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Master's thesis

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The Segal Conjecture for Finite Groups

A Beginner's Guide

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Contents

1 Introduction

One of the great triumphs of algebraic topology in the 1980's was the resolution of the Segal conjecture for finite groups, the precise formulation of which will be given in §4. Its formulation and proof is the concern of the present note. The interest in this problem before and after its resolution led to a flurry of work and developments in algebraic topology in the 1970's and 1980's by many people, culminating in Gunnar Carlsson's paper [Car84]. One implication of this is that even the literature for the full treatment of the core story for finite groups is quite scattered and the inevitable amount of forward references as well as folklore logical jumps left unsaid can be quite exhausting for the uninitiated. It is therefore the purpose of this note to gather and flesh out the various strands in the formulation and proof of this conjecture for finite groups, and we hope that this will provide an accessible one-stop resource for fellow graduate students and/or the interested non-specialist mathematician to what we think is a very beautiful story in algebraic topology.

A historical context

We hope that this short historical tour provides not just a motivating context for the problem, but also as a collection of references to the literature. Another good (but inevitably less updated) source for this material is Carlsson's paper [Car84].

It all began with Michael Atiyah's seminal 1961 paper [Ati61], where among other things he computed the complex K-theory of the classifying space BG of a finite group G in terms of the complex representation ring $R(G)$. More precisely, we have the isomorphism $R(G)^{\wedge}_{I(G)} \cong KU^0(BG) = [BG, BU \times \mathbb{Z}]$ coming from the association $V \mapsto \Big(EG \times_G V \to BG\Big)$ where V is a finite-dimensional complex representation of G. Here $I(G) \leq R(G)$ is the augmentation ideal and the left hand side is the completion at this ideal. This is supposed to be taken as an amazing result for (at least) two reasons: (1) the usual technology of complex K-theory is good at handling compact spaces only, and BG is an infinite space when G is a finite group (for example, $BC_2 \simeq \mathbf{R}P^{\infty}$); (2) we have related a slightly mysterious topological thing $KU^*(BG)$ to a reasonably well-understood algebraic gadget $R(G)$. This result has come to be known as the Atiyah-Segal completion theorem.

Now in the same way that representation rings $R(G)$ are naturally related to $BU \times \mathbb{Z}$, which "controls symmetries of complex vector spaces stably," as explained above, Graeme Segal wondered if something similar might be true for the *Burnside ring* $A(G)$ of the group G - this is just the ring generated by the finite G-sets under disjoint unions and products, and it also has an augmentation ideal. More precisely, a famous result of Barratt-Priddy and Quillen says that $QS^0 \simeq B\Sigma_\infty^+ \times \mathbf{Z}$ where $(-)^+$ is Quillen's plus construction and QS_0 is the infinite loop space representing the cohomology theory of stable comotopy π_S^*. And so we see that QS^0 "controls the symmetries of finite sets stably," and we can then wonder about the analogy

$$R(G)^\wedge_{I(G)} \xleftrightarrow{\text{Atiyah}} KU^0(BG)$$

$$A(G)^\wedge_{I(G)} \xleftrightarrow{?} \pi_S^0(BG)$$

This was Segal's Burnside ring conjecture.

The early attempts at attacking the conjecture was via a nonequivariant approach using the Adams spectral sequence. The first success was in the work of Lin Wen-Hsiung [Lin] who proved it for the case $G = \mathbb{Z}/2$. This depended on a hard Ext group calculation which was simplified in the paper by Lin, Don Davis, Mark Mahowald, and Frank Adams [LDMA80]. Jeremy Gunawardena [Gun80] pushed this method and proved it for the case of $G = \mathbb{Z}/p$ where p is odd. Doug Ravenel [Rav81] built on these and proved it for all cyclic groups.

In another vein, Erkki Laitinen [Lai79] proved that the map $A(G)^\wedge_{I(G)} \to \pi_S^0(BG_+)$ is injective for elementary abelians $G = (\mathbb{Z}/p)^r$, and Segal and C.T. Stretch [SS81], [Str81] extended this to all abelian groups. Using Brown-Gitler spectra and the Adams spectral sequence, Carlsson proved in another paper [Car83] for the case of $G = (\mathbb{Z}/2)^k$. Adams, Gunawardena, and Haynes Miller [AGM85] then generalised Lin's methods to settle the case of all elementary abelian p-groups.

At some point, however, it was realised that a purely computational approach was not feasible for general groups, and this is the next part of the story. We look back again to the Atiyah-Segal completion theorem for inspiration. As with many good theorems, there have been many proofs for the completion theorem for KU (a possibly non-exhaustive list being [Ati61], [AS69], [Jac85], [Hae83], [AHJM88a], [Gre93]). What is of note, however, is that the first proof given in [Ati61] proceeded without using genuine equivariant stable homotopy theory and instead used the Atiyah-Hirzebruch spectral sequence, and it was the landmark paper of Atiyah-Segal [AS69] that formulated and solved the problem more slickly in this language - in fact, the theorem was generalised to all compact Lie groups there. Here the formulation is strengthened to the assertion that there is a natural map

$$KU_G^*(S^0)^\wedge_{I(G)} \to KU_G^*(EG_+)$$

which is an isomorphism, where KU_G is the equivariant complex K-theory of [Seg68]. The main lesson, for which all the subsequent proofs have adhered to, is that it is better to formulate a stronger statement at the level of genuine equivariant spectra, where many more tools such as long exact sequences and subgroup inductions/restrictions are available. See [AHJM88a] for a very short and purely homotopy-theoretic proof of the Atiyah-Segal completion theorem starting from equivariant Bott periodicity of KU_G.

There is a notion of genuine sphere G-spectrum representing equivariant stable cohomotopy π_G^*, and Segal [Seg70] and Tammo tom Dieck [tD75] showed that $\pi_G^0(S^0) \cong A(G)$. So by analogy we're led to a stronger form of Segal's Burnside ring conjecture, asserting that there is a natural map

$$\pi_G^*(S^0)^\wedge_{I(G)} \to \pi_G^*(EG_+)$$

which is an isomorphism. Working in this setting, Peter May and Jim McClure [MM82] showed that to prove the conjecture for general groups, it is enough to show it for p-groups and with p-completions instead. Furthermore, the Adams-Gunawardena-Miller [AGM85] paper mentioned above in fact proved the stronger form of the Segal conjecture for elementary abelians. Finally, building on these two works, Carlsson [Car84] proved the strong form of the Segal conjecture for general groups using various ingenious inductive techniques via genuine equivariant stable homotopy theory, invoking along the way such classical results as Quillen's F-isomorphism theorem and Quillen's homotopical analyses of subgroup posets.

To end this subsection, we mention other similar problems in homotopy theory. One general formulation of such problems is the so-called *homotopy limit problem* of Bob Thomason: there is always a map from the limit to the homotopy limit and one wants to understand under what circumstances this map is an equivalence - these are usually very difficult and deep problems in homotopy theory. For example, in the Segal conjecture case, one corollary is that the map $(\mathbb{S}_G)^G \to (\mathbb{S}_G)^{hG}$ exhibits the augmentation completion of $(\mathbb{S}_G)^G$. See [Tho83] or [Car85]. Other spectra for which such homotopy limit/completion theorems hold are $K\mathbb{F}_q$ due to D.L. Rector [Rec74], MU_G due to John Greenlees and May [GM97], and KO_G [AHJM88a]. There is also an unstable analogue of the problem, namely the Sullivan conjecture, the generalised version of which says that for G a p-group and X a finite G-complex, there is a natural map $(X^G)^\wedge_p \to (X^\wedge_p)^{hG}$ which is an equivalence. The original version was proved in a celebrated paper by Miller [Mil84] and later generalised by [Lan92]. More recent related works include [BBLNR14], [NS18], and [HW19]: the first two streamlined Ravenel's inductive procedure for cyclic groups from [Rav81] and the last proves the Segal conjecture for $G = \mathbb{Z}/2$ using new methods from genuine equivariant stable homotopy theory that is less homological algebra heavy than Lin's approach. We mention also that there are various homotopy limit theorems in other areas such as motivic homotopy theory.

Organisation

The backbone reference for this document is the paper of Caruso-May-Priddy [CMP87], which we follow closely, where they've simplified and slightly generalised some of Carlsson's original arguments. Other main references are [Car84], [AGM85], and [MM82]. Sections 2 and 3 introduce the prerequisites on genuine G-spectra and Mackey functors, respectively. Section 4 gives a precise formulation of the conjecture and section 5 presents the reduction procedure to the case of p-groups and p-completions, following [MM82]. In section 6 we introduce the algebraic completions that we'd be working concretely with. Section 7 will then state all the main theorems and give an overview of Carlsson's inductive strategy. Section 8 will solve the "singular" part of the problem, and sections 9 and 10 the "free" part. Section 11 deals with the case of elementary abelian groups. Finally, the appendix gives the proof for the main algebraic ingredient in the reduction to p-groups step.

Disclaimer: At many points in the document I have chosen to err on the side of being careful and explicit in arguments both for my own benefit and also for the benefit of those who are less familiar with the things presented but still want to follow the proofs in detail. For example, I found the relationship between the various types of completions (different ones are used in the statement and the proof of the theorem) quite confusing at first, and so have expended some effort in being very explicit about these. Consequently, many of the proofs appear slightly longer than they should be, as in the original sources, and I apologise for these in advance to the experts in some or all of the areas. Lastly, while I've tried to present complete proofs to most things, the Ext group calculation of [AGM85] is only sketched as it is a whole other (interesting) computational story in its own right.

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2 Genuine equivariant stable homotopy theory

There are many distinct ways to encode the notion of a "spectrum with a symmetry by a group G," or equivariant stable homotopy theory. The first definition that one might conceivably come up with is $\mathrm{Fun}(BG, \mathrm{Sp})$, which we call spectra with G-action following current conventions. Here, a G-map $f : X \to Y$ in $\mathrm{Fun}(BG, \mathrm{Sp})$ is an equivalence if the underlying map of spectra is (that is, it is a π_*-isomorphism). A more sophisticated notion, however, is that of genuine G-spectra, where part of the data of a G-spectrum are its various genuine fixed points and equivalences are tested more stringently againts all these fixed points. This is the setting that will allow us to prove the Segal conjecture. As a point on terminology, we will freely interchange between genuine equivariant stable homotopy theory, genuine G-spectra, and G-spectra.

What makes G-spectra such a useful technology is the presence of different types of fixed points and transfer maps which provide very powerful inductive methods by relating the information between the various subgroups - we will see this in action in at least two places in the proof of the Segal conjecture: the first in §5 where we perform reduction to the case of p-groups and p-completions using the transfer (or Mackey) structure of G-spectra, and the second in §8 where we solve the "singular" part of the problem by a clever induction via geometric fixed points.

In this section we briefly review genuine equivariant stable homotopy theory via orthogonal spectra - this will be the foundation to the rest of the document. We've chosen this model simply because we feel that it's the most commonly used one these days and which has many good sources out there. We assume that the reader is more or less familiar with stable homotopy theory. There are many models for genuine equivariant stable homotopy theory: one of the first ones being the Lewis-May model [LMS86] on which our main reference [CMP87] is based; orthogonal spectra (see [HHR16], [Sch], or [Sch18] for good references); and as spectral Mackey functors (see [Bar17] for an ∞-categorical treatment of this).

The orthogonal spectra model

Let G be a finite group. We now define orthogonal G-spectra following Schwede's book [Sch18]. The main reason for specifying a model is that we will need to have a concrete definition of equivariant cohomology theories and geometric fixed points in order to make various calculations in the rest of the document. Since the precise definitions are quite technical and long-winded, we've chosen only to give enough detail for our purposes, and refer the reader to the book for details. All spaces will be pointed, unless stated otherwise.

Construction 2.1 ([Sch18] 3.1.1)**.** Let V, W be inner product spaces (ie. finite-dimensional $\mathbb{R}$-vector spaces with inner product). Write $\mathbb{L}(V, W)$ for the space of linear isometric embeddings. There is an "orthogonal complement" vector bundle $\xi(V, W) \to \mathbb{L}(V, W)$ given by

$$\xi(V, W) := \{(w, \varphi) \in W \times \mathbb{L}(V, W) \mid w \perp \varphi(V)\}$$

and projection onto the second factor. Write $\mathbb{O}(V, W)$ for the Thom space of the bundle, ie. the one-point compactification of the total space $\xi(V, W)$. Given a third inner product space U, there is an obvious map

$$\xi(V, W) \times \xi(U, V) \to \xi(U, W)$$

which induces an associative "composition"

$$\circ : \mathbb{O}(V, W) \wedge \mathbb{O}(U, V) \to \mathbb{O}(U, W)$$

Now define $\mathbb{O}$ to be the topological category with objects inner product spaces and morphism space from V to W given by $\mathbb{O}(V, W)$.

Definition 2.2. Define $\mathcal{S}_*^G$ to be the (topological or ∞-)category of based G-complexes and based G-maps. A map $f : X \to Y$ is an equivalence iff $f^H : X^H \to Y^H$ is an ordinary based equivalence for all $H \leq G$.

Definition 2.3. An orthogonal G-spectrum is a based continuous functor from $\mathbb{O}$ to the category of based G-spaces $\mathcal{S}_*^G$, and a morphism of orthogonal G-spectra is just a natural transformation of functors. Write Sp^G for the (topological or ∞-)category of orthogonal G-spectra.

Remark 2.4. The definition above is basically a very compact way to encode the various structures we expect from a spectrum. For us the important points will be the following:

- Given any G-representation V, $X(V)$ is then a based $G \times G$-space, coming from the G-action on $X(V) \in \mathcal{S}_*^G$ and the G-action $G \to O(V)$ together with the $O(V)$ functoriality of $X(V)$. We then consider $X(V)$ as a G-space via the diagonal action.

- Given two inner product spaces V and W, we always have the suspension structure maps

$$\sigma_{V,W} : S^V \wedge X(W) \to X(V \oplus W)$$

using the canonical inclusion $S^V \to \mathbb{O}(W, V \oplus W)$ and functoriality of X, thinking of $\mathbb{O}(W, V \oplus W)$ as the morphism space between W and $V \oplus W$. See [Sch18] 3.1.4. When V, W are G-representations, then these structure maps automatically become G-equivariant.

Definition 2.5. For any $A \in \mathcal{S}_*^G$, we can define the suspension G-spectrum $\Sigma^\infty A$ given as $(\Sigma^\infty A)(V) := S^V \wedge A$ with $O(V)$-action on S^V and G-action on A. The structure maps $\sigma_{V,W} : S^V \wedge S^W \wedge A \to S^{V\oplus W} \wedge A$ is just given by the canonical homeomorphism $S^V \wedge S^W \cong S^{V \oplus W}$. We will also write Σ_G^∞ when we want to emphasise that we're taking the suspension spectrum for the group G.

Definition 2.6. The sphere G-spectrum $\mathbb{S}_G$ is given by $\Sigma^\infty S^0$, where $S^0 \in \mathcal{S}_*^G$ has trivial G-action.

Warning 2.7. Even though $\mathbb{S}_G$ was defined in terms of spaces with trivial G-actions, it is far from being equivariantly uninteresting. The point is that when we evaluate at a G-representation V, $\mathbb{S}_G(V)$ has an interesting G-equivariance coming from V.

Fact 2.8. It turns out that Sp^G is a closed symmetric monoidal stable (topological or ∞-)category with unit $\mathbb{S}_G$, smash product denoted by $\otimes$, and the function spectrum denoted by $F(-,-)$ (or $F_H(-,-)$ if we want to emphasise that we're taking it in Sp^H for $H \leq G$). The suspension spectrum is then a strong monoidal functor, ie. for $X, Y \in \mathcal{S}_*^G$ we have

$$\Sigma^\infty(X \wedge Y) \simeq \Sigma^\infty X \otimes \Sigma^\infty Y$$

Definition 2.9. An orthogonal ring G-spectrum is then just a monoid object in Sp^G. In other words, an orthogonal ring G-spectrum is some $R \in \mathrm{Sp}^G$ equipped with a multiplication $R \otimes R \to R$ and a unit $\mathbb{S}_G \to R$ such that the associativity and unit diagrams commute. We write $\mathrm{CAlg}(\mathrm{Sp}^G)$ for the category of commutative ring G-spectra with ring morphisms.

Notation 2.10. We will denote by $\mathrm{Map}_{\mathrm{Sp}^G}(-,-)$ the mapping space in the (topological or ∞-)category Sp^G - this is canonically enriched as a spectrum because Sp^G was a stable category, and we write $\mathrm{map}_G(-,-)$ for the mapping spectrum. Finally, by limits and colimits, we always mean homotopy limits and homotopy colimits.

Restrictions, inductions and finite G-sets

Let $H \leq G$. Then we have the following spectral induction-restriction-coinduction adjunction

$$\mathrm{Ind}_H^G : \mathrm{Sp}^H \rightleftarrows \mathrm{Sp}^G : \mathrm{Res}_H^G$$

$$\mathrm{Res}_H^G : \mathrm{Sp}^G \rightleftarrows \mathrm{Sp}^H : \mathrm{Coind}_H^G$$

Here Res_H^G is strong monoidal. In terms of our concrete model, Res_H^G is just given by restricting the G-action to the H-action on our based G-spaces. We also have the adjunction

$$\mathrm{Ind}_H^G = G_+ \wedge_H - : \mathcal{S}_*^H \rightleftarrows \mathcal{S}_*^G : \mathrm{Res}_H^G$$

at the level of based G-complexes, such that if $X \in \mathcal{S}_*^G$, then $\mathrm{Ind}_H^G \mathrm{Res}_H^G X$ is G-homeomorphic to $G/H_+ \wedge X$. Furthermore, we also have $\mathrm{Map}_{\mathcal{S}_*^G}(G/H_+, X) \simeq X^H \in \mathcal{S}_*$.

One of the bread and butter toolbox in manipulating G-spectra is the following omnibus theorem, all of whose parts are intimately related.

Theorem 2.11. *Let $H \leq G$ be a subgroup.*

1. *(Wirthmuller isomorphism) $\mathrm{Ind}_H^G \simeq \mathrm{Coind}_H^G$. In particular, Ind_H^G and Res_H^G are both left and right adjoints, and so preserve all small limits and small colimits.*
2. *For $X \in \mathrm{Sp}^G$, $\mathrm{Ind}_H^G \mathrm{Res}_H^G X \simeq G/H_+ \otimes X$.*

3. *Finite G-sets are canonically self-dual in Sp^G. In particular, for $X, Y \in Sp^G$ we have*

$$Map_{Sp^G}(X, G/H_+ \otimes Y) \simeq Map_{Sp^G}(G/H_+ \otimes X, Y)$$

4. *(Frobenius formula) $Ind_H^G(X \otimes Res_H^G Y) \simeq Ind_H^G X \otimes Y$*

Corollary 2.12. *Let $C \in Sp^H$ and $B, Y, Z \in Sp^G$. Then we have*

$$Res_H^G F_G(Y, Z) \simeq F_H(Res_H^G Y, Res_H^G Z) \quad \text{and} \quad Ind_H^G F_H(Res_H^G B, C) \simeq F_G(B, Ind_H^G C)$$

Proof. Let $X \in \mathrm{Sp}^H$. Then

$$\begin{aligned} \mathrm{Map}_{\mathrm{Sp}^H}(X, \mathrm{Res}_H^G F_G(Y, Z)) &\simeq \mathrm{Map}_{\mathrm{Sp}^G}(\mathrm{Ind}_H^G X \otimes Y, Z) \\ &\simeq \mathrm{Map}_{\mathrm{Sp}^G}(\mathrm{Ind}_H^G(X \otimes \mathrm{Res}_H^G Y), Z) \\ &\simeq \mathrm{Map}_{\mathrm{Sp}^H}(X, F_H(\mathrm{Res}_H^G Y, \mathrm{Res}_H^G Z)) \end{aligned}$$

The other one is done similarly, using the Wirthmuller isomorphism to say Ind_H^G is right adjoint to Res_H^G. □

Genuine fixed points and geometric fixed points

One of the slightly daunting things for those who are seeing G-spectra for the first time is the multitude of fixed points, namely that of homotopy fixed points $(-)^{hG}$, genuine fixed points $(-)^G$, and geometric fixed points $(-)^{\Phi G}$. The notion of homotopy fixed points is already available at the level of spectra with G-action $\mathrm{Fun}(BG, \mathrm{Sp})$: for $X \in \mathrm{Fun}(BG, \mathrm{Sp})$, this is just taking the limit $X^{hG} := \lim_{BG} X \in \mathrm{Sp}$.

Definition 2.13. A complete G-universe is a G-representation $\mathcal{U}$ of countable real dimension that contains infinitely many copies of every irreducible G-representation. One concrete model for it is $\oplus^\infty \rho_G$ where ρ_G is the regular representation of G. Complete universes satisfy the following closure properties:

- For $H \leq G$, the infinite-dimensional H-representation $\mathrm{Res}_H^G \mathcal{U}$ is also a complete H-universe.
- For $K \triangleleft H \leq G$, the infinite-dimensional (H/K)-representation $\mathcal{U}^K = (\mathrm{Res}_H^G \mathcal{U})^K$ is a complete H/K-universe.

While using a particular complete G-universe involves a choice, all notions in sight will turn out to be independent of this choice.

Definition 2.14. Let $X \in \mathrm{Sp}^G$, and $K \triangleleft H \leq G$. Fix a complete G-universe $\mathcal{U}$. We define the H/K-geometric fixed point of X, $\Phi^{H/K} X \in \mathrm{Sp}^{H/K}$ as follows: for any $V^K \in \mathcal{U}^K$ define

$$(\Phi^{H/K} X)(V^K) := (\mathrm{Res}_H^G X(V))^K \in \mathcal{S}_*^{H/K}$$

When the context of $K \triangleleft H$ is clear (for example when $H = G$), we simply write $\Phi^K X$, and to save on notation, if we denote H/K by J, we also write $\Phi^J X := \Phi^{H/K} X$.

Warning 2.15. The convenient notation $\Phi^J X$ hides the fact that this really depends on $K \triangleleft H$. For example, in general $\Phi^{G/G} X \not\simeq \Phi^e X$ even though $G/G \cong e$.

