_{\mathrm{dR}}(B)$$

where d is the fiber dimension of p, see [GHV72, Ch VII]. The family Stokes theorem is a folklore formula relating fiber integration along the bundle p and its associated boundary bundle $p\,i$ (see e.g.[GHV72, Ch VII Prob 4]), which we can express as a commutative diagram

$$(0.2) \qquad \begin{tikzcd} \Omega^{*+d-1}_{\mathrm{dR}}(\partial E) \arrow[d, "\delta"] \arrow[dr, "\int_{pi}"] & \\ \Omega^{*+d}_{\mathrm{dR}}(E, \partial E) \arrow[r, "\int_p"'] & \Omega^*_{\mathrm{dR}}(B) \end{tikzcd}$$

in the derived category of real vector spaces.

The usual family Stokes theorem is not sensitive to information about the fiber of p. We correct this defect here by lifting (0.2) to a formula in the derived category of sheaves of real vector spaces on the base space B of the bundle. Recall that a map of topological spaces $f \colon X \to Y$ induces two pushforward functors on derived categories of sheaves f_* and $f_! \colon D(\mathrm{Shv}(X; \mathrm{Vect}_{\mathbb{R}})) \to D(\mathrm{Shv}(Y; \mathrm{Vect}_{\mathbb{R}}))$, defined as the total derived functors of ordinary pushforward and compactly-supported pushforward, respectively (see [KS90, Ch II §§ 2.5–6]). We let $\mathbb{R}_X$ denote the constant sheaf on X with value $\mathbb{R}$.

Theorem A (Stokes theorem for sheaves)**.** *Let $W \to E \xrightarrow{p} B$ be an oriented fiber bundle, where W is a compact topological manifold with boundary. Let $i \colon \partial^p E \hookrightarrow E$ and $j \colon E \setminus \partial^p E \hookrightarrow E$ denote the inclusions of the fiberwise boundary and interior, respectively.*

There is a commutative diagram

$$(0.3) \qquad \begin{tikzcd} \Sigma^{d-1}(p\,i)_* \mathbb{R}_{\partial^p E} \arrow[d, "\delta"] \arrow[dr, "\int_p"] & \\ \Sigma^d (p\,j)_! \mathbb{R}_{E \setminus \partial^p E} \arrow[r, "\int_{pi}"'] & \mathbb{R}_B \end{tikzcd}$$

in the derived category of sheaves $D(\mathrm{Shv}(B; \mathrm{Vect}_{\mathbb{R}}))$, such that the derived global sections of this diagram is equivalent to (0.2) *if B happens to be a closed smooth manifold.*

Furthermore, the diagram (0.3) *is natural in maps of bundles, in the sense that if $f \colon B' \to B$ is a map of topological spaces, then applying the derived pullback $f^* \colon D(\mathrm{Shv}(B; \mathrm{Vect}_{\mathbb{R}})) \to D(\mathrm{Shv}(B'; \mathrm{Vect}_{\mathbb{R}}))$ to* (0.3) *produces a commutative diagram which is canonically identified with the version of* (0.3) *for the pullback bundle $f^{-1}E \to B'$.*

By naturality of the diagram (0.3) in maps of bundles, taking stalks at some $b \in B$ recovers the ordinary Stokes theorem for the fiber W_b. This makes it possible to extract properties of the fiber integration maps (0.1) in terms of the cohomology of the fiber. We will give an application of this idea to the study of characteristic classes of bundles of manifolds with boundary.

In fact, we prove a version of this theorem for sheaves with coefficients in an arbitrary presentably symmetric monoidal stable ∞-category (Theorem 2.29), provided that p is oriented with respect to this category. For instance, the notion of orientation associated with spectral sheaves[1] is that of a fiberwise stable framing. Given such an orientation of p, we get an analog of (0.3) for stable cohomotopy. Crucially, however, our version of Stokes theorem is already an improvement on the usual family Stokes theorem (that is, for singular cohomology with coefficients in a field of characteristic zero such as $\mathbb{R}$ or $\mathbb{Q}$), and it is this case that is relevant for our application.

0.1. **Characteristic classes of handlebody bundles.** Let $V_g = (S^n \times D^{n+1})^{\natural g}$ be a $(2n+1)$-dimensional genus g handlebody. We consider the topological group $\mathrm{Diff}^+(V_g)$ of orientation-preserving diffeomorphisms. We will be especially interested in the cohomology $H^*(B\,\mathrm{Diff}^+(V_g); k)$ of the classifying space of $\mathrm{Diff}^+(V_g)$ with coefficients in some ring k, or equivalently in characteristic classes

$$\zeta\colon (V_g \to E \xrightarrow{p} B) \mapsto \zeta(p) \in H^*(B; k)$$

of smooth V_g-bundles. We also consider the closed subgroup $\mathrm{Diff}(V_g, D) \subseteq \mathrm{Diff}^+(V_g)$ of diffeomorphisms which restrict to the identity on a neighborhood of a disk $D = D^{2n} \subseteq \partial V_g$.[2] There are maps

$$B\,\mathrm{Diff}(V_g, D) \to B\,\mathrm{Diff}(V_{g+1}, D) \tag{0.4}$$

defined by extending diffeomorphisms by the identity. Letting g go to infinity along the maps (0.4), there is also a scanning map

$$\mathrm{colim}_g\, B\,\mathrm{Diff}(V_g, D) \to \Omega_0^\infty \Sigma_+^\infty BSO(2n+1)\langle n\rangle \tag{0.5}$$

which is a homology equivalence for $n = 1$ according to a theorem whose proof was sketched by Hatcher [Hat12] and which is proved in upcoming work of Barkan–Steinebrunner (personal communication), and for $n \geq 4$ and by work of Botvinnik–Perlmutter [BP17]. (Here $X\langle n\rangle$ denotes the nth stage of the Whitehead tower of X, see [Hat01, Exmp 4.20].) Computing the cohomology of the right side of (0.5) is an easy task using standard tools from homotopy theory, especially with coefficients in a field k. Together with homological stability results of Hatcher–Wahl [HW10] and the author [Har25b] in the $n = 1$ case and Perlmutter [Per18] for $n \geq 4$, this allows us to completely describe the cohomology $H^*(B\,\mathrm{Diff}^+(V_g); k)$ in the *stable range*, which is $* \leq \frac{2}{3}(g-1)$ if $n = 1$ and $* \leq \frac{1}{2}(g-4)$ if $n \geq 4$.

Outside of this range, the cohomology of the moduli space $B\,\mathrm{Diff}^+(V_g)$ remains largely mysterious.[3] We focus on the case of rational coefficients $k = \mathbb{Q}$. As an approximation to $H^*(B\,\mathrm{Diff}^+(V_g); \mathbb{Q})$, one can consider the

[1] i.e. sheaves valued in the ∞-category of spectra

[2] Up to conjugation, this subgroup does not depend on the choice of D, which we therefore sweep under the rug for the purpose of exposition.

[3] For instance, the second cohomology of $B\,\mathrm{Diff}^+(V_g)$ is unknown outside the stable range even for $n = 1$.

image of the scanning map $B\operatorname{Diff}^+(V_g) \to \Omega_0^\infty \Sigma_+^\infty BSO(2n+1)\langle n\rangle$ in rational cohomology. This is a subalgebra $R^*(V_g) \subseteq H^*(B\operatorname{Diff}^+(V_g);\mathbb{Q})$, which we refer to as the *tautological algebra of* V_g. It follows from [BP17, Cor D] that it is generated by the κ-classes (aka Miller–Morita–Mumford classes) pulled back from the usual κ-classes $\kappa_c \in H^*(B\operatorname{Diff}^+(\partial V_g);\mathbb{Q})$ along the restriction map

$$B\operatorname{Diff}^+(V_g) \to B\operatorname{Diff}^+(\partial V_g),$$

which classifies the procedure of replacing a smooth V_g-bundle with its fiberwise boundary bundle. Describing $R^*(V_g)$ is thus equivalent to determining the relations between κ-classes that are satisfied by every smooth V_g-bundle. Recall that if W^d is an orientable closed smooth manifold, k is a commutative ring, and $c \in H^*(BSO(d);k)$, the corresponding κ-class is the characteristic class defined by sending a smooth fiber bundle $W \to E \xrightarrow{p} B$ to

$$\kappa_c(p) = \int_p c(T^pE) \in H^{*-d}(B;k), \tag{0.6}$$

where T^pE is the vertical tangent bundle of p. In order to study these classes, we prove the following general theorem about fiber integration:

Theorem B (Theorem 4.2)**.** *Let $W \to E \xrightarrow{p} B$ be an oriented fiber bundle, where W is a compact odd-dimensional topological manifold with boundary. Let $i\colon \partial^pE \hookrightarrow E$ denote its fiberwise boundary.*

Suppose that $\widetilde{H}_*(W;\mathbb{Q})$ is concentrated in odd degrees. If $a \in H^*(\partial^pE;\mathbb{Q})$ is an even-degree cohomology class such that the fiber integral $\int_{p\,i} a$ is zero and $b \in H^*(E;\mathbb{Q})$ is arbitrary, then

$$\left(\int_{p\,i} a \cdot i^*(b)\right)^{g+1} = 0,$$

where $g = \dim_{\mathbb{Q}} \widetilde{H}_*(W;\mathbb{Q})$.

Note that the $(2n+1)$-dimensional handlebody V_g satisfies the assumptions of the theorem if n is odd, and there is furthermore no conflict of notation in this case since the genus g is equal to the dimension $\dim_{\mathbb{Q}} \widetilde{H}_*(V_g;\mathbb{Q})$.

Theorem B complements analogous results for bundles of even-dimensional manifolds without boundary by Grigoriev [Gri17, Thm 2.7], Randal-Williams [RW18, Thm 2.8], and others. As with these results, Theorem B is well-suited for studying the cohomology of moduli spaces of manifolds. We will give examples of this in § 4.3.

The definition of the tautological algebra $R^*(V_g)$ is inspired by the following definition, which is originally due to Mumford [Mum83] in the surface case. If W is a closed orientable smooth manifold, the *tautological ring of* W, denoted $R^*(W)$, is the subalgebra $R^*(W) \subseteq H^*(B\operatorname{Diff}^+(W);\mathbb{Q})$ generated by the κ-classes (0.6).

The first subtlety in defining tautological rings for manifolds with boundary is that (0.6) no longer makes sense. Indeed, if $W \to E \xrightarrow{p} B$ is an oriented smooth bundle where W is a compact smooth manifold with boundary, then the associated fiber integration map is only defined on the relative cohomology $H^*(E, \partial^pE;k)$, where $\partial^pE \subseteq E$ is the fiberwise boundary along p.

A natural substitute for this definition is to use the Becker–Gottlieb transfer $\mathrm{trf}_p\colon H^*(E;k) \to H^*(B;k)$ in lieu of the fiber integral in (0.6), thereby defining the ρ-class

$$\rho_c(p) = \mathrm{trf}_p(c) \in H^*(B;k) \tag{0.7}$$

associated with $c \in H^*(BSO(d);k)$. This gives another potential definition of the tautological algebra of a manifold with boundary, namely as the subalgebra of $H^*(B\operatorname{Diff}^+(W);\mathbb{Q})$ generated by the ρ-classes. In order to compare this definition with the definition of the tautological algebra $R^*(V_g)$ given above, we prove:

Theorem C (Theorem 3.12)**.** *Let W^d be a compact smooth orientable manifold with boundary, let*

$$r\colon B\operatorname{Diff}^+(W) \to B\operatorname{Diff}^+(\partial W)$$

denote the map induced by restriction to the boundary, and let k be a commutative ring.

If $c \in H^*(BSO(d);k)$ is a stable class, in the sense that it lies in the image of $H^*(BO;k) \to H^*(BSO(d);k)$, then

$$r^*\kappa_c = 0. \tag{0.8}$$

If in addition d is odd and $2 \in k$ is either zero or invertible, then

$$r^*\kappa_{ec} = 2\rho_c, \tag{0.9}$$

where $e \in H^d(BSO(d);k)$ is the Euler class.

Here (0.8) is a theorem of Giansiracusa and Tillmann [GT11]. We give a new proof of this theorem using the ordinary family Stokes theorem, giving further evidence of the usefulness of Stokes theorem for the study of moduli spaces of manifolds with boundary. We take the opportunity to also answer a question posed in the same article of Giansiracusa and Tillmann:

Theorem D (Theorem 5.2)**.** *Let $g \geq 2$ and let $\mathrm{LMod}_g \leq \mathrm{Mod}_g$ denote the Lagrangian mapping class group (see § 5). Then $\kappa_{2i+1} = 0 \in H^*(\mathrm{LMod}_g;\mathbb{Q})$ for each $i \geq 0$.*

Remark 0.1. Crucially, we do not limit ourselves to considering diffeomorphisms $f \in \operatorname{Diff}^+(W)$ which fix the boundary pointwise; in the language of [Ebe13], we consider bundles of manifolds with a *free boundary* as opposed to a *fixed boundary*. Indeed, if we insist on the condition that the boundary is fixed, then there are no nontrivial bundles of this kind in the case of the three-dimensional handlebody, as Hatcher has shown that the fibers of the restriction map $\operatorname{Diff}(V_g) \to \operatorname{Diff}(\partial V_g)$ are empty or contractible [Hat99].

0.2. **Relation to the work of Randal-Williams.** The proof of Theorem B is closely inspired by the proof of the analogous theorem for closed manifolds by Randal-Williams [RW18, Thm 2.8]:

Theorem. *Let $W^{2n} \to E \xrightarrow{p} B$ be an oriented fiber bundle, where W is a closed even-dimensional topological manifold such that $H^{0<*<2n}(W;\mathbb{Q})$*

is concentrated in odd degrees. Let $d = \dim_{\mathbb{Q}} H^{0<*<2n}(W;\mathbb{Q})$*. If* $a, b \in H^*(E;\mathbb{Q})$ *have* $\int_p a = \int_p b = 0$ *and* $|a|$ *is even, then*

$$\left(\int_p ab\right)^{d+1} = 0.$$

We briefly review his strategy here. Let $W \to E \xrightarrow{p} B$ be an oriented smooth bundle, where W is a closed manifold. Let $\mathrm{Loc}(E;\mathbb{Q})$ and $\mathrm{Loc}(B;\mathbb{Q})$ denote the derived categories of local systems of $\mathbb{Q}$-vector spaces on E and B, respectively. The map p induces a functor

$$p_* \colon \mathrm{Loc}(E;\mathbb{Q}) \to \mathrm{Loc}(B;\mathbb{Q}),$$

namely the total derived functor of the usual pushforward of local systems. Poincare duality for W can be generalized to a self-duality property for the local system $p_*\mathbb{Q}_E$, where $\mathbb{Q}_E \in \mathrm{Loc}(E;\mathbb{Q})$ denotes the constant local system at $\mathbb{Q}$.

The fiber integration map $\int_p \colon H^*(E;\mathbb{Q}) \to H^{*-d}(B;\mathbb{Q})$ has a straightforward interpretation in terms of the self-duality of $p_*\mathbb{Q}_E$. This perspective has the advantage that $p_*\mathbb{Q}_E$ also contains cohomological information about the fiber W; namely, the stalk $(p_*\mathbb{Q}_E)_b$ at some $b \in B$ is equivalent via a Beck–Chevalley map to the cochains $C^*(W_b;\mathbb{Q})$ of the fiber $W_b = p^{-1}(b)$. Informally, the local system $p_*\mathbb{Q}_E$ is $C^*(W;\mathbb{Q})$ parametrized over B via the bundle p. It is by exploiting the tension between stalkwise information and the total derived global sections functor $\Gamma \colon \mathrm{Loc}(B;\mathbb{Q}) \to \mathrm{Mod}_{\mathbb{Q}}$ that Randal-Williams proves the closed manifold counterpart to Theorem B.

If W^d is instead an orientable compact manifold with boundary, duality becomes more subtle. Instead of a perfect pairing between the cohomology and homology of W, Poincare duality now supplies a perfect pairing

$$H^*(W;\mathbb{Z}) \otimes H_{d-*}(W, \partial W;\mathbb{Z}) \to \mathbb{Z}.$$

In order to capture the interactions between W and ∂W—or rather, between the total space of a bundle and its fiberwise boundary—we work with the derived categories of all sheaves, not just local systems. The key to proving Theorem B is to give a kind of compatibility in this setting between the duality of a smooth bundle and the duality of its fiberwise boundary. This is exactly the content of our sheaf-theoretic enhancement of the family Stokes theorem (Theorem A).

0.3. **Organization of the article.** In § 2, we prove the general form of Theorem A. We have aimed to make this section as self-contained as possible by giving a summary of the yoga of six operations for sheaves on locally compact Hausdorff spaces, and the interpretation of orientations and integration in this framework. A reader who is familiar with these ideas may wish to jump directly to the proof of the Stokes theorem for sheaves in § 2.5. In § 3, we prove Theorem C (including the theorem of Giansiracusa–Tillmann); in § 4, we prove Theorem B; and in § 5 we prove Theorem D.

