PhD Thesis

Norms and periodicities in genuine equivariant hermitian K-theory

University of Copenhagen

Kaif Hilman bin Muhammad Borhan Tan

August 2022

Kaif Hilman
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
DK–2100 København Ø
Denmark
kaif.tan@math.ku.dk

PhD thesis submitted to:
PhD School of Science
Faculty of Science
University of Copenhagen

Date of submission:	29th August 2022
Date of defence:	14th October 2022
Advisors:	Jesper Grodal, *University of Copenhagen, Denmark*
	Markus Land, *L.M.U. Munich, Germany*
Assessment committe:	Søren Galatius (chair), *University of Copenhagen, Denmark*
	John Greenlees, *University of Warwick, United Kingdom*
	Thomas Nikolaus, *University of Münster, Germany*

This work was supported by the Danish National Research Foundation through the Centre for Symmetry & Deformation (CPH–SYM–DNRF92) and the Copenhagen Centre for Geometry & Topology (CPH-GEOTOP-DNRF151). Part of the work was also supported by the Swedish Research Council (grant no. 2016-06596) during the research program “Higher algebraic structures in algebra, topology and geometry” held at Institut Mittag–Leffler, Sweden in the spring of 2022.

ISBN 978-87-7125-059-6

Kalam madahmu berpancar terang,
    Maka tertambatlah hatiku ini;
Hamba datang sehelai sepinggang,
    Dan hamba sembah sehampar ufti.

Abstract

In this thesis, we explore various aspects of genuine G–equivariant K–theory on stable ∞–categories for finite groups G. The main theme is establishing equivariant multiplicative norms - in the sense of Hill–Hopkins–Ravenel - on these K–theories and our formalism of choice is that of the parametrised higher category theory by Barwick–Dotto–Glasman–Nardin–Shah. This thesis is divided into three parts, each one building up towards the next.

In Part I, we develop the theory of G–presentable and G–perfect–stable ∞–categories. This will serve as the technical underpinnings for our investigations on G–equivariant algebraic K–theory in Part II where we show that when G is a 2–group, algebraic K–theory refines to the structure of a ring G–spectrum equipped with the Hill–Hopkins–Ravenel norms. Along the way, we will obtain a "multiplicative Borelification principle" via a simple categorification–decategorification procedure which provides a huge source of examples of equivariant K–theory with norms. Finally, in Part III, we initiate the study of genuine equivariant hermitian K–theory by introducing the notion of G–Poincaré ∞–categories, generalising in the equivariant direction the recent advances made by Calmés–Dotto–Harpaz–Hebestreit–Land–Moi–Nardin–Nikolaus–Steimle. Among other things, we refine Borel equivariant Grothendieck–Witt theory to the structure of a normed ring G–spectrum when G is a 2–group, and we also obtain a new source of equivariantly periodic ring G–spectra in the form of equivariant L–theory.

Resumé

I denne afhandling udforsker vi nogle aspekter af ægte G–ækvivariant K–teori for stabile ∞–kategorier, når G er en endelig gruppe. Det overordnede mål er at etablere ækvivariante multiplikative normer - i betydningen af Hill–Hopkins–Ravenel - på disse K–teorier. Til den ende bruger vi formalismen af parametriseret højere kategoriteori af Barwick–Dotto–Glasman–Nardin–Shah. Afhandlingen er opdelt i tre dele, hvor hver del bygger op til den næste.

I Del I udvikler vi teorien om G–præsentable og G–perfekte–stabile ∞–kategorier. Målet er at udvikle nødvendige tekniker til vores undersøgelser ved G–ækvivariant algebraisk K–teori. I Del II, vi beviser at algebraisk K–teori kan gives strukturen af et G–ringspektrum udstyret med Hill–Hopkins–Ravenel normer, når G er en endelig 2–gruppe. Undervejs viser vi et "multiplikativt Borelificeringsprincip", som følger af et enkelt kategorifiseringafkategoriseringsargument, og som giver mange eksempler på G–ækvivariante K–teorier med normer. I den sidste Del III indleder vi en undersøgelse af ægte ækvivariant hermitisk K–teori ved at indføre begrebet en G–Poincaré ∞–kategori, som er en ækvivariant generalisering af teorien af Calmés–Dotto–Harpaz–Hebestreit–Land–Moi–Nardin–Nikolaus–Steimle. Blandt andet forfiner vi Borelækvivariant Grothendieck–Witt teori med strukturen af et G–ringspektrum udstyret med multiplikative normer, når G er en endelig 2–gruppe. Det leder også til en ny kilde til ækvivariante periodiske G–ringspektre i form af ækvivariant L–teori.

Acknowledgements

> Das Alter macht nicht kindisch, wie man spricht, es findet uns nur noch als wahre Kinder.
>
> Faust, *Johann Wolfgang von Goethe*

This PhD has been a long journey of myself both as a mathematician and as a person, and I would like to honour here those who have, in one way or another, nourished me and eased my growing pains. Truly, this is as much their achievement as it is mine.

First of all, I owe my heartiest gratitude to my advisors Jesper Grodal and Markus Land from whom I have learnt so much on just about everything. Thank you Jesper for making me into a G–prepender, for sharing your boundless optimism on dark days and your constant cheerleading, for giving me the space to grow independently and yet keeping me grounded in my work. But perhaps most of all, thank you for having created such a vibrant and cutting–edge centre for homotopy theory here in Copenhagen: anywhere else and I think I would have been the lesser for it. Markus, thank you for guiding me in treading the daunting world of K–theory with steady hands and fearless enthusiasm in equal measure. Your friendliness and approachability make it easy to forget how much cool and profound mathematics you have bequeathed to the community for which I can never thank you enough. Much more than just putting food on my table, your work and insights have never ceased to fill me with hope and give me reason to look forward to another day. For these, and a whole lot more, I am very grateful and honoured to have you both as my advisors: I would not have had it any other way.

To Søren Galatius, John Greenlees, and Thomas Nikolaus: thank you for agreeing to be in the thesis assessment committee. This document did not fare well in concision by any standards, and so I really appreciate your willingness to stand as academic gatekeepers - surely an under–rewarded community service - amidst your hectic schedules.

As mentioned above, Copenhagen is a homotopy theory haven and I was also fortunate to have been a participant at the research program “Higher algebraic structures in algebra, topology and geometry" held in spring 2022 at Institut Mittag–Leffler. As such, I have benefited, mathematically or otherwise, from many inspiring conversations with many people over the years, including (in roughly chronological order): Oscar Randal–Williams, Elden Elmanto, Jonas McCandless, Markus Hausmann, Simon Gritschacher, Lars Hesselholt, Arpon Raksit, Tobias Barthel, Piotr Pstragowski, Achim Krause, Fabian Hebestreit, Søren Galatius, Nathalie Wahl, Dustin Clausen, Shachar Carmeli, Thomas Nikolaus, Greg Arone, Yajit Jain, Asaf Horev, Emanuele Dotto, Lennart Meier, Joachim Kock, Robert Burklund, and Jan Steinebrunner. A special thanks goes to Maxime Ramzi who has helped and taught me so much in the way of clean and "correct" categorical arguments: one certainly cannot ask for a better academic brother.

Four years ago, the idea that the mathematics research community was a formidable crowd to be around was a given; but that it could also be a wholesome and fun one was completely unthinkable. When the maths got wild (and it always gets wild), it was precisely this warmth that has kept me coming back. For this, I have Robin Sroka, Malte Leip, Philipp Schmitt, Riccardo Pengo, Francesco Campagna, Marie Fialova, David Bauer, Luigi Pagano, Josh Hunt, Calista Bernard, Mikala Ørsnes–Jansen, Clemens Borys, Asbjørn Nordentoft, Severin Mejak, Nanna Aamand, Jeroen van der Meer, Alex Frei, Alexis Aumonier, Vignesh Subramanian, Maxime Ramzi, Pierre Elis, Adriano Cordova, Dagur Ásgeirsson, Nuno Hultberg, Marie–Camille Delarue, Jingxuan Zhang, Andrea Bianchi, Florian Naef, and Ronno Das to thank. Among these, two special thanks are in order: firstly, to my co–organisers (together with Jan Tapdrup of course) of the Young Topologists Meeting 2022 for being such a dependable and intelligent team; and secondly to my degenerate office mates who provided the perfect foil of daily stupidity. The point is that doing mathematics can be emotionally draining, and I think that it is beneficial to have a safe space to just be dumb and laugh like a kid. Sometimes I wonder if there isn't a secret camera crew following us like the one in Dunder–Mifflin Scranton.

But even before Copenhagen, I have been blessed with a strong and supportive network of family and old friends. While being some of the best years of my life, the PhD also showed me the deepest depths of despairs, and I would not have survived to tell the tale without my friends Paul, Dakshan, Derk, Jou, and Wen Yao. To them, and to all my friends, I owe a big thank you for standing by my side when things got rough. Last but not least, to my aunts and uncles 二叔, 大姑, 二姑, 三姑, 四姑, 小叔, and to Iwan, Izad, mama, and my late papa: if I have managed to achieve anything in life, it is because I had your loving and unconditional support every step of the way. Trite as it may sound, words can do no justice to my gratitude. However, I believe that deeds begin where words end, and so I dedicate this thesis to you all.

Contents

Introduction

Som kun den sande kunstner ved:
den sande kunst er kunstløshed.

Piet Hein

Contextual overview

It can be argued that the two most fundamental structures in algebra are rings and modules - which govern additions and multiplications - on the one hand, and on the other, groups - which govern symmetries of mathematical objects. In some sense, the latter notion is much harder to study owing to the noncommutativity of abstract symmetries, whereas the former is amenable to powerful linear algebraic techniques. A desire to understand groups via linear methods leads us naturally to the rich discipline of representation theory, which roughly speaking, animates the abstract symmetries encoded in a group by a ring or a module which it symmetrises, bringing to bear a suite of module-theoretic ideas to the study of groups. However, this relationship between groups and modules is far from a unilateral one as many naturally occurring rings and modules admit important symmetries: the case of the complex numbers $\mathbb{C}$ with its conjugate action immediately comes to mind. Consequently, representation theory should rather be viewed as the study of rings and modules equipped with symmetries by groups as interesting objects in their own right.

Now, enter topology. One of the great triumphs of early twentieth century mathematics was the birth of *algebraic topology*, whose basic insight was that many properties of a geometric space are reflected in the ring/module structures of various algebraic constructions one can associate to it. As with the case of rings and modules, many geometric spaces that arise in nature come with certain symmetry groups which can be exploited to understand the space better. For example, the circle tautologically has circular symmetry. This means that the module-theoretic algebraic invariants we attach to these spaces also attain a natural symmetry coming from the symmetry of the input space, making a connection to representation theory that we would do well not to overlook. The study of such situations is usually called *equivariant homotopy theory* (the adjective "equivariant" being a standard indication that the situation considered has an additional symmetry by a group).

As it was with representation theory, the relationship between topology and algebra is very much a two-way street. One particularly deep approach to studying modules, due to Alexander Grothendieck, is by assembling all the isomorphism classes of modules over a particular ring R into another module called its *K–group* $\mathrm{K}_0(R)$ and studying this object instead; this can be seen as a "global" way to study a ring R as it involves detecting large-scale structures present on the collection of *all* R–modules. This is a very robust formalism which applies to many module-theoretic settings, and in particular applies to representation theory where one can assemble all the isomorphism classes of representations of a group G over a ring R. And yet, it soon became apparent that considering these K–groups as bare modules was not enough as it did not exploit all the higher structures involved. Thus, in the 1970's and 80's, Daniel Quillen and Friedhelm Waldhausen pioneered the field of *higher algebraic K–theory* where the K–groups are no longer just a module but a module-like *space*. This approach quickly turned into a whole industry as it allowed one to attack these K–groups with the combined powers of topology and algebra. And as before, in the presence of a symmetry by a group G on the ring R, the associated K–theory space also inherits this symmetry and it governs the representation theory of the group G over the ring R. In this way, we can fairly term as higher representation theory the study of K–theory spaces of rings in the presence of equivariance by a group.

With the advent of ∞–categories as developed by Lurie in [Lur09; Lur17], we now have the correct language in which to speak of Quillen and Waldhausen's higher K–theories, and many mathematical luminaries - too many to mention here - have worked in recent decades to fully realise the higher algebraic K–theory program. As hinted at by the preceding paragraphs, one subprogram of this is the study of equivariant higher algebraic K–theory and we offer a small contribution to this story in this thesis.

In slightly more detail, a recent series of papers by Calmés-Dotto-Harpaz-Hebestreit-Land-Moi-Nardin-Nikolaus-Steimle [CDH+20a; CDH+20b; CDH+20c] further developed the ideas of Lurie [Lur11], which was in turn based on ideas of Andrew Ranicki from the 1980's, into a fully-fledged theory of higher algebraic K–theory for hermitian forms, or *hermitian K–theory* for short. This series of work can be viewed as the distillation of decades of insights from many mathematicians such as Max Karoubi, Marco Schlichting, and others. Hermitian K–theory is a deep invariant which governs many disparate fields of mathematics, among others, the surgery theory of manifolds. In this particular setting, equivariance by a group feature prominently via the fundamental group of the manifold, and so an understanding of equivariant hermitian K–theory is an integral part of perfoming surgeries on manifolds, and much more so when the manifolds themselves are endowed with a symmetry by a group.

One of the main themes of this thesis is to establish the important and subtle extra structure of G–equivariant power operations - a structure which featured crucially in the stunnning resolution of the Kervaire invariant one problem by Hill, Hopkins, and Ravenel [HHR16] - on algebraic and hermitian K–theory where G is a finite group. To achieve this, we will introduce and develop the foundations of hermitian K–theory for so-called G–stable ∞–categories using the formalism of *parametrised homotopy theory* by Barwick-Dotto-Glasman-Nardin-Shah [BDG+16a], which should be of independent interest. Along the way, we will also further develop the parametrised formalism, obtaining various fundamental results in this line.

Technical overview

This thesis consists of seven chapters distributed over three parts, each one building towards the next. The first two parts roughly correspond to the two articles [Hil22b; Hil22a]. Part I will be concerned with further developing the parametrised homotopy theory formalism of [BDG+16a] for our purposes.

In Part II we will use the theory developed in the previous part to investigate equivariant algebraic K–theory for G–stable ∞–categories and show that when G is a finite 2–group, the pointwise version of G–equivariant algebraic K–theory refines to the structure of a G–ring spectrum equipped with the Hill-Hopkins-Ravenel norms. To prevent potential confusion, we point out that there are two versions of higher algebraic K–theory: on the one hand, there is the *group-completion* K-*theory* whose input is a small symmetric monoidal ∞–category $\mathcal{C}$ and one group completes the $\mathbb{E}_\infty$–space $\mathcal{C}^{\simeq}$ to obtain a connective spectrum - classically, this is related to Quillen's +-construction and the reader is referred to [GGN15] for an ∞–categorical treatment; on the other hand, there is the *stable* K-*theory* whose input is a small stable ∞–category - this corresponds to Quillen's Q-construction and Segal and Waldhausen's $\mathrm{S}_\bullet$–construction. Most of the work on equivariant algebraic K–theory in the literature [Mer17; BMM+21; Sch19; Len21] deal with the former version; this thesis is rather in the company of [BGS20; CMN+20] in treating the latter.

In the final Part III, we prove that Borel equivariant GW–theory canonically admits the structure of the multiplicative norms when G is a 2–group, similar to the case of algebraic K–theory above. We then introduce a genuine equivariant refinement of the hermitian K–theory of [CDH+20a] by overlaying the theory of the prior part with a notion of G–quadratic structures, and we then explore some applications of this point of view. The remainder of this introduction will provide a more detailed overview of the entire thesis, progressing according to the themes of the respective parts.

Parametrised homotopy theory is the study of higher categories fibred over a base ∞–category. This is a generalisation of the usual theory of higher categories, which can be viewed as the parametrised homotopy theory over a point. The advantage of this approach is that many structures can be cleanly encoded by the morphisms in the base ∞–category. For example, in the algebro-geometric world, various forms of pushforwards exist for various classes of scheme morphisms (see [BH21] for more details). Another example, which is the main motivation of our work, is that of genuine equivariant homotopy theory for a finite group G - here the base ∞–category would be $\mathcal{O}_G^{\mathrm{op}}$, the opposite of the G–orbit category. In this case, for subgroups $H \leq K \leq G$, important and fundamental constructions such as *indexed coproducts*, *indexed products*, and *indexed tensors*

$$\coprod_{K/H} \qquad \prod_{K/H} \qquad \bigotimes_{K/H}$$

can be encoded by the morphisms in $\mathcal{O}_G^{\mathrm{op}}$. One framework in which to study this is the series of papers following [BDG+16a] and the results in Part I should be viewed as a continuation of the vision from the aforementioned series. We refer the reader to these papers for more motivations and examples.

For an ∞–category to admit all small colimits and limits is a very desirable property as it means that many constructions can be done in it. However, this property entails that it has to be large enough and we might lose control of it due to size issues. Fortunately, there is a fix to this problem in the form of the very well-behaved class of *presentable* ∞–categories: these are cocomplete ∞–categories that are "essentially generated" by a small subcategory. One of the most important features of presentable ∞–categories is the adjoint functor theorem which says that one can test whether or not a functor between presentables is right or left adjoint by checking that it preserves limits or colimits respectively. The ∞–categorical theory of presentability was developed by Lurie in [Lur09, Chapter 5], generalising the classical 1-categorical notion of *locally presentable categories*.

One of the goals of Part I of the thesis is to translate the above-mentioned theory of presentable ∞–categories to the parametrised setting and to understand the relationship between the notion of parametrised presentability and its unparametrised analogue in [Lur09]. We will adopt the convention of [Nar17] by defining a $\mathcal{T}$–category, for a fixed based ∞–category $\mathcal{T}$, to be a cocartesian fibration over the *opposite*, $\mathcal{T}^{\mathrm{op}}$. This convention is geared towards equivariant homotopy theory as introduced in the motivation above where $\mathcal{T} = \mathcal{O}_G$. Note that by the straightening-unstraightening equivalence of [Lur09], a $\mathcal{T}$–category can equivalently be thought of as an object in $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \widehat{\mathrm{Cat}}_\infty)$. The first main result of Chapter 2 is then the following straightened characterisation of parametrised presentable ∞–categories.

Theorem A (Full version in Theorem 2.2.2)**.** *Let $\underline{\mathcal{C}}$ be a $\mathcal{T}$–category. Then it is $\mathcal{T}$–presentable if and only if the associated straightening $C\colon \mathcal{T}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ factors through the non-full subcategory $\mathrm{Pr}_{\mathrm{L}} \subset \widehat{\mathrm{Cat}}_\infty$ of presentable categories and left adjoint functors, and morevoer these functors themselves have left adjoints satisfying certain Beck-Chevalley conditions (1.2.8).*

In the full version, we also give a complete parametrised analogue of the characterisations of presentable ∞–categories due to Lurie and Simpson (cf. [Lur09, Thm. 5.5.1.1]), which in particular shows that the notion defined in [Nar17, §1.4] satisfies all the expected descriptions. While it is generally expected that the theory of ∞–cosmoi in [RV22] should absorb the statement and proof of the Lurie-Simpson-style characterisations of presentability, the value of the theorem above is in clarifying the relationship between the notion of presentability in the parametrised sense and in the unparametrised sense. Indeed, the description in Theorem A is a genuinely parametrised statement that is not seen in the unparametrised realm where $\mathcal{T} = *$. One consequence of this is that we can easily deduce the parametrised adjoint functor theorem *from* the unparametrised version instead of repeating the same arguments:

Theorem B (Parametrised adjoint functor theorem, Theorem 2.2.3)**.** *Let $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a $\mathcal{T}$–functor between $\mathcal{T}$–presentable ∞–categories. Then:*

(i) If F strongly preserves $\mathcal{T}$–colimits, then F admits a $\mathcal{T}$–right adjoint.
(ii) If F strongly preserves $\mathcal{T}$–limits and is $\mathcal{T}$–accessible, then F admits a $\mathcal{T}$–left adjoint.

Another application of Theorem A is the construction of *presentable Dwyer-Kan localisations*, Theorem 2.2.10. This is deduced essentially by performing fibrewise localisations, which are in turn furnished by [Lur09]. Other highlights include the *localisation-cocompletions* construction in Theorem 2.2.12, the idempotent-complete-presentables correspondence Theorem 2.2.16, as well as studying the various interactions between presentability and functor categories in §2.2.7.

Having set up the theory of parametrised presentability, we then move on to studying parametrised semiadditive-presentable categories, in preparation for our K–theoretic investigations in Part II. Along these lines, we will use Theorem A to deduce the following:

Theorem C (Precise version in Theorem 2.3.4)**.** *We have a fully faithful inclusion of $\mathcal{T}$–presentable-stable ∞–categories into $\mathcal{T}$–Mackey functors valued in presentable-stable ∞–categories. The essential image consists of the $\mathcal{T}$–Mackey functors such that the Mackey semiadditivity norm map is an equivalence and the Mackey unit map exhibits the transfer $f_!$ as being left adjoint to f^*.*

This theorem says that the genuinely parametrised notion of presentable stability considered in [Nar17] can be viewed as Mackey functors valued in ordinary presentable-stable ∞–categories satisfying some further conditions. One argument as to why this internal notion of $\mathcal{T}$–presentable-stabilility is better than just presentable-stable-valued $\mathcal{T}$–Mackey functors is that as far as we know, it is the internal notion that admits a G–symmetric monoidal structure (due to [Nar17]), ie. the structure of categorical Hill-Hopkins-Ravenel norms. This structure will play a crucial role in our formulation for the norm structures on equivariant algebraic K–theory.

Before moving on to describing the next part, we first comment on the methods and philosophy of our work on parametrised homotopy theory. The approach taken here is an axiomatic one, and is slightly different in flavour from the series of papers in [BDG+16a] in that we freely pass between the viewpoint of parametrised ∞–categories as cocartesian fibrations and as ∞–category-valued functors via the straightening-unstraightening equivalence of Lurie. This allows us to work model-independently, ie. without thinking of our ∞–categories as simplicial sets. The point is that, as far as presentability and adjunctions are concerned, the foundations laid in [BDG+16b; BGN14; Sha22a; Sha22b; Nar17] are sufficient for us to make model-independent formulations and proofs via universal properties. Indeed, a recurring method here is to say that relevant universal properties guarantee the existence of certain functors, and then we can just check that certain diagrams commute by virtue of the essential uniqueness of left/right adjoints.

As mentioned above, Part II will be concerned with equivariant algebraic K–theory where we specialise the parametrised theory to the equivariant situation by setting $\mathcal{T} = \mathcal{O}_G$ for a finite group G. As preparation, we will provide a brief summary of the structures available in the notion of a G–category in Chapter 3. There are two findings that we think deserve being highlighted here. The first is the following very general "monoidal Borelification principle":

Theorem D (Precise version in Theorem 3.3.4)**.** *Any symmetric monoidal ∞–category $\underline{\mathcal{D}}$ induces a G–symmetric monoidal ∞–category $\underline{\mathrm{Bor}}(\underline{\mathcal{D}})$ which is fibrewise given by $\mathrm{Fun}(BH, \underline{\mathcal{D}})$ for $H \leq G$. Moreover, any G–symmetric monoidal G–category $\underline{\mathcal{C}}$ gives rise to a G–symmetric monoidal functor $\underline{\mathcal{C}} \to \underline{\mathrm{Bor}}(\mathrm{Res}^G_e \underline{\mathcal{C}})$. In particular, a right adjoint canonically refines to a G–lax symmetric monoidal functor.*

As far as we are aware, this is the first treatment of the relationship between G–symmetric monoidal categories and their Borelifications, which clarifies the link between a G–symmetric monoidal structure on a G–category and the one on the Borel objects induced by the symmetric monoidal structure of the underlying ∞–category with G–action. The final statement in the theorem is a situation that is often satisfied, and therefore gives us a very general procedure to produce G–commutative algebra objects by endowing an ordinary commutative algebra object with a G–action (Proposition 3.3.6). As we shall soon point out, this will be an ingredient in obtaining a large source of examples for normed equivariant algebraic K–theory. We should comment here that this was one of the problems that we were stuck with for the longest time for the special case of $\underline{\mathcal{C}} = \underline{\mathrm{Cat}}^{\mathrm{perf}}_{G}$, and the solution turned out to be much easier to solve in the vast generality of Theorem D, proceeding by first categorifying the formulation and then decategorifying it to obtain the desired statement. In hindsight, it was very much inspired by the philosophy of [GGN15] in dealing with monoidal *structures* via the *properties* of categorical products.

The second is a genuine equivariant refinement of the Nikolaus-Scholze Tate diagonal in the case when G is odd and $p = 2$. Let $\underline{\mathrm{T}}_2 \colon \underline{\mathrm{Sp}}_G \to \underline{\mathrm{Sp}}_G$ be the functor $X \mapsto (X \otimes X)^{t\Sigma_2}$.

Theorem E (Precise version in Theorem 3.6.5)**.** *Let G be an odd group. Then $\underline{\mathrm{T}}_2$ is G–linear and there is a natural transformation of G–linear functors*

$$\mathrm{id} \xRightarrow{\Delta_2} \underline{\mathrm{T}}_2$$

which refines the Nikolaus-Scholze Tate diagonal to genuine G–spectra.

This will be a corollory of a more general statement about G–linearity of diagonalisations of G–bilinear functors upon passing to $(-)^{t\Sigma_2}$ when G is odd (Corollary 3.5.3); it will involve some double-coset counting arguments. Much like in the nonequivariant theory of [CDH+20a], this will be an input in constructing the *universal G–Poincaré category* in §7.1.7.

We now discuss the approach to algebraic K–theory that we have taken in this thesis. As a functor $\mathrm{K} \colon \mathrm{Cat}^{\mathrm{perf}}_{\infty} \to \mathrm{Sp}$, algebraic K–theory is the universal additive spectral invariant on the ∞–category $\mathrm{Cat}^{\mathrm{perf}}_{\infty}$ of small perfect stable ∞–categories by the work of [BGT13] and moreover behaves well with respect to symmetric monoidal structures by [BGT14]. The methods of these papers were to construct the initial stable ∞–category receiving an additive functor (in the sense of sending exact sequences of ∞–categories to exact sequences in the target category) called the ∞–category of *noncommutative motives* NMot through which the functor K above factors.

It is then natural to ask for an analogue of this in the equivariant setting where the objects in $\mathrm{Cat}^{\mathrm{perf}}_{\infty}$ are moreover equipped with actions by a finite group G and their algebraic K-theories sometimes admit "equivariant power operations" known as the *multiplicative norms*. As noted above, this extra structure - first enlisted into stable homotopy theory by Greenlees and May [GM97] where it was used to prove a completion theorem for equivariant MU - has most famously led to the stunning resolution of the Kervaire invariant one problem in [HHR16] and is well-known to be very tricky to construct. For example, by way of the Day convolution, [BGS20] only constructed a symmetric monoidal structure on equivariant algebraic

K–theory without these norms, which they termed *Green functors* (in analogy with the classical theory of Mackey functors).

Something interesting happens when one pursues this line of thought: it turns out that there are *two* natural candidates for a definition of equivariant algebraic K–theory. For one, we can just tack on G–Mackey objects on the functor $\mathrm{K}\colon \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Sp}$ to obtain

$$\underline{\mathrm{K}}_G^{\mathrm{pw}}\colon \underline{\mathrm{Mack}}_G(\mathrm{Cat}_\infty^{\mathrm{perf}}) \longrightarrow \underline{\mathrm{Mack}}_G(\mathrm{Sp}) \simeq \underline{\mathrm{Sp}}_G$$

where the decoration $(-)^{\mathrm{pw}}$ indicates that this is a pointwise construction (this is for example the construction considered in [BGS20; CMN+20]). Theorem C then tells us that this is a reasonable definition of equivariant algebraic K–theory for G–perfect-stable ∞–categories which after all sits in $\underline{\mathrm{Mack}}_G(\mathrm{Cat}_\infty^{\mathrm{perf}})$ as a full subcategory. But then one can also mimic [BGT13] in carrying out a genuine motivic construction to obtain $\underline{\mathrm{K}}_G\colon \underline{\mathrm{Cat}}_G^{\mathrm{perf}} \longrightarrow \underline{\mathrm{Sp}}_G$. The advantage of this definition is it admits the sought-after multiplicative norms by design. There is then a canonical comparison $\underline{\mathrm{K}}_G^{\mathrm{pw}} \Rightarrow \underline{\mathrm{K}}_G$, and the main theorem of Chapter 4 is that this is an equivalence when G is a finite 2–group:

Theorem F (cf. Corollary 4.3.20 and Corollary 4.3.21)**.** *Let G be a 2–group. Then $\underline{\mathrm{K}}_G^{\mathrm{pw}}$ canonically refines to a G–lax symmetric monoidal functor. Together with Theorem D, this implies that for any small symmetric monoidal perfect-stable ∞–category $\mathcal{C}^{\otimes} \in \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{perf}})$, the collection $\{\mathrm{K}(\mathrm{Fun}(BH,\mathcal{C}))\}_{H\leq G}$ assembles to a G–normed ring spectrum when G is a 2–group.*

This theorem should be read as a normed refinement of the Green functors considered in [BGS20] in the case G is a finite 2–group. As of now, we do not know if the comparison $\underline{\mathrm{K}}_G^{\mathrm{pw}} \Rightarrow \underline{\mathrm{K}}_G$ is an equivalence for a general group G, but our expectation is that it *is* so.

Having obtained a good understanding of G–perfect-stable ∞–categories and their algebraic K–theories, we are now ready to overlay these with the structure of hermitian forms: this is the subject of Part III. In [CDH+20b], we learn that hermitian K–theory is the addition of two extra spectral invariants, called the *Grothendieck-Witt spectrum* GW and *L–theory* L respectively, to the algebraic K–theory functor K. These are both functors $\mathrm{Cat}_\infty^{\mathrm{p}} \to \mathrm{Sp}$ where $\mathrm{Cat}_\infty^{\mathrm{p}}$ is the ∞–category of Poincaré ∞–categories - here Poincaré ∞–categories refer to a refinement of small stable ∞–categories with duality which was introduced by Lurie in [Lur11] and further developed in [CDH+20a]. Roughly speaking, the datum of a Poincaré ∞–category is a pair $(\mathcal{C}, \text{Ϙ})$ where $\mathcal{C}$ is a small stable ∞–category, Ϙ is a functor $\text{Ϙ}\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ which is quadratic (ie. it is reduced and 2–excisive in the sense of Goodwillie calculus), and that a canonically constructed duality functor $D_{\text{Ϙ}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ associated to Ϙ is an equivalence.

Similarly as explained above in the case of algebraic K–theory, the first natural candidates one might come up with as a genuine equivariant refinement of these are gotten by applying the functor Mack_G to these spectral invariants, yielding

$$\underline{\mathrm{GW}}_G(-),\, \underline{\mathrm{L}}_G \colon \underline{\mathrm{Mack}}_G(\mathrm{Cat}^{\mathrm{p}}_\infty) \longrightarrow \underline{\mathrm{Mack}}_G(\mathrm{Sp}) \simeq \underline{\mathrm{Sp}}_G$$

However, when one seriously considers the possibility of multiplicative norms, an immediate problem arises: the data structure of $\mathrm{Mack}_G(\mathrm{Cat}^{\mathrm{p}}_\infty)$ cannot see this in general, essentially because an object in this category is a collection of Poincaré ∞–categories

$$\left\{(\mathcal{C}_H, \Omega_H \colon \mathcal{C}_H^{\mathrm{op}} \to \mathrm{Sp})\right\}_{H \leq G}$$

equipped with restriction and transfer maps, satisfying some properties. Nowhere in this are genuine G–spectra featured, and these are of course *sine qua non* in any theory that features the multiplicative norms on genuine G–spectra.

Before proceeding further with this line of investigation, we nevertheless show how *Borel* equivariant GW–theory attains the structure of the multiplicative norms when G is a 2–group. Here, by Borel equivariant GW, we mean the composite functor

$$\mathrm{Fun}(BG, \mathrm{Cat}^{\mathrm{p}}_\infty) \xhookrightarrow{\mathrm{Bor}} \mathrm{Mack}_G(\mathrm{Cat}^{\mathrm{p}}_\infty) \xrightarrow{\mathrm{GW}} \mathrm{Mack}_G(\mathrm{Sp}) = \mathrm{Sp}_G \tag{0.1}$$

so that it is the *input* that is Borel equivariant, and *not* the output. In general, the output is very far from being Borel equivariant, and it is indeed the business of descent theory to study situations where this might be the case, up to various kinds of completions. In any case, augmenting the methods used to obtain Theorem F with some understanding of the quadratic structures, we obtain the following:

Theorem G (Precise version in Corollary 5.4.6)**.** *Let G be a 2–group. Then the Borel equivariant GW–theory canonically refines to a G–lax symmetric monoidal functor. Consequently, for any small symmetric monoidal Poincaré ∞–category $(\mathcal{C}, \Omega)^{\otimes} \in \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{p}})$, the collection of spectra*

$$\{\mathrm{GW}(\mathrm{Fun}(BH, \mathcal{C}), \Omega^{hH})\}_{H \leq G}$$

assembles canonically to a G–normed ring spectrum.

Coming back to considering a truly genuine refinement of hermitian K–theory, we are led to define the notion of *G–Poincaré ∞–categories* which are pairs $(\underline{\mathcal{C}}, \underline{\Omega})$ where $\underline{\mathcal{C}}$ is now a small G–stable G–category and $\underline{\Omega}$ is a G–functor $\underline{\Omega} \colon \underline{\mathcal{C}}^{\underline{\mathrm{op}}} \to \underline{\mathrm{Sp}}_G$ which is *G–quadratic* - this notion will involve Dotto's theory of equivariant Goodwillie calculus [Dot17] and requires slightly more than just being pointed and $2G$–excisive to guarantee in our setting the quadratic-linear-bilinear stable recollement so crucial in [CDH+20a; CDH+20b; CDH+20c]. Unfortunately, and importantly, the theory we develop so far only works for when G is an odd group, essentially by the same reason that Theorem E also only works for odd groups.

Notwithstanding, in order to realise this goal, in Chapter 6 we will translate Dotto's model category approach to equivariant Goodwillie calculus into the parametrised homotopy theory framework, whose methods will roughly follow the scheme

$$\begin{tikzcd} \{\text{Goodwillie}\} \arrow[r] \arrow[d] & \{\text{Lurie}\} \arrow[d] \\ \{\text{Dotto}\} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \{\text{Chapter 6}\} \end{tikzcd}$$

We think it worth pointing out that, while we claim no originality in the results for this chapter, we have written arguments down in a manner slightly more axiomatic than that of Lurie's (thanks to our standing assumption that the relevant Kan extensions exist), reducing many proofs to the same circle of Kan extension yoga and spotting appropriate poset adjunctions which might be useful in other settings.

With the parametrised theory of equivariant Goodwillie calculus in place, we are then ready to define G–Poincaré categories in the final Chapter 7. Via Theorem D, our definition will be seen in §7.3.2 to contain $\mathrm{Fun}(BG, \mathrm{Cat}^{\mathrm{p}}_{\infty})$ as a full subcategory, situating our notions as a faithful enlargement of those of [CDH+20a]. While most of the proof methods will be a direct mimicry of those of [CDH+20a], the section on G–symmetric monoidal matters §7.2 will involve new arguments owing to the seemingly essential and unavoidable fact that currying of tensor products does not work in the equivariant setting: this is because when we take an equivariant tensor $\bigotimes_{G/H}$, there is no way to induct by currying over separate components in the same way that we are used to in the nonequivariant setting, where $\mathrm{Map}(X \otimes Y, Z) \simeq \mathrm{Map}(X, \mathrm{map}(Y, Z))$. We think that these methods give a more general explanation as to why the linear and bilinear parts of a quadratic functor commutes with tensor products - an observation first made, as far as we are aware, in [CDH+20a, Prop. 5.1.3] and whose importance is difficult to overstate. We are extremely grateful to Maxime Ramzi who provided a crucial perspective in allowing these arguments to go through. Among other things, we prove that our G–category of G–Poincaré categories $\underline{\mathrm{Cat}}^{\mathrm{p}}_{G}$ is G–semiadditive-presentable and refines to the structure of a G–symmetric monoidal category (cf. §7.3).

In the last §7.4, we will assume a much more exploratory tone to indicate potential applications of the general approach of "genuinising" equivariant hermitian K–theory. In the first subsection, we analyse the particularly concrete case of $G = C_2$ when 2 is inverted in the input: recall that while our theory above only works when G is odd, if we invert 2 in our quadratic structures, then it will also work for even groups G. This can be a potentially confusing situation as we have the equivariance coming from $G = C_2$ but *also* a C_2–equivariance coming from the hermitian structure, which is already present even in the nonequivariant case. We emphasise that this distinction is not something new: it is analogous to the difference between Segal's equivariant complex K–theory KU_G for $G = C_2$ and Atiyah's C_2–real K–theory $\mathrm{K}\mathbb{R}$, and our setting for the case $G = C_2$ is then a combination of these two structures analogous to the combination in the complex K–theory case which yields $\mathrm{K}\mathbb{R}_G$ for $G = C_2$. For this reason, we have opted to denote by Σ_2 the equivariance coming from the hermitian structure to distinguish it from $G = C_2$. In any case, one consequence of the theory is the following:

Theorem H (Theorem 7.4.11)**.** *Suppose we have a C_2–Poincaré category $(\underline{\mathcal{C}}, \underline{\Omega})$ where 2 is inverted (ie. $\underline{\Omega}$ is a C_2–quadratic functor $\underline{\Omega}\colon \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Sp}}_{C_2}[\frac{1}{2}]$) such that we have an equivalence $(\mathcal{C}_e, \Omega^q_e) \simeq (\mathcal{C}_e, \Sigma^2\Omega^q_e)$ on the underlying Poincaré ∞–category, for instance, when it is induced by a C_2–ring spectrum which is 2–periodic away from 2. Then there is a natural equivalence*

$$\Omega^{2\sigma}\underline{\mathrm{L}}(\underline{\mathcal{C}}, \underline{\Omega}) \simeq \underline{\mathrm{L}}(\underline{\mathcal{C}}, \Omega^{2\sigma}\underline{\Omega})$$

where σ is the real sign representation. In particular, since Segal's equivariant complex K–theory KU_{C_2} satisfies equivariant complex Bott periodicity and since $2\sigma \cong_{C_2} 1 + \sigma \cong_{C_2} \mathbb{C}$, we obtain an equivalence $\Omega^{2\sigma}\underline{\mathrm{L}}^{\mathrm{s}}(\mathrm{KU}_{C_2}[\frac{1}{2}]) \simeq \underline{\mathrm{L}}^{\mathrm{s}}(\mathrm{KU}_{C_2}[\frac{1}{2}])$.

This seems to suggest that our genuine equivariant L–theory might be a good factory to manufacture potentially interesting equivariant rings with equivariant periodicity. Finally, we cannot resist but to come full circle and give a conjectural descent application of our theory in §7.4.2 which was the original motivation for the materials in this thesis: we will indicate how one can exploit the multiplicative norms on L–theory, if they exist, to obtain enough equivariant periodicity so as to be able to run the argument of [Gre93] in proving some completion theorems for equivariant L–theory. This concludes the general introduction and we refer the reader to the schematic summary on the next page for a bird's eye view of this thesis.

Conventions and assumptions From now on we will drop the adjective ∞– and mean ∞–categories when we say categories. Accordingly, we will write Cat in lieu of Cat_∞ to denote the ∞–category of small ∞–categories, and we will denote by $\mathrm{Cat}^{(1)} \subseteq \mathrm{Cat}$ the full ∞–subcategory of 1–categories (ie. the ∞–categories with the property that the mapping spaces are in fact mapping sets). Recall that $\mathrm{Cat}^{(1)}$ is itself a 1–category. The advantage of this choice of notation will be clear in the parametrised setting where it is necessary to carry many extra decorations. Furthermore, $\widehat{\mathrm{Cat}}$ will be used to denote the ∞–category of large ∞–categories.