While we won't really be needing all the categorical properties of these various fixed points, we think it's helpful to lay out the organising principles for them and summarise the situation abstractly by stating the following omnibus result. See [NS18] II.2 for a good reference and [Wil17] Notation 1.33 for an abstract but very general treatment of these adjunctions from the spectral Mackey functor point of view.

Theorem 2.16 (Fixed points)**.** *Let $N \triangleleft G$ be a normal subgroup.*

(a) *(Genuine fixed points) We have the following adjunctions*

$$i_! : Sp^{G/N} \rightleftarrows Sp^G : (-)^N$$

$$(-)^N : Sp^G \rightleftarrows Sp^{G/N} : i_*$$

The functor $i_!$ is strong monoidal, and so by abstract nonsense, the genuine fixed point functor $(-)^N$ is lax monoidal. When $N = G$, $i_!$ is the functor that associates to an ordinary spectrum the genuine G-spectrum with trivial action.

While it won't be important to know what the functor i_* is, the point of the second adjunction is that it shows that $(-)^N$ is also a left adjoint, and so $(-)^N$ preserves all small limits and small colimits.

Here we see that for $X \in Sp^G$, X^N is not just an ordinary spectrum, but also has a residual equivariance on the group G/N. Forgetting the residual action, the genuine fixed points are corepresentable: that is, we have $X^H \simeq map_{Sp^G}(G/H_+, X) \in Sp$.

Fixed points are transitive: if $K \triangleleft H \triangleleft G$ and $K \triangleleft G$, then $((-)^K)^{H/K} \simeq (-)^H$.

(b) *(Geometric fixed points) We have the following adjunction*

$$\Phi^N : Sp^G \rightleftarrows Sp^{G/N} : \Xi^N$$

where the geometric fixed point functor Φ^N is strong monoidal, and being a left adjoint, preserves small colimits.

As above, note that for $X \in Sp^G$, $\Phi^N X$ has a residual G/N-equivariance and not just an ordinary spectrum. Following [Wil17] we use the notation $X^{\Phi N}$ to denote the underlying nonequivariant spectrum of $\Phi^N X$.

Finally, a very important property of geometric fixed points is that for $X \in \mathcal{S}_*^G$, we have that

$$\Phi^N(\Sigma_G^\infty X) \simeq \Sigma_{G/N}^\infty X^N$$

Geometric fixed points are transitive: if $K \triangleleft H \triangleleft G$ and $K \triangleleft G$, then $\Phi^{H/K}\Phi^K \simeq \Phi^H$.

(c) *(Homotopy fixed points) There is a natural functor $Sp^G \to Fun(BG, Sp)$ given by $X \mapsto X^e$ and remembering the G-action which induces, for each $H \leq G$, a small limit-preserving functor*

$$(-)^{hH} : Sp^G \to Fun(BG, Sp) \to Sp$$

A concrete model for X^{hH} when $X \in Sp^G$ is given by

$$X^{hH} = F(EG_+, X)^H$$

Remark 2.17. The category Sp^G is compactly generated by the transitive orbits $\{G/H_+\}_{H \leq G}$, and this in particular means that equivalences in Sp^G can be tested by applying $\mathrm{map}_{\mathrm{Sp}^G}(G/H_+, -)$ for all $H \leq G$. By the corepresentability of genuine fixed points stated above, we get the statement that a map of G-spectra $X \to Y$ is an equivalence iff all the induced maps $X^H \to Y^H$ for all $H \leq G$ is an equivalence.

Observation 2.18. For $\mathbb{S}_G$ and $K \triangleleft H \leq G$, we have $\Phi^K(\mathrm{Res}_H^G \mathbb{S}_G) = \Sigma_{H/K}^\infty S^0 = \mathbb{S}_{H/K}$, and so the geometric fixed points of sphere spectra are the sphere spectra for subquotient groups. This closure property will be crucial to Carlsson's inductive proof of the Segal conjecture.

Theorem 2.19 (Segal-tom Dieck splitting)**.** *For $X \in \mathcal{S}_*^G$, we have an equivalence*

$$(\Sigma_G^\infty X)^G \simeq \bigoplus_{(H)} \Sigma^\infty \left(EWH_+ \wedge_{WH} X^H\right)$$

where the sum runs over conjugacy classes (H) of subgroups of G and $WH := NH/H$ is the Weyl group of H. In particular when $X = S^0$ we get

$$(\mathbb{S}_G)^G \simeq \bigoplus_{(H)} \Sigma^\infty BWH_+$$

Remark 2.20. This theorem highlights the subtlety of the notion of genuine fixed points: even for $\mathbb{S}_G$ which comes from the G-space S^0 with trivial G-action, the genuine fixed points is anything but simple. In contrast, as stated in Theorem 2.16, geometric fixed points interact very nicely with the suspension functor.

Remark 2.21. In the proof of the Segal conjecture, this theorem will be important to us since it guarantees that the equivariant homotopy groups of $\mathbb{S}_G$ are finitely generated abelian groups.

Equivariant homotopy groups and cohomology theories

Fix a complete G-universe $\mathcal{U}_G$ once and for all, and write $\mathcal{U}_H := \mathrm{Res}^G_H \mathcal{U}_G$. For $K \lhd H \leq G$, write $\mathcal{U}_{H/K} = (\mathrm{Res}^G_H \mathcal{U}_G)^K$. For $\mathcal{U}$ a complete G-universe, we denote by $s(\mathcal{U})$ the poset of finite-dimensional G-subrepresentations under inclusions.

Notation 2.22. Let A, B be pointed G-spaces. Then we denote by $[A, B]_G$ the set of G-homotopy classes of based G-maps from A to B.

Definition 2.23. Let $k \in \mathbb{Z}$, $X \in \mathrm{Sp}^G$, and $H \leq G$. We define the k-th H-equivariant stable homotopy group of X as

$$\pi_k^H(X) := \begin{cases} \mathrm{colim}_{V \in s(\mathcal{U}_H)} [S^V \wedge S^k, \mathrm{Res}^G_H X(V)]_H & \text{if } k \geq 0 \\ \mathrm{colim}_{V \in s(\mathcal{U}_H)} [S^V, \mathrm{Res}^G_H X(V \oplus \mathbb{R}^{-k})]_H & \text{if } k < 0 \end{cases}$$

We collect some standard properties of equivariant stable homotopy groups in the following.

Proposition 2.24. *Let $X \in Sp^G$ and $H \leq G$.*

(a) The equivariant stable homotopy group really does have a canonical abelian group structure, justifying the name.

(b) They are independent of the choice of a cofinal family, and so in particular the cofinal family $\{n\rho_G\}_n$ can be used.

(c) There is a natural isomorphism $\pi_k^H(X) \cong \pi_k(X^H)$ where the latter is just the usual stable homotopy group of a spectrum.

(d) $\pi_k^{(-)}(X)$ for the various subgroups of G collect into a Mackey functor, which we shall introduce in the next section.

Example 2.25. For M a $\mathbb{Z}G$-module, we can always construct an Eilenberg-MacLane G-spectrum $\underline{HM}$ satisfying

$$\pi_q^H \underline{HM} = \begin{cases} M^H & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, evaluating at a G-representation V satisfies $\underline{HM}(V) \simeq K(M, n)$ nonequivariantly, where $n = \dim V$. See example 2.13 of [Sch] for details.

Definition 2.26. Let G be a finite group. The Burnside ring $A(G)$ is the commutative unital ring that is finitely generated as an abelian group by the finite G-sets under disjoint union, and multiplication is given by taking products of G-sets.

Theorem 2.27 (Segal)**.** $\pi_0^G(\mathbb{S}_G) \cong A(G)$.

Definition 2.28. Let $k \in \mathbb{Z}$, $E \in \mathrm{Sp}^G$, and $H \leq G$. Then E defines a cohomology theory $E^*_H : \mathcal{S}^H_* \to \prod_{\mathbb{Z}} \mathcal{A}b$ given as follows: for $X \in \mathcal{S}^H_*$

$$E^k_H(X) := \begin{cases} \mathrm{colim}_{V \in s(\mathcal{U}_H)} [S^V \wedge X, \mathrm{Res}^G_H E(V \oplus \mathbb{R}^k)]_H & \text{if } k \geq 0 \\ \mathrm{colim}_{V \in s(\mathcal{U}_H)} [S^V \wedge S^k \wedge X, \mathrm{Res}^G_H E(V)]_H & \text{if } k < 0 \end{cases}$$

We also define $E_k^H(X) := E_H^{-k}(X)$. When we take the underlying spectrum of E and consider the ordinary cohomology theory, we write it as $E^k(-)$.

Notation 2.29. The cohomology theories above can be interpreted as follows: for $E \in \mathrm{Sp}^G$ and $X \in \mathcal{S}^G_*$ we have for $k \in \mathbb{Z}$

$$E^k_G(X) \cong \pi_0 \mathrm{Map}_{\mathrm{Sp}^G}(\Sigma^\infty_G X, S^k \otimes E)$$

where S^k for $k < 0$ is the tensor inverse of S^{-k}. Because of this, we will also often write $[X, \Sigma^k E]_G$ for $E^k_G(X)$ and so on.

Notation 2.30. We write $\pi^q_H(X) := (\mathbb{S}_G)^q_H(X)$ for the equivariant stable cohomotopy groups.

Proposition 2.31. *For $X \in \mathcal{S}_*^G$ and $E \in Sp^G$, we have*

$$E_G^q(G/H_+ \wedge X) \cong E_H^q(X)$$

Proof. Using Theorem 2.11 we get

$$E_G^q(G/H_+ \wedge X) = [\mathrm{Ind}_H^G \mathrm{Res}_H^G X, \Sigma^q E]_G \cong [\mathrm{Res}_H^G X, \Sigma^q \mathrm{Res}_H^G E]_H = E_H^q(X)$$

□

Universal spaces and isotropy separation

For $\mathcal{F}$ a family of subgroups of G, that is, a collection of subgroups of G closed under subconjugations, there is an associated universal space $E\mathcal{F} \in \mathcal{S}^G$ uniquely characterised

$$E\mathcal{F}^H \simeq \begin{cases} * & \text{if } H \in \mathcal{F} \\ \emptyset & \text{if } H \notin \mathcal{F} \end{cases}$$

We can then define the pointed G-space $\widetilde{E\mathcal{F}}$ as the cofibre in $\mathcal{S}_*^G$ of

$$E\mathcal{F}_+ \to S^0 \to \widetilde{E\mathcal{F}}$$

This will then be uniquely characterised by

$$\widetilde{E\mathcal{F}}^H \simeq \begin{cases} * & \text{if } H \in \mathcal{F} \\ S^0 & \text{if } H \notin \mathcal{F} \end{cases}$$

Two families will be important for our purposes, namely $\{e\}$ the trivial family and $\mathcal{P}$ the family of proper subgroups. For the case of $\mathcal{F} = \{e\}$ we write $EG := E\{e\}$ and $\widetilde{EG} := \widetilde{E\{e\}}$ instead. Just like in other parts of genuine equivariant stable homotopy theory, the cofibre sequence displayed will be one of the key ideas in the proof of the Segal conjecture, as it separates the problem into the "free" part EG_+ and the "singular" part $\widetilde{EG}$.

Construction 2.32 (Carlsson's model)**.** We introduce a particularly convenient model for $\widetilde{E\mathcal{P}}$ following Carlsson. Let V be the reduced regular complex representation of G, that is, $V = \rho_G - \mathbb{R}\{\sum_{g \in G} g\}$. Let $\mathcal{X} = \bigcup S^{nV}$. Since $V^G = 0$ we get $\mathcal{X}^G = S^0$. If $H \lneq G$ then V has a trivial H-summand (namely the one-dimensional subspace generated by the sum of all elements of H) and so $\mathcal{X}^H \simeq S^\infty$ is contractible.

Split theories

For $E \in \mathrm{Sp}^G$ there is always a canonical map of ordinary spectra $E^G \to E$ (where the target is considered as the underlying spectrum of E). We say that E is *split* if there is a map of ordinary spectra $E \to E^G$ such that the composite

$$E \to E^G \to E$$

is homotopic to the identity. For our purposes, the importance of this notion comes from the following property:

Proposition 2.33. *If E is a split G-spectrum and $X \in \mathcal{S}_*^G$ is a free based G-space, then $E_G^*(X) \cong E^*(X/G)$.*

Example 2.34. $\mathbb{S}_G$ is a split theory since the Segal-tom Dieck splitting 2.19 gives $(\mathbb{S}_G)^G \simeq \bigoplus_{(H)} \Sigma^\infty BWH_+$, and then it can be checked that the inclusion of the summand $\mathbb{S} = \Sigma^\infty BWG_+ \to \bigoplus_{(H)} \Sigma^\infty BWH_+ \simeq (\mathbb{S}_G)^G$ gives the required splitting.

Completions

For this part we will summarise the notions we need from [GM92]. The point is that for any ideal $I \leq A(G)$ there is a concrete and easily manipulated functor $X \mapsto X_I^\wedge$ such that:

- $X_I^\wedge$ is local in the categorical sense of Bousfield (which we shall explain below).
- When X is sufficiently finite, then $\pi_*^G(X_I^\wedge) \cong (\pi_*^G X)_I^\wedge$.

Throughout this section let $I \leq A(G) \cong \pi_0^G(\mathbb{S}_G)$ be an ideal, and since $A(G)$ is Noetherian, we have that $I = (a_1, \cdots, a_n)$ for some choice of generators.

Definition 2.35. We define

$$\mathbb{S}_G/\underline{a} := \mathbb{S}_G/a_1 \otimes \cdots \otimes \mathbb{S}_G/a_n$$

and $M(a_i)$ as the fibre in the sequence

$$M(a_i) \to \mathbb{S}_G \to \mathbb{S}_G[a_i^{-1}]$$

We then define $M(\underline{a}) := M(a_1) \otimes \cdots \otimes M(a_n)$.

Remark 2.36. It turns out that $M(\underline{a})$ is independent of the choice of generators, and so we can also write $M(I)$ instead. See [GM95a] page 11, for example.

Definition 2.37. By the remark above, the following is a well-defined notion: a G-spectrum $W \in \mathrm{Sp}^G$ is said to be I-acyclic if $W \otimes M(I) \simeq *$.

Definition 2.38 (Bousfield I-completeness)**.** A G-spectrum $X \in \mathrm{Sp}^G$ is said to be I-complete if for any I-acyclic spectrum $W \in \mathrm{Sp}^G$, we have that $\mathrm{Map}_{\mathrm{Sp}^G}(W, X) \simeq *$. A map of G-spectra $X \to Y$ is said to be an I-completion if Y is I-complete and the fibre is I-acyclic (equivalently, if the map becomes an equivalence after tensoring with $M(I)$).

Theorem 2.39 ([GM92] 1.6 and 2.3)**.** *Let $X \in Sp^G$.*

(a) The natural map $M(I) \to \mathbb{S}_G$ induces the map

$$X \simeq F(\mathbb{S}_G, X) \to F(M(I), X)$$

exhibiting the I-completion of X.

(b) If X was bounded below and finite type (that is $\pi_n^H(X) = 0$ for all H for small enough n and each of them are finitely generated abelian groups), then $\pi_n^H(X) \to \pi_n^H(X_I^\wedge)$ exhibits the I-completion in the usual algebraic sense, that is, it is the canonical map where $\pi_n^H(X_I^\wedge) \cong \lim_n \pi_n^H(X)/I^n \cdot \pi_n^H(X)$.

Remark 2.40. In particular, we now know what it means to complete a G-spectrum at the augmentation ideal $I(G) \leq A(G)$ and to p-complete it, namely using $I = (p) \leq A(G)$, and we even have a concrete model for these completions using function spectra. Furthermore, since the sphere G-spectrum is bounded below and finite type, these completions behave as expected on the equivariant homotopy groups.

Notation 2.41. For the augmentation ideal $I(G) \leq A(G)$ we denote $I(G)$-completion by $(-)_{I(G)}^\wedge$; we denote p-completion by $(-)_p^\wedge$.

The remaining part of this section will not strictly be needed to understand the rest of the document and can be safely skipped, but we've included it just to clarify that p-completions commute with genuine fixed points.

Theorem 2.42 ([GM92] 2.2)**.** *For $W \in Sp^G$, we have that $W \otimes \mathbb{S}_G/\underline{a} \simeq *$ iff $W \otimes M(\underline{a}) \simeq *$. That is, I-acyclicity can equally well be tested with $\mathbb{S}_G/\underline{a}$.*

Proposition 2.43. *Let $X \in Sp^G$.*

(a) If X was p-complete as a G-spectrum, then X^H is also p-complete as an ordinary spectrum for all $H \leq G$.

(b) Furthermore, the map $X^H \to (X_p^\wedge)^H$ exhibits $(X_p^\wedge)^H$ as p-completion of X^H in Sp.

Proof. We will apply Corollary 2.12 at various points in the proof to manipulate restrictions and function spectra. The main point to note is that if $i_! : \mathrm{Sp} \to \mathrm{Sp}^H$ is the strong monoidal left adjoint to $(-)^H$ then we have an identification of cofibre sequences

$$i_!\Big(\mathbb{S} \xrightarrow{p} \mathbb{S} \to \mathbb{S}/p\Big) \simeq \Big(\mathbb{S}_H \xrightarrow{p} \mathbb{S}_H \to \mathbb{S}_H/p\Big)$$

since $i_!$ is left adjoint, and so preserves cofibre sequences. Besides that, Res^G_H commutes with colimits and so commutes with $(-)[p^{-1}]$ and hence

$$\mathrm{Res}^G_H\Big(M_G((p)) \to \mathbb{S}_G \to \mathbb{S}_G[p^{-1}]\Big) \simeq \Big(M_H((p)) \to \mathbb{S}_H \to \mathbb{S}_H[p^{-1}]\Big)$$

Also note that $\mathrm{Res}^G_H(X^\wedge_p)$ is a p-complete genuine H-spectrum since $X^\wedge_p \xrightarrow{\simeq} F_G(M_G((p)), X^\wedge_p)$ gives

$$\mathrm{Res}^G_H(X^\wedge_p) \xrightarrow{\simeq} \mathrm{Res}^G_H F_G(M_G((p)), X^\wedge_p) \simeq F_H(M_H((p)), \mathrm{Res}^G_H(X^\wedge_p))$$

and moreover $\mathrm{Res}^G_H X \to \mathrm{Res}^G_H(X^\wedge_p)$ exhibits the p-completion of $\mathrm{Res}^G_H X$ since

$$F_H(M_H((p)), \mathrm{Res}^G_H X) \simeq \mathrm{Res}^G_H F_G(M_G((p)), X) \simeq \mathrm{Res}^G_H(X^\wedge_p)$$

(a) To see $X^H = (\mathrm{Res}^G_H X)^H$ is p-complete if X is, let $Z \in \mathrm{Sp}$ be such that $Z \otimes \mathbb{S}/p \simeq *$. We need to show that $\mathrm{Map}_{\mathrm{Sp}}(Z, X^H) \simeq *$. Now

$$\mathrm{Map}_{\mathrm{Sp}}(Z, X^H) \simeq \mathrm{Map}_{\mathrm{Sp}^H}(i_!Z, \mathrm{Res}^G_H X)$$

and $i_!Z \otimes \mathbb{S}_H/p \simeq i_!Z \otimes i_!\mathbb{S}/p \simeq i_!(Z \otimes \mathbb{S}/p) \simeq *$, so the latter mapping space is contractible since $\mathrm{Res}^G_H X$ was a p-complete H-spectrum.

(b) Let us note that

$$i_!\Big(M((p)) \to \mathbb{S} \to \mathbb{S}[p^{-1}]\Big) \simeq \Big(M_H((p)) \to \mathbb{S}_H \to \mathbb{S}_H[p^{-1}]\Big)$$

since $i_!$ preserves colimits so commutes with $(-)[p^{-1}]$. By part (a), to see that $X^H \to (X^\wedge_p)^H$ exhibits p-completion of X^H we just need to show that

$$F(M((p)), X^H) \to F(M((p)), (X^\wedge_p)^H)$$

is an equivalence. Let $Z \in \mathrm{Sp}$, and testing against this by applying $\mathrm{Map}_{\mathrm{Sp}}(Z, -)$ and unwinding adjunctions we get the equivalence (since $\mathrm{Res}^G_H(X^\wedge_p)$ was p-completion of $\mathrm{Res}^G_H X$ in Sp^H)

$$\mathrm{Map}_{\mathrm{Sp}^H}(i_!Z, F_H(M_H((p)), \mathrm{Res}^G_H X)) \to \mathrm{Map}_{\mathrm{Sp}^H}(i_!Z, F_H(M_H((p)), \mathrm{Res}^G_H(X^\wedge_p)))$$

where again we've used $i_!M((p)) \simeq M_H((p))$. Hence $F(M((p)), X^H) \to F(M((p)), (X^\wedge_p)^H)$ is equivalence as required.

□

3 Mackey functors

We now introduce Mackey functors. One of the canonical references for this is chapter 6 of [tD79], and everything in this section (except possibly the last lemma) are standard. We provide proofs to some of them to give a taste of how things go with Mackey functors.

Definition 3.1. Let G be a finite group and $G\mathrm{Set}$ be the category of finite G-sets and G-maps. A bifunctor

$$M = (M^*, M_*) : G\mathrm{Set} \to \mathcal{A}b$$

is a pair of functors with M^* contravariant and M_* covariant which agree on objects. For a G-map $f : S \to T$ write $f^* := M^*f$ and $f_* := M_*f$. A bifunctor M is called a *Mackey functor* if it satisfies the following two properties:

(a) (Additivity) The homomorphism $M^*(S\bigsqcup T)\to M^*(S)\oplus M^*(T)$ induced from $S\to S\bigsqcup T\leftarrow T$ is an isomorphism.

(b) (Double coset axiom) For any pullback diagram in GSet

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "h"'] & S \arrow[d, "g"] \\ T \arrow[r, "k"'] & V \end{tikzcd}$$

the diagram

$$\begin{tikzcd} M(U) \arrow[r, "f_*"] & M(S) \\ M(T) \arrow[u, "h^*"] \arrow[r, "k_*"'] & M(V) \arrow[u, "g^*"'] \end{tikzcd}$$

commutes.