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1. Notation and terminology

In this article, we use the word "category" to mean an ∞-category. Thus for us categories in the ordinary sense are simply those categories whose mapping spaces are all homotopy discrete. We let $\mathrm{Pr}^L_{\mathrm{st}}$ denote the category of presentable stable categories, considered as a symmetric monoidal category under the Lurie tensor product. We let Sp denote the category of spectra, viewed as a symmetric monoidal category under the smash product $\otimes$. For a commutative ring spectrum $R \in \mathrm{CAlg}(\mathrm{Sp})$, we let Mod_R denote the (stable) category of R-modules. If R is an ordinary commutative ring, our convention means that the category that we write as Mod_R is equal to the category which classically is known as the derived category of R. We let $\mathcal{S}$ denote the category of "spaces" in the sense of Lurie [Lur09], i.e. $\mathcal{S}$ is the category of homotopy types/∞-groupoids. It is unfortunate that the word "space" is overloaded in mathematics; we use it here both in the sense of Lurie and in the sense of topological spaces. We have tried to make it clear when the word is used in the second sense, e.g. by prepending the adjectives "topological" or "(locally compact) Hausdorff," or by appending "having the homeomorphism type of a (locally finite) CW complex".

We let pt denote the one-point topological space. Given a topological space X and a point $x \in X$, we will abuse notation by writing $X\colon X \to \mathrm{pt}$ for the unique map and $x\colon \mathrm{pt} \to X$ for the map which picks out x.

2. Stokes theorem revisited

The family Stokes theorem computes the fiber integral of the exterior derivative of a cohomology class on a family of manifolds with boundary in terms of the fiber integral along the boundary family. The usual formula is not sensitive to cohomological information about the fiber of the family. We correct this defect here by lifting the family Stokes theorem to an equation in the derived category of sheaves on the family's base space, which interpolates between Stokes theorem for the fiber and the family Stokes theorem for the bundle.

2.1. **Recollection I: Stokes theorem.** Let M^d be an oriented d-dimensional smooth compact manifold with boundary ∂M. Restricting differential forms along the inclusion $i\colon \partial M \hookrightarrow M$ defines a map of de Rham chain complexes

$$\Omega^*_{\mathrm{dR}}(M) \to \Omega^*_{\mathrm{dR}}(\partial M).$$

Furthermore, this map is an epimorphism in each degree e.g. by the existence of collar neighborhoods and bump functions. Putting

$$\Omega^*_{\mathrm{dR}}(M, \partial M) = \ker(\Omega^*_{\mathrm{dR}}(M) \to \Omega^*_{\mathrm{dR}}(\partial M)),$$

we get a long exact sequence of de Rham cohomology groups

$$\cdots \xrightarrow{i^*} H^{k-1}_{\mathrm{dR}}(\partial M) \xrightarrow{\delta} H^k_{\mathrm{dR}}(M,\partial M) \to H^k_{\mathrm{dR}}(M) \xrightarrow{i^*} H^k_{\mathrm{dR}}(\partial M) \xrightarrow{\delta} \cdots .$$

The orientation of M induces an orientation of the boundary ∂M, referred to as the *Stokes orientation*, by defining $v_1 \wedge \cdots \wedge v_{d-1} \in \bigwedge^{d-1}(T_x\partial M)$ to be positively-oriented if $u \wedge v_1 \wedge \cdots \wedge v_{d-1} \in \bigwedge^d(T_xM)$ is positively-oriented, where u is an outwards-pointing vector at x, see Figure 1.

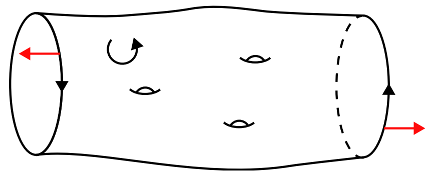

FIGURE 1. The Stokes orientation on the boundary of an oriented surface.

We then have the following avatar of Stokes theorem:

Theorem (Stokes theorem). *If $a \in H^{d-1}_{\mathrm{dR}}(\partial M)$ is a top-degree class, then*

$$\int_M \delta a = \int_{\partial M} a.$$

Remark 2.1. Stokes theorem is often stated in terms of integrating the exterior derivative of a (non-closed) differential form on M. The fact that these statements are equivalent follows from the snake lemma description of the connecting homomorphism δ. To see that the usual formulation implies the statement above, let $a \in H^{d-1}_{\mathrm{dR}}(\partial M)$. Then we can pick a differential form $\alpha \in \Omega^d_{\mathrm{dR}}(\partial M)$ representing a, and a form $\beta \in \Omega^{d-1}_{\mathrm{dR}}(M)$ with $\beta|_{\partial M} = \alpha$. The exterior derivative $d\beta$ is a representative of δa, and the theorem above becomes

$$\int_M d\beta = \int_{\partial M} \alpha = \int_{\partial M} \beta|_{\partial M},$$

which is the usual formulation of Stokes theorem. On the other hand, the usual formulation of Stokes theorem follows from the statement given here by well-definedness of the connecting homomorphism.

Folklore extends Stokes theorem to a family of manifolds parametrized over a smooth base manifold B, see e.g. Problem 4 in [GHV72, Ch VII]. Namely, suppose M is as above and let $p\colon E \to B$ be a smooth submersion with fiber M, equipped with a fiberwise orientation. Let $i\colon \partial^p E \hookrightarrow E$ denote the inclusion of the fiberwise boundary, see Figure 2. Integrating differential forms against the fiberwise orientation defines a *fiber integration* map

$$\int_p \colon H^*_{\mathrm{dR}}(E, \partial^p E) \to H^{*-d}_{\mathrm{dR}}(B).$$

Note that the composition $p\,i$ is again a smooth submersion, and equipping each fiber ∂M_b with the Stokes orientation defines a fiberwise orientation of pi. Thus we have an analogous fiber integration map

$$\int_{pi} \colon H^*_{\mathrm{dR}}(\partial^p E) \to H^{*-d}_{\mathrm{dR}}(B).$$

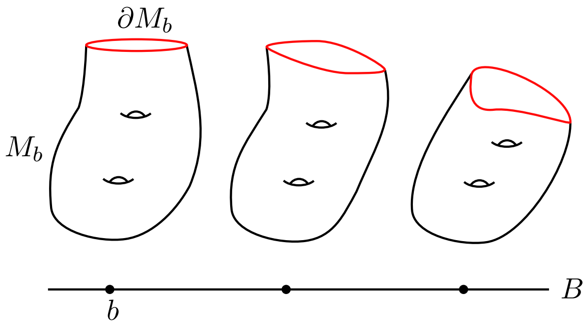

FIGURE 2. Three fibers in a family of surfaces $p\colon E \to B$, with the fiberwise boundary $\partial^p E$ highlighted in red.

The family-version of Stokes theorem relates these integrals:

Theorem (Family Stokes theorem)**.** *For each* $a \in H^k_{\mathrm{dR}}(\partial^p E)$*,*

$$\int_p \delta a = \int_{pi} a.$$

Our goal in the next few subsections is to provide a sheaf-theoretic interpretation of the family Stokes theorem. This perspective has several advantages over the usual approach via differential forms.

First, by de Rham's theorem we can interpret the family Stokes theorem above as having to do with singular cohomology with coefficients in $\mathbb{R}$. Our approach has no bias when it comes to coefficients, and hence we get a version of the family Stokes theorem with coefficients in an arbitrary cohomology theory, provided the family is oriented with respect to this theory. (E.g. we get a family Stokes theorem in stable cohomotopy for families of manifolds equipped with a fiberwise stable framing.)

Second, the sheaf-theoretic perspective works just as well for topological manifolds, or even homology manifolds, as it does for smooth manifolds.

Most importantly (at least for our purposes), we improve the family Stokes theorem even for singular cohomology with coefficients in $\mathbb{R}$. In fact, this is the case which is relevant for the main results of this article.

In order to explain our improvement, note that the usual Stokes theorem can be seen as giving a commutative diagram in the derived category of sheaves on a point:

$$\begin{tikzcd}
\Omega^{*+d}_{\mathrm{dR}}(E, \partial^p E) \arrow[d, "\delta"] \arrow[dr, "\int_p"] & \\
\Omega^{*+d-1}_{\mathrm{dR}}(\partial^p E) \arrow[r, "\int_{pi}"'] & \Omega^*_{\mathrm{dR}}(B),
\end{tikzcd}$$

where δ is the connecting homomorphism associated to the fiber sequence $\Omega^*_{\mathrm{dR}}(E, \partial^p E) \to \Omega^*_{\mathrm{dR}}(E) \xrightarrow{i^*} \Omega^*_{\mathrm{dR}}(\partial^p E)$. We show that this commutative diagram arises as the global sections of a commutative diagram in the derived category of sheaves on the base space of the family. This latter commutative diagram ("sheafy Stokes") is richer than the usual formulation of Stokes theorem. In the opposite direction to taking global sections, taking stalks of our sheafy Stokes theorem recovers the usual Stokes theorem for each individual fiber. The situation is summarized in the following heuristic diagram:

$$\text{Stokes for the fiber} \xleftarrow{\text{stalks}} \text{sheafy Stokes} \xrightarrow{\text{global sections}} \text{family Stokes}$$

The dialectic between the two directions in this diagram allows us to prove new quantitative results about fiber integration in terms of the cohomology of the fiber.

We hope that the sheafy Stokes theorem will be useful beyond the applications in this article. Because of this, we have tried to make the exposition as self-contained as possible.

2.2. **Recollection II: The six operations in topology.** We briefly recall some basic facts about the six-functor formalism for derived sheaves on locally compact Hausdorff spaces. In its original form, this theory is due to Verdier [Ver65]. The six-functor formalism has been enhanced and vastly generalized by Lurie [Lur17, Lur14] and Volpe [Vol23]. We present here a brief summary of the relevant parts of Volpe's article. Textbook references for the classical theory are [KS90, Ive86].

Convention 2.2. For the remainder of this subsection, fix a presentably symmetric monoidal stable category $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$.[4]

Given a topological space X, we consider the category of $\mathcal{C}$-valued sheaves $\mathrm{Shv}(X;\mathcal{C})$. Just as for sheaves valued in a 1-category, the category $\mathrm{Shv}(X;\mathcal{C})$ is defined as the full subcategory of the presheaf category $\mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}},\mathcal{C})$ spanned by those F that satisfy the *sheaf condition*, meaning that for every open $U \subseteq X$ and every open cover $\{V_i\}_{i\in I}$ of U, the canonical map

$$F(U) \to \lim\nolimits_V F(V)$$

is an equivalence, where the limit ranges over open subsets $V \subseteq X$ such that $V \subseteq V_i$ for some $i \in I$. The inclusion $\mathrm{Shv}(X;\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}},\mathcal{C})$ has a left adjoint, which we call *sheafification.*

Remark 2.3. If X is a sufficiently nice topological space (e.g. a CW complex) and $\mathcal{C} = \mathrm{Mod}_k$ for some ordinary commutative ring $k \in \mathrm{CAlg}(\mathrm{Ab})$, then the category $\mathrm{Shv}(X;\mathrm{Mod}_k)$ is canonically equivalent to the derived category $D\,\mathrm{Shv}(X;\mathrm{Mod}_k^{\heartsuit})$ of the category of sheaves of ordinary k-modules [Sch23, Prop 7.1]. In particular, the triangulated homotopy category of $\mathrm{Shv}(X;\mathrm{Mod}_k)$ is equivalent to the triangulated category of derived sheaves considered by pre-Lurie authors.

Observe that a map of topological spaces $f\colon X \to Y$ gives rise to a map of posets $f^{-1}\colon \mathrm{Open}(Y) \to \mathrm{Open}(X)$, defined by $U \mapsto f^{-1}(U)$. Restriction along f^{-1} defines a functor on presheaf categories $\mathrm{Fun}(\mathrm{Open}(Y)^{\mathrm{op}},\mathcal{C}) \to \mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}},\mathcal{C})$. This functor preserves the sheaf condition, and hence restricts to a *pushforward* functor

$$f_*\colon \mathrm{Shv}(X;\mathcal{C}) \to \mathrm{Shv}(Y;\mathcal{C}).$$

[4] For instance, one can take $\mathcal{C}$ to be the derived category $\mathrm{Mod}_{\mathbb{Q}}$ or $\mathrm{Mod}_{\mathbb{Z}}$, or the category of spectra Sp.

For formal reasons this functor admits a left adjoint

$$f^*\colon \operatorname{Shv}(Y;\mathcal{C}) \to \operatorname{Shv}(X;\mathcal{C}),$$

which we refer to as *pullback*. We summarize the situation in the diagram

$$\begin{tikzcd} \operatorname{Shv}(X;\mathcal{C}) \arrow[r, shift right, "f_*"'] \arrow[r, phantom, "\perp"] & \operatorname{Shv}(Y;\mathcal{C}). \arrow[l, shift right, "f^*"'] \end{tikzcd} \tag{2.1}$$

If X and Y are both locally compact Hausdorff spaces, Verdier realized that the adjunction (2.1) is complemented by a further adjunction

$$\begin{tikzcd} \operatorname{Shv}(X;\mathcal{C}) \arrow[r, shift left, "f_!"] \arrow[r, phantom, "\perp"] & \operatorname{Shv}(Y;\mathcal{C}), \arrow[l, shift left, "f^!"] \end{tikzcd} \tag{2.2}$$

where $f_!$ and $f^!$ are the so-called *exceptional pushforward* and *exceptional pullback*, respectively. The exceptional pushforward $f_!F$ of a sheaf $F \in \operatorname{Shv}(X;\mathcal{C})$ is defined by sending an open subset $U \subseteq Y$ to the sections of F on $f^{-1}(U)$ having compact support, in a way that can be made precise. The functor $f^!$ is more mysterious, but will be described under the assumptions relevant for our work in § 2.3 below. However, when X is a locally compact Hausdorff space having the homeomorphism type of a CW complex[5], we have the following useful description of the stalks of the dualizing sheaf:

Proposition 2.4. *Let X be a locally compact Hausdorff space having the homeomorphism type of a CW complex, and suppose that $\mathcal{C} = \operatorname{Mod}_R$ is the category of modules over some commutative ring spectrum $R \in \operatorname{CAlg}(\mathrm{Sp})$. For each $x \in E$, the stalk $x^*\omega_p$ is equivalent to the local homology*

$$C_*(X, X \setminus \{x\}; R) = \operatorname{cofib}\left(\Sigma^\infty_+ X \setminus \{x\} \to \Sigma^\infty_+ X\right) \otimes R.$$

Proof. This is a straightforward argument using the recollement fiber sequences, see [KNP24, Rem 4.6.19]. $\square$

Crucially, the invariants classically studied by topologists can be described in terms of these functors in the special case where $f = X\colon X \to \mathrm{pt}$ is the projection to a point, we have the following dictionary:

cohomology	X_*X^*M
homology	$X_!X^!M$
compactly supported cohomology	$X_!X^*M$
Borel–Moore homology	$X_*X^!M$

where everything is understood to have coefficients in some $M \in \mathcal{C} = \operatorname{Shv}(\mathrm{pt};\mathcal{C})$.

Notation 2.5. Given a topological space X and an object M, we write

$$M_X = X^*M \in \operatorname{Shv}(X;\mathcal{C}),$$

and refer to this sheaf as the *constant sheaf (on X with value M)*.

[5]Equivalently, assume X has the homeomorphism type of a locally finite CW complex.

Notation 2.6. Let X be a locally compact Hausdorff space. We refer to the functors X_* and $X_!\colon \mathrm{Shv}(X;\mathcal{C}) \to \mathrm{Shv}(\mathrm{pt};\mathcal{C}) \simeq \mathcal{C}$ as *global sections* and *compactly supported global sections*, respectively. We also write

$$\Gamma(X,-) = X_* \quad \text{and} \quad \Gamma_c(X,-) = X_!.$$

Note that by definition $\Gamma(X,F) = F(X)$. We also write

$$\Gamma(U,-) = X_*j^* \quad \text{and} \quad \Gamma_c(X,-) = X_!j^*$$

for $j\colon U \hookrightarrow X$ the inclusion of some open subset.

If $x\colon \mathrm{pt} \to X$ is the inclusion of a point, we refer to the functors x^* and $x^!\colon \mathrm{Shv}(X;\mathcal{C}) \to \mathrm{Shv}(\mathrm{pt};\mathcal{C}) \simeq \mathcal{C}$ as the *stalk* and *costalk (at x)*, respectively.

Remark 2.7. To be precise, the invariants in the left column above should be prepended by the word "sheaf". For a general locally compact Hausdorff space, sheaf cohomology does not agree with singular cohomology, although there is always a comparison map. This comparison is an equivalence under mild assumptions on the space, see e.g. [Pet22] for for the case where $\mathcal{C} = \mathrm{Mod}_k$ is the derived category of an ordinary ring k. Since all the spaces that we care about in the present article have the homeomorphism types of CW complexes, it follows from Lurie's monodromy theorem [Lur17, Thm A.1.15] that their sheaf-theoretical invariants agree with their singular invariants with arbitrary coefficients, or in other words for $\mathcal{C} = \mathrm{Sp} = \mathrm{Mod}_{\mathbb{S}}$.

Most properties of the functors defined above are captured by saying that they assemble into a six-functor formalism, as formalized by Liu–Zheng [LZ24] (see also [Man22] and [Sch23]). In order to state this, recall that a category $\mathcal{T}$ with finite limits has an associated *category of correspondences* $\mathrm{Corr}(\mathcal{T})$. This category has the same objects as $\mathcal{T}$, but morphisms from X to $Y \in \mathcal{T}$ are instead given by correspondences

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

in $\mathcal{T}$, with composition given by pullback

$$\begin{tikzcd} & & W \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \\ & Z \arrow[dl] \arrow[dr] & & Z' \arrow[dl] \arrow[dr] & \\ X & & Y & & Y' \end{tikzcd},$$

i.e. the composition of the two correspondences $X \leftarrow Z \rightarrow Y$ and $Y \leftarrow Z' \rightarrow Y'$ is given by the outer correspondence $X \leftarrow W \rightarrow Y'$. Note that if $\mathcal{T}$ is a 1-category, the category of correspondences will be a 2-category since pullback is only defined up to canonical equivalence. The cartesian symmetric monoidal structure induces a symmetric monoidal structure on $\mathrm{Corr}(\mathcal{T})$, which we also write $\times$.