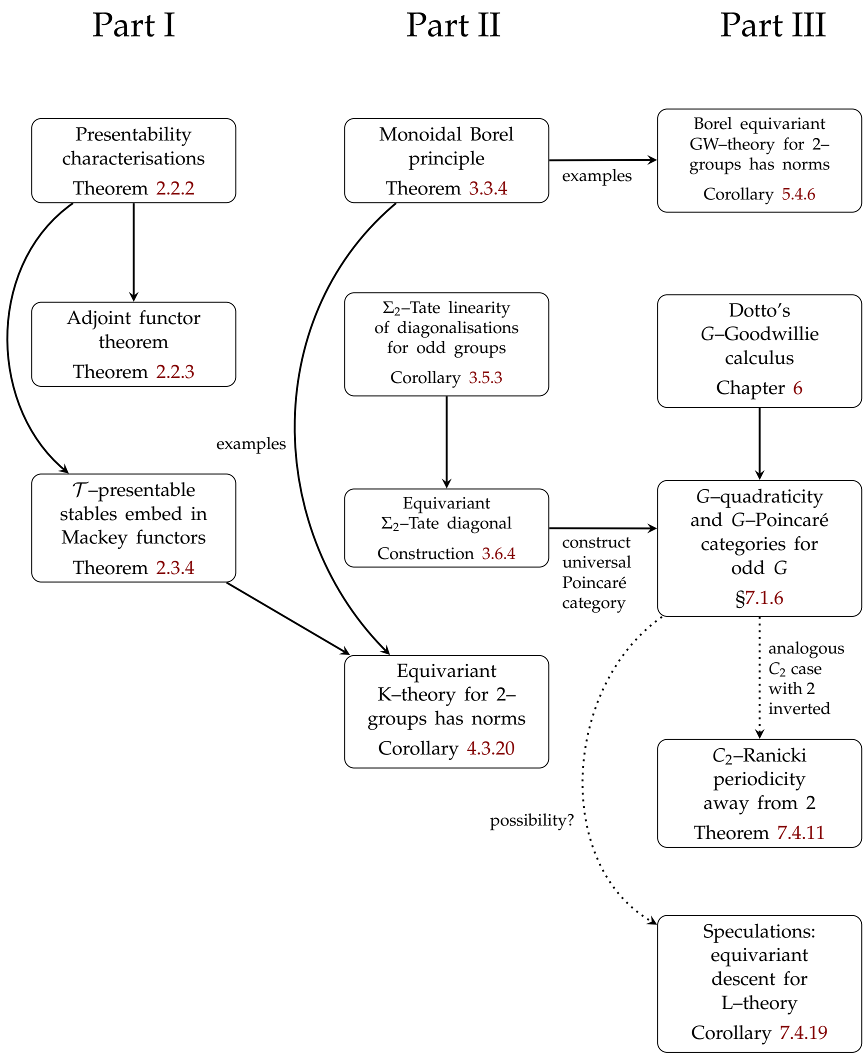

Figure 1: Leitfaden.

Part I

Foundations

Chapter 1

Elements of parametrised homotopy theory

We collect here basic definitions and foundational results from [Sha22a; Sha22b; Nar17; BDG+16b] for the convenience of the reader interested in using this formalism. We also develop some basic theory to the level of detail that we will be needing. To orientate the reader with our notational convention, we will always think of $\mathcal{T} = \mathcal{O}_G$ and that everything is parametrised over $\mathcal{T}^{\mathrm{op}} = \mathcal{O}_G^{\mathrm{op}}$, so that we mean working with $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \widehat{\mathrm{Cat}})$ where $\widehat{\mathrm{Cat}}$ is the huge category of large categories. We gradually increase the restrictions on our base categories, starting with general base categories in §1.1, imposing orbitality in §1.2, and further imposing atomicity in §1.3 (cf. Definition 1.1.11 for the definitions of these terms). A general guideline for this is that orbitality is required for the theory of (co)limits to go smoothly, and atomicity is required for algebraic notions such as semiadditivity, stability, and operads. In both sections, we have denoted by "Recollections" those subsections that contain mostly statements already proved in the literature and are included in order to establish notational consistency as well as to make the reading of this chapter as self-contained as possible. We point out that, in this chapter, we have organised these results thematically instead of chronologically, and so we will occasionally refer ahead to results from subsequent (sub)sections.

1.1 Preliminaries: general base categories

1.1.1 Recollections: basic objects and constructions

Recollections 1.1.1. For a category $\mathcal{T}$, there is Lurie's straightening-unstraightening equivalence $\mathrm{coCart}(\mathcal{T}^{\mathrm{op}}) \simeq \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat})$ (cf. for example [HW21, Thm. I.23]). The category of *$\mathcal{T}$–categories* is then defined simply as $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat})$ and we also write this as $\mathrm{Cat}_{\mathcal{T}}$. We will always denote a $\mathcal{T}$–category with an underline $\underline{\mathcal{C}}$. Under the equivalence above, the datum of a $\mathcal{T}$–category is equivalent to the datum of a cocartesian fibration $p\colon \mathrm{Total}(\underline{\mathcal{C}}) \to \mathcal{T}^{\mathrm{op}}$, and a $\mathcal{T}$–functor is defined just to be a morphism of $\mathcal{T}$–categories $\underline{\mathcal{C}} \to \underline{\mathcal{D}}$, is then equivalently a map of cocartesian fibrations $\mathrm{Total}(\underline{\mathcal{C}}) \to \mathrm{Total}(\underline{\mathcal{D}})$ over $\mathcal{T}^{\mathrm{op}}$. For an object $V \in \mathcal{T}$, we will write $\mathcal{C}_V$ or $\underline{\mathcal{C}}_V$ for the fibre of $\mathrm{Total}(\underline{\mathcal{C}}) \to \mathcal{T}^{\mathrm{op}}$ over V.

Remark 1.1.2. The product $\underline{\mathcal{C}} \times \underline{\mathcal{D}}$ in $\mathrm{Cat}_{\mathcal{T}}$ of two $\mathcal{T}$–categories $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ is given as the pullback $\mathrm{Total}(\underline{\mathcal{C}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Total}(\underline{\mathcal{D}})$ in the cocartesian fibrations perspective. We will always denote with $\times$ when we are viewing things as $\mathcal{T}$–categories and we reserve $\times_{\mathcal{T}^{\mathrm{op}}}$ for when we are viewing things as total categories. In this way, there will be no confusion as to whether or not $\times_{\mathcal{T}^{\mathrm{op}}}$ denotes a pullback in $\mathrm{Cat}_{\mathcal{T}}$: this will never be the case.

Notation 1.1.3. Since $\mathrm{Cat}_{\mathcal{T}} = \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat})$ is naturally even a 2–category, for $\underline{\mathcal{C}}, \underline{\mathcal{D}} \in \mathrm{Cat}_{\mathcal{T}}$, we have the *category* of $\mathcal{T}$–functors from $\underline{\mathcal{C}}$ to $\underline{\mathcal{D}}$: this we write as $\mathrm{Fun}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$. Unstraightening, we obtain $\mathrm{Fun}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \simeq \mathrm{Fun}^{\mathrm{cocart}}(\mathrm{Total}(\underline{\mathcal{C}}), \mathrm{Total}(\underline{\mathcal{D}})) \times_{\mathrm{Fun}(\mathrm{Total}(\underline{\mathcal{C}}), \mathcal{T}^{\mathrm{op}})} \{p\}$ where $\mathrm{Fun}^{\mathrm{cocart}}$ is the full subcategory of functors preserving $\mathcal{T}^{\mathrm{op}}$–cocartesian morphisms.

Example 1.1.4. We now give some basic examples of $\mathcal{T}$–categories to set notation.

- (Fibrewise $\mathcal{T}$–categories) Let $K \in \mathrm{Cat}$. Write $\underline{\mathrm{const}}_{\mathcal{T}}(K) \in \mathrm{Cat}_{\mathcal{T}}$ for the constant K–valued diagram. In other words, $\mathrm{Total}(\underline{\mathrm{const}}_{\mathcal{T}}(K)) \simeq K \times \mathcal{T}^{\mathrm{op}}$.
- We write $\underline{*} := \underline{\mathrm{const}}_{\mathcal{T}}(*)$. This is clearly a final object in $\mathrm{Cat}_{\mathcal{T}} = \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat})$.
- (Corepresentable $\mathcal{T}$–categories) Let $V \in \mathcal{T}$. Then we can consider the left (and so cocartesian) fibration associated to the functor $\mathrm{Map}_{\mathcal{T}}\colon \mathcal{T}^{\mathrm{op}} \to \mathcal{S}$ and denote this $\mathcal{T}$–category by $\underline{V}$. Note that $\mathrm{Total}(\underline{V}) \simeq (\mathcal{T}_{/V})^{\mathrm{op}}$. By corepresentability of $\underline{V}$, we have $\mathrm{Fun}_{\mathcal{T}}(\underline{V}, \underline{\mathcal{C}}) \simeq \mathcal{C}_V$. To wit, for $K \in \mathrm{Cat}$, by Construction 1.1.13, we have

$$\begin{aligned} \mathrm{Map}_{\mathrm{Cat}}\left(K, \mathrm{Fun}_{\mathcal{T}}(\underline{V}, \underline{\mathcal{C}})\right) &\simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}}}\left(\underline{V}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{const}}(K), \underline{\mathcal{C}})\right) \\ &\simeq \mathrm{Map}_{\mathrm{Cat}}(K, \mathcal{C}_V) \end{aligned}$$

Definition 1.1.5. The category of $\mathcal{T}$–*objects of* $\underline{\mathcal{C}}$ is defined to be $\mathrm{Fun}_{\mathcal{T}}(\underline{*}, \underline{\mathcal{C}})$.

Remark 1.1.6. If $\mathcal{T}^{\mathrm{op}}$ has an initial object $T \in \mathcal{T}^{\mathrm{op}}$, then this means that the category of $\mathcal{T}$–objects in $\underline{\mathcal{C}}$ is just $\mathcal{C}_T$.

Construction 1.1.7 (Parametrised opposites)**.** For a $\mathcal{T}$–category $\underline{\mathcal{C}}$, its $\mathcal{T}$*–opposite* $\underline{\mathcal{C}}^{\mathrm{op}}$ is defined to be the image under the functor obtained by applying $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, -)$ to $(-)^{\mathrm{op}}\colon \mathrm{Cat} \to \mathrm{Cat}$. In the unstraightened view, this is given by taking fibrewise opposites in the total category. In [BDG+16b] this was called vertical opposites $(-)^{\mathrm{vop}}$ to invoke just such an impression.

Observation 1.1.8. Let $\underline{V}$ be a corepresentable $\mathcal{T}$–category. Then $\underline{V}^{\underline{\mathrm{op}}} \simeq \underline{V}$ since the functor $(-)^{\mathrm{op}}\colon \mathrm{Cat} \to \mathrm{Cat}$ restricts to the identity on $\mathcal{S}$.

Construction 1.1.9. The cone and cocone are functors $(-)^{\triangleleft}, (-)^{\triangleright}\colon \mathrm{Cat} \to \mathrm{Cat}$ which add a (co)cone point to a category. Applying $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, -)$ to this functor yields the $\mathcal{T}$*–cone* and *–cocone* functors $(-)^{\underline{\triangleleft}}$ and $(-)^{\underline{\triangleright}}$ respectively. We refer to [Sha22a] for more on this.

Definition 1.1.10. A $\mathcal{T}$–functor is $\mathcal{T}$*-fully faithful* (resp. $\mathcal{T}$*-equivalence*) if it is so fibrewise. There is the expected characterisation of $\mathcal{T}$–fully faithfulness in terms of $\mathcal{T}$–mapping spaces, see Remark 1.2.19.

Definition 1.1.11. We say that the category $\mathcal{T}$ is *orbital* if the finite coproduct cocompletion $\mathrm{Fin}_{\mathcal{T}}$ admits finite pullbacks. Here, by finite coproduct cocompletion, we mean the full subcategory of the presheaf category $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{S})$ spanned by finite coproduct of representables. We say that it is *atomic* if every retraction is an equivalence.

Notation 1.1.12 (Basechange)**.** As in [Nar17], we will write $\underline{\mathcal{C}}_{\underline{V}} := \underline{\mathcal{C}} \times \underline{V} = \mathrm{Total}(\underline{\mathcal{C}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Total}(\underline{V})$ for the basechanged parametrised category, which is now viewed as a $\mathcal{T}_{/V}$–category. The $(-)_{\underline{V}}$ is a useful reminder that we have basechanged to V, and so for example we will often use the notation $\mathrm{Fun}_{\underline{V}}$ to mean $\mathrm{Fun}_{\mathcal{T}_{/V}}$ and *not* $\mathrm{Fun}_{\mathrm{Total}(\underline{V})} \simeq \mathrm{Fun}_{(\mathcal{T}_{/V})^{\mathrm{op}}}$.

Construction 1.1.13 (Internal $\mathcal{T}$–functor category, [BDG+16b, §9])**.** For $\underline{\mathcal{C}}, \underline{\mathcal{D}} \in \mathrm{Cat}_{\mathcal{T}}$, there is a $\mathcal{T}$–category $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ such that

$$\mathrm{Fun}_{\mathcal{T}}(\underline{\mathcal{E}}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})) \simeq \mathrm{Fun}_{\mathcal{T}}(\underline{\mathcal{E}} \times \underline{\mathcal{C}}, \underline{\mathcal{D}})$$

This is because $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat})$ is presentable and the endofunctor $- \times \underline{\mathcal{C}}$ has a right adjoint since it preserves colimits. In particular, by a Yoneda argument we get $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{*}, \underline{\mathcal{D}}) \simeq \underline{\mathcal{D}}$. Moreover, plugging in $\underline{\mathcal{E}} = \underline{*}$ we see that $\mathcal{T}$–objects of the internal $\mathcal{T}$–functor object are just $\mathcal{T}$–functors. Furthermore, the $\mathcal{T}$–functor categories basechange well in that

$$\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})_{\underline{V}} \simeq \underline{\mathrm{Fun}}_{\underline{V}}(\underline{\mathcal{C}}_{\underline{V}}, \underline{\mathcal{D}}_{\underline{V}})$$

so the fibre over $V \in \mathcal{T}^{\mathrm{op}}$ is given by $\mathrm{Fun}_{\underline{V}}(\underline{\mathcal{C}}_{\underline{V}}, \underline{\mathcal{D}}_{\underline{V}})$. To wit, for any $\mathcal{T}_{/V}$–category $\underline{\mathcal{E}}$,

$$\begin{aligned}
\mathrm{Map}_{(\mathrm{Cat}_{\mathcal{T}})_{/\underline{V}}}(\underline{\mathcal{E}}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})_{\underline{V}}) &\simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}}}(\underline{\mathcal{E}}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})) \\
&\simeq \mathrm{Map}_{(\mathrm{Cat}_{\mathcal{T}})_{/\underline{V}}}(\underline{\mathcal{E}} \times \underline{\mathcal{C}}, \underline{\mathcal{D}}_{\underline{V}}) \\
&\simeq \mathrm{Map}_{(\mathrm{Cat}_{\mathcal{T}})_{/\underline{V}}}(\underline{\mathcal{E}} \times_{\underline{V}} \mathcal{C}_{\underline{V}}, \mathcal{D}_{\underline{V}}) \\
&\simeq \mathrm{Map}_{(\mathrm{Cat}_{\mathcal{T}})_{/\underline{V}}}(\underline{\mathcal{E}}, \underline{\mathrm{Fun}}_{\underline{V}}(\mathcal{C}_{\underline{V}}, \mathcal{D}_{\underline{V}}))
\end{aligned}$$

Notation 1.1.14 (Parametrised cotensors)**.** Let I be a small unparametrised category. Then the adjunction $- \times I : \mathrm{Cat} \rightleftarrows \mathrm{Cat} : \mathrm{Fun}(I, -)$ induces the adjunction

$$(- \times I)_* : \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat}) \rightleftarrows \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat}) : \mathrm{Fun}(I, -)_*$$

Under the identification $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat}) \simeq \mathrm{Cat}_{\mathcal{T}}$ where $\mathrm{Cat}_{\mathcal{T}}$ is the category of $\mathcal{T}$-categories, it is clear that $(- \times I)_*$ corresponds to the $\mathcal{T}$-functor $\underline{\mathrm{const}}_{\mathcal{T}}(I) \times -$, whose right adjoint we know is $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{const}}_{\mathcal{T}}(I), -)$. Therefore $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{const}}_{\mathcal{T}}(I), -)$ implements the *fibrewise functor construction*. We will introduce the notation $\mathrm{fun}(I, -)$ for $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{const}}_{\mathcal{T}}(I), -)$. This satisfies the following properties whose proofs are immediate.

(i) $\underline{\mathrm{Cat}}_{\mathcal{T}}$ is cotensored over Cat in the sense that for any $\mathcal{T}$-categories $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ we have
$$\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \mathrm{fun}(I, \underline{\mathcal{D}})) \simeq \mathrm{fun}(I, \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}))$$

(ii) $\mathrm{fun}(I, -)$ preserves $\mathcal{T}$-adjunctions.

Observation 1.1.15. There is a natural equivalence of $\mathcal{T}$–categories

$$\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})^{\underline{\mathrm{op}}} \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathcal{D}}^{\underline{\mathrm{op}}})$$

This is because $(-)^{\underline{\mathrm{op}}} : \mathrm{Cat}_{\mathcal{T}} \to \mathrm{Cat}_{\mathcal{T}}$ is an involution, and so for any $\mathcal{T}$–category $\underline{\mathcal{E}}$,

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}}}(\underline{\mathcal{E}}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})^{\underline{\mathrm{op}}}) &\simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}}}(\underline{\mathcal{E}}^{\underline{\mathrm{op}}}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})) \\
&\simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}}}(\underline{\mathcal{E}}^{\underline{\mathrm{op}}} \times \underline{\mathcal{C}}, \underline{\mathcal{D}}) \\
&\simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}}}(\underline{\mathcal{E}} \times \underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathcal{D}}^{\underline{\mathrm{op}}}) \\
&\simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}}}(\underline{\mathcal{E}}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathcal{D}}^{\underline{\mathrm{op}}}))
\end{aligned}$$

Construction 1.1.16 (Cofree parametrisation, [Nar17, Def. 1.10])**.** Let $\mathcal{D}$ be a category. There is a $\mathcal{T}$–category $\underline{\mathrm{Cofree}}(\mathcal{D}) : \mathcal{T}^{\mathrm{op}} \to \mathrm{Cat}$ classified by $V \mapsto \mathrm{Fun}((\mathcal{T}_{/V})^{\mathrm{op}}, \mathcal{D})$. This has the following universal property: if $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$, then there is a natural equivalence

$$\mathrm{Fun}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{Cofree}}(\mathcal{D})) \simeq \mathrm{Fun}(\mathrm{Total}(\underline{\mathcal{C}}), \mathcal{D})$$

of ordinary ∞-categories. This construction is of foundational importance and it allows us to define the following two fundamental $\mathcal{T}$–categories.

Notation 1.1.17. We will write $\underline{\mathrm{Cat}}_{\mathcal{T}} := \underline{\mathrm{Cofree}}_{\mathcal{T}}(\mathrm{Cat})$ for the *$\mathcal{T}$–category of $\mathcal{T}$–categories*; we write $\underline{\mathcal{S}}_{\mathcal{T}} := \underline{\mathrm{Cofree}}_{\mathcal{T}}(\mathcal{S})$ for the *$\mathcal{T}$–category of $\mathcal{T}$–spaces*.

Theorem 1.1.18 (Parametrised straightening-unstraightening, [BDG+16b, Prop. 8.3])**.** *Let $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$. Then there are equivalences*

$$\mathrm{Fun}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{Cat}}_{\mathcal{T}}) \simeq \mathrm{coCart}(\mathrm{Total}(\underline{\mathcal{C}})) \qquad \mathrm{Fun}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{S}}_{\mathcal{T}}) \simeq \mathrm{Left}(\mathrm{Total}(\underline{\mathcal{C}}))$$

Proof. This is an immediate consequence of the usual straightening-unstraightening and the universal property of $\mathcal{T}$–categories of $\mathcal{T}$–objects above. For example,

$$\mathrm{Fun}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{Cat}}_{\mathcal{T}}) \simeq \mathrm{Fun}(\mathrm{Total}(\underline{\mathcal{C}}), \mathrm{Cat}) \simeq \mathrm{coCart}(\mathrm{Total}(\underline{\mathcal{C}}))$$

and similarly for spaces. □

1.1.2 Parametrised adjunctions

$\mathcal{T}$–adjunctions as introduced in [Sha22a] is based on the *relative adjunctions* of [Lur17].

Definition 1.1.19 ([Lur17, Def. 7.3.2.2])**.** Suppose we have diagrams of categories

$$\begin{tikzcd} \mathcal{C} \arrow[dr, "q"'] & & \mathcal{D} \arrow[ll, "G"'] \arrow[dl, "p"] \\ & \mathcal{E} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "q"'] & & \mathcal{D} \arrow[dl, "p"] \\ & \mathcal{E} & \end{tikzcd}$$

Then we say that:

- For the first diagram, G admits a *left adjoint F relative to $\mathcal{E}$* if G admits a left adjoint F such that for every $C \in \mathcal{C}$, q sends the unit $\eta : C \to GFC$ to an equivalence in $\mathcal{E}$ (equivalently, if $q\eta : q \Rightarrow p \circ F$ exhibits a commutation $p \circ F \simeq q$ by [Lur17, Prop. 7.3.2.1]).
- For the second diagram, F admits a *right adjoint G relative to $\mathcal{E}$* if F admits a right adjoint G such that for every $D \in \mathcal{D}$, p maps the counit $\varepsilon : FGD \to D$ to an equivalence in $\mathcal{E}$ (equivalently if $p\varepsilon : q \circ G \Rightarrow p$ exhibits $q \circ G \simeq p$ by [Lur17, Prop. 7.3.2.1]).

Observe that when $\mathcal{E} \simeq *$, this specialises to the usual notion of adjunctions.

Remark 1.1.20. These two definitions are compatible. To see this, assume the first condition for example, ie. that G has a left adjoint F relative to $\mathcal{E}$. We need to see that F then admits a right adjoint G relative to $\mathcal{E}$ in the sense of the second condition, ie. that p sends the counit $\varepsilon : FGD \to D$ to an equivalence in $\mathcal{E}$. For this just consider the commutative diagram

$$\begin{tikzcd} qG \arrow[r, "q\eta_G", "\simeq"'] \arrow[dr, equal] & qGFG \arrow[r, "\simeq"] \arrow[d, "qG\varepsilon"] & pFG \arrow[d, "p\varepsilon"] \\ & qG \arrow[r, "\simeq"] & p \end{tikzcd}$$

where the triangle is by the adjunction, and the square is by the natural equivalence $qG \simeq p$.

Definition 1.1.21. Let $\underline{\mathcal{C}}, \underline{\mathcal{D}} \in \mathrm{Cat}_{\mathcal{T}}$. Then a *$\mathcal{T}$-adjunction* $F : \underline{\mathcal{C}} \rightleftarrows \underline{\mathcal{D}} : G$ is defined to be a relative adjunction such that F, G are $\mathcal{T}$–functors. A *$\mathcal{T}$-Bousfield localisation* is a $\mathcal{T}$–adjunction where the $\mathcal{T}$–right adjoint is $\mathcal{T}$–fully faithful.

Proposition 1.1.22 (Stability of relative adjunctions under pullbacks, [Lur17, Prop. 7.3.2.5])**.** *Suppose we have a relative adjunction*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, shift left, "F"] \arrow[dr, "q"'] & & \mathcal{D} \arrow[ll, shift left, "G"] \arrow[dl, "p"] \\ & \mathcal{E} & \end{tikzcd}$$

Then for any functor $\mathcal{E}' \to \mathcal{E}$ the diagram of pullbacks

$$\begin{tikzcd} \mathcal{C} \times_{\mathcal{E}} \mathcal{E}' \arrow[rr, shift left, "F"] \arrow[dr, "q"'] & & \mathcal{D} \times_{\mathcal{E}} \mathcal{E}' \arrow[ll, shift left, "G"] \arrow[dl, "p"] \\ & \mathcal{E}' & \end{tikzcd}$$

is again a relative adjunction.

We now have the following criteria to obtain relative adjunctions - these are just modified from Lurie's more general assumptions.

Proposition 1.1.23 (Criteria for relative adjunctions, [Lur17, Prop. 7.3.2.6])**.** *Suppose $p : \mathcal{C} \to \mathcal{E}, q : \mathcal{D} \to \mathcal{E}$ are cocartesian fibrations. If we have a map of cocartesian fibrations F*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "q"'] & & \mathcal{D} \arrow[dl, "p"] \\ & \mathcal{E} & \end{tikzcd}$$

Then:

(1) *F admits a right adjoint G relative to $\mathcal{E}$ if and only if for each $E \in \mathcal{E}$ the map of fibres $F_E : \mathcal{C}_E \to \mathcal{D}_E$ admits a right adjoint G_E. The right adjoint need no longer be a map of cocartesian fibrations.*

(2) *F admits a left adjoint L relative to $\mathcal{E}$ if and only if for each $E \in \mathcal{E}$ the map of fibres $F_E : \mathcal{C}_E \to \mathcal{D}_E$ admits a left adjoint L_E and the canonical comparison maps (constructed in [Lur17, Prop. 7.3.2.11])*

$$Lf^* \to LFf^*L \xrightarrow{\varepsilon} f^*L$$

constructed from the fibrewise adjunction are equivalences - here f^* is the pushforward given by the cocartesian lift along some $f : E' \to E$ in $\mathcal{E}$. The *relative left adjoint, if it exists, must necessarily be a map of cocartesian fibrations.*

Proof. We prove each in turn. To see (1), suppose F has an $\mathcal{E}$-right adjoint G. Then for each $e \in \mathcal{E}$ the inclusion $\{e\} \hookrightarrow \mathcal{E}$ induces a pullback relative adjunction over the point $\{e\}$ by Proposition 1.1.22, and so we get the statement on fibres. Conversely, suppose we have fibrewise right adjoints. To construct an $\mathcal{E}$-right adjoint G, since adjunctions can be constructed objectwise by the unparametrised version of Proposition 1.2.26 below, we need to show that for each $e \in \mathcal{E}$ and $d \in \mathcal{D}_e$, there is a $Gd \in \mathcal{C}_e$ and a map $\varepsilon : FGd \to d$ such that:

(a) For every $c \in \mathcal{C}$ the following composition is an equivalence

$$\mathrm{Map}_{\mathcal{C}}(c, Gd) \xrightarrow{F} \mathrm{Map}_{\mathcal{D}}(Fc, FGd) \xrightarrow{\varepsilon} \mathrm{Map}_{\mathcal{D}}(Fc, d)$$

(b) The morphism $p\varepsilon : pFGd \to pd$ is an equivalence in $\mathcal{E}$.

We can just define $Gd := G_e(d) \in \mathcal{C}_e$ given by the fibrewise right adjoint and let $\varepsilon : FGd \to d$ be the fibrewise counit. Since these are fibrewise, point (b) is automatic. To see point (a), let $c \in \mathcal{C}_{e'}$ for some $e' \in \mathcal{E}$. Since the mapping space in the total category of cocartesian fibrations are just disjoint unions over the components lying under $\mathrm{Map}_{\mathcal{C}}(c, Gd)$, we can work over some $f \in \mathrm{Map}_{\mathcal{E}}(e', e)$. Consider

$$\begin{tikzcd}
\mathrm{Map}^f_{\mathcal{C}}(c, Gd) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Map}_{\mathcal{D}_e}(f^*Fc, d) \simeq \mathrm{Map}^f_{\mathcal{D}}(Fc, d) \\
\mathrm{Map}_{\mathcal{C}_e}(f^*c, Gd) \arrow[r, "F"] \arrow[ur, "\simeq"] & \mathrm{Map}_{\mathcal{D}_e}(f^*Fc, FGd) \arrow[u, "\varepsilon"]
\end{tikzcd}$$

where we have used also that F was a map of cocartesian fibrations so that $f^*F \simeq Ff^*$ and that the diagonal map is an equivalence since we had a fibrewise adjunction $F_e \dashv G_e$ by hypothesis. This completes the proof of part (1).

For case (2), to see the cocartesianness of a relative left adjoint L, note

$$\begin{aligned}
\mathrm{Map}_{\mathcal{C}}(Lf^*d, c) \simeq \mathrm{Map}_{\mathcal{D}}(f^*d, Fc) &\simeq \mathrm{Map}^f_{\mathcal{D}}(d, Fc) \\
&\simeq \mathrm{Map}^f_{\mathcal{C}}(Ld, c) \simeq \mathrm{Map}_{\mathcal{C}}(f^*Ld, c)
\end{aligned}$$

The proof for right adjoints in (1) go through in this case but now we use

$$\begin{tikzcd}
\mathrm{Map}^f_{\mathcal{C}}(Ld, c) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Map}_{\mathcal{D}_E}(f^*d, Fc) \simeq \mathrm{Map}^f_{\mathcal{D}}(d, Fc) \\
\mathrm{Map}_{\mathcal{C}_E}(f^*Ld, c) \arrow[r] & \mathrm{Map}_{\mathcal{C}_E}(Lf^*d, c) \arrow[u, "\simeq"]
\end{tikzcd}$$

and so the technical condition in the statement says that there is a canonical map inducing the bottom map in the square which must necessarily be an equivalence. □

Remark 1.1.24. One might object to the notation we have adopted for the pushforward being f^* instead of $f_!$. This convention is standard in the framework of [BDG+16a] because the latter notation is reserved for the *left adjoint* of f^* (the so-called $\mathcal{T}$–coproducts) that will be recalled later.

Corollary 1.1.25 (Fibrewise criteria for $\mathcal{T}$–adjunctions)**.** *Let $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a $\mathcal{T}$–functor. Then it admits a $\mathcal{T}$–right adjoint if and only if it has fibrewise right adjoints G_V for all $V \in \mathcal{T}$ and*

$$\begin{tikzcd} \mathcal{C}_W & \mathcal{D}_W \arrow[l, "G_W"'] \\ \mathcal{C}_V \arrow[u, "f^*"] & \mathcal{D}_V \arrow[l, "G_V"] \arrow[u, "f^*"'] \end{tikzcd}$$

commutes for all $f : W \to V$ in $\mathcal{T}$. Similarly for left $\mathcal{T}$–adjoints.

Proof. The commuting square ensures that the relative right adjoint is a $\mathcal{T}$–functor. □

Proposition 1.1.26 (Criteria for $\mathcal{T}$–Bousfield localisations, "[Lur09, Prop. 5.2.7.4]")**.** *Let $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$ and $L : \underline{\mathcal{C}} \to \underline{\mathcal{C}}$ a $\mathcal{T}$–functor equipped with a fibrewise natural transformation $\eta : \mathrm{id} \Rightarrow L$. Let $j : L\underline{\mathcal{C}} \subseteq \underline{\mathcal{C}}$ be the inclusion of the $\mathcal{T}$–full subcategory spanned by the image of L. Suppose the transformations $L\eta, \eta_L : L \Longrightarrow L \circ L$ are equivalences. Then the pair (L, j) constitutes a $\mathcal{T}$–Bousfield localisation with unit η.*

Proof. We want to apply Corollary 1.1.25. Since we are already provided with the fact that L was a $\mathcal{T}$–functor, all that is left to show is that it is fibrewise left adjoint to the inclusion $L\underline{\mathcal{C}} \subseteq \underline{\mathcal{C}}$. But this is guaranteed by [Lur09, Prop. 5.2.7.4], and so we are done. □

Finally, we show that parametrised adjunctions have the expected internal characterisation in terms of the parametrised mapping spaces recalled in Construction 1.2.18.

Lemma 1.1.27 (Mapping space characterisation of $\mathcal{T}$–adjunctions)**.** *Let $F : \underline{\mathcal{C}} \leftrightarrows \underline{\mathcal{D}} : G$ be a pair of $\mathcal{T}$–functors. Then there is a $\mathcal{T}$–adjunction $F \dashv G$ if and only if we have a natural equivalence*

$$\underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(F-,-) \simeq \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(-,G-) : \underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{D}} \longrightarrow \underline{\mathcal{S}}_{\mathcal{T}}$$

Proof. The if direction is clear: since F and G were already $\mathcal{T}$–functors, by Corollary 1.1.25 the only thing left to do is to show fibrewise adjunction, and this is easily implied by the equivalence which supplies the unit and counits. For the only if direction, by definition of a relative adjunction, we have a fibrewise natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ (ie. a morphism in $\mathrm{Fun}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{C}})$) and so we obtain a natural comparison

$$\underline{\mathrm{Map}_{\mathcal{D}}}(F-,-) \xrightarrow{G} \underline{\mathrm{Map}_{\mathcal{C}}}(GF-,G-) \xrightarrow{\eta^*} \underline{\mathrm{Map}_{\mathcal{C}}}(-,G-)$$

Since equivalences between $\mathcal{T}$–functors are checked fibrewise, let $c \in \mathcal{C}_V, d \in \mathcal{D}_V$. Then

$$\begin{aligned} \underline{\mathrm{Map}_{\mathcal{D}}}(F-,-): \quad (c,d) &\mapsto \quad \Big((W \xrightarrow{f} V) \mapsto (\mathrm{Map}_{\mathcal{D}_V}(Fc,d) \to \mathrm{Map}_{\mathcal{D}_W}(f^*Fc, f^*d)\Big) \in \underline{\mathcal{S}_V} \\ \underline{\mathrm{Map}_{\mathcal{C}}}(-,G-): \quad (c,d) &\mapsto \quad \Big((W \xrightarrow{f} V) \mapsto (\mathrm{Map}_{\mathcal{C}_V}(c,Gd) \to \mathrm{Map}_{\mathcal{C}_W}(f^*c, f^*Gd)\Big) \in \underline{\mathcal{S}_V} \end{aligned}$$

Since F, G were $\mathcal{T}$–functors, we have $Ff^* \simeq f^*F$ and $Gf^* \simeq f^*G$, and so the natural comparison coming from the relative adjunction unit given above exhibits a pointwise equivalence between $\underline{\mathrm{Map}_{\mathcal{D}}}(F-,-)$ and $\underline{\mathrm{Map}_{\mathcal{C}}}(-,G-)$ by Corollary 1.1.25. □

1.2 Preliminaries: orbital base categories

Some of the notions here still make sense for general $\mathcal{T}$, but we want orbitality in order to make formulations involving Beck-Chevalley conditions. Hence, from now on, we assume that $\mathcal{T}$ is orbital.

1.2.1 Recollections: colimits and Kan extensions

Definition 1.2.1. Let $\underline{K} \in \mathrm{Cat}_{\mathcal{T}}$ and $q \colon \underline{K} \to \underline{*}$ be the unique map. Then precomposition induces the $\mathcal{T}$–functor $q^* : \underline{\mathcal{D}} \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{*}, \underline{\mathcal{D}}) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{K}, \underline{\mathcal{D}})$. The $\mathcal{T}$–left adjoint $q_!$, if it exists, is called the $\underline{K}$–indexed *$\mathcal{T}$–colimit*, and similarly for *T-limits* q_*.

Example 1.2.2. Here are some special and important classes of these:

- A $\mathcal{T}$–(co)limit indexed by $\underline{\mathrm{const}}_{\mathcal{T}}(K)$ for some ordinary ∞–category K is called a *fibrewise $\mathcal{T}$-(co)limit*.
- A $\mathcal{T}$–(co)limit indexed by a corepresentable $\mathcal{T}$–category $\underline{V}$ (cf. Example 1.1.4) of some $V \in \mathcal{T}$ is called the *T-(co)product*.

Definition 1.2.3 ([Sha22a, Def. 11.2])**.** Let $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a $\mathcal{T}$–functor.

- We say that it *preserves $\mathcal{T}$-colimits* if for all $\mathcal{T}$–colimit diagrams $\overline{d} : K^{\rhd} \to \underline{\mathcal{C}}$, the post-composed diagram $F \circ \overline{d} : K^{\rhd} \to \underline{\mathcal{D}}$ is a $\mathcal{T}$–colimit. Similarly for $\mathcal{T}$–limits.

- We say that F *strongly preserves* $\mathcal{T}$*-colimits* if for all $V \in \mathcal{T}$, $F_{\underline{V}} : \underline{\mathcal{C}}_{\underline{V}} \to \underline{\mathcal{D}}_{\underline{V}}$ preserves $\mathcal{T}_{/V}$–colimits. Similarly for $\mathcal{T}$–limits.

Warning 1.2.4 ([Sha22a, Rmk. 5.14])**.** Note that being $\mathcal{T}$–cocomplete is much stronger than just admitting all $\mathcal{T}$–colimits. This is because admitting all $\mathcal{T}$–colimits just means that any $\mathcal{T}_{/V}$–diagram $\underline{K}_{\underline{V}} \to \underline{\mathcal{C}}_{\underline{V}}$ pulled back from a $\mathcal{T}$–diagram $\underline{K} \to \underline{\mathcal{C}}$ admits a $\mathcal{T}_{/V}$–colimit. However not every $\mathcal{T}_{/V}$–diagram is pulled back as such. We will elaborate on the distinction of these definitions in the next subsection. In this document, we will never consider preservations, but only strong preservations.

Definition 1.2.5 ([Sha22a, Def. 5.13])**.** Let $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$. Then we say $\underline{\mathcal{C}}$ is $\mathcal{T}$–*(co)complete* if for all $V \in \mathcal{T}$ and $\mathcal{T}_{/V}$–diagram $p : \underline{K} \to \underline{\mathcal{C}}_{\underline{V}}$ with $\underline{K}$ small, p admits a $\mathcal{T}_{/V}$–(co)limit.

Terminology 1.2.6. When we want to specify particular kinds of parametrised (co)limits that a $\mathcal{T}$–category admits, it is convenient to use the following terminology: for $\mathcal{K} = \{\mathcal{K}_V\}_{V \in \mathcal{T}}$ some collection of diagrams varying over $V \in \mathcal{T}$, we say that $\underline{\mathcal{C}}$ *strongly admits* $\mathcal{K}$–(co)limits if for all $V \in \mathcal{T}$, $\underline{\mathcal{C}}_{\underline{V}}$ admits $\underline{K}$–colimits for all $\underline{K} \in \mathcal{K}_V$. Examples include:

- $\underline{\mathcal{C}}$ strongly admits all $\mathcal{T}$–(co)limits means that it is $\mathcal{T}$–(co)cocomplete,
- Let κ be a regular cardinal. We say that $\underline{\mathcal{C}}$ strongly admits κ–small $\mathcal{T}$–(co)products to mean that it has $\mathcal{T}$–(co)limits for any diagram indexed over $\coprod_{a \in A} \underline{V_a}$ where A is a κ–small set. Hence, strongly admitting finite $\mathcal{T}$–(co)products means admitting finite fibrewise (co)products and (co)limits for all corepresentable diagrams $\underline{V}$.