Remark 3.2. We've followed the definition in §6 of [tD79], but we point out that there are at least three different ways to define Mackey functors, some better for calculations and some for a more conceptual understanding. We refer the reader to [Luc96] for example for a nice exposition on this.

Notation 3.3. We will often use the underline notation $\underline{M}$ to emphasise that something is a Mackey functor. As is common in the literature, we will also freely interchange between writing $M(H)$ and $M(G/H)$ for $H\leq G$ - this will usually not cause any confusions.

Definition 3.4. Let M be a Mackey functor for the group G and S a finite G-set.

(a) Define the Mackey functor M_S by $M_S(T):=M(S\times T)$. We then have a map

$$\theta_S : M_S\to M \quad \text{and} \quad \theta^S : M\to M_S$$

given by $\theta_S(T)=\pi_*$ and $\theta^S(T)=\pi^*$ where $\pi:S\times T\to T$ is the projection.

(b) We say that M is S-projective if $\theta_S:M_S\to M$ is split-surjective.

(c) We say that M is S-injective if $\theta^S:M\to M_S$ is split-injective.

Proposition 3.5 ([tD79] 6.1.3). *A Mackey functor M is S-projective iff it is S-injective.*

Construction 3.6. For $S\in G$Set, let $S^0=*$ and $S^k=\prod_{i=0}^{k-1}S$ and $\pi_i:S^{k+1}\to S^k$ denote the projection omitting the i-th factor for $0\leq i\leq k$. For M a Mackey functor we have two chain complexes

$$0\to M(S^0)\xrightarrow{d^0}M(S^1)\xrightarrow{d^1}M(S^2)\xrightarrow{d^2}\cdots$$

$$0\leftarrow M(S^0)\xleftarrow{d_0}M(S^1)\xleftarrow{d_1}M(S^2)\xleftarrow{d_2}\cdots$$

where $d^k=\sum_{i=0}^{k}(-1)^i\pi_i^*$ and $d_k=\sum_{i=0}^{k}(-1)^i\pi_{i*}$.

The following proposition says that these chain complexes give us "S-injective and S-projective resolutions".

Proposition 3.7 ([tD79] 6.1.6). *Let M be a Mackey functor. Then*

(a) M_S is always S-injective and S-projective.

(b) If M is S-injective then the chain complexes above are exact.

Proof. We show the exactness of the first chain complex in part (b). Let $\psi:M_S\to M$ be a splitting. We use it as a contracting homotopy for the chain complex as follows: consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & M(S^0) \arrow[r, "d^0"] \arrow[d, equal] & M(S^1) \arrow[r, "d^1"] \arrow[d, equal] \arrow[dl, "\psi"'] & M(S^2) \arrow[r, "d^2"] \arrow[d, equal] \arrow[dl, "\psi"'] & \cdots \arrow[dl, "\psi"'] \\
0 \arrow[r] & M(S^0) \arrow[r, "d^0"] & M(S^1) \arrow[r, "d^1"] & M(S^2) \arrow[r, "d^2"] & \cdots
\end{tikzcd}$$

This is a contracting homotopy since for example

$$\psi \circ (\pi_0^* - \pi_1^*) + \pi_0^* \psi = id - \psi \pi_1^* + \pi_0^* \psi = id$$

where we've used that we have diagrams like

$$\begin{tikzcd}
M(S \times S^{k-1}) \arrow[r, "\pi_{j+1}^*"] \arrow[d, "\psi"] & M(S \times S^k) \arrow[d, "\psi"] \\
M(S^{k-1}) \arrow[r, "\pi_j^*"] & M(S^k)
\end{tikzcd}$$

by the naturality of the splitting ψ. □

We now introduce Green functors which are the algebra objects in the abelian category of Mackey functors.

Definition 3.8. A Green functor $U : G\text{Set} \to \mathcal{A}b$ is a Mackey functor together with the data of a collection of bilinear maps

$$U(S) \times U(S) \to U(S) \quad :: \quad (x, y) \mapsto x \cdot y$$

for each $S \in G$Set satisfying:

(a) Each of these maps are bilinear.

(b) For each $S \in G$Set these maps make $U(S)$ into a unital associative ring.

(c) For $f : S \to T$ a G-map, $f^* : U(T) \to U(S)$ is a unital ring map.

(d) (Frobenius conditions) For any G-map $f : S \to T$ we have

$$f_*(f^* x \cdot y) = x \cdot f_* y$$

$$f_*(x \cdot f^* y) = f_* x \cdot y$$

Remark 3.9. If U is a Green functor then there is an obvious notion of a left U-module Mackey functor: that is, a pairing $U \times M \to M$ making each $M(S)$ into a left $U(S)$-module.

Definition 3.10. The Burnside ring Green functor $\underline{A}$ for a group G is defined as $\underline{A}(G/H) := A(H)$. Restrictions, inductions, and conjugations of H-sets induce the Mackey structure on $\underline{A}$, and the levelwise ring structure induces the Green functor structure.

Fact 3.11. It turns out that all Mackey functors admit a unique structure as modules over the Burnside ring Green functor - this is analogous to the fact that all abelian groups are modules over $\mathbb{Z}$ in a unique way. See [tD79] Proposition 6.2.3.

The following theorem says that the splitness condition in the notion of S-projectivity for Green functors is redundant, and the proof is an archetypal use of the properties in the definition of Green functors.

Theorem 3.12 ([tD79] 6.2.2)**.** *Let U be a Green functor and S a finite G-set. Then the following are equivalent:*

(a) For $f : S \to *$ the unique map, the map $f_* : U(S) \to U(*)$ is surjective.

(b) U is S-projective.

(c) All U-modules are S-projective.

Proof. The implications (c) $\Rightarrow$ (b) $\Rightarrow$ (a) are clear. To see that (a) $\Rightarrow$ (c), let M be a U-module and we want to show that

$$\pi_* : M_S \to M$$

is split surjective. Since f_* is surjective, there exists $u \in U(S)$ such that $f_* u = 1 \in U(*)$. Now define

$$\psi(T) : M(T) \to M_S(T) \quad :: \quad m \mapsto p^* u \cdot \pi^* m$$

where

$$p : S \times T \to S \qquad \pi : S \times T \to T$$

are the projections. Then

$$\pi_* \psi(m) = \pi_*(p^* u \cdot \pi^* m) = (\pi_* p^* u) \cdot m = (g^* f_* u) \cdot m = 1 \cdot m = m$$

where we've used the double coset formula associated to the pullback

$$\begin{tikzcd} S \times T \arrow[r, "\pi"] \arrow[d, "p"'] & T \arrow[d, "g"] \\ S \arrow[r, "f"'] & * \end{tikzcd} \qquad \Rightarrow \qquad \begin{tikzcd} U(S \times T) \arrow[r, "\pi_*"] & U(T) \\ U(S) \arrow[u, "p^*"] \arrow[r, "f_*"'] & U(*) \arrow[u, "g^*"'] \end{tikzcd}$$

$\square$

The next lemma, which appeared as Lemma 5 in [MM82], is of a sufficiently general nature that we've included it in this section. But it is one of the key lemmas in the reduction of the Segal conjecture to p-groups that we'll work on in §5.

Construction 3.13. Let $C(G) := \prod_{(H)} \mathbb{Z}$ where the product runs over conjugacy classes of subgroups of G, and write $IC(G) \subset C(G)$ for the ideal with 0 for the e-th coordinate. This is sometimes called the ghost ring. We then always have a homomorphism

$$\chi : A(G) \to C(G)$$

where for S a G-set, the H-th coordinate is given by $\chi_H(S) := |S^H|$. Then χ is a monomorphism with finite cokernel and $|G| \cdot C(G)$ being in the image of χ (and so also $|G| \cdot IC(G) \subset \chi I(G)$). See [tD79] §1 for these.

Lemma 3.14. *Let M be a Mackey functor and let $\pi : G/e \to G/G$ which induces $\pi^* : M(G/G) \to M(G/e)$. Then $|G| \cdot \ker \pi^* \subset I(G) \ker \pi^*$. If G is a p-group, then the p-adic topology $\{p^r \ker \pi^*\}_r$ and the augmentation topology $\{I(G)^r \ker \pi^*\}_r$ on $\ker \pi^*$ coincide.*

Proof. First recall that any Mackey functor is a module over the Burnside ring Mackey functor and in fact multiplication by $G/e \in A(G)$ on $M(G/G)$ is given by

$$M(G/G) \xrightarrow{\pi^*} M(G/e) \xrightarrow{\pi_*} M(G/G)$$

since for $1 \in A(G)$ and $m \in M(G/G)$ we have

$$\pi_* \pi^*(1 \cdot m) = \pi_*(\pi^* 1 \cdot \pi^* m) = (\pi_* \pi^* 1) \cdot m = G/e \cdot m$$

This means that $G/e \cdot \ker \pi^* = 0$, and so

$$|G| \cdot \ker \pi^* = (|G| - G/e) \cdot \ker \pi^* \in I(G) \ker \pi^*$$

Now let G be a p-group with $|G| = p^n$. Then the above clearly gives us

$$p^{nm} \ker \pi^* = |G|^m \ker \pi^* \subset I(G)^m \ker \pi^*$$

so now we claim that $I(G)^{n+1} \subset pI(G)$ which would imply that

$$I(G)^{m(n+1)} \ker \pi^* \subset (pI(G))^m \ker \pi^* \subset p^m \ker \pi^*$$

giving the other direction. To see the claim, let $H, K \le G$ with $H \neq e$. Then note that $\chi_H(G/K - |G/K|)$ is divisible by p since $G/K - (G/K)^H$ consists of a disjoint union of nontrivial H-orbits and H is a p-group. Therefore $\chi_H I(G) \subset p\mathbb{Z}$ and so $\chi I(G) \subset pIC(G)$, and

$$\chi I(G)^{n+1} \subset p^{n+1} IC(G) = p|G| IC(G) \subset p \chi I(G)$$

By injectivity of χ, we're done. $\square$

4 Motivation and formulation of the conjecture

We explain here why completion theorems are natural questions to ask and give a general formulation of the problem at the level of G-spectra.

Let $R \in \mathrm{CAlg}(\mathrm{Sp}^G)$. Then $R^0_G(S^0)$ acts on $R^*_G(G/e_+) = \pi_0\mathrm{Map}_{\mathrm{Sp}^G}(\mathrm{Ind}_e^G\mathrm{Res}_e^G\mathbb{S}_G, R) \cong \pi_0\mathrm{Map}_{\mathrm{Sp}}(\mathbb{S}, \mathrm{Res}_e^G R) \cong R^*(S^0)$ via the restriction homomorphism

$$\mathrm{Res}_e^G : R^0_G(S^0) \to R^0_G(G/e_+)$$

and so if we define I to be the kernel of this homomorphism, then by definition I acts as zero on $R^*_G(G/e_+)$.

Proposition 4.1. *I^n as defined above acts as zero on $R^*_G(X)$ for X an $(n-1)$-dimensional finite free G-complex.*

Proof. We show this by induction on n, where the case $n = 1$ is as above. Suppose true for n and let X be an n-dimensional finite free G-complex. Then the cofibre sequence

$$X^{(n-1)} \xrightarrow{i} X^{(n)} \xrightarrow{q} \bigvee(S^n \wedge G/e_+)$$

gives the long exact sequence

$$\cdots \leftarrow [S^k \wedge X^{(n-1)}, R]_G \xleftarrow{i^*} [S^k \wedge X^{(n)}, R]_G \xleftarrow{q^*} \bigoplus [S^{k+n} \wedge G/e_+, R]_G \leftarrow \cdots$$

By induction the left hand terms are annihilated by I^n, and the base case says that the right hand terms are annihilated by I, and hence by exactness the middle term is killed by I^{n+1}. $\square$

Given these observations, and noting that EG_+ has finite free skeleta we get that the maps $\{EG^n_+ \to S^0\}$ induce a factorisation

$$\begin{tikzcd}
R^*_G(S^0) \arrow[r] \arrow[d] & \lim_n R^*_G(EG^n_+) \\
\lim_n R^*_G(S^0)/I^n \arrow[ur, dashed] &
\end{tikzcd}$$

On the other hand the Milnor sequence

$$0 \to \lim_n{}^1 R^{*-1}_G(EG^n_+) \to R^*_G(EG_+) \to \lim_n R^*_G(EG^n_+) \to 0$$

says that in good cases where the $\lim^1$ term vanishes, we have a comparison map

$$\lim_n R^*_G(S^0)/I^n \to R^*_G(EG_+)$$

and so in the context where there are no $\lim^1$ issues in comparing $R^*_G(EG_+)$ and $\lim_n R^*_G(EG^n_+)$, we can ask when

$$R^*_G(S^0)^\wedge_I \to R^*_G(EG_+)$$

becomes an isomorphism. To put it more memorably, we want to know when an “algebraic completion” on the left becomes the same as a “geometric completion” on the right.

In fact, we even have the following $\lim^1$-free result purely at the level of spectra crystallising the discussions above, although we won’t be needing it in this work.

Proposition 4.2. *Suppose $X \in \mathrm{Sp}^G$ is Borel complete, that is $X \simeq F(\mathbb{S}_G, X) \to F(EG_+, X)$ is an equivalence. Then it is also complete with respect to any (finitely generated) ideal $I \leq I(G) \leq A(G)$ where $I(G)$ is the augmentation ideal.*

Proof. Since $I = (a_1, \cdots, a_r)$ is finitely generated, and $M(I) = M(a_1) \otimes \cdots \otimes M(a_r)$, we get that $Y_I^\wedge = (Y_{I'}^\wedge)_{a_1}^\wedge$, and so by induction, it is enough to show the case of $I = (a)$ a principal ideal. We think of $I \leq A(G) = \pi_0^G(\mathbb{S}_G)$ as representing self maps of S_G^0. Since it's in the augmentation ideal, ie. in the kernel of $\mathrm{Res}_e^G : \pi_0^G(\mathbb{S}_G) \to \pi_0^S(\mathbb{S})$, we know $a : \mathbb{S}_G \to \mathbb{S}_G$ is nonequivariantly trivial. And so $M(a) \to \mathbb{S}_G$ is a nonequivariant equivalence since $\mathbb{S}_G[a^{-1}]$ is nonequivariantly trivial. Therefore $M(a) \otimes EG_+ \to EG_+$ is a G-equivalence (see [GM95b] Proposition 1.1 for example), and so $F(\mathbb{S}_G, F(EG_+, X)) \to F(M(a), F(EG_+, X))$ is a G-equivalence. □

These considerations together with the spectral completions from §2 lead us naturally to formulate the following version of the Segal conjecture.

Theorem 4.3 (Segal conjecture for finite groups)**.** *For G a finite group, the comparison map*

$$\xi_G : (\mathbb{S}_G)_{I(G)}^\wedge \to F(EG_+, \mathbb{S}_G)$$

is an equivalence of G-spectra. Equivalently, the comparison map

$$\pi_G^*(S^0)_{I(G)}^\wedge \to \pi_G^*(EG_+)$$

is an isomorphism for all finite groups G.

Proof of equivalence of formulations: A map of G-spectra being an equivalence can be checked on the equivariant homotopy groups for all subgroups of G. Also, recall that

$$\mathrm{Res}_H^G F(EG_+, \mathbb{S}_G) \simeq F(\mathrm{Res}_H^G EG_+, \mathrm{Res}_H^G \mathbb{S}_G) \simeq F(EH_+, \mathbb{S}_H)$$

Finally, note from the function spectrum model of completions that

$$\mathrm{Res}_H^G\big((\mathbb{S}_G)_{I(G)}^\wedge\big) \simeq (\mathbb{S}_H)_{\mathrm{Res}_H^G I(G)}^\wedge$$

Therefore, the map of G-spectra ξ_G is an equivalence iff

$$\pi_*^H((\mathbb{S}_H)_{\mathrm{Res}_H^G I(G)}^\wedge) \to \pi_*^H F(EH_+, \mathbb{S}_H)$$

is an isomorphism for all $H \leq G$. Now the Segal-tom Dieck splitting 2.19 says that $\pi_*^H \mathbb{S}_H$ are all bounded below and finite type, and so Theorem 2.39 says that $\pi_*^H\big((\mathbb{S}_H)_{\mathrm{Res}_H^G I(G)}^\wedge\big) \cong \pi_*^H(\mathbb{S}_H)_{\mathrm{Res}_H^G I(G)}^\wedge$. On the other hand, Lemma A.5 gives that $\pi_*^H(\mathbb{S}_H)_{\mathrm{Res}_H^G I(G)}^\wedge \cong \pi_*^H(\mathbb{S}_H)_{I(H)}^\wedge$. And so switching to the cohomology notation we get that ξ_G is an equivalence iff

$$\pi_H^*(S^0)_{I(H)}^\wedge \to \pi_H^*(EH_+)$$

is an isomorphism for all $H \leq G$, as required. □

Remark 4.4. This agrees with Segal's original Burnside ring formulation since Theorem 2.27 says that $\pi_G^0(S^0) \cong A(G)$ and splitness of $\mathbb{S}_G$ and Proposition 2.33 give that $\pi_G^0(EG_+) \cong \pi_S^0(BG_+)$.

The next section will show how, using the theory of Mackey functors, we can reduce this to the case of p-groups with p-completions instead of augmentation completions. But before ending this section, let us see how, following [GM92], the language of completions of G-spectra allow us to generalise this result by purely formal reasons.

Theorem 4.5. *For X any G-spectrum, the comparison map*

$$F(X, \mathbb{S}_G)_{I(G)}^\wedge \to F(EG_+, F(X, \mathbb{S}_G))$$

is a G-equivalence. In particular, since finite G-spectra are the dualisable ones, we have that

$$Y_{I(G)}^\wedge \to F(EG_+, Y)$$

is a G-equivalence if Y is a finite G-spectrum.

Proof. Recall that for any G-spectrum Z, we have $Z_{I(G)}^\wedge \simeq F(M(I(G)), Z)$. And so the G-equivalence is obtained just by applying $F(X, -)$ to the equivalence in the Segal conjecture. □

5 Reduction to the case of p-groups

This is based on [MM82]. Here is the first instance where we see the power of working with the genuine equivariant formulation where we use that the homotopy groups of genuine objects naturally admit the structure of Mackey functors which encode the relationships between the group and all its subgroups, and these structures give very rigid restrictions to what might happen. There will be two reductions: first to the case of p-groups, and second, replacing augmentation completions by the strictly more drastic and better understood p-completions.

Reduction to p-groups

The following is the induction theorem for completed Burnside rings provided in [MM82] which is the key to this step. We point out here that [AHJM88b] has generalised this reduction step with a topological transfer argument in their §5 to generalise the Segal conjecture to include localisations.

Theorem 5.1. *Let G be a finite group, and $\widehat{\underline{A}}$ the Burnside ring Green functor completed at the augmentation ideal $I(G)$. For each prime p, let G_p denote a representative Sylow p-subgroup of G. Then the sum*

$$\bigoplus_p i_* : \bigoplus_p \widehat{A}(G_p) \to \widehat{A}(G)$$

is an epimorphism. In other words, together with Theorem 3.12 this implies that all $\widehat{\underline{A}}$-modules are projective with respect to the set of all Sylow p-subgroups for all p dividing $|G|$.

Proof. See appendix B. □

Our first aim now is to show that the Segal conjecture, as formulated in Theorem 4.3, can equivalently be stated as

$$\pi^*_G(S^0)^\wedge_{I(G)} \xrightarrow{\cong} \pi^*_G(EG_+)^\wedge_{I(G)}$$

being an isomorphism. To this end, we need to know that $\pi^*_G(EG_+)^\wedge_{I(G)} \cong \pi^*_G(EG_+)$, and from the $\lim^1$ discussion of §4 and Proposition 4.1 we just need to show that $\lim^1 \pi^*_G(EG^n_+) = 0$. We have not been able to find a source for this folklore result and so have supplied a proof inspired by the statement of Corollary 4.7 in Atiyah's paper [Ati61]. There might be a much simpler way of showing it which we are not aware of.

Proposition 5.2. *The system $\{\pi^*_G(EG^n_+)\}$ satisfies $\lim^1 \pi^*_G(EG^n_+) = 0$.*

Proof. Fix a $k \neq 0$. We show that for each n, there is an $m \geq n$ such that

$$\mathrm{Im}\big(\pi^k_G(EG^m_+) \to \pi^k_G(EG^n_+)\big)$$

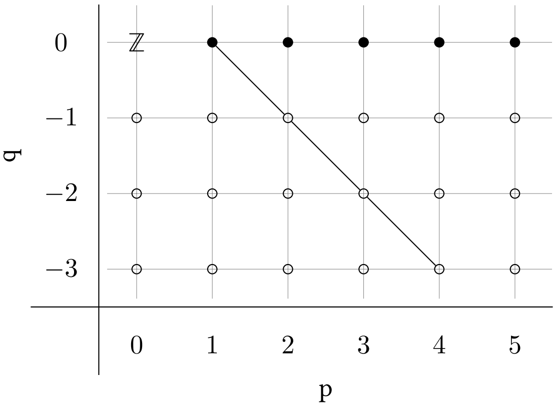
is finite, and then the vanishing of $\lim^1$ will follow by Mittag-Leffler. We'll say what happens for the case $k = 0$ later. To do this, we note that $\pi^*_G(EG^n_+) \cong \pi^*_S(BG^n_+)$ where the latter is the nonequivariant stable cohomotopy of a skeleton of BG_+ since EG^n are all free G-complexes and $EG^n/G \simeq BG^n$. These can in turn can be analysed by the Atiyah-Hirzebruch spectral sequence

$$E_2^{p,q}(n) = \widetilde{H}^p(BG^n_+, \pi^q_S(S^0)) \Rightarrow \pi^{p+q}_S(BG^n_+)$$

We now recall a couple of basic facts we'll use:

- $$\pi^k_S(S^0) = \begin{cases} 0 & \text{if } k \geq 1 \\ \mathbf{Z} & \text{if } k = 0 \\ \text{finite} & \text{if } k \leq -1 \end{cases}$$
- $\widetilde{H}^p(BG_+, \mathbf{Z})$ is finite when $p \geq 1$ (from the theory of group cohomology, for example), and so for a fixed p, we can always find an m ($m = p + 2$ say) such that for A any finitely generated abelian group, $H^s(BG^m_+, A)$ is finite for all $1 \leq s \leq p$ and becomes 0 for $s \geq p + 2$ for skeletal reasons.