A ($\mathcal{C}$-linear symmetric monoidal[6]) *six-functor formalism* on $\mathcal{T}$ is a a symmetric monoidal functor

$$D\colon \operatorname{Corr}(\mathcal{T})^{\times} \to \operatorname{Mod}_{\mathcal{C}}(\operatorname{Pr}^L_{\mathrm{st}})^{\otimes_{\mathcal{C}}},$$

where $\otimes_{\mathcal{C}}$ is the $\mathcal{C}$-linear Lurie tensor product.

Theorem 2.8 (Lurie [Lur14], Volpe [Vol23])**.** *Let* LCHaus *denote the* 1-*category of locally compact Hausdorff spaces and continuous maps. There is a six-functor formalism*

$$\operatorname{Shv}(-;\mathcal{C})\colon \operatorname{Corr}(\mathrm{LCHaus})^{\times} \to \operatorname{Mod}_{\mathcal{C}}(\operatorname{Pr}^L_{\mathrm{st}})^{\otimes_{\mathcal{C}}}$$

given on objects by $X \mapsto \operatorname{Shv}(X;\mathcal{C})$ *and on morphisms by sending a correspondence*

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd} \mapsto g_! f^*\colon \operatorname{Shv}(X;\mathcal{C}) \to \operatorname{Shv}(Y;\mathcal{C}).$$

Furthermore,

(1) *There is a canonical natural transformation* $f_! \to f_*$ *which is an equivalence if* f *is proper.*

(2) *Similarly, there is a canonical equivalence* $f^* \simeq f^!$ *if* f *is an open inclusion.*

(3) *Let* $\operatorname{Shv}^*(-;\mathcal{C})\colon \mathrm{LCHaus}^{\mathrm{op}} \to \operatorname{Mod}_{\mathcal{C}}\left(\operatorname{Pr}^L_{\mathrm{st}}\right)$ *denote the restriction of* $\operatorname{Shv}(-;\mathcal{C})$ *to the wide subcategory of* $\operatorname{Corr}(\mathrm{LCHaus})$ *which only has morphisms of the form*

$$\begin{tikzcd} & X \arrow[dl, "f"] \arrow[dr, equal] & \\ Y & & X. \end{tikzcd}$$

Then $\operatorname{Shv}^*(-;\mathcal{C})$ *satisfies descent with respect to open covers, in the sense that for every open cover* $\{U_i\}_{i\in I}$ *of a space* X*, the canonical map*

$$\operatorname{Shv}(X;\mathcal{C}) \to \lim_V \operatorname{Shv}^*(U;\mathcal{C})$$

is an equivalence, where the limit ranges over open subsets $U \subseteq X$ *such that* $U \subseteq U_i$ *for some* $i \in I$.

(4) *Let* X *be a locally compact Hausdorff space and let* $Z \subseteq X$ *be a closed subspace. Let* $i\colon Z \hookrightarrow X$ *denote the inclusion, and let* $j\colon U \hookrightarrow X$ *denote the inclusion of the open complement* $U = X \setminus Z$*. The pushforwards* i_* *and* j_* *are both fully faithful and exhibit* $\operatorname{Shv}(X;\mathcal{C})$ *as the recollement of* $\operatorname{Shv}(Z;\mathcal{C})$ *and* $\operatorname{Shv}(U;\mathcal{C})$ *in the sense of* [Lur17, Sec A.8]. *In particular, there are fiber sequences*

$$j_! j^* F \to F \to i_* i^* F \quad \textit{and} \quad i_* i^! F \to F \to j_* j^* F,$$

where the maps are the units and counits of the obvious adjunctions.

(5) *The canonical map* $f^!(\mathbf{1}) \otimes f^* \to f^!$ *is an equivalence if* f *is a fiber bundle whose fiber has the homeomorphism type of a CW complex, or more generally if* f *is a shape submersion in the sense of* [Vol23, Defn 3.21].

[6] A general six-functor formalism is only required to be lax symmetric monoidal, to encompass the case of etale sheaves.

Notation 2.9. Given a map $f\colon X \to Y$ of locally compact Hausdorff spaces, we write

$$\omega_f = f^! \mathbf{1} \in \operatorname{Shv}(X;\mathcal{C}),$$

and refer to this sheaf as the *relative dualizing sheaf (of f)*. If $f = X\colon X \to \mathrm{pt}$ is the projection to a point, we refer to ω_X simply as the *dualizing sheaf (of X)*.

Remark 2.10 (Base change and projection formula)**.** We will frequently use the following two consequences of the theorem above. Suppose

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g'"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

is a pullback diagram in the category of locally compact Hausdorff spaces. We can also interpret this as a composition

$$\begin{tikzcd} & & X \arrow[dl, "g"'] \arrow[dr, "f"] \arrow[dd, phantom, "\vee", very near start] & & \\ & X' \arrow[dl, equal] \arrow[dr, "f'"] & & Y \arrow[dl, "g'"] \arrow[dr, equal] & \\ X' & & Y' & & Y' \end{tikzcd},$$

Functoriality now gives a canonical equivalence

$$(g')^* f'_! \simeq f_! g^*.$$

There is an analogous equivalence for pushforward and exceptional pullback. These equivalences are collectively known as *base change*. Note that if f is proper we recover Lurie's proper base change theorem $(g')^* f'_* \simeq f_* g^*$ [Lur09].

The correspondences

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr, equal] & \\ \mathrm{pt} & & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & X \times X \arrow[dl] \arrow[dr, equal] & \\ X & & X \times X \end{tikzcd}$$

endow X with the structure of an algebra object in the category of correspondences. (In fact, the symmetry of the category of correspondences imply that X is canonically a *Frobenius algebra* in this category, but we will not go into this here.) From the theorem above, it therefore follows that $\operatorname{Shv}(X;\mathcal{C})$ is canonically a $\mathcal{C}$-algebra, i.e. it has a symmetric monoidal structure compatible with the $\mathcal{C}$-linear structure. This is the so-called *pointwise tensor product* on sheaves, given informally by defining the tensor product $F \otimes G$ of sheaves F and $G \in \operatorname{Shv}(X;\mathcal{C})$ to be the sheafification of the presheaf $U \mapsto F(U) \otimes G(U)$, for $U \subseteq X$ open.

Given a map $f\colon X \to Y$ in $\mathcal{C}$, the correspondence

$$\begin{tikzcd} & X \arrow[dl, "f"] \arrow[dr, equal] & \\ Y & & X \end{tikzcd} \tag{2.3}$$

defines a map of algebras in the category of correspondences. Hence the pullback functor $f^*\colon \mathrm{Shv}(Y;\mathcal{C}) \to \mathrm{Shv}(X;\mathcal{C})$ has a canonical symmetric monoidal structure. The map of algebras (2.3) also endows X with a Y-module structure, and with respect to this structure the correspondence

$$\begin{tikzcd} & X \arrow[dl, equal] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

is a map of Y-modules. Symmetric monoidality of our six-functor formalism then means that the pushforward $f_!\colon \mathrm{Shv}(X;\mathcal{C}) \to \mathrm{Shv}(Y;\mathcal{C})$ is a map of $\mathrm{Shv}(Y;\mathcal{C})$-modules. In particular, we have a canonical equivalence

$$f_!(f^*F \otimes G) \simeq F \otimes f_!G$$

for sheaves $F \in \mathrm{Shv}(X;\mathcal{C})$ and $G \in \mathrm{Shv}(Y;\mathcal{C})$; this equivalence is called the *projection formula*.

Remark 2.11. By the definition of $\mathrm{Mod}_{\mathcal{C}}\left(\mathrm{Pr}^L_{\mathrm{st}}\right)$, a commutative algebra $\mathcal{C}^{\otimes} \in \mathrm{CAlg}\left(\mathrm{Mod}_{\mathcal{C}}\left(\mathrm{Pr}^L_{\mathrm{st}}\right)\right)$ also has an internal Hom, i.e. a right adjoint to the tensor product. We denote the internal Hom corresponding to the pointwise tensor product on $\mathrm{Shv}(X;\mathcal{C})$ by $\underline{\mathrm{Hom}}_X$.

The projection formulae from the previous remark then give rise to canonical equivalences

$$f_*\underline{\mathrm{Hom}}_X(F, f^!G) \simeq \underline{\mathrm{Hom}}_Y(f_!F, G), \qquad \text{and} \tag{2.4}$$

$$f^!\underline{\mathrm{Hom}}_Y(G, G') \simeq \underline{\mathrm{Hom}}_X(f^*G, f^!G') \tag{2.5}$$

for a map $f\colon X \to Y$ and sheaves G and $G' \in \mathrm{Shv}(Y;\mathcal{C})$ and $F \in \mathrm{Shv}(X;\mathcal{C})$ [Vol23, Prop 6.12].

2.3. **Orientations.** In order to have a well-defined notion of integration, one first needs to choose some orientation. In the setting of sheaves, this is captured by the following definition:

Definition 2.12. Let $W^d \to E \xrightarrow{p} B$ be a fiber bundle, where W is a topological manifold with $\partial W = \varnothing$. Assume that B is locally compact Hausdorff. A *$\mathcal{C}$-orientation* of p is an equivalence

$$\Sigma^d \mathbf{1}_E \xrightarrow{\sim} \omega_p.$$

If on the other hand $\partial W \neq \varnothing$, a $\mathcal{C}$-orientation is defined to be an orientation of the associated interior bundle $W \setminus \partial W \to E \setminus \partial^p E \to B$.

If $\mathcal{C} = \mathrm{Mod}_R$ is the category of modules over a commutative ring spectrum $R \in \mathrm{CAlg}(\mathrm{Sp})$, we will abbreviate by referring to a Mod_R-orientation simply as an *R-orientation*.

In the remainder of this section, we will compare the definition given above with classical notions of orientation.

2.3.1. *Smooth submersions.* In the smooth case, recall that a smooth submersion $p\colon E \to B$ has an associated vertical tangent bundle T^pE on E, defined by the exact sequence

$$0 \to T^pE \to TE \xrightarrow{Dp} p^*(TB) \to 0$$

of vector bundles. That is, $T^p_x E \subseteq T_x E$ is the subspace consisting of vectors that are tangent to the fiber. Classically, the notion of an orientation of p is defined in terms of the vertical tangent bundle $T^p E$. The equivalence between the classical notion of orientation and Definition 2.12 is then provided by Volpe's version of relative Atiyah duality, which we state here for completeness.

Before stating Volpe's theorem, we need will need a construction. Given a paracompact Hausdorff space X, Volpe defines a *Thom functor*

$$\mathrm{Th}\colon \mathrm{Vect}_X^{\mathrm{gp}} \to \mathrm{Pic}\left(\mathrm{Shv}(X;\mathrm{Sp})\right),$$

natural in X [Vol23]. Here Vect_X is the symmetric monoidal groupoid of real vector bundles over X (under direct sum) and $\mathrm{Vect}_X^{\mathrm{gp}}$ is its group completion, and $\mathrm{Pic}\left(\mathrm{Shv}(X;\mathrm{Sp})\right) \subseteq \mathrm{Shv}(X;\mathrm{Sp})$ is the Picard subgroupoid of the sheaf category, i.e. the (non-full) subcategory on invertible sheaves (with respect to the pointwise tensor product) and equivalences between these. Informally, the Thom functor is defined by sending a vector bundle classified by a map $X \to BGL_d(\mathbb{R}) \simeq BO(d)$ to its underlying stable spherical fibration.

The following is [Vol23, Thm 7.11]:

Theorem 2.13 (relative Atiyah duality)**.** *Let $p\colon E \to B$ be a smooth submersion between smooth closed manifolds.*

There is a canonical equivalence

$$\omega_p \simeq \mathrm{Th}(-T^p E) \otimes \mathbf{1}_E.$$

It is well-known that the notion of vertical tangent bundles

2.3.2. *Bundles of topological manifolds.* We now deal more generally with the topological case. Here the classical notion of an orientation can also be phrased in terms of vertical tangent bundles.

Definition 2.14. Let $W \to E \xrightarrow{p} B$ be a fiber bundle, where W is a topological manifold. The *vertical tangent microbundle* of p, denoted $T^p E$, is the microbundle

$$(E \times_B E, \pi_1\colon E \times_B E \to E, \Delta\colon E \hookrightarrow E \times_B E), \tag{2.6}$$

where π_1 is the projection onto the first factor and Δ is the diagonal inclusion.

By the Kister–Mazur, the vertical tangent microbundle $T^p E$ corresponds to an essentially unique $\mathbb{R}^d$-bundle on E, where d is the dimension of W [Kis64]. Namely, there is an open neighborhood U of the diagonal in $E \times_B E$ such that the restriction $\pi_1\colon U \to E$ is an $\mathbb{R}^d$ bundle, with zero section given by Δ.

The key to comparing Definition 2.12 with classical notions of orientations is the following calculation, done for smooth s:

Theorem 2.15. *Let $W \to E \xrightarrow{p} B$ be a fiber bundle, where W is a topological manifold. Let $\Delta\colon E \hookrightarrow E \times_B E$ be the diagonal inclusion.*

- *(a) There is a canonical equivalence $\underline{\mathrm{Hom}}(\omega_p, \mathbf{1}_E) \simeq \Delta^! \mathbf{1}_{E \times_B E}$.*
- *(b) If $\partial W = \varnothing$, then ω_p is invertible and $\omega_p^{-1} = \Delta^! \mathbf{1}_{E \times_B E}$.*

Proof. Let $\pi_2\colon E \times_B E \to E$ denote the projection onto the second factor. We start by proving (a). By [Vol23, Lem 3.24], the pullback functors p^* and π_2^* both have left adjoints, which we denote $p_\#$ and $(\pi_2)_\#$ respectively. In the language of [HM24], the map p is suave. Lemma 4.5.6 in [HM24] gives an equivalence

$$\omega_p \simeq (\pi_2)_\# \Delta_! \mathbf{1}.$$

(This is a completely formal calculation, true for a suave map in any six-functor formalism.) Using the projection formula (see Remark 2.11) and the smooth projection formula [Vol23, Cor 3.26], we then calculate

$$\begin{aligned} \underline{\mathrm{Hom}}((\pi_2)_\# \Delta_! \mathbf{1}_E, \mathbf{1}_E) &\simeq (\pi_2)_* \underline{\mathrm{Hom}}(\Delta_! \mathbf{1}_E, \pi_2^* \mathbf{1}_E) \\ &\simeq (\pi_2)_* \Delta_* \underline{\mathrm{Hom}}(\mathbf{1}_E, \Delta^! \mathbf{1}_{E\times_B E}) \\ &\simeq \Delta^! \mathbf{1}_{E\times_B E}, \end{aligned}$$

where the last equivalence uses that $\pi_2 \Delta = \mathrm{id}$.

In (b), the claim that $\omega_p = p^! \mathbf{1}$ is invertible is local on B, and for this we may therefore assume that p is a trivial fiber bundle. But then we may even assume that p is the projection from W to a point by [Vol23, Prop 6.16]. The statement is again local on W, and thus follows from the case $W = \mathbb{R}^d$, where it is [Vol23, Prop 6.18]. But if ω_p is invertible, then its inverse must be given by $\underline{\mathrm{Hom}}(\omega_p, \mathbf{1}_E)$. □

Remark 2.16. Let X be a locally compact Hausdorff space. A sheaf $F \in \mathrm{Shv}(X; \mathcal{C})$ is invertible if and only if F is locally constant and has invertible stalks. This is due to [MW22] if $\mathcal{C} = \mathrm{Mod}_R$ is the category of modules over a commutative ring spectrum $R \in \mathrm{CAlg}(\mathrm{Sp})$, and to [Har25a] in the general case.

Corollary 2.17. *Let $W \to E \xrightarrow{p} B$ be a fiber bundle, where W is a topological manifold without boundary. Let $\Delta\colon E \hookrightarrow E \times_B E$ denote the diagonal inclusion. There is a canonical equivalence*

$$\mathrm{Map}^{\simeq}(\Sigma^d \mathbf{1}_E, \omega_p) \simeq \mathrm{Map}^{\simeq}(\Delta^! \mathbf{1}_{E\times_B E}, \Sigma^{-d} \mathbf{1}_E),$$

where we denote by $\mathrm{Map}^{\simeq}(F, G) \subseteq \mathrm{Map}(F, G)$ the subspace spanned by equivalences. That is, the space of orientations of p is canonically equivalent to the space of equivalences $\Delta^! \mathbf{1}_{E\times_B E} \xrightarrow{\sim} \Sigma^{-d} \mathbf{1}_E$.