Lemma 1.2.7 (Decomposition of indexed coproducts)**.** *Let $R_a, V \in \mathcal{T}$ and $\coprod_a f_a : \coprod_a R_a \to V$ be a map where the coproduct is not necessarily finite. Suppose $\underline{\mathcal{C}}$ strongly admits finite $\mathcal{T}$–coproducts and arbitrary fibrewise coproducts. Then $\underline{\mathcal{C}}$ admits $\coprod_a f_a$-coproducts and this is computed by composing fibrewise $\mathcal{T}$–coproduct $\coprod_a$ with the individual indexed $\mathcal{T}$–coproducts.*

Proof. We will in fact show that we have $\mathcal{T}_{/V}$–adjunctions

$$\underline{\mathrm{Fun}}_{\underline{V}}(\textstyle\coprod_a \underline{R_a}, \underline{\mathcal{C}}_{\underline{V}}) = \prod_a \underline{\mathrm{Fun}}_{\underline{V}}(\underline{R_a}, \underline{\mathcal{C}}_{\underline{V}}) \underset{\prod_a (f_a)^*}{\overset{\prod_a (f_a)_!}{\rightleftarrows}} \prod_a \underline{\mathrm{Fun}}_{\underline{V}}(\underline{V}, \underline{\mathcal{C}}_{\underline{V}}) \underset{\Delta}{\overset{\coprod_a}{\rightleftarrows}} \underline{\mathrm{Fun}}_{\underline{V}}(\underline{V}, \underline{\mathcal{C}}_{\underline{V}})$$

That these $\mathcal{T}_{/V}$–adjunctions exist is by our hypotheses, and all that is left to do is check that $\prod_a (f_a)^* \circ \Delta \simeq (\coprod_a f_a)^*$. But this is also clear since we have the commuting diagram

$$\begin{tikzcd} \coprod_a \underline{R_a} \arrow[r, "\coprod_a f_a"] \arrow[d, "f_a"'] & \underline{V} \\ \coprod_a \underline{V} \arrow[ur, "\coprod_a \mathrm{id}_V"'] & \end{tikzcd}$$

Applying $(-)^*$ to this triangle completes the proof. $\square$

Terminology 1.2.8 (Beck-Chevalley conditions)**.** Let $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$ that admits finite fibrewise coproducts (resp. products) and such that for each $f : W \to V$ in $\mathcal{T}$, $f^* : \mathcal{C}_V \to \mathcal{C}_W$ admits a left adjoint $f_!$ (resp. right adjoint f_*). We say that $\underline{\mathcal{C}}$ satisfies the *left Beck-Chevalley condition* (resp. *right Beck-Chevalley condition*) if for every pair of morphisms $f : W \to V$ and $g : Y \to V$ in $\mathcal{T}$: in the pullback (whose orbital decomposition exists by orbitality of $\mathcal{T}$)

$$\begin{tikzcd} \coprod_a R_a = Y \times_V W \arrow[r, "\coprod_a f_a"] \arrow[d, "\coprod_a g_a"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ W \arrow[r, "f"'] & V \end{tikzcd}$$

the canonical basechange transformation

$$\coprod_a g_{a!} f_a^* \Longrightarrow f^* g_! \quad \left(\text{resp. } f^* g_* \Longrightarrow \prod_a g_{a*} f_a^*\right)$$

is an equivalence.

Here is an omnibus of results due to Jay Shah.

Theorem 1.2.9 ([Sha22a, 5.5-5.12 and §12], [Nar17, Prop. 1.16])**.** *Let $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$. Then:*

(1) (Fibrewise criterion) $\underline{\mathcal{C}}$ strongly admits $\mathcal{T}$–colimits indexed by $\underline{\mathrm{const}}_{\mathcal{T}}(K)$ if and only if for every $V \in \mathcal{T}$ the fibre $\mathcal{C}_V$ has all colimits indexed by K and for every morphism $f : W \to V$ in $\mathcal{T}$ the cocartesian lift $f^* : \mathcal{C}_V \to \mathcal{C}_W$ preserves colimits indexed by K. A cocone diagram $\overline{p} : \underline{\mathrm{const}}_{\mathcal{T}}(K)^{\trianglerighteq} \to \underline{\mathcal{C}}$ is a $\mathcal{T}$–colimit if and only if it is so fibrewise.

(1) (T-coproducts criteria) $\underline{\mathcal{C}}$ strongly admits finite $\mathcal{T}$–coproducts if and only if we have:

(a) For every $W \in \mathcal{T}$ the fibre $\mathcal{C}_W$ has all finite coproducts and for every $f : W \to V$ in $\mathcal{T}$ the map $f^* : \mathcal{C}_V \to \mathcal{C}_W$ preserves finite coproducts,

(b) $\mathcal{C}$ satisfies the left Beck-Chevalley condition (cf. Terminology 1.2.8).

(3) (Decomposition principle) $\underline{\mathcal{C}}$ is $\mathcal{T}$–cocomplete if and only if it has all fibrewise colimits and strongly admits finite $\mathcal{T}$–coproducts.

Similar statements hold for $\mathcal{T}$–limits, and the right adjoint to f^* will be denoted f_*.

Theorem 1.2.10 (Omnibus $\mathcal{T}$–adjunctions, [Sha22a, §8])**.** *Let $F : \underline{\mathcal{C}} \rightleftarrows \underline{\mathcal{D}} : G$ be a $\mathcal{T}$–adjunction and $\underline{I}$ be a $\mathcal{T}$–category. Then:*

(1) *We get adjunctions* $F_*\colon \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I},\underline{\mathcal{C}}) \rightleftarrows \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I},\underline{\mathcal{D}}) : G_*$, $G^*\colon \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}},\underline{I}) \rightleftarrows \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{D}},\underline{I}) : F^*$. *By Corollary 1.1.25 this implies ordinary adjunctions when we replace* $\underline{\mathrm{Fun}}_{\mathcal{T}}$ *by* $\mathrm{Fun}_{\mathcal{T}}$.

(2) F *strongly preserves* $\mathcal{T}$*–colimits and* G *strongly preserves* $\mathcal{T}$*–limits.*

Proof. In [Sha22a, Cor. 8.9], part (2) was stated only as ordinary preservation, not strong preservation. But then strong preservation was implicit since relative adjunctions are stable under pullbacks by Proposition 1.1.22, and the statement in [Sha22a] also holds after pulling back to $- \times \underline{V}$ for all $V \in \mathcal{T}$. □

Proposition 1.2.11 ($\mathcal{T}$–cocompleteness of Bousfield local subcategories). *If* $L : \underline{\mathcal{C}} \rightleftarrows \underline{\mathcal{D}} : j$ *is a* $\mathcal{T}$*–Bousfield localisation where* $\underline{\mathcal{C}}$ *is* $\mathcal{T}$*–cocomplete, then* $\underline{\mathcal{D}}$ *is too and* $\mathcal{T}$*–colimits in* $\underline{\mathcal{D}}$ *is computed as* L *applied to the* $\mathcal{T}$*–colimit computed in* $\underline{\mathcal{C}}$.

Proof. This is an immediate consequence of Lemma 1.1.27. □

Proposition 1.2.12 ($\mathcal{T}$–(co)limits of functor categories is pointwise). *Let* $\underline{K}, \underline{I}, \underline{\mathcal{C}}$ *be* $\mathcal{T}$*–categories. Suppose* $\underline{\mathcal{C}}$ *strongly admits* $\underline{K}$*–indexed diagrams. Then so does* $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I},\underline{\mathcal{C}})$ *and the parametrised (co)limits are inherited from that of* $\underline{\mathcal{C}}$.

Proof. This is a direct consequence of the adjunction $(\underline{\mathrm{colim}}_{\underline{K}})_* : \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{K},\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I},\underline{\mathcal{C}})) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I},\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{K},\underline{\mathcal{C}})) \rightleftarrows \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I},\underline{\mathcal{C}}) : \mathrm{const}$ for $\mathcal{T}$–colimits. The other case is similar. □

Definition 1.2.13 ($\mathcal{T}$–Kan extensions). Let $j : \underline{I} \to \underline{K}$ be a $\mathcal{T}$–functor. If $j^* : \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{K},\underline{\mathcal{D}}) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I},\underline{\mathcal{D}})$ has a $\mathcal{T}$–left adjoint, then we denote it by $j_!$ and call it the *T-left Kan extension*. Similarly for $\mathcal{T}$–right Kan extensions.

Proposition 1.2.14 (Fully faithful $\mathcal{T}$–Kan extensions, [Sha22a, Prop. 10.6]). *Let* $i : \underline{\mathcal{C}} \hookrightarrow \underline{\mathcal{D}}$ *be a* $\mathcal{T}$*–fully faithful functor and* $F : \underline{\mathcal{C}} \to \underline{\mathcal{E}}$ *be another* $\mathcal{T}$*–functor. If the* $\mathcal{T}$*–left Kan extension* $i_!F$ *exists, then the adjunction unit* $F \Rightarrow i^*i_!F\colon \underline{\mathcal{C}} \to \underline{\mathcal{E}}$ *is an equivalence.*

Theorem 1.2.15 (Omnibus $\mathcal{T}$–Kan extensions, [Sha22a, Thm. 10.5]). *Let* $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$ *be* $\mathcal{T}$*–cocomplete. Then for every* $\mathcal{T}$*–functor of small* $\mathcal{T}$*–categories* $f : \underline{I} \to \underline{K}$*, the* $\mathcal{T}$*–left Kan extension* $f_! : \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I},\underline{\mathcal{C}}) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{K},\underline{\mathcal{C}})$ *exists.*

1.2.2 Strong preservation of $\mathcal{T}$-colimits

We now explain in more detail the notion of strong preservation. In particular, the reader may find Proposition 1.2.17 to be a convenient alternative description, and we will have many uses of it in the coming sections.

Observation 1.2.16 (Strong preservations vs preservations). Here are some comments for the distinction. Proposition 1.2.17 will then characterise strong preservations more concretely.

(1) Recall Warning 1.2.4 that admitting $\mathcal{T}$–colimits is weaker than being $\mathcal{T}$–cocomplete. In the proof of the Lurie-Simpson characterisation Theorem 2.2.2, we will see that we really need $\mathcal{T}$–cocompleteness via Proposition 1.2.17.

(2) However, $\underline{\mathcal{C}}$ admitting $\mathcal{T}$–colimits indexed by $p : \underline{K} \to \mathcal{T}^{\mathrm{op}}$ does imply $\underline{\mathcal{C}}_{\underline{V}}$ admits $\mathcal{T}_{/V}$–colimits indexed by $\underline{K}_{\underline{V}}$. This is because the adjunction $p_! \colon \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{K}, \underline{\mathcal{C}}) \rightleftarrows \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{*}, \underline{\mathcal{C}}) : p^*$ pulls back to the $p_! \colon \underline{\mathrm{Fun}}_{\underline{V}}(\underline{K}_{\underline{V}}, \underline{\mathcal{C}}_{\underline{V}}) \rightleftarrows \underline{\mathrm{Fun}}_{\underline{V}}(\underline{V}, \underline{\mathcal{C}}_{\underline{V}}) : p^*$ adjunction by Proposition 1.1.22. We have also used that functor $\mathcal{T}$–categories basechange well by Construction 1.1.13.

(3) Strongly preserving fibrewise $\mathcal{T}$–(co)limits is equivalent to preserving these (co)limits on each fibre since by Theorem 1.2.9 fibrewise (co)limits are constructed fibrewise.

The following result was also recorded in the recent [Sha22b, Thm. 8.6].

Proposition 1.2.17 (Characterisation of strong preservations)**.** *Let $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ be $\mathcal{T}$–cocomplete categories and $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ a $\mathcal{T}$–functor. Then F strongly preserves $\mathcal{T}$–colimits if and only if it preserves colimits in each fibre and for all $f : W \to V$ in $\mathcal{T}$, the following square commutes (and similarly for $\mathcal{T}$–limits)*

$$\begin{tikzcd}
\mathcal{C}_W \arrow[r, "f_!"] \arrow[d, "F_W"'] & \mathcal{C}_V \arrow[d, "F_V"] \\
\mathcal{D}_W \arrow[r, "f_!"] & \mathcal{D}_V
\end{tikzcd}$$

Proof. To see the only if direction, that F preserves colimits in each fibre is clear since F in particular preserves fibrewise $\mathcal{T}$–colimits. Now for $f : W \to V$, we basechange to $\underline{V}$. Since F strongly preserves $\mathcal{T}$–colimits, we get commutative squares

$$\begin{tikzcd}
\underline{\mathrm{Fun}}_{\underline{V}}(\underline{W}, \underline{\mathcal{C}} \times \underline{V}) \arrow[r, "f_!"] \arrow[d, "(F_{\underline{V}})_*"'] & \underline{\mathrm{Fun}}_{\underline{V}}(\underline{V}, \underline{\mathcal{C}} \times \underline{V}) \arrow[d, "(F_{\underline{V}})_*"] \\
\underline{\mathrm{Fun}}_{\underline{V}}(\underline{W}, \underline{\mathcal{D}} \times \underline{V}) \arrow[r, "f_!"] & \underline{\mathrm{Fun}}_{\underline{V}}(\underline{V}, \underline{\mathcal{D}} \times \underline{V})
\end{tikzcd}$$

Taking global sections by using that $\mathrm{Fun}_{\underline{V}}(\underline{W}, \underline{\mathcal{C}} \times \underline{V}) \simeq \mathrm{Fun}_{\mathcal{T}}(\underline{W}, \underline{\mathcal{C}}) \simeq \mathcal{C}_W$ from Example 1.1.4, we get the desired square.

For the if direction, we know by Theorem 1.2.9 that all $\mathcal{T}$–colimits can be decomposed as fibrewise $\mathcal{T}$–colimits and indexed $\mathcal{T}$–coproducts, and so if we show strong preservation of these we would be done. By Observation 1.2.16 (3) strong preservation of fibrewise $\mathcal{T}$–colimits is the same as preserving colimits in each fibre, so this case is covered. Since arbitrary indexed $\mathcal{T}$–coproducts are just compositions of orbital $\mathcal{T}$–coproducts and arbitrary fibrewise coproducts by Lemma 1.2.7, we need only show for orbital $\mathcal{T}$–coproducts, so let $f : W \to V$ be a morphism in $\mathcal{T}$. We need to show that the canonical comparison in

$$\begin{tikzcd}
\underline{\mathrm{Fun}}_{\underline{V}}(\underline{W}, \underline{\mathcal{C}} \times \underline{V}) \arrow[r, "f_!"] \arrow[d, "(F_{\underline{V}})_*"'] & \underline{\mathrm{Fun}}_{\underline{V}}(\underline{V}, \underline{\mathcal{C}} \times \underline{V}) \arrow[d, "(F_{\underline{V}})_*"] \\
\underline{\mathrm{Fun}}_{\underline{V}}(\underline{W}, \underline{\mathcal{D}} \times \underline{V}) \arrow[r, "f_!"] & \underline{\mathrm{Fun}}_{\underline{V}}(\underline{V}, \underline{\mathcal{D}} \times \underline{V})
\end{tikzcd}$$

is a natural equivalence. Since equivalences is by definition a fibrewise notion, we can check this on each fibre. So let $\varphi : Y \to V$ be in $\mathcal{T}$, and consider the pullback

$$\begin{tikzcd}
\coprod_a R_a \arrow[r, "\coprod f_a"] \arrow[d, "\coprod \varphi_a"'] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "\varphi"] \\
W \arrow[r, "f"] & V
\end{tikzcd}$$

by orbitality of $\mathcal{T}$. We need to show that

$$\begin{tikzcd}
\underline{\mathrm{Fun}}_{\underline{V}}(\underline{W}, \underline{\mathcal{C}} \times \underline{V})_Y \arrow[r, "f_!"] \arrow[d, "(F_{\underline{V}})_*"'] & \underline{\mathrm{Fun}}_{\underline{V}}(\underline{V}, \underline{\mathcal{C}} \times \underline{V})_Y \arrow[d, "(F_{\underline{V}})_*"] \\
\underline{\mathrm{Fun}}_{\underline{V}}(\underline{W}, \underline{\mathcal{D}} \times \underline{V})_Y \arrow[r, "f_!"] & \underline{\mathrm{Fun}}_{\underline{V}}(\underline{V}, \underline{\mathcal{D}} \times \underline{V})_Y
\end{tikzcd}$$

commutes. But then by the universal property of the internal functor $\mathcal{T}$–categories from Construction 1.1.13, this is the same as

$$\begin{tikzcd}
\mathrm{Fun}_{\underline{Y}}(\coprod_a \underline{R_a}, \underline{\mathcal{C}} \times \underline{Y}) \simeq \mathrm{Fun}_{\underline{Y}}(\underline{W} \times_{\underline{V}} \underline{Y}, \underline{\mathcal{C}} \times \underline{Y}) \arrow[r, "f_!"] \arrow[d, "(F_{\underline{Y}})_*"'] & \mathrm{Fun}_{\underline{Y}}(\underline{Y}, \underline{\mathcal{C}} \times \underline{Y}) \arrow[d, "(F_{\underline{Y}})_*"] \\
\mathrm{Fun}_{\underline{Y}}(\coprod_a \underline{R_a}, \underline{\mathcal{D}} \times \underline{Y}) \simeq \mathrm{Fun}_{\underline{Y}}(\underline{W} \times_{\underline{V}} \underline{Y}, \underline{\mathcal{D}} \times \underline{Y}) \arrow[r, "f_!"] & \mathrm{Fun}_{\underline{Y}}(\underline{Y}, \underline{\mathcal{D}} \times \underline{Y})
\end{tikzcd}$$

and this is in turn

$$\begin{tikzcd}
\prod_a \mathcal{C}_{R_a} \arrow[r, "\coprod (f_a)_!"] \arrow[d, "\prod F_a"'] & \mathcal{C}_Y \arrow[d, "F_V"] \\
\prod_a \mathcal{D}_{R_a} \arrow[r, "\coprod (f_a)_!"'] & \mathcal{D}_Y
\end{tikzcd}$$

which commutes by hypothesis together with that F commutes with fibrewise $\mathcal{T}$–colimits (and so in particular finite fibrewise coproducts). This finishes the proof of the result. □

1.2.3 Recollections: mapping spaces and Yoneda

Construction 1.2.18 (Parametrised mapping spaces and Yoneda, [BDG+16b, Def. 10.2])**.** Let $\underline{\mathcal{C}}$ be a $\mathcal{T}$–category. Then the $\mathcal{T}$–twisted arrow construction gives us a left $\mathcal{T}$–fibration

$$(s,t) : \underline{\mathrm{TwAr}}_{\mathcal{T}}(\underline{\mathcal{C}}) \longrightarrow \underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}}$$

$\mathcal{T}$–straightening this via Theorem 1.1.18 we get a $\mathcal{T}$–functor

$$\underline{\mathrm{Map}}_{\underline{\mathcal{C}}} : \underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}} \longrightarrow \underline{\mathcal{S}}_{\mathcal{T}}$$

By [BGN14, §5] we know that $\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(-,-) : \underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}} \to \underline{\mathcal{S}}_{\mathcal{T}}$ is given on fibre over V by the map $\mathcal{C}_V^{\mathrm{op}} \times \mathcal{C}_V \to \mathrm{Fun}((\mathcal{T}_{/V})^{\mathrm{op}}, \mathcal{S})$

$$(c,c') \mapsto \Big((W \xrightarrow{f} V) \mapsto (\mathrm{Map}_{\mathcal{C}_V}(c,c') \to \mathrm{Map}_{\mathcal{C}_W}(f^*c, f^*c')\Big)$$

Moreover, by currying we obtain the $\mathcal{T}$–Yoneda embedding

$$j : \underline{\mathcal{C}} \longrightarrow \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}) = \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}})$$

which on level $V \in \mathcal{T}$ is given by

$$\begin{aligned} j_V : \mathcal{C}_V \hookrightarrow \mathrm{Total}(\underline{\mathcal{C}}^{\mathrm{op}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Total}(\underline{V}) &\hookrightarrow \mathrm{Fun}_{\underline{V}}(\underline{\mathcal{C}}^{\mathrm{op}} \times \underline{V}, \underline{\mathcal{S}}_{\underline{V}}) \\ &\simeq \mathrm{Fun}(\mathrm{Total}(\underline{\mathcal{C}}^{\mathrm{op}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Total}(\underline{V}), \mathcal{S}) \end{aligned}$$

Remark 1.2.19. By the explicit fibrewise description of the parametrised mapping spaces above, we see immediately that a $\mathcal{T}$–functor $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is $\mathcal{T}$–fully faithful if and only if it induces equivalences on $\underline{\mathrm{Map}}(-,-)$.

Lemma 1.2.20 ($\mathcal{T}$–Yoneda Lemma, [BDG+16b, Prop. 10.3])**.** *Let $\underline{\mathcal{C}}$ be a $\mathcal{T}$–category and let $X \in \mathcal{C}_V$ for some $V \in \mathcal{T}$. Then for any $\mathcal{T}_{/V}$–functor $F : \underline{\mathcal{C}}^{\mathrm{op}} \times \underline{V} \longrightarrow \underline{\mathcal{S}}_{\underline{V}}$, we have an equivalence of $\mathcal{T}_{/V}$–spaces*

$$F(X) \simeq \underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\underline{V}}(\underline{\mathcal{C}}_{\underline{V}})}(j_V(X),\ F)$$

In particular, the $\mathcal{T}$–Yoneda embedding $j : \underline{\mathcal{C}} \longrightarrow \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ is $\mathcal{T}$–fully faithful.

Proof. First of all note that the V-fibre Yoneda map above factors as

$$\begin{tikzcd} \mathcal{C}_V \arrow[r, "j_V"] \arrow[d, hook] & \mathrm{Fun}(\mathrm{Total}(\underline{\mathcal{C}}^{\mathrm{op}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Total}(\underline{V}), \mathcal{S}) \\ (\mathrm{Total}(\underline{\mathcal{C}}^{\mathrm{op}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Total}(\underline{V}))^{\mathrm{op}} \arrow[ur, hook, "\tilde{j}_V"'] & \end{tikzcd}$$

This already gives that j_V is fully faithful, and so by definition of parametrised fully faithfulness, the $\mathcal{T}$–yoneda functor $j : \underline{\mathcal{C}} \to \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ is $\mathcal{T}$–fully faithful. On the other hand, by the universal property of the $\mathcal{T}$–category of $\mathcal{T}$–objects from Construction 1.1.16, we can regard F as an ordinary functor $F : \mathrm{Total}(\underline{\mathcal{C}}^{\mathrm{op}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Total}(\underline{V}) \to \mathcal{S}$. And so by ordinary Yoneda we get

$$\begin{aligned}\mathrm{Map}_{\mathrm{Fun}_{\underline{V}}(\underline{\mathcal{C}}^{\mathrm{op}} \times \underline{V}, \underline{\mathcal{S}}_{\underline{V}})}\big(j_V(X), F\big) &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{Total}(\underline{\mathcal{C}}^{\mathrm{op}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Total}(\underline{V}), \mathcal{S})}\big(j_V(X), F\big) \\ &\simeq F(X) \in \mathcal{S}\end{aligned}$$

as required. □

Theorem 1.2.21 (Continuity of $\mathcal{T}$–Yoneda, [Sha22a, Cor. 11.10])**.** *Let $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$. The $\mathcal{T}$–yoneda embedding $j : \underline{\mathcal{C}} \to \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ strongly preserves and detects $\mathcal{T}$–limits.*

Corollary 1.2.22. *Let $f : V \to W$ be a map in $\mathcal{T}$. Let $B \in \mathcal{C}_V$, $X \in \mathcal{C}_W$, and $f_! \dashv f^* \dashv f_*$. Then*

$$\underline{\mathrm{Map}}_{\underline{\mathcal{C}}_W}(f_!B, X) \simeq f_* \underline{\mathrm{Map}}_{\underline{\mathcal{C}}_V}(B, f^*X) \in \underline{\mathcal{S}}_W$$

Proof. Applying Theorem 1.2.21 on $\underline{\mathcal{C}}^{\mathrm{op}}$, we see that

$$\underline{\mathcal{C}}^{\mathrm{op}} \hookrightarrow \underline{\mathrm{Fun}}(\underline{\mathcal{C}}, \underline{\mathcal{S}}) \quad :: \quad A \mapsto \underline{\mathrm{Map}}_{\underline{\mathcal{C}}^{\mathrm{op}}}(-, A) \simeq \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(A, -)$$

strongly preserves $\mathcal{T}$–limits. Hence, since f_* in $\underline{\mathcal{C}}^{\mathrm{op}}$ is given by $f_!$ in $\underline{\mathcal{C}}$, we see that

$$\underline{\mathrm{Map}}_{\underline{\mathcal{C}}_W}(f_!B, -) \simeq \underline{\mathrm{Map}}_{\underline{\mathcal{C}}_W^{\mathrm{op}}}(-, f_!B) \simeq f_* \underline{\mathrm{Map}}_{\underline{\mathcal{C}}_V^{\mathrm{op}}}(-, B) \simeq f_* \underline{\mathrm{Map}}_{\underline{\mathcal{C}}_V}(B, -)$$

as required. □

Theorem 1.2.23 ($\mathcal{T}$–Yoneda density, [Sha22a, Lem. 11.1])**.** *Let $j : \underline{\mathcal{C}} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ be the $\mathcal{T}$–yoneda embedding. Then $\mathrm{id}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})} \simeq j_!j$, that is, everything in the $\mathcal{T}$–presheaf is a $\mathcal{T}$–colimit of representables.*

Theorem 1.2.24 (Universal property of $\mathcal{T}$–presheaves, [Sha22a, Thm. 11.5])**.** *Let $\underline{\mathcal{C}}, \underline{\mathcal{D}} \in \mathrm{Cat}_{\mathcal{T}}$ and suppose $\underline{\mathcal{D}}$ is $\mathcal{T}$–cocomplete. Then the precompositions $j^* : \mathrm{Fun}^L_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}), \underline{\mathcal{D}}) \longrightarrow \mathrm{Fun}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ and $j^* : \underline{\mathrm{Fun}}^L_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}), \underline{\mathcal{D}}) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ are equivalences with the inverse given by left Kan extensions. Here $\mathrm{Fun}^L_{\mathcal{T}}$ means those functors which strongly preserve $\mathcal{T}$–colimits (cf. Notation 1.2.27).*

We learnt of the following useful procedure from Fabian Hebestreit.

Definition 1.2.25 (Adjoint objects)**.** Let $R : \underline{\mathcal{D}} \to \underline{\mathcal{C}}$ be a $\mathcal{T}$–functor. Let $x \in \underline{\mathcal{C}}$ and $y \in \underline{\mathcal{D}}$ and $\eta : x \to R(y)$. We say that η *witnesses y as a left adjoint object to x under R* if

$$\underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(y, -) \xrightarrow{R} \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(Ry, R-) \xrightarrow{\eta^*} \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(x, R-)$$

is an equivalence of $\mathcal{T}$–functors $\underline{\mathcal{D}} \to \underline{\mathcal{S}}_{\mathcal{T}}$.

The following observation, due to Lurie, is quite surprising for ∞-categories: adjunctions can be constructed objectwise, ie. to check that we have an adjunction, it is enough to construct a left adjoint object for each object.

Proposition 1.2.26 (Pointwise construction of adjunctions)**.** *$R : \underline{\mathcal{D}} \to \underline{\mathcal{C}}$ admits a left adjoint $L : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ if and only if all objects in $\underline{\mathcal{C}}$ admits a left adjoint object. More generally, writing $\underline{\mathcal{C}}_R$ for the full subcategory of objects admitting left adjoint objects, we obtain a $\mathcal{T}$–functor $L : \underline{\mathcal{C}}_R \to \underline{\mathcal{D}}$ that is $\mathcal{T}$–left adjoint to the restriction of $R : \underline{\mathcal{D}} \to \underline{\mathcal{C}}$ to the subcategory of $\underline{\mathcal{D}}$ landing in $\underline{\mathcal{C}}_R$.*

Proof. The trick is to use the $\mathcal{T}$–Yoneda lemma to help us assemble the various left adjoint objects into a coherent $\mathcal{T}$–functor. We consider $\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(-, R-) : \underline{\mathcal{C}}^{\underline{\mathrm{op}}} \times \underline{\mathcal{D}} \to \underline{\mathcal{S}}_{\mathcal{T}}$ as a $\mathcal{T}$–functor $H : \underline{\mathcal{C}}^{\underline{\mathrm{op}}} \to \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{D}}, \underline{\mathcal{S}}_{\mathcal{T}})$. Hence by definition of $\underline{\mathcal{C}}_R$, the bottom left composition lands in the essential image of the Yoneda embedding and so we obtain a lift $L^{\underline{\mathrm{op}}}$ in the commuting square

$$\begin{tikzcd}
\underline{\mathcal{C}}_R^{\underline{\mathrm{op}}} \arrow[r, "L^{\underline{\mathrm{op}}}"] \arrow[d, hook] & \underline{\mathcal{D}}^{\underline{\mathrm{op}}} \arrow[d, hook, "y"] \\
\underline{\mathcal{C}}^{\underline{\mathrm{op}}} \arrow[r, "H"] & \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{D}}, \underline{\mathcal{S}}_{\mathcal{T}})
\end{tikzcd}$$

To see that when $\underline{\mathcal{C}}_R = \underline{\mathcal{C}}$, we get a $\mathcal{T}$–left adjoint, note that by construction $y \circ L^{\underline{\mathrm{op}}} \simeq H$ in $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \mathrm{Fun}(\underline{\mathcal{D}}, \underline{\mathcal{S}}_{\mathcal{T}}))$, and hence $\underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(L-, -) \simeq \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(-, R-)$ in $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}} \times \underline{\mathcal{D}}, \underline{\mathcal{S}}_{\mathcal{T}})$. By the characterisation of $\mathcal{T}$–adjunctions from Lemma 1.1.27, we are done. □

Notation 1.2.27. We write $\underline{\mathrm{RFun}}_{\mathcal{T}}$ (resp. $\underline{\mathrm{LFun}}_{\mathcal{T}}$) for the $\mathcal{T}$–full subcategories in $\underline{\mathrm{Fun}}_{\mathcal{T}}$ of $\mathcal{T}$–right adjoint functors (resp. $\mathcal{T}$–left adjoint functors). This is distinguished from the notations $\underline{\mathrm{Fun}}^R_{\mathcal{T}}$ (resp. $\underline{\mathrm{Fun}}^L_{\mathcal{T}}$) by which we mean the $\mathcal{T}$–full subcategories in $\underline{\mathrm{Fun}}_{\mathcal{T}}$ of strongly $\mathcal{T}$–limit-preserving functors (resp. strongly $\mathcal{T}$–colimit-preserving functors).

Proposition 1.2.28 ("[Lur09, Prop. 5.2.6.2]")**.** *Let $\underline{\mathcal{C}}, \underline{\mathcal{D}} \in \mathrm{Cat}_{\mathcal{T}}$. Then there is a canonical equivalence $\underline{\mathrm{LFun}}_{\mathcal{T}}(\underline{\mathcal{D}}, \underline{\mathcal{C}}) \simeq \underline{\mathrm{RFun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})^{\underline{\mathrm{op}}}$.*

Proof. Let $j : \underline{\mathcal{D}} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})$ be the $\mathcal{T}$–Yoneda embedding. Then the $\mathcal{T}$–functor

$$j_* : \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \hookrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}} \times \underline{\mathcal{D}}^{\underline{\mathrm{op}}}, \underline{\mathcal{S}}_{\mathcal{T}})$$

which is $\mathcal{T}$–fully faithful by Corollary 1.2.34 has essential image consisting of those parametrised functors $\varphi : \underline{\mathcal{C}} \times \underline{\mathcal{D}}^{\underline{\mathrm{op}}} \to \underline{\mathcal{S}}_{\mathcal{T}}$ such that for all $c \in \underline{\mathcal{C}}$, $\varphi(c, -) : \underline{\mathcal{D}}^{\underline{\mathrm{op}}} \to \underline{\mathcal{S}}_{\mathcal{T}}$ is representable. The essential image under j_* of $\underline{\mathrm{RFun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \subseteq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ will then be those parametrised functors as above which moreover satisfy that for all $d \in \underline{\mathcal{D}}$, $\varphi(-, d) : \underline{\mathcal{C}} \to \underline{\mathcal{S}}_{\mathcal{T}}$ is corepresentable - this is since $\mathcal{T}$–adjunctions can be constructed objectwise by Proposition 1.2.26. Let $\underline{\mathcal{E}} \subseteq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}} \times \underline{\mathcal{D}}^{\underline{\mathrm{op}}}, \underline{\mathcal{S}}_{\mathcal{T}})$ be the

$\mathcal{T}$–full subcategory spanned by those functors satisfying these two properties, so that $\underline{\mathrm{RFun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \xrightarrow{\simeq} \underline{\mathcal{E}}$.

On the other hand, repeating the above for $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{D}}^{\underline{\mathrm{op}}}, \underline{\mathcal{C}}^{\underline{\mathrm{op}}})$ gives

$$\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{D}}^{\underline{\mathrm{op}}}, \underline{\mathcal{C}}^{\underline{\mathrm{op}}}) \hookrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{D}}^{\underline{\mathrm{op}}} \times \underline{\mathcal{C}}, \underline{\mathcal{S}}_{\mathcal{T}})$$

where the essential image of $\underline{\mathrm{RFun}}_{\mathcal{T}}(\underline{\mathcal{D}}^{\underline{\mathrm{op}}}, \underline{\mathcal{C}}^{\underline{\mathrm{op}}})$ will be precisely those that satisfy the two properties, and so also $\underline{\mathrm{RFun}}_{\mathcal{T}}(\underline{\mathcal{D}}^{\underline{\mathrm{op}}}, \underline{\mathcal{C}}^{\underline{\mathrm{op}}}) \xrightarrow{\simeq} \underline{\mathcal{E}}$. Thus, combining with $\underline{\mathrm{RFun}}_{\mathcal{T}}(\underline{\mathcal{D}}^{\underline{\mathrm{op}}}, \underline{\mathcal{C}}^{\underline{\mathrm{op}}}) \simeq \underline{\mathrm{LFun}}_{\mathcal{T}}(\underline{\mathcal{D}}, \underline{\mathcal{C}})^{\underline{\mathrm{op}}}$ from Observation 1.1.15, we obtain the desired result. □

1.2.4 (Full) faithfulness

In this subsection we provide the parametrised analogue of the Lurie-Thomason formula for limits in categories, Theorem 1.2.30, as well as show that parametrised functor categories preserve (fully) faithfulness in Corollary 1.2.34.

Notation 1.2.29. For $p : \underline{\mathcal{C}} \to \underline{I}$ a $\mathcal{T}$–functor which is also a cocartesian fibration, we will write $\underline{\Gamma}^{\mathrm{cocart}}_{\mathcal{T}}(p)$ for the $\mathcal{T}$–category of cocartesian sections of p. In other words, it is the $\mathcal{T}$–category $\underline{\mathrm{Fun}}^{\mathrm{cocart}}_{\mathcal{T}}(\underline{I}, \underline{\mathcal{C}}) \times_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I}, \underline{I})} \underline{*}$ where $\underline{\mathrm{Fun}}^{\mathrm{cocart}}_{\mathcal{T}}(\underline{I}, \underline{\mathcal{C}})$ means the full $\mathcal{T}$–subcategory of those that parametrised functors that preserve cocartesian morphisms over $\underline{I}$, and the $\mathcal{T}$–functor $\underline{*} \to \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I}, \underline{I})$ is the section corresponding to the identity on $\underline{I}$.

The following proof is just a parametrisation of the unparametrised proof that we learnt from [HW21, Prop. I.36].

Theorem 1.2.30 (Lurie-Thomason formula)**.** *Given a $\mathcal{T}$–diagram $F : \underline{I} \to \underline{\mathrm{Cat}}_{\mathcal{T}}$, we get*

$$\underline{\lim}_{\underline{I}} F \simeq \underline{\Gamma}^{\underline{I}-\mathrm{cocart}}_{\mathcal{T}}(\mathrm{UnStr}^{\mathrm{cocart}}(F))$$

In particular, if it factors through $F : \underline{I} \to \underline{\mathcal{S}}_{\mathcal{T}}$, then we have $\underline{\lim}_{\underline{I}} F \simeq \underline{\Gamma}_{\mathcal{T}}(\mathrm{UnStr}^{\mathrm{cocart}}(F))$.

Proof. Let $d : \underline{I} \to \underline{*}$ be the unique map. Since $\underline{\mathrm{Cat}}_{\mathcal{T}}$ has all $\mathcal{T}$–limits, we know abstractly that we have the $\mathcal{T}$–right adjoint

$$d^* : \underline{\mathrm{Cat}}_{\mathcal{T}} \rightleftarrows \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I}, \underline{\mathrm{Cat}}_{\mathcal{T}}) : d_* =: \underline{\lim}_{\mathcal{T}}$$

so now we just need to understand the fibrewise right adjoint formula (by virtue of Corollary 1.1.25). Without loss of generality, we work with global sections and we want to describe the right adjoint in

$$d^* : \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat}) \rightleftarrows \mathrm{Fun}_{\mathcal{T}}(\underline{I}, \underline{\mathrm{Cat}}_{\mathcal{T}}) \simeq \mathrm{Fun}(\mathrm{Total}(\underline{I}), \mathrm{Cat}) : d_*$$

We can now identify d^* concretely via the straightening-unstraightening equivalence to get $d^*\colon \mathrm{coCart}(\mathcal{T}^{\mathrm{op}}) \to \mathrm{coCart}(\mathrm{Total}(\underline{I}))$ given by

$$\underline{\mathcal{C}} \mapsto \big(\pi_{\mathcal{C}} : \mathrm{Total}(\underline{\mathcal{C}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Total}(\underline{I}) \to \mathrm{Total}(\underline{I})\big)$$

Let $(p : \mathcal{E}_F \to \mathrm{Total}(\underline{I})) := \mathrm{UnStr}^{\mathrm{coCart}}(F)$ be the cocartesian fibration associated to $F : \mathrm{Total}(\underline{I}) \to \mathrm{Cat}$. We need to show that $\underline{\Gamma}_T^{\underline{I}-\mathrm{cocart}}(p)$ satisfies a natural equivalence

$$\mathrm{Map}_{\mathrm{coCart}(\mathrm{Total}(\underline{I}))}(\pi_{\mathcal{C}}, p) \simeq \mathrm{Map}_{\mathrm{coCart}(\mathcal{T}^{\mathrm{op}})}(\underline{\mathcal{C}}, \underline{\Gamma}_T^{\underline{I}-\mathrm{cocart}}(p))$$

for all $\underline{\mathcal{C}} \in \mathrm{coCart}(\mathcal{T}^{\mathrm{op}})$. First of all, by definition we have the pullback

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{coCart}(\mathcal{T}^{\mathrm{op}})_{/\mathrm{Total}(\underline{I})}}(\pi_{\mathcal{C}}, p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Map}_{\mathrm{coCart}(\mathcal{T}^{\mathrm{op}})}(\mathrm{Total}(\underline{\mathcal{C}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Total}(\underline{I}), \mathcal{E}_F) \arrow[d] \\
* \arrow[r, "\pi_{\mathcal{C}}"] & \mathrm{Map}_{\mathrm{coCart}(\mathcal{T}^{\mathrm{op}})}(\mathrm{Total}(\underline{\mathcal{C}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Total}(\underline{I}), \mathrm{Total}(\underline{I}))
\end{tikzcd}$$

which by currying is the same as the pullback

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{coCart}(\mathcal{T}^{\mathrm{op}})_{/\mathrm{Total}(\underline{I})}}(\pi_{\mathcal{C}}, p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Map}_{\mathrm{coCart}(\mathcal{T}^{\mathrm{op}})}(\underline{\mathcal{C}}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I}, \mathcal{E}_F)) \arrow[d] \\
* \arrow[r, "\mathrm{id}_{\underline{I}}"] & \mathrm{Map}_{\mathrm{coCart}(\mathcal{T}^{\mathrm{op}})}(\underline{\mathcal{C}}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I}, \underline{I}))
\end{tikzcd}$$

Now recall that $\mathrm{Map}_{\mathrm{coCart}(\mathrm{Total}(\underline{I}))}(\pi_{\mathcal{C}}, p) \subseteq \mathrm{Map}_{\mathrm{coCart}(\mathcal{T}^{\mathrm{op}})_{\mathrm{Total}(\underline{I})}}(\pi_{\mathcal{C}}, p)$ consists precisely of those components of functors over $\mathrm{Total}(\underline{I})$ (in the left diagram)

$$\begin{tikzcd}
\mathrm{Total}(\underline{\mathcal{C}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Total}(\underline{I}) \arrow[r] \arrow[dr, "\pi_{\mathcal{C}}"'] & \mathcal{E}_F \arrow[d, "p"] \\
& \mathrm{Total}(\underline{I})
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[dr, "\pi_{\mathcal{C}}"'] & \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I}, \mathcal{E}_F) \arrow[d, "p"] \\
& \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I}, \underline{I})
\end{tikzcd}$$

preserving cocartesian morphisms over $\mathrm{Total}(\underline{I})$. Since the cocartesian morphisms in the cocartesian fibration $\pi_{\mathcal{C}} : \mathrm{Total}(\underline{\mathcal{C}}) \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Total}(\underline{I}) \to \mathrm{Total}(\underline{I})$ are precisely the morphisms of $\mathrm{Total}(\underline{I})$ and an equivalence in $\mathcal{C}$, we see that this condition corresponds in the curried version on the right to those functors landing in $\underline{\mathrm{Fun}}_{\mathcal{T}}^{\underline{I}-\mathrm{cocart}}(\underline{I}, \mathcal{E}_F)$. Finally for the statement about the case of factoring over $\underline{\mathcal{S}}_{\mathcal{T}}$ recall that unstraightening brings us to left fibrations $\mathcal{E}_F \to \mathrm{Total}(\underline{I})$, and since in left fibrations all morphisms are cocartesian, we need not have imposed the condition above. This shows us that we have a bijection of components

$$\pi_0 \mathrm{Map}_{\mathrm{coCart}(\mathrm{Total}(\underline{I}))}(\pi_{\mathcal{C}}, p) \simeq \pi_0 \mathrm{Map}_{\mathrm{coCart}(\mathcal{T}^{\mathrm{op}})}(\underline{\mathcal{C}}, \underline{\Gamma}_T^{\underline{I}-\mathrm{cocart}}(p))$$

We now need to show that this would already imply that we have an equivalence of mapping *spaces*. For this, we will need to first construct a map of spaces realising the bijection above. First note that we have a map of cocartesian fibrations over $\mathrm{Total}(\underline{I})$

$$\varepsilon : \underline{\Gamma}_{\mathcal{T}}^{\underline{I}-\mathrm{cocart}}(p) \times \underline{I} \longrightarrow \mathcal{E}_F$$

from the evaluation. Therefore we get the following maps of spaces

$$\begin{aligned}
&\mathrm{Map}_{\mathrm{coCart}(\mathcal{T}^{\mathrm{op}})}(-, \underline{\Gamma}_{\mathcal{T}}^{\underline{I}-\mathrm{cocart}}(p)) \\
&\xrightarrow{\underline{I}\times -} \mathrm{Map}_{\mathrm{coCart}(\mathrm{Total}(\underline{I}))}(\underline{I} \times -, \underline{I} \times \underline{\Gamma}_{\mathcal{T}}^{\underline{I}-\mathrm{cocart}}(p)) \\
&\xrightarrow{\varepsilon_*} \mathrm{Map}_{\mathrm{coCart}(\mathrm{Total}(\underline{I}))}(\underline{I} \times -, \mathcal{E}_F)
\end{aligned} \tag{1.1}$$

On the other hand, we know by the pullback definition of $\underline{\Gamma}_{\mathcal{T}}$ that

$$\mathrm{Map}_{\mathrm{coCart}(\mathcal{T}^{\mathrm{op}})}(-, \underline{\Gamma}_{\mathcal{T}}(p)) \simeq \mathrm{Map}_{\mathrm{coCart}(\mathcal{T}^{\mathrm{op}})_{/\,\mathrm{Total}(\underline{I})}}(\underline{I} \times -, \mathcal{E}_F) \tag{1.2}$$

and so the comparison map Eq. (1.1) is induced by this equivalence. Our bijection on components then gives that the equivalence Eq. (1.2) restricts to an equivalence of spaces Eq. (1.1). This completes the proof of the result. $\square$

As far as we are aware the following proof strategy first appeared in [GHN17, §5].