The spectral sequence for $\pi_S^{p+q}(BG_+^n)$ then looks as follows:



where for example the line consists of the contributions to $\pi_S^1(BG_+^n)$. Here the solid dots are groups that are finitely generated abelian groups that are possibly infinite, and the hollow ones are finite. By the second fact above, all the terms vanish for $p \gg 0$.

- For a fixed $k > 0$ and n, it is easy to see by using the second fact that we can just take $m \gg 0$ so that $E_2^{k,0}(m)$ is finite, and then $\pi_S^k(BG_+^m)$ itself would be a finite extension of finite groups, so finite. Hence indeed $\mathrm{Im}\big(\pi_G^k(EG_+^m) \to \pi_G^k(EG_+^n)\big)$ is finite as required.
- The case of $k < 0$ is similar but even easier since all the E_2 terms were finite to begin with.
- Finally, for the case of $k = 0$, note that $\pi_S^0(BG_+^n) \cong \mathbb{Z} \oplus A_n$ for some A_n finite, where the $\mathbb{Z}$ splitting is the natural one coming from choices of

  $$S^0 \to BG_+^n \to S^0$$

  compatible across different n's. Here the A_n's are finite as in the arguments above. And so the maps

  $$\pi_S^0(BG_+^m) \to \pi_S^0(BG_+^n)$$

  are going to the identities on the copies of $\mathbf{Z}$, and hence for a fixed n, the groups $\mathrm{Im}\big(\pi_G^k(EG_+^m) \to \pi_G^k(EG_+^n)\big)$ stabilises as $m \gg 0$. So this also satisfies the Mittag-Leffler condition.

$\square$

Proposition 5.3. *The Segal conjecture for finite groups is true iff it is true for all p-groups for all primes p.*

Proof. Let $S = \{G_p\}_p$ be the set of Sylow subgroups of G, and let $\mathcal{S} = \coprod_p G/G_p$. We know that both Mackey functors $\underline{\pi}^*(S^0)^\wedge_{I(G)}$ and $\underline{\pi}^*(EG_+)^\wedge_{I(G)}$ are $I(G)$-complete, and so they are both $\widehat{\underline{A}}$-module Mackey functors. Theorem 5.1 then says that $\underline{\pi}^*(S^0)^\wedge_{I(G)}$ and $\underline{\pi}^*(EG_+)^\wedge_{I(G)}$ are S-projective, and so Proposition 3.7 says that we have a map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \underline{\pi}^*(S^0)^\wedge_{I(G)}(*) \arrow[r] \arrow[d] & \underline{\pi}^*(S^0)^\wedge_{I(G)}(\mathcal{S}) \arrow[r] \arrow[d] & \underline{\pi}^*(S^0)^\wedge_{I(G)}(\mathcal{S} \times \mathcal{S}) \arrow[d] \\
0 \arrow[r] & \underline{\pi}^*(EG_+)^\wedge_{I(G)}(*) \arrow[r] & \underline{\pi}^*(EG_+)^\wedge_{I(G)}(\mathcal{S}) \arrow[r] & \underline{\pi}^*(EG_+)^\wedge_{I(G)}(\mathcal{S} \times \mathcal{S})
\end{tikzcd}$$

But then Mackey functors turn coproducts into direct sums, and $\mathcal{S}$ and $\mathcal{S} \times \mathcal{S}$ consist of orbits of the form G/H where $H \leq G$ is a p-group for some prime p. And so if the Segal conjecture holds for all p-groups for all primes p, then

$$\pi_G^*(S^0)^\wedge_{I(G)} = \underline{\pi}^*(S^0)^\wedge_{I(G)}(G/G) \xrightarrow{\cong} \underline{\pi}^*(EG_+)^\wedge_{I(G)}(G/G) = \pi_G^*(EG_+)^\wedge_{I(G)}$$

as required since the two right vertical maps are isomorphisms. $\square$

Reduction to p-completions

Let G be a p-group from now on. We show here that instead of working with $(\pi_G^*)^\wedge_{I(G)}$, we can work with $(\pi_G^*)^\wedge_p$, and this is gotten as Proposition 14 of [MM82].

Warning 5.4. Even though the usual slogan is that "the p-adic topology and the $I(G)$-adic topology are the same for p-groups," this hides the distinction between two different p-adic topologies, namely $\{p^r I(G)\}_r$ and $\{p^r A(G)\}_r$ - it is the former that agrees with the $I(G)$-adic topology $\{I(G)^r\}_r$, but it is the latter which we want to work with. That is, we have the correspondences

$$\begin{array}{ccc} (\pi_G^*)^\wedge_{I(G)} \cong (\pi_G^*)^\wedge_{pI(G)} & \longleftrightarrow & \{ \text{I(G)}^r\}_r = \{p^r I(G)\}_r \\ (\pi_G^*)^\wedge_p & \longleftrightarrow & \{\text{p}^r A(G)\}_r \end{array}$$

To illustrate the difference, consider the Burnside ring $A(G) \cong \mathbb{Z} \oplus I(G)$. Then $A(G)^\wedge_{pI(G)} \cong \mathbb{Z} \oplus I(G)^\wedge_p$ but $A(G)^\wedge_p \cong \mathbb{Z}_p \oplus I(G)^\wedge_p$.

Proposition 5.5. *Let G be a p-group. Then*

$$\pi_G^*(S^0)^\wedge_{I(G)} \to \pi_G^*(EG_+)^\wedge_{I(G)}$$

is an isomorphism iff

$$\pi_G^*(S^0)^\wedge_p \to \pi_G^*(EG_+)^\wedge_p$$

is.

Proof. Consider the diagram of $A(G)$-modules

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "k"] & \pi_G^*(S^0) \arrow[r] \arrow[d, "c"] & \pi_e^*(S^0) \arrow[r] \arrow[d, "\cong"] \arrow[l, dashed, bend left] & 0 \\ 0 \arrow[r] & L \arrow[r] & \pi_G^*(EG_+) \arrow[r] & \pi_e^*(EG_+) \arrow[r] \arrow[l, dashed, bend left] & 0 \end{tikzcd}$$

where the natural splitting dashed maps come from the fact that $\mathbb{S}_G$ is a split G-spectrum as introduced in §2. This is good because while the completion functors $(-)^\wedge_{I(G)}$ and $(-)^\wedge_p$ are only left exact in general, here they preserve the short exact sequences by virtue of the splitting. By Lemma 3.14 we get that the topologies $\{p^r K\}_r$ and $\{I(G)^r K\}_r$ agree on K, and similarly for L. And so

$$c^\wedge_{I(G)} \text{ is iso } \Leftrightarrow k^\wedge_{I(G)} \text{ is iso } \Leftrightarrow k^\wedge_p \text{ is iso } \Leftrightarrow c^\wedge_p \text{ is iso}$$

as required. □

Hence we've reduced the original formulation for all finite groups G in Theorem 4.3 into the following version.

Theorem 5.6 (Segal conjecture for p-groups)**.** *Let G be a p-group. Then the comparison map*

$$\pi_G^*(S^0)^\wedge_p \to \pi_G^*(EG_+)^\wedge_p$$

is an isomorphism. In other words, $(\mathbb{S}_G)^\wedge_p \to F(EG_+, \mathbb{S}_G)^\wedge_p$ is an equivalence of G-spectra.

Remark 5.7. Sometimes people also talk about the Segal conjecture as saying that the Tate construction of the sphere spectrum exhibits its p-completion. This concretely means the following: in the case of $G = C_p$, the Tate diagram in fact becomes

$$\begin{tikzcd} \mathbb{S}_{hC_p} \arrow[r] \arrow[d, equal] & \mathbb{S}^{C_p} \arrow[r] \arrow[d] & \mathbb{S}^{\Phi C_p} \arrow[d] \\ \mathbb{S}_{hC_p} \arrow[r] & \mathbb{S}^{hC_p} \arrow[r] & \mathbb{S}^{tC_p} \end{tikzcd}$$

Now p-completion is a left adjoint, and so is an exact functor (ie. preserves stable (co)fibre sequences). Hence the middle vertical is an equivalence after p-completion iff the right vertical is. But recall that $\mathbb{S}^{\Phi C_p} \simeq \mathbb{S}$ in Sp, and so the Segal conjecture for $G = C_p$ is equivalent to saying that $\mathbb{S} \to \mathbb{S}^{tC_p}$ is an equivalence after p-completion. But then we know that $\mathbb{S}^{tC_p}$ is p-complete since the Tate construction of any bounded below G-spectrum is p-complete when G is a p-group (see [NS18] I.2.9 for the case of $G = C_p$ or [GM95b] 4.1 for the general case), and so in fact the map exhibits the p-completion of $\mathbb{S}$. See for example Remark III.1.6 of [NS18] for this point of view.

Remark 5.8. From Theorem 5.6 we can also get the Sullivan conjecture type statement that $(X^G)^\wedge_p \simeq (X^\wedge_p)^{hG}$ when G is a p-group and X a finite G-spectrum. Similarly as in Theorem 4.5, we can get that for $X \in \mathrm{Sp}^G$ finite, $X^\wedge_p \xrightarrow{\simeq} F(EG_+, X)^\wedge_p \simeq F(EG_+, X^\wedge_p)$ from Theorem 5.6. Now taking genuine fixed points $(-)^G$ and using that it commutes with p-completions (see Proposition 2.43), we get the required statement.

6 Operational formulation of the conjecture for p-groups

The version in Theorem 5.6 is still not exactly the version we'll be working with, and the issue is a familiar one: we want to have long exact sequences of cohomology groups associated to cofibre sequences, but p-completion is not an exact functor in general. However, all is not lost since it is exact when the modules involved are finitely generated by the Artin-Rees Lemma (see Proposition 10.12 of [AM69]), and this section is concerned with introducing a variant of $(\pi^*_G(-))^\wedge_p$ that will be sufficient for our purposes, and in giving the form of the conjecture that we shall be working with in Theorem 6.10.

Remark 6.1. In what follows we could equally well have worked with pro-groups as was first done in [AS69] in their proof of the Atiyah-Segal completion theorem and the philosophy is basically the same. We have however chosen to follow the treatment in [CMP87] so as to avoid the technicalities of the pro-category.

The workable variant of algebraic completion

We work now with general G-spectra. Since we will want to use the isotropy separation sequence to resolve the cohomology theory represented by the sphere spectrum, it will be convenient to introduce the following notation.

Definition 6.2. Recall the definition of cohomology theories in Definition 2.28. For $\mathcal{E} \in \mathrm{Sp}^G$, $X, Y \in \mathcal{S}^G_*$ with X finite, and $q \in \mathbb{Z}$ we define

$$\mathcal{E}^G_{-q}(X;Y) = \mathcal{E}^q_G(X;Y) := (\Sigma^\infty Y \otimes E)^q_G(X)$$

So this is a cohomology theory in X and a homology theory in Y.

Now we introduce the variant of completion that we shall be working with in the sequel.

Definition 6.3. When $X \in \mathcal{S}^G_*$ we define

$$\widehat{\mathcal{E}}^*_G(X;Y) := \lim_a \left(\mathcal{E}^*_G(X_a;Y)^\wedge_p\right)$$

where the limit runs over X_a finite G-subcomplexes of X.

Warning 6.4. It is of course not true that $[X, \mathcal{E}^\wedge_p]_G \cong \lim_a [X_a, \mathcal{E}]_G{}^\wedge_p$ - the latter is not even a cohomology theory in general since inverse limits do not always preserve exactness. The point is that while we're interested in studying the spectrum $\mathcal{E}^\wedge_p$, we need a concrete way to work with it algebraically, and the latter is an approximation to this end.

Remark 6.5. Greenlees and May [GM95a] have later on developed the theory of derived completions to deal with the completions happening at the level of G-spectra, in which case we have instead the exact sequence

$$0 \to L^I_1(\mathcal{E}^{n+1}_G(X)) \to (\mathcal{E}_G{}^\wedge_I)^n(X) \to L^I_0(\mathcal{E}^n_G(X)) \to 0$$

where the functors L^I_0 and L^I_1 are exact, and one could conceivably work with these instead. In any case, we will be working with the classical formulation using inverse limits.

The following lemma shows that the notion of completion via inverse limits is good enough to preserve long exact sequences for our purposes.

Lemma 6.6. *Suppose $\mathcal{E}_q^G(Y)$ is finitely generated if Y has finite skeleta. Then associated to the cofibering $EG_+ \to S^0 \to \widetilde{EG}$ and X a finite skeleta G-complex, we have the two long exact sequences*

$$\cdots \to \widehat{\mathcal{E}}^q_G(X \wedge \widetilde{EG}; Y) \to \widehat{\mathcal{E}}^q_G(X; Y) \to \widehat{\mathcal{E}}^q_G(X \wedge EG_+; Y) \to \cdots$$
$$\cdots \to \widehat{\mathcal{E}}^q_G(X; EG_+ \wedge Y) \to \widehat{\mathcal{E}}^q_G(X; Y) \to \widehat{\mathcal{E}}^q_G(X; \widetilde{EG} \wedge Y) \to \cdots$$

Proof. We break the proof into a series of steps.

1) Apply the cohomology theory $\mathcal{E}^q_G(-; Y)$ to the cofibering $X^k \to X^{k+1} \to \bigvee^{n_k} S^{k+1}$ and use the assumption to get inductively that $\mathcal{E}^q_G(X^k; Y)$ is finitely generated.

2) Since p-completion is exact on finitely generated things, we have the exact sequence

$$\cdots \to \mathcal{E}^q_G(X^k \wedge EG^k_+; Y)^\wedge_p \to \mathcal{E}^q_G(X^k; Y)^\wedge_p \to \mathcal{E}^q_G(X^k \wedge \widetilde{EG}^k; Y)^\wedge_p \to \cdots$$

3) Finally we know that inverse limits of compact Hausdorff abelian groups have trivial $\lim^1$, so by the six term $\lim^1$ sequence we get that inverse limit is exact on exact sequences consisting of compact Hausdorff groups. And since the sequence in step (2) was a sequence of compact Hausdorff groups (since they were $\otimes \mathbb{Z}_p$ of finitely generated groups), we're free to take inverse limit to pass to $\widehat{\mathcal{E}}^q_G(-; Y)$.

The proof for the other sequence is similar. □

How this relates to the original question

It's all well and good to have Lemma 6.6 to guarantee exactness in cases we care about, but we still need to relate $\widehat{\pi}^*_G(X) = \lim_a \left(\pi^*_G(X_a)^\wedge_p\right)$ to what we're actually interested in, namely $\pi^*_G(X)^\wedge_p$.

Lemma 6.7. *Let $\{A_n\}$ be an inverse sequence of finitely generated abelian groups such that $\lim^1 A_n = 0$. Then the natural map*

$$(\lim A_n)^\wedge_p \to \lim((A_n)^\wedge_p)$$

is an isomorphism.

Proof. Let ${}_qA_n = \{a \in A_n \mid p^q a = 0\}$. Consider the sequences

$$0 \longrightarrow {}_qA_n \longrightarrow A_n \longrightarrow p^q A_n \longrightarrow 0 \qquad\qquad 0 \longrightarrow p^q A_n \longrightarrow A_n \longrightarrow A_n/p^q A_n \longrightarrow 0$$

Since ${}_qA_n$ are finite, we have $\lim^1 {}_qA_n = 0$, and from six-term derived limit sequence from the first short exact sequence, we get that $\lim^1 p^q A_n$ is a quotient of $\lim^1 A_n = 0$, so vanishes also. So both the displayed sequences remain exact after passing to inverse limits over n. Hence the first inverse limit sequence gives $\lim_n p^q A_n = p^q \lim_n A_n$, and the second gives $\lim_n(A_n/p^q A_n) \cong \lim_n A_n / p^q \lim_n A_n$. Now just take inverse limits over q. □

Lemma 6.8. *If X and Y have finite skeleta and $\lim^1_n \mathcal{E}^*_G(X^n; Y) = 0$ then*

$$\mathcal{E}^*_G(X; Y)^\wedge_p \cong \lim(\mathcal{E}^*_G(X^n; Y)^\wedge_p)$$

Proof. Just apply the preceding lemma with $A_n = \mathcal{E}^*_G(X^n, Y)$, where we've used the Milnor sequence

$$0 \longrightarrow \lim^1_n \mathcal{E}^*_G(X^n; Y) \longrightarrow \mathcal{E}^*_G(X; Y) \longrightarrow \lim_n \mathcal{E}^*_G(X^n; Y) \longrightarrow 0$$

and the vanishing of the left hand term by hypothesis. □

Corollary 6.9. $\lim \left(\pi^*_G(EG^n_+)^\wedge_p\right) \cong \pi^*_G(EG_+)^\wedge_p$.

Proof. Apply Proposition 5.2 to the lemma above. □

Now putting together Lemma 6.6 applied on the cofibre sequence $EG_+ \to S^0 \to \widetilde{EG}$ in the first variable, that $\pi^*_G(S^0)^\wedge_p = \widehat{\pi}^*_G(S^0)$, and the corollary above, we finally arrive at the operational formulation of the conjecture as given in Carlsson's paper [Car84].

Theorem 6.10 (Segal Conjecture)**.** $\widehat{\pi}^*_G(\widetilde{EG}) = 0$ *for all finite p-groups G.*

7 An overview of the inductive strategy

In this section we present the organising roadmap of Carlsson's induction. By and large, it consists of very clever variations on the theme of harnessing the isotropy separation sequence

$$EG_+ \to S^0 \to \widetilde{EG}$$

The first lemma uses another isotropy separation to allow us flexibility in working with $\widehat{\pi}^*_G(\widetilde{E\mathcal{P}})$ instead of $\widehat{\pi}^*_G(\widetilde{EG})$.

Definition 7.1. By a contractible based G-space X, we mean an $X \in \mathcal{S}^G_*$ such that $X^e \simeq *$ in $\mathcal{S}_*$.

Note that this is not the same as a G-contractible based space, which requires that $X^H \simeq *$ in $\mathcal{S}_*$ for all $H \leq G$.

Lemma 7.2. *Assume $\widehat{\mathcal{E}}^*_H$ vanishes on contractible H-spaces for all proper subgroups $H \lneq G$. If $\widehat{\mathcal{E}}^*_G(X) = 0$ for any contractible based G-space X such that $X^G \simeq S^0$, then $\widehat{\mathcal{E}}^*_G$ vanishes on all contractible based G-spaces.*

Proof. Since $X^G \simeq S^0$, we have a cofibre sequence $S^0 \to X \to X/S^0$. Let X' be any other contractible G-space and consider the cofibre sequence

$$X' \to X' \wedge X \to X' \wedge (X/S^0)$$

Plugging this into the cohomology theory $\widehat{\mathcal{E}}^*_G$, we see that it's enough to show that $\widehat{\mathcal{E}}^*_G(X' \wedge X)$ and $\widehat{\mathcal{E}}^*_G(X' \wedge (X/S^0))$ vanish.

For the first case, we claim the $\widehat{\mathcal{E}}^*_G(W \wedge X) = 0$ for any G-complex W. Since by definition $\widehat{\mathcal{E}}^*_G$ was defined as an inverse limit over finite skeleta, we might as well prove it for the case W finite. Since W^n/W^{n-1} for $n \geq 0$ are all just wedges of $S^n \wedge (G/H)_+$, by the long exact sequence associated to cofibres, we might as well just deal with the case $W = S^n \wedge (G/H)_+$. Since we're working with cohomology theories, and so are free to translate by suspensions, we might as well prove for the case $W = G/H_+$. If $H = G$, this holds by the hypothesis $\widehat{\mathcal{E}}^*_G(X) = 0$, and if $H \neq G$, then Proposition 2.31 gives $\widehat{\mathcal{E}}^*_G((G/H)_+ \wedge X) \cong \widehat{\mathcal{E}}^*_H(X)$ and this together with our hypothesis gives the claim.

For the second case, we show that $\widehat{\mathcal{E}}^*_G(X' \wedge Z) = 0$ for any Z, such as X/S^0, such that $Z^G = *$. Arguing as above, we can reduce this to case $Z = (G/H)_+$. Since $Z^G = *$, $H \neq G$ necessarily, and for this case we can use Proposition 2.31 again. ☐

Remark 7.3. This is good since instead of proving the vanishing of various variants of the completed cohomotopy on $\widetilde{EG}$ directly, we will prove these vanishings on $\widetilde{E\mathcal{P}}$ instead, which has much better properties for carrying out inductions, namely that $\widetilde{E\mathcal{P}}^H \simeq *$ for all $H \lneq G$, allowing us to have many vanishing statements when we pass to proper subgroups during an inductive step. Furthermore, it also has a concrete model in terms of unions of spheres as in Construction 2.32 - this will be used in analysing the free part of the problem.

We are now ready to state the four main theorems that will organise the inductive procedure. Recall from §2 that for $K \lhd H \leq G$ and writing $J := H/K$, we write $\Phi^J\mathcal{E}$ for the H/K-geometric fixed point $\Phi^{H/K}\mathcal{E}$.

Theorem A. *Let $K \lhd H \leq G$ and write $J = H/K$. Suppose $(\widehat{\Phi^J\mathcal{E}})^*_J$ vanishes on contractible J-spaces for all proper subquotients J. Let X be a G-complex such that $X^G \simeq S^0$ and X^H contractible for all proper subgroups. Then*

i) If G is not elementary abelian then $\widehat{\mathcal{E}}^*_G(X; \widetilde{EG}) = 0$.

ii) If $G = \mathbb{F}^r_p$ then $\widehat{\mathcal{E}}^*_G(X; \widetilde{EG})$ is the direct sum of $p^{r(r-1)/2}$ copies of $\Sigma^{r-1}(\widehat{\Phi^G\mathcal{E}})^*(S^0)$.