Remark 2.18. An equivalence $\Delta^! \mathbf{1}_{E\times_B E} \xrightarrow{\sim} \Sigma^{-d} \mathbf{1}_E$ is the sheaf-theoretic incarnation of a Thom class for the microbundle $T^p E$. Indeed, assume that B has the homeomorphism type of a CW complex and that $\mathcal{C} = \mathrm{Mod}_R$ is the category of modules over a commutative ring spectrum $R \in \mathrm{CAlg}(\mathrm{Sp})$. An argument using recollements (as in [KNP24, Rem 4.6.19]) and Lurie's monodromy theorem [Lur17, Thm A.1.15] shows that $\Delta^! \mathbf{1}_{E\times_B E}$ is the sheafification of the presheaf

$$U \mapsto C^*(p^{-1}(U), p^{-1}(U) \setminus \big(\Delta(E) \cap p^{-1}(U)\big); R), \tag{2.7}$$

see e.g. [Ive86, p 333]. By excision, we can replace $E \times_B E$ with the neighborhood $V \supset \Delta(E)$ supplied by the Kister–Mazur theorem, and the sheafification of (2.7) is then the local system given informally by $b \mapsto C_*(V_b, V_b \setminus \{b\}; R)$; an equivalence between this local system and $\Sigma^{-d} \mathbf{1}_E$ is precisely a Thom class for the $\mathbb{R}^d$-bundle $\pi_1\colon V \to E$.

It follows from this description that a $\mathbb{Z}$-orientation of a fiber bundle p (as in Definition 2.12) is the same as an orientation of p in the usual sense, i.e. a coherent choice of orientations of the vertical tangent spaces. On the other hand, an $\mathbb{S}$-orientation is a fiberwise stable framing of p, or in other words a reduction of the structure group for the map $E \to B\,\mathrm{Top}(d) \to BG$ classifying the underlying stable spherical fibration of T^pE along the map $BSG \to BG$, where $B\,\mathrm{Top}(d)$, $BG = \mathrm{colim}_n\, B\,\mathrm{hAut}(S^n)$, and $BSG = \mathrm{colim}_n\, B\,\mathrm{hAut}^+(S^n)$ are the classifying spaces for $\mathbb{R}^d$-bundles, stable spherical fibrations, and oriented stable spherical fibrations respectively.

2.4. **Integration.** We can now define the notion of integration with coefficients in the category $\mathcal{C}$.

Construction 2.19. Let W^d be a closed topological manifold, and let $W \to E \xrightarrow{p} B$ be a fiber bundle such that B is locally compact Hausdorff. Suppose that
$$\theta\colon \Sigma^d \mathbf{1}_E \xrightarrow{\sim} \omega_p$$
is an orientation of p. We define the map
$$\int_p d\theta\colon p_*\Sigma^d\mathbf{1}_E \to \mathbf{1}_B \tag{2.8}$$
to be the composite of $p_*\theta\colon p_*\Sigma^d\mathbf{1}_E \xrightarrow{\sim} p_*\omega_p = p_*p^!\mathbf{1}_B$ and the counit $p_*p^!\mathbf{1}_B \to \mathbf{1}_B$, where we have used that W is compact (or in other words that p is proper) to get a canonical identification $p_! \xrightarrow{\sim} p_*$.

By abuse of notation, we also let
$$\int_p d\theta\colon \Sigma^d E_*\mathbf{1}_E \to B_*\mathbf{1}_B \tag{2.9}$$
denote the map defined by applying the global sections functor $\Gamma(B, -) = B_*$ to (2.8).

Remark 2.20. Given an oriented fiber bundle $W \to E \xrightarrow{p} B$ with W a closed topological manifold and B a space having the homeomorphism type of a CW complex, there is a fiber integration map
$$\int_p\colon H^*(E;\mathbb{Z}) \to H^{*-d}(B;\mathbb{Z}),$$
generalizing the fiber integration in de Rham cohomology for smooth submersions [Gri17, Defn 3.5], see also [Boa70]. If B is also locally compact (i.e. if it has the homeomorphism type of a locally finite CW complex), this map agrees with the one we have defined. Indeed, Grigoriev defines fiber integration using the Leray–Serre spectral sequence, which is the Grothendieck spectral sequence for the composition of $B_* = \Gamma(B, -)$ and p_*. Passing to parametrized homotopy groups in our construction exactly recovers his definition.

In the non-closed case, we need the following lemma:

Lemma 2.21. *Let W^d be a manifold with boundary, and let $W \to E \xrightarrow{p} B$ be a smooth fiber bundle such that B is locally compact Hausdorff. Let*

$j\colon E \setminus \partial^p E \hookrightarrow E$ denote the inclusion of the fiberwise interior. Then the counit map

$$j_! j^* \omega_p \to \omega_p$$

is an equivalence.

Proof. By Theorem 2.8, there is a fiber sequence

$$j_! j^* p^! \mathbf{1}_B \to p^! \mathbf{1}_B \to i_* i^* p^! \mathbf{1}_B,$$

where $i\colon \partial^p E \hookrightarrow E$ is the inclusion of the fiberwise boundary. We must show that $i_* i^* p^! \mathbf{1}_B \simeq 0$. Since E has the homeomorphism type of a CW complex, it follows from [Hoy16] that it suffices to show that the stalk $x^* i_* i^* p^! \mathbf{1}_B$ vanishes for every $x \in E$. By proper base change, it is enough to show that $x^* p^! \mathbf{1}_B \simeq 0$ for every $x \in \partial^p E$. This is a local condition, so we can assume that $p = p_1\colon B \times W \to B$ is a projection. Then $p^! \mathbf{1}_B \simeq p_2^*(\omega_W)$ by [Vol23, Prop 6.16]. Thus we are reduced to the absolute statement $x^* \omega_W$ for $x \in \partial W$. Since this statement is local, we may even assume that

$$W = \mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\} \quad \text{and} \quad x = 0.$$

But the statement now follows from Proposition 2.4, since $\mathbb{H}^n \setminus \{0\} \hookrightarrow \mathbb{H}^n$ is a homotopy equivalence. □

Remark 2.22. In the absolute case where $p = W\colon W \to \mathrm{pt}$ is the projection to a point, the lemma above recovers the useful calculation $\omega_W \simeq j_! \omega_{W \setminus \partial W}$, where $j\colon W \setminus \partial W \hookrightarrow W$ is the inclusion of the interior [Ive86, p. 298]. That is, the dualizing sheaf of W is the dualizing sheaf of its interior extended by zero to all of W.

Using the previous lemma, we can now define fiber integration for a family of manifolds with boundary.

Construction 2.23. Let W^d be a compact topological manifold with boundary, and let $W \to E \xrightarrow{p} B$ be a fiber bundle such that B is locally compact Hausdorff.

Let $j\colon E \setminus \partial^p E \hookrightarrow E$ denote inclusion of the fiberwise interior, and suppose that

$$\theta\colon \Sigma^d \mathbf{1}_{E \setminus \partial^p E} \xrightarrow{\sim} \omega_{pj}$$

is an orientation of the interior bundle pj. We define the map

$$\int_p d\theta\colon p_* j_! \Sigma^d \mathbf{1}_E \to \mathbf{1}_B \tag{2.10}$$

to be the composite of

$$p_* j_! \Sigma^d \mathbf{1}_E \xrightarrow[p_* j_! \theta]{\sim} p_* j_! \omega_{pj} \xrightarrow[\text{counit}]{\sim} p_* \omega_p$$

and the counit $p_* \omega_p = p_* p^! \mathbf{1}_B \to \mathbf{1}_B$, where we have also used that p is proper to get a canonical identification $p_! \xrightarrow{\sim} p_*$.

By abuse of notation, we also let

$$\int_p d\theta\colon \Sigma^d E_* j_! \mathbf{1}_E \to B_* \mathbf{1}_B \tag{2.11}$$

denote the map defined by applying the global sections functor $\Gamma(B, -) = B_*$ to (2.10).

Remark 2.24. Given $b \in B$, the proper base change lets us identify the stalk b^* of the map (2.10) with a map

$$\Sigma^d \Gamma(W_b, j'_! \mathbf{1}_{W_b \setminus \partial W_b}) \to \mathbf{1},$$

where $j' \colon W_b \setminus \partial W_b \hookrightarrow W_b$ is the inclusion of the interior. If $\mathcal{C} = \mathrm{Mod}_{\mathbb{R}}$ is the derived category of $\mathbb{R}$, Lurie's monodromy theorem [Lur17, Thm A.1.15] lets us identify this with a map

$$\Sigma^d C^*(W_b, \partial W_b; \mathbb{R}) \to \mathbb{R}$$

implementing integration for W_b. The same theorem lets us identify (2.11) with a map

$$\Sigma^d C^*(E, \partial^p E; \mathbb{R}) \to C^*(B; \mathbb{R})$$

implementing fiber integration.

2.5. **The sheafy Stokes theorem.** In the classical (family) Stokes theorem described in § 2.1, one must first define the Stokes orientation on the boundary, and then prove the theorem by calculating integrals (essentially by repeated application of the fundamental theorem of calculus). In our sheafy Stokes theorem, all the mathematical work goes towards defining the sheaf-theoretic Stokes orientation, which then gives rise to our sheafy Stokes theorem by construction.

Let X be a topological space, and let $i \colon Z \hookrightarrow X$ and $j \colon U \hookrightarrow X$ denote the inclusions of a closed subspace Z and its complement $U = X \setminus Z$. We have the recollement

$$\begin{tikzcd} \mathrm{Shv}(Z;\mathcal{C}) \arrow[r, "i_*"] & \mathrm{Shv}(X;\mathcal{C}) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathrm{Shv}(U;\mathcal{C}). \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} \tag{2.12}$$

In particular, we have the functors

$$i^* j_* \quad \text{and} \quad i^! j_! \colon \mathrm{Shv}(U;\mathcal{C}) \to \mathrm{Shv}(Z;\mathcal{C}).$$

Conceptually, the gluing functor $i^* j_*$ is a sheaf-theoretic incarnation of taking the closure of a subset $A \subseteq U$ inside X, and then intersecting with Z. The functor $i^! j_!$ is more mysterious, but is related to the procedure of interior multiplication by a normal tensor field to Z. We will use this functor to produce the Stokes orientation.

Convention 2.25. Throughout this subsection, let $W^d \to E \xrightarrow{p} B$ be a fiber bundle where W is a compact topological manifold and B is locally compact Hausdorff, and let $i \colon \partial^p E \hookrightarrow E$ and $j \colon E \setminus \partial^p E \hookrightarrow E$ denote the inclusions of its fiberwise boundary and interior, respectively.

Lemma 2.26. *If $F \in \mathrm{Shv}(E;\mathcal{C})$ is locally constant, then*

$$i^! F \simeq 0.$$

Proof. Since the statement is local on E, we may suppose that $p = p_1 \colon B \times W \to B$ is a trivial fiber bundle. But then by [Vol23, Prop 6.16] we are reduced to the absolute case $p = W \colon W \to \mathrm{pt}$.

We have the recollement fiber sequence

$$i_* i^! F \to F \to j_* j^* F,$$

so it will suffice to show that $F \to j_* j^* F$ is an equivalence. For this it suffices to observe that the restriction map

$$\Gamma(U, F) \to \Gamma(U, j_* j^* F) = \Gamma(U \setminus \partial W, F)$$

is an equivalence for every open subset $U \subseteq W$. By the sheaf condition, it suffices to show this for small U, whence we are reduced to considering the restriction map

$$\Gamma(\mathbb{H}^n, F) \to \Gamma(\mathbb{H}^n \setminus \partial \mathbb{H}^n, F)$$

where $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. Here the statement follows from homotopy invariance of locally constant sheaves (see Appendix A in [Lur17]) and the fact that $\mathbb{H}^n \setminus \partial \mathbb{H}^n \hookrightarrow \mathbb{H}^n$ is a homotopy equivalence. □

Lemma 2.27. *Let $F \in \operatorname{Shv}(E; \mathcal{C})$ be a locally constant sheaf on E, and let*

$$\delta \colon \Sigma^{-1} i_* i^* F \to j_! j^* F$$

denote the connecting homomorphism associated to the recollement fiber sequence $j_! j^* F \to F \to i_* i^* F$.

The exceptional pullback

$$i^!(\delta) \colon \Sigma^{-1} i^! i_* i^* F \to i^! j_! j^* F$$

is an equivalence.

Proof. We have the fiber sequence

$$\Sigma^{-1} i^! i_* i^* F \xrightarrow{i^!(\delta)} i^! j_! j^* F \to i^! F.$$

The previous lemma shows that $i^! F \simeq 0$, proving that $i^!(\delta)$ is an equivalence as desired. □

Construction 2.28. Suppose $\theta \colon \Sigma^d \mathbf{1}_{E \setminus \partial^p E} \xrightarrow{\sim} \omega_{pj}$ is an orientation of p. We define the associated *Stokes orientation* of the boundary bundle $pi \colon \partial^p E \to B$ to be the equivalence

$$\theta_s \colon \Sigma^{d-1} \mathbf{1}_{\partial^p E} \xrightarrow{\sim} \omega_{pi}$$

defined as the composite

$$\Sigma^{d-1} \mathbf{1}_{\partial^p E} \xrightarrow[i^!(\delta)]{\sim} \Sigma^d i^! j_! \mathbf{1}_{E \setminus \partial^p E} \xrightarrow[i^! j_!(\theta)]{\sim} i^! j_! \omega_{pj} \xrightarrow[i^!(\text{counit})]{\sim} \omega_{pi},$$

where $i^!(\delta)$ is an equivalence by Lemma 2.27 and the rightmost map is an equivalence by Lemma 2.21.

Theorem 2.29 (Sheafy Stokes)**.** *Suppose θ is an orientation of the fiber bundle $W \to E \xrightarrow{p} B$, and let θ_s denote the associated Stokes orientation of the boundary bundle $pi \colon \partial^p E \to B$.*

There is a commutative diagram

$$\begin{tikzcd} \Sigma^{d-1} p_* i_* \mathbf{1}_{\partial^p E} \arrow[d, "p_*(\delta)"'] \arrow[dr, "\int_{pi} d\theta_s"] & \\ \Sigma^d p_* j_! \mathbf{1}_{E \setminus \partial^p E} \arrow[r, "\int_p d\theta"'] & \mathbf{1}_B, \end{tikzcd} \tag{2.13}$$

in $\operatorname{Shv}(B; \mathcal{C})$*, where the arrows are as indicated. Furthermore, this diagram is natural in bundle maps.*

Note that in the setting of higher categories, commutativity of a diagram is structure and not just property, hence the awkward phrasing "there is a commutative diagram ... where the arrows are as indicated." The proof below supplies a specific 2-cell which fills the map $\partial\Delta^2 \to \mathrm{Shv}(B;\mathcal{C})$ indicated by (2.13), and the naturality is respect to these 2-cells.

Proof. By construction, we have the commutative diagram

$$\begin{tikzcd}
\Sigma^{d-1} i_* \mathbf{1}_{\partial^p E} \arrow[r, "\sim", "i_*i^!\delta"'] \arrow[dr, "\delta"'] & \Sigma^d i_* i^! j_! \mathbf{1}_{E\setminus\partial^p E} \arrow[r, "\sim", "i_*i^!j_!(\theta)"'] \arrow[d] & i_* i^! j_! \omega_{pj} \arrow[r, "\sim"] \arrow[d] & i_* \omega_{pi} \arrow[d] \\
 & \Sigma^d j_! \mathbf{1}_{E\setminus\partial^p E} \arrow[r, "\sim", "j_!(\theta)"'] & j_! \omega_{pj} \arrow[r, "\sim"] & \omega_p
\end{tikzcd} \tag{2.14}$$

where the vertical maps are the counits for the $i_* \dashv i^!$ adjunction and the rightmost horizontal maps are the counits for the $j_! \dashv j^!$ adjunction. The composition of the upper horizontal arrows is the Stokes orientation θ_s. Applying the pushforward p_* and postcomposing with the counit $p_*p^!\mathbf{1}_B \to \mathbf{1}_B$ produces the desired commutative diagram. Note that all of the 2-cells in (2.14) are compatible with base change along bundle maps, as they themselves are given by unit and counit maps, whence we get the desired naturality of (2.13). $\square$

Remark 2.30. Suppose that $\mathcal{C} = \mathrm{Mod}_k$ is the category of modules over an ordinary commutative ring $k \in \mathrm{CAlg}(\mathrm{Ab})$. Passing to pointwise homology groups in (2.13), we get a commutative diagram

$$\begin{tikzcd}
\mathcal{H}^{*-d+1}(\partial W_b;k) \arrow[d, "\delta"] \arrow[dr, "\int_{\partial W_b}"] & \\
\mathcal{H}^{*-d}(W_b, \partial W_b;k) \arrow[r, "\int_{W_b\setminus\partial W_b}"'] & k.
\end{tikzcd} \tag{2.15}$$

in the 1-category of ordinary local systems $\mathrm{Fun}(\Pi_1 B, \mathrm{Mod}_k^{\heartsuit})$. In other words, the ordinary Stokes theorem for the fiber is compatible with the monodromy of the bundle p. For some readers, it might be tempting to try to replace (2.13) with (2.15) in the arguments appearing in subsequent sections. Here Grothendieck's philosophy is pertinent: the total derived functors p_*i_* and $p_*j_!$ are both richer and much easier to work with than their homology groups. Indeed, in order to apply (2.15) to questions about the fiberwise integration map $\int_p \colon H^{*-d}(E, \partial^p E;k) \to H^*(B;k)$ such as those considered in this article, one must go through the Leray–Serre spectral sequence for the fiber bundle p. Already for bundles of closed manifolds, the analogous spectral sequence arguments are complicated, see [Gri17]. The situation seems significantly worse for bundles of manifolds with boundary, as one would need to keep track of pairings of various spectral sequences.