Proposition 1.2.31 (Mapping space formula in $\mathcal{T}$–functor categories). *Let $\underline{\mathcal{C}}, \underline{\mathcal{D}} \in \mathrm{Cat}_{\mathcal{T}}$ and $F, G : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be $\mathcal{T}$–functors. Then we have an equivalence of $\mathcal{T}$–spaces*

$$\underline{\mathrm{Nat}}_{\mathcal{T}}(F, G) \simeq \underline{\lim}_{(x\to y)\in \underline{\mathrm{TwAr}}_{\mathcal{T}}(\underline{\mathcal{C}})} \underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(F(x), G(y)) \in \underline{\mathcal{S}}_{\mathcal{T}}$$

Proof. Recall from [BDG+16b, §10] that by definition, the parametrised mapping spaces are classified by the parametrised twisted arrow categories. By Theorem 1.2.30 we have

$$\underline{\lim}_{(x\to y)\in \underline{\mathrm{TwAr}}(\underline{\mathcal{C}})} \underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(F(x), G(y)) \simeq \underline{\Gamma}_{\mathcal{T}}(\underline{P} \to \underline{\mathrm{TwAr}}_{\mathcal{T}}(\underline{\mathcal{C}}))$$

where $p : \underline{P} \to \underline{\mathrm{TwAr}}_{\mathcal{T}}(\underline{\mathcal{C}})$ is the associated unstraightening. By considering the pullbacks

$$\begin{tikzcd}
& \underline{P} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner" very near start] & & \underline{P}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \underline{\mathrm{TwAr}}_{\mathcal{T}}(\underline{\mathcal{D}}) \arrow[d] & \\
& \underline{\mathrm{TwAr}}_{\mathcal{T}}(\underline{\mathcal{C}}) \arrow[rr, "(s,t)"'] & & \underline{\mathcal{C}}^{\underline{\mathrm{op}}} \times \underline{\mathcal{C}} \arrow[r, "F^{\underline{\mathrm{op}}} \times G"'] & \underline{\mathcal{D}}^{\underline{\mathrm{op}}} \times \underline{\mathcal{D}} \arrow[r, "\underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(-,-)"'] & \underline{\mathcal{S}}_{\mathcal{T}}
\end{tikzcd}$$

we get that

$$\underline{\Gamma}_{\mathcal{T}}(\underline{P} \to \underline{\mathrm{TwAr}}_{\mathcal{T}}(\underline{\mathcal{C}})) \simeq \underline{\mathrm{Map}}_{/\underline{\mathcal{C}}^{\underline{\mathrm{op}}} \times \underline{\mathcal{C}}}(\underline{\mathrm{TwAr}}_{\mathcal{T}}(\underline{\mathcal{C}}), \underline{P}')$$

Now by the parametrised straightening of Theorem 1.1.18 we see furthermore that

$$\underline{\mathrm{Map}}_{/\underline{\mathcal{C}}^{\mathrm{op}}\times\underline{\mathcal{C}}}(\underline{\mathrm{TwAr}}_{\mathcal{T}}(\underline{\mathcal{C}}), \underline{P}') \simeq \underline{\mathrm{Nat}}_{\mathcal{T}}\big(\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}, \underline{\mathrm{Map}}_{\underline{\mathcal{D}}} \circ (F^{\underline{\mathrm{op}}} \times G)\big)$$

Currying $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}} \times \underline{\mathcal{C}}, \underline{\mathcal{S}}_{\mathcal{T}}) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}))$ we see that

$$\underline{\mathrm{Nat}}_{\mathcal{T}}\big(\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}, \underline{\mathrm{Map}}_{\underline{\mathcal{D}}} \circ (F^{\underline{\mathrm{op}}} \times G)\big) \simeq \underline{\mathrm{Nat}}_{\mathcal{T}}\big(y_{\underline{\mathcal{C}}}, F^* \circ y_{\underline{\mathcal{D}}} \circ G)\big)$$

But then now we have the sequence of equivalences

$$\begin{aligned}
\underline{\mathrm{Nat}}_{\mathcal{T}}\big(y_{\underline{\mathcal{C}}}, F^* \circ y_{\underline{\mathcal{D}}} \circ G)\big) &\simeq \underline{\mathrm{Nat}}_{\mathcal{T}}\big(F_! \circ y_{\underline{\mathcal{C}}}, y_{\underline{\mathcal{D}}} \circ G)\big) \\
&\simeq \underline{\mathrm{Nat}}_{\mathcal{T}}\big(y_{\underline{\mathcal{D}}} \circ F, y_{\underline{\mathcal{D}}} \circ G)\big) \\
&\simeq \underline{\mathrm{Nat}}_{\mathcal{T}}(F, G)
\end{aligned}$$

where the last equivalence is by Lemma 1.2.20 and the second by the square

$$\begin{tikzcd}
\underline{\mathcal{C}} \arrow[r, "F"] \arrow[d, hook, "y_{\underline{\mathcal{C}}}"'] & \underline{\mathcal{D}} \arrow[d, hook, "y_{\underline{\mathcal{D}}}"] \\
\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}) \arrow[r, "F_!"] & \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})
\end{tikzcd}$$

which commutes by functoriality of presheaves. □

Definition 1.2.32. A $\mathcal{T}$–functor is called $\mathcal{T}$-faithful if it is so fibrewise, where an ordinary functor is called faithful if it induces component inclusions on mapping spaces.

Observation 1.2.33. For $f : X \to Y$ a map of spaces, it being an inclusion of components is equivalent to the condition that for each $x \in X$, the fibre $\mathrm{fib}_{f(x)}\big(X \to Y\big)$ is contractible. On the other hand, it is an equivalence if and only if for each $y \in Y$, the fibre $\mathrm{fib}_y\big(f : X \to Y\big)$ is contractible. We learnt of this formulation and of the following proof in the unparametrised case from [Lei22, Appendix B].

Corollary 1.2.34. *Let $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a $\mathcal{T}$–(fully) faithful functor and $\underline{I}$ another $\mathcal{T}$–category. Then $F_* : \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I}, \underline{\mathcal{C}}) \to \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I}, \underline{\mathcal{D}})$ is again $\mathcal{T}$–(fully) faithful.*

Proof. Since $\mathcal{T}$–(fully) faithfulness was defined as a fibrewise condition, we just assume without loss of generality that $\mathcal{T}$ has a final object and work on global sections. In the faithful case, let $\varphi, \psi : \underline{I} \to \underline{\mathcal{C}}$ be two $\mathcal{T}$–functors. We need to show that

$$\underline{\mathrm{Nat}}_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I},\underline{\mathcal{C}})}(\varphi, \psi) \longrightarrow \underline{\mathrm{Nat}}_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I},\underline{\mathcal{D}})}(F\varphi, F\psi)$$

is an inclusion of components. By the preceeding observation, we need to show that for each $\eta \in \mathrm{Nat}_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I},\underline{\mathcal{C}})}(\varphi, \psi)$, the fibre

$$\underline{\mathrm{fib}}_{\eta}\big(\underline{\mathrm{Nat}}_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I},\underline{\mathcal{C}})}(\varphi, \psi) \to \underline{\mathrm{Nat}}_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I},\underline{\mathcal{D}})}(F\varphi, F\psi)\big) \in \Gamma(\underline{\mathcal{S}}_{\mathcal{T}} \to \mathcal{T}^{\mathrm{op}})$$

is contractible. But then we are now in position to use Proposition 1.2.31:

$$\begin{aligned}
&\underline{\mathrm{fib}}_\eta\Big(\underline{\mathrm{Nat}}_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I},\underline{\mathcal{C}})}(\varphi,\psi) \to \underline{\mathrm{Nat}}_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{I},\underline{\mathcal{D}})}(F\varphi,F\psi)\Big)\\
&\simeq \underline{\lim}_{(x\to y)\in\underline{\mathrm{TwAr}}_{\mathcal{T}}(\underline{I})}\underline{\mathrm{fib}}_\eta\Big(\underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(\varphi(x),\psi(y)) \to \underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(F\varphi(x),G\psi(y))\Big)\\
&\simeq \underline{\lim}_{(x\to y)\in\underline{\mathrm{TwAr}}_{\mathcal{T}}(\underline{I})} *_{\mathcal{T}} \simeq *_{\mathcal{T}}
\end{aligned}$$

as was to be shown, where the second last step is by our hypothesis that F was $\mathcal{T}$–faithful. The case of $\mathcal{T}$–fully faithfulness can be done similarly. □

1.2.5 Recollections: filtered colimits and Ind-completions

Construction 1.2.35. Let κ be a regular cardinal. We define the *$\mathcal{T}$-Ind-completion* functor $\underline{\mathrm{Ind}}_\kappa\colon \mathrm{Cat}_{\mathcal{T}} \to \mathrm{Cat}_{\mathcal{T}}$ to be the one obtained by applying $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}},-)$ to the ordinary functor $\mathrm{Ind}_\kappa\colon \mathrm{Cat} \to \mathrm{Cat}$.

Notation 1.2.36. We will write $\underline{\mathrm{Fun}}^{\mathrm{filt}}_{\mathcal{T}}$ for the full $\mathcal{T}$–subcategory of parametrised functors preserving fibrewise ω–filtered colimits, and similarly $\underline{\mathrm{Fun}}^{\kappa\text{-filt}}_{\mathcal{T}}$ for those that preserve fibrewise κ-filtered colimits.

Remark 1.2.37. This agrees with the definition given in the recent paper [Sha22b] by virtue of the paragraph after Theorem D therein. As indicated there, $\underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}})$ is the minimal $\mathcal{T}$–subcategory of $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ generated by $\underline{\mathcal{C}}$ under fibrewise κ-filtered colimits. In more detail, [Sha22b, Rmk. 9.4] showed that the fibrewise presheaf construction $\underline{\mathrm{PSh}}^{\mathrm{fb}}_{\mathcal{T}}(\underline{\mathcal{C}})$ is a $\mathcal{T}$–full subcategory of the $\mathcal{T}$–presheaf $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ via the fibrewise left Kan extension. In particular, this means that $\underline{\mathrm{PSh}}^{\mathrm{fb}}_{\mathcal{T}}(\underline{\mathcal{C}}) \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ preserves fibrewise colimits. On the other hand, by construction and [Lur09, Cor. 5.3.5.4], $\underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}}) \subseteq \underline{\mathrm{PSh}}^{\mathrm{fb}}_{\mathcal{T}}(\underline{\mathcal{C}})$ is the minimal $\mathcal{T}$–subcategory generated by $\underline{\mathcal{C}}$ under fibrewise κ-filtered colimits. Therefore, in total, we see that $\underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}}) \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ is the $\mathcal{T}$–subcategory generated by $\mathcal{C}$ under fibrewise κ-filtered colimits.

Proposition 1.2.38 (Universal property of $\underline{\mathrm{Ind}}$, "[Lur09, Prop. 5.3.5.10]")**.** *Let $\underline{\mathcal{C}},\underline{\mathcal{D}}$ be $\mathcal{T}$–categories where $\underline{\mathcal{C}}$ is small and $\underline{\mathcal{D}}$ has fibrewise small κ-filtered colimits. Then:*

(1) $\underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}}) \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ is the $\mathcal{T}$–subcategory generated by $\mathcal{C}$ under fibrewise κ-filtered colimits.

(2) The $\mathcal{T}$–inclusion $i : \underline{\mathcal{C}} \hookrightarrow \underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}})$ induces an equivalence

$$i^* : \underline{\mathrm{Fun}}^{\kappa\text{-filt}}_{\mathcal{T}}(\underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}}),\underline{\mathcal{D}}) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}},\underline{\mathcal{D}})$$

Proof. Part (1) is by the remark above. For part (2), we show that the $\mathcal{T}$–left Kan extension functor $i_! : \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}},\underline{\mathcal{D}}) \longrightarrow \underline{\mathrm{Fun}}^{\kappa\text{-filt}}_{\mathcal{T}}(\underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}}),\underline{\mathcal{D}})$ exists and is an inverse to i^*. To do this, it will be enough to show that functors $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ can be $\mathcal{T}$–left Kan extended to $i_!F : \underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}}) \to \underline{\mathcal{D}}$ and that functors $F : \underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}}) \to \underline{\mathcal{D}}$ which preserves fibrewise κ-filtered colimits satisfy that $i_!i^*F \Rightarrow F$ is an equivalence. This will be enough since we would have shown the natural equivalence $i_!i^* \simeq \mathrm{id}$, and Proposition 1.2.14 gives that $i^*i_! \simeq \mathrm{id}$ always.

To show that the $\mathcal{T}$–left Kan extension exists, consider the diagram

$$\begin{tikzcd}
\underline{\mathcal{C}} \arrow[d, hook] \arrow[dr, "f"] & & \\
\underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}}) \arrow[r, dashed] \arrow[d, hook] & \underline{\mathcal{D}} \arrow[dr, hook] & \\
\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}) \arrow[rr, dashed] & & \underline{\mathcal{D}}'
\end{tikzcd}$$

where $\underline{\mathcal{D}} \subseteq \underline{\mathcal{D}}'$ is a strongly $\mathcal{T}$–colimit preserving inclusion into a $\mathcal{T}$–cocomplete $\underline{\mathcal{D}}'$ using the opposite $\mathcal{T}$–Yoneda embedding. In particular by hypothesis $\underline{\mathcal{D}}$ is closed under κ-filtered colimits in $\underline{\mathcal{D}}'$. The bottom dashed map is gotten from Theorem 1.2.24, and so strongly preserves $\mathcal{T}$–colimits. Hence restriction to $\underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}})$ lands in $\underline{\mathcal{D}}$ so we get middle dashed map, and by the following Lemma 1.2.39, this is a left Kan extension.

Now we show that if F preserves fibrewise κ-filtered colimits, then the canonical comparison $i_!i^*F \Rightarrow F$ is an equivalence. Again, by Proposition 1.2.14 we know that both sides agree on $\underline{\mathcal{C}} \subseteq \underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}})$. Also, both sides preserve κ-filtered colimits by assumption. Hence, by statement (1) of the proposition, we see that it must be an equivalence as was to be shown. □

Lemma 1.2.39. *Suppose we have fully faithful functors* $\underline{\mathcal{C}} \xhookrightarrow{i} \underline{\mathcal{D}} \xhookrightarrow{j} \underline{\mathcal{E}}$ *and functors* $\underline{\mathcal{C}} \xrightarrow{f} \underline{\mathcal{A}} \xhookrightarrow{y} \underline{\mathcal{B}}$*, where* $\underline{\mathcal{B}}$ *is* $\mathcal{T}$*–(co)complete. Suppose we have a factorisation* $j^*j_!i_!(y \circ f) \colon \underline{\mathcal{C}} \xrightarrow{\overline{f}} \underline{\mathcal{A}} \xhookrightarrow{y} \underline{\mathcal{B}}$*. Then* $\overline{f} \simeq i_!f \colon \underline{\mathcal{D}} \to \underline{\mathcal{A}}$*.*

Proof. Let $\varphi \colon \underline{\mathcal{D}} \to \underline{\mathcal{A}}$. We need to show that $\underline{\mathrm{Nat}}(\overline{f}, \varphi) \simeq \underline{\mathrm{Nat}}(f, i^*\varphi)$. We compute:

$$\begin{aligned}
\underline{\mathrm{Nat}}(\overline{f}, \varphi) &\simeq \underline{\mathrm{Nat}}(y \circ \overline{f}, y \circ \varphi) \\
&= \underline{\mathrm{Nat}}(j^*j_!i_!(y \circ f), y \circ \varphi) \\
&\simeq \underline{\mathrm{Nat}}(y \circ f, i^*j^*j_*(y \circ \varphi)) \\
&\simeq \underline{\mathrm{Nat}}(y \circ f, i^*(y \circ \varphi) \simeq \underline{\mathrm{Nat}}(f, i^*\varphi)
\end{aligned}$$

where the first and last equivalences are since y was fully faithful; the fourth equivalence is since j was fully faithful and so Proposition 1.2.14 applies. The relevant Kan extensions exist since $\underline{\mathcal{B}}$ was assumed to be $\mathcal{T}$–(co)complete. □

We learnt of the following proof method from Markus Land.

Lemma 1.2.40. *For $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}$, we have a functor* $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{D}))$ *that takes* $F : \mathcal{C} \to \mathcal{D}$ *to* $\mathrm{Ind}(F) : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$.

Proof. We know that $\mathrm{Ind}(\mathcal{E} \times \mathcal{C}) \simeq \mathrm{Ind}(\mathcal{E}) \times \mathrm{Ind}(\mathcal{C})$. In particular, we get functors

$$\Delta^n \times \mathrm{Ind}(\mathcal{C}) \longrightarrow \mathrm{Ind}(\Delta^n \times \mathcal{C}) \simeq \mathrm{Ind}(\Delta^n) \times \mathrm{Ind}(\mathcal{C})$$

natural in both Δ^n and $\mathcal{C}$. These then induce a map of simplicial spaces

$$\mathrm{Fun}(\Delta^\bullet \times \mathcal{C}, \mathcal{D})^\simeq \longrightarrow \mathrm{Fun}(\mathrm{Ind}(\Delta^\bullet \times \mathcal{C}), \mathrm{Ind}(\mathcal{D}))^\simeq \longrightarrow \mathrm{Fun}(\Delta^\bullet \times \mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{D}))^\simeq$$

where the first map is just by the $(\infty, 1)$–functoriality of Ind. Via the complete Segal space model of ∞-categories, we see that we have the desired functor which behaves as in the statement by looking at the case $\bullet = 0$. ☐

Lemma 1.2.41 (Ind adjunctions)**.** *Let $f : \mathcal{C} \rightleftarrows \mathcal{D} : g$ be an adjunction. Then we also have an adjunction $F := \mathrm{Ind}(f) : \mathrm{Ind}(\mathcal{C}) \rightleftarrows \mathrm{Ind}(\mathcal{D}) : \mathrm{Ind}(g) =: G$.*

Proof. By [RV19, Def. 1.1.2] we know that such an adjunction is tantamount to the data of $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow gf$ and $\varepsilon : fg \Rightarrow \mathrm{id}_{\mathcal{D}}$ such that we have the triangle identities

$$\begin{tikzcd} & \mathcal{C} \arrow[rr, equal] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\varepsilon"] & & \mathcal{C} \\ \mathcal{D} \arrow[ur, "g"] \arrow[rr, equal] & & \mathcal{D} \arrow[ur, "g"'] \arrow[u, phantom, "\Downarrow\eta"] & \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{C} \\ \mathcal{D} \arrow[u, bend left, "g"] \arrow[u, bend right, "g"'] \arrow[u, phantom, "\overset{\Rightarrow}{\mathrm{id}_g}"] \end{tikzcd}$$

and the analogous other triangle. Now, we have $\mathrm{Fun}(\mathcal{C}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{C}))$ by Lemma 1.2.40 and so the the triangle identity on the source gets sent to a triangle identity on the target. ☐

Theorem 1.2.42 (Diagram decomposition, [Sha22b, Thm. 8.1])**.** *Let $\underline{\mathcal{C}}$ be a $\mathcal{T}$–category, J a category, and $p_\bullet : J \to (\mathrm{Cat}_{\mathcal{T}})_{/\underline{\mathcal{C}}}$ a functor with colimit the $\mathcal{T}$–functor $p : \underline{K} \to \underline{\mathcal{C}}$ and suppose that for all $j \in J$, the $\mathcal{T}$–functor $p_j : \underline{K_j} \to \underline{\mathcal{C}}$ admits a $\mathcal{T}$–colimit σ_j. Then the σ_j's assemble to a $\mathcal{T}$–functor $\sigma_\bullet : \underline{\mathrm{const}}_{\mathcal{T}}(J) \to \underline{\mathcal{C}}$ so that if $\sigma_\bullet$ admits a $\mathcal{T}$–colimit σ, then p admits a $\mathcal{T}$–colimit given by σ.*

Corollary 1.2.43 (Parametrised filtered colimit decomposition, "[Lur09, Cor. 4.2.3.11]")**.** *Let $\tau \ll \kappa$ be regular cardinals and $\underline{\mathcal{C}}$ be a $\mathcal{T}$–category admitting τ-small $\mathcal{T}$–colimits and fibrewise colimits indexed by κ-small τ-filtered posets. Then for any κ-small $\mathcal{T}$–diagram $d : \underline{K} \to \underline{\mathcal{C}}$, its $\mathcal{T}$–colimit in $\underline{\mathcal{C}}$ exists and can be decomposed as a fibrewise κ-small τ-filtered colimit whose vertices are τ-small $\mathcal{T}$–colimits of $\underline{\mathcal{C}}$.*

Proof. Let J denote the poset of τ-small $\mathcal{T}$–subcategories of $\underline{K}$. It is clearly τ-filtered and moreover it is κ-small by the hypothesis that $\tau \ll \kappa$. We can therefore apply the theorem above since the associated $\sigma_\bullet : \underline{\mathrm{const}}_{\mathcal{T}}(J) \to \underline{\mathcal{C}}$ will admit a $\mathcal{T}$–colimit by hypothesis. $\square$

Theorem 1.2.44 (Limit-filtered colimit exchange, special case of [Sha22b, Thm. C]). *Let κ be a regular cardinal and J a κ–filtered category. Then $\underline{\mathrm{colim}}_{\underline{\mathrm{const}}_{\mathcal{T}}(J)} : \underline{\mathrm{Fun}}(\underline{\mathrm{const}}_{\mathcal{T}}(J), \underline{\mathcal{S}}_{\mathcal{T}}) \longrightarrow \underline{\mathcal{S}}_{\mathcal{T}}$ strongly preserves $\mathcal{T}$–κ–small $\mathcal{T}$–colimits.*

1.3 Preliminaries: atomic orbital base categories

Finally, we begin to impose the strictest conditions on our base category $\mathcal{T}$. From here on, $\mathcal{T}$ will be assumed to be both orbital *and* atomic.

1.3.1 Recollections: parametrised semiadditivity and stability

In this subsection we recall the algebraic constructions and results of [Nar17].

Construction 1.3.1. The following list of constructions will be important in discussing $\mathcal{T}$–semiadditivity and $\mathcal{T}$–stability. See [Nar17, §2.2] for the original source on these constructions. Note that we have adopted the notation of Span instead of the original notation of effective Burnside categories A^{eff}.

(1) Write $\mathrm{Span}(\mathcal{T}) := \mathrm{Span}(\mathrm{Fin}_{\mathcal{T}})$.

(2) By [Nar17, Cons. 2.11], there is a $\mathcal{T}$–category $p : \underline{\mathrm{Span}}(T) \to \mathcal{T}^{\mathrm{op}}$ whose objects are morphisms $[U \to V]$ in $\mathrm{Fin}_{\mathcal{T}}$ where $V \in \mathcal{T}$ and the cocartesian fibration p sends $[U \to V]$ to V. The morphisms in this category are spans

$$\begin{tikzcd} U \arrow[d] & W \arrow[l] \arrow[r] \arrow[d] & U' \arrow[d] \\ V & V' \arrow[l] \arrow[r, equal] & V' \end{tikzcd} \tag{1.3}$$

(3) From this we obtain a wide $\mathcal{T}$–subcategory $\underline{\mathrm{Fin}}_{*\mathcal{T}} \subset \underline{\mathrm{Span}}(T)$ whose morphisms are spans as in Eq. (1.3) such that the map $W \to U \times_V V'$ in $\mathrm{Fin}_{\mathcal{T}}$ is a summand inclusion: this makes sense since $\mathcal{T}$ was assumed to be orbital and so $\mathrm{Fin}_{\mathcal{T}}$ admits the pullback $U \times_V V'$ which will be a finite coproduct of objects of V.

(4) There is a canonical inclusion $\underline{*} \hookrightarrow \underline{\mathrm{Fin}}_{*\mathcal{T}}$ given by sending $W \to V$ to

$$\begin{tikzcd} V \arrow[d, equal] & W \arrow[l] \arrow[r, equal] \arrow[d, equal] & W \arrow[d, equal] \\ V & W \arrow[l] \arrow[r, equal] & W \end{tikzcd}$$

Definition 1.3.2. Let $\underline{\mathcal{C}}$ strongly admit finite $\mathcal{T}$–coproducts and $\underline{\mathcal{D}}$ strongly admit finite $\mathcal{T}$–products. Then we say that a $\mathcal{T}$–functor $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is *$\mathcal{T}$-semiadditive* if it sends finite $\mathcal{T}$–coproducts to finite $\mathcal{T}$–products. We say that a $\mathcal{T}$–category $\underline{\mathcal{C}}$ strongly admitting finite $\mathcal{T}$–products and $\mathcal{T}$–coproducts is *$\mathcal{T}$–semiadditive* if the identity functor is $\mathcal{T}$–semiadditive. If moreover $\underline{\mathcal{C}}$ has fibrewise pushouts and $\underline{\mathcal{D}}$ has fibrewise pullbacks, then we say that F is *$\mathcal{T}$–linear* if it is $\mathcal{T}$–semiadditive and sends fibrewise pushouts to fibrewise pullbacks. We write $\underline{\mathrm{Fun}}^{\mathrm{sadd}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ (resp. $\underline{\mathrm{Lin}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$) for the $\mathcal{T}$–full subcategories of $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ consisting of the $\mathcal{T}$–semiadditive functors (resp. $\mathcal{T}$–linear functors).

Notation 1.3.3. For $\underline{\mathcal{C}}$ strongly admitting finite $\mathcal{T}$–limits we will denote *$\mathcal{T}$–Mackey functors* by $\underline{\mathrm{Mack}}_{\mathcal{T}}(\underline{\mathcal{C}}) := \underline{\mathrm{Fun}}^{\mathrm{sadd}}_{\mathcal{T}}(\underline{\mathrm{Span}}(\mathcal{T}), \underline{\mathcal{C}})$ and *$\mathcal{T}$–commutative monoids* by $\underline{\mathrm{CMon}}_{\mathcal{T}}(\underline{\mathcal{C}}) := \underline{\mathrm{Fun}}^{\mathrm{sadd}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{C}})$.

Proposition 1.3.4 ($\mathcal{T}$–semiadditivisation, [Nar17, Prop. 2.27])**.** *Let $\underline{\mathcal{C}}$ be a $\mathcal{T}$–category strongly admitting finite $\mathcal{T}$–products. Then the functor $\underline{\mathrm{CMon}}_{\mathcal{T}}(\underline{\mathcal{C}}) \to \underline{\mathcal{C}}$ induced by the inclusion $\underline{*} \hookrightarrow \underline{\mathrm{Fin}}_{*\mathcal{T}}$ from Construction 1.3.1 (4) is an equivalence if and only if $\underline{\mathcal{C}}$ were $\mathcal{T}$–semiadditive.*

Theorem 1.3.5 ("CMon = Mackey", [Nar17, Thm. 2.32])**.** *Let $\underline{\mathcal{C}}$ strongly admit finite $\mathcal{T}$–limits. Then the defining inclusion $j : \underline{\mathrm{Fin}}_{*\mathcal{T}} \hookrightarrow \underline{\mathrm{Span}}(\mathcal{T})$ induces an equivalence*

$$j^* : \underline{\mathrm{Fun}}^{\mathrm{sadd}}_{\mathcal{T}}(\underline{\mathrm{Span}}(\mathcal{T}), \underline{\mathcal{C}}) \longrightarrow \underline{\mathrm{CMon}}_{\mathcal{T}}(\underline{\mathcal{C}})$$

Notation 1.3.6. We write $\mathrm{Fun}^{\mathrm{ex}}_{\mathcal{T}}, \mathrm{Fun}^{\mathrm{lex}}_{\mathcal{T}}$, and $\mathrm{Fun}^{\mathrm{rex}}_{\mathcal{T}}$ for the category of $\mathcal{T}$–functors which strongly preserve finite $\mathcal{T}$–(co)limits, strongly preserve finite $\mathcal{T}$–limits, and strongly preserve finite $\mathcal{T}$–colimits, respectively.

Construction 1.3.7. Let $\underline{\mathrm{Sp}}^{\mathrm{pw}} \colon \mathrm{Cat}^{\mathrm{lex}}_{\mathcal{T}} \to \mathrm{Cat}^{\mathrm{lex}}_{\mathcal{T}}$ be the functor obtained by applying $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, -)$ to $\mathrm{Sp} \colon \mathrm{Cat}^{\mathrm{lex}} \to \mathrm{Cat}^{\mathrm{lex}}$. Now let $\underline{\mathcal{D}} \in \mathrm{Cat}_{\mathcal{T}}$ strongly admitting finite $\mathcal{T}$–limits. Then we can define its *$\mathcal{T}$–stabilisation* to be $\underline{\mathrm{Sp}}_{\mathcal{T}}(\underline{\mathcal{D}}) := \underline{\mathrm{CMon}}_{\mathcal{T}}(\underline{\mathrm{Sp}}^{\mathrm{pw}}(\underline{\mathcal{D}}))$. In particular, applying this to the case $\underline{\mathcal{D}} = \underline{\mathcal{S}}_{\mathcal{T}}$, we get $\underline{\mathrm{Sp}}_{\mathcal{T}} := \underline{\mathrm{CMon}}_{\mathcal{T}}(\underline{\mathrm{Sp}}^{\mathrm{pw}}(\underline{\mathcal{S}}_{\mathcal{T}}))$ which is called *$\mathcal{T}$–category of genuine $\mathcal{T}$–spectra*. Note that this is different from the notation in [Nar17, Defn. 2.35] where he used $\underline{\mathrm{Sp}}^{\mathcal{T}}$ instead, and reserved $\underline{\mathrm{Sp}}_{\mathcal{T}}$ for what we wrote as $\underline{\mathrm{Sp}}^{\mathrm{pw}}$. We prefer the notation we have adopted as it aligns well with all the parametrised subscripts $(-)_{\mathcal{T}}$ and the superscripts are reserved for modifiers such as $(-)^{\underline{\omega}}$ or $(-)^{\Delta^1}$ that we will need later.

Theorem 1.3.8 (Universal property of $\mathcal{T}$–stabilisations, [Nar17, Thm. 2.36])**.** *Let $\underline{\mathcal{C}}$ be a pointed $\mathcal{T}$–category strongly admitting finite $\mathcal{T}$–colimits and $\underline{\mathcal{D}}$ a $\mathcal{T}$–category strongly admitting finite $\mathcal{T}$–limits. Then the functor $\underline{\Omega}^{\infty} : \underline{\mathrm{Fun}}^{\mathrm{rex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{Sp}}_{\mathcal{T}}(\underline{\mathcal{D}})) \longrightarrow \underline{\mathrm{Lin}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ is an equivalence of $\mathcal{T}$–categories. In particular, we see that $\underline{\mathrm{Sp}}_{\mathcal{T}}(\underline{\mathcal{D}}) \simeq \underline{\mathrm{Lin}}_{\mathcal{T}}(\underline{\mathcal{S}}^{\mathrm{fin}}_{*\mathcal{T}}, \underline{\mathcal{D}})$.*

1.3.2 Parametrised symmetric monoidality and commutative algebras

Recollections 1.3.9. There is a notion of $\mathcal{T}$–operads mimicking the notion of ∞-operads, in the sense of [Lur17, §2.1], due to Nardin in [Nar17, §3]. A $\mathcal{T}$–symmetric monoidal category is then a $\mathcal{T}$–category $\underline{\mathcal{C}}^{\otimes}$ equipped with a cocartesian fibration over $\underline{\mathrm{Fin}}_{*\mathcal{T}}$ satisfying the $\mathcal{T}$–operad axioms analogous to the operad axioms of [Lur17, Definition 2.1.1.10]. Alternatively, the $\mathcal{T}$–category of $\mathcal{T}$–symmetric monoidal categories is also given as $\underline{\mathrm{CMon}}(\underline{\mathrm{Cat}})$ much like in the unparametrised setting. Furthermore, there is also the attendant notion of *$\mathcal{T}$-inert morphisms* defined as those morphisms in $\underline{\mathrm{Fin}}_{*\mathcal{T}}$ where the the map $W \to U'$ is an equivalence (cf. the span notation in Eq. (1.3)). The $\mathcal{T}$–category of *$\mathcal{T}$-commutative algebras* $\underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\otimes})$ of a $\mathcal{T}$–symmetric monoidal category $\underline{\mathcal{C}}^{\otimes}$ is then defined to be $\underline{\mathrm{Fun}}^{\mathrm{inert}}_{\underline{\mathrm{Fin}}_{*\mathcal{T}}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{C}}^{\otimes})$ where $\underline{\mathrm{Fun}}^{\mathrm{inert}}_{\underline{\mathrm{Fin}}_{*\mathcal{T}}} \subseteq \underline{\mathrm{Fun}}_{\underline{\mathrm{Fin}}_{*\mathcal{T}}}$ is the $\mathcal{T}$–full subcategory of functors over $\underline{\mathrm{Fin}}_{*\mathcal{T}}$ preserving $\mathcal{T}$–inert morphisms. We refer the reader to the original source [Nar17, §3.1] for details on this.

Terminology 1.3.10. Let $\underline{\mathcal{C}}^{\otimes}, \underline{\mathcal{D}}^{\otimes}$ be $\mathcal{T}$–symmetric monoidal categories. By a *$\mathcal{T}$-symmetric monoidal localisation* $\underline{L}^{\otimes} : \underline{\mathcal{C}}^{\otimes} \to \underline{\mathcal{D}}^{\otimes}$ we mean a $\mathcal{T}$–symmetric monoidal functor whose underlying $\mathcal{T}$–functor is a $\mathcal{T}$–Bousfield localisation. By the proof of [Nar17, Prop. 3.5], we see that the $\mathcal{T}$–right adjoint canonically refines to a $\mathcal{T}$–lax symmetric functor. Hence in this situation we obtain a relative adjunction over $\underline{\mathrm{Fin}}_{*\mathcal{T}}$

$$\begin{tikzcd} \underline{\mathcal{C}}^{\otimes} \arrow[rr, shift left, "\underline{L}^{\otimes}"] \arrow[dr] & & \underline{\mathcal{D}}^{\otimes} \arrow[ll, shift left] \arrow[dl] \\ & \underline{\mathrm{Fin}}_{*\mathcal{T}} & \end{tikzcd}$$

in the sense of [Lur17, §7.3.2] whose counit is moreover an equivalence.