Theorem B. *Suppose $\mathcal{E} \in Sp^G$ is split and the underlying spectrum is bounded below. Let X be a G-space such that $X^G \simeq S^0$ and X^H contractible for all proper subgroups. Then*

i) *If G is not elementary abelian then $\widehat{\mathcal{E}}^*_G(X; EG_+) = 0$.*

ii) *If $G = \mathbb{F}_p^r$ and $H^q(\mathcal{E}, \mathbb{F}_p) = 0$ for all sufficiently large q then $\widehat{\mathcal{E}}^*_G(X; EG_+)$ is the direct sum of $p^{r(r-1)/2}$ copies of $\Sigma^r \widehat{\mathcal{E}}^*(S^0)$.*

Remark 7.4. As we shall see in the subsequent sections, once the right ideas have been set up, the case of non-elementary abelian p-groups in Theorems A and B is not so hard. In contrast to that, the case of elementary abelian p-groups is riddled with technical fixes and hard computations.

Remark 7.5. In the proof of Theorem B we will see that it is equivalent to a problem in nonequivariant topology, whereas the $\widetilde{EG}$ part is genuinely equivariant. This makes sense since the free part EG_+ kills information away from the trivial subgroup, whereas $\widetilde{EG}$ keeps information everywhere except at the trivial subgroup. This illustrates the general philosophy of using this kind of isotropy separation: we shave off one irreducible slice of the problem in EG_+ and deal with the rest by subgroup induction in the $\widetilde{EG}$ part.

These theorems together with Lemma 7.2 allow us to finish off the non-elementary abelian case by induction in the form of the following theorem - the sphere G-spectrum $\mathbb{S}_G$ obviously satisfies all the hypotheses.

Theorem C. *Let G be a finite p-group which is not elementary abelian. Let $\mathcal{E} \in Sp^G$ satisfying*

i) *$(\widehat{\Phi^J \mathcal{E}})^*_J$ vanishes on contractible J-spaces for all elementary abelian subquotients J of G*

ii) *$\Phi^J \mathcal{E}$ is split for all non-elementary abelian subquotients $J = H/K$ of G and $\Phi^{K/K}\mathcal{E}$ is bounded below for all $K \leq G$.*

Then $(\widehat{\Phi^J \mathcal{E}})^*_J$ vanishes on contractible J-spaces for all subquotients $J = H/K$ of G, including G itself.

Proof. By hypothesis (i) we need only deal with subquotients $K \triangleleft H \leq G$ for which $J = H/K$ are not elementary abelian. By induction on subquotients we can assume that all proper subquotients of H/K (meaning those coming from extensions $K \leq L \triangleleft M \lneq H$ or $K \lneq L \triangleleft M \leq H$) have the required property. Now consider the subquotient coming from an extension $K \triangleleft H$ such that $J = H/K$ is not elementary abelian. Let X be a contractible J-space such that $X^J = S^0$ and the other fixed points are contractible. By Theorem A(i) and induction we have $(\widehat{\Phi^J \mathcal{E}})^*_J(X; \widetilde{EJ}) = 0$. By hypothesis (ii) and Theorem B(i) we have $(\widehat{\Phi^J \mathcal{E}})^*_J(X; EJ_+) = 0$. And so in total we have $(\widehat{\Phi^J \mathcal{E}})^*_J(X) = 0$. Lemma 7.2 applied to the group J then says that $(\widehat{\Phi^J \mathcal{E}})^*_J(Y) = 0$ for all contractible based J-spaces Y. □

And finally to deal with the elementary abelian cases we have the following theorem.

Theorem D. *Let $G = \mathbb{F}_p^r$, and X a G-space such that $X^G = S^0$ and $X^H \simeq *$ for all proper subgroups H of G. Assume that the Segal conjecture holds for $\mathbb{F}_p^s$ for all $s < r$, ie. $\widehat{\pi}^*_{\mathbb{F}_p^s}(X) = 0$. Then*

$$\delta : \widehat{\pi}^q_G(X; \widetilde{EG}) \to \widehat{\pi}^{q+1}_G(X; EG_+)$$

is an isomorphism for all q.

Idea 7.6. Carlsson's idea to do this is as follows: looking at the diagram below, part (ii) of Theorems A and B say that the top δ is a morphism of free $\widehat{\pi}^*(S^0)$-modules, so it's enough to show bijection on generators, that is, isomorphism when $q = r - 1$. In this degree, δ is just a morphism of free $\mathbb{Z}_p$-modules with the same number of generators, so it will suffice to show that it's an isomorphism upon reduction mod p. And for this case, it's enough to show it's injective mod p. In order to do this, the idea is to compare the theory $\widehat{\pi}$ with a test cohomology theory $\mathcal{K} = F(EG_+, \underline{H\mathbb{F}_p})$ where $\underline{H\mathbb{F}_p}$ is the Eilenberg-MacLane G-spectrum.

$$\begin{tikzcd}
\widehat{\pi}^{r-1}_G(X; \widetilde{EG}) \arrow[r, "\delta"] \arrow[d, "\eta"] & \widehat{\pi}^r_G(X; EG_+) \arrow[d, "\eta"] \\
\widehat{\mathcal{K}}^{r-1}_G(X; \widetilde{EG}) \arrow[r, "\delta"] & \widehat{\mathcal{K}}^r_G(X; EG_+)
\end{tikzcd}$$

We will show that the bottom δ is an isomorphism and that the left vertical η is injective mod p.

Remark 7.7. These theorems show us that there is a fundamental distinction between non-elementary abelian p-groups and elementary abelian p-groups in the Segal conjecture story. Both satisfy the conjecture, but they do so for different reasons: the former because all other terms in the isotropy long exact sequence vanish; the latter because the boundary map in the sequence is a nontrivial isomorphism.

We end this section with a sketch of the ideas involved in proving Theorems A and B.

(A) For a based G-complex X let $SX \subset X$ the singular subcomplex of X, ie. the points of X with nontrivial isotropy groups. Then we have $[X, \widetilde{EG} \wedge Y]_G \cong [SX, Y]_G$, and we want to show that $[X, \widetilde{EG} \wedge Y]_G = 0$ when X satisfies $X^G \simeq S^0$ and $X^H \simeq *$ for $H \lneq G$. The idea is to perform a "blow-up" of SX, that is, to enlarge SX into a G-equivalent space AX by remembering at each point of SX the chains of subgroups fixing that point. This space AX will admit a nice filtration by the lengths of the chains, whose subquotients are induced G-spaces, and so by adjunction become problems in the proper subquotients of G where the statement is true by the inductive hypothesis.

(B) This is where we use the model $\widetilde{E\mathcal{P}} = \cup_n S^{nV}$ where V is the reduced regular complex representation of G. The theory of equivariant Thom spectra will give the identification $\mathcal{E}_*^G(S^V; EG_+) \cong \mathcal{E}_*(BG^{-V})$ and so we're in the realm of nonequivariant algebraic topology. The latter groups will be analysed using the Adams spectral sequence, and we use complex representations in order to have a good theory of Euler classes. The non-elementary abelian case will be relatively painless, essentially just applying Quillen's F-isomorphism theorem on the Euler class. The elementary abelian case, on the other hand, will involve a difficult Ext group calculation due to [AGM85], of which we provide a sketch.

8 S-functors and the proof of Theorem A

In this section, we will see how Carlsson used the extra structure available on G-spaces, namely that they have point-set models and so admit the singular set functor, to inductively analyse the spectrum $\widetilde{EG} \otimes \mathcal{E}$ associated to some $\mathcal{E} \in \mathrm{Sp}^G$ via his notion of S-functors.

Lemma 8.1. *Let $X, Y \in \mathcal{S}_*^G$ with X finite and $\mathcal{F}$ a family. Denote by $X_{\mathcal{F}}$ the G-subcomplex of X consisting of cells of orbit types away from $\mathcal{F}$. Then the inclusions $X_{\mathcal{F}} \hookrightarrow X$ and $S^0 \hookrightarrow \widetilde{E\mathcal{F}}$ induce bijections*

$$[X, \widetilde{E\mathcal{F}} \wedge Y]_G \xrightarrow{\cong} [X_{\mathcal{F}}, \widetilde{E\mathcal{F}} \wedge Y]_G \xleftarrow{\cong} [X_{\mathcal{F}}, Y]_G$$

Proof. The first bijection uses the long exact sequence associated to the cofibration of based G-complexes $X_{\mathcal{F}} \to X \to X/X_{\mathcal{F}}$ where $X/X_{\mathcal{F}}$ is finite with only cells of orbit types in $\mathcal{F}$, giving an exact sequence of sets

$$[X_{\mathcal{F}}, \widetilde{E\mathcal{F}} \wedge Y]_G \leftarrow [X, \widetilde{E\mathcal{F}} \wedge Y]_G \leftarrow [X/X_{\mathcal{F}}, \widetilde{E\mathcal{F}} \wedge Y]_G \leftarrow \cdots$$

Now use cellular induction and the fact that

$$[G/H_+ \wedge S^n, \widetilde{E\mathcal{F}} \wedge Y]_G \cong [S^n, \mathrm{Res}^G_H(\widetilde{E\mathcal{F}} \wedge Y)]_H = *$$

for $H \in \mathcal{F}$ since $\mathrm{Res}^G_H \widetilde{E\mathcal{F}} \simeq *$ to get that the third term is zero, giving that the first map is injective. This map is also surjective since there is no obstruction to extending a map $X_{\mathcal{F}} \to \widetilde{E\mathcal{F}} \wedge Y$ to $X \to \widetilde{E\mathcal{F}} \wedge Y$ because for $H \in \mathcal{F}$, $[G/H_+, \widetilde{E\mathcal{F}} \wedge Y]_G = \pi_0(\widetilde{E\mathcal{F}}^H \wedge Y^H) = *$. The second bijection comes from the cofibration sequence

$$E\mathcal{F}_+ \wedge Y \to Y \to \widetilde{E\mathcal{F}} \wedge Y$$

and the fact that $[X_{\mathcal{F}}, E\mathcal{F}_+ \wedge Y]_G = 0$ because $E\mathcal{F}_+ \wedge Y$ only have cells of orbit type in $\mathcal{F}$ and $[G/K_+, G/H_+]_G = *$ if $K \notin \mathcal{F}$ and $H \in \mathcal{F}$, just by definition of families being closed under subconjugations. □

Remark 8.2. Applying Lemma 8.1 to the family $\mathcal{F} = \{e\}$ we get that for finite G-complexes X

$$\mathcal{E}^q_G(X, \widetilde{EG}) = \begin{cases} \operatorname*{colim}_{V \in s(\mathcal{U}_G)} [S(\Sigma^V X), \mathcal{E}_G(V \oplus \mathbb{R}^q)]_G & \text{if } q \geq 0 \\ \operatorname*{colim}_{V \in s(\mathcal{U}_G)} [S(\Sigma^V \Sigma^{-q} X), \mathcal{E}_G V]_G & \text{if } q < 0 \end{cases}$$

where S is the singular set functor, ie. $SX = X_{\{e\}}$ denotes the G-subcomplex of cells with orbit type not of the form G/e.

We now axiomatise the properties of the singular set functor in the notion of S-functors.

Definition 8.3. a) An S-functor consists of the data (T, τ) where T is an endofunctor of the category of based G-complexes $\mathcal{S}_*^G$ and natural maps $\tau : T(X \wedge Y) \to (TX) \wedge Y$ satisfying

i) $\tau = id$ when $Y = S^0$

ii) τ transitive

iii) τ is a homeomorphism when G acts trivially on Y

b) A map of S-functors is just a natural transformation of S-functors making the structure maps commute strictly. We say that a map of S-functors is an equivalence or a cofibration if it is componentwise G-equivalence or G-cofibration.

c) For an S-functor, we can define the groups

$$\mathcal{E}_G^q(X, T) = \begin{cases} \operatorname*{colim}_{V \in s(\mathcal{U}_G)} [T(\Sigma^V X), \mathcal{E}_G(V \oplus \mathbb{R}^q)]_G & \text{if } q \geq 0 \\ \operatorname*{colim}_{V \in s(\mathcal{U}_G)} [T(\Sigma^V \Sigma^{-q} X), \mathcal{E}_G V]_G & \text{if } q < 0 \end{cases}$$

using the structure maps τ. For example in case $q = 0$ and $V \hookrightarrow W$, we have

$$\begin{aligned} [T(S^V \wedge X), \mathcal{E}_G V]_G &\to [S^{W-V} \wedge T(S^V \wedge X), S^{W-V}\mathcal{E}_G V]_G \\ &\xrightarrow{\tau^*} [T(S^W \wedge X), S^{W-V}\mathcal{E}_G V]_G \\ &\xrightarrow{\sigma_*} [T(S^W \wedge X), \mathcal{E}_G W]_G \end{aligned}$$

Fact 8.4. S-functors have all the usual operations we like to take on spaces - see [Car84] §IV.

1. We can take wedges, smash product with spaces, pushouts, cofibres in the category of S-functors. When $\varphi : T \to T'$ is a cofibration, we can take the quotient S-functor T'/T as the cofibre defined by
$$T'/T(X) = T'X/TX$$
2. An equivalence of S-functors induces an isomorphism of the associated groups.
3. A cofibration of S-functors induces the long exact sequence
$$\cdots \to \mathcal{E}_G^q(X; T'/T) \to \mathcal{E}_G^q(X; T') \to \mathcal{E}_G^q(X; T) \to \cdots$$
4. Clearly $\mathcal{E}_G^q(X; T' \vee T) \cong \mathcal{E}_G^q(X; T') \oplus \mathcal{E}_G^q(X; T)$

Definition 8.5. Suppose given $K \triangleleft H \leq G$, define an S-functor $C(K, H)$ by letting

$$C(K, H)(X) = G_+ \wedge_H X^K$$

The structure map $G_+ \wedge_H (X^K \wedge Y^K) \to (G_+ \wedge_H X^K) \wedge Y$ is the one induced by inclusion $X^K \wedge Y^K \to (G_+ \wedge_H X^K) \wedge Y$ using that $G_+ \wedge_H -$ is the left adjoint to Res_H^G.

The following is the reason why geometric fixed points appear in the proof.

Lemma 8.6. *For $K \triangleleft H \subset G$ and $J := H/K$, $\widehat{\mathcal{E}}_G^*(X; C(K, H)) \cong (\widehat{\Phi^J \mathcal{E}})_J^*(X^K)$.*

Proof. Since we're comparing the p-adic theories on both sides, we might as well work with X finite. For notational simplicity we just show for the degree 0 case.

$$\begin{aligned} \mathcal{E}_G^0(X; C(K, H)) &= \operatorname*{colim}_V [G_+ \wedge_H \Sigma^{V^K} X^K, \mathcal{E}_G V]_G \\ &= \operatorname*{colim}_V [\Sigma^{V^K} X^K, \mathcal{E}_G V]_H \\ &= \operatorname*{colim}_V [\Sigma^{V^K} X^K, (\mathcal{E}_G V)^K]_{H/K} \\ &= (\Phi^J \mathcal{E})_J^0(X^K) \end{aligned}$$

Here we've used the fact that if $\mathcal{U}_G$ was a complete G-universe, then $\mathrm{Res}_H^G \mathcal{U}_G$ is a complete H-universe and $(\mathrm{Res}_H^G \mathcal{U}_G)^K$ is a complete H/K-universe. $\square$

The main theorem of this section is the following, which basically says that we can reconstruct SX, up to G-homotopy type, from its fixed point sets X^H for H elementary abelian subgroups of G - this should be quite surprising, and is the content of the extra structure on $\mathcal{S}_*^G$ needed in the proof of the Segal conjecture that wouldn't have been present had we worked purely with G-spectra. This is where we will apply Quillen's homotopical analysis of subgroups posets.

Theorem 8.7 (The S-functor approximation)**.** *Let G be a finite p-group of rank r. Then*

(a) There is an S-functor A and an equivalence $\phi : A \to S$, where A has filtration

$$F_0A \subset F_1A \subset \cdots \subset F_{r-1}A = A$$

by successive cofibrations. Writing $B_0 = F_0A$ and $B_q = F_qA/F_{q-1}A$ for $0 < q < r$, there are isomorphisms of S-functors

$$B_q \cong \bigvee_{[\omega]} \Sigma^q C(A(\omega), H(\omega))$$

where ω are strictly ascending chains of nontrivial elementary abelians subgroups of G, $(A_0 \lneq \ldots \lneq A_q)$, $A(\omega) = A_q$, and $H(\omega) = \{g \in G \mid gA_ig^{-1} = A_i \text{ for } 0 \le i \le q\}$. The wedge runs over orbits of such chains under the conjugation action of G on the set of such chains.

(b) If $G = \mathbb{F}_p^r$ is elementary abelian, then there is another S-functor $\widetilde{A}$ with a filtration

$$F_0\widetilde{A} \subset F_1\widetilde{A} \subset \cdots \subset F_{r-2}\widetilde{A} = \widetilde{A}$$

by successive cofibrations. Writing $\widetilde{B}_0 = F_0\widetilde{A}$ and $\widetilde{B}_q = F_q\widetilde{A}/F_{q-1}\widetilde{A}$ for $0 < q < r-1$, there are isomorphisms of S-functors

$$\widetilde{B}_q \cong \bigvee_{\omega} \Sigma^q C(A(\omega), G)$$

with notation as before. Moreover, there is a cofibration $\widetilde{A} \to A$ such that the quotient $A/\widetilde{A}$ is equivalent to the wedge of $p^{r(r-1)}$ copies of the S-functor $\Sigma^{r-1}C(G,G) : X \mapsto \Sigma^{r-1}X^G$.

Definition 8.8. Let $\mathcal{A} = \mathcal{A}(G)$ be the poset of nontrivial elementary abelian p-subgroups of G with opposite inclusion, ie. $A \to B$ if $B \le A$. Let $\widetilde{\mathcal{A}} \subset \mathcal{A}$ be the poset of nontrivial proper subgroups of G. These are G-categories where G acts by conjugation.

In order to proceed, we'll need to introduce a couple of constructions.

Construction 8.9. Let X be G-space and SX denote the singular subspace.

1. We can consider X as a topological category discretely, ie. the space of objects is X and the space of morphisms is also X (ie. only identities). The topological classifying space BX of this topological category will be X again since the simplicial space will be $(NX)_0 = X, (NX)_1 = X, (NX)_2 = X \times_X X = X, \ldots$ with structural maps all id_X, so taking the diagonal gives X again.

2. Let $\mathcal{A}[X]$ denote the topological G-category with objects (A, x) where $A \in \mathcal{A}$ and $x \in X^A$, and morphisms $(A, x) \to (B, y)$ if $B \le A$ and $x = y \in X^A$. The space of objects is topologised as the disjoint union of X^A and morphism space topologised as disjoint union of X^A indexed over inclusions $B \le A$. The G-action is by $g \cdot (A, x) = (gAg^{-1}, gx)$.

3. This is equipped with a functor $\psi : \mathcal{A}[X] \to SX$ given by projecting onto the second coordinate. When X is a based, then the G-cofibration $* \hookrightarrow X$ induces a G-cofibration $B\mathcal{A}[*] \hookrightarrow B\mathcal{A}[X]$, and then clearly $B\psi$ factors through $\phi : B\mathcal{A}[X]/B\mathcal{A}[*] \to SX$.

4. Given a second based G-complex Y, we can consider it as a G-category as above, and then define the G-category $\mathcal{A}[X] \wedge Y$ as the (pointed) product of G-categories.

Proposition 8.10 (Carlsson's Singular Blow Up)**.** *For any X the map $B\psi : B\mathcal{A}[X] \to BSX = SX$ is a G-homotopy equivalence.*

Proof. We show that $(B\psi)^H : (B\mathcal{A}[X])^H \to (SX)^H$ is equivalence for all $H \leq G$. Note that fixed points commute with B so we can bring $(-)^H$ in into the level of categories. We want to use Quillen's theorem A, so let $x \in (SX)^H$ and consider the overcategory ψ^H/x. Note that the reason we need the singular locus SX is to ensure that the overcategory ψ/x is not empty for any $x \in SX$ in the case $H = e$.

Observe that ψ^H/x has objects (A, x) where $x \in X^A$, $hx = x$ for all $h \in H$, and A fixed by H. Write G_x for the stabiliser of x in G. Then $H \leq G_x$ and ψ^H/x is just $(\mathcal{A}(G_x))^H$, which is nonempty contractible by Lemma 8.11 below. □

Lemma 8.11 (Quillen's Lemma). *If $G \neq e$ then $B\mathcal{A}$ is G-contractible. In particular $(B\mathcal{A})^H$ is nonempty contractible for all $H \leq G$.*

Proof. Let $C \leq G$ be a central subgroup of order p. Then for any $A \in \mathcal{A}$ we get $AC \in \mathcal{A}$ and $A \subset AC \supset C$. So we get natural transformations of functors on the category $\mathcal{A}$

$$id \Leftarrow (-) \cdot C \quad \text{and} \quad (-) \cdot C \Rightarrow c_C$$

where c_C is the constant functor valued at C. Note that these are G-equivariant natural transformations since C central. So passing to classifying space gives us equivariant homotopy between $id_{B\mathcal{A}}$ and the constant functor. □

Definition 8.12. For X a based G-complex, we define

$$AX := B\mathcal{A}[X]/B\mathcal{A}[*] \quad \text{and} \quad F_qAX := F_qB\mathcal{A}[X]/F_qB\mathcal{A}[*]$$

where the filtrations come from the usual skeletal filtrations on realisations of simplicial G-spaces, namely for X_* a simplicial G-space, $F_q|X_*|$ is the geometric realisation of the smallest subsimplicial G-space containing the first q levels.

Fact 8.13. For X_* a simplicial G-space, we have $F_0|X_*| = X_0$ and for $q > 0$ we have that $F_q|X_*|/F_{q-1}|X_*|$ is G-homeomorphic to $\Sigma^q(X_q/sX_{q-1})$ where $sX_{q-1} \subset X_q$ is the G-space of degenerate q-simplices.

Remark 8.14. We can define a natural transformation of topological categories

$$\mathcal{A}[X \wedge Y] \to \mathcal{A}[X] \wedge Y \quad :: \quad (A, x \wedge y) \mapsto (A, x) \wedge y$$

Since classifying spaces commute with products, and since $BY \cong Y$ when Y is considered as a discrete topological category, this passes to

$$A[X \wedge Y] \to A[X] \wedge Y$$

This is easily seen to give an S-functor structure, since for example, when G acts trivially on Y, $Y^A = Y$ for all $A \leq G$ and so $\mathcal{A}[X \wedge Y] \to \mathcal{A}[X] \wedge Y$ is already an isomorphism of G-categories.