Corollary 2.31. *Suppose θ is an orientation of the fiber bundle $W \to E \xrightarrow{p} B$, and let θ_s denote the associated Stokes orientation of the boundary bundle $pi \colon \partial^p E \to B$.*

There is a commutative diagram

$$\begin{tikzcd}
\Sigma^{d-1}\Gamma(\partial^p E; \mathbf{1}_{\partial^p E}) \arrow[d, "\delta"] \arrow[dr, "\int_{pi} d\theta_s"] & \\
\Sigma^d \Gamma(E; j_! \mathbf{1}_{E \setminus \partial^p E}) \arrow[r, "\int_p d\theta"'] & \Gamma(B; \mathbf{1}_B),
\end{tikzcd}$$

where the arrows are as indicated. In particular, if B has the homeomorphism type of a CW complex and $\mathcal{C} = \mathrm{Mod}_R$ is the category of modules over a commutative ring spectrum $R \in \mathrm{CAlg}(\mathrm{Sp})$, we get the commutative diagram

$$\begin{tikzcd}
\Sigma^{d-1} C^*(\partial^p E; R) \arrow[d, "\delta"] \arrow[dr, "\int_{pi} d\theta_s"] & \\
\Sigma^d C^*(E, \partial^p E; R) \arrow[r, "\int_p d\theta"'] & C^*(B; R).
\end{tikzcd}$$

3. Tautological classes and the boundary bundle

In this section, we define the tautological ring of a compact smooth manifold with boundary. The definition is modelled on the definition of the tautological ring of a closed manifold. For an odd-dimensional manifold with boundary, we show that its tautological ring is generated by the tautological ring of its boundary.

3.1. **Smooth bundles and tautological classes.**

We start by recalling some basic notions.

Definition 3.1. Let W be a smooth manifold. A *smooth W-bundle* is a fiber bundle $W \to E \xrightarrow{p} B$ with structure group $\mathrm{Diff}^+(W)$ (see e.g. [Spa81, p. 90]), which we assume to have a numerable local trivialization.

Equivalently, a smooth bundle with base space B is classified by the homotopy class of a map $f\colon B \to B\,\mathrm{Diff}(W)$. The underlying fiber bundle associated with f is pulled back from the *universal smooth W-bundle*

$$W \to W /\!\!/ \mathrm{Diff}(W) \to B\,\mathrm{Diff}(W).$$

For instance, a smooth submersion $p\colon E \to B$ between smooth manifolds can be viewed as a smooth bundle in a canonical way by Ehresmann's fibration theorem.

For the rest of this subsection, fix an ordinary commutative ring $k \in \mathrm{CAlg}(\mathrm{Ab})$. Classically $k = \mathbb{Q}$, and we will restrict ourselves to this case later.

3.1.1. *The tautological ring of a closed manifold.* We wish to describe characteristic classes of oriented smooth bundles

$$\eta\colon \left(W \to E \xrightarrow{p} B\right) \mapsto \eta(p) \in H^*(B; k).$$

Equivalently, we will study the cohomology ring $H^*(B\,\mathrm{Diff}^+(W); k)$ of the classifying space for such bundles. When W is closed, a systematic source of such characteristic classes are those coming from the characteristic classes associated to the vertical tangent bundle, namely the so-called (generalized)

Miller–Morita–Mumford classes. We recall their definition here for completeness.

In § 2.3.2, we defined the vertical tangent $\mathbb{R}^d$-bundle associated to a fiber bundle $W \to E \to B$, where W was a topological manifold and B was an arbitrary space. On the other hand, a smooth submersion $p\colon E \to B$ has a vertical tangent bundle, which is an actual vector bundle on the total space E. It is well-known among topologists that such a vertical tangent bundle can be defined more generally for a smooth bundle (see e.g. [BG76, § 4]):

Construction 3.2. Let $W \to E \xrightarrow{p} B$ be a smooth fiber bundle over B, classified by a map $f\colon B \to B\,\mathrm{Diff}(W)$. Assume that B has the homeomorphism type of a CW complex. We can identify $B\,\mathrm{Diff}(W)$ with the groupoid having one object W whose automorphism space is the space of diffeomorphisms of W. Since diffeomorphisms induce maps of tangent bundles, there is a forgetful functor

$$B\,\mathrm{Diff}(W) \xrightarrow{\mathrm{fgt}} \mathcal{S}_{/BO(d)}.$$

The unstraightening theorem provides a canonical homotopy equivalence

$$|E| \simeq \operatorname{colim}\left(B \xrightarrow{f} B\,\mathrm{Diff}(W) \xrightarrow{\mathrm{fgt}} \mathcal{S}_{/BO(d)} \xrightarrow{\mathrm{fgt}} \mathcal{S}\right).$$

There is an obvious natural transformation from $B \xrightarrow{f} B\,\mathrm{Diff}(W) \to \mathcal{S}_{/BO(d)} \to \mathcal{S}$ to the constant functor with value $BO(d)$. This classifies a d-dimensional vector bundle on E, which we denote by T^pE and refer to as a the *vertical tangent bundle* of p.

Given an oriented smooth bundle, the vertical tangent bundle refines to an oriented vector bundle by replacing $\mathrm{Diff}(W)$ with $\mathrm{Diff}^+(W)$ and $BO(d)$ with $BSO(d)$ above.

We leave it as an exercise to the reader to check that if $W \to E \xrightarrow{p} B$ is a smooth bundle, then the underlying $\mathbb{R}^d$-bundle of the vertical tangent bundle T^pE as constructed above agrees with the $\mathbb{R}^d$-bundle of the underlying bundle of topological manifolds, as given in Definition 2.14.

In particular, the universal oriented smooth bundle

$$W \to W/\!\!/\,\mathrm{Diff}^+(W) \xrightarrow{p} B\,\mathrm{Diff}^+(W),$$

has an associated oriented vertical tangent bundle.

Definition 3.3. Suppose W is a closed smooth manifold. Given a cohomology class $c \in H^*(BSO(d); k)$, the associated *κ-class*[7] is given by

$$\kappa_c = \int_p c(T^pE) \in H^{*-d}(B\,\mathrm{Diff}^+(W); k), \tag{3.1}$$

where $p\colon E = W/\!\!/\,\mathrm{Diff}^+(W) \to B\,\mathrm{Diff}^+(W)$ is the universal smooth bundle with fiber W.

The *tautological ring* of W is the k-subalgebra

$$R^*(W; k) \subseteq H^*(B\,\mathrm{Diff}^+(W); k), \tag{3.2}$$

generated by the κ-classes. We are especially interested in the case $k = \mathbb{Q}$, so we abbreviate $R^*(W) = R^*(W; \mathbb{Q})$ in this case.

[7] aka *Miller–Morita–Mumford class*

Although (3.3) may seem ad hoc, the study of $R^*(W)$ is vindicated by the Madsen–Weiss theorem and its many variants, which roughly state that the inclusion (3.3) becomes an equality as the genus of W tends to infinity.

Remark 3.4. Ebert and Randal-Williams have shown that κ-classes can be defined more generally for fiber bundles $W \to E \to B$, where W is a closed topological manifold, and even more generally for block bundles [ERW14]. It would be interesting to apply our methods to this more general situation.

3.1.2. *The tautological ring of a manifold with boundary.* Note that (3.1) does not make sense if W is a manifold with boundary, since fiber integration is only defined on relative classes. Instead, recall that for any fiber bundle $F \to E \xrightarrow{p} B$ with finitely-dominated fiber, there is an associated Becker–Gottlieb transfer $\mathrm{trf}_p\colon \Sigma^\infty_+ B \to \Sigma^\infty_+ E$ [BG76]. This transfer induces wrong-way maps in cohomology that we also denote $\mathrm{trf}_p\colon H^*(E;k) \to H^*(B;k)$.

Definition 3.5. Let W^d be a compact smooth manifold with boundary and let $k \in \mathrm{CAlg}(\mathrm{Ab})$ be an ordinary commutative ring. For any $c \in H^*(BSO(d);k)$, we define the associated *ρ-class* by the formula

$$\rho_c = \mathrm{trf}_p(c(T^p E)) \in H^*(B\,\mathrm{Diff}^+(W);k),$$

where

$$W \to E = W/\!\!/\,\mathrm{Diff}^+(W) \xrightarrow{p} B\,\mathrm{Diff}^+(W)$$

is the universal oriented smooth bundle with fiber W and T^pE is its vertical tangent bundle.

The *tautological ring* of W is the k-subalgebra

$$R^*(W;k) \subseteq H^*(B\,\mathrm{Diff}^+(W);k), \tag{3.3}$$

generated by the ρ-classes. As in the closed case, we put $R^*(W) = R^*(W;\mathbb{Q})$.

Remark 3.6. The definition of $R^*(W;k) \subseteq H^*(B\,\mathrm{Diff}^+(W);k)$ can be rewritten in the following way. The spectrum-level Becker–Gottlieb transfer $\Sigma^\infty_+ B\,\mathrm{Diff}^+(W) \to \Sigma^\infty_+ W_{h\,\mathrm{Diff}^+(W)}$ is adjoint to a map

$$B\,\mathrm{Diff}^+(W) \to \Omega^\infty\Sigma^\infty_+ W_{h\,\mathrm{Diff}^+(W)}.$$

By postcomposing this with (the functor $\Omega^\infty\Sigma^\infty_+$ applied to) the map $W_{h\,\mathrm{Diff}^+(W)} \to BSO(d)$ classifying the vertical tangent bundle, we get a map

$$\alpha_W\colon B\,\mathrm{Diff}^+(W) \to \Omega^\infty\Sigma^\infty_+ BSO(d). \tag{3.4}$$

The tautological ring $R^*(W;k)$ is simply the image of this map in cohomology with k-coefficients.

Remark 3.7. Let $V_g = (S^n \times D^{n+1})^{\natural g}$ denote the $(2n+1)$-dimensional handlebody of genus g, and suppose that $n = 1$ or $n \geq 4$. By Remark 3.6 and the work of Hatcher [Hat12], Barkan–Steinebrunner (personal communication), and Botvinnik–Perlmutter [BP17] cited in the introduction, we can view

$$R^*(V_g, D) = \mathrm{img}\left(R^*(W) \subseteq H^*(B\,\mathrm{Diff}^+(V_g);\mathbb{Q}) \to H^*(B\,\mathrm{Diff}(V_g, D);\mathbb{Q})\right)$$

as the subring spanned by stable classes under genus-stabilization, i.e.

$$R^*(V_g, D) = \mathrm{img}\left(H^*(\mathrm{colim}_h\, B\,\mathrm{Diff}(V_h, D);\mathbb{Q}) \to H^*(B\,\mathrm{Diff}(V_g, D);\mathbb{Q})\right).$$

Homological stability results of Hatcher–Wahl [HW10] as well as the author [Har25b] in the $n = 1$ case and Perlmutter [Per18] in the $n \geq 4$ case imply that the map $H^*(\mathrm{colim}_h B\,\mathrm{Diff}(V_h, D); \mathbb{Z}) \to H^*(B\,\mathrm{Diff}(V_g, D); \mathbb{Z})$ is an isomorphism for $* \leq \frac{2}{3}(g-1)$ if $n = 1$ and $* \leq \frac{1}{2}(g-4)$ otherwise. In this *stable range* of degrees, we therefore have a complete description of $H^*(B\,\mathrm{Diff}(V_g, D); \mathbb{Q})$. The tautological ring $R^*(V_g, D)$ can therefore be seen as sitting somewhere between the stable cohomology, of which we have a complete description, and the unstable cohomology, about which we know very little.

Remark 3.8. The definition of the ρ-classes make sense more generally if W is a finitely-dominated smooth manifold, e.g. if W is the interior of a compact manifold with boundary.

3.2. **A theorem of Giansiracusa and Tillmann.** Let W be a compact orientable manifold with boundary. Restricting to the boundary defines a map of classifying spaces

$$r\colon B\,\mathrm{Diff}^+(W) \to B\,\mathrm{Diff}^+(\partial W). \tag{3.5}$$

In terms of fiber bundles, the map r classifies the operation of replacing a smooth fiber bundle $W \to E \to B$ with its associated boundary bundle $W \to \partial^p E \to B$.

Informally, one can view tautological rings as consisting of "geometrically defined classes." Since the boundary of a manifold is certainly a geometric notion, we would expect r to be compatible with tautological rings. In order to show that this is the case, we will need a theorem of Giansiracusa and Tillmann [GT11]. The original proof of this theorem proceeds via an analysis of Madsen–Tillmann spectra. We will instead derive it as a consequence of the ordinary family Stokes theorem.

We work over a commutative ring k. Let us say that a characteristic class $c \in H^*(BSO(d); k)$ is *stable* if it lies in the image of the map $H^*(BO; k) \to H^*(BSO(d); k)$. For instance, it is well-known that if $2 \in k$ is invertible, then

$$H^*(BSO(d); k) \cong \begin{cases} k[p_1, \ldots, p_{d/2}, e]/(e^2 - p_{d/2}), & \text{if } d \text{ is even,} \\ k[p_1, \ldots, p_{(d-1)/2}], & \text{if } d \text{ is odd,} \end{cases}$$

where the p_i are the Pontrjagin classes and e is the Euler class, see [MS74, Thm 15.9]. In this case, a class $c \in H^*(BSO(d); k)$ is stable if and only if it is a polynomial in Pontryagin classes. On the other hand, if $2 = 0 \in k$, then

$$H^*(BSO(d); k) \cong k[w_2, \ldots, w_d],$$

where the w_i are the Stiefel–Whitney classes, see [MS74, Thm 12.4]. In this case $H^*(BSO(d); k)$ consists entirely of stable classes.

The integral cohomology ring $H^*(BSO(d); \mathbb{Z})$ is more complicated, but has been completely computed by Brown [Bro82]. In particular, this ring is generated by

(1) the Pontrjagin classes $p_i \in H^{4i}(BSO(d); \mathbb{Z})$ for $0 < i \leq \lfloor d/2 \rfloor$;

(2) the classes of the form $\widetilde{\beta}(w_{i_1} \dots w_{i_n})$ for $0 < i_1 < \dots < i_n \leq \lfloor d/2 \rfloor$, where $w_i \in H^i(BSO(d); \mathbb{F}_2)$ is the ith Stiefel–Whitney class and

$$\widetilde{\beta}\colon H^*(BSO(d); \mathbb{F}_2) \to H^{*+1}(BSO(d); \mathbb{Z})$$

is the Bockstein homomorphism; and

(3) the Euler class $e_d \in H^d(BSO(d); \mathbb{Z})$.

For our purposes, it will not be necessary to know the relations between these classes, except that $e_d = \widetilde{\beta}(w_{(d-1)/2})$ if d is odd. In particular, $H^*(BSO(d); \mathbb{Z})$ consists entirely of stable classes if d is odd, whereas if d is even then the stable classes are precisely those that are polynomials in Pontryagin classes and Bockstein images of Stiefel–Whitney classes.

We will use the following basic observation:

Lemma 3.9. *Let $W \to E \xrightarrow{p} B$ be an oriented fiber bundle, where W is a compact manifold with boundary. Let $i\colon \partial^p E \hookrightarrow E$ denote the inclusion of the fiberwise boundary.*

If $a \in H^*(\partial^p E; k)$ extends to a cohomology class on all of E, in the sense that $a = i^*(b)$ for some $b \in H^*(E; k)$, then

$$\int_{pi} a = 0.$$

Proof. By the family Stokes theorem, we have

$$\int_{pi} a = \int_p \delta i^* b = 0,$$

where $\delta\colon H^*(\partial^p E; k) \to H^*(E, \partial^p E; k)$ is the connecting homomorphism. □

Theorem 3.10 (Giansiracusa–Tillmann [GT11])**.** *Let W^d be smooth compact orientable manifold with boundary. If $c \in H^*(BSO(d-1); k)$ is stable, then*

$$r^* \kappa_c = 0 \in H^*(B\operatorname{Diff}^+(W); k).$$

In particular, if d is even then $r^*\kappa_c = 0$ for all $c \in H^*(BSO(d); k)$.

Proof. Let $W \to E \xrightarrow{p} B$ be an arbitrary smooth bundle, and let $i\colon \partial^p E \hookrightarrow E$ denote the inclusion of the fiberwise boundary. By picking an outwards-pointing vector field for $T^p E$ along $\partial^p E$, we get a splitting

$$i^* T^p E \cong T^{pi} \partial^p E \oplus \mathbb{R}_{\partial^p E}, \tag{3.6}$$

where $\mathbb{R}_{\partial^p E}$ denotes the trivial line bundle on $\partial^p E$.