Lemma 1.3.11 ($\mathcal{T}$–adjunction on $\mathcal{T}$–commutative algebras, "[GGN15, Lem. 3.6]"). *Let $\underline{\mathcal{C}}^{\otimes}, \underline{\mathcal{D}}^{\otimes}$ be $\mathcal{T}$–symmetric monoidal categories and $\underline{L}^{\otimes} : \underline{\mathcal{C}}^{\otimes} \to \underline{\mathcal{D}}^{\otimes}$ a $\mathcal{T}$–symmetric monoidal localisation. Then there is an induced $\mathcal{T}$–Bousfield localisation $L' : \underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{C}}) \to \underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{D}})$ such that the diagram*

$$\begin{tikzcd} \underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{C}}) \arrow[r, shift left, "L'"] \arrow[d] & \underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{D}}) \arrow[l, shift left, "R'"] \arrow[d] \\ \underline{\mathcal{C}} \arrow[r, shift left, "L"] & \underline{\mathcal{D}} \arrow[l, shift left, "R"] \end{tikzcd}$$

commutes, where the vertical maps are given by

$$\underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{C}}) \coloneqq \underline{\mathrm{Fun}}^{\mathrm{inert}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{C}}^{\otimes}) \times_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathrm{Fin}}_{*\mathcal{T}})} \underline{*} \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{*}, \underline{\mathcal{C}}) \simeq \underline{\mathcal{C}}$$

induced by the inclusion $\underline{*} \hookrightarrow \underline{\mathrm{Fin}}_{*\mathcal{T}}$*, which lands in the* $\mathcal{T}$*–inerts. Moreover, given* $A \in \underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{C}})$ *there is a unique* $\mathcal{T}$*–commutative algebra structure on* RLA *such that the unit map* $A \to RLA$ *enhances to a morphism of* $\mathcal{T}$*–commutative algebras.*

Proof. First note that we have the adjunction squares

$$\begin{tikzcd}
\underline{\mathrm{Fun}}^{\mathrm{inert}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{C}}^{\otimes}) \arrow[r, shift left, "L'"] \arrow[d, hook] & \underline{\mathrm{Fun}}^{\mathrm{inert}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{D}}^{\otimes}) \arrow[l, shift left, "R'"] \arrow[d, hook] \\
\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{C}}^{\otimes}) \arrow[r, shift left, "L_*^{\otimes}"] & \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{D}}^{\otimes}) \arrow[l, shift left, "R_*^{\otimes}"]
\end{tikzcd}$$

where the bottom $\mathcal{T}$–adjunction is by Theorem 1.2.10 and has the property that the counit is an equivalence. Now [Lur17, Prop. 7.3.2.5] says that relative adjunctions are stable under pullbacks and the property of being $\mathcal{T}$–functors is of course preserved by pullbacks too, and so we get the square

$$\begin{tikzcd}
\underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{C}}) \arrow[d, equal] & \underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{D}}) \arrow[d, equal] \\
\underline{\mathrm{Fun}}^{\mathrm{inert}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{C}}^{\otimes}) \times_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathrm{Fin}}_{*\mathcal{T}})} \underline{*} \arrow[r, shift left, "L'"] \arrow[d, hook] & \underline{\mathrm{Fun}}^{\mathrm{inert}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{D}}^{\otimes}) \times_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathrm{Fin}}_{*\mathcal{T}})} \underline{*} \arrow[l, shift left, "R'"] \arrow[d, hook] \\
\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{C}}^{\otimes}) \times_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathrm{Fin}}_{*\mathcal{T}})} \underline{*} \arrow[r, shift left, "L_*^{\otimes}"] & \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{D}}^{\otimes}) \times_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathrm{Fin}}_{*\mathcal{T}})} \underline{*} \arrow[l, shift left, "R_*^{\otimes}"]
\end{tikzcd}$$

Then the square in the statement of the result is just composition of this square with the one induced by the inclusion $\underline{*} \hookrightarrow \underline{\mathrm{Fin}}_{*\mathcal{T}}$ namely

$$\begin{tikzcd}
\underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{C}}) \arrow[r, shift left, "L'"] \arrow[d, hook] & \underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{D}}) \arrow[l, shift left, "R'"] \arrow[d, hook] \\
\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{C}}) \times_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathrm{Fin}}_{*\mathcal{T}})} \underline{*} \arrow[r, shift left, "L_*^{\otimes}"] \arrow[d] & \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathcal{D}}) \times_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, \underline{\mathrm{Fin}}_{*\mathcal{T}})} \underline{*} \arrow[l, shift left, "R_*^{\otimes}"] \arrow[d] \\
\underline{\mathcal{C}} = \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{*}, \underline{\mathcal{D}}) \arrow[r, shift left, "L_*"] & \underline{\mathcal{D}} = \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{*}, \underline{\mathcal{D}}) \arrow[l, shift left, "R_*"]
\end{tikzcd}$$

For the next part, we know already that $R'L'A$ comes with a canonical $\mathcal{T}$–commutative algebra map $\eta' : A \to R'L'A$ given by the $L' \dashv R'$ unit evaluated at A. By the square in the statement we see that this forgets to the $L \dashv R$ unit $\eta : A \to RLA$. Now if $\eta'' : A \to R'B$ is another such map of $\mathcal{T}$–commutative algebras, then by universality of η' we have an essentially unique factorisation $\phi \circ \eta' : A \to R'L'A \to R'B$. Now fgt: $\underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{C}}) \to \underline{\mathcal{C}}$ is conservative by [Lur17, Lem. 3.2.2.6], thus since ϕ forgets to the identity, ϕ must have been an equivalence in $\underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathcal{C}})$ as required. ∎

1.3.3 Indexed (co)products of categories

We now investigate various permanence properties of indexed products on categories. To begin with, recall the following from [Nar17, Cons. 3.14].

Construction 1.3.12 (Indexed products of categories). Let $f : U \to U'$ be a map of finite $\mathcal{T}$–sets. Then Construction 1.1.16 gives us the equivalences in

$$f^* : \mathrm{Fun}(\mathrm{Total}(\underline{U}'), \mathrm{Cat}) \simeq \mathrm{Fun}_{\mathcal{T}}(\underline{U}', \underline{\mathrm{Cat}}_{\mathcal{T}}) \to \mathrm{Fun}_{\mathcal{T}}(\underline{U}, \underline{\mathrm{Cat}}_{\mathcal{T}}) \simeq \mathrm{Fun}(\mathrm{Total}(\underline{U}), \mathrm{Cat})$$

This has a right adjoint f_* (also written $\prod_f$). Thus, for $\underline{\mathcal{C}} \in \underline{\mathrm{Cat}}_{\underline{U}}$ and $\underline{\mathcal{D}} \in \underline{\mathrm{Cat}}_{\underline{\mathcal{D}}}$ we have

$$\underline{\mathrm{Fun}}_{\underline{U}'}\left(\underline{\mathcal{D}}, f_*\underline{\mathcal{C}}\right) \simeq \underline{\mathrm{Fun}}_{\underline{U}}(f^*\underline{\mathcal{D}}, \underline{\mathcal{C}})$$

By setting $\underline{\mathcal{D}} = \underline{V}$ we see that $f_*\underline{\mathcal{C}}$ is a $\mathcal{T}_{/U'}$–category with V–fibre given by

$$\mathrm{Fun}_{\underline{U}}(\underline{U}_V, \underline{\mathcal{C}}) \simeq \prod_{O \in \mathrm{Orbit}(U \times_{U'} V)} \mathcal{C}_O$$

where $\underline{U}_V$ is the model for the corepresentable $\mathcal{T}$–category associated to $U \times_{U'} V$ whose fibre over $[W \to U]$ is given by the space of commutative squares in $\mathrm{Fin}_{\mathcal{T}}$

$$\begin{array}{ccc} W & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & U' \end{array}$$

Lemma 1.3.13 (Indexed constructions preserve adjunctions). *Let $f : W \to V$ be in $\mathcal{T}$. Let $L : \underline{\mathcal{C}} \rightleftarrows \underline{\mathcal{D}} : R$ be a $\mathcal{T}_{/W}$–adjunction and $M : \underline{\mathcal{A}} \rightleftarrows \underline{\mathcal{B}} : N$ be a $\mathcal{T}_{/V}$–adjunction. Then*

$$f_*L : f_*\underline{\mathcal{C}} \rightleftarrows f_*\underline{\mathcal{D}} : f_*R \qquad f^*M : f^*\underline{\mathcal{A}} \rightleftarrows f^*\underline{\mathcal{B}} : f^*N$$

are $\mathcal{T}_{/V}$– and $\mathcal{T}_{/W}$–adjunctions respectively.

Proof. By Corollary 1.1.25, we need to show that these induce fibrewise adjunctions. This is clear for the pair (f^*M, f^*N) since fibrewise they are the same as (M, N); for (f_*L, f_*R), we use that (unparametrised) products of adjunctions are again adjunctions. □

Lemma 1.3.14 ((Co)unit of indexed products). *The $\mathcal{T}$–cofree category $\underline{\mathrm{Cat}}_{\mathcal{T}}$ strongly admits $\mathcal{T}$–products, and for $f : W \to V$, $X \in \mathcal{T}_{/W}$, and $Y \in \mathcal{T}_{/V}$, we have that $(f_*\underline{\mathcal{D}})_Y \simeq \prod_{M \in \mathrm{Orbit}(Y \times_V W)} \mathcal{D}_M$ and moreover:*

- *The unit is given by $\eta = F^* : \mathcal{C}_Y \longrightarrow (f_*f^*\underline{\mathcal{C}})_Y = \prod_{M \in \mathrm{Orbit}(Y \times_V W)} \mathcal{C}_M$ where $F : Y \times_V W \to Y$ is the structure map from the pullback,*
- *The counit is given by $\varepsilon = \mathrm{proj} : (f^*f_*\underline{\mathcal{C}})_X = \prod_{N \in \mathrm{Orbit}(X \times_V W)} \mathcal{D}_N \longrightarrow \mathcal{D}_X$ the component projection (see the proof for why we have this).*

Proof. We know that $f^* : \mathrm{Fun}((\mathcal{T}_{/V})^{\mathrm{op}}, \mathrm{Cat}) \longrightarrow \mathrm{Fun}((\mathcal{T}_{/W})^{\mathrm{op}}, \mathrm{Cat})$ abstractly has a right adjoint f_* via right Kan extension, and the formula for ordinary right Kan extensions gives us the required description (which is also gotten from Construction 1.3.12).

To describe the (co)units, we have to check the triangle identities

$$\begin{tikzcd} f^* \arrow[r, "f^*\eta"] \arrow[dr, equal] & f^*f_*f^* \arrow[d, "\varepsilon_{f^*}"] \\ & f^* \end{tikzcd} \qquad \begin{tikzcd} f_* \arrow[r, "\eta_{f_*}"] \arrow[dr, equal] & f_*f^*f_* \arrow[d, "f_*\varepsilon"] \\ & f_* \end{tikzcd} \tag{1.4}$$

First of all we clarify why we have the counit map as stated. For this it will be helpful to write carefully the datum $\varphi : X \to W$ instead of just X. Consider

$$\begin{tikzcd} X \arrow[dr, dashed] \arrow[drr, equal, bend left] \arrow[ddr, bend right, "\varphi"'] & & \\ & X \times_V W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f\varphi"] \\ & W \arrow[r, "f"] & V \end{tikzcd}$$

This shows that X is a retract of $X \times_V W$, and so by atomicity, we get that X was an orbit in the orbit decomposition of $X \times_V W$, and so the component projection $\varepsilon : (f^*f_*\underline{\mathcal{D}})_X = \prod_{N \in \mathrm{Orbit}(X \times_V W)} \mathcal{D}_N \longrightarrow \mathcal{D}_X$ is well-defined

To check the first triangle identity, let $(\varphi : X \to W) \in \mathcal{T}_{/W}$ and consider

$$\begin{tikzcd} \coprod_a N_a \arrow[r, "\coprod_a \xi_a"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f\varphi"] \\ W \arrow[r, "f"] & V \end{tikzcd}$$

where one of the N_a's is X, by the argument above. Then we have that the composition in the first triangle in Eq. (1.4) is

$$\Big((f^*\underline{\mathcal{C}})_X \xrightarrow{f^*\eta} (f^*f_*f^*\underline{\mathcal{C}})_X \xrightarrow{\varepsilon_{f^*}} (f^*\underline{\mathcal{C}})_X\Big) \simeq \Big(\mathcal{C}_X \xrightarrow{\prod_a \xi_a^*} \prod_a \mathcal{C}_{N_a} \xrightarrow{\mathrm{proj}} \mathcal{C}_X\Big)$$

which is of course the identity since $\xi_a = \mathrm{id}$ in the case $N_a = X$.

The second triangle identity is slightly more intricate. Let $(\psi : Y \to V) \in \mathcal{T}_{/V}$. We consider two pullbacks (where the right square is for each b appearing in the left square)

$$\begin{tikzcd} \coprod_b M_b \arrow[r, "\coprod_b \zeta_b"] \arrow[d, "\coprod_b \rho_b"'] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "\psi"] \\ W \arrow[r, "f"] & V \end{tikzcd} \qquad \begin{tikzcd} \coprod_{c_b} \widetilde{M}_{c_b} \arrow[r, "\coprod_{c_b} \ell_{c_b}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & M_b \arrow[d, "f\rho_b"] \\ W \arrow[r, "f"] & V \end{tikzcd}$$

From this, the composition in the second triangle in Eq. (1.4) is

$$\begin{aligned} &\Big((f_*\underline{\mathcal{D}})_Y \xrightarrow{\eta_{f_*}} (f_*f^*f_*\underline{\mathcal{D}})_Y \xrightarrow{f_*\varepsilon} (f_*\underline{\mathcal{D}})_Y\Big) \\ \simeq &\Big(\prod_b \mathcal{D}_{M_b} \xrightarrow{\prod_b \prod_{c_b} \ell^*_{c_b}} \prod_b \prod_{c_b} \mathcal{D}_{\widetilde{M}_{c_b}} \xrightarrow{\prod_b \mathrm{proj}} \prod_b \mathcal{D}_{M_b}\Big) \end{aligned}$$

which is the identity map as wanted since M_b is one of the orbits in $\coprod_{c_b} \widetilde{M}_{c_b}$ by the argument above. Here we have used the diagram

$$\begin{tikzcd} (f_*\underline{\mathcal{D}})_Y \arrow[r, "\eta_{f_*} = \prod_b \zeta_b^*"] \arrow[d, equal] & \prod_b (f_*\underline{\mathcal{D}})_{M_b} \arrow[d, equal] \\ \prod_b \mathcal{D}_{M_b} \arrow[r, "\prod_b \prod_{c_b} \ell^*_{c_b}"] & \prod_b \prod_{c_b} \mathcal{D}_{\widetilde{M}_{c_b}} \end{tikzcd}$$

to analyse the map η_{f_*}, which in turn comes from the top square in

$$\begin{tikzcd} & \coprod_{c_b} \widetilde{M}_{c_b} \arrow[rr, "\coprod_{c_b} \ell_{c_b}"] \arrow[dl, "\coprod_{c_b} \ell_{c_b}"'] \arrow[dd] \arrow[drr, phantom, "\lrcorner"] & & M_b \arrow[dl, "\zeta_b"] \arrow[dd] \\ \coprod_b M_b \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner"] & & Y \arrow[dd] & \\ & W \arrow[rr, "f"] \arrow[dl, equal] & & V \arrow[dl, equal] \\ W \arrow[rr, "f"] & & V & \end{tikzcd}$$

This finishes the proof. □

1.3.4 Norms and adjunctions

We now recall the notion of $\mathcal{T}$–distributivity and indexed tensor products (also termed *norms*) of categories introduced in [Nar17, §3.3 and §3.4].

Definition 1.3.15. Let $f : U \to V$ be a map in $\mathrm{Fin}_{\mathcal{T}}$, $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}_{/U}}$, $\underline{\mathcal{D}} \in \mathrm{Cat}_{\mathcal{T}_{/V}}$, and $F : f_*\underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a $\mathcal{T}_{/V}$–functor. Then we say that F is $\mathcal{T}_{/V}$*–distributive* if for every pullback

$$\begin{tikzcd} U' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & V' \arrow[d, "g"] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

in $\mathrm{Fin}_{\mathcal{T}}$ and $\mathcal{T}_{/U'}$–colimit diagram $p : \underline{K}^{\triangleright} \to g'^*\underline{\mathcal{C}}$, the $\underline{V}'$–functor

$$(f'_*\underline{K})^{\triangleright} \xrightarrow{\text{can}} f'_*(\underline{K}^{\triangleright}) \xrightarrow{f'_*p} f'_*p'^*\underline{\mathcal{C}} \simeq g^*f_*\underline{\mathcal{C}} \xrightarrow{g^*F} g^*\underline{\mathcal{D}}$$

is a $\mathcal{T}_{/V'}$–colimit. We write $\mathrm{Fun}^{\delta}_{\underline{V}}(f_*\underline{\mathcal{C}}, \underline{\mathcal{D}})$ for the subcategory of $\mathcal{T}_{/V}$–distributive functors.

Construction 1.3.16 (Norms of categories)**.** Let $f : U \to V$ be a map in $\mathrm{Fin}_{\mathcal{T}}$ and $\underline{\mathcal{C}}$ a $\mathcal{T}_{/U}$–category which is $\mathcal{T}_{/U}$–cocomplete. Then we define the *f-norm* $f_{\otimes}\underline{\mathcal{C}}$, if it exists, to be a $\mathcal{T}_{/V}$–cocomplete category admitting a $\mathcal{T}_{/V}$–distributive functor $\tau : f_*\underline{\mathcal{C}} \to f_{\otimes}\underline{\mathcal{C}}$ such that for any other $\mathcal{T}_{/V}$–cocomplete category, the following functor is an equivalence

$$\tau^* : \mathrm{Fun}^{L}_{\underline{V}}(f_{\otimes}\underline{\mathcal{C}}, \underline{\mathcal{D}}) \to \mathrm{Fun}^{\delta}_{\underline{V}}(f_*\underline{\mathcal{C}}, \underline{\mathcal{D}})$$

We also write this as $f_{\otimes} = \bigotimes_f$.

Lemma 1.3.17 (Norms preserve adjunctions)**.** *Let $F : \underline{\mathcal{C}} \rightleftarrows \underline{\mathcal{D}} : G$ be a $\mathcal{T}_{/U}$–adjunction such that G itself admits a right adjoint and $f : U \to V$ be a map in* $\mathrm{Fin}_{\mathcal{T}}$. *Then this induces a $\mathcal{T}_{/V}$–adjunction*

$$f_{\otimes}F : f_{\otimes}\underline{\mathcal{C}} \rightleftarrows f_{\otimes}\underline{\mathcal{D}} : f_{\otimes}G$$

Proof. Recall from Lemma 1.3.13 that we have a $\mathcal{T}_{/V}$–adjunction $f_*F : f_*\underline{\mathcal{C}} \rightleftarrows f_*\underline{\mathcal{D}} : f_*G$ and since G itself has a right adjoint, both f_*F and f_*G strongly preserve $\mathcal{T}_{/V}$–colimits. Now observe that this adjunction can equivalently be encoded by the data of morphisms

$$\big(\eta\colon \mathrm{id} \Rightarrow (f_*G) \circ (f_*F)\big) \in \underline{\mathrm{Fun}}^{L}_{\underline{V}}(f_*\underline{\mathcal{C}}, f_*\underline{\mathcal{C}})$$

$$\big(\varepsilon\colon (f_*F) \circ (f_*G) \Rightarrow \mathrm{id}\big) \in \underline{\mathrm{Fun}}^{L}_{\underline{V}}(f_*\underline{\mathcal{D}}, f_*\underline{\mathcal{D}})$$

whose images under the functors

$$(f_*F)_*\colon \mathrm{Fun}^{L}_{\underline{V}}(f_*\underline{\mathcal{C}}, f_*\underline{\mathcal{C}}) \to \mathrm{Fun}^{L}_{\underline{V}}(f_*\underline{\mathcal{C}}, f_*\underline{\mathcal{D}})$$

$$(f_*F)^*\colon \mathrm{Fun}^{L}_{\underline{V}}(f_*\underline{\mathcal{D}}, f_*\underline{\mathcal{D}}) \to \mathrm{Fun}^{L}_{\underline{V}}(f_*\underline{\mathcal{C}}, f_*\underline{\mathcal{D}})$$

respectively compose to a morphism equivalent to the identity

$$\begin{tikzcd} f_*F \arrow[r, Rightarrow, "f_*F(\eta)"] \arrow[dr, equal] & (f_*F) \circ (f_*G) \circ (f_*F) \arrow[d, Rightarrow, "\varepsilon_{f_*F}"] \\ & f_*F \end{tikzcd}$$

and similarly for the other triangle identity. Now, we have commutative squares

$$\begin{tikzcd} f_*\underline{\mathcal{C}} \arrow[r, shift left, "f_*F"] \arrow[d, "\varphi"'] & f_*\underline{\mathcal{D}} \arrow[l, shift left, "f_*G"] \arrow[d, "\psi"] \\ f_\otimes\underline{\mathcal{C}} \arrow[r, shift left, "f_\otimes F"] & f_\otimes\underline{\mathcal{D}} \arrow[l, shift left, "f_\otimes G"] \end{tikzcd}$$

where $\varphi : f_*\underline{\mathcal{C}} \to f_\otimes\underline{\mathcal{C}}$, $\psi : f_*\underline{\mathcal{D}} \to f_\otimes\underline{\mathcal{D}}$ are the universal distributive functors: this is since G strongly preserves $\mathcal{T}$–colimits by hypothesis. This yields

$$\begin{tikzcd} \mathrm{Fun}^L_{\underline{V}}(f_*\underline{\mathcal{C}}, f_*\underline{\mathcal{C}}) \arrow[r, "(f_*F)_*"] \arrow[d, "\varphi_*"'] & \mathrm{Fun}^L_{\underline{V}}(f_*\underline{\mathcal{C}}, f_*\underline{\mathcal{D}}) \arrow[d, "\psi_*"] & \mathrm{Fun}^L_{\underline{V}}(f_*\underline{\mathcal{D}}, f_*\underline{\mathcal{D}}) \arrow[l, "(f_*F)^*"'] \arrow[d, "\psi_*"] \\ \mathrm{Fun}^\delta_{\underline{V}}(f_*\underline{\mathcal{C}}, f_\otimes\underline{\mathcal{C}}) \arrow[r, "(f_\otimes F)_*"] & \mathrm{Fun}^\delta_{\underline{V}}(f_*\underline{\mathcal{C}}, f_\otimes\underline{\mathcal{D}}) & \mathrm{Fun}^\delta_{\underline{V}}(f_*\underline{\mathcal{D}}, f_\otimes\underline{\mathcal{D}}) \arrow[l, "(f_\otimes F)^*"'] \\ \mathrm{Fun}^L_{\underline{V}}(f_\otimes\underline{\mathcal{C}}, f_\otimes\underline{\mathcal{C}}) \arrow[r, "(f_\otimes F)_*"] \arrow[u, "\varphi^*", "\simeq"'] & \mathrm{Fun}^L_{\underline{V}}(f_\otimes\underline{\mathcal{C}}, f_\otimes\underline{\mathcal{D}}) \arrow[u, "\varphi^*", "\simeq"'] & \mathrm{Fun}^L_{\underline{V}}(f_\otimes\underline{\mathcal{D}}, f_\otimes\underline{\mathcal{D}}) \arrow[l, "(f_\otimes F)^*"'] \arrow[u, "\psi^*", "\simeq"'] \end{tikzcd}$$

Then the morphism $\big(\eta : \mathrm{id} \Rightarrow (f_*G) \circ (f_*F)\big) \in \underline{\mathrm{Fun}}^L_{\underline{V}}(f_*\underline{\mathcal{C}}, f_*\underline{\mathcal{C}})$ in the top left corner gets sent to a morphism $\big(\widetilde{\eta} : \mathrm{id} \Rightarrow (f_\otimes G) \circ (f_\otimes F)\big) \in \underline{\mathrm{Fun}}^L_{\underline{V}}(f_\otimes\underline{\mathcal{C}}, f_\otimes\underline{\mathcal{C}})$ in the bottom left, and similarly for ε. Then by the characterisation of adjunctions above, since the composition of the images in the middle top term is equivalent to the identity, so is the image in the middle bottom term, that is, we have the commuting diagram

$$\begin{tikzcd} f_\otimes F \arrow[r, Rightarrow, "f_\otimes F(\widetilde{\eta})"] \arrow[dr, equal] & (f_\otimes F) \circ (f_\otimes G) \circ (f_\otimes F) \arrow[d, Rightarrow, "\widetilde{\varepsilon}_{f_\otimes F}"] \\ & f_\otimes F \end{tikzcd}$$

and similarly for the other triangle identity. This witnesses that we have a $\mathcal{T}_{/V}$–adjunction $f_\otimes F \dashv f_\otimes G$ as required. □

Recollections 1.3.18 ($\mathcal{T}$–tensors and norms, [Nar17, pg. 37])**.** Let $V, W \in \mathcal{T}$ and $\underline{\mathcal{C}}^{\otimes}$ a $\mathcal{T}$–symmetric monoidal category. Then we get the structure of tensor products and norm functors as follows:

- (Tensor functor): Consider the morphism in $\underline{\mathrm{Fin}}_{*\mathcal{T}}$ given by

$$\begin{tikzcd} V \coprod V \arrow[r, equal] \arrow[d, "\nabla"] & V \coprod V \arrow[r, "\nabla"] \arrow[d, "\nabla"] & V \arrow[d, equal] \\ V \arrow[r, equal] & V \arrow[r, equal] & V \end{tikzcd}$$

The cocartesian lifts along this morphism give us the tensor product

$$\otimes : \mathcal{C}_V \times \mathcal{C}_V \simeq \mathcal{C}_{V \coprod V} \longrightarrow \mathcal{C}_V$$

- (Norm functor): Suppose $f : V \to W$ is a morphism in $\mathcal{T}$. Consider

$$\begin{tikzcd} V \arrow[r, equal] \arrow[d, "f"] & V \arrow[r, "f"] \arrow[d, "f"] & W \arrow[d, equal] \\ W \arrow[r, equal] & W \arrow[r, equal] & W \end{tikzcd}$$

The cocartesian lifts along this morphism give us the norm functor

$$\mathrm{N}^f : \mathcal{C}_V \simeq \mathcal{C}^{\otimes}_{[f:V\to W]} \longrightarrow \mathcal{C}^{\otimes}_{[W=W]} \simeq \mathcal{C}_W$$

Note that it might have been tempting to define the norm functor as the pushforward along the more obvious morphism

$$\begin{tikzcd} V \arrow[r, equal] \arrow[d, equal] & V \arrow[r, "f"] \arrow[d, equal] & W \arrow[d, equal] \\ V \arrow[r, equal] & V \arrow[r, "f"] & W \end{tikzcd}$$

instead, but the problem is that this is not a morphism in $\underline{\mathrm{Fin}}_{*\mathcal{T}}$ because by definition the bottom right map needs to be the identity!

Chapter 2

Parametrised presentability

We now investigate the notion of compactness and presentability in the parametrised context. Since this topic is tightly intertwined with colimits, the base category $\mathcal{T}$ will always be orbital in this chapter (cf. the guide in the overview of Chapter 1). It turns out that $\mathcal{T}$–presentable categories are nothing but $\mathcal{T}$–cocomplete categories which are fibrewise presentable in the unparametrised sense of [Lur09, Ch. 5]. Two highlights are seven characterisations of parametrised presentability Theorem 2.2.2 analogous to [Lur09, Thm. 5.5.1.1] and the adjoint functor Theorem 2.2.3. We then use these as tools to deduce standard results about presentable localisations and closure of presentability under various categorical constructions.

After that, we turn to study semiadditive and stable presentables in §2.3. As commented in the overview of Chapter 1, we will further stipulate that $\mathcal{T}$ is also atomic. Among other things, we prove Theorem 2.3.4 which shows that the $\mathcal{T}$–semiadditive-presentables embed fully faithfully in $\mathcal{T}$–Mackey functors valued in semiadditive presentables. Moreover, we even characterise the essential image as the $\mathcal{T}$–Mackey functors whose abstract transfers are equivalent to the left and right adjoints of the restrictions via the canonical Beck-Chevalley maps (which exist by our crucial hypothesis of atomic orbitality of $\mathcal{T}$): we expect, but have not proven, that this should be encoded by the *2–category* of spans in $\mathcal{T}$ where the 2–morphisms would allow us to encode the required adjunctions. In any case, this theorem gives a direct comparison between the internal and external notions of parametrised semiadditive-presentability, as it were, and we hope that it clarifies the interplay between the parametrised and the unparametrised concepts. These will prepare the ground for our study of equivariant algebraic K–theory of G–perfect-stable categories in the next part of the thesis.

2.1 Parametrised smallness adjectives

We now introduce the notion of $\mathcal{T}$–compactness and $\mathcal{T}$–idempotent-completeness. Not only are these notions crucial in proving the characterisations of $\mathcal{T}$–presentables in Theorem 2.2.2, they are also fundamental for the applications we have in mind for parametrised algebraic K–theory in Part II. The moral of this section is that these are essentially fibrewise notions and should present no conceptual difficulties to those already familiar with the unparametrised versions. Recall that we will assume throughout that $\mathcal{T}$ is orbital.

2.1.1 Parametrised compactness

Recall that an object X in a category $\mathcal{C}$ is compact if $\mathrm{Map}_{\mathcal{C}}(X,-) : \mathcal{C} \to \mathcal{S}$ commutes with filtered colimits (cf. [Lur09, §5.3.4]). In this subsection we introduce the parametrised analogue of this notion and study its interaction with Ind-completions.

Definition 2.1.1. Let $\underline{\mathcal{C}}$ be a $\mathcal{T}$–category and $V \in \mathcal{T}$. A V-object in $\underline{\mathcal{C}}$ (ie. an object in $\mathrm{Fun}_{\mathcal{T}}(\underline{V}, \underline{\mathcal{C}})$) is $\mathcal{T}_{/V}$–κ*-compact* if it is fibrewise κ-compact. We will also use the terminology *parametrised-*κ*-compact objects* when we allow V to vary. We write $\underline{\mathcal{C}}^{\underline{\kappa}}$ for the $\mathcal{T}$–subcategory of parametrised-κ-compact objects, that is, $(\underline{\mathcal{C}}^{\underline{\kappa}})_V$ is given by the full subcategory of $\mathcal{T}_{/V}$–κ-compact objects.

Notation 2.1.2. We write $\underline{\mathrm{Fun}}^{\underline{\kappa}}_{\mathcal{T}}$ for the full $\mathcal{T}$–subcategory of parametrised functors preserving parametrised κ-compact objects.

Warning 2.1.3. In general, for $V \in \mathcal{T}^{\mathrm{op}}$, the inclusion $(\underline{\mathcal{C}}^{\underline{\kappa}})_V \subseteq (\mathcal{C}_V)^{\kappa}$ is not an equivalence - the point is that parametrised-κ-compactness must be preserved under the cocartesian lifts $f^* : \mathcal{C}_V \to \mathcal{C}_W$ for all $f : W \to V$, but these do not preserve κ-compactness in general.

This definition of compactness makes sense by virtue of the following:

Proposition 2.1.4 (Characterisation of parametrised-compactness)**.** *Let $\underline{\mathcal{C}}$ admit fibrewise κ-filtered $\mathcal{T}$–colimits. A $\mathcal{T}$–object $C \in \mathrm{Fun}_{\mathcal{T}}(\underline{*}, \underline{\mathcal{C}})$ is κ–$\mathcal{T}$–compact in the sense above if and only if for all $V \in \mathcal{T}$ and all fibrewise κ-filtered $\mathcal{T}_{/V}$–diagram $d : \underline{\mathrm{const}}_V(K) \to \underline{\mathcal{C}}_{\underline{V}}$ the comparison*

$$\underline{\mathrm{colim}}_{\underline{\mathrm{const}}_V(K)} \underline{\mathrm{Map}}_{\underline{\mathcal{C}}_V}(C_{\underline{V}}, d) \to \underline{\mathrm{Map}}_{\underline{\mathcal{C}}_V}(C_{\underline{V}}, \underline{\mathrm{colim}}_{\underline{\mathrm{const}}_V(K)} d)$$

is an equivalence.

Proof. Suppose C is κ–$\mathcal{T}$–compact. We are already provided with the comparison map above, and we just need to check that it is an equivalence, which can be done by checking fibrewise. Since $\mathrm{Total}(\underline{V}) = (\mathcal{T}_{/V})^{\mathrm{op}}$ has an initial object, we can assume that $\mathcal{T}$ has a final object. So let $W \in \mathcal{T}$. Recall that as in the proof of Lemma 1.1.27 we have

$$\Big(\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(C,d)\Big)_W \simeq \Big(\mathrm{Map}_{\mathcal{C}_\bullet}(C_\bullet,d_\bullet)\Big)_{\bullet\in(\mathcal{T}_{/W})^{\mathrm{op}}} \in \mathrm{Fun}((\mathcal{T}_{/W})^{\mathrm{op}},\mathcal{S})$$

Then

$$\begin{aligned}\Big(\underline{\mathrm{colim}}_{\underline{\mathrm{const}}_V(K)}\underline{\mathrm{Map}}_{\underline{\mathcal{C}}_{\underline{V}}}(C_{\underline{V}},d)\Big)_W &\simeq \operatorname*{colim}_K \Big(\mathrm{Map}_{\mathcal{C}_\bullet}(C_\bullet,d_\bullet)\Big)_{\bullet\in(\mathcal{T}_{/W})^{\mathrm{op}}}\\ &\xrightarrow{\simeq} \Big(\mathrm{Map}_{\mathcal{C}_\bullet}(C_\bullet,\operatorname*{colim}_K d_\bullet)\Big)_{\bullet\in(\mathcal{T}_{/W})^{\mathrm{op}}}\end{aligned}$$

where the first equivalence is since fibrewise parametrised colimits are computed fibrewise, and the comparison map is an equivalence since colimits in $\mathrm{Fun}((\mathcal{T}_{/V})^{\mathrm{op}},\mathcal{S})$ are computed pointwise, and C is pointwise κ-compact by hypothesis.

Now for the reverse direction, let $C \in \underline{\mathcal{C}}$ satisfy the property in the statement and $V \in \mathcal{T}$ arbitrary. We want to show that $C_V \in \mathcal{C}_V$ is κ-compact, that is: for any ordinary small κ-filtered diagram $d : K \to \mathcal{C}_V$, we have that

$$\operatorname*{colim}_K \mathrm{Map}_{\mathcal{C}_V}(C_V,d) \to \mathrm{Map}_{\mathcal{C}_V}(C_V,\operatorname*{colim}_K d)$$

is an equivalence. Now recall that $\mathcal{C}_V = \mathrm{Fun}_{\underline{V}}(\underline{V},\underline{\mathcal{C}}_{\underline{V}})$ by Example 1.1.4 and so by adjunction we obtain from $d : K \to \mathcal{C}_V$ a $\mathcal{T}_{/V}$–functor $\overline{d} : \underline{\mathrm{const}}_V(K) \longrightarrow \underline{\mathcal{C}}_{\underline{V}}$. In this case the desired comparison is an equivalence by virtue of the following diagram

$$\begin{tikzcd} \mathrm{colim}_K \mathrm{Map}_{\mathcal{C}_V}(C_V,d) \arrow[r] \arrow[d, equal] & \mathrm{Map}_{\mathcal{C}_V}(C_V,\mathrm{colim}_K d) \arrow[d, equal] \\ \big(\underline{\mathrm{colim}}_{\underline{\mathrm{const}}_V(K)}\underline{\mathrm{Map}}_{\underline{\mathcal{C}}_{\underline{V}}}(C_{\underline{V}},d)\big)_V \arrow[r, "\simeq"] & \big(\underline{\mathrm{Map}}_{\underline{\mathcal{C}}_{\underline{V}}}(C_{\underline{V}},\underline{\mathrm{colim}}_{\underline{\mathrm{const}}_V(K)}\overline{d})\big)_V \end{tikzcd}$$

where the bottom map is an equivalence by hypothesis. This finishes the proof. □

Observation 2.1.5. By the characterisation of $\mathcal{T}$–compactness above together with the $\mathcal{T}$–Yoneda Lemma 1.2.20, and that $\mathcal{T}$–colimits in $\mathcal{T}$–functor categories are computed in the target by Proposition 1.2.12 we see that the $\mathcal{T}$–Yoneda embedding lands in $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})^{\underline{\kappa}}$.

Proposition 2.1.6 ($\mathcal{T}$–compact closure, "[Lur09, Cor. 5.3.4.15]")**.** *Let κ be a regular cardinal and $\underline{\mathcal{C}}$ be $\mathcal{T}$–cocomplete. Then $\underline{\mathcal{C}}^{\underline{\kappa}}$ is closed under κ-small $\mathcal{T}$–colimits in $\underline{\mathcal{C}}$, and hence is κ–$\mathcal{T}$–cocomplete.*

Proof. Let $d : K \to \underline{\mathcal{C}}^{\underline{\kappa}}$ be a κ-small $\mathcal{T}$–diagram. Since all κ-small $\mathcal{T}$–colimits can be decomposed as κ-small fibrewise $\mathcal{T}$–colimits and $\mathcal{T}$–coproducts by the decomposition principle in Theorem 1.2.9 (3), we just have to treat these two special cases. The former case is clear by [Lur09, Cor. 5.3.4.15] since everything is fibrewise. For the latter case, let $\underline{V}$ be a corepresentable $\mathcal{T}$–category, A be a κ-filtered category, and $f : \underline{\mathrm{const}}_{\mathcal{T}}(A) \to \underline{\mathcal{C}}$ be a κ-filtered fibrewise $\mathcal{T}$–diagram. We need to show that the map in $\underline{\mathcal{S}}_{\mathcal{T}}$

$$\underline{\mathrm{colim}}_{\underline{\mathrm{const}}_{\mathcal{T}}(A)}\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(\underline{\mathrm{colim}}_{\underline{V}}d, f) \longrightarrow \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(\underline{\mathrm{colim}}_{\underline{V}}d, \underline{\mathrm{colim}}_{\underline{\mathrm{const}}_{\mathcal{T}}(A)}f)$$

is an equivalence. In this case, since we have for the source

$$\underline{\mathrm{colim}}_{\underline{\mathrm{const}}_{\mathcal{T}}(A)}\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(\underline{\mathrm{colim}}_{\underline{V}}d, f) \simeq \underline{\mathrm{colim}}_{\underline{\mathrm{const}}_{\mathcal{T}}(A)}\underline{\lim}_{\underline{V}^{\mathrm{op}}}\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(d, f)$$

and for the target

$$\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(\underline{\mathrm{colim}}_{\underline{V}}d, \underline{\mathrm{colim}}_{\underline{\mathrm{const}}_{\mathcal{T}}(A)}f) \simeq \underline{\lim}_{\underline{V}^{\mathrm{op}}}\underline{\mathrm{colim}}_{\underline{\mathrm{const}}_{\mathcal{T}}(A)}\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(d, f),$$

Theorem 1.2.44 gives the required equivalence, using also that $\underline{V}^{\mathrm{op}}$ is still corepresentable by Observation 1.1.8. $\square$

2.1.2 Parametrised Ind-completions and accessibility

Proposition 2.1.7 ($\underline{\mathrm{Ind}}$ fully faithfulness, "[Lur09, Prop. 5.3.5.11]")**.** *Let $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$ and $\underline{\mathcal{D}} \in \widehat{\mathrm{Cat}}_{\mathcal{T}}$ which strongly admits fibrewise κ-filtered colimits. Suppose $F : \underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{C}} \to \underline{\mathcal{D}}$ strongly preserves fibrewise κ-filtered colimits and $f = F \circ j : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$.*

(i) If f is $\mathcal{T}$–fully faithful and the $\mathcal{T}$–essential image lands in $\underline{\mathcal{D}}^{\underline{\kappa}}$, then F is $\mathcal{T}$–fully faithful.

(ii) If f is $\mathcal{T}$–fully faithful, lands in $\underline{\mathcal{D}}^{\underline{\kappa}}$, and the $\mathcal{T}$–essential image of f generates $\underline{\mathcal{D}}$ under fibrewise κ-filtered colimits, then F is moreover a $\mathcal{T}$–equivalence.

Proof. We prove (i) two steps. The goal is to show that

$$\underline{\mathrm{Map}}_{\underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{C}}}(A, B) \to \underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(FA, FB)$$

is an equivalence. First suppose $A \in \underline{\mathcal{C}}$ and write $B \simeq \underline{\mathrm{colim}}_i B_i$ as a fibrewise filtered colimit where $B_i \in \underline{\mathcal{C}}$. We can equivalently compute $\underline{\mathrm{Map}}_{\underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{C}}}(A, B)$ as $\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})}(A, B)$, and so

$$\begin{aligned} \underline{\mathrm{Map}}_{\underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{C}}}(A, B) \simeq \underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})}(A, \underline{\mathrm{colim}}_i B_i) &\simeq \underline{\mathrm{colim}}_i \underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})}(A, B_i) \\ &\simeq \underline{\mathrm{colim}}_i \underline{\mathrm{Map}}_{\underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{C}}}(A, B_i) \end{aligned}$$

and

$$\underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(FA, F\underline{\mathrm{colim}}_i B_i) \simeq \underline{\mathrm{colim}}_i \underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(fA, fB_i)$$

where for the second equivalence we have used both hypotheses that F preserves fibrewise κ-filtered colimits and that the image lands in $\underline{\mathcal{D}}^{\underline{\kappa}}$. This completes this case. For a general $A \simeq \underline{\mathrm{colim}}_i A_i$ where $A_i \in \underline{\mathcal{C}}$ and the $\mathcal{T}$–colimit is fibrewise κ-filtered, we have

$$\begin{aligned}\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})}(A,B) \simeq \underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})}(\underline{\mathrm{colim}}_i A_i, B) &\simeq \underline{\lim}_i \underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})}(A_i, B)\\ &\xrightarrow{\simeq} \underline{\lim}_i \underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(FA_i, FB)\\ &\simeq \underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(FA, FB)\end{aligned}$$

where the third equivalence is by the special case above, and so we are done. For (ii), we have shown $\mathcal{T}$–fully faithfulness, and $\mathcal{T}$–essential surjectivity is by hypothesis. □

Lemma 2.1.8. *Let $\underline{\mathcal{D}} \in \mathrm{Cat}_{\mathcal{T}}$. Then the $\mathcal{T}$–Yoneda embedding $y\colon \underline{\mathcal{D}} \hookrightarrow \underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}})$ strongly preserves finite $\mathcal{T}$–colimits.*

Proof. Suppose $k : \underline{K} \to \underline{\mathcal{D}}$ is a finite $\mathcal{T}$–diagram. We need to show that the map

$$\underline{\mathrm{colim}}_{\underline{K}} \underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(-,k) \to \underline{\mathrm{Map}}_{\underline{\mathcal{D}}}(-, \underline{\mathrm{colim}}_{\underline{K}} k)$$

in $\underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}})$ is an equivalence. So let $\varphi \in \underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}})$ be an arbitrary object. Then mapping the morphism above into this and using Yoneda, we obtain

$$\varphi(\underline{\mathrm{colim}}_{\underline{K}} k) \longrightarrow \underline{\lim}_{\underline{K}^{\mathrm{op}}} \varphi(k)$$

which is an equivalence since φ is a $\mathcal{T}$–left exact functor. □

We thank Maxime Ramzi for teaching us the following slick proof, which is different from the standard one from [BGT13, Prop. 3.2], for instance.