We're now ready to prove the S-functor approximation theorem - the proof is not hard given all the ingredients.

Proof of Theorem 8.7. Let's work on part (a) first. The nondegenerate simplices of $\mathcal{A}[X]$ consist of pairs (ω, x) where $\omega = (A_0 \lneq \ldots \lneq A_q)$ is a chain of strictly ascending nontrivial elementary abelian subgroups of G and $x \in X^{A_q}$. Recall the notations $A(\omega)$ and $H(\omega)$ from the statement of the theorem, and note that $A(\omega) \lhd H(\omega)$. Observe that if G has p-rank r then $F_{r-1}AX = AX$ since there are no nondegenerate q-simplices for $q \geq r$. By Fact 8.13 we get that with the G-actions ignored,

$$B_qX = F_qAX/F_{q-1}AX \cong_G \bigvee_{\omega} \Sigma^q X^{A(\omega)}$$

and this is because, for example,

$$B\mathcal{A}[X]_q = (\coprod_{A \leq B} X^B) \times_{(\coprod_C X^C)} (\coprod_{A \leq B} X^B) \times_{(\coprod_C X^C)} \cdots \times_{(\coprod_C X^C)} (\coprod_{A \leq B} X^B)$$

and each of the pullbacks for $A \leq B$ look like

$$\begin{tikzcd}
 & X^B \arrow[dl, equal] \arrow[dr, hook, "incl"] & \\
X^B \arrow[dr, hook, "incl"'] & & X^A \arrow[dl, equal] \\
 & X^A &
\end{tikzcd}$$

and so taking iterated pullbacks for any choice of chain $A_0 \leq \cdots \leq A_q$ always gives us X^{A_q}. We're indexed over the strictly ascending chains because we've divided out by the nondegenerate simplices as in Fact 8.13, and we take wedge sum because we've divided out by $B\mathcal{A}[*]$. It is then easy to see that incorporating the G-actions we have a G-homeomorphism

$$\bigvee_{[\omega]} G_+ \wedge_{H(\omega)} \Sigma^q X^{A(\omega)} \xrightarrow{\cong_G} B_q X$$

Since everything was natural we get that this induces an isomorphism of S-functors

$$B_q \cong \bigvee_{[\omega]} \Sigma^q C(A(\omega), H(\omega))$$

as required.

As for part (b) in the case of $G = \mathbb{F}_p^r$, we can similarly define $\widetilde{A}$ using $\widetilde{\mathcal{A}}$ instead and run similar arguments as above. For the statement about comparing $\widetilde{A}$ and A, note that since the chains ω with $A(\omega) = G$ are all parametrised by X^G in AX, we easily get that

$$AX/\widetilde{A}X \cong_G (B\mathcal{A}/B\widetilde{\mathcal{A}}) \wedge X^G$$

Now the sequence $B\widetilde{\mathcal{A}} \hookrightarrow B\mathcal{A} \to B\mathcal{A}/B\widetilde{\mathcal{A}}$ together with G-contractibility of $B\mathcal{A}$ by Lemma 8.11 gives that $B\mathcal{A}/B\widetilde{\mathcal{A}} \simeq_G \Sigma B\widetilde{\mathcal{A}}$. Applying the following Lemma 8.15 finishes the proof. □

The following proof is based on Quillen's arguments at the end of §8 of [Qui78], but we follow and flesh out the presentation by Benson [Ben91] Theorem 6.8.5 where it is clearer how we get the number $p^{r(r-1)/2}$. This actually follows from a more general result in the theory of Tits buildings, but we've chosen just to present the following argument since it's so elementary and does not require the introduction of more new concepts. We've included it here not only for the convenience of readers not familiar with Tits buildings, but also because it's such a crucial number that makes the Segal conjecture for elementary abelian p-groups true as was explained in §6.

Lemma 8.15. *If $G = \mathbb{F}_p^r$, then $B\widetilde{\mathcal{A}}$ is equivalent to the wedge of $p^{r(r-1)/2}$ copies of the sphere S^{r-2}.*

Proof. We write $G = V$ since we want to think of these as vector spaces. We prove by induction on r. When $r = 2$, $\widetilde{\mathcal{A}}(V)$ is just the discrete category of one-dimensional subspaces, for which there are $p + 1$ many, so is clearly a wedge sum of $p^{r(r-1)/2}$ copies of S^0. Now suppose $r \geq 3$. Choose any one-dimensional subspace L and let $\mathcal{H}$ be the set of all $(r-1)$-dimensional complements of L in V. By $GL_r(V)$-symmetry of V, we might as well assume $L = \mathbb{F}_p\{(1, 0, \ldots, 0)\}$. Note that $|\mathcal{H}| = p^{r-1}$ since all such will admit a basis of the form

$$\{(a_1, 1, 0, 0, \ldots, 0), (a_2, 0, 1, 0, \ldots, 0), \ldots, (a_{r-1}, 0, 0, \ldots, 0, 1)\}$$

where $a_i \in \mathbb{F}_p$.

Now let $Y = \widetilde{\mathcal{A}}(V) \backslash \mathcal{H}$ considered as a subposet of $\widetilde{\mathcal{A}}(V)$. We show that the simplicial set associated to it is contractible. Consider the quotient

$$q : Y \to \widetilde{\mathcal{A}}_0(V/L)$$

induced by $V \to V/L$, where $\widetilde{\mathcal{A}}_0$ denotes the poset of all proper subspaces (including 0). The target is of course contractible since 0 is initial. We use Quillen's Theorem A to show that q is an equivalence. Let $W \in \widetilde{\mathcal{A}}_0(V/L)$. Then the fibre category q/W is clearly contractible since it contains a terminal object, namely the preimage of $W \subset V/L$ in V, and so Quillen's theorem A implies that $NY \simeq N\widetilde{\mathcal{A}}_0(V/L) \simeq *$. Here we really needed to remove $\mathcal{H}$ from $\widetilde{\mathcal{A}}(V)$ for otherwise we will get instead a functor $q : \widetilde{\mathcal{A}}(V) \to \mathcal{A}_0(V/L)$ where the target is the full poset of V/L, and then the fibre category over V/L would just be $\widetilde{\mathcal{A}}(V)$ again, which is not contractible.

Hence $N\widetilde{\mathcal{A}}(V) \simeq N\widetilde{\mathcal{A}}(V)/NY$, where N is the nerve functor. We now describe the latter - $N\widetilde{\mathcal{A}}(V)_n/NY_n$ consists of the n-chains

$$(0 \neq V_0 \leq \cdots \leq V_n)$$

where $V_n \in \mathcal{H}$. Thus we see that $N\widetilde{\mathcal{A}}(V)/NY$ is just the (pointed simplicial set) wedge sum of subsimplicial sets S_H consisting of chains ending in H, for all $H \in \mathcal{H}$. But then each of these is nothing but $\Sigma\widetilde{\mathcal{A}}(H)$ since the datum of the top subspace being H is superfluous, and the suspension is just because $\widetilde{\mathcal{A}}(H)_n \cong (S_H)_{n+1}$. Hence by induction the S_H are wedges of $p^{(r-1)(r-2)/2}$ copies of $(r-3)+1$ spheres. But we've argued above that $|\mathcal{H}| = p^{r-1}$, and so we're done. $\square$

We are now ready to apply the filtration theorem 8.7 to prove Theorem A.

Theorem A. *Let $K \lhd H \leq G$ and write $J = H/K$. Suppose $(\widehat{\Phi^J\mathcal{E}})^*_J$ vanishes on contractible J-spaces for all proper subquotients J. Let X be a G-complex such that $X^G \simeq S^0$ and X^H contractible for all proper subgroups. Then*

i) If G is not elementary abelian then $\widehat{\mathcal{E}}^*_G(X;\widetilde{EG}) = 0$.

ii) If $G = \mathbb{F}_p^r$ then $\widehat{\mathcal{E}}^*_G(X;\widetilde{EG})$ is the direct sum of $p^{r(r-1)/2}$ copies of $\Sigma^{r-1}(\widehat{\Phi^G\mathcal{E}})^*(S^0)$.

Proof. i) By Remark 8.2 and the equivalence $A \to S$ we have

$$\widehat{\mathcal{E}}^*_G(X;\widetilde{EG}) \cong \widehat{\mathcal{E}}^*_G(X;S) \cong \widehat{\mathcal{E}}^*_G(X;A)$$

By Lemma 8.6, and writing $J(\omega) = H(\omega)/A(\omega)$, we have

$$\widehat{\mathcal{E}}^n_G(X;B_q) \cong \bigoplus_{[\omega]} \widehat{\mathcal{E}}^{n-q}_G(X;C(A(\omega),H(\omega))) \cong \bigoplus_{[\omega]} (\widehat{\Phi^{J(\omega)}\mathcal{E}})^{n-q}_{J(\omega)}(X^{A(\omega)})$$

Now G is assumed not to be elementary abelian, and so $A(\omega) \neq G$, hence by all our hypotheses we get that the right-hand side vanishes, and so $\widehat{\mathcal{E}}^n_G(X;B_q) = 0$. And so by induction up the filtration we get $\widehat{\mathcal{E}}^n_G(X;A) = 0$.

ii) Let $G = \mathbb{F}_p^r$. If $r = 1$ then $B_0 = A$ and $\widehat{\mathcal{E}}^*_G(X;B_0) \cong (\widehat{\Phi^G\mathcal{E}})^*(S^0)$ since $\omega = (G)$ is the only possible chain, so this proves it in this case. Now suppose $r \geq 2$. By the arguments of part (i) we get $\widehat{\mathcal{E}}^*_G(X;\widetilde{A}) = 0$, and so by the cofibration $\widetilde{A} \to A \to A/\tilde{A}$, we have $\widehat{\mathcal{E}}^*_G(X;A) \cong \widehat{\mathcal{E}}^*_G(X;A/\widetilde{A})$. And now since $\widehat{\mathcal{E}}^*_G(X;C(G,G)) = (\widehat{\Phi^G\mathcal{E}})^*(S^0)$ we get that $\widehat{\mathcal{E}}^*_G(X;\widetilde{EG})$ is the sum of $p^{r(r-1)/2}$ copies of $\Sigma^{r-1}(\widehat{\Phi^G\mathcal{E}})^*(S^0)$.

$\square$

9 The Ext group calculation of Adams-Gunawardena-Miller

We include here a sketch of the hard calculational input from [AGM85] in the Segal conjecture story. We have chosen to omit the many intricate technical details and only highlight some of the beautiful ideas involved.

Let p be a prime, and let A be the mod p Steenrod algebra (where we've suppressed writing p since we've fixed a prime once and for all). In this section, let $G = V = (\mathbb{F}_p)^r$ be an elementary abelian p-group, where the notation is supposed to suggest that we should think of it as a finite-dimensional $\mathbb{F}_p$-vector space instead. We will write $H^*(X)$ to mean $H^*(X, \mathbb{F}_p)$.

Fact 9.1. Recall that the cohomology ring structure of $H^*(BV, \mathbb{F}_p)$ is given as follows

$$H^*(BV, \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p[h_1, \ldots, h_r] & \text{if } p = 2 \\ \mathbb{F}_p[\beta h_1, \ldots, \beta h_r] \otimes \Lambda(h_1, \ldots, h_r) & \text{if } p > 2 \end{cases}$$

Here $\{h_i\}_i$ forms a basis for $H^1(BV, \mathbb{F}_p) \cong V^*$, β is the Bockstein Steenrod operation, and Λ is the exterior algebra.

We now give the definitions of some basic notions.

Definition 9.2. (a) Let $L = \chi(V) = (\beta \cdot h_1) \cdots (\beta \cdot h_r) \in H^{2r}(BV)$ be the Euler class, where $\{h_i\}_i$ is a basis for $H^1(BV) \cong V^*$.

(b) Write $H^*(BV)_{loc}$ for $H^*(BV)$ after inverting

$$\prod_{0 \neq h \in H^1(BV) \cong V^*} \beta h$$

Note that this is the same as $H^*(BV)[L^{-1}]$. In this section we prefer to work with the former since it makes certain things clearer, for example, that fixed points under the action of $GL(V)$ commute with this localisation, since the product is invariant under the action of $GL(V)$.

(c) We say that a homomorphism of A-modules $M \to N$ is a Tor-equivalence if the induced map

$$\mathrm{Tor}^A_{**}(\mathbb{F}_p, M) \to \mathrm{Tor}^A_{**}(\mathbb{F}_p, N)$$

is an isomorphism.

(d) Denote by $Syl(V) \leq GL(V)$ the subgroup of upper unitriangular matrices, that is, upper triangular matrices with 1's on the diagonal. Note that $|Syl(V)| = p^{r(r-1)/2}$.

The following is then the main computational input that we will need to complete the proof of the Segal conjecture for elementary abelian p-groups.

Theorem 9.3. *Consider the $\mathbb{F}_p$-module $\mathbb{F}_p \otimes_A H^*(BV)_{loc}$ of A-indecomposable elements of $H^*(BV)_{loc}$ and regard it as a trivial A-module (since $\mathbb{F}_p$ was a trivial A-module). Then*

(a) The quotient homomorphism

$$\theta : H^*(BV)_{loc} \to \mathbb{F}_p \otimes_A H^*(BV)_{loc}$$

is a Tor-equivalence.

(b) $\mathbb{F}_p \otimes_A H^*(BV)_{loc}$ is concentrated in degree $-r$ where the rank is $p^{r(r-1)/2}$.

Corollary 9.4. *The quotient homomorphism $\theta : H^*(BV)_{loc} \to \mathbb{F}_p \otimes_A H^*(BV)_{loc}$ induces an isomorphism*

$$Ext^{**}_A(K \otimes \mathbb{F}_p \otimes_A H^*(BV)_{loc}, \mathbb{F}_p) \to Ext^{**}_A(K \otimes H^*(BV)_{loc}, \mathbb{F}_p)$$

for any finite dimensional A-module K.

Remark 9.5. These results are really surprising since they allow us to identify the Tor/Ext group of a big and scary A-module with one of a much smaller and manageable A-module, and the heart of the matter is the result of Gunawardena and Miller in Theorem 9.6 below.

The proof of this will involve (one of the many variants of) a very important construction in the study of Steenrod modules, namely the Singer construction. This is a functor $T : \mathrm{Mod}_A \to \mathrm{Mod}_A$ that additively looks like $M \mapsto H^*(B\mathbb{F}_p)_{loc} \otimes M$ equipped with a natural map $\varepsilon : TM \to M$. See [AGM85] §2 for more details. The two main results on the Singer construction that we will need are the following:

Theorem 9.6 (Gunawardena, Miller)**.** *The map $\varepsilon : T(M) \to M$ is a Tor-equivalence.*

Theorem 9.7. *There is an isomorphism of A-algebras $T^r(\mathbb{F}_p) \cong H^*(BV)^{Syl(V)}_{loc}$.*

We are now ready to prove Theorem 9.3, and the clever idea is to prove it by downward induction on certain carefully chosen normal subgroups G of $Syl(V)$. More precisely [AGM85] §6 shows that there exists a chain of normal subgroups

$$e = G_n \lhd G_{n-1} \lhd \cdots \lhd G_1 \lhd G_0 = Syl(V)$$

satisfying $G_i/G_{i+1} \cong \mathbb{F}_p$ and such that for each i there is a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_p = H^*(BV)^{G_i}_{loc}$$

by A-submodules in which each subquotient M_j/M_{j-1} is isomorphic to $H^*(BV)^{G_{i-1}}_{loc}$. The formulation of the induction hypothesis is then as follows:

Theorem 9.8. *For each i the following are true.*

(a) The quotient map

$$H^*(BV)^{G_i}_{loc} \xrightarrow{q} \mathbb{F}_p \otimes_A H^*(BV)^{G_i}_{loc}$$

is a Tor-equivalence.

(b) $\mathbb{F}_p \otimes_A H^*(BV)^{G_i}_{loc}$ is concentrated in degree $-r$, where it is of rank $|Syl(V) : G_i|$.

The case of $G_n = e$ will then give us Theorem 9.3.

Proof of Theorem 9.8. The case of $i = 0$ is given by the two preceding theorems: $H^*(BV)^{Syl(V)}_{loc} \cong T^r(\mathbb{F}_p)$ by Theorem 9.7, and by Theorem 9.6 we have the sequence of Tor-equivalences

$$T^r\mathbb{F}_p \to T^{r-1}\mathbb{F}_p \to \cdots \to T\mathbb{F}_p \to \mathbb{F}_p$$

and so these combine to give part (a) for the $i = 0$ case. Furthermore, we know already that (b) is also true for this case.

Now suppose (a) and (b) are true for the case $i \geq 0$ and we want to prove them for $i + 1$. Suppose as the hypothesis of a subsidiary induction over j, that the quotient map $M_j \xrightarrow{q} \mathbb{F}_p \otimes_A M_j$ is a Tor-equivalence and that $\mathbb{F}_p \otimes_A M_j$ is zero except in degree $-r$. Now consider the following diagram

$$\begin{tikzcd}
& 0 \arrow[r] & M_j \arrow[r] \arrow[d, "q_i"] & M_{j+1} \arrow[r] \arrow[d, "q_{j+1}"] & H^*(BV)^{G_i}_{loc} \arrow[r] \arrow[d, "q"] & 0 \\
\mathrm{Tor}^A_{1,*}(\mathbb{F}_p, H^*(BV)^{G_i}_{loc}) \arrow[rr] & & \mathbb{F}_p \otimes_A M_j \arrow[r] & \mathbb{F}_p \otimes_A M_{j+1} \arrow[r] & \mathbb{F}_p \otimes_A H^*(BV)^{G_i}_{loc} \arrow[r] & 0
\end{tikzcd}$$

By our main inductive hypothesis (b) we have

$$\mathrm{Tor}^A_{1,-r}(\mathbb{F}_p, H^*(BV)^{G_i}_{loc}) \cong \bigoplus \mathrm{Tor}^A_{1,0}(\mathbb{F}_p, \mathbb{F}_p) = 0$$

so the bottom row is short exact. The outer vertical maps are Tor-equivalences by induction, so by the five lemma, the middle one is too. Going up in this way we obtain part (a) of the theorem for the case of $i+1$.

Furthermore, the inductive hypothesis also says that $\mathbb{F}_p \otimes_A M_j$ and $\mathbb{F}_p \otimes_A H^*(BV)^{G_i}_{loc}$ are concentrated in degree $-r$, so the middle term is too. Finally, using that $|G_i/G_{i+1}| = p$, that $|Syl(V) : G_{i+1}| = |Syl(V) : G_i| \cdot |G_i : G_{i+1}|$, and that the subsidiary filtration of $H^*(BV)^{G_i}_{loc}$ is of length p, we also get the rank size in part (b) for the case $i + 1$. ☐

10 Nonequivariant theory and the proof of Theorem B

The idea of this section is to translate the G-cohomology theories accounting for the "free" part of the Segal conjecture problem into ordinary cohomology theories via Thom spectra, and these in turn can be analysed with an inverse limit Adams spectral sequence (ASS) using the Ext group calculation from the previous section as an input. All cohomologies here will be cohomology mod p.

Theorem 10.1 ([CMP87] 8.1). *There are Thom spectra BG^{-V} for complex representations V of G and maps $f : BG^{-W} \to BG^{-V}$ for inclusions $V \subset W$ which satisfy*

i) $H^*(BG^{-V})$ is a free $H^*(BG)$-module on one generator ι_V of degree $-dim_R V$ and $f^* : H^*(BG^{-V}) \to H^*(BG^{-W})$ is the morphism of $H^*(BG)$-modules specified by $f^*(\iota_V) = \chi(W - V)\iota_W$ where $\chi(W - V) \in H^*(BG)$ is the Euler class of the representation bundle $EG \times_G (W - V) \to BG$.

ii) If $\mathcal{E} \in Sp^G$ is split then $\mathcal{E}_*(BG^{-V})$ is isomorphic to $\mathcal{E}^G_*(S^V; EG_+)$ and we have the commuting diagram where $e : S^V \to S^W$ is the inclusion

$$\begin{tikzcd} \mathcal{E}_*(BG^{-W}) \arrow[r, "f_*"] \arrow[d, "\cong"] & \mathcal{E}_*(BG^{-V}) \arrow[d, "\cong"] \\ \mathcal{E}^G_*(S^W; EG_+) \arrow[r, "e^*"] & \mathcal{E}^G_*(S^V; EG_+) \end{tikzcd}$$

Corollary 10.2. *If $\mathcal{E} \in Sp^G$ is split and $\mathcal{X} = \bigcup S^{nV}$ Carlsson's model, then*

$$\widehat{\mathcal{E}}_G^{-q}(\mathcal{X}; EG_+) \cong \lim_n \mathcal{E}_q(BG^{-nV})^\wedge_p = \lim_n \pi_q(\mathcal{E} \otimes BG^{-nV})^\wedge_p$$

Proof. Just recall notation that $\mathcal{E}^G_q = \mathcal{E}^{-q}_G$, the inverse limit definition of $\widehat{\mathcal{E}}$ theories, and (ii) of the preceding theorem. □

These inverse limit homotopy groups can in turn be understood using a certain inverse limit Adams spectral sequence given as follows.

Proposition 10.3. *Assume we have a sequence of spectra*

$$\cdots \to X_{n+1} \to X_n \to \cdots \to X_0$$

such that each X_n is p-complete, bounded below, and of finite type over $\mathbb{Z}_p$. Let $\{E_r X\}$ be the inverse limit of the spectral sequences $\{E_r X_n\}_n$ where the E_2 pages are

$$E_2^{s,t} X_n = Ext_A^{s,t}(H^*(X_n), \mathbb{F}_p) \Rightarrow \pi_{t-s}(X_n) \qquad d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$$

Then

1. *$E_2 X = Ext_A(\mathrm{colim}_n H^*(X_n), \mathbb{F}_p)$*
2. *$E_r X$ is a differential $E_r S^0$-module*
3. *The spectral sequence converges strongly to $\lim_n \pi_*(X_n)$*

Proof. See Proposition 7.1 of [CMP87]. □

Note that Definition 9.2 and Theorem 10.1 give us that $\mathrm{colim}_n H^*(BG^{-nV}) = H^*(BG)[L^{-1}] = H^*(BG)_{loc}$. Having laid out all the general machinery, let's recall Theorem B.