Since $BSO(d-1) \to BO$ factors through $BSO(d) \to BO$, our assumption implies that c has a canonical lift $\widetilde{c} \in H^*(BSO(d); k) \to H^*(BSO(d-1); k)$. Since $BSO(d-1) \to BSO(d)$ classifies the procedure of replacing a $(d-1)$-dimensional vector bundle V with the Whitney sum of V and a trivial line bundle, we find

$$c(T^{pi} \partial^p E) = \widetilde{c}(T^{pi} \partial^p E \oplus \mathbb{R}_{\partial^p E}) = \widetilde{c}(i^*(T^p E)) = i^* \widetilde{c}(T^p E).$$

The statement now follows from the preceding lemma. □

Remark 3.11. A crucial step in the preceding proof was the choice of an outwards-pointing vector field along the boundary bundle. Such a choice also exists for bundles of topological manifolds. Suppose $W \to E \xrightarrow{p} B$ is a fiber bundle, where W is a topological manifold with boundary and B is a paracompact Hausdorff space. As in the smooth case, the vertical tangent microbundle T^pE admits a nonzero section when restricted to the fiberwise boundary $\partial^p E$. Indeed, let $i\colon \partial^p E \hookrightarrow E$ denote the inclusion. The pasting lemma for pullbacks implies that the restriction i^*T^pE is the microbundle

$$(\partial^p E \times_B E, \pi_1\colon \partial^p E \times_B E \to \partial^p E, \Delta\colon \partial^p E \hookrightarrow \partial^p E \times_B E).$$

Since the space of collars of a manifold with boundary is contractible, we can pick a fiberwise collar neighborhood $\varphi\colon \mathbb{R}_{\geq 0} \times \partial^p E \to E$. This defines a splitting of the restriction i^*T^pE.

3.3. **Boundary tautological classes.** We can now prove the main result of this section, comparing the tautological ring of a manifold with boundary and the tautological ring of its boundary. We thank Oscar Randal-Williams for suggesting the method of proof.

Theorem 3.12. *Let W^d be a compact smooth orientable manifold with boundary. Let*

$$r\colon B\operatorname{Diff}^+(W) \to B\operatorname{Diff}^+(\partial W)$$

denote the map induced by restriction to the boundary. Assume that $2 \in k$ is invertible or zero, or otherwise that $k = \mathbb{Z}$.

If $c \in H^*(BSO(d-1); k)$ is a stable class and d is odd, then

$$2\rho_c = r^*\kappa_{ec} \in H^*(B\operatorname{Diff}^+(W); k).$$

Proof. Let $W \to E \xrightarrow{p} B$ be a smooth bundle with fiber W, and let $i\colon \partial^p E \hookrightarrow E$ denote the inclusion of its fiberwise boundary. We must show that

$$\operatorname{trf}_p(c(T^pE)) = \int_{pi} e(T^{pi}\partial^p E)c(T^{pi}\partial^p E).$$

Here

$$\int_{pi} e(T^{pi}\partial^p E)c(T^{pi}) = \operatorname{trf}_{pi}(c(T^{pi}\partial^p E))$$

by [BG76, Thm 4.3]. In order to compare the two Becker–Gottlieb transfers, we will use that every bundle of manifolds with boundary has an associated double bundle. Recall that the double of W can be identified with (a canonical smoothing of) the boundary $\partial(W \times [0,1])$. We consider the map

$$B\operatorname{Diff}^+(W) \to B\operatorname{Diff}^+(W \times [0,1]) \xrightarrow{r'} B\operatorname{Diff}^+(\partial\,(W \times [0,1])), \tag{3.7}$$

where the first map is induced by sending $\varphi \in \operatorname{Diff}^+(W)$ to $\varphi \times \mathrm{id} \in \operatorname{Diff}^+(W \times [0,1])$ and the second map is given by restricting to the boundary. The map (3.7) classifies the procedure of replacing a bundle $W \to E \xrightarrow{p} B$ with the double bundle

$$W \cup_{\partial W} \overline{W} \to E \cup_{\partial^p E} \overline{E} \xrightarrow{\pi} B,$$

where $\overline{W} \to \overline{E} \xrightarrow{\overline{p}} B$ denotes the oriented smooth bundle which has the same underlying smooth bundle as p, but where the orientation of p has been reversed.

We write $T^\pi = T^\pi(E \cup_{\partial^p E} \overline{E})$ for the vertical tangent bundle of π. The excision property of the Becker–Gottlieb transfer [BS98, Thm 1.3] now implies that

$$\begin{aligned}\mathrm{trf}_\pi c(T^\pi) &= \mathrm{trf}_p\left(c(T^\pi)|_E\right) + \mathrm{trf}_{\overline{p}}\left(c(T^\pi)|_{\overline{E}}\right).\mathrm{trf}_{pi}\left(c(T^\pi)|_{\partial^p E}\right)\\ &= \mathrm{trf}_p\left(c(T^\pi|_E)\right) + \mathrm{trf}_{\overline{p}}\left(c(T^\pi|_{\overline{E}})\right) - \mathrm{trf}_{pi}\left(c(T^\pi|_{\partial^p E})\right).\end{aligned}$$

Here $T^\pi|_E \cong T^p E$, whereas $T^\pi|_{\overline{E}}$ is isomorphic to $T^p E$ with the reversed orientation. On the other hand, the subbundle $pi\colon \partial^p E \to B$ has an essentially unique bicollaring, which gives an isomorphism $T^\pi|_{\partial^p E} \cong T^{pi}\partial^p E \oplus \mathbb{R}_{\partial^p E}$. Since c is independent of orientation, we have

$$\mathrm{trf}_p\left(c(T^\pi|_E)\right) = \mathrm{trf}_{\overline{p}}\left(c(T^\pi|_{\overline{E}})\right) = \rho_c(p).$$

But then we get

$$2\rho_c(p) + (r^*\kappa_{ec})(p) = \mathrm{trf}_\pi c(T^\pi) = ((r')^*\kappa_{ec})(\pi),$$

and $(r')^*\kappa_{ec} = 0$ by Theorem 3.10, since $H^*(BSO(d);k)$ consists entirely of stable classes on account of d being odd. □

Corollary 3.13. *Let W^d be a compact smooth orientable manifold with boundary, and let*

$$r\colon B\,\mathrm{Diff}^+(W) \to B\,\mathrm{Diff}^+(\partial W)$$

denote the map induced by restriction to the boundary. Let $k \in \mathrm{CAlg}(\mathrm{Ab})$ be an ordinary commutative ring, with $\mathbb{Z}[\frac{1}{2}] \subseteq k \subseteq \mathbb{Q}$.

If d is odd, then

$$R^*(W;k) = \mathrm{img}\left(R^*(\partial W;k) \subseteq H^*(B\,\mathrm{Diff}^+(\partial W;k) \xrightarrow{r^*} H^*(B\,\mathrm{Diff}^+(W;k)\right).$$

Example 3.14. In the proof of Theorem 3.12, one might have hoped instead to prove that the double of a bundle of manifolds with boundary is trivial, or even just that it has a trivial vertical tangent bundle. Here we show that this is not true in the case of three-dimensional handlebodies.

Let $V_g = (S^1 \times D^2)^{\natural g}$ be a genus g three-dimensional handlebody. The double $V_g \cup_{\partial V_g} V_g$ is diffeomorphic to $(S^1 \times S^2)^{\# g}$. Put $W_g = (S^1 \times S^2)^{\# g}$ for short. By passing to fundamental groups, the doubling map (3.7) induces a homomorphism of mapping class groups[8]

$$ð\colon \pi_0\,\mathrm{Diff}^+(V_g) \to \pi_0\,\mathrm{Diff}^+(W_g).$$

We claim that this is an epimorphism. First note that there is a commutative diagram

$$\begin{array}{ccc} \pi_0\,\mathrm{Diff}^+(V_g) & \longrightarrow & \mathrm{Out}(\pi_1 V_g) \\ \big\downarrow{\scriptstyle ð} & & \big\downarrow{\scriptstyle \simeq} \\ \pi_0\,\mathrm{Diff}^+(W_g) & \longrightarrow & \mathrm{Out}(\pi_1 W_g), \end{array} \tag{3.8}$$

e.g. by the van Kampen theorem. Both horizontal maps are epimorphisms, by [Zie62, McM63] in the handlebody case and [Whi36a, Whi36b] in the case of W_g.

[8]We use the Norse/Old English letter ð (pronounced "eth") for the **d**oubling map so as not to overload the more familiar symbols $d, \delta, \partial, \ldots$

Luft has shown that the kernel of the top horizontal map in (3.8) is equal to the subgroup $\mathrm{Tw}(V_g) \trianglelefteq \pi_0 \operatorname{Diff}^+(V_g)$ generated by meridian Dehn twists [Luf78]. For W_g, a similar description was given by Laudenbach. Let $S \subseteq W_g$ be an embedded sphere and let $[0,1] \times S \subseteq W_g$ be a bicollaring, such that $\frac{1}{2} \times S$ corresponds to the original copy of S. Fix a loop $\gamma \in \Omega_{\mathrm{id}} SO(3)$ which is not homotopic to the constant loop.[9] The *sphere twist* associated to this data is the isotopy class of the diffeomorphism $\tau \colon W_g \to W_g$ which is given by $(t, x) \mapsto (t, \gamma(e^{2\pi i t})x)$ on $[0,1] \times S$, and by the identity elsewhere. It is not hard to show that the isotopy class of τ only depends on the homotopy class of the sphere S. Laudenbach proved that the kernel of the bottom horizontal map in (3.8) is the subgroup $\mathrm{Tw}(W_g) \trianglelefteq \pi_0 \operatorname{Diff}^+(W_g)$ consisting of products of sphere twists, and furthermore that this subgroup is generated by the g many sphere twists corresponding to the $S^1 \times S^2$-factors in W_g [Lau73, Lau74], see also [BBP23]. To prove that $\eth$ is surjective, it therefore suffices by the five lemma to show that $\eth(\mathrm{Tw}(V_g)) = \mathrm{Tw}(W_g)$.

Let α be an oriented meridian curve in V_g. The meridian Dehn twist t_α is supported on a solid cylinder $[0,1] \times D \subseteq V_g$ with $[0,1] \times \partial D \subseteq \partial V_g$ and $\frac{1}{2} \times \partial D = \alpha$, where it is given by $(t, x) \mapsto (t, e^{2\pi i t}x)$. The doubled mapping class $\eth(t_\alpha)$ is therefore supported in $[0,1] \times D \cup_{[0,1]\times\partial D} [0,1] \times D \cong [0,1] \times S^2 \subseteq W_g$, which we depict in Figure 3 as a continuous family of concentric spheres with $0 \times S^2$ being the outermost sphere. One copy of

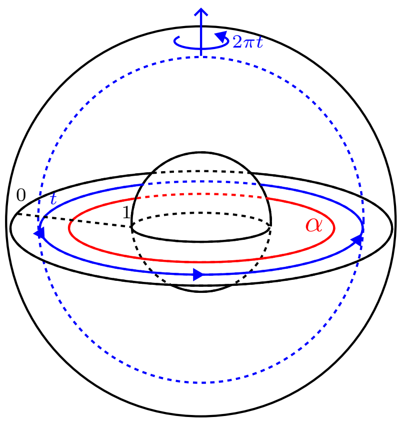

FIGURE 3. The sphere twist $\eth(t_\alpha)$ rotates the layer $t \times S^2 \subseteq I \times S^2$ by a $2\pi t$ radian rotation around the central vertical axis.

the handlebody contributes the family of northern hemispheres, whereas the other contributes the family of southern hemispheres. On this family of spheres, the mapping class $\eth(t_\alpha)$ acts by rotating $t \times S^2$ by $2\pi t$ radians. Hence $\eth(t_\alpha)$ is the sphere twist associated to the embedded sphere $\frac{1}{2} \times S^2 \subseteq W_g$. In particular, taking α to be the curve $1 \times \partial D^2$ corresponding to one of the $S^1 \times D^2$-summands of V_g, we find that $\eth(t_\alpha)$ is the sphere twist associated to

[9] In order to ensure that the construction we are giving defines a diffeomorphism, one should require $\gamma \colon (S^1, 1) \to (SO(3), \mathrm{id})$ to be constant at the identity in a neighborhood of $1 \in S^1$. We ignore this issue in this example, since it is easily fixed using bump functions.

the corresponding $S^1 \times S^2$-summand of W_g, whence $\eth(\mathrm{Tw}(V_g)) = \mathrm{Tw}(W_g)$ as desired.

4. Relations in the tautological ring of a manifold with boundary

Our goal in this section is to prove Theorem B. Like its analogs for bundles of closed manifolds [Gri17, RW18], our theorem can be used to produce relations between characteristic classes of manifold bundles, and we also give some examples of this here.

4.1. **Recollection III: Schur functors.** In the article of Randal-Williams [RW18], a key part is played by the Schur functors acting on the category of derived local systems of $\mathbb{Q}$-vector spaces on a space. We briefly recall the construction of these functors given there, together with some basic properties that we will need.

Let n be a non-negative integer. For every (unordered) partition λ of n, there is a corresponding irreducible representation V_λ of the symmetric group Σ_n. For instance, one can take $V_\lambda \subseteq \mathbb{Q}[\Sigma_n]$ to be the right principal ideal generated by the Young symmetrizer $c_\lambda \in \mathbb{Q}[\Sigma_n]$ corresponding to a Young tableau for λ. Different choices of Young tableaux give rise to isomorphic representations. Young showed that the rule $\lambda \mapsto V_\lambda$ defines a bijection between partitions of n and isomorphism classes of irreducible Σ_n-representations. Furthermore, the Young symmetrizer c_λ is idempotent after rescaling; that is, there is some $n_\lambda \in \mathbb{Q}_{>0}$ such that

$$e_\lambda = \frac{c_\lambda}{n_\lambda} \in \mathbb{Q}[\Sigma_n]$$

is an idempotent element. We refer to [FH91, p. 46] for all of these statements.

Suppose now that $(\mathcal{C}, \otimes, \mathbf{1})$ is a $\mathbb{Q}$-linear symmetric monoidal category. For any object $X \in \mathcal{C}$, the Σ_n-action on $X^{\otimes n}$ defined by permuting factors defines a morphism of $\mathbb{Q}$-algebras

$$\mathbb{Q}[\Sigma_n] \to \mathrm{End}(X^{\otimes n}).$$

In particular, the idempotent e_λ mentioned above is sent to an idempotent $e_\lambda(X) \in \mathrm{End}(X^{\otimes n})$. If $\mathcal{C}$ is idempotent-complete (e.g. if $\mathcal{C}$ is cocomplete), then we define the *Schur functor* $S_\lambda \colon \mathcal{C} \to \mathcal{C}$ informally by sending X to the retract associated to the idempotent $e_\lambda(X)$. (See Appendix A for a precise and highly coherent definition.)

The Schur functors associated to the partitions (n) and $(1, \ldots, 1)$ are particularly interesting for us, and we write

$$\mathrm{Sym}^n X = S_{(n)}(X) \quad \text{and} \quad \wedge^n X = S_{(1,\ldots,1)}(X).$$

This recovers the usual definitions of symmetric and alternating powers when $\mathcal{C}$ is the ordinary category of $\mathbb{Q}$-vector spaces $\mathrm{Vect}_\mathbb{Q} = \mathrm{Mod}_\mathbb{Q}^\heartsuit$. On the other hand, if $\mathcal{C}$ is the category of graded $\mathbb{Q}$-vector spaces $\mathrm{Vect}_\mathbb{Q}^{\mathrm{gr}} = \mathrm{Fun}(\mathbb{Z}, \mathrm{Vect}_\mathbb{Q})$ (with usual symmetric monoidal structure, namely Day convolution), then the Schur functors are sensitive to parity due to the Koszul sign rule, e.g. if

$V \in \mathrm{Vect}_{\mathbb{Q}}$ and $V[d] \in \mathrm{Vect}_{\mathbb{Q}}^{\mathrm{gr}}$ is the associated graded vector space concentrated in degree d, then

$$\mathrm{Sym}^n(V[d]) \simeq \begin{cases} \mathrm{Sym}^n(V)[nd], & \text{if } d \text{ is even,} \\ \wedge^n(V)[nd], & \text{if } d \text{ is odd.} \end{cases}$$

Lemma 4.1. *Let X be a space having the homeomorphism type of a CW complex. If $F \in \mathrm{Shv}(X; \mathrm{Mod}_{\mathbb{Q}})$ has $S^\lambda(H_*(x^*F)) \simeq 0$ for every $x \in X$, then $S^\lambda(F) \simeq 0$.*

Proof. The proof is the same as for the analogous result for local systems [RW18, Lem 2.4]. The assumption on X implies that $S^\lambda(F) \simeq 0$ if and only if $x^* S^\lambda(F) \simeq 0$ for each $x \in X$. But the pullback x^* is symmetric monoidal, so $x^* S^\lambda(F) \simeq S^\lambda(x^*F)$. Since taking homology is a symmetric monoidal functor $\mathrm{Mod}_{\mathbb{Q}} \to \mathrm{Vect}_{\mathbb{Q}}^{\mathrm{gr}}$ (by the Kunneth theorem), the result now follows. $\square$

4.2. **Proof of Theorem B.** We now prove Theorem B, whose statement we recall here:

Theorem 4.2. *Let $W \to E \xrightarrow{p} B$ be an oriented fiber bundle, where W is a compact odd-dimensional topological manifold with boundary. Let $i\colon \partial^p E \hookrightarrow E$ denote its fiberwise boundary.*

Suppose that $\widetilde{H}_*(W; \mathbb{Q})$ is concentrated in odd degrees. If $a \in H^*(\partial^p E; \mathbb{Q})$ is an even-degree cohomology class such that

$$\int_{pi} a = 0$$

and $b \in H^*(E; \mathbb{Q})$ is arbitrary, then

$$\left(\int_{pi} a \cdot i^*(b) \right)^{g+1} = 0, \tag{4.1}$$

where $g = \dim_{\mathbb{Q}} \widetilde{H}_*(W; \mathbb{Q})$.