Proposition 2.1.9. *Let $\underline{\mathcal{D}} \in \mathrm{Cat}_{\mathcal{T}}$. Then $\underline{\mathrm{Ind}}(\underline{\mathcal{D}}) \simeq \underline{\mathrm{Fun}}^{\mathrm{lex}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}})$. In particular, if $\underline{\mathcal{D}}$ were $\mathcal{T}$–stable, then $\underline{\mathrm{Ind}}(\underline{\mathcal{D}}) \simeq \underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_{\mathcal{T}})$.*

Proof. First of all, note that $\underline{\mathrm{Fun}}^{\mathrm{lex}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}}) \subseteq \underline{\mathrm{Fun}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}})$ is closed under fibrewise filtered colimits since fibrewise filtered colimits commutes with finite $\mathcal{T}$–limits in $\underline{\mathcal{S}}_{\mathcal{T}}$ by Theorem 1.2.44. Hence $y\colon \underline{\mathcal{D}} \hookrightarrow \underline{\mathrm{Fun}}^{\mathrm{lex}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}})$ induces $\overline{y}\colon \underline{\mathrm{Ind}}(\underline{\mathcal{D}}) \longrightarrow \underline{\mathrm{Fun}}^{\mathrm{lex}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}})$ which we then know is $\mathcal{T}$–fully faithful by Proposition 2.1.7. Moreover, since y strongly preserves finite $\mathcal{T}$–colimits by Lemma 2.1.8, $\overline{y}$ strongly preserves small $\mathcal{T}$–colimits. Hence, by Theorem 2.2.3, it has a right adjoint $R : \underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{D}}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}}) \to \underline{\mathrm{Ind}}(\underline{\mathcal{D}})$ (we are free to use this result here since the present situation will not feature anywhere in the proof of adjoint functor theorem). If we can show that this right adjoint is conservative, then we would have shown that $\overline{y}$ and R are inverse equivalences. But conservativity is clear by mapping from representable functors and an immediate application of Yoneda. Finally, the statement for the $\mathcal{T}$–stable case is a straightforward consequence of Theorem 1.3.8. □

Proposition 2.1.10 ("[Lur09, Prop. 5.3.5.12]"). *Let $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$ and κ a regular cardinal. Then the canonical functor $F : \underline{\mathrm{Ind}}_{\kappa}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})^{\underline{\kappa}}) \to \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ is an equivalence.*

Proof. To see that F is an equivalence, we want to apply Proposition 2.1.7. Let $j : \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})^{\underline{\kappa}} \hookrightarrow \underline{\mathrm{Ind}}_{\kappa}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})^{\underline{\kappa}})$ be the canonical embedding. That the composite $f := F \circ j$ is $\mathcal{T}$–fully faithful and lands in $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})^{\underline{\kappa}}$ is clear. To see that the essential image of f generates $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ under fibrewise κ-filtered colimits, recall that any $X \in \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ can be written as a small $\mathcal{T}$–colimit of a diagram valued in $\underline{\mathcal{C}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ by Theorem 1.2.23. Then Corollary 1.2.43 gives that X can be written as a fibrewise κ-filtered colimit taking values in $\underline{\mathcal{E}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ where each object of $\underline{\mathcal{E}}$ is itself a κ-small $\mathcal{T}$–colimit of some diagram taking values in $\underline{\mathcal{C}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})^{\underline{\kappa}}$. But then by Proposition 2.1.6 we know that $\underline{\mathcal{E}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})^{\underline{\kappa}}$, and so this completes the proof. □

Proposition 2.1.11 (Characterisation of $\mathcal{T}$–compacts in $\mathcal{T}$–presheaves, "[Lur09, Prop. 5.3.4.17]"). *Let $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$ and κ a regular cardinal. Then a $\mathcal{T}$–object $C \in \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ is κ–$\mathcal{T}$–compact if and only if it is a retract of a κ-small $\mathcal{T}$–colimit indexed in $\underline{\mathcal{C}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$.*

Proof. The if direction is clear since $\underline{\mathcal{C}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})^{\underline{\kappa}}$ and by the compact closure of Proposition 2.1.6 we know that κ–$\mathcal{T}$–compacts are closed under κ-small $\mathcal{T}$–colimits and retracts.

Now suppose C is κ–$\mathcal{T}$–compact. First of all recall by Theorem 1.2.23 that $C \simeq \underline{\mathrm{colim}}_a j(B_a)$ where $j : \underline{\mathcal{C}} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ is the $\mathcal{T}$–Yoneda embedding and $B_a \in \underline{\mathcal{C}}$. Combining this with Corollary 1.2.43 yields

$$C = \underline{\mathrm{colim}}_a j(B_a) \simeq \underline{\mathrm{colim}}_{f \in \underline{\mathrm{const}}_{\mathcal{T}}(F)} \underline{\mathrm{colim}}_{\left(p_f : K_f \to \underline{\mathcal{C}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})\right)} p_f$$

where F is a κ-filtered category. But then by Proposition 2.1.4 we then have that

$$\mathrm{id}_C \in \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(C, C) \simeq \underline{\mathrm{colim}}_{f \in \underline{\mathrm{const}}_{\mathcal{T}}(F)} \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(C, \underline{\mathrm{colim}}_{\left(p_f : K_f \to \underline{\mathcal{C}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})\right)} p_f)$$

Hence we see that C is a retract of some $\underline{\mathrm{colim}}_{\left(p_f : K_f \to \underline{\mathcal{C}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})\right)} p_f$ as required. □

Definition 2.1.12. Let κ be a regular cardinal and $\underline{\mathcal{C}}$ a $\mathcal{T}$–category. We say that $\underline{\mathcal{C}}$ is *κ–$\mathcal{T}$–accessible* if there is a small $\mathcal{T}$–category $\underline{\mathcal{C}}^0$ and a $\mathcal{T}$–equivalence $\underline{\mathrm{Ind}}_{\kappa}(\underline{\mathcal{C}}^0) \to \underline{\mathcal{C}}$. We say that $\underline{\mathcal{C}}$ is *$\mathcal{T}$–accessible* if it is κ–$\mathcal{T}$–accessible for some regular cardinal κ. A $\mathcal{T}$–functor out of a $\mathcal{T}$–accessible $\underline{\mathcal{C}}$ is said to be *$\mathcal{T}$-accessible* if it strongly preserves all fibrewise κ-filtered colimits for some regular cardinal κ.

Lemma 2.1.13 ($\mathcal{T}$–accessibility of $\mathcal{T}$–adjoints, "[Lur09, Prop. 5.4.7.7]"). *Let $G : \underline{\mathcal{C}} \to \underline{\mathcal{C}}'$ be a $\mathcal{T}$–functor between $\mathcal{T}$–accessibles. If G admits a right or a left $\mathcal{T}$–adjoint, then G is $\mathcal{T}$–accessible.*

Proof. The case of left $\mathcal{T}$–adjoints is clear since these strongly preserve all $\mathcal{T}$–colimits, so suppose $G \dashv F$. Choose a regular cardinal κ so that $\underline{\mathcal{C}}'$ is κ-accessible, ie. $\underline{\mathcal{C}}' = \underline{\mathrm{Ind}}_\kappa\underline{\mathcal{D}}$ for some $\underline{\mathcal{D}}$ small. Consider the composite $\underline{\mathcal{D}} \xrightarrow{j} \underline{\mathrm{Ind}}_\kappa\underline{\mathcal{D}} \xrightarrow{F} \underline{\mathcal{C}}$. Since $\underline{\mathcal{D}}$ is small there is a regular cardinal $\tau \gg \kappa$ so that both $\underline{\mathcal{C}}$ is τ-accessible and the essential image of $F \circ j$ consists of τ-$\mathcal{T}$–compact objects of $\underline{\mathcal{C}}$. We will show that G strongly preserves fibrewise τ-filtered colimits.

Since $\underline{\mathrm{Ind}}_\kappa\underline{\mathcal{D}} \subseteq \underline{\mathrm{PSh}}_\mathcal{T}(\underline{\mathcal{D}})$ is stable under small τ-filtered colimits by Proposition 1.2.38 it will suffice to prove that

$$G' : \underline{\mathcal{C}} \xrightarrow{G} \underline{\mathrm{Ind}}_\kappa\underline{\mathcal{D}} \to \underline{\mathrm{PSh}}_\mathcal{T}(\underline{\mathcal{D}})$$

preserves fibrewise τ-filtered colimits. Since colimits in presheaf categories are computed pointwise by Proposition 1.2.12 it suffices to show this when evaluated at each $D \in \mathcal{D}_V$ for all $V \in \mathcal{T}$. Without loss of generality we just work with $D \in \mathcal{D}$, ie. a $\mathcal{T}$–object $D \in \mathrm{Fun}_\mathcal{T}(\underline{*}, \underline{\mathcal{D}})$. In other words, by the $\mathcal{T}$–Yoneda lemma we just need to show that

$$G'_D : \underline{\mathcal{C}} \xrightarrow{G} \underline{\mathrm{Ind}}_\kappa\underline{\mathcal{D}} \hookrightarrow \underline{\mathrm{PSh}}_\mathcal{T}(\underline{\mathcal{D}}) \xrightarrow{\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_\mathcal{T}(\underline{\mathcal{D}})}(j(D),-)} \underline{\mathcal{S}}_\mathcal{T}$$

preserves fibrewise τ-filtered colimits. But G is a right adjoint and so by Lemma 1.1.27

$$\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_\mathcal{T}(\underline{\mathcal{D}})}(j(D), G(-)) \simeq \underline{\mathrm{Map}}_{\underline{\mathrm{Ind}}_\kappa\underline{\mathcal{D}}}(j(D), G(-)) \simeq \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(Fj(D), -)$$

By assumption on τ, Fj lands in τ-compact objects, completing the proof. □

2.1.3 Parametrised idempotent-completeness

Recall that every retraction $r : X \rightleftarrows M : i$ gives rise to an idempotent self-map $i \circ r$ of X since $(i \circ r) \circ (i \circ r) \simeq i \circ (r \circ i) \circ r \simeq i \circ r$. On the other hand, in general, not every idempotent self-map of an object in a category arises in this way, and a category is defined to be idempotent-complete if every idempotent self-map of an object arises from a retraction (cf. [Lur09, §4.4.5]). We now introduce the parametrised version of this.

Definition 2.1.14. A $\mathcal{T}$–category is said to be *$\mathcal{T}$-idempotent-complete* if it is so fibrewise. A $\mathcal{T}$–functor $f : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is said to be a *$\mathcal{T}$-idempotent-completion* if it is fibrewise an idempotent-completion (cf. [Lur09, Def. 5.1.4.1]).

Observation 2.1.15 (Consequences of fibrewise definitions)**.** Here are some facts we can immediately glean from our fibrewise definitions.

(i) We know that for $\underline{\mathcal{C}}$ small, $\mathrm{Ind}_\kappa(\mathrm{Ind}_\kappa(\mathcal{C})^\kappa) \simeq \mathrm{Ind}_\kappa(\mathcal{C})$, and so since $\mathcal{T}$–compactness and $\mathcal{T}$–Ind objects are fibrewise notions, we also get that for any small $\mathcal{T}$–category $\underline{\mathcal{C}}$ we have $\underline{\mathrm{Ind}}_\kappa(\underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}})^{\underline{\kappa}}) \simeq \underline{\mathrm{Ind}}_\kappa\underline{\mathcal{C}}$. Here we have used crucially that $\underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}})^{\underline{\kappa}}$ is really just fibrewise compact, that is, that the cocartesian lifts of the cocartesian fibration $\underline{\mathrm{Ind}}_\kappa\underline{\mathcal{C}} \to \mathcal{T}^{\mathrm{op}}$ preserve κ-compact objects. This is because $\mathrm{Ind}_\kappa(-)^\kappa$ computes the idempotent-completion by [Lur09, Lem. 5.4.2.4], which is a functor.

(ii) By the same token, $\underline{\mathcal{C}} \to (\underline{\mathrm{Ind}}_\kappa\underline{\mathcal{C}})^{\underline{\kappa}}$ exhibits the $\mathcal{T}$–idempotent-completion of $\underline{\mathcal{C}}$ for any small $\mathcal{T}$–category $\underline{\mathcal{C}}$.

The following result will be crucial in the proof of Theorem 2.2.2.

Proposition 2.1.16 ($\mathcal{T}$–Yoneda of idempotent-complete, "[Lur09, Prop. 5.3.4.18]")**.** *Let $\underline{\mathcal{C}}$ be a small $\mathcal{T}$–idempotent-complete $\mathcal{T}$–category which is κ–$\mathcal{T}$–cocomplete. Then the $\mathcal{T}$–Yoneda embedding $j : \underline{\mathcal{C}} \to \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})^{\underline{\kappa}}$ has a $\mathcal{T}$–left adjoint.*

Proof. By Proposition 1.2.26 we construct the adjunction objectwise. Let $\underline{\mathcal{D}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ be the full subcategory generated by all presheaves M where there exists $\ell M \in \underline{\mathcal{C}}$ satisfying

$$\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})}(M, j(-)) \simeq \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(\ell M, -)$$

By definition, the desired left adjoint exists on this full subcategory, and hence it would suffice now to show that $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})^{\underline{\kappa}} \subseteq \underline{\mathcal{D}}$.

We first claim that $\underline{\mathcal{D}}$ is closed under retracts and inherits κ–$\mathcal{T}$–cocompleteness from $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$. If $\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})}(N, j(-))$ is a retract of $\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})}(M, j(-))$ inside $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$. But then $\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})}(M, j(-))$ is in the Yoneda image from $\underline{\mathcal{C}}$, which is idempotent-complete, and hence its retract is also in the Yoneda image.

To see that $\underline{\mathcal{D}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ inherits κ–$\mathcal{T}$–cocompleteness, consider

$$\begin{aligned}\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})}(\underline{\mathrm{colim}}_{\underline{K}} M_k, j(-)) &\simeq \underline{\lim}_{\underline{K}^{\mathrm{op}}} \underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})}(M_k, j(-)) \\ &\simeq \underline{\lim}_{\underline{K}^{\mathrm{op}}} \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(\ell M_k, -) \\ &\simeq \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(\underline{\mathrm{colim}}_{\underline{K}} \ell M_k, -)\end{aligned}$$

where the last is since $\underline{\mathcal{C}}$ is κ–$\mathcal{T}$–cocomplete by hypothesis.

Now Proposition 2.1.11 says that everything in $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ is a retract of κ-small $\mathcal{T}$–colimits of the Yoneda image $\underline{\mathcal{C}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$. Hence, since $\underline{\mathcal{C}} \subseteq \underline{\mathcal{D}}$ clearly, the paragraphs above yield that $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})^{\underline{\kappa}} \subseteq \underline{\mathcal{D}}$ as required. □

2.2 Parametrised presentability

We are now ready to formulate and prove two of the main results in this paper, namely the characterisations of $\mathcal{T}$–presentables in Theorem 2.2.2 and the $\mathcal{T}$–adjoint functor theorem, Theorem 2.2.3. As we shall see, given all the technology that we have, the proofs for these parametrised versions will present us with no especial difficulties either because we can mimic the proofs of [Lur09] almost word-for-word, or because we can deduce them from the unparametrised versions (as in the cases of the adjoint functor theorem or the presentable Dwyer-Kan localisation Theorem 2.2.10). In subsections §2.2.3 and §2.2.4 we will also develop the important construction of *localisation–cocompletions*. We will then prove the parametrised analogue of the correspondence between presentable categories and small idempotent-complete ones in Theorem 2.2.16 as well as record the various expected permanence properties for parametrised presentability in §2.2.7 and §2.2.6.

2.2.1 Characterisations of parametrised presentability

Definition 2.2.1. A $\mathcal{T}$–category $\underline{\mathcal{C}}$ is $\mathcal{T}$–*presentable* if $\underline{\mathcal{C}}$ is $\mathcal{T}$–accessible and is $\mathcal{T}$–cocomplete.

We are now ready for the Lurie-Simpson-style characterisations of parametrised presentability. Note that characterisation (7) is a purely parametrised phenomenon and has no analogue in the unparametrised world. The proofs for the equivalences between the first six characterisations is exactly the arguments in [Lur09] and so the expert reader might want to jump ahead to the parts that concern point (7).

Theorem 2.2.2 (Characterisations for parametrised presentability, "[Lur09, Thm. 5.5.1.1]")**.** *Let $\underline{\mathcal{C}}$ be a $\mathcal{T}$–category. Then the following are equivalent:*

(1) $\underline{\mathcal{C}}$ is $\mathcal{T}$–presentable.

(2) $\underline{\mathcal{C}}$ is $\mathcal{T}$–accessible, and for every regular cardinal κ, $\underline{\mathcal{C}}^{\underline{\kappa}}$ is κ–$\mathcal{T}$–cocomplete.

(3) There exists a regular cardinal κ such that $\underline{\mathcal{C}}$ is κ–$\mathcal{T}$–accessible and $\underline{\mathcal{C}}^{\underline{\kappa}}$ is κ–$\mathcal{T}$–cocomplete

(4) There exists a regular cardinal κ, a small $\mathcal{T}$–idempotent-complete and κ–$\mathcal{T}$–cocomplete category $\underline{\mathcal{D}}$, and an equivalence $\underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{D}} \to \underline{\mathcal{C}}$. In fact, this $\underline{\mathcal{D}}$ can be chosen to be $\underline{\mathcal{C}}^{\underline{\kappa}}$.

(5) There exists a small $\mathcal{T}$–idempotent-complete category $\underline{\mathcal{D}}$ such that $\underline{\mathcal{C}}$ is a κ–$\mathcal{T}$–accessible Bousfield localisation of $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})$. By definition, this means that the image is κ–$\mathcal{T}$–accessible, and so by Lemma 2.1.13 the $\mathcal{T}$–right adjoint is also a κ–$\mathcal{T}$–accessible functor and hence the Bousfield localisation preserves κ–$\mathcal{T}$–compacts.

(6) $\underline{\mathcal{C}}$ is locally small and is $\mathcal{T}$–cocomplete, and there is a regular cardinal κ and a small set $\mathcal{G}$ of T-κ-compact objects of $\underline{\mathcal{C}}$ such that every $\mathcal{T}$–object of $\underline{\mathcal{C}}$ is a small $\mathcal{T}$–colimit of objects in $\mathcal{G}$.

(7) $\underline{\mathcal{C}}$ satisfies the left Beck-Chevalley condition (Terminology 1.2.8) and there is a regular cardinal κ such that the straightening $\mathcal{C} : \mathcal{T}^{\mathrm{op}} \longrightarrow \widehat{\mathrm{Cat}}$ factors through $\mathcal{C} : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{Pr}_{L,\kappa}$.

Proof. That (1) implies (2) is immediate from Proposition 2.1.6. That (2) implies (3) is because by definition of $\mathcal{T}$–accessibility, there is a κ such that $\underline{\mathcal{C}}$ is κ–$\mathcal{T}$–accessible, and since the second part of (2) says that $\underline{\mathcal{C}}^{\underline{\tau}}$ is τ-$\mathcal{T}$–cocomplete for all τ, this is true in particular for $\tau = \kappa$ so chosen. To see (3) implies (4), note that accessibility is a fibrewise condition and so we can apply the characterisation of accessibility in [Lur09, Prop. 5.4.2.2 (2)]. To see (4) implies (5), let $\underline{\mathcal{D}}$ be given by (4). We want to show that $\underline{\mathcal{C}}$ is a $\mathcal{T}$–accessible Bousfield localisation of $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})$. Consider the $\mathcal{T}$–Yoneda embedding (it lands in κ–$\mathcal{T}$–compacts by Observation 2.1.5)

$$j : \underline{\mathcal{D}} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})^{\underline{\kappa}}$$

This has a $\mathcal{T}$–left adjoint ℓ by Proposition 2.1.16. Define $L := \underline{\mathrm{Ind}}_\kappa(\ell)$ and $J := \underline{\mathrm{Ind}}_\kappa(j)$, so that, since $\underline{\mathrm{Ind}}_\kappa$ is a fibrewise construction, we have a $\mathcal{T}$–adjunction by Lemma 1.2.41

$$L : \underline{\mathrm{Ind}}_\kappa(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})^{\underline{\kappa}}) \rightleftarrows \underline{\mathrm{Ind}}_\kappa\underline{\mathcal{D}} : J$$

where J is $\mathcal{T}$–fully faithful by Proposition 2.1.7. But then by Proposition 2.1.10, we get $\underline{\mathrm{Ind}}_\kappa(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})^{\underline{\kappa}}) \simeq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})$ and this completes this implication.

To see (5) implies (6), first of all $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})$ is locally small and so $\underline{\mathcal{C}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})$ is too. Moreover, Bousfield local $\mathcal{T}$–subcategories always admit $\mathcal{T}$–colimits admitted by the ambient category and so $\underline{\mathcal{C}}$ is $\mathcal{T}$–cocomplete. For the last assertion, consider the composite

$$\varphi : \underline{\mathcal{D}} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}}) \xrightarrow{L} \underline{\mathcal{C}}$$

Since $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})$ is generated by $\underline{\mathcal{D}}$ under small $\mathcal{T}$–colimits by Theorem 1.2.23 and since L preserves $\mathcal{T}$–colimits, we see that $\underline{\mathcal{C}}$ is generated under $\mathcal{T}$–colimits by $\mathrm{Im}\,\varphi$. To see that $\mathrm{Im}\,\varphi \subseteq \underline{\mathcal{C}}^{\underline{\kappa}}$, note that since by hypothesis $\underline{\mathcal{C}}$ was κ–$\mathcal{T}$–accessible, we know from Lemma 2.1.13 that the $\mathcal{T}$–right adjoint of L is automatically $\mathcal{T}$–accessible, and so L preserves κ–$\mathcal{T}$–compacts, and we are done.

To see (6) implies (1), by definition, we just need to check that $\underline{\mathcal{C}}$ is κ–$\mathcal{T}$–accessible. Assumption (6) says that everything is a $\mathcal{T}$–colimit of $\mathcal{T}$–compacts, but we need to massage this to say that everything is a fibrewise κ-filtered $\mathcal{T}$–colimit of an essentially small subcategory - note this is where we need the assumption about $\mathcal{G}$ and not just use all of $\underline{\mathcal{C}}^{\underline{\kappa}}$, the problem being that the latter is not necessarily small. Let $\underline{\mathcal{C}}' \subseteq \underline{\mathcal{C}}^{\underline{\kappa}}$ be generated by $\mathcal{G}$ and $\underline{\mathcal{C}}' \subseteq \underline{\mathcal{C}}'' \subseteq \underline{\mathcal{C}}^{\underline{\kappa}}$ be the κ–$\mathcal{T}$–colimit closure of $\underline{\mathcal{C}}'$: here we are using that $\underline{\mathcal{C}}'' \subseteq \underline{\mathcal{C}}^{\underline{\kappa}}$ since κ–$\mathcal{T}$–compacts are closed under κ-small $\mathcal{T}$–colimits Proposition 2.1.6. Then since small $\mathcal{T}$–colimits decompose as κ-small $\mathcal{T}$–colimits and fibrewise κ-filtered colimits, we get that $\underline{\mathcal{C}}$ is generated by $\underline{\mathcal{C}}'' \subseteq \underline{\mathcal{C}}$ under κ-filtered colimits, as required.

Now to see (5) implies (7), suppose we have a $\mathcal{T}$–Bousfield localisation $F : \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}) \rightleftarrows \underline{\mathcal{D}} : G$. For $f : W \to V$ in $\mathcal{T}$ we have

$$\begin{tikzcd}
\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_V = \mathrm{Fun}(\mathrm{Total}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}} \times \underline{V}), \mathcal{S}) \arrow[r, shift left, "F_V"] \arrow[d, shift left, "f^*"] & \mathcal{D}_V \arrow[l, shift left, "G_V"] \arrow[d, shift left, "f^*"] \\
\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_W = \mathrm{Fun}(\mathrm{Total}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}} \times \underline{W}), \mathcal{S}) \arrow[u, shift left=3, "f_!"] \arrow[u, shift right=3, "f_*"'] \arrow[r, shift left, "F_W"] & \mathcal{D}_W \arrow[l, shift left, "G_W"] \arrow[u, dashed, shift left=3, "f_!"] \arrow[u, dashed, shift right=3, "f_*"']
\end{tikzcd}$$

where all the solid squares commute. We need to show a few things, namely:

- That the dashed adjoints exist.
- That $f^* : \mathcal{D}_V \to \mathcal{D}_W$ preserves κ-compacts.
- That $f_! \dashv f^*$ on $\underline{\mathcal{D}}$ satisfies the left Beck-Chevalley conditions.

To see that the dashed arrows exist, define $f_!$ to be $F_V \circ f_! \circ G_W$. This works since

$$\begin{aligned}
\mathrm{Map}_{\mathcal{D}_V}(F_V \circ f_! \circ G_W -, -) &\simeq \mathrm{Map}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_W}(G_W -, f^* \circ G_V -) \\
&\simeq \mathrm{Map}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_W}(G_W -, G_W \circ f^* -) \\
&\simeq \mathrm{Map}_{\mathcal{D}_W}(-, f^* -)
\end{aligned}$$

To see that f_* exists, we need to see that f^* preserves ordinary colimits. For this, we use the description of colimits in Bousfield local subcategories. So let $\varphi : K \to \mathcal{D}_V$ be a diagram. Then

$$\begin{aligned}
f^* \operatorname*{colim}_{K \subseteq \mathcal{D}_V} \varphi &\simeq f^* F_V \big(\operatorname*{colim}_{K \subseteq \mathrm{PSh}_V} G_V \circ \varphi \big) \\
&\simeq F_W f^* \big(\operatorname*{colim}_{K \subseteq \mathrm{PSh}_V} G_V \circ \varphi \big) \\
&\simeq F_W \big(\operatorname*{colim}_{K \subseteq \mathrm{PSh}_W} f^* \circ G_V \circ \varphi \big) \\
&\simeq F_W \big(\operatorname*{colim}_{K \subseteq \mathrm{PSh}_W} G_W \circ f^* \circ \varphi \big) \\
&=: \operatorname*{colim}_{K \subseteq \mathcal{D}_W} f^* \circ \varphi
\end{aligned}$$

And hence f^* preserves colimits as required, and so by presentability, we obtain a right adjoint f_*. This completes the first point. Now to see that $f^* : \mathcal{D}_V \to \mathcal{D}_W$ preserves κ-compacts, note that $f^* : \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_V \to \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_W$ does since $f_* : \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_W \to \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_V$ is κ-accessible by Lemma 2.1.13. Hence since $f^* F_V \simeq F_W f^*$, taking right adjoints we get $f_* G_W \simeq G_V f_*$. By hypothesis (5), G was κ-accessible and so since it is also fully faithful fibrewise, we get that $f_* : \mathcal{D}_W \to \mathcal{D}_V$ is κ-accessible, as required. For the third point, we already know from Proposition 1.2.11 that $\underline{\mathcal{D}}$ is $\mathcal{T}$–cocomplete, and so $f_!$ must *necessarily* give the indexed coproducts which satisfy the left Beck-Chevalley condition by Theorem 1.2.9.

Finally to see (7) implies (1), Theorem 1.2.9 says that $\underline{\mathcal{C}}$ is $\mathcal{T}$–cocomplete, and so we are left to show that it is κ–$\mathcal{T}$–accessible. But then this is just because

$\underline{\mathcal{C}} \simeq \underline{\mathrm{Ind}}_\kappa(\underline{\mathcal{C}}^{\underline{\kappa}})$ by [Lur09, Prop. 5.3.5.12] (since parametrised-compacts and ind-completion is just fibrewise ordinary compacts/ind-completion because the straightening lands in $\mathrm{Pr}_{L,\kappa}$). This completes the proof for this step and for the theorem. □

2.2.2 The adjoint functor theorem

We now deduce the parametrised version of the adjoint functor theorem from the unparametrised version using characterisation (7) of Theorem 2.2.2. Interestingly, and perhaps instructively, the proof shows us precisely where we need the notion of strong preservation and not just preservation (cf. Definition 1.2.3 and the discussion in Observation 1.2.16).

Theorem 2.2.3 (Parametrised adjoint functor theorem)**.** *Let $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a $\mathcal{T}$–functor between $\mathcal{T}$–presentable categories. Then:*

(1) If F strongly preserves $\mathcal{T}$–colimits, then F admits a $\mathcal{T}$–right adjoint.

(2) If F strongly preserves $\mathcal{T}$–limits and is $\mathcal{T}$–accessible, then F admits a $\mathcal{T}$–left adjoint.

Proof. We want to apply Corollary 1.1.25. To see (1), observe that the ordinary adjoint functor theorem gives us fibrewise right adjoints $F_V : \mathcal{C}_V \rightleftarrows \mathcal{D}_V : G_V$. To see that this assembles to a $\mathcal{T}$–functor G, we just need to check that the dashed square in the diagram

$$\begin{tikzcd}
\mathcal{C}_V \arrow[r, shift left, "F_V"] \arrow[d, shift right, "f_!"'] & \mathcal{D}_V \arrow[l, shift left, dashed, "G_W"] \arrow[d, shift right, "f_!"'] \\
\mathcal{C}_W \arrow[r, shift left, "F_W"] \arrow[u, shift right, dashed, "f^*"'] & \mathcal{D}_W \arrow[l, shift left, dashed, "G_W"] \arrow[u, shift right, dashed, "f^*"']
\end{tikzcd}$$

commutes. But then the left adjoints of the dashed compositions are the solid ones, which we know to be commutative by hypothesis that F strongly preserves $\mathcal{T}$–colimits (and so in particular indexed coproducts, see Observation 1.2.16). Hence we are done for this case and part (2) is similar. □

We will need the following characterisation of functors that strongly preserve $\mathcal{T}$–colimits between $\mathcal{T}$–presentables in order to understand the correspondence between $\mathcal{T}$–presentable categories and small $\mathcal{T}$–idempotent-complete ones.

Proposition 2.2.4 ("[Lur09, Prop. 5.5.1.9]")**.** *Let $f : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a $\mathcal{T}$–functor between $\mathcal{T}$–presentables and suppose $\underline{\mathcal{C}}$ is κ–$\mathcal{T}$–accessible. Then the following are equivalent:*

(a) The functor f strongly preserves $\mathcal{T}$–colimits

(b) *The functor f strongly preserves fibrewise κ-filtered colimits, and the restriction $f|_{\underline{\mathcal{C}}^{\kappa}}$ strongly preserves κ–$\mathcal{T}$–colimits.*

Proof. That (a) implies (b) is clear since $\underline{\mathcal{C}}^{\kappa} \subseteq \underline{\mathcal{C}}$ creates $\mathcal{T}$–colimits by Proposition 2.1.6. Now to see (b) implies (a), let $\underline{\mathcal{C}} = \underline{\mathrm{Ind}}_{\kappa}(\underline{\mathcal{C}}^{\kappa})$ where $\underline{\mathcal{C}}^{\kappa}$ is κ–$\mathcal{T}$–cocomplete and $\mathcal{T}$–idempotent-complete category by Proposition 2.1.6. Now by the proof of (4) implies (5) in Theorem 2.2.2 we have a $\mathcal{T}$–Bousfield adjunction

$$L : \underline{\mathrm{PSh}}(\underline{\mathcal{C}}^{\kappa}) \rightleftarrows \underline{\mathcal{C}} : k$$

Now consider the composite

$$j^*f : \underline{\mathcal{C}}^{\kappa} \xrightarrow{j} \underline{\mathcal{C}} \xrightarrow{f} \underline{\mathcal{D}}$$

By the universal property of $\mathcal{T}$–presheaves we get a strongly $\mathcal{T}$–colimit-preserving functor F fitting into the diagram

$$\begin{tikzcd}
\underline{\mathcal{C}}^{\kappa} \arrow[r, "j^*f"] \arrow[d, hook, "j"'] \arrow[dd, bend right=60, "y"'] & \underline{\mathcal{D}} \\
\underline{\mathcal{C}} \arrow[ur, "f"] \arrow[d, hook, "k"'] & \\
\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\kappa}) \arrow[uur, "F:=y_!j^*f"'] &
\end{tikzcd}$$

We know then that $f \simeq k^*y_!j^*f = k^*F$. On the other hand, we can define a functor

$$F' := f \circ L \simeq F \circ k \circ L : \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\kappa}) \longrightarrow \underline{\mathcal{C}} \longrightarrow \underline{\mathcal{D}}$$

The $\mathcal{T}$–Bousfield adjunction unit $\mathrm{id}_{\underline{\mathrm{PSh}}_{\mathcal{T}}} \Rightarrow k \circ L$ gives us a natural transformation

$$\beta : F \Longrightarrow F' = F \circ k \circ L$$

If we can show that β is an equivalence then we would be done, since F, and so $F' = f \circ L$, strongly preserves $\mathcal{T}$–colimits. Hence since L was a $\mathcal{T}$–Bousfield localisation, f also strongly preserves $\mathcal{T}$–colimits, as required.

To see that β is an equivalence, let $\underline{\mathcal{E}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\kappa})$ be the full $\mathcal{T}$–subcategory on which β is an equivalence. Since both F and F' strongly preserve fibrewise κ-filtered colimits, we see that $\underline{\mathcal{E}}$ is stable under such. Hence it suffices to show that $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\kappa})^{\underline{\kappa}} \subseteq \underline{\mathcal{E}}$ since the inclusion will then induce the $\mathcal{T}$–functor $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\kappa}) \simeq \underline{\mathrm{Ind}}_{\kappa}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\kappa})^{\underline{\kappa}}) \to \underline{\mathcal{E}}$ which is an equivalence by Proposition 2.1.7 (2).

Since $L \circ k \simeq \mathrm{id}$ we clearly have $\underline{\mathcal{C}}^{\kappa} \subseteq \underline{\mathcal{E}}$, ie. that $\beta : F \Rightarrow F'$ is an equivalence on $\underline{\mathcal{C}}^{\kappa} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\kappa})$. On the other hand, by Proposition 2.1.6 we know that $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\kappa})^{\underline{\kappa}}$ is κ–$\mathcal{T}$–cocomplete, and its objects are retracts of κ-small $\mathcal{T}$–colimits valued in $\underline{\mathcal{C}}^{\kappa} \subseteq$

$\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\kappa}})$ by Proposition 2.1.11. Thus it suffices to show that F and F' strongly preserve κ-small $\mathcal{T}$–colimits when restricted to $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\kappa}})^{\underline{\kappa}}$. That F does is clear since it in fact strongly preserves all small $\mathcal{T}$–colimits. That F' does is because it can be written as the composition

$$F'|_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\kappa}})^{\underline{\kappa}}} : \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\kappa}})^{\underline{\kappa}} \xrightarrow{L} \underline{\mathcal{C}}^{\underline{\kappa}} \xrightarrow{f} \underline{\mathcal{D}}$$

where L is a $\mathcal{T}$–left adjoint and f strongly preserves κ-small $\mathcal{T}$–colimits by assumption. Here we have crucially used that L lands in $\underline{\mathcal{C}}^{\underline{\kappa}}$ since this category is $\mathcal{T}$–idempotent-complete and κ–$\mathcal{T}$–cocomplete. □

2.2.3 Dwyer-Kan localisations

Terminology 2.2.5. We recall the clarifying terminology of [Hin16] in distinguishing between Bousfield localisations, as defined in Definition 1.1.21, and *Dwyer-Kan localisations*. By the latter, we will mean the following: let $\underline{\mathcal{C}}$ be a $\mathcal{T}$–category and S a class of morphisms in $\underline{\mathcal{C}}$ such that $f^*(S_W) \subseteq S_V$ for all $f\colon V \to W$ in $\mathcal{T}$. Suppose a $\mathcal{T}$–category $S^{-1}\underline{\mathcal{C}}$ exists and is equipped with a map $f : \underline{\mathcal{C}} \to S^{-1}\underline{\mathcal{C}}$ inducing the equivalence

$$f^* : \underline{\mathrm{Fun}}_{\mathcal{T}}(S^{-1}\underline{\mathcal{C}}, \underline{\mathcal{D}}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}_{\mathcal{T}}^{S^{-1}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$$

for all $\mathcal{T}$–categories $\underline{\mathcal{D}}$, where $\underline{\mathrm{Fun}}_{\mathcal{T}}^{S^{-1}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \subseteq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ is the full subcategory of parametrised functors sending morphisms in S to equivalences. Such a $\mathcal{T}$–category must necessarily be unique if it exists, and this is then defined to be *the $\mathcal{T}$–Dwyer-Kan localisation of $\underline{\mathcal{C}}$ with respect to S*. The following proposition shows that being a $\mathcal{T}$–Bousfield localisation is stronger than that of being a $\mathcal{T}$–Dwyer-Kan localisation.

Proposition 2.2.6 (Bousfield implies Dwyer-Kan)**.** *Let $\underline{\mathcal{C}}, L\underline{\mathcal{C}}$ be $\mathcal{T}$–categories and $L : \underline{\mathcal{C}} \rightleftarrows L\underline{\mathcal{C}} : i$ be a $\mathcal{T}$–Bousfield localisation. Let S be the collection of morphisms in $\underline{\mathcal{C}}$ that are sent to equivalences under L. Then the functor L induces an equivalence $L^* : \underline{\mathrm{Fun}}_{\mathcal{T}}(L\underline{\mathcal{C}}, \underline{\mathcal{D}}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}_{\mathcal{T}}^{S^{-1}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ for any $\mathcal{T}$–category $\underline{\mathcal{D}}$ so that $L\underline{\mathcal{C}}$ is a Dwyer-Kan localisation against S.*

Proof. Since $L \dashv i$ was a $\mathcal{T}$–Bousfield localisation, we know that $i^* : \underline{\mathrm{Fun}}_{\mathcal{T}}(L\underline{\mathcal{C}}, \underline{\mathcal{D}}) \rightleftarrows \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) : L^*$ is also a $\mathcal{T}$–Bousfield localisation by Theorem 1.2.10, and so in particular L^* is $\mathcal{T}$–fully faithful. The image of L^* also clearly lands in $\underline{\mathrm{Fun}}_{\mathcal{T}}^{S^{-1}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$, and so we are left to show $\mathcal{T}$–essential surjectivity. By basechanging if necessary, we just show this on $\mathrm{Fun}_{\mathcal{T}}^{S^{-1}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$. Let $\varphi : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a $\mathcal{T}$–functor that inverts morphisms in S. We aim to show that $\varphi \Rightarrow \varphi \circ i \circ L$ is an equivalence. Since $L \dashv i$ was a $\mathcal{T}$–Bousfield localisation, the unit $\eta : \mathrm{id} \Rightarrow i \circ L$ gets sent to an equivalence under L, and so $\eta \in S$. Since φ inverts S by assumption, in particular it inverts η. □

Proposition 2.2.7. *$\mathcal{T}$–presentable categories are $\mathcal{T}$–complete.*

Proof. Let $\underline{\mathcal{C}}$ be $\mathcal{T}$–presentable so that it is a $\mathcal{T}$–Bousfield localisation of some $\mathcal{T}$–presheaf category $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})$ by description (5) of Theorem 2.2.2. We know that $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}})$ is $\mathcal{T}$–complete and so all we need to show is that $\mathcal{T}$–Bousfield local subcategories are closed under $\mathcal{T}$–limits which exist in the ambient category. But this is clear since $\mathcal{T}$–Bousfield local subcategories can be described by a mapping-into property. □

Observation 2.2.8. We can define strong saturation in the parametrised setting just to be strong saturation fibrewise, and clearly this condition is closed under arbitrary intersections as in [Lur09, Rmk. 5.5.4.7]. Therefore it still makes sense to talk about the strongly saturated class generated by a class of morphisms.