Theorem B. *Suppose $\mathcal{E} \in Sp^G$ is split and the underlying spectrum is bounded below. Let X be a G-space such that $X^G \simeq S^0$ and X^H contractible for all proper subgroups. Then*

i) If G is not elementary abelian then $\widehat{\mathcal{E}}^*_G(X; EG_+) = 0$.

ii) If $G = \mathbb{F}_p^r$ and $H^q(\mathcal{E}, \mathbb{F}_p) = 0$ for all sufficiently large q then $\widehat{\mathcal{E}}^*_G(X; EG_+)$ is the direct sum of $p^{r(r-1)/2}$ copies of $\Sigma^r \widehat{\mathcal{E}}^*(S^0)$.

By the results above we have the strongly convergent inverse limit ASS $\{E_r\} = \{E_r(\mathcal{E} \otimes BG^{-nV\wedge}_{\ \ p})\}$

$$E_2^{s,t} = Ext_A^{s,t}(H^*(\mathcal{E}) \otimes H^*(BG^{-nV})[L^{-1}], \mathbb{F}_p) \Rightarrow \widehat{\mathcal{E}}_G^{s-t}(\mathcal{X}; EG_+)$$

where here we've used the Kunneth theorem for mod p cohomology.

Proof of Theorem B(i). We observe that in general, if $i : H \hookrightarrow G$ is a proper subgroup, then $\chi(V) \in H^*(BG)$ restricts to zero in $H^*(BH)$. To see this, we record the argument given in [Car84] Lemma III.1. Now the regular complex representation of G has $|G|/|H|$ trivial H-summands, so the reduced representation V has $|G|/|H| - 1$ trivial H-summands. But then $H \neq G$ so $|G|/|H| \geq 2$, so i^*V has a trivial H-summand. Thus by the standard characteristic class result that says a complex vector bundle has zero Euler class if it contains a trivial line bundle, we get $\chi(i^*V) = 0$. Hence by naturality of characteristic classes, $\chi(V)$ restricts to $\chi(i^*V) = 0$.

Coming back to our case, since we suppose G is not elementary abelian, we get that $\chi(V)$ restricts to zero in every elementary abelian subgroup of G. So by Quillen's F-isomorphism theorem ([Qui71] 6.2), $\chi(V) \in H^*(BG)$ is nilpotent. Together with Theorem 10.1(i) this implies $H^*(BG)[L^{-1}] = 0$ in this case, so by the spectral sequence above, we get $\widehat{\mathcal{E}}_G^*(\mathcal{X}; EG_+) = 0$. □

Unlike the relatively painless proof of Theorem B(i), the proof of part (ii) utilises the hard computation result from the previous section. We use it to prove the following theorem which immediately implies Theorem B(ii) by Corollary 10.2.

Fact 10.4 (See [Ada66] for the case $p = 2$ and [Liu63] for $p > 2$). For $h_0 \in \mathrm{Ext}_A^{1,1}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathrm{Ext}_A^{1,1}(H^*(\mathbb{S}), \mathbb{F}_p)$ the generator corresponding to the unit $1 \in \pi_0(\mathbb{S}_p^\wedge)$, we have for $p = 2$

$$\mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} 0 & \text{if } t - s < 0 \\ \mathbb{F}_2\{h_0^s\} & \text{if } t = s \\ \text{Annihilated by } h_0 & \text{if } t = s + 1 \end{cases}$$

and for p odd

$$\mathrm{Ext}_A^{s,t}(\mathbb{F}_p, \mathbb{F}_p) = \begin{cases} 0 & \text{if } t - s < 0 \\ \mathbb{F}_p\{h_0^s\} & \text{if } t = s \\ 0 & \text{if } t = s + 1 \end{cases}$$

Theorem 10.5. *Let $\mathcal{E}$ be a p-complete ordinary spectrum which is bounded below, of finite type over $\mathbb{Z}_p$, and cohomologically bounded above, ie. $H^*(\mathcal{E}, \mathbb{F}_p) = 0$ for $*$ sufficiently large. Let Y be the wedge of $p^{r(r-1)/2}$ copies of $\mathbb{S}^{-r}$. Then there is a compatible system of maps $\alpha : Y \to BG^{-nV}$ which induces an isomorphism*

$$\pi_*(\mathcal{E} \otimes Y) \to \lim \pi_*(\mathcal{E} \otimes BG^{-nV})$$

Remark 10.6. We have enough finiteness in the homotopy groups of $\mathcal{E}$ and BG^{-nV}, and therefore also in $\mathcal{E} \otimes BG^{-nV}$, to guarantee that the p-completion of all these spectra is just gotten from smashing with $\mathbb{S}_p^\wedge$. In particular, ${BG^{-nV}}_p^\wedge \simeq \mathbb{S}_p^\wedge \otimes BG^{-nV}$ and $\mathcal{E} \otimes BG^{-nV} \simeq (\mathcal{E} \otimes \mathbb{S}_p^\wedge) \otimes BG^{-nV} \simeq (\mathcal{E} \otimes BG^{-nV})_p^\wedge$.

Proof. First consider case $\mathcal{E} = \mathbb{S}_p^\wedge$. Theorem 9.3 and Corollary 9.4 gives us

$$\mathrm{Ext}_A(H^*(BG)[L^{-1}], \mathbb{F}_p) \xleftarrow{\cong} \mathrm{Ext}_A(H^*(BG)[L^{-1}] \otimes_A \mathbb{F}_p, \mathbb{F}_p) \cong \mathrm{Ext}_A(\Sigma^{-r} \oplus^{p^{r(r-1)/2}} \mathbb{F}_p, \mathbb{F}_p)$$

and so by the facts 10.4 we get for $E_2^{s,t} := E_2^{s,t}(\lim_n BG^{-nV})$ that $E_2^{s,t} = 0$ if $t - s < -r$, $E_2^{0,-r} = \oplus^{p^{r(r-1)/2}} \mathbb{F}_p$, $E_2^{s,s-r} = h_0^s \cdot E_2^{0,-r}$, and $E_2^{s,s-r+1} = 0$ when p odd and is annihilated by h_0 when $p = 2$ and $s = 1$. Hence the elements of $E_2^{s,s-r}$ are all non-bounding permanent cycles, where in the $p = 2$ case, this is because if there exists a minimal $k \in \mathbb{Z}$, an $s \in \mathbb{Z}$, and $x \in E_k^{s,s-r+1}$ such that $d_k(x) = l \neq 0 \in E_k^{s+k,s-r+k}$, then

$$0 = d_k(h_0 x) = h_0 \cdot d_k(x) = h_0 \cdot l \neq 0$$

where the last term is not 0 by minimality of k - this is a contradiction. Furthermore, we know by the usual ASS for $\mathbb{S}_p^\wedge$ that multiplication by h_0 induces multiplication by p on the group converged to by the vertical line $t - s = 0$ of the spectral sequence, and so since the spectral sequence converges strongly to $\lim_n \pi_*({BG^{-nV}}_p^\wedge)$, we get that $\lim_n \pi_{-r}({BG^{-nV}}_p^\wedge)$ is a free $\mathbb{Z}_p$-module on $p^{r(r-1)/2}$ generators, in the same way that $\pi_0(\mathbb{S}_p^\wedge) \cong \mathbb{Z}_p$.

Choose the $\mathbb{Z}_p$ generators $\{\alpha_i\}_i$ of $\lim_n \pi_{-r}({BG^{-nV}}_p^\wedge)$. Each of these can be thought of as a compatible sequence of maps $\mathbb{S}^{-r} \to {BG^{-nV}}_p^\wedge$ over the n's. And now define $Y := \bigoplus^{p^{r(r-1)/2}} \mathbb{S}^{-r}$ and define

$$\alpha = \oplus \alpha_i : Y \to {BG^{-nV}}_p^\wedge$$

We claim that this α induces the map $\theta : H^*(BG)[L^{-1}] \to H^*(BG)[L^{-1}] \otimes_A \mathbb{F}_p$ and then we'd be done, even for the case of general $\mathcal{E}$, since by the naturality of the ASS we have the following comparison

$$\begin{array}{ccc} \mathrm{Ext}_A(H^*\mathcal{E} \otimes H^*(BG)[L^{-1}] \otimes_A \mathbb{F}_p, \mathbb{F}_p) & \Longrightarrow & \pi_*(\mathcal{E} \otimes Y) \\ \big\downarrow \cong & & \big\downarrow \\ \mathrm{Ext}_A(H^*\mathcal{E} \otimes H^*(BG)[L^{-1}], \mathbb{F}_p) & \Longrightarrow & \lim_n \pi_*(\mathcal{E} \otimes BG^{-nV}) \end{array}$$

where the vertical isomorphism on the left is by our claim and Corollary 9.4.

To see the claim, note that by the construction of the inverse limit ASS we have

$$\lim_n \pi_{-r}(BG^{-nV}{}^\wedge_p) \to E_2^{0,-r} = \mathrm{Hom}_A^{-r}(H^*(BG)[L^{-1}], \mathbb{F}_p) \xleftarrow{\theta^*} \mathrm{Hom}_A^{-r}(\oplus \Sigma^{-r}\mathbb{F}_p, \mathbb{F}_p)$$

which is given by passing to cohomology, so indeed our choices of $\{\alpha_i\}$ were precisely the ones realising the Ext-equivalence θ on cohomology. $\square$

11 The case of elementary abelians and the proof of Theorem D

This section is based on §6 of [CMP87]. Unless otherwise stated, G will denote an elementary abelian p-group $\mathbb{F}_p^r$ of rank r. We recall the statement of Theorem D and the aims in the proof.

Theorem D. *Let $G = \mathbb{F}_p^r$, and X a G-space such that $X^G = S^0$ and $X^H \simeq *$ for all proper subgroups H of G. Assume that the Segal conjecture holds for $\mathbb{F}_p^s$ for all $s < r$, ie. $\widehat{\pi}^*_{\mathbb{F}_p^s}(X) = 0$. Then*

$$\delta : \widehat{\pi}^q_G(X; \widetilde{EG}) \to \widehat{\pi}^{q+1}_G(X; EG_+)$$

is an isomorphism for all q.

The idea was to show that the bottom δ in the following diagram is an isomorphism and that the left η is injective mod p, where $\mathcal{K}^*_G$ is the G-cohomology theory represented by the Borel completed Eilenberg-MacLane spectrum $\mathcal{K} := F(EG_+, \underline{H\mathbb{F}_p})$.

$$\begin{array}{ccc} \widehat{\pi}^{r-1}_G(X; \widetilde{EG}) & \xrightarrow{\delta} & \widehat{\pi}^r_G(X; EG_+) \\ \big\downarrow \eta & & \big\downarrow \eta \\ \widehat{\mathcal{K}}^{r-1}_G(X; \widetilde{EG}) & \xrightarrow{\delta} & \widehat{\mathcal{K}}^r_G(X; EG_+) \end{array}$$

Remark 11.1. Note that the G-spectrum associated to $\mathcal{K}_G(-; \widetilde{EG})$ is the Tate G-spectrum $\widetilde{EG} \otimes F(EG_+, \underline{H\mathbb{F}_p})$, and as remarked on page 4 of [GM95b], this was one of the early instances of the Tate construction in equivariant stable homotopy theory. In hindsight, once we've had the idea of showing mod p isomorphism of the top δ map by comparison with a test theory, the spectrum $F(EG_+, \underline{H\mathbb{F}_p})$ presents itself quite naturally as a candidate since $\underline{H\mathbb{F}_p}$ detects mod p equivalences of spectra, and we Borel complete it to force the isomorphism of the bottom map.

Remark 11.2. While the general strategy above and most parts of its proof are interesting, this section also contains some of the most annoying bits of this whole document (Lemma 11.10) with lots of close-call technical fixes, so readers be warned!

Bottom δ is an isomorphism

The idea is to justify that we indeed have the long exact sequence of $\widehat{\mathcal{K}}_G(X; -)$ associated to the isotropy cofibre sequence $EG_+ \to S^0 \to \widetilde{EG}$. This is not obvious since it's not true that $\mathcal{K}^G_*(Y)$ is finite type for all G-complexes Y with finite skeleta, so we couldn't have just appealed to Lemma 6.6 for exactness.

Proposition 11.3. *For any G-complex X we have a long exact sequence*

$$\cdots \to \widehat{\mathcal{K}}^q_G(X; \widetilde{EG}) \to \widehat{\mathcal{K}}^{q+1}_G(X; EG_+) \to \widehat{\mathcal{K}}^{q+1}_G(X) \to \widehat{\mathcal{K}}^{q+1}_G(X; \widetilde{EG}) \to \cdots$$

Proof. Note that $\mathcal{K}^G_*(Y)$ is of finite type when Y is finite by cellular induction starting from the fact that $\mathcal{K}^G_{-q}(G/H_+) = \mathcal{K}^q_G(G/H_+) = H^q(G/H \times_G EG) = H^q(BH)$. In general we have the long exact sequence

$$\cdots \to \mathcal{K}^G_{q+1}(Y^n/Y^{n-1}) \to \mathcal{K}^G_q(Y^{n-1}) \to \mathcal{K}^G_q(Y^n) \to \mathcal{K}^G_q(Y^n/Y^{n-1}) \to \cdots$$

where Y^n/Y^{n-1} are wedges of $\Sigma^n G/H_+$, so for a fixed q there's no reason to expect that $\mathcal{K}^G_q(Y) = \operatorname{colim}_n \mathcal{K}^G_q(Y^n)$ to be attained for a fixed n. However, if $Y = Y'_+$ for Y' free, then the only orbit type that can occur is $H = e$, so $H^*(BH) = 0$.

The upshot of this is that $\mathcal{K}^G_q(EG_+)$ is in particular of finite type. Hence $\mathcal{K}^G_q(X; EG_+)$ is too, for X finite, again by cellular induction. By the long exact sequence associated to the isotropy cofibre sequence, the same is true of $\mathcal{K}^G_q(X; \widetilde{EG})$. So everything in the following long exact sequence

$$\cdots \to \mathcal{K}^G_{q+1}(X; \widetilde{EG}) \to \mathcal{K}^G_q(X; EG_+) \to \mathcal{K}^G_q(X) \to \mathcal{K}^G_q(X; \widetilde{EG}) \to \cdots$$

is finitely generated for finite X, and so p-completion is exact, and produces finitely generated $\mathbb{Z}_p$-modules. As in the proof of Lemma 6.6 passing to inverse limits over finite subcomplexes is still exact, and we get the fundamental exact sequence for $\widehat{\mathcal{K}}_G$ as required. $\square$

Lemma 11.4. *For every subquotient J of G, including G itself, $\widehat{\mathcal{K}}^*_J$ vanishes on contractible J-spaces.*

Proof. Let $J = H/K$. For a finite J-complex X and for $q \geq 0$,

$$\begin{aligned}
(\Phi^J\mathcal{K})^q_J(X) &= \operatorname*{colim}_V [\Sigma^{V^K} X, \mathcal{K}(V \oplus \mathbb{R}^q)^K]_J \\
&= \operatorname*{colim}_V [\Sigma^{V^K} X, F(EG_+, \underline{H\mathbb{F}}_p(V \oplus \mathbb{R}^q))]_H \\
&= \operatorname*{colim}_V [\Sigma^{V^K} X \wedge EG_+, \underline{H\mathbb{F}}_p(V \oplus \mathbb{R}^q)]_H \\
&= \operatorname*{colim}_V \lim_n [\Sigma^{V^K} X \wedge EG^n_+, \underline{H\mathbb{F}}_p(V \oplus \mathbb{R}^q)]_H
\end{aligned}$$

Here the last step is just because the $\lim^1$ term vanishes since everything is finite. We can do a similar thing for $q < 0$. All in all, for a general X we have

$$\widehat{(\Phi^J\mathcal{K})}^q_J(X) = \lim_\alpha \operatorname*{colim}_V \lim_n [\Sigma^{V^K} X_\alpha \wedge EG^n_+, \underline{H\mathbb{F}}_p(V \oplus \mathbb{R}^q)]_H$$

Note that here we didn't have to p-complete anything on the right-hand side since everything is already an $\mathbb{F}_p$-vector space. Now if X was contractible, then $X \wedge EG_+$ is H-contractible since $(X \wedge EG_+)^L = X^L \wedge * = *$ for all $\{e\} \neq L \subset H$, and $X \wedge EG_+ \simeq X \wedge S^0 = X \simeq *$. Hence, for a fixed pair (α, n), there exists a pair (β, m) for $n \leq m$ such that the inclusion $X_\alpha \wedge EG^n_+ \subset X_\beta \wedge EG^m_+$ is null H-homotopic. This is just by a compactness argument of the following form where the bottom horizontal is a H-nullhomotopy

$$\begin{tikzcd}
X_\alpha \wedge EG^n_+ \wedge I_+ \arrow[r, hook, dashed] \arrow[d, hook] & X_\beta \wedge EG^m_+ \arrow[d, hook] \\
X \wedge EG_+ \wedge I_+ \arrow[r] & X \wedge EG_+
\end{tikzcd}$$

So $[\Sigma^{V^K} X_\beta \wedge EG^m_+, \underline{H\mathbb{F}}_p(V \oplus \mathbb{R}^q)]_H \to [\Sigma^{V^K} X_\alpha \wedge EG^n_+, \underline{H\mathbb{F}}_p(V \oplus \mathbb{R}^q)]_H$ is zero for every V, and so $\widehat{\mathcal{K}}^q_J(X) = 0$. $\square$

The following is then an easy combination of the preceding two results.

Corollary 11.5. *The map $\widehat{\mathcal{K}}^{r-1}_G(X; \widetilde{EG}) \xrightarrow{\delta} \widehat{\mathcal{K}}^r_G(X; EG_+)$ is an isomorphism when X is a contractible G-space.*

Injectivity of η

We want to show that $\eta : \widehat{\pi}_G^{r-1}(X; \widetilde{EG}) \to \widehat{\mathcal{K}}_G^{r-1}(X; \widetilde{EG})$ is injective mod p. To this end, we use Theorem 8.7 from §8 again. We have the diagram

$$\begin{array}{ccc}
\bigoplus \Sigma^{r-1}\widehat{\pi}_G^0(S^0) \cong \widehat{\pi}_G^{r-1}(X; A/\widetilde{A}) & \xrightarrow{\cong} & \widehat{\pi}_G^{r-1}(X; A) \cong \widehat{\pi}_G^{r-1}(X; \widetilde{EG}) \\
\Big\downarrow \bigoplus \Sigma^{r-1}\eta & & \Big\downarrow \eta \\
\bigoplus \Sigma^{r-1}(\widehat{\Phi^G\mathcal{K}})^0(S^0) \cong \widehat{\mathcal{K}}_G^{r-1}(X; A/\widetilde{A}) & \longrightarrow & \widehat{\mathcal{K}}_G^{r-1}(X; A) \cong \widehat{\mathcal{K}}_G^{r-1}(X; \widetilde{EG})
\end{array}$$

where the top arrow is an isomorphism by the argument in the proof of Theorem A(ii). In this subsection we shall do two things:

- Show that the left vertical arrow is an injection mod p in Lemma 11.7 below.
- As in the previous subsection, we don't have enough finiteness to guarantee long exact sequences for $\widehat{\mathcal{K}}_G$, so we cannot just argue as for $\widehat{\pi}_G$ that the bottom map is an isomorphism in general - we shall show instead that the bottom map is injective on the image of $\oplus\Sigma^{r-1}\eta$ in Lemma 11.10 below.

Let $j := \Phi^G F(EG_+, \underline{H\mathbb{F}}_p) \in \mathrm{Sp}$. Using that $\underline{H\mathbb{F}}_p$ was a G-ring spectrum and that Φ^G is strong monoidal, we get a unit map in Sp

$$\eta : \mathbb{S} \to j$$

Remark 11.6. We thank Markus Land for the following observation: of course since this is the unit map for a ring spectrum, it is going to send $1 \in \pi_0\mathbb{S}$ to $1 \in \pi_0 j$. The following lemma that we prove is however not vacuous since we want to show that $1 \in \pi_0 j$ is not trivial mod p, and not just that $1 \in \pi_0 j$ is nontrivial, which is obvious. For example, for $q \neq p$ another prime, we even have $\pi_0 H\mathbb{F}_q$ vanishes mod p.

Lemma 11.7. *The element* $1 \in \pi_0 j$ *is nonzero mod* p*.*

Proof. We first unravel to see what $1 \in \pi_0 j$ is represented by. Let $\varepsilon : EG_+ \to S^0$ be the obvious map and $e : S^{V^G} \to S^V$ be induced by the inclusion. By definition,

$$\pi_0 j = \operatorname*{colim}_V [S^{V^G}, F(EG_+, \underline{H\mathbb{F}}_p(V))^G] = \operatorname*{colim}_V [S^{V^G} \wedge EG_+, \underline{H\mathbb{F}}_p(V)]_G$$

Then $1 \in \pi_0 j$ being the image of $1 \in \pi_0\mathbb{S}$ under η is just represented by the following G-map of G-spaces

$$\alpha_V : S^{V^G} \wedge EG_+ \xrightarrow{1\wedge\varepsilon} S^{V^G} \xrightarrow{1\wedge e} S^V \xrightarrow{\eta} \underline{H\mathbb{F}}_p(V)$$

We want to show that this map is not trivial mod p, that is, it is not of the form $p \cdot f$ for some other map $f : S^{V^G} \wedge EG_+ \to \underline{H\mathbb{F}}_p(V)$. To do this, we just show that, choosing V a complex representation with $V \neq V^G$, α_V induces a nontrivial map upon taking mod p Borel cohomology, namely that $(\alpha_V)_{hG}$ induces a nontrivial map on mod p cohomology: this is enough since then the cohomology groups are p-torsion, and so if $\alpha_V = p \cdot f$, then of course the induced map on cohomology is going to be trivial.

Now $(EG_+ \wedge EG_+) \wedge_G S^{V^G} \cong EG_+ \wedge_G (S^{V^G} \wedge EG_+) \xrightarrow{1\wedge_G(1\wedge\varepsilon)} EG_+ \wedge_G S^{V^G}$ is of course an equivalence since $EG_+ \wedge EG_+ \to EG_+$ is a G-equivalence, so we can ignore this part of α_V and we focus now on the other two maps in α_V.