Proof. We may assume that B is locally compact Hausdorff. Let $j\colon E \setminus \partial^p E \hookrightarrow E$ denote the inclusion of the fiberwise interior. In the sheaf category $\mathrm{Shv}(B; \mathrm{Mod}_{\mathbb{Q}})$, the cohomology classes a and b correspond to homotopy classes

$$a\colon \Sigma^k \mathbf{1}_B \to (pi)_* \mathbf{1}_{\partial^p E} \quad \text{and} \quad b\colon \Sigma^l \mathbf{1}_B \to p_* \mathbf{1}_E,$$

where $k, l \in \mathbb{Z}$ and k is even. By the sheafy Stokes theorem (Theorem 2.29), our assumption on a means that there is a dashed arrow making the diagram

$$\begin{tikzcd} \Sigma^{k+d-1}\mathbf{1}_B \arrow[r, "\Sigma^{d-1}a"] \arrow[d, dashed] & \Sigma^{d-1}(pi)_*\mathbf{1}_{\partial^p E} \arrow[d, "p_*(\delta)"] \arrow[dr, "\int_{pi} d\theta_s"] & \\ M \arrow[r] & \Sigma^d (pj)_! \mathbf{1}_{E \setminus \partial^p E} \arrow[r, "\int_p d\theta"'] & \mathbf{1}_B, \end{tikzcd} \tag{4.2}$$

commute, where

$$M = \mathrm{fib}\left(\int_p d\theta\colon \Sigma^d p_* j_! \mathbf{1}_{E \setminus \partial^p E} \to \mathbf{1}_B \right).$$

Let us write $\overline{a}\colon \Sigma^{k+d-1}\mathbf{1}_B \to M$ for this dashed arrow.

The connecting homomorphism $\delta\colon \Sigma^{d-1}i_*\mathbf{1}_{\partial^p E} \to \Sigma^d j_!\mathbf{1}_{E\setminus\partial^p E}$ is (trivially) a map of $\mathbf{1}_E$-modules, and so by lax symmetric monoidality of the pushforward p_*, the map $p_*(\delta)$ is a map of $p_*\mathbf{1}_E$-modules. The fiber integral $\int_{pi} a\cdot i^*(b)$ is thus represented by the map $\Sigma^{k+l+d-1}\mathbf{1}_B \simeq \Sigma^{k+d-1}\mathbf{1}_B\otimes\Sigma^l\mathbf{1}_B \to \mathbf{1}_B$ from the commutative diagram

$$\begin{tikzcd}
\Sigma^{k+d-1}\mathbf{1}_B \otimes \Sigma^l\mathbf{1}_B \arrow[r, "\Sigma^d a\otimes b"] \arrow[d, "\overline{a}\otimes b"] & \Sigma^{d-1}(pi)_*\mathbf{1}_{\partial E}\otimes p_*\mathbf{1}_E \arrow[r] \arrow[d, "\delta\otimes\mathrm{id}"] & \Sigma^{d-1}(pi)_*\mathbf{1}_{\partial E} \arrow[r] \arrow[d, "\delta"] & \mathbf{1}_B \\
M\otimes p_*\mathbf{1}_E \arrow[r] & \Sigma^d p_*j_!\mathbf{1}_{E\setminus\partial E}\otimes p_*\mathbf{1}_E \arrow[r] & \Sigma^d p_*j_!\mathbf{1}_{E\setminus\partial E}. \arrow[ur] &
\end{tikzcd}$$

As in the proof of [RW18, Thm 2.8], we find that the class $\left(\int_{pi} a\cdot i^*(b)\right)^N$, $N\geq 0$, is represented by the composition

$$\left(\Sigma^{k+d-1}\mathbf{1}_B\right)^{\otimes N}\otimes\left(\Sigma^l\mathbf{1}_B\right)^{\otimes N} \xrightarrow{\overline{a}^{\otimes N}\otimes b^{\otimes N}} M^{\otimes N}\otimes\left(\Sigma^l\mathbf{1}_B\right)^{\otimes N} \to \mathbf{1}_B^{\otimes N} \xrightarrow{\sim} \mathbf{1}_B,$$

where we have used that $k+d-1$ is even to rearrange the factors without picking up a sign. Invoking the parity assumption again, we find that the map $\overline{a}^{\otimes N}$ factors through $\mathrm{Sym}^N(M)$. Hence it will suffice to show that $\mathrm{Sym}^{g+1}(M)\simeq 0$.

Let $b\in B$. Base change identifies the stalk b^* of the map $\int_p d\theta$ with the integration map for the fiber $W_b = p^{-1}(b)$, which is a map

$$\int_{[W_b]}\colon \Sigma^d C^*(W_b,\partial W_b;\mathbb{Q}) \to \mathbb{Q}$$

inducing an isomorphism $H^d(W_b,\partial W_b;\mathbb{Q})\to\mathbb{Q}$ in degree zero. It follows from the long exact sequence in homology that the fiber $M_b = b^*M$ of this map has

$$H_*(M_b)\simeq\begin{cases} H_*\left(\Sigma^d C^*(W_b,\partial W_b;Q)\right), & \text{for } *<0\leq d,\\ 0, & \text{otherwise.}\end{cases}$$

Here

$$H_*\left(\Sigma^d C^*(W_b,\partial W_b;Q)\right)\simeq H^{d-*}(W_b,\partial W_b;\mathbb{Q})\simeq H_*(W_b;\mathbb{Q})$$

by Poincare duality. Since $H_{*>0}(W_b;\mathbb{Q}) = \widetilde{H}_*(W_b;\mathbb{Q})$ is concentrated in odd degrees and has total dimension g, it follows that

$$\mathrm{Sym}^{g+1} H_*(M_b)\simeq 0,$$

and we thus have $\mathrm{Sym}^{g+1} M\simeq 0$ by Lemma 4.1. $\square$

Remark 4.3. Let n be an odd positive integer, and let $V_g = (S^n\times D^{n+1})^{\natural g}$ be a genus g handlebody of dimension $2n+1$. Then V_g satisfies the conditions of the preceding theorem. Note also that the genus g is equal to $\dim_{\mathbb{Q}}\widetilde{H}_*(V_g;\mathbb{Q})$, justifying the choice of notation.

By Theorem 3.12, tautological classes for V_g-bundles are pulled back along the map

$$B\,\mathrm{Diff}^+(V_g)\to B\,\mathrm{Diff}^+(\partial V_g)$$

induced by restricting a diffeomorphism to the boundary. The boundary ∂V_g is canonically diffeomorphic to $W_g = (S^n \times S^n)^{\# g}$. The analog of Theorem 4.2 for W_g was first proved by Grigoriev [Gri17] (and reproved using different methods by [RW18]). For these manifolds, the exponent in the counterpart of (4.1) is $2g+1$ rather than $g+1$. Our result therefore gives sharper relations for V_g-bundles than the relations which are known for W_g-bundles; these sharper relations can be seen as obstructions for a smooth W_g-bundle to admit a fiberwise nullcobordism, or in other words a reduction of the structure group along the canonical map $B\operatorname{Diff}^+(W_g) \to B\operatorname{Diff}^+(V_g)$.

4.3. **Example computations.** We give here some examples of new relations among characteristic classes for bundles of manifolds with boundary coming from Theorem 4.2. For moduli spaces of closed manifolds, the standard procedure (starting with the original article of Mumford [Mum83]) has two steps:

(1) first prove relations in the cohomology of various pointed versions of the moduli space $B\operatorname{Diff}^+(W)$;
(2) then use these to derive equations in the cohomology of $B\operatorname{Diff}^+(W)$.

The pointed moduli spaces and their tautological cohomology rings are interesting in their own right. In this subsection, we will carry out step (1) and leave step (2) for future work.

Let W be an oriented smooth compact manifold with boundary. We write

$$\mathcal{M}_W = B\operatorname{Diff}^+(W),$$

which we may think of as the moduli space of manifolds diffeomorphic to W. We consider also the moduli space of manifolds diffeomorphic to W together with n ordered marked points on their boundary (that are allowed to coincide); that is

$$\mathcal{M}_W(n) = \operatorname{Map}\left(\{1,\dots,n\}, \partial W\right)_{h\operatorname{Diff}^+(W)},$$

where $\operatorname{Diff}^+(W)$ acts on $\operatorname{Map}(\{1,\dots,n\},\partial W)$ via its action on ∂W. Here we adopt the convention that $\mathcal{M}_W(0) = \mathcal{M}_W$. For each $1 \le i \le n$, there is a projection map

$$p_i\colon \mathcal{M}_W(n) \to \mathcal{M}_W(n-1)$$

given by forgetting the ith marked point. This is an oriented smooth bundle with fiber ∂W, which is the fiberwise boundary the oriented smooth bundle

$$\pi_i\colon \mathcal{M}^i_W(n) \to \mathcal{M}_W(n-1),$$

where

$$\mathcal{M}^i_W(n) = \operatorname{Map}((\{1,\dots,n\}, \{1,\dots,n\}\setminus i), (W,\partial W))_{h\operatorname{Diff}^+(W)},$$

i.e. $\mathcal{M}^i_W(n)$ is the moduli space of manifolds diffeomorphic to W with n ordered marked points of which all but the ith is required to lie on the boundary.

For each n, there is a canonical map

$$\mathcal{M}_W(n) \to \mathcal{M}_{\partial W}(n),$$

and the bundles p_i have been constructed precisely so that the diagram

$$\begin{tikzcd} \mathcal{M}_W(n) \arrow[r] \arrow[d, "p_i"] & \mathcal{M}_{\partial W}(n) \arrow[d, "q_i"] \\ \mathcal{M}_W(n-1) \arrow[r] & \mathcal{M}_{\partial W}(n-1) \end{tikzcd}$$

is a pullback diagram (and in particular commutes), where the map q_i is defined analogously to p_i. In particular q_1 is the universal oriented smooth bundle with fiber ∂W and π_1 is the universal oriented smooth bundle with fiber W.

Definition 4.4. Let n be a positive integer. For $1 \leq i \leq n$, the associated *ψ-class*[10], denoted ψ_i, is given by

$$\psi_i = e(T^{p_i}\mathcal{M}_W(n)) \in H^d(\mathcal{M}_W(n); \mathbb{Q}).$$

We will also consider the cohomology class associated to the locus in $\mathcal{M}_W(n)$ on which the ith and jth marked points coincide. These classes are also defined, for instance, in [Gri17]. We give another definition here to emphasize (again) the convenience of working with sheaves. In order to define this cohomology class, suppose that $\partial W \to E \xrightarrow{p} B$ is an oriented smooth bundle together with a section $s\colon B \to E$. Assume for now that B has the homeomorphism type of a locally finite CW complex. Applying $s_! s^!$ to the inverse of the orientation $\theta\colon : \Sigma^d \mathbf{1}_E \xrightarrow{\sim} p^! \mathbf{1}_B$ defines a map

$$s_* \mathbf{1}_B \simeq s_! s^! p^! \mathbf{1}_B \xrightarrow{\sim} \Sigma^d s_! s^! \mathbf{1}_E,$$

where we have used that s is proper (since any section of a map of Hausdorff spaces is automatically a closed inclusion) and that $ps = \mathrm{id}$. By composing with the counit of the $s_! \dashv s^!$ adjunction, we get a map

$$s_* \mathbf{1}_B \to \Sigma^d \mathbf{1}_E,$$

which descends to the *Gysin map*

$$H^*(B; \mathbb{Q}) \to H^{*+d}(E; \mathbb{Q}).$$

Under this map, the unit $1 \in H^0(B; \mathbb{Q})$ is sent to a class that we abusively denote $[s(B)] \in H^d(E; \mathbb{Q})$. We can remove the assumption on B by defining $[s(B)]$ to be the uniquely determined cohomology class such that $f^*[s(B)] = [s'(B')]$ for every map of oriented smooth bundles

$$\begin{tikzcd} E' \arrow[r, "f"] \arrow[d] & E \arrow[d] \\ B' \arrow[r] & B \end{tikzcd}$$

for which B' has the homeomorphism type of a locally finite CW complex, where s' is defined by pulling back s.

Definition 4.5. For $1 \leq i, j \leq n$ such that $i \neq j$, the associated *ν-class*[11], denoted $\nu_{i,j}$, is given by

$$\nu_{i,j} = [s_i(\mathcal{M}_W(n-1))] \in H^*(\mathcal{M}_W(n); \mathbb{Q}),$$

[10] aka *(gravitational) descendant class*

[11] aka *diagonal class*

where s_i is the section of p_j induced by the map

$$\mathrm{Map}(\{1,\dots,n\}\setminus j, \partial W) \to \mathrm{Map}(\{1,\dots,n\}, \partial W)$$

given by precomposing with the map $\{1,\dots,n\} \to \{1,\dots,n\}\setminus j$ defined by

$$k \mapsto \begin{cases} i, & \text{if } k = j, \\ k, & \text{else.} \end{cases}$$

Definition 4.6. Let n be a non-negative integer. The *tautological ring* of $\mathcal{M}_W(n)$ is the $\mathbb{Q}$-subalgebra

$$R^*(\mathcal{M}_W(n);\mathbb{Q}) \subseteq H^*(\mathcal{M}_W(n);\mathbb{Q})$$

generated by the κ-classes, ψ-classes, and ν-classes.

For the remainder of this section, let W be smooth manifold satisfying the hypotheses of Theorem 4.2 and let n be a positive integer. We abbreviate $p = p_{n+1} \colon \mathcal{M}_W(n+1) \to \mathcal{M}_W(n)$ and $\pi = \pi_{n+1} \colon \mathcal{M}_W^{n+1}(n+1) \to \mathcal{M}_W(n)$. Put $g = \dim_{\mathbb{Q}} \widetilde{H}_*(W;\mathbb{Q})$.

Example 4.7. Let $(A_i)_1^n \in \mathbb{Q}^n$ be a tuple of rational numbers such that $\sum_1^n A_i = 0$. The class

$$a = \sum_{i=1}^{n} A_i \nu_{i,n+1} \in H^*(\mathcal{M}_W(n+1);\mathbb{Q})$$

has $\int_p a = 0$, e.g. by [Gri17, Lem 5.11]. Since $\psi_{n+1}^2 = p_1(T^\pi \mathcal{M}_W^{n+1}(n))|_{\mathcal{M}_W(n+1)}$, Theorem 4.2 gives

$$0 = \left(\sum_{i=1}^{n} \int_p A_i \nu_{i,n+1} \psi_{n+1}^2\right)^{g+1} = \left(\sum_{i=1}^{n} \int_p A_i \nu_{i,n+1} \psi_i^2\right)^{g+1} = \left(\sum_{i=1}^{n} A_i \psi_i^2\right)^{g+1},$$

where we have again used [Gri17, Lem 5.11].

Example 4.8. Let $\chi = \chi(\partial W)$. By [Gri17, Lem 5.2], we have $\int_p \psi_{n+1} = \chi(\partial W)$. Hence the class

$$a = \chi \nu_{i,n+1} - \psi_{n+1} \in H^*(\mathcal{M}_W(n+1);\mathbb{Q})$$

satisfies $\int_p a = 0$. As in the previous example, we thus find

$$\begin{aligned} 0 &= \left(\int_p (\chi \nu_{i,n+1} - \psi_{n+1})\, \psi_{n+1}^2\right)^{g+1} \\ &= \left(\chi \int_p \psi_i^2 \nu_{i,n+1} - \int_p \psi_{n+1}^3\right)^{g+1} \\ &= \left(\chi \psi_i^2 - \kappa_2\right)^{g+1} \\ &= \sum_{k=0}^{g+1} \binom{g+1}{k} \chi^k \psi_i^{2k} (-\kappa_2)^{g+1-k} \end{aligned}$$

Example 4.9. Let $V_h = (S^n \times D^{n+1})^{\natural h}$ be a genus h handlebody of dimension $2n+1$, where n is odd. Then V_h satisfies the hypotheses of Theorem 4.2, and

$$g = \dim_{\mathbb{Q}} \widetilde{H}_*(V_h;\mathbb{Q}) = h.$$

We have $\chi(\partial V_g) = 2 - 2g$. The class

$$a = \psi_{n+1} + 2\sum_{i=1}^{n} \nu_{i,n+1} \in H^*(\mathcal{M}_{V_g}(n+1);\mathbb{Q})$$

satisfies $\int_p a = 0$. Thus

$$0 = \left(\int_p (\psi_{n+1} + 2\sum_{i=1}^{n} \nu_{i,n+1})\psi_{n+1}^2\right)^{g+1} = \left(\kappa_2 + 2\sum_{i=1}^{n} \psi_i^2\right)^{g+1}.$$

5. Vanishing results for Lagrangian mapping class groups

Let $V_g = (S^1 \times D^2)^{\natural g}$ be a three-dimensional handlebody of genus g. The boundary $\Sigma_g = \partial V_g$ is a genus g surface. We write $\mathrm{Mod}_g = \pi_0 \mathrm{Diff}^+(\Sigma_g)$ for the usual (surface) mapping class group and $\mathrm{HMod}_g = \pi_0 \mathrm{Diff}^+(V_g)$ for the handlebody mapping class group. It is well-known that the restriction map $\mathrm{HMod}_g \to \mathrm{Mod}_g$ is injective, and by identifying Mod_g with its image under this map, we can therefore view HMod_g as a subgroup of Mod_g.