Terminology 2.2.9. For $S \subseteq \underline{\mathcal{C}}$ a collection of morphisms, an object $X \in \underline{\mathcal{C}}$ is said to be S–local if $\underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(-, X)$ sends morphisms in S to equivalences.

The following result, which will be crucial for our application in Part II, is another example of the value of characterisation (7) from Theorem 2.2.2. The proof of the unparametrised result, given by Lurie in [Lur09, §5.5.4], is long and technical, and characterisation (7) allows us to obviate this difficulty by bootstrapping from Lurie's statement.

Theorem 2.2.10 (Parametrised presentable Dwyer-Kan localisations)**.** *Let $\underline{\mathcal{C}}$ be a T-presentable category and S a small collection of $\mathcal{T}$-morphisms of $\underline{\mathcal{C}}$ (ie. if $f : V \to W$ in $\mathcal{T}$ and $y \to z$ a morphism in S_W, then $f^*y \to f^*z$ is in S_V). Let $S^{-1}\underline{\mathcal{C}} \subseteq \underline{\mathcal{C}}$ be the full subcategory of S-local objects. Then:*

(1) We have a $\mathcal{T}$–accessible $\mathcal{T}$–Bousfield localisation $L : \underline{\mathcal{C}} \rightleftarrows S^{-1}\underline{\mathcal{C}} : i$.

(2) For any $\mathcal{T}$–category $\underline{\mathcal{C}}$, the $\mathcal{T}$–functors $L^* : \underline{\mathrm{Fun}}_{\mathcal{T}}(S^{-1}\underline{\mathcal{C}}, \underline{\mathcal{D}}) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}^{S^{-1}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ and $L^* : \underline{\mathrm{Fun}}_{\mathcal{T}}^{L}(S^{-1}\underline{\mathcal{C}}, \underline{\mathcal{D}}) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}^{L,S^{-1}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ are equivalences.

Proof. For (1), we know from [Lur09, Prop. 5.5.4.15] that we already have fibrewise Bousfield localisations, and all we need to do is show that these assemble to a $\mathcal{T}$–Bousfield localisation via Corollary 1.1.25. Let $f : V \to W$ be in $\mathcal{T}$. We need to show that

$$\begin{tikzcd}
\mathcal{C}_V \arrow[r, "L_V"] & S^{-1}\mathcal{C}_V \\
\mathcal{C}_W \arrow[u, "f^*"] \arrow[r, "L_W"] & S^{-1}\mathcal{C}_W \arrow[u, "f^*"']
\end{tikzcd}$$

commutes, and for this, we first note that the diagram

$$\begin{tikzcd} \mathcal{C}_V \arrow[d, "f_*"'] & S^{-1}\mathcal{C}_V \arrow[l, hook', "i_V"'] \arrow[d, "f_*"] \\ \mathcal{C}_W & S^{-1}\mathcal{C}_W \arrow[l, hook', "i_W"] \end{tikzcd}$$

commutes where here f_* exists since $\underline{\mathcal{C}}$ is $\mathcal{T}$–complete by Proposition 2.2.7. Now recall by definition that $f^*(S_W) \subseteq S_V$ and so for $y \to z$ in S_W the map

$$\mathrm{Map}_{\mathcal{C}_W}(z, f_*x) \simeq \mathrm{Map}_{\mathcal{C}_V}(f^*z, x) \longrightarrow \mathrm{Map}_{\mathcal{C}_V}(f^*y, x) \simeq \mathrm{Map}_{\mathcal{C}_V}(y, f_*x)$$

is an equivalence, which implies that f_* takes S-local objects to S-local objects. Now by uniqueness of left adjoints, the first diagram commutes, as required. Now (2) is just a consequence of Proposition 2.2.6. □

2.2.4 Localisation–cocompletions

In this subsection we formulate and prove the construction of *localisation–cocompletions* whose proof is exactly analogous to that of [Lur09]. As far as we can see, unfortunately the proof cannot be bootstrapped from the unparametrised statement as with the proof of Theorem 2.2.10 because the notion of a parametrised collection of diagrams might involve diagrams that are not fibrewise in the sense of Example 1.2.2.

Definition 2.2.11 (Parametrised collection of diagrams)**.** Let $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}}$. A *parametrised collection of diagrams in* $\underline{\mathcal{C}}$ is defined to be a triple $(\underline{\mathcal{C}}, \mathcal{K}, \mathcal{R})$ where:

- $\mathcal{K}$ is a collection of small categories parametrised over $\mathcal{T}^{\mathrm{op}}$, ie. a collection $\mathcal{K}_V$ of small $\mathcal{T}_{/V}$–categories for each $V \in \mathcal{T}$.
- $\mathcal{R}$ is a parametrised collection of diagrams in $\underline{\mathcal{C}}$ whose indexing categories belong to $\mathcal{K}$, ie. for each $V \in \mathcal{T}$ a collection of coconed diagrams $\mathcal{R}_V$ indexed over categories in $\mathcal{K}_V$.

Theorem 2.2.12 ($\mathcal{T}$–localisation–cocompletions, "[Lur09, Prop. 5.3.6.2]")**.** *Let* $(\underline{\mathcal{C}}, \mathcal{K}, \mathcal{R})$ *be a parametrised collection of diagrams in* $\underline{\mathcal{C}}$*. Then there is a* $\mathcal{T}$*–category* $\underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}})$ *and a* $\mathcal{T}$*–functor* $j : \underline{\mathcal{C}} \to \underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}})$ *such that:*

(i) The category $\underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}})$ *is* $\mathcal{K}$*–*$\mathcal{T}$*–cocomplete, ie. it strongly admits* $\mathcal{K}$*–indexed* $\mathcal{T}$*–colimits,* $\underline{\mathcal{C}_V}$ *admits K–indexed* $\mathcal{T}_{/V}$*–colimits.*

(ii) For every $\mathcal{K}$*-*$\mathcal{T}$*–cocomplete category* $\underline{\mathcal{D}}$*, the map j induces an equivalence of* $\mathcal{T}$*–categories*

$$j^* : \underline{\mathrm{Fun}}^{\mathcal{K}}_{\mathcal{T}}(\underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}}), \underline{\mathcal{D}}) \longrightarrow \underline{\mathrm{Fun}}^{\mathcal{R}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$$

where the source denotes the $\mathcal{T}$*–category of functors which strongly preserve* $\mathcal{K}$*–indexed colimits and the target consists of those functors carrying each diagram in* $\mathcal{R}$ *to a parametrised colimit diagram in* $\underline{\mathcal{D}}$*.*

(iii) *If each member of $\mathcal{R}$ were already a $\mathcal{T}$–colimit diagram in $\underline{\mathcal{C}}$, then in fact j is $\mathcal{T}$–fully faithful.*

Proof. We give first all the constructions. By enlarging the universe, if necessary, we may reduce to the case where:

- Every element of $\mathcal{K}$ is small
- That $\underline{\mathcal{C}}$ is small
- The collection of diagrams $\mathcal{R}$ is small

Let $y : \underline{\mathcal{C}} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ be the $\mathcal{T}$–yoneda embedding and let $V \in \mathcal{T}$. For a $\mathcal{T}_{/V}$–diagram $\bar{p} : K^{\triangleright} \to \underline{\mathcal{C}}_{\underline{V}}$ with cone point Y, let X denote the $\mathcal{T}_{/V}$–colimit of $y \circ p : K \to \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_{\underline{V}}$. This induces a $\mathcal{T}_{/V}$–morphism in $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_{\underline{V}}$

$$s : X \to y(Y)$$

Here we have used that $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_{\underline{V}} \simeq \underline{\mathrm{PSh}}_{\underline{V}}(\underline{\mathcal{C}}_{\underline{V}})$ by Construction 1.1.13. Now let S be the set of all such $\mathcal{T}_{/V}$–morphisms running over all $V \in \mathcal{T}$. This is small by our assumption and so let $L : \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}) \to S^{-1}\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ denote the $\mathcal{T}$–Bousfield localisation from Theorem 2.2.10. Now we define $\underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}}) \subseteq S^{-1}\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ to be the smallest $\mathcal{K}$–cocomplete full $\mathcal{T}$–subcategory containing the image of $L \circ y : \underline{\mathcal{C}} \to \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}) \to S^{-1}\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$. We show that this works and prove each point in turn.

Point (i) is true by construction, and so there is nothing to do. For point (ii), let $\underline{\mathcal{D}}$ be $\mathcal{K}$–$\mathcal{T}$–cocomplete. We now perform a reduction to the case when $\underline{\mathcal{D}}$ is $\mathcal{T}$–cocomplete. By taking the opposite Yoneda embedding we see that $\underline{\mathcal{D}}$ sits $\mathcal{T}$–fully faithfully in a $\mathcal{T}$–cocomplete category $\underline{\mathcal{D}}'$ and the inclusion strongly preserves $\mathcal{K}$–colimits. We now have a square of $\mathcal{T}$–categories (where the vertical functors are $\mathcal{T}$–fully faithful by Corollary 1.2.34)

$$\begin{tikzcd}
\underline{\mathrm{Fun}}^{\mathcal{K}}_{\mathcal{T}}(\underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}}), \underline{\mathcal{D}}) \arrow[r, "\phi := j^*"] \arrow[d, hook] & \underline{\mathrm{Fun}}^{\mathcal{R}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \arrow[d, hook] \\
\underline{\mathrm{Fun}}^{\mathcal{K}}_{\mathcal{T}}(\underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}}), \underline{\mathcal{D}}') \arrow[r, "\phi' := j^*"] & \underline{\mathrm{Fun}}^{\mathcal{R}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}')
\end{tikzcd}$$

We claim this is cartesian in $\widehat{\mathrm{Cat}}_{\mathcal{T}}$ if ϕ' were an equivalence: given this, to prove that ϕ is an equivalence, it suffices to prove that ϕ' is an equivalence. For this, we need to show that the map into the pullback is an equivalence. That ϕ' is an equivalence ensures that the map into the pullback is fully faithful. To see essential surjectivity, let $F : \underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}}) \to \underline{\mathcal{D}}'$ be a strongly $\mathcal{K}$–colimit preserving functor that restricts to $\underline{\mathcal{C}} \to \underline{\mathcal{D}}$. Then in fact F lands in $\underline{\mathcal{D}} \subseteq \underline{\mathcal{D}}'$ since $\underline{\mathcal{D}} \subseteq \underline{\mathcal{D}}'$ is stable under $\mathcal{K}$–indexed colimits, and by construction, $\underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\mathcal{C})$ is generated under $\mathcal{K}$–indexed colimits by $\underline{\mathcal{C}}$.

Now we turn to showing ϕ is an equivalence in the case $\underline{\mathcal{D}}$ is $\mathcal{T}$–cocomplete. Let $\underline{\mathcal{E}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ be the inverse image $L^{-1}\underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}})$ and $\overline{S}$ be the collection of all morphisms α in $\underline{\mathcal{E}}$ such that $L\alpha$ is an equivalence. Since the $\mathcal{T}$–Bousfield localisation $L : \underline{\mathrm{PSh}}(\underline{\mathcal{C}}) \rightleftarrows S^{-1}\underline{\mathrm{PSh}}(\underline{\mathcal{C}}) : i$ induces a $\mathcal{T}$–Bousfield localisation $L : \underline{\mathcal{E}} \rightleftarrows \underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}}) : i$ we see by Proposition 2.2.6 that $L^* : \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}}), \underline{\mathcal{D}}) \to \underline{\mathrm{Fun}}^{\overline{S}^{-1}}_{\mathcal{T}}(\underline{\mathcal{E}}, \underline{\mathcal{D}})$ is an equivalence. Furthermore, by using the description of colimits in $\mathcal{T}$–Bousfield local subcategories as being given by applying the localisation L to the colimit in the ambient category, we see that $f : \underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}}) \to \underline{\mathcal{D}}$ strongly preserves $\mathcal{K}$–colimits if and only if $f \circ L : \underline{\mathcal{E}} \to \underline{\mathcal{D}}$ does. This gives us the following factorisation of ϕ

$$\phi : \underline{\mathrm{Fun}}^{\mathcal{K}}_{\mathcal{T}}(\underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}}), \underline{\mathcal{D}}) \xrightarrow[\simeq]{L^*} \underline{\mathrm{Fun}}^{\overline{S}^{-1},\mathcal{K}}_{\mathcal{T}}(\underline{\mathcal{E}}, \underline{\mathcal{D}}) \xrightarrow{j^*} \underline{\mathrm{Fun}}^{\mathcal{R}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$$

and hence we need to show that the functor j^* is an equivalence. Since $\underline{\mathcal{D}}$ is $\mathcal{T}$–cocomplete, we can consider the $\mathcal{T}$–adjunction $j_! : \underline{\mathrm{Fun}}^{\mathcal{R}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \rightleftarrows \underline{\mathrm{Fun}}^{\mathcal{K}}_{\mathcal{T}}(\underline{\mathcal{E}}, \underline{\mathcal{D}}) : j^*$. We need to show:

- that $j_!$ lands in $\underline{\mathrm{Fun}}^{\overline{S}^{-1},\mathcal{K}}_{\mathcal{T}}(\underline{\mathcal{E}}, \underline{\mathcal{D}})$,
- that $j_! \circ j^* \simeq \mathrm{id}$ on $\underline{\mathrm{Fun}}^{\overline{S}^{-1},\mathcal{K}}_{\mathcal{T}}(\underline{\mathcal{E}}, \underline{\mathcal{D}})$ and $j^* \circ j_! \simeq \mathrm{id}$.

For the first point, fix a $V \in \mathcal{T}$. Since relative adjunctions are closed under pullbacks by Proposition 1.1.22 and since $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})_{\underline{V}} \simeq \underline{\mathrm{Fun}}_{\underline{V}}(\underline{\mathcal{C}}_{\underline{V}}, \underline{\mathcal{D}}_{\underline{V}})$ by Construction 1.1.13, we also get a $\mathcal{T}_{/V}$-adjunction $j_! : \underline{\mathrm{Fun}}^{\mathcal{R}_{\underline{V}}}_{\underline{V}}(\underline{\mathcal{C}}_{\underline{V}}, \underline{\mathcal{D}}_{\underline{V}}) \rightleftarrows \underline{\mathrm{Fun}}^{\mathcal{K}_{\underline{V}}}_{\underline{V}}(\underline{\mathcal{E}}_{\underline{V}}, \underline{\mathcal{D}}_{\underline{V}}) : j^*$. Suppose $F : \underline{\mathcal{C}}_{\underline{V}} \to \underline{\mathcal{D}}$ is a $\underline{V}$–functor that sends $\mathcal{R}_{\underline{V}}$ to $\underline{V}$–colimit diagrams. We want to show that $j_!F : \underline{\mathcal{E}}_{\underline{V}} \to \underline{\mathcal{D}}_{\underline{V}}$ inverts maps in $\overline{S}$, ie. those maps that get inverted by $L_{\underline{V}}$. Consider

$$\begin{tikzcd}
\underline{\mathcal{C}}_{\underline{V}} \arrow[d, hook, "j"'] \arrow[dr, "F"] \arrow[dd, bend right=60, "y"'] & \\
\underline{\mathcal{E}}_{\underline{V}} \arrow[d, hook, "k"'] & \underline{\mathcal{D}}_{\underline{V}} \\
\underline{\mathrm{PSh}}_{\underline{V}}(\underline{\mathcal{C}}_{\underline{V}}) \arrow[ur, "y_!F"'] &
\end{tikzcd}$$

Note that $j_!F \simeq k^*y_!F$ since $j_! = \mathrm{id} \circ j_! \simeq k^*k_!j_! \simeq k^*y_!$. Now since $y_!F : \underline{\mathrm{PSh}}_{\underline{V}}(\underline{\mathcal{C}}_{\underline{V}}) \to \underline{\mathcal{D}}_{\underline{V}}$ strongly preserves $\underline{V}$–colimits and since $\underline{\mathcal{E}}_{\underline{V}}$ is stable under $\mathcal{K}$–indexed colimits in $\underline{\mathrm{PSh}}_{\underline{V}}(\underline{\mathcal{C}}_{\underline{V}})$ (since $\underline{\mathrm{PSh}}^{\mathcal{K}_{\underline{V}}}_{\mathcal{R}_{\underline{V}}}(\underline{\mathcal{C}}_{\underline{V}})$ was closed under $\mathcal{K}$–colimits by construction) it follows that $j_!F \simeq k^*y_!F$ strongly preserves $\mathcal{K}$–colimits. Now note that the maps in $S \subseteq \underline{\mathrm{PSh}}_{\underline{V}}(\underline{\mathcal{C}}_{\underline{V}})$ are inverted by $y_!F$ since these were the maps comparing colimit in $\underline{\mathrm{PSh}}_{\underline{V}}(\underline{\mathcal{C}}_{\underline{V}})$ and cone point in $\underline{\mathcal{C}}_{\underline{V}}$, and by hypothesis, F, and hence $y_!F$ turns these into equivalences. Therefore, by the universal property of Dwyer-Kan localisations Theorem 2.2.10, $y_!F : \underline{\mathrm{PSh}}_{\underline{V}}(\underline{\mathcal{C}}_{\underline{V}}) \to \underline{\mathcal{D}}$ factors through the Bousfield localisation L, and so in particular inverts $\overline{S}$, so that $j_!F \simeq k^*y_!F$ does too. Also $y_!F$ strongly preserves all $\underline{V}$–colimits by the universal property of presheaves, and so $j_!F \simeq k^*y_!F$ strongly preserves $\mathcal{K}$–colimits since the inclusion $k : \underline{\mathcal{E}}_{\underline{V}} \hookrightarrow \underline{\mathrm{PSh}}_{\underline{V}}(\underline{\mathcal{C}}_{\underline{V}})$ does.

For the second point, since j was $\mathcal{T}$–fully faithful, we have that $j^* \circ j_! \simeq \mathrm{id}$ as usual by Proposition 1.2.14. For the equivalence $j_! \circ j^* \simeq \mathrm{id}$, suppose $F \in \underline{\mathrm{Fun}}_{\mathcal{T}}^{\overline{S}^{-1},\mathcal{K}}(\underline{\mathcal{E}}, \underline{\mathcal{D}})$. Write $F' := j_!j^*F$. By universal property of Kan extensions we have $\alpha : F' = j_!j^*F \to F$ and we want to show this is an equivalence. Since F inverts $\overline{S}$ by hypothesis and $j_!j^*F$ also inverts $\overline{S}$ by the claim of the previous paragraph, we get the diagram

$$\begin{tikzcd}
\underline{\mathcal{C}}_{\underline{V}} \arrow[d, hook, "j"'] & \\
\underline{\mathcal{E}}_{\underline{V}} \arrow[r, "F"] \arrow[r, bend left=50, "F'"] \arrow[dr, "L_{\underline{V}}"'] & \underline{\mathcal{D}}_{\underline{V}} \\
& \underline{\mathrm{PSh}}^{\mathcal{K}_{\underline{V}}}_{\mathcal{R}_{\underline{V}}}(\underline{\mathcal{C}}_{\underline{V}}) \arrow[u, "f'"] \arrow[u, bend right=50, "f"']
\end{tikzcd}$$

The transformation α induces a transformation $\beta : f' \to f$ since $\underline{\mathrm{Fun}}_{\underline{V}}^{\overline{S}^{-1}}(\underline{\mathcal{E}}_{\underline{V}}, \underline{\mathcal{D}}_{\underline{V}}) \simeq \underline{\mathrm{Fun}}_{\underline{V}}(\underline{\mathrm{PSh}}^{\mathcal{K}_{\underline{V}}}_{\mathcal{R}_{\underline{V}}}(\underline{\mathcal{C}}_{\underline{V}}), \underline{\mathcal{D}}_{\underline{V}})$ and we want to show that β is an equivalence. To begin with, note that it is an equivalence on the image of the embedding $j : \underline{\mathcal{C}}_{\underline{V}} \hookrightarrow \underline{\mathrm{PSh}}^{\mathcal{K}_{\underline{V}}}_{\mathcal{R}_{\underline{V}}}(\underline{\mathcal{C}}_{\underline{V}})$. Since F and F' strongly preserve $\mathcal{K}$–colimits, hence so do f' and f. Therefore, $f' \to f$ is an equivalence on all of $\underline{\mathrm{PSh}}^{\mathcal{K}_{\underline{V}}}_{\mathcal{R}_{\underline{V}}}(\underline{\mathcal{C}}_{\underline{V}})$ since this $\underline{V}$-category was by construction generated under these colimits by $\underline{\mathcal{C}}$. This completes the proof of point (ii).

Finally, for point (iii), suppose every element of $\mathcal{R}$ were already a colimit diagram in $\underline{\mathcal{C}}$. The Yoneda map can be factored, by construction, as $j : \underline{\mathcal{C}} \hookrightarrow \underline{\mathcal{E}} \xrightarrow{L} \underline{\mathrm{PSh}}^{\mathcal{K}}_{\mathcal{R}}(\underline{\mathcal{C}})$ where the first map is $\mathcal{T}$–fully faithful. Since the restriction $L|_{S^{-1}\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})} \simeq \mathrm{id}$, it will suffice to show that j lands in $S^{-1}\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$. That is, that $\underline{\mathcal{C}}$ is S-local, ie. for each $V \in \mathcal{T}$ and $C \in \mathcal{C}_V$, and for each $f : W \to V$ in $\mathcal{T}$ and $s : X \to jY$ in S_W, we need to see that

$$s^* : \mathrm{Map}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_W}(jY, jf^*C) \longrightarrow \mathrm{Map}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_W}(X, jf^*C)$$

is an equivalence. To see this, the hypothesis of (iii) gives $Y = \underline{\mathrm{colim}}_{K \subseteq \underline{\mathcal{C}}_{\underline{W}}} \varphi$. Then

$$\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_{\underline{W}}}(jY, jf^*C) \simeq \underline{\mathrm{Map}}_{\underline{\mathcal{C}}_{\underline{W}}}(\underline{\mathrm{colim}}_{K \subseteq \underline{\mathcal{C}}_{\underline{W}}} \varphi, f^*C) \simeq \underline{\mathrm{lim}}_{K^{\underline{\mathrm{op}}} \subseteq \underline{\mathcal{C}}_{\underline{W}}^{\underline{\mathrm{op}}}} \mathrm{Map}_{\mathcal{C}}(\varphi, f^*C)$$

where the first equivalence is by Yoneda. On the other hand,

$$\begin{aligned}\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_{\underline{W}}}(X, jf^*C) &\simeq \underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_{\underline{W}}}(\underline{\mathrm{colim}}_K j\circ\varphi, jf^*C)\\ &\simeq \underline{\lim}_{K^{\mathrm{op}}}\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})_{\underline{W}}}(j\varphi, jf^*C)\\ &\simeq \underline{\lim}_{K^{\mathrm{op}}}\underline{\mathrm{Map}}_{\underline{\mathcal{C}}_{\underline{W}}}(\varphi, f^*C)\end{aligned}$$

and so taking the section over W, one checks that these two identifications are compatible with the map s^*. This completes the proof of (iii). □

2.2.5 The presentables–idempotents equivalence

We want to formulate the equivalence between presentables and idempotent-completes in the parametrised world, and so we need to introduce some definitions. To avoid potential confusion, we will for example use the terminology *parametrised*-accessibles instead of $\mathcal{T}$–accessibles to indicate that we take $\mathcal{T}_{/V}$–accessibles in the fibre over V.

Definition 2.2.13. Let κ be a regular cardinal.

- Let $\underline{\mathrm{Acc}}_{\mathcal{T},\kappa} \subset \widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}$ be the non-full $\mathcal{T}$–subcategory of κ-parametrised-accessible categories and κ-parametrised-accessible functors preserving κ-parametrised-compacts.
- Let $\underline{\mathrm{Cat}}^{\mathrm{Idem}}_{\mathcal{T}} \subseteq \widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}$ be the full $\mathcal{T}$–subcategory on the small parametrised-idempotent-complete categories.
- Let $\underline{\mathrm{Cat}}^{\mathrm{rex}(\kappa)}_{\mathcal{T}} \subset \widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}$ be the non-full subcategory whose objects are κ-parametrised–cocomplete small categories and morphisms those parametrised–functors that strongly preserve κ-small parametrised–colimits.
- Let $\underline{\mathrm{Cat}}^{\mathrm{Idem}(\kappa)}_{\mathcal{T}} \subseteq \underline{\mathrm{Cat}}^{\mathrm{rex}(\kappa)}_{\mathcal{T}}$ be the full subcategory whose objects are κ-parametrised–cocomplete small parametrised-idempotent-complete categories.
- Let $\underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa} \subset \underline{\mathrm{Acc}}_{\mathcal{T},\kappa}$ be the non-full $\mathcal{T}$–subcategory whose objects are parametrised-presentables and whose morphisms are parametrised-left adjoints that preserve κ-parametrised-compacts.
- Let $\underline{\mathrm{Pr}}_{\mathcal{T},R,\kappa\text{-filt}} \subset \widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}$ be the non-full $\mathcal{T}$–subcategory of parametrised presentable categories and morphisms the parametrised κ-accessible functors which strongly preserve parametrised limits.

Notation 2.2.14. Let $\underline{\mathrm{Fun}}^{\kappa}_{\mathcal{T}} \subseteq \underline{\mathrm{Fun}}_{\mathcal{T}}$ be the full subcategory of κ–$\mathcal{T}$–compact-preserving functors.

Lemma 2.2.15 ("[Lur09, Prop. 5.4.2.17]")**.** *Let κ be a regular cardinal. Then $(-)^{\underline{\kappa}}$: $\underline{\mathrm{Acc}}_{\mathcal{T},\kappa} \longrightarrow \widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}$ induces an equivalence to $\underline{\mathrm{Cat}}^{\mathrm{Idem}}_{\mathcal{T}}$, whose inverse $\underline{\mathrm{Cat}}^{\mathrm{Idem}}_{\mathcal{T}} \to \underline{\mathrm{Acc}}_{\mathcal{T},\kappa}$ is $\underline{\mathrm{Ind}}_{\kappa}$.*

Proof. To see $\mathcal{T}$–fully faithfulness, Proposition 1.2.38 gives

$$\begin{aligned}\underline{\mathrm{Fun}}_{\mathcal{T}}^{\underline{\kappa}\text{-filt},\underline{\kappa}}(\underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{C}},\underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{D}}) &\xrightarrow{\simeq} \underline{\mathrm{Fun}}_{\mathcal{T}}^{\underline{\kappa}}(\underline{\mathrm{Ind}}_{\kappa}(\underline{\mathcal{C}})^{\underline{\kappa}},\underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{D}})\\ &\xrightarrow{\simeq} \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Ind}}_{\kappa}(\underline{\mathcal{C}})^{\underline{\kappa}},\underline{\mathrm{Ind}}_{\kappa}(\underline{\mathcal{D}})^{\underline{\kappa}})\end{aligned}$$

where we have also used, by Observation 2.1.15 (1), that $\underline{\mathrm{Ind}}_{\kappa}(\underline{\mathrm{Ind}}_{\kappa}(\underline{\mathcal{C}})^{\underline{\kappa}}) \simeq \underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{C}}$. As for the essential image, let $\underline{\mathcal{C}}$ be a small $\mathcal{T}$–idempotent-complete category. Then by Observation 2.1.15 (2) we know that $\underline{\mathcal{C}} \simeq \underline{\mathrm{Ind}}_{\kappa}(\underline{\mathcal{C}})^{\underline{\kappa}}$, and so it is in the essential image as required. Finally to see the statement about the inverse, just note that we already have the functors and the appropriate natural transformations on compositions. Then using Observation 2.1.15 again, we see that the transformations are pointwise equivalences, and so equivalences. □

Theorem 2.2.16 ($\mathcal{T}$–presentable-idempotent correspondence, "[Lur09, Prop. 5.5.7.8 and Rmk. 5.5.7.9]")**.** *Let κ be a regular cardinal. Then $(-)^{\underline{\kappa}} : \underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa} \longrightarrow \widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\underline{\mathrm{rex}}(\kappa)}$ is $\mathcal{T}$–fully faithful with essential image $\underline{\mathrm{Cat}}_{\mathcal{T}}^{\underline{\mathrm{Idem}}(\kappa)}$, and inverse $\underline{\mathrm{Cat}}_{\mathcal{T}}^{\underline{\mathrm{Idem}}(\kappa)} \to \underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa}$ given by $\underline{\mathrm{Ind}}_{\kappa}$.*

Proof. That it is $\mathcal{T}$–fully faithful with the specified essential image is by Lemma 2.2.15 together with Proposition 2.1.6 and Proposition 2.2.4. That the inverse from Lemma 2.2.15 via $\underline{\mathrm{Ind}}_{\kappa}$ lands in $\mathcal{T}$–presentables is by Theorem 2.2.2 (4). □

2.2.6 Indexed products of presentables

The purpose of this subsection is to show that the (non-full) inclusions $\underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa}, \underline{\mathrm{Pr}}_{\mathcal{T},R,\kappa\text{-filt}} \subset \widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}$ create indexed products.

Lemma 2.2.17 (Indexed products of $\mathcal{T}$–presentables)**.** *Let $f : W \to V$ be in $\mathcal{T}$ and $\underline{\mathcal{C}}$ be a $\mathcal{T}_{/W}$–presentable category. Then $f_*\underline{\mathcal{C}}$ is a $\mathcal{T}_{/V}$–presentable category.*

Proof. We first note that if $\underline{\mathcal{D}}$ is a $\mathcal{T}_{/W}$–category, then $f_*\underline{\mathrm{Fun}}_{\underline{W}}(\underline{\mathcal{D}},\underline{\mathcal{S}}_{\underline{W}}) \simeq \underline{\mathrm{Fun}}_{\underline{V}}(f_!\underline{\mathcal{D}},\underline{\mathcal{S}}_{\underline{V}})$. To see this, let $\underline{\mathcal{E}}$ be a $\mathcal{T}_{/V}$–category. Then

$$\begin{aligned}\mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}_{/V}}}(\underline{\mathcal{E}}, f_*\underline{\mathrm{Fun}}_{\underline{W}}(\underline{\mathcal{D}},\underline{\mathcal{S}}_{\underline{W}})) &\simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}_{/W}}}(f^*\underline{\mathcal{E}}, \underline{\mathrm{Fun}}_{\underline{W}}(\underline{\mathcal{D}},\underline{\mathcal{S}}_{\underline{W}}))\\ &\simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}_{/W}}}(\underline{\mathcal{D}}, \underline{\mathrm{Fun}}_{\underline{W}}(f^*\underline{\mathcal{E}},\underline{\mathcal{S}}_{\underline{W}}))\\ &\simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}_{/W}}}(\underline{\mathcal{D}}, f^*\underline{\mathrm{Fun}}_{\underline{V}}(\underline{\mathcal{E}},\underline{\mathcal{S}}_{\underline{V}}))\\ &\simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}_{/V}}}(f_!\underline{\mathcal{D}}, \underline{\mathrm{Fun}}_{\underline{V}}(\underline{\mathcal{E}},\underline{\mathcal{S}}_{\underline{V}}))\\ &\simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{T}_{/V}}}(\underline{\mathcal{E}}, \underline{\mathrm{Fun}}_{\underline{V}}(f_!\underline{\mathcal{D}},\underline{\mathcal{S}}_{\underline{V}}))\end{aligned}$$

By Theorem 2.2.2 we have a accessible $\mathcal{T}_{/W}$–Bousfield localisation $\underline{\mathrm{Fun}}_W(\underline{\mathcal{D}}, \underline{\mathcal{S}}_W) \rightleftarrows \underline{\mathcal{C}}$ for some small $\mathcal{T}_{/W}$–category $\underline{\mathcal{D}}$. Hence by Lemma 1.3.13, we obtain the accessible adjunction

$$\underline{\mathrm{Fun}}_V(f_!\underline{\mathcal{D}}, \underline{\mathcal{S}}_V) \simeq f_*\underline{\mathrm{Fun}}_W(\underline{\mathcal{D}}, \underline{\mathcal{S}}_W) \rightleftarrows f_*\underline{\mathcal{C}}$$

Therefore, $f_*\underline{\mathcal{C}}$ must be $\mathcal{T}_{/V}$–presentable, again by Theorem 2.2.2. □

Proposition 2.2.18 (Creation of indexed products for presentables)**.** *The (non-full) inclusions* $\underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa}, \underline{\mathrm{Pr}}_{\mathcal{T},R,\kappa\text{-filt}} \subset \widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}$ *create indexed products.*

Proof. Let $f : W \to V$ be in $\mathcal{T}$ and $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ be $\mathcal{T}_{/V}$– and $\mathcal{T}_{/W}$–presentables, respectively. We know from Lemma 1.3.14 that $\widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}$ has indexed products. We need to show that

$$\mathrm{Map}^L_{\underline{V}}(\underline{\mathcal{C}}, f_*\underline{\mathcal{D}}) \simeq \mathrm{Map}^L_{\underline{W}}(f^*\underline{\mathcal{C}}, \underline{\mathcal{D}})$$

$$\mathrm{Map}^{R,\kappa\text{-filt}}_{\underline{V}}(\underline{\mathcal{C}}, f_*\underline{\mathcal{D}}) \simeq \mathrm{Map}^{R,\kappa\text{-filt}}_{\underline{W}}(f^*\underline{\mathcal{C}}, \underline{\mathcal{D}})$$

We claim that the unit and counit in $\widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}$ are already in both $\underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa}$ and $\underline{\mathrm{Pr}}_{\mathcal{T},R,\kappa\text{-filt}}$. If we can show this then we would be done by the following pair of diagrams

$$\begin{tikzcd}
\mathrm{Map}^L_{\underline{V}}(\underline{\mathcal{C}}, f_*\underline{\mathcal{D}}) \arrow[r, "f^*"] \arrow[d, hook] & \mathrm{Map}^L_{\underline{W}}(f^*\underline{\mathcal{C}}, f^*f_*\underline{\mathcal{D}}) \arrow[r, "\varepsilon_*"] \arrow[d, hook] & \mathrm{Map}^L_{\underline{W}}(f^*\underline{\mathcal{C}}, \underline{\mathcal{D}}) \arrow[d, hook] \\
\mathrm{Map}_{\underline{V}}(\underline{\mathcal{C}}, f_*\underline{\mathcal{D}}) \arrow[r, "f^*"] \arrow[rr, bend right, "\simeq"'] & \mathrm{Map}_{\underline{W}}(f^*\underline{\mathcal{C}}, f^*f_*\underline{\mathcal{D}}) \arrow[r, "\varepsilon_*"] & \mathrm{Map}_{\underline{W}}(f^*\underline{\mathcal{C}}, \underline{\mathcal{D}})
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Map}^L_{\underline{V}}(\underline{\mathcal{C}}, f^*\underline{\mathcal{D}}) \arrow[d, hook] & \mathrm{Map}^L_{\underline{V}}(f_*f^*\underline{\mathcal{C}}, f^*\underline{\mathcal{D}}) \arrow[l, "\eta^*"'] \arrow[d, hook] & \mathrm{Map}^L_{\underline{W}}(f^*\underline{\mathcal{C}}, \underline{\mathcal{D}}) \arrow[l, "f_*"'] \arrow[d, hook] \\
\mathrm{Map}_{\underline{V}}(\underline{\mathcal{C}}, f^*\underline{\mathcal{D}}) & \mathrm{Map}_{\underline{V}}(f_*f^*\underline{\mathcal{C}}, f^*\underline{\mathcal{D}}) \arrow[l, "\eta^*"'] & \mathrm{Map}_{\underline{W}}(f^*\underline{\mathcal{C}}, \underline{\mathcal{D}}) \arrow[l, "f_*"'] \arrow[ll, bend left, "\simeq"]
\end{tikzcd}$$

and similarly when we replace Map^L by $\mathrm{Map}^{R,\kappa\text{-filt}}$: that the (co)units are in $\underline{\mathrm{Pr}}_{\mathcal{T},R,\kappa\text{-filt}}$ and $\underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa}$ imply that the maps ε_* and η^* above takes Map^L to Map^L; that f^* and f_* also do these is by Lemma 1.3.13; and finally the bottom equivalences are inverse to each other, and so restrict to inverse equivalences to the top row of each diagram.

We now prove the claims. That they preserve κ–$\mathcal{T}$–compact objects is clear by Lemma 1.3.14 and Theorem 2.2.2. To see that the counit $\varepsilon : f^*f_*\underline{\mathcal{D}} \to \underline{\mathcal{D}}$ strongly preserves $\mathcal{T}$–(co)limits, since it is clear that they preserve fibrewise $\mathcal{T}$–(co)limits, by Proposition 1.2.17 we are left to show that they preserve the indexed (co)products.

So let $\xi : Y \to Z$ be in $\mathcal{T}_{/W}$. For this we will need to know that $\underline{\mathcal{D}}$ has indexed coproducts and products (for the latter, see Proposition 2.2.7). We need to show that the squares with the dashed arrows in

$$\begin{tikzcd}
(f^*f_*\underline{\mathcal{D}})_Z \arrow[r, "\varepsilon"] \arrow[d, "\xi^*"] & \underline{\mathcal{D}}_Z \arrow[d, "\xi^*"] \\
(f^*f_*\underline{\mathcal{D}})_Y \arrow[r, "\varepsilon"] \arrow[u, bend left, dashed, "\xi_!"] \arrow[u, bend right, dashed, "\xi_*"'] & \underline{\mathcal{D}}_Y \arrow[u, bend left, dashed, "\xi_!"] \arrow[u, bend right, dashed, "\xi_*"']
\end{tikzcd} \tag{2.1}$$

commute. We analyse this in terms of the counit formula from Lemma 1.3.14. For this, consider the diagram of orbits

$$\begin{tikzcd}
& \coprod_b R_b \arrow[rr] \arrow[dl, "\coprod_b \xi_{a_b}"'] \arrow[dd] \arrow[drr, phantom, "\lrcorner"] & & Y \arrow[dl, "\xi"] \arrow[dd] \\
\coprod_a S_a \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner"] & & Z \arrow[dd] & \\
& W \arrow[rr, "f"] \arrow[dl, equal] & & V \arrow[dl, equal] \\
W \arrow[rr, "f"] & & V &
\end{tikzcd} \tag{2.2}$$

where the top square is also a pullback since we can view this diagram as

$$\begin{tikzcd}
\coprod_b R_b \arrow[r] \arrow[d] & \coprod_a S_a \arrow[r] \arrow[d] & W \arrow[d, "f"] \\
Y \arrow[r] & Z \arrow[r] & V
\end{tikzcd}$$

with the right square and the outer rectangle being pullbacks. From this we obtain that the diagram Eq. (2.1) is equivalent to

$$\begin{tikzcd}
\prod_a \mathcal{D}_{S_a} \arrow[r, "\pi_Z"] \arrow[d, "\prod_b \xi^*_{a_b}"] & \mathcal{D}_Z \arrow[d, "\xi^*"] \\
\prod_b \mathcal{D}_{R_b} \arrow[r, "\pi_Y"] \arrow[u, bend left, dashed, "\xi_!"] \arrow[u, bend right, dashed, "\xi_*"'] & \mathcal{D}_Y \arrow[u, bend left, dashed, "\xi_!"] \arrow[u, bend right, dashed, "\xi_*"']
\end{tikzcd}$$

where the counits have been identified with the projections π_Z (resp. π_Y) onto the $\underline{\mathcal{D}}_Z$ (resp. $\underline{\mathcal{D}}_Y$) components by virtue of Lemma 1.3.14. Here $\prod_b \xi^*_{a_b}$ is supposed to mean forgetting about the components of $\coprod_a S_a$ that do not receive a map from $\coprod_b R_b$ and the functor $\xi^*_{a_b}$ for the other components: this makes sense because an orbit in a coproduct can only map to a unique orbit. Since $\underline{\mathcal{C}}$ was $\mathcal{T}_{/W}$–presentable, it in particular admits an $\mathcal{T}_{/W}$–initial object. And so we can easily use these, together with the adjoints $(\xi_{a_b})_!$ and fibrewise coproducts to obtain a left adjoint $\xi_!$ of $\prod_b \xi^*_{a_b}$, and similarly a right adjoint ζ_*. It is then immediate that the dashed squares also commute since the counits just project left/right adjoints from the left vertical to those on the right.