Recall that nonequivariantly $\eta : S^V \to \underline{H\mathbb{F}}_p(V) = K(\mathbb{F}_p, \dim V)$ represents the fundamental class $1 \in H^{\dim V}(S^V, \mathbb{F}_p) \cong \mathbb{F}_p$. Write $\mu_V \in \widetilde{H}^*(EG_+ \wedge_G S^V)$ for the Thom class, $W := V - V^G$, and $n := \dim V^G$. We claim that $\mu_V = (1 \wedge_G \eta)^*(v)$ for some $v \in \widetilde{H}^*(EG_+ \wedge_G \underline{H\mathbb{F}}_p(V))$ and will prove this in Lemma 11.8 below. Naturality of Thom classes says that $\mu_V = \Sigma^n \mu_W$. On the other hand, by definition the Euler class $\chi(W) = (1\wedge_G e)^*(\mu_W)$. And so applying mod p cohomology to $S^{V^G} \xrightarrow{1\wedge e} S^V \xrightarrow{\eta} \underline{H\mathbb{F}}_p(V)$ gives

$$v \mapsto \mu_V = \Sigma^n \mu_W \mapsto \Sigma^n \chi(W)$$

But then W was elementary abelian and so since $W^G = 0$, we get that $\chi(W) \neq 0$ by Lemma 11.9 below. $\square$

Lemma 11.8. *Let G be a finite group, V a complex representation (so that Thom classes always make sense in mod p cohomology), and $\eta : S^V \to \underline{H\mathbb{F}}_p(V)$ the fundamental class. For $\mu_V \in \widetilde{H}^{\dim V}(EG_+ \wedge_G S^V)$ the Thom class, there exists some $v \in \widetilde{H}^*(EG_+ \wedge_G \underline{H\mathbb{F}}_p(V))$ such that $(1 \wedge_G \eta)^* v = \mu_V$.*

Proof. We look at the map between the spectral sequences associated to mod p Borel cohomology

$$\begin{tikzcd} \widetilde{H}^s(BG_+, \widetilde{H}^t \underline{H\mathbb{F}}_p(V)) \arrow[r, Rightarrow] \arrow[d, "\eta^*"] & \widetilde{H}^{s+t}(EG_+ \wedge_G \underline{H\mathbb{F}}_p(V)) \arrow[d, "\eta^*"] \\ \widetilde{H}^s(BG_+, \widetilde{H}^t(S^V)) \arrow[r, Rightarrow] & \widetilde{H}^{s+t}(EG_+ \wedge_G S^V) \end{tikzcd}$$

Now since η was the fundamental class, we know that it induces isomorphism $\widetilde{H}^{\dim V}(\underline{H\mathbb{F}}_p(V)) \to \widetilde{H}^{\dim V}(S^V)$. By definition of Thom classes, for any point $x \in BG$, μ_V needs to restrict to a generator $\widetilde{H}^{\dim V}(G/e_+ \wedge_G S^V) = \widetilde{H}^{\dim V}(S^V) \cong \mathbb{F}_p$. And so looking at the spectral sequence for $EG_+ \wedge_G S^V$ shows that the Thom class had to have come from the $\widetilde{H}^0(BG_+, \widetilde{H}^{\dim V}(S^V))$ term in the spectral sequence, which in turn is isomorphic via η^* to $\widetilde{H}^0(BG_+, \widetilde{H}^{\dim V} \underline{H\mathbb{F}}_p(V))$ from the spectral sequence for $EG_+ \wedge_G \underline{H\mathbb{F}}_p(V)$. This gives the required preimage in $\widetilde{H}^{\dim V}(EG_+ \wedge_G \underline{H\mathbb{F}}_p(V))$. □

Lemma 11.9. *Let $G = \mathbb{F}_p^r$ be elementary abelian and W be a complex representation without a trivial summand, that is, $W^G = 0$. Then the Euler class $\chi(W)$ is nonzero.*

Proof. Since Euler classes satisfy $e(V \oplus V') = e(V)e(V')$, we might as well suppose W was nontrivial irreducible. Schur's lemma and abelianness of G say that W must be one-dimensional, and so W is given by a homomorphism $\rho : G \to \mathbb{C}^\times$, which is determined by $(a_1, \dots, a_r) \in \mathbb{F}_p^r$ with $\rho : (g_1, \dots, g_r) \mapsto \xi^{a_1 g_1} \cdots \xi^{a_r g_r}$ where $\xi = e^{2\pi i/p}$. But we have a well known natural isomorphism (see [Ati61] point (3) of the appendix, for example) given by taking the Euler class

$$\mathrm{Hom}(G, \mathbb{C}^\times) \xrightarrow{\cong} H^2(BG, \mathbb{Z}) \cong \mathbb{F}_p\{c_1, \dots, c_r\} \quad :: \quad (a_1, \dots, a_r) \mapsto a_1 c_1 + \cdots + a_r c_r$$

And so if $W^G = 0$ then $(a_1, \dots, a_r) \neq (0, \dots, 0)$, so $\chi(W) = a_1 c_1 + \cdots + a_r c_r \neq 0$. □

Lemma 11.10. *Let X be a based G-space such that $X^G = S^0$ and $X^H \simeq *$ for all $H \lneq G$. The map $\widehat{\mathcal{K}}_G^{r-1}(X; A/\widetilde{A}) \to \widehat{\mathcal{K}}_G^{r-1}(X; A)$ restricts to an injection on the image of $\oplus \Sigma^{r-1}\eta$.*

Proof. The proof will consist of two main steps, and we set up some notations first before proceeding. For an S-functor T, define

$$T^j_{a,V,m} = \begin{cases} [T(\Sigma^V X_a) \wedge EG^m_+, \underline{H\mathbb{F}}_p(V \oplus \mathbb{R}^j)]_G & \text{if } j \geq 0 \\ [T(\Sigma^V \Sigma^{-j} X_a) \wedge EG^m_+, \underline{H\mathbb{F}}_p(V)]_G & \text{if } j < 0 \end{cases}$$

Note that finiteness of $[X, \underline{H\mathbb{F}}_p(V)]_G$ when X is a finite G-complex implies that $\lim^1_m [T(\Sigma^V X_a) \wedge EG^m_+, \underline{H\mathbb{F}}_p(V \oplus \mathbb{R}^j)]_G = 0$, and so we get that

$$\widehat{\mathcal{E}}^j_G(X; T) = \lim_a \operatorname*{colim}_V \lim_m T^j_{a,V,m}$$

Note that, as in the proof of Lemma 11.4, we didn't need to p-complete anything on the right-hand side since they're all already $\mathbb{F}_p$-vector spaces. We're now ready to proceed with the proof.

- Recall from Theorem 8.7 (b) that we have a filtration of S-functors

  $$F_0\widetilde{A} \subset F_1\widetilde{A} \subset \cdots F_{r-2}\widetilde{A} = \widetilde{A}$$

  where $\widetilde{B}_q = F_q\widetilde{A}/F_{q-1}\widetilde{A}$ is a wedge of suspensions of S-functors $C(K, G)$ where $\widehat{\mathcal{K}}^*_G(X, C(K, G)) = (\widehat{\Phi^{G/K}\mathcal{K}})^*_{G/K}(X^K)$. And so by Lemma 11.4 we have that

  $$\widehat{\mathcal{K}}^*_G(X, \widetilde{B}_q) = 0 \quad \text{for } 0 \leq q \leq r-2$$

We claim that $\widehat{\mathcal{K}}^*_G(\mathcal{X};\widetilde{A})=0$. This is not obvious since, as indicated at the beginning of this subsection, we don't have enough finiteness to guarantee that passing from $\mathcal{K}^*_G(X_a;T)$ to $\widehat{\mathcal{K}}^*_G(X;T)$ by inverse limits preserve exactness of long exact sequences coming from cofibrations of S-functors. But the situation is saved by the method in the proof of Lemma 11.4 as follows.

When $T=C(K,H)$ we see from Lemma 8.6 which says that $\widehat{\mathcal{K}}^*_G(X;C(K,H))\cong(\widehat{\Phi^{H/K}\mathcal{K}})^*_{H/K}(X^K)$ and from the proof of Lemma 11.4 that for each pair (a,m) there is a pair (b,n) with $X_a\subset X_b$ and $m\le n$ such that $T^*_{b,V,n}\to T^*_{a,V,m}$ is zero for every V. This property is obviously closed under taking wedges of S-functors, and it is also easy to see that if $T'\to T\to T''$ is a cofibration of S-functors and the property holds for T',T'', then it holds also for T. And so since this property holds for $\widetilde{B}_q$, we inductively get that it holds for all $F_q\widetilde{A}$, and in particular for $F_{r-2}\widetilde{A}=\widetilde{A}$.

- Consider the system of exact sequences

$$\widetilde{A}^{r-2}_{a,V,m}\to(A/\widetilde{A})^{r-1}_{a,V,m}\to A^{r-1}_{a,V,m}$$

from the cofibration $A\to A/\widetilde{A}\to\Sigma\widetilde{A}$. Recall from the proof of Theorem 8.7 (b) that we have

$$(A/\widetilde{A})(X)=(B\mathcal{A}/B\widetilde{\mathcal{A}})\wedge X^G$$

where $B\mathcal{A}/B\widetilde{\mathcal{A}}$ is equivalent to a wedge of $p^{r(r-1)/2}$ of $(r-1)$-spheres. Since $X^G=S^0$, we may set $X_0=S^0$ and restrict to $X_a\supset X_0$ so that $X_a^G=S^0$ for all a. Then the system $\{(A/\widetilde{A})^{r-1}_{a,V,m}\}$ is constant in a with $(A/\widetilde{A})^{r-1}_{a,V,m}$ being the sum of $p^{r(r-1)/2}$ copies of $[S^{V^G}\wedge EG^m_+,\underline{H\mathbb{F}}_p(V)]_G$.

Now let x be a nonzero element of

$$\mathrm{Im}\eta\subset\widehat{\mathcal{K}}^{r-1}_G(\mathcal{X};A/\widetilde{A})=\lim_a\operatorname*{colim}_V\lim_m(A/\widetilde{A})^{r-1}_{a,V,m}=\operatorname*{colim}_V\lim_m(A/\widetilde{A})^{r-1}_{0,V,m}$$

We want to show that x gets mapped to something nonzero in $\widehat{\mathcal{K}}^*_G(\mathcal{X};A)$. Let V be a large enough complex representation such that x is represented by $x_{V,m}\in(A/\widetilde{A})^{r-1}_{0,V,m}$. Since $x\in\mathrm{Im}\eta$ we know from the proof of Lemma 11.7 that without loss of generality $x_{V,m}$ looks like

$$\alpha_{V,m}:S^{V^G}\wedge EG^m_+\xrightarrow{1\wedge\varepsilon}S^{V^G}\xrightarrow{e}S^V\xrightarrow{\eta}\underline{H\mathbb{F}}_p(V)$$

By that lemma we also know that α_V was nontrivial mod p for complex representations $V\neq V^G$, and so since the vanishing of $\lim^1$ (since all groups involved are finite $\mathbb{F}_p$-vector spaces) implies that $[S^{V^G}\wedge EG_+,\underline{H\mathbb{F}}_p(V)]_G\cong\lim_m[S^{V^G}\wedge EG^m_+,\underline{H\mathbb{F}}_p(V)]_G$, we get that $\alpha_{W,m}$ are nontrivial for all m and $W\supset V$. So in total, we obtain that for all $W\supset V$ and all m, $x_{W,m}\neq 0$ in $(A/\widetilde{A})^{r-1}_{0,W,m}$.

Now suppose for a contradiction that x maps to zero in $\widetilde{\mathcal{K}}^{r-1}_G(X;A)$. This means that, writing $x_{a,V,m}$ for $x_{V,m}$ considered as an element in $(A/\widetilde{A})^{r-1}_{a,W,m}$, we get that for all a there exists $V_a\supset V$ such that $x_{a,V,m}$ maps to $0\in A^{r-1}_{a,V_a,m}$. On the other hand, $\widehat{\mathcal{K}}^*_G(\mathcal{X};\widetilde{A})=0$ from the previous step gives us that for each m we can choose $a\ge 0$ and $n\ge m$ such that the map $\widetilde{A}^{r-2}_{a,V,n}\to\widetilde{A}^{r-2}_{0,V,m}$ is zero for all V. Write W for V_a. Now chasing the following diagram, starting at $(A/\widetilde{A})^{r-1}_{a,V,m}$ and following the arrows in sequence, we get that $x_{a,W,m}=0\in(A/\widetilde{A})^{r-1}_{a,W,m}$, contradicting the previous paragraph.

$$\begin{tikzcd}
 & (A/\widetilde{A})^{r-1}_{a,V,n} \arrow[dr, "(1)"] \arrow[dd, "(2)"] & & \\
 & & (A/\widetilde{A})^{r-1}_{a,V,m} \arrow[dd, "(3)"] & \\
\widetilde{A}^{r-2}_{a,W,n} \arrow[r, "(4)"] \arrow[d, "0"', "(5)"] & (A/\widetilde{A})^{r-1}_{a,W,n} \arrow[rr, "(7)"] \arrow[dr] & & A^{r-1}_{a,W,n} \\
\widetilde{A}^{r-2}_{0,W,m} \arrow[rr, "(6)"] & & (A/\widetilde{A})^{r-1}_{0,W,m}=(A/\widetilde{A})^{r-1}_{a,W,m} &
\end{tikzcd}$$

$\square$

Appendix A Induction theorem for the completed Burnside ring Green functor

Here we give the proof of the following theorem that we used for the reduction to p-groups. The proof will highlight two standard philosophies in working with the Burnside ring, namely the importance (1) of understanding the prime ideals of $A(G)$ (which is analogous to the importance of understanding primes in $\mathbb{Z}$); and (2) of the embedding of $A(G)$ into the ghost ring $C(G)$.

Theorem 5.1. *Let G be a finite group, and $\widehat{\underline{A}}$ the Burnside ring Green functor completed at the augmentation ideal $I(G)$. For each prime p, let G_p denote a representative Sylow p-subgroup of G. Then the sum*

$$\bigoplus_p i_* : \bigoplus_p \widehat{A}(G_p) \to \widehat{A}(G)$$

is an epimorphism. In other words, together with Theorem 3.12 this implies that all $\widehat{\underline{A}}$-modules are projective with respect to the set of all Sylow p-subgroups for all p dividing $|G|$.

Observation A.1. Note that $\widehat{A}(G) = \mathbb{Z} \oplus \widehat{I}(G)$ and so the natural composite

$$\mathbb{Z} \hookrightarrow \widehat{A}(H) \xrightarrow{i_*} \widehat{A}(G) \xrightarrow{\chi_e} \mathbb{Z}$$

is multiplication by $|G/H|$. And so since the greatest common divisor of $\{|G/G_p|\}_p$ is one, we see that the homomorphism in the theorem is always surjective on the $\mathbb{Z}$ copy, and so it's enough to show surjectivity with $\widehat{A}$ replaced by $\widehat{I}$.

Idea A.2. Vaguely speaking, the point of the proof will be that $I(H)/I(H)^n$ appearing as the terms of the completion will all be finite, and so it will be enough to show surjectivity at each of these finite terms. The finiteness of these terms, in turn, allows us to work p-locally, where we will want to combine the transfer

$$G/G_p \cdot - : I(G)/I(G)^n \cdot I(G) \xrightarrow{i^*} I(G_p)/I(G)^n \cdot I(G_p) \xrightarrow{i_*} I(G)/I(G)^n \cdot I(G)$$

with the fact that $|G/G_p|$ is prime to p to show that i_* is p-locally even a split surjection.

The proof of the theorem will depend on knowledge of the prime ideals of $A(G)$ and three lemmas building on that, and we discuss them now.

Fact A.3 ([tD79] §1)**.** It turns out that we know all the prime ideals of $A(G)$ and they are of form

$$q(H,0) := \ker(A(G) \xrightarrow{\chi_H} \mathbb{Z}) \quad \text{or} \quad q(H,p) := \ker(A(G) \xrightarrow{\chi_H} \mathbb{Z} \to \mathbb{Z}/p)$$

Write H^p for the smallest normal subgroup of H such that H/H^p a p-group. These prime ideals satisfy

$$q(H,0) \subset q(H,p)$$

$$q(H,0) = q(K,0) \text{ if } H \text{ is conjugate to } K$$

$$q(H,p) = q(K,p) \text{ if } H^p \text{ is conjugate to } K^p$$

Note in particular that $q(e,p) = q(H,p)$ iff H is a p-group.

Construction A.4. Let $H \xrightarrow{i} G$ be the inclusion of a subgroup. Then via $i^* : A(G) \to A(H)$, any $A(H)$-module is also an $A(G)$-module. Via $\chi : A(H) \to C(H)$, any $C(H)$-module is an $A(H)$-module.

The key to many of the arguments is the following lemma, which is the Burnside ring version of a result for representation rings going back to [Ati61] Theorem 6.1. The equivalence of the first two topologies is due to Laitinen [Lai79].

Lemma A.5. *The following topologies on $A(H)$ and $I(H)$ coincide.*

(a) The $I(G)$-adic topology.

(b) The $I(H)$-adic topology.

(c) The subspace topology induced from the $I(H)$-adic topology on $C(H)$ (recall from Construction 3.13 that χ was an injection).

Proof. See Lemma 6 of [MM82]. $\square$

Definition A.6. Let $H \le G$, N be some $A(H)$-module, and $n \ge 1$.

(a) Define $P_n(N, H) := N/I(H)^n N$. We simply write $P_n N$ for $P_n(N, G)$ for short.

(b) Define $J^n(H) := \chi^{-1}(I(H)^n C(H)) \subset A(H)$.

(c) Define $Q_n(N, H) := N/J^n(H)N$.

Remark A.7. From these we can get the following easy observations.

(a) By Lemma A.5 we get that $P_n(A(H), G)$ is a quotient of $P_m(A(H), H)$ for some m.

(b) Since $I(H)^n$ is contained in $J^n(H)$ we have a natural surjection $P_n(N, H) \to Q_n(N, H)$, and is just the identity if $N = C(H)$.

(c) The injection χ induces an injection

$$Q_n(A(H), H) \rightarrowtail Q_n(C(H), H) = P_n(C(H), H) = \prod_{(K)} P_n \mathbb{Z}_K$$

where $\mathbb{Z}_K$ is just $\mathbb{Z}$ considered as an $A(H)$-module via χ_K.

Lemma A.8. *(a) If $K = e$ then $P_n \mathbb{Z}_K = \mathbb{Z}$ for all n.*

(b) If K is not a p-group for any p, then $P_n \mathbb{Z}_K = 0$ for all n.

(c) If K is a p-group, then $P_n \mathbb{Z}_K$ is a p-group for all n.

Proof. See Lemma 7 of [MM82]. $\square$

Lemma A.9. *The group $P_n I(H)$ is finite for all $n \ge 1$.*

Proof. The equivalence of (b) and (c) from Lemma A.5 gives us that $P_n(I(H), H)$ is a quotient of $Q_m(I(H), H)$ for some m. On the other hand, the injection χ induces an injection

$$Q_m(I(H), H) \rightarrowtail P_m(IC(H), H) = \prod_{(K) \neq e} P_m \mathbb{Z}_K$$

and the latter is finite by Lemma A.8. $\square$

We are now ready to prove the theorem.

Proof of Theorem 5.1. The equivalence of (a) and (b) from Lemma A.5 gives us the diagram

$$\begin{tikzcd}
\bigoplus_p \widehat{I}(G_p) \arrow[r, "\oplus_p i_*"] \arrow[d, "\cong"] & \widehat{I}(G) \arrow[d, "\cong"] \\
\lim_n \bigoplus_p P_n(I(G_p), G) \arrow[r, "\oplus_p i_*"] & \lim_n P_n I(G)
\end{tikzcd}$$

And so our task now is to show that the bottom map is surjective. Now Lemma A.9 gives that each component of the bottom map is finite, and so by the usual $\lim^1$ exact sequence and Mittag-Leffler we're reduced to showing that $\bigoplus_p P_n(I(G_p), G) \to P_n I(G)$ is surjective. Again by finiteness, this will hold provided the p-th map i_* is surjective on the p-primary component of $P_n I(G)$.

To this end, we'll leverage on the transfer maps and show the stronger statement that

$$P_n I(G) \xrightarrow{i^*} P_n(I(G_p), G) \xrightarrow{i_*} P_n I(G)$$

becomes an isomorphism when localised at p. As in the proof of Lemma 3.14 we know that this is just multiplication by G/G_p, now viewed as an element of $P_n A(G)$. But then Lemma A.5 again gives us that $P_n A(G)$ is a quotient of $Q_m A(G)$ for some m, and so it suffices to show that G/G_p is a unit in $Q_m A(G)_{(p)}$.

On the other hand, recall from Construction 3.13 that $Q_m C(G)/Q_m A(G)$ is finite, and so $Q_m C(G)_{(p)}$ is an integral extension of $Q_m A(G)_{(p)}$. And so it suffices to check that G/G_p is a unit in $Q_m C(G)_{(p)}$, since if $S \subset R$ was an integral extension and $u \in S$ is a unit in R, say with $v \in R$ the inverse, then a monic polynomial with coefficients in S

$$v^n + a_{n-1}v^{n-1} + \cdots + a_0 = 0$$

then gives

$$v = -(a_{n-1} + a_{n-2}u + \cdots + a_0 u^{n-1}) \in S$$

Now Lemma A.8 shows that

$$Q_m C(G)_{(p)} = \mathbb{Z}_{(p)} \times \prod_{(K)} Q_m \mathbb{Z}_K$$

where the product on the right is restricted to the conjugacy classes of p-groups $K \leq G$. Recall from Fact A.3 that $q(K,p) = q(e,p)$ in $A(G)$. Since $\chi_e(G/G_p) = |G/G_p|$ is prime to p, we have that $G/G_p \notin q(e,p) = q(K,p)$. Hence we must also have that $\chi_K(G/G_p)$ is prime to p, and so G/G_p is a unit in $Q_m C(G)_{(p)}$ as required. $\square$

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