Suppose that V_g is equipped with an orientation. The resulting Stokes orientation on Σ_g gives rise to a non-degenerate intersection pairing

$$\lambda\colon H_1(\Sigma_g;\mathbb{Z}) \otimes H_1(\Sigma_g;\mathbb{Z}) \to \mathbb{Z}. \tag{5.1}$$

Let L denote the kernel of the map $H_1(\Sigma_g;\mathbb{Z}) \to H_1(V_g;\mathbb{Z})$ induced by the inclusion. A straightforward calculation shows that L is a *Lagrangian* for λ, meaning a maximal summand in $H_1(\Sigma_g;\mathbb{Z})$ on which λ restricts to a trivial pairing. By construction, if $f \in \mathrm{HMod}_g$, then $f_*L = L$. The subgroup of Mod_g consisting of all mapping classes that have this property was first studied by Hirose [Hir06], who referred to it as the "homological handlebody group." We follow instead the terminology of Sakasai:

Definition 5.1. The *Lagrangian mapping class group*, denoted LMod_g, is the subgroup

$$\mathrm{LMod}_g = \{f \in \mathrm{Mod}_g \mid f_*L = L\}.$$

Sakasai [Sak12, Thm 7.3] showed that the κ-class $\kappa_{2i-1} \in H^*(\mathrm{LMod}_g;\mathbb{Q})$ vanishes in the stable range. Giansiracusa and Tillmann [GT11] asked whether this vanishing holds outside of the stable range, and with integral rather than rational coefficients. In this appendix, we answer the first question in the affirmative.

Let $\mathrm{Sp}_{2g}^L(\mathbb{Z}) \leq \mathrm{Sp}_{2g}(\mathbb{Z})$ denote the subgroup consisting of transformations which fix the Lagrangian $L \subseteq H_1(\Sigma_g;\mathbb{Z}) = \mathbb{Z}^{\oplus 2g}$, and similarly for $\mathbb{R}$ instead of $\mathbb{Z}$. Consider the commutative diagram

$$\begin{tikzcd} B\mathrm{LMod}_g \arrow[r] \arrow[d] & B\mathrm{Sp}_{2g}^L(\mathbb{Z}) \arrow[r] \arrow[d] & B\mathrm{Sp}_{2g}^L(\mathbb{R}) \arrow[d] & BO(g) \arrow[l, "\sim"'] \arrow[d] \\ B\mathrm{Mod}_g \arrow[r] & B\mathrm{Sp}_{2g}(\mathbb{Z}) \arrow[r] & B\mathrm{Sp}_{2g}(\mathbb{R}) & BU(g), \arrow[l, "\sim"'] \end{tikzcd} \tag{5.2}$$

where the vertical maps are induced by inclusions; the (top and bottom) rightmost horizontal maps are induced by the standard symplectic representation of the mapping class group; the middle horizontal maps are induced by inclusions; and the rightmost horizontal maps are the homotopy equivalences induced by including maximal compact subgroups.

Assume that $g \geq 2$, so that $B\mathrm{Mod}_g$ is homotopy equivalent to the moduli space $\mathcal{M}_g = B\,\mathrm{Diff}^+(\Sigma_g)$ of genus g surfaces [EE69]. The bottom horizontal composition in (5.2) classifies a rank g complex vector bundle on $\mathcal{M}_g$ of genus g surfaces, and it is a standard fact that this agrees with the *Hodge bundle* considered by algebraic geometers, which is traditionally denoted $\mathbb{E}$. Using the Grothendieck–Riemann–Roch formula, Mumford [Mum83] calculated the Chern character

$$\mathrm{ch}(\mathbb{E}) = g + \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!}\kappa_{2i-1} \in H^*(\mathcal{M}_g;\mathbb{Q}), \tag{5.3}$$

where $B_k = -k\zeta(1-k)$ denotes the kth Bernoulli number, $k \geq 2$.

Theorem 5.2. *Let $g \geq 2$. Then $\kappa_{2i+1} = 0 \in H^*(\mathrm{LMod}_g;\mathbb{Q})$ for each $i \geq 0$.*

Proof. Let $\lambda_i = c_i(\mathbb{E})$ denote the ith Chern class of $\mathbb{E}$, $1 \leq i \leq g$. It follows by comparing sides in (5.3) degreewise that every "even" Chern class $\lambda_{2i} \in H^*(\mathcal{M}_g;\mathbb{Q})$, $i \geq 1$, is a polynomial in the "odd" Chern classes that precede it, i.e. the classes $\{\lambda_{2k+1} \mid 0 \leq k < i\}$.

Let $\mathcal{L}_g = B\mathrm{LMod}_g$. The rightmost vertical arrow in (5.3) classifies the procedure of replacing a real vector bundle with its complexification. Hence the complex vector bundle $\mathbb{E}|_{\mathcal{L}_g}$ is isomorphic to the complexification of a real vector bundle. It follows from [MS74, Lem 14.9] that

$$\lambda_{2i+1}|_{\mathcal{L}_g} = c_{2i+1}\left(\mathbb{E}|_{\mathcal{L}_g}\right) = 0 \in H^*(\mathcal{L}_g;\mathbb{Q})$$

for each $i \geq 0$. But since the even Chern classes are polynomials in the odd ones, it then follows that $\lambda_i = 0 \in H^*(\mathcal{L}_g;\mathbb{Q})$ for all $i > 0$. Since $B_{2i} \neq 0$ for $i \geq 1$, it follows from (5.3) that $\kappa_{2i+1} = 0 \in H^*(\mathcal{L}_g;\mathbb{Q})$ for all $i \geq 0$ also. □

Since the handlebody mapping class group HMod_g is a subgroup of LMod_g by construction, Theorem 5.2 also gives another proof of Giansiracusa and Tillmann's vanishing result in the handlebody case [GT11, Thm A], albeit only with rational coefficients.

With integral coefficients, the fact that $\mathbb{E}|_{\mathcal{L}_g}$ is the complexification of a real vector bundle only implies that the odd integral Chern classes $\lambda_{2i+1} \in H^*(\mathcal{L}_g;\mathbb{Z})$ are 2-torsion [MS74, Lem 14.9]. By combining this with the integral Grothendieck–Riemann–Roch formula of [Pap07] or [Mad10] applied to the Hodge bundle, one can determine explicit positive integers n_i such that $n_i\kappa_{2i+1} = 0 \in H^*(\mathcal{L}_g;\mathbb{Z})$, $i \geq 0$. For instance, using [Pap07, Thm 2.2], we compute

$$4\kappa_1 = 0 \in H^2(\mathcal{L}_g;\mathbb{Z}).$$

This method cannot answer the question of whether the odd κ-classes vanish in the integral cohomology $H^*(\mathcal{L}_g;\mathbb{Z})$.

Let W^{2n+2} be an oriented smooth closed manifold. As in § 4.3, we write $\mathcal{M}_W = B\,\mathrm{Diff}^+(W)$. We let $\lambda_W\colon H_{n+1}(W;\mathbb{Z}) \otimes H_{n+1}(W;\mathbb{Z}) \to \mathbb{Z}$ denote the intersection pairing, and consider the group $\mathrm{Aut}(H_{n+1}(W;\mathbb{Z}), \lambda_W)$ of automorphisms of this formed space. Note that λ_W is skew-symmetric due to the Koszul sign rule, so $(H_{n+1}(W;\mathbb{Z}), \lambda_W)$ is a symplectic space. Pick a Lagrangian $L \subseteq H_{n+1}(W;\mathbb{Z})$, and let $\mathrm{Aut}^L(H_{n+1}(W;\mathbb{Z}), \lambda_W) \leq$

$\mathrm{Aut}(H_{n+1}(W;\mathbb{Z}),\lambda_W)$ denote the subroup of automorphisms that fix L as a subspace.

Definition 5.3. The *Lagrangian moduli space of* W, denoted $\mathcal{L}_W$, is defined by the following pullback

$$\begin{tikzcd} \mathcal{L}_W \arrow[r] \arrow[d] & B\,\mathrm{Aut}^L(H_{n+1}(W;\mathbb{Z}),\lambda_W) \arrow[d] \\ \mathcal{M}_W \arrow[r] & B\,\mathrm{Aut}(H_{n+1}(W;\mathbb{Z}),\lambda_W) \end{tikzcd}$$

in the category of spaces $\mathcal{S}$.

We thank Oscar Randal-Williams for pointing out the following generalization of Theorem 5.2:

Theorem 5.4. *Let* $W \to \mathcal{L}_W(1) \xrightarrow{p} \mathcal{L}_W$ *be the pullback of the universal oriented smooth bundle along the canonical map* $\mathcal{L}_W \to \mathcal{M}_W$*, and let* $\mathscr{L}_i(T^p\mathcal{L}_W(1))$ *denote the ith Hirzebruch L-class of its vertical tangent bundle. Then*

$$\int_p \mathscr{L}_i(T^p\mathcal{L}_W(1)) = 0 \in H^*(\mathcal{L}_W;\mathbb{Q}).$$

Proof. Let $\mathcal{H}$ be the complex vector bundle classified by the composition

$$B\,\mathrm{Aut}(H_{n+1}(W;\mathbb{Z}),\lambda_W) \to B\,\mathrm{Aut}(H_{n+1}(W;\mathbb{R}),\lambda_W\otimes\mathbb{R}) \xleftarrow{\sim} BU(d),$$

where $d = \dim_{\mathbb{R}} H_{n+1}(W;\mathbb{R})$. Let $\xi = \overline{\mathcal{H}} - \mathcal{H} \in K^0(B\,\mathrm{Aut}(H_{n+1}(W;\mathbb{Z}),\lambda_W))$ denote the difference of $\mathcal{H}$ and its complex conjugate in complex K-theory. By the rational family signature theorem [RW24, Thm 2.6], we have

$$\int_p \mathscr{L}_i(T^p\mathcal{L}_W(1)) = 2^{2i-n-1}\mathrm{ch}_{2i-n-1}(\phi^*(\xi)),$$

where $\phi\colon \mathcal{L}_W \to B\,\mathrm{Aut}(H_{n+1}(W;\mathbb{Z}),\lambda_W)$ is the composition

$$\mathcal{L}_W \to \mathcal{M}_W \to B\,\mathrm{Aut}(H_{n+1}(W;\mathbb{Z}),\lambda_W),$$

But by construction the map ϕ factors through $B\,\mathrm{Aut}^L(H_{n+1}(W;\mathbb{R}),\lambda_W\otimes\mathbb{R})$, and as in the proof of Theorem 5.2 we then find that $\phi^*(\mathcal{H})$ is the complexification of a real vector bundle. It follows that $\phi^*(\mathcal{H}) = \overline{\phi^*(\mathcal{H})} = \phi^*(\overline{\mathcal{H}}) \in K^0(\mathcal{L}_W)$, whence $\phi^*(\xi) = 0$, giving the desired result. ☐

As with Theorem 5.2, a more careful application of the integral family signature theorem of Randal-Williams [RW24] yields upper bounds on the orders of characteristic classes in integral cohomology.

Appendix A. Schur functors as power operations

In this appendix, we will briefly explain a more categorical perspective on the Schur functors appearing in § 4 above. In the setting of ordinary categories, this perspective is worked out in great detail by Baez, Moeller, and Trimble [BMT24].

Schur functors are certain non-monoidal functors that operate on the underlying category of a symmetric monoidal linear category. To motivate the description given below, recall that the *ring of stable power operations* associated to a commutative ring spectrum $k \in \mathrm{CAlg}(\mathrm{Sp})$ is the endomorphism ring of the forgetful functor

$$\mathrm{fgt}\colon \mathrm{CAlg}(\mathrm{Mod}_k) \to \mathrm{Sp}.$$

(We refer to [GL20] for a treatment of power operations from this perspective.) Since the forgetful functor is represented by the free k-algebra $k\{\sigma\}$ on a single generator, the spectrum-enriched Yoneda lemma implies that the ring of power operations can be identified with the homology of $k\{\sigma\}$. The description of this ring when $k \in \mathrm{CAlg}(\mathrm{Ab})$ is a finite prime field by Mandell [Man01]—as well as the earlier closely related work of Araki–Kudo [KA56], Dyer–Lashof [DL62], Cohen [CLM76], and others—are among the most useful results in homotopy theory.

The notion of power operations also makes sense after going one categorical level up. Let Pr^L denote the (very large) category of presentable categories and colimit-preserving functors between them equipped with the Lurie tensor product. Let $k \in \mathrm{CAlg}(\mathrm{Sp}_{\geq 0})$ be a connective ring spectrum. The symmetric monoidal category $\mathrm{Mod}_k^{\mathrm{cn}}$ of connective k-modules is an algebra object of Pr^L. Write

$$\mathrm{LinCat}_k = \mathrm{Mod}_{\mathrm{Mod}_k^{\mathrm{cn}}}(\mathrm{Pr}^L)$$

for the symmetric monoidal category of k-linear cocomplete categories. Let $\widehat{\mathrm{Cat}}$ denote the (very large) category of large categories. We consider the functor

$$\mathrm{fgt}_{\mathrm{cat}}\colon \mathrm{CAlg}(\mathrm{LinCat}_k) \to \widehat{\mathrm{Cat}}, \tag{A.1}$$

which sends $\mathcal{C} \in \mathrm{CAlg}(\mathrm{LinCat}_k)$ to its underlying category. Both the source and target of this functor admit in a canonical way the structure of 2-categories, and the functor $\mathrm{fgt}_{\mathrm{cat}}$ extends to a functor between these 2-categories.

Recall that the symmetric monoidal groupoid $(\mathrm{Fin}, \sqcup, \varnothing)$ of finite sets under disjoint union is the free symmetric monoidal category on a single generator. Let $(\mathrm{Fun}(\mathrm{Fin}^{\mathrm{op}}, \mathrm{Mod}_k^{\mathrm{cn}}), \otimes_{\mathrm{day}}, \mathbf{1}) \in \mathrm{CAlg}(\mathrm{LinCat}_k)$ be the category of functors with the symmetric monoidal structure of Day convolution. The universal property of Day convolution [Lur17, Exmp 2.2.6.9] implies that $(\mathrm{Fun}(\mathrm{Fin}^{\mathrm{op}}, \mathrm{Mod}_k^{\mathrm{cn}}), \otimes_{\mathrm{day}}, \mathbf{1})$ represents the functor (A.1) in the 2-categorical sense. By the 2-categorical Yoneda lemma [GR17, p. 485], we thus get an equivalence of categories

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}(\mathrm{LinCat}_k), \widehat{\mathrm{Cat}})}(\mathrm{fgt}_{\mathrm{cat}}, \mathrm{fgt}_{\mathrm{cat}}) \simeq \mathrm{Fun}(\mathrm{Fin}^{\mathrm{op}}, \mathrm{Mod}_k^{\mathrm{cn}}). \tag{A.2}$$

Assume now that $k \in \mathrm{CAlg}(\mathrm{Ab})$ is a field of characteristic zero. In this case, the right side of (A.2) is well-understood. Specifically, the full subcategory

$$\mathrm{Fun}(\mathrm{Fin}^{\mathrm{op}}, \mathrm{Perf}_k^{\mathrm{cn}}) \subseteq \mathrm{Fun}(\mathrm{Fin}^{\mathrm{op}}, \mathrm{Mod}_k^{\mathrm{cn}})$$

of finite-dimensional representations is semisimple, and its irreducible objects were classified by Young. Namely, for each unordered partition λ of n, one can construct an irreducible representation V_λ of the symmetric group Σ_n, and this defines a bijection

$$\{\text{unordered partitions of } n\} \to \{\text{irreducible representations of } \Sigma_n\}/\cong,$$

see [JK81, Thm 2.1.11]. Hence irreducible objects of $\mathrm{Fun}(\mathrm{Fin}^{\mathrm{op}}, \mathrm{Perf}_k^{\mathrm{cn}})$ correspond (up to shift) to pairs (n, λ) consisting of a non-negative integer n and an unordered partition λ of n. Namely, to such a pair (n, λ), we associate the functor

$$S_\lambda = (i_n)_! V_\lambda,$$

where i_n is the inclusion into Fin of the full subcategory spanned by the object $\{1, \ldots, n\}$, and $(i_n)_!$ denotes left Kan extension along this inclusion.

Definition A.1. Let $k \in \mathrm{CAlg}(\mathrm{Ab})$ be a field of characteristic zero. The *Schur operation* associated to an unordered partition λ of a non-negative integer n is the natural transformation

$$S^\lambda \in \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}(\mathrm{LinCat}_k), \widehat{\mathrm{Cat}})}(\mathrm{fgt}_{\mathrm{cat}}, \mathrm{fgt}_{\mathrm{cat}})$$

corresponding to

$$S_\lambda \in \mathrm{Fun}(\mathrm{Fin}^{\mathrm{op}}, \mathrm{Mod}_k^{\mathrm{cn}})$$

under (A.2). Given a k-linear presentably symmetric monoidal stable category $\mathcal{C} \in \mathrm{CAlg}(\mathrm{LinCat}_k)$, the Schur operation S^λ restricts to an endofunctor

$$S^\lambda \colon \mathcal{C} \to \mathcal{C},$$

which we refer to as the *Schur functor* on $\mathcal{C}$ associated to λ.

Unwinding definitions, we see that $S^\lambda \colon \mathcal{C} \to \mathcal{C}$ is given on objects by

$$X \mapsto (V_\lambda \otimes X^{\otimes n})_{h\Sigma_n}, \tag{A.3}$$

where Σ_n acts on $X^{\otimes n}$ by permuting the factors. It follows from this description that S^λ agrees with the definition of Schur functors given in § 4.1 above.

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