To see that the unit strongly preserves $\mathcal{T}$–(co)limits, similarly as above, we are reduced to the case of showing that it preserves indexed (co)products. Let $\zeta : U \to X$ be in $\mathcal{T}_{/V}$. And so we want the squares with the dashed arrows

$$\begin{tikzcd}
\underline{\mathcal{C}}_X \arrow[r, "\eta"] \arrow[d, "\zeta^*"] & (f_*f^*\underline{\mathcal{C}})_X \arrow[d, "\zeta^*"'] \\
\underline{\mathcal{C}}_U \arrow[r, "\eta"] \arrow[u, dashed, bend left, "\zeta_!"] \arrow[u, dashed, bend right, "\zeta_*"'] & (f_*f^*\underline{\mathcal{C}})_U \arrow[u, dashed, bend left, "\zeta_!"] \arrow[u, dashed, bend right, "\zeta_*"']
\end{tikzcd}$$

to commute. For this consider the pullback comparison

$$\begin{tikzcd}
& \coprod_b M_b \arrow[rr] \arrow[dl, "\coprod_b \zeta_{a_b}"'] \arrow[dd] \arrow[drr, phantom, "\lrcorner"] & & U \arrow[dl, "\zeta"] \arrow[dd] \\
\coprod_a N_a \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner"] & & X \arrow[dd] & \\
& W \arrow[rr, "f"] \arrow[dl, equal] & & V \arrow[dl, equal] \\
W \arrow[rr, "f"] & & V &
\end{tikzcd}$$

where the top square is also a pullback by the argument for the previous case. Since

$$(f_*f^*\underline{\mathcal{C}})_X = \prod_a \mathcal{C}_{N_a} \quad \text{and} \quad (f_*f^*\underline{\mathcal{C}})_U = \prod_b \mathcal{C}_{M_b}$$

we see that the units η arise as restrictions along the maps $\coprod_a N_a \to X$ and $\coprod_b M_b \to U$ respectively. Then the required dashed squares commute by the Beck-Chevalley property of indexed (co)products of $\underline{\mathcal{C}}$ associated to the top pullback square. This completes the proof. $\square$

2.2.7 Functor categories and tensors of presentables

In this final subsection, we record several basic results about the interaction between parametrised-presentability and functor categories, totally analogous to the unparametrised setting.

Lemma 2.2.19 (Small cotensors preserve $\mathcal{T}$–presentability)**.** *Let $\underline{\mathcal{C}}$ be a small $\mathcal{T}$-category and $\underline{\mathcal{D}}$ be $\mathcal{T}$–presentable. Then $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ is also $\mathcal{T}$–presentable.*

Proof. As a special case, suppose first that $\underline{\mathcal{D}} \simeq \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}}')$ for a small $\mathcal{T}$–category $\underline{\mathcal{D}}'$. Then $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}},\underline{\mathcal{D}}) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}} \times \underline{\mathcal{D}}'^{\underline{\mathrm{op}}}, \underline{\mathcal{S}}_{\mathcal{T}})$, and so it is also a $\mathcal{T}$–presheaf category, and so is $\mathcal{T}$–presentable. For a general $\mathcal{T}$–presentable $\underline{\mathcal{D}}$, we know that we have a κ–$\mathcal{T}$–accessible Bousfield localisation $L : \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}}') \rightleftarrows \underline{\mathcal{D}} : i$ for some small $\mathcal{T}$–category $\underline{\mathcal{D}}'$. Then we get a κ–$\mathcal{T}$–accessible Bousfield localisation $L_* : \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}}')) \rightleftarrows \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}},\underline{\mathcal{D}}) : i_*$ and so since $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{D}}'))$ was $\mathcal{T}$–presentable by the first part above, by characterisation Theorem 2.2.2 (5) we get that $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}},\underline{\mathcal{D}})$ is too. □

Lemma 2.2.20 ("[Lur09, Lem. 5.5.4.17]"). *Let $F : \underline{\mathcal{C}} \rightleftarrows \underline{\mathcal{D}} : G$ be a $\mathcal{T}$–adjunction between $\mathcal{T}$–presentables. Suppose we have a $\mathcal{T}$–accessible Bousfield localisation $L : \underline{\mathcal{C}} \rightleftarrows \underline{\mathcal{C}}^0 : i$. Let $\underline{\mathcal{D}}^0 := G^{-1}(\underline{\mathcal{C}}^0) \subseteq \underline{\mathcal{D}}$. Then we have a $\mathcal{T}$–accessible Bousfield localisation $L' : \underline{\mathcal{D}} \rightleftarrows \underline{\mathcal{D}}^0 : i'$.*

Proof. The $\mathcal{T}$–accessibility of the Bousfield localisation $L : \underline{\mathcal{C}} \rightleftarrows \underline{\mathcal{C}}^0 : i$ ensures that there is a small set of morphisms of $\underline{\mathcal{C}}$ such that $\underline{\mathcal{C}}^0$ are precisely the S-local objects. Then it is easy to see that $\underline{\mathcal{D}}^0 \subseteq \underline{\mathcal{D}}$ is precisely the $F(S)$-local $\mathcal{T}$–subcategory by using the adjunction. □

Lemma 2.2.21 ("[Lur09, Lem. 5.5.4.18]"). *Let $\underline{\mathcal{C}}$ be a $\mathcal{T}$–presentable category and $\{\underline{\mathcal{C}}_a\}_{a\in A}$ be a family of $\mathcal{T}$–accessible Bousfield local subcategories indexed by a small set A. Then $\bigcap_{a\in A} \underline{\mathcal{C}}_a$ is also a $\mathcal{T}$–accessible Bousfield local subcategory.*

Proof. This is because, if we write $S(a)$ for the morphisms of $\underline{\mathcal{C}}$ such that $\underline{\mathcal{C}}_a$ is the $S(a)$-local objects, then $\bigcap_{a\in A} \underline{\mathcal{C}}_a$ are the $\bigcup_{a\in A} S(a)$-local objects. □

For the remaining results, recall from Notation 1.2.27 that $\underline{\mathrm{Fun}}^R_{\mathcal{T}}$ and $\underline{\mathrm{Fun}}^L_{\mathcal{T}}$ denote strongly $\mathcal{T}$–limit- and $\mathcal{T}$–colimit-preserving functors, respectively, and $\underline{\mathrm{RFun}}_{\mathcal{T}}$ and $\underline{\mathrm{LFun}}_{\mathcal{T}}$ denote $\mathcal{T}$–right and $\mathcal{T}$–left adjoint functors, respectively.

Lemma 2.2.22 (Presentable functor categories, "[Lur17, Lem. 4.8.1.16]"). *Let $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ be $\mathcal{T}$–presentables. Then $\underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathcal{D}})$ and $\underline{\mathrm{Fun}}^L_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$ are also $\mathcal{T}$–presentable.*

Proof. By characterisation (5) of Theorem 2.2.2 and that Bousfield localisations are Dwyer-Kan Proposition 2.2.6, we know that $\underline{\mathcal{C}} \simeq S^{-1}\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}')$ for some small $\mathcal{T}$–category $\underline{\mathcal{C}}'$ and S a small collection of morphisms in $\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}')$. Then we have

$$\begin{aligned}\underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}')^{\underline{\mathrm{op}}}, \underline{\mathcal{D}}) \simeq \underline{\mathrm{Fun}}^L_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}'), \underline{\mathcal{D}}^{\underline{\mathrm{op}}})^{\underline{\mathrm{op}}} &\simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}', \underline{\mathcal{D}}^{\underline{\mathrm{op}}})^{\underline{\mathrm{op}}} \\ &\simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}'^{\underline{\mathrm{op}}}, \underline{\mathcal{D}})\end{aligned}$$

where the first and last equivalence is by Observation 1.1.15, and the second by Proposition 1.2.38 and since $\mathcal{T}$–presentables are also $\mathcal{T}$–complete by Proposition 2.2.7. The right hand term is $\mathcal{T}$–presentable by Lemma 2.2.19, and so $\underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}')^{\underline{\mathrm{op}}}, \underline{\mathcal{D}})$ is too by the equivalence above. Now note that we have

$\underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathcal{D}}) \simeq \underline{\mathrm{Fun}}^{R,S^{-1}}_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}')^{\underline{\mathrm{op}}}, \underline{\mathcal{D}})$: this is by virtue of the following diagram

$$\begin{aligned}\underline{\mathrm{Fun}}^R_{\mathcal{T}}((S^{-1}\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}'))^{\underline{\mathrm{op}}}, \underline{\mathcal{D}}) &\simeq \underline{\mathrm{Fun}}^L_{\mathcal{T}}(S^{-1}\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}'), \underline{\mathcal{D}}^{\underline{\mathrm{op}}})^{\underline{\mathrm{op}}} \\ &\xrightarrow[\simeq]{L^*} \underline{\mathrm{Fun}}^{L,S^{-1}}_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}'), \underline{\mathcal{D}}^{\underline{\mathrm{op}}})^{\underline{\mathrm{op}}} \\ &\simeq \underline{\mathrm{Fun}}^{R,S^{-1}}_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}')^{\underline{\mathrm{op}}}, \underline{\mathcal{D}})\end{aligned}$$

where we have the equivalence L^* owing to the formula for $\mathcal{T}$–colimits in $\mathcal{T}$–Bousfield local subcategories. Therefore, if for each $\alpha \in S$ we write $\underline{\mathcal{E}}(\alpha) \subseteq \underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}')^{\underline{\mathrm{op}}}, \underline{\mathcal{D}})$ to be the $\mathcal{T}$–full subcategory of those functors which carry α to an equivalence in $\underline{\mathcal{D}}$, then $\underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathcal{D}}) \simeq \bigcap_{\alpha\in S}\underline{\mathcal{E}}(\alpha) \subseteq \underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}')^{\underline{\mathrm{op}}}, \underline{\mathcal{D}})$. Hence to show $\underline{\mathrm{Fun}}^R(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathcal{D}})$ is a $\mathcal{T}$–accessible Bousfield localisation of $\underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}')^{\underline{\mathrm{op}}}, \underline{\mathcal{D}})$, it will be enough to show it, by Lemma 2.2.21, for each $\underline{\mathcal{E}}(\alpha)$. Now these α's are morphisms in the various fibres over $\mathcal{T}^{\mathrm{op}}$ but since everything interacts well with basechanges, we can just assume without loss of generality that $\mathcal{T}^{\mathrm{op}}$ has an initial object and that α is a morphism in the fibre of this initial object. Given this, it is clear that we have the pullback

$$\begin{tikzcd} \underline{\mathcal{E}}(\alpha) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}')^{\underline{\mathrm{op}}}, \underline{\mathcal{D}}) \arrow[d, "\mathrm{ev}_\alpha"] \\ \underline{\mathcal{E}} \arrow[r, hook] & \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{const}}_{\mathcal{T}}(\Delta^1), \underline{\mathcal{D}}) \end{tikzcd}$$

where $\underline{\mathcal{E}}$ is the full subcategory spanned by the equivalences. Hence by Lemma 2.2.20 it will suffice to show that $\underline{\mathcal{E}} \subseteq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{const}}_{\mathcal{T}}(\Delta^1), \underline{\mathcal{D}})$ is a $\mathcal{T}$–accessible Bousfield localisation. But this is clear since it is just given by the $\mathcal{T}$–left Kan extension along $* \to \Delta^1$.

The statement for $\underline{\mathrm{Fun}}^L_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ is proved analogously, but without having to take opposites in showing that $L^* : \underline{\mathrm{Fun}}^L_{\mathcal{T}}(S^{-1}\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}'), \underline{\mathcal{D}}) \to \underline{\mathrm{Fun}}^{L,S^{-1}}_{\mathcal{T}}(\underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}}'), \mathcal{D})$ is an equivalence. □

The following result was stated as Example 3.26 in [Nar17] without proof, and so we prove it here. Here the tensor product is the one constructed in [Nar17, §3.4].

Proposition 2.2.23 (Formula for presentable $\mathcal{T}$–tensors)**.** *Let $\mathcal{T}$ be an atomic orbital category, and let $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ be $\mathcal{T}$–presentable categories. Then $\underline{\mathcal{C}} \otimes \underline{\mathcal{D}} \simeq \underline{\mathrm{Fun}}^R_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathcal{D}})$.*

Proof. This is just a consequence of the universal property of the tensor product. To wit, let $\underline{\mathcal{E}}$ be an arbitrary $\mathcal{T}$–presentable category and write $\underline{\mathrm{Fun}}^{R,\mathrm{acc}}_{\mathcal{T}}$ for $\mathcal{T}$–

accessible strongly $\mathcal{T}$–limit preserving functors. Then

$$\begin{aligned}
\underline{\mathrm{Fun}}^{L,L}(\underline{\mathcal{C}} \times \underline{\mathcal{D}}, \underline{\mathcal{E}}) &\simeq \underline{\mathrm{Fun}}^{L}(\underline{\mathcal{C}}, \underline{\mathrm{Fun}}^{L}(\underline{\mathcal{D}}, \underline{\mathcal{E}})) \\
&\simeq \underline{\mathrm{Fun}}^{R}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathrm{Fun}}^{L}_{\mathcal{T}}(\underline{\mathcal{D}}, \underline{\mathcal{E}})^{\underline{\mathrm{op}}})^{\underline{\mathrm{op}}} \\
&\simeq \underline{\mathrm{Fun}}^{R}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathrm{LFun}}_{\mathcal{T}}(\underline{\mathcal{D}}, \underline{\mathcal{E}})^{\underline{\mathrm{op}}})^{\underline{\mathrm{op}}} \\
&\simeq \underline{\mathrm{Fun}}^{R}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathrm{RFun}}_{\mathcal{T}}(\underline{\mathcal{E}}, \underline{\mathcal{D}}))^{\underline{\mathrm{op}}} \\
&\simeq \underline{\mathrm{Fun}}^{R}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathrm{Fun}}^{R,\mathrm{acc}}_{\mathcal{T}}(\underline{\mathcal{E}}, \underline{\mathcal{D}}))^{\underline{\mathrm{op}}} \\
&\simeq \underline{\mathrm{Fun}}^{R,\mathrm{acc}}_{\mathcal{T}}(\underline{\mathcal{E}}, \underline{\mathrm{Fun}}^{R}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathcal{D}}))^{\underline{\mathrm{op}}} \\
&\simeq \underline{\mathrm{RFun}}_{\mathcal{T}}(\underline{\mathcal{E}}, \underline{\mathrm{Fun}}^{R}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathcal{D}}))^{\underline{\mathrm{op}}} \\
&\simeq \underline{\mathrm{LFun}}_{\mathcal{T}}(\underline{\mathrm{Fun}}^{R}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathcal{D}}), \underline{\mathcal{E}}) \\
&\simeq \underline{\mathrm{Fun}}^{L}_{\mathcal{T}}(\underline{\mathrm{Fun}}^{R}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathcal{D}}), \underline{\mathcal{E}})
\end{aligned}$$

where the second equivalence is by Observation 1.1.15; the third, fifth, seventh, and ninth equivalence is by the adjoint functor Theorem 2.2.3; the fourth and eighth are from Proposition 1.2.28. In the seventh and ninth equivalence, we have also used that $\underline{\mathrm{Fun}}^{R}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathcal{D}})$ is $\mathcal{T}$–presentable, which is provided by Lemma 2.2.22. Therefore, $\underline{\mathrm{Fun}}^{R}_{\mathcal{T}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathcal{D}})$ satisfies the universal property of $\underline{\mathcal{C}} \otimes \underline{\mathcal{D}}$. □

2.3 Parametrised presentable-stable theory

We are now ready to initiate the study of $\mathcal{T}$–presentable-stable categories for atomic orbital categories $\mathcal{T}$. We first state and prove Theorem 2.3.4, the comparison between $\mathcal{T}$–presentable-stables and $\mathcal{T}$–Mackey functors valued in presentable stables, in §2.3.1. In the remaining subsections we will then analyse aspects of the "closed $\mathcal{T}$–symmetric monoidality" of $\underline{\mathrm{Pr}}_{\mathcal{T},\mathrm{st},L}$ in preparation for Part II.

2.3.1 Embedding into Mackey functors

We begin with the following basic observation.

Proposition 2.3.1. *The $\mathcal{T}$–categories $\underline{\mathrm{Pr}}_{\mathcal{T},\mathrm{st},L,\kappa}$, $\underline{\mathrm{Pr}}_{\mathcal{T},\mathrm{sadd},L,\kappa}$ and $\underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa}$ are $\mathcal{T}$–semiadditive.*

Proof. We only show that $\underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa}$ is $\mathcal{T}$–semiadditive. This would then imply that the $\mathcal{T}$–full subcategory $\underline{\mathrm{Pr}}_{\mathcal{T},\mathrm{st},L,\kappa}$ is too, since $\mathcal{T}$–presentable-stables are closed under $\mathcal{T}$–products. Now to see that $\underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa}$ is $\mathcal{T}$–semiadditive, we just need to show that the $\mathcal{T}$–products, which by definition are the right adjoints of restrictions, happen also to be the left adjoints to the restrictions. For this, let $f : W \to V$ be in $\mathrm{Fin}_{\mathcal{T}}$. We then observe that

$$\mathrm{Map}^{L,\kappa}_{\underline{V}}(f_*\underline{\mathcal{D}}, \underline{\mathcal{C}}) \simeq \mathrm{Map}^{R,\kappa\text{-filt}}_{\underline{V}}(\underline{\mathcal{C}}, f_*\underline{\mathcal{D}}) \simeq \mathrm{Map}^{R,\kappa\text{-filt}}_{\underline{W}}(f^*\underline{\mathcal{C}}, \underline{\mathcal{D}}) \simeq \mathrm{Map}^{L,\kappa}_{\underline{W}}(\underline{\mathcal{D}}, f^*\underline{\mathcal{C}})$$

where the first and last equivalences is by the adjoint functor Theorem 2.2.3 and Proposition 1.2.28, and the middle equivalence is by Proposition 2.2.18. This shows that $f_* \dashv f^*$, and so $f_* \simeq f_!$ as was to be shown. □

The following considerations will elaborate on some structural consequences inherent in a Mackey functor valued in presentable categories.

Construction 2.3.2. Let $\underline{\mathcal{C}} \in \mathrm{Fun}^\times(\mathrm{Span}(\mathcal{T}), \mathrm{Pr}_{\mathrm{st},L,\kappa})$ and $f : W \to V$ be in $\mathcal{T}$. Let the Mackey transfer map be $f_! : \mathcal{C}_W \to \mathcal{C}_V$ (which need not necessarily be a left adjoint to f^*) - this is by definition the image of the span morphism $(W \xleftarrow{\mathrm{id}} W \xrightarrow{f} V)$ under the functor $\underline{\mathcal{C}} : \mathrm{Span}(\mathcal{T}) \to \mathrm{Pr}_{\mathrm{st},L,\kappa}$. Let $f_* : \mathcal{C}_W \to \mathcal{C}_V$ be the right adjoint of f^* (this exists since we are landing in $\mathrm{Pr}_{\mathrm{st},L,\kappa}$). Now the pullback of orbits

$$\begin{tikzcd} \coprod_a S_a \arrow[r, "\coprod_a f_a"] \arrow[d, "\coprod_a f_a"'] \arrow[dr, phantom, "\lrcorner"] & W \arrow[d, "f"] \\ W \arrow[r, "f"'] & V \end{tikzcd}$$

gives us that $f^* f_! \simeq \bigoplus_a (f_a)_!(f_a)^*$. Crucially, the hypothesis of atomic orbitality guarantees that one of the orbits S_a in the decomposition is equivalent to W by the argument in the proof of Lemma 1.3.14. From this we can obtain two canonical transformations:

(i) Projecting onto the component $f_a = \mathrm{id} : S_a = W \longrightarrow W$ yields a transformation
$$f^* f_! \simeq \bigoplus_a (f_a)_!(f_a)^* \Longrightarrow \mathrm{id}$$
which together with the $f^* \dashv f_*$ adjunction gives us a transformation
$$f_! \Longrightarrow f_*$$
We call this the *Mackey semiadditivity norm map.*

(ii) Inclusion of the component $f_a = \mathrm{id} : S_a = W \longrightarrow W$ yields a transformation
$$\mathrm{id} \Longrightarrow \bigoplus_a (f_a)_!(f_a)^* \simeq f^* f_!$$
We call this the *Mackey unit map.*

These will allow us to describe in what way the parametrised presentable-stables embed in presentable-stable-valued Mackey functors. We will provide some comments about the theorem after the proof.

Notation 2.3.3. Since $\underline{\mathrm{Pr}}_{\mathcal{T},\mathrm{st},L,\omega} \simeq \underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{st},\mathrm{idem}(\omega)}$ by Theorem 2.2.16, we will use the notation $\underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{perf}} \simeq \underline{\mathrm{Pr}}_{\mathcal{T},\mathrm{st},L,\omega}$ for the $\mathcal{T}$–perfect-stable categories, where the word perfect is standard terminology for being idempotent-complete.

Theorem 2.3.4. *We have $\mathcal{T}$–fully faithful inclusions $\underline{\mathrm{Pr}}_{\mathcal{T},\mathrm{sadd},L,\kappa} \subseteq \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Pr}_{\mathrm{sadd},L,\kappa})$ and $\underline{\mathrm{Pr}}_{\mathcal{T},\mathrm{st},L,\kappa} \subseteq \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Pr}_{\mathrm{st},L,\kappa})$ whose essential images consist of the Mackey functors such that:*

- *the Mackey semiadditivity norm map is an equivalence,*
- *the Mackey unit map exhibits the transfer $f_!$ as being left adjoint to f^*.*

Hence, via $\underline{\mathrm{Cat}}^{\mathrm{perf}} \simeq \underline{\mathrm{Pr}}_{\mathrm{st},L,\omega}$, we also have an inclusion $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \subseteq \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$. Similarly, we also have a fully faithful inclusion $\underline{\mathrm{Pr}}_{\mathcal{T},L,\kappa} \subseteq \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Pr}_{L,\kappa})$ whose essential image are precisely the Mackey functors such that Mackey unit map exhibits $f_! \dashv f^*$.

Proof. We only prove the stable case as the others are similar. By definition we have the following solid non-full $\mathcal{T}$–faithful inclusions

$$\begin{tikzcd}
\underline{\mathrm{Pr}}_{\mathcal{T},\mathrm{st},L,\kappa} \arrow[r, hook] \arrow[dr, hook, dashed] & \widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}} \\
& \underline{\mathrm{Cofree}}_{\mathcal{T}}(\mathrm{Pr}_{\mathrm{st},L,\kappa}) \arrow[u, hook]
\end{tikzcd}$$

which strongly preserve finite $\mathcal{T}$–products: the top horizontal inclusion by Proposition 2.2.18 and the vertical inclusion since $\mathrm{Pr}_{\mathrm{st},L,\kappa} \subset \widehat{\mathrm{Cat}}$ preserves limits. By Theorem 2.2.2 (7) and the characterisation of strong preservations Proposition 1.2.17 we in fact have the dashed factorisation which must, by the preceding points, also strongly preserve finite $\mathcal{T}$–products. Now by definition $\underline{\mathrm{CMon}}_{\mathcal{T}}(-) := \underline{\mathrm{Fun}}^{\mathrm{sadd}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, -) \subseteq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Fin}}_{*\mathcal{T}}, -)$ and so applying $\underline{\mathrm{CMon}}_{\mathcal{T}}(-)$ and invoking Corollary 1.2.34 we get a $\mathcal{T}$–faithful inclusion

$$\underline{\mathrm{Pr}}_{\mathcal{T},\mathrm{st},L,\kappa} \subset \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Pr}_{\mathrm{st},L,\kappa})$$

where we can dispense with the $\mathcal{T}$–semiadditivisation of the source by virtue of Proposition 1.3.4 and Proposition 2.3.1.

We are now left to show that the $\mathcal{T}$–faithful inclusion is in fact $\mathcal{T}$–fully faithful and has the prescribed essential image. For this recall that for all $V \in \mathcal{T}$, $(\underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Pr}_{\mathrm{st},L,\kappa}))_V = \mathrm{Fun}^{\times}(\mathrm{Span}(\mathcal{T}_{/V}), \mathrm{Pr}_{\mathrm{st},L,\kappa})$ from Theorem 1.3.5. The $\mathcal{T}$–faithful inclusion above is then just given by sending a $\mathcal{T}$–presentable-stable category to a $\mathcal{T}$–Mackey functor where we have chosen the indexed biproducts as the transfers in the $\mathcal{T}$–Mackey functor (there is a contractible space of choice of left/right adjoints of a specified functor): in fact the essential image is easily seen to be characterised by those Mackey functors as in the statement of the theorem because a $\mathcal{T}$–category being $\mathcal{T}$–presentable-stable is a property and this property is satisfied by $\mathcal{T}$–Mackey functors with the prescribed conditions since these conditions guarantee that the fibrewise presentable $\mathcal{T}$–category is $\mathcal{T}$–cocomplete and is

$\mathcal{T}$–semiadditive. From this identification, we see by the characterisation of strong preservation Proposition 1.2.17 that $\mathcal{T}$–functors strongly preserving $\mathcal{T}$–colimits are precisely natural transformations of $\mathcal{T}$–Mackey functors valued in $\mathrm{Pr}_{\mathrm{st},L,\kappa}$, whence the $\mathcal{T}$–fully faithfulness. □

Remark 2.3.5. This embedding is perhaps slightly surprising at first glance since in Mackey functors on the right-hand side, we provide a *structure* in the form of transfers, whereas on the left-hand side, a $\mathcal{T}$–category being $\mathcal{T}$–presentable-stable is a *property*. The point here is that, in our relatively restrictive case of atomic orbital base categories and the fact that morphisms in Pr_L have right adjoints, the situation is sufficiently rigid so that a natural transformation of Mackey functors, which would ordinarily be *extra structure* that one has to supply, becomes now a *property* about colimit-preservation when restricted to the Mackey functors coming from $\mathcal{T}$–presentable-stable categories. In the case of $T = *$, this inclusion degenerates to the equivalence $\mathrm{Pr}_{\mathrm{st},L,\kappa} \simeq \mathrm{CMon}(\mathrm{Pr}_{\mathrm{st},L,\kappa})$ by virtue of the semiadditivity of $\mathrm{Pr}_{\mathrm{st},L,\kappa}$.

Remark 2.3.6. Intuitively, this theorem says that there are only two possible points of failure for a Mackey functor in $\mathrm{Mack}_T(\mathrm{Cat}^{\mathrm{perf}})$ to being a genuinely parametrised object, namely: (1) that the transfer maps might be arbitrary and need not have been left adjoints; (2) if they were left adjoints, they need not have been equivalent to the right adjoints of the restriction maps in the Mackey structure. This is essentially because the notion of Mackey functors that we have been considering is built on the $(\infty, 1)$-categorical version of the span category $\mathrm{Span}(\mathcal{T})$. While this is sufficient to encode the structures in Construction 2.3.2, it cannot enforce that these be equivalences. We expect that an $(\infty, 2)$*-categorical* version of the span category and of Mackey functors should yield $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$ since the adjointness of functors can be encoded by the available 2-morphisms.

Proposition 2.3.7. *The inclusion* $\underline{\mathrm{Pr}}_{\mathcal{T},\mathrm{st},L,\kappa} \subseteq \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Pr}_{\mathrm{st},L,\kappa})$ *creates fibres and cofibres.*

Proof. The case of fibres is clear since the solid arrows in the following preserve these.

$$\begin{tikzcd} \underline{\mathrm{Pr}}_{\mathcal{T},\mathrm{st},L,\kappa} \arrow[r, hook] \arrow[rr, hook, dashed, bend right] & \widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}} & \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Pr}_{\mathrm{st},L,\kappa}) \arrow[l, hook'] \end{tikzcd}$$

Now for the cofibre case, let $i : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a morphism in $\underline{\mathrm{Pr}}_{\mathcal{T},\mathrm{st},L,\kappa}$ and $p\colon \underline{\mathcal{D}} \to \underline{\mathcal{E}}$ be the cofibre in $\underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Pr}_{\mathrm{st},L,\kappa})$. By [NS18, §I.3], p is fibrewise a Bousfield localisation, so let j be the Bousfield inclusion. The $\mathcal{T}$–category $\underline{\mathcal{E}}$ is fibrewise stable, and we need to show that it has indexed (co)products $f_!$ and f_*, and that $f_! \simeq f_*$. So consider the diagram

$$\begin{tikzcd}
\mathcal{C}_W \arrow[r, "i_W"] \arrow[d, shift right=6, "f_!"'] \arrow[d, shift left=6, "f_*"] & \mathcal{D}_W \arrow[r, two heads, shift left, "p_W"] \arrow[d, shift right=6, "f_!"'] \arrow[d, shift left=6, "f_*"] & \mathcal{E}_W \arrow[l, hook', shift left, "j_W"] \arrow[d, dashed, shift right=6, "\overline{f}_!"'] \arrow[d, dashed, shift left=6, "\overline{f}_*"] \\
\mathcal{C}_V \arrow[r, "i_V"] \arrow[u, "f^*"] & \mathcal{D}_V \arrow[r, two heads, shift left, "p_V"] \arrow[u, "f^*"] & \mathcal{E}_V \arrow[l, hook', shift left, "j_V"] \arrow[u, "f^*"]
\end{tikzcd}$$

where the dashed maps $\overline{f}_!$ and $\overline{f}_*$ are induced by the cofibreness of $\mathcal{E}_W$. If we can show that $j_V \circ \overline{f}_* \simeq f_* \circ j_W$ then we would be done since the j's were fully faithful and so the $f^* \dashv f_*$ adjunction on $\underline{\mathcal{D}}$ restricts to an $f^* \dashv \overline{f}_*$ adjunction on $\mathcal{E}$; moreover, since $f_! \simeq f_*$ on $\underline{\mathcal{D}}$, this also means that the $f_! \dashv f^*$ adjunction on $\underline{\mathcal{D}}$ induces one on $\underline{\mathcal{E}}$. Now to see the desired commutation, the universal property gives that $\overline{f}_* \circ p_W \simeq p_V \circ f_*$, and hence

$$j_V \circ \overline{f}_* \simeq j_V \circ \overline{f}_* \circ p_W \circ j_W \simeq j_V \circ p_V \circ f_* \circ j_W$$

and so if we can show that f_* preserves Bousfield completeness then we would further obtain $j_V \circ p_V \circ f_* \circ j_W \simeq f_* \circ j_W$. So suppose we have $y_V \to z_V$ in $\mathcal{D}_V$ that is an $\mathcal{E}_V$-local equivalence. Let $x_W \in \mathcal{E}_W$. Then

$$\mathrm{Map}_{\mathcal{D}_V}(z_V, f_* x_W) \simeq \mathrm{Map}_{\mathcal{D}_W}(f^* z_V, x_W) \xrightarrow{\simeq} \mathrm{Map}_{\mathcal{D}_W}(f^* y_V, x_W) \simeq \mathrm{Map}_{\mathcal{D}_V}(y_V, f_* x_W)$$

where the second equivalence is because $p_W \circ f^* \simeq f^* \circ p_V$, and so f^* preserves Bousfield local equivalences. This completes the proof. □

2.3.2 Symmetric monoidality and presentable–stability

The goal of this subsection is to show that the $\mathcal{T}$–presentable-stables are a $\mathcal{T}$–smashing localisation of all $\mathcal{T}$–presentables. One upshot of this is that the $\mathcal{T}$–symmetric monoidal structure on $\underline{\mathrm{Pr}}_{\mathcal{T},L}$ constructed in [Nar17] then induces a $\mathcal{T}$–symmetric monoidal structure on the $\mathcal{T}$–presentable-stables.

Proposition 2.3.8. *For $f : U \to W$ a map in $\mathrm{Fin}_{\mathcal{T}}$ and $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{T}/U}$, there is a natural equivalence*

$$\bigotimes_f \underline{\mathrm{PSh}}_U(\underline{\mathcal{C}}) \simeq \underline{\mathrm{PSh}}_W\Big(\prod_f \underline{\mathcal{C}}\Big)$$

Proof. Let $\underline{\mathcal{D}}$ be a $\mathcal{T}_{/W}$–presentable category. By [Nar17, Prop. 3.19], the following map

$$\underline{\mathrm{Fun}}^L_W\Big(\bigotimes_f \underline{\mathrm{PSh}}_U(\underline{\mathcal{C}}), \mathcal{D}\Big) \longrightarrow \underline{\mathrm{Fun}}_W\Big(\prod_f \underline{\mathcal{C}}, \underline{\mathcal{D}}\Big)$$

is an equivalence. But then the target is naturally equivalent to $\underline{\mathrm{Fun}}^L_W(\underline{\mathrm{PSh}}_W(\prod_f \underline{\mathcal{C}}), \underline{\mathcal{D}})$ by Theorem 1.2.24 and so we are done. □

The following two results have also been obtained independently in [NS22, §6] and so we content ourselves with just a sketch of their proofs.

Proposition 2.3.9. *The presheaf functor* $\underline{\mathrm{PSh}}_{\mathcal{T}}(-) : \underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{Idem}(\omega)} \to \underline{\mathrm{Pr}}_{\mathcal{T},L}$ *refines to a* $\mathcal{T}$*–lax symmetric monoidal functor. Hence, this induces the functor* $\underline{\mathrm{PSh}}_{\mathcal{T}}(-) : \underline{\mathrm{CAlg}}_{\mathcal{T}}((\underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{Idem}(\omega)})^{\underline{\otimes}}) \to \underline{\mathrm{CAlg}}_{\mathcal{T}}((\underline{\mathrm{Pr}}_{\mathcal{T},L})^{\underline{\otimes}})$.

Proof sketch. The $\mathcal{T}$–symmetric monoidal structure on $\underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{Idem}(\omega)}$ is the one induced by $\underline{\mathrm{Pr}}_{\mathcal{T},L}$ constructed by Nardin in [Nar17] under the equivalence Theorem 2.2.16. By construction of the $\mathcal{T}$–symmetric monoidal structure $\underline{\mathrm{Pr}}_{\mathcal{T},L}^{\underline{\otimes}}$ as a $\mathcal{T}$–suboperad of the $\mathcal{T}$–cartesian symmetric monoidal structure $\widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\underline{\times}}$ in [Nar17, §3], we get the $\mathcal{T}$–suboperad inclusion $(\underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{Idem}(\omega)})^{\underline{\otimes}} \subset \underline{\mathrm{Cat}}_{\mathcal{T}}^{\underline{\times}}$. On the other hand, the $\underline{\mathrm{PSh}}_{\mathcal{T}}(-) : \underline{\mathrm{Cat}}_{\mathcal{T}} \hookrightarrow \underline{\mathrm{Pr}}_{\mathcal{T},L}$ canonically refines to a $\mathcal{T}$–symmetric monoidal functor $\underline{\mathrm{PSh}}_{\mathcal{T}}(-)^{\underline{\otimes}} : \underline{\mathrm{Cat}}_{\mathcal{T}}^{\underline{\times}} \hookrightarrow \underline{\mathrm{Pr}}_{\mathcal{T},L}^{\underline{\otimes}}$: one can construct this $\mathcal{T}$–symmetric monoidal functor by mimicking the proof of [Lur17, Prop. 4.8.1.3] by taking the appropriate $\mathcal{T}$–suboperad of $\underline{\mathrm{const}}_{\mathcal{T}}(\Delta^1) \times \widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\underline{\times}}$ and checking that it is cocartesian over $\underline{\mathrm{const}}_{\mathcal{T}}(\Delta^1) \times \underline{\mathrm{Fin}}_{*\mathcal{T}}$ by using Proposition 2.3.8 to see the compositions of locally cocartesian morphisms are locally cocartesian. We thus obtain the refinement to a map of $\mathcal{T}$–operads

$$\underline{\mathrm{PSh}}_{\mathcal{T}}(-) : \underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{Idem}(\omega)} \subset \underline{\mathrm{Cat}}_{\mathcal{T}}^{\underline{\times}} \longrightarrow \underline{\mathrm{Pr}}_{\mathcal{T},L}^{\underline{\otimes}}$$

which is by definition, a $\mathcal{T}$–lax symmetric monoidal functor. □

The following can in principle be deduced from the method of proof above, provided we first construct the $\mathcal{T}$–symmetric monoidal structure $\widehat{\underline{\mathrm{Cat}}}_{\mathcal{T},L}^{\underline{\otimes}}$ on large $\mathcal{T}$–cocomplete categories and functors which strongly preserve these. By Lemma 1.3.11 we have

$$\underline{\mathrm{PSh}}_{\mathcal{T}} : \underline{\mathrm{CAlg}}_{\mathcal{T}}(\widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\underline{\times}}) \rightleftarrows \underline{\mathrm{CAlg}}_{\mathcal{T},L}(\widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\underline{\otimes}}) : \mathrm{incl}$$

and inspecting the adjunction unit yields the following desired conclusion.

Corollary 2.3.10 ($\mathcal{T}$–symmetric monoidality of Yoneda)**.** *If* $\underline{\mathcal{C}}^{\underline{\otimes}}$ *is a* $\mathcal{T}$*–symmetric monoidal category, then the Yoneda embedding* $\underline{\mathcal{C}} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{T}}(\underline{\mathcal{C}})$ *refines to a* $\mathcal{T}$*–symmetric monoidal functor.*

Proposition 2.3.11. *For* $\underline{\mathcal{C}}$ *a* $\mathcal{T}$*–presentable category, we have that* $\underline{\mathrm{Sp}}_{\mathcal{T}}(\mathcal{C}) \simeq \underline{\mathcal{C}} \otimes \underline{\mathrm{Sp}}_{\mathcal{T}}$.

Proof. Consider the sequence of equivalences

$$\begin{aligned}
\underline{\mathcal{C}} \otimes \underline{\mathrm{Sp}}_{\mathcal{T}} &\simeq \underline{\mathrm{Fun}}_{\mathcal{T}}^{R}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_{\mathcal{T}}) \\
&\simeq \underline{\mathrm{Fun}}_{\mathcal{T}}^{R}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Lin}}_{\mathcal{T}}(\underline{\mathcal{S}}_{*\mathcal{T}}^{\mathrm{fin}}, \underline{\mathcal{S}}_{\mathcal{T}})) \\
&\simeq \underline{\mathrm{Lin}}_{\mathcal{T}}(\underline{\mathcal{S}}_{*\mathcal{T}}^{\mathrm{fin}}, \underline{\mathrm{Fun}}_{\mathcal{T}}^{R}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}})) \\
&\simeq \underline{\mathrm{Lin}}_{\mathcal{T}}(\underline{\mathcal{S}}_{*\mathcal{T}}^{\mathrm{fin}}, \underline{\mathcal{C}} \otimes \underline{\mathcal{S}}_{\mathcal{T}}) \\
&\simeq \underline{\mathrm{Lin}}_{\mathcal{T}}(\underline{\mathcal{S}}_{*\mathcal{T}}^{\mathrm{fin}}, \underline{\mathcal{C}}) \simeq \underline{\mathrm{Sp}}_{\mathcal{T}}(\underline{\mathcal{C}})
\end{aligned}$$

where the first equivalence is by Proposition 2.2.23. We have also used Nardin's formula for $\mathcal{T}$–stabilisation from Theorem 1.3.8. $\square$

Proposition 2.3.12 (Parametrised stabilisation is smashing, "[GGN15, Thm. 4.6]")**.** *The association $\underline{\mathcal{C}} \mapsto \underline{\mathrm{Sp}}_{\mathcal{T}}(\underline{\mathcal{C}})$ refines to a $\mathcal{T}$–symmetric monoidal localisation $\underline{\mathrm{Sp}}_{\mathcal{T}} \otimes - : \underline{\mathrm{Pr}}_{\mathcal{T},L} \longrightarrow \underline{\mathrm{Pr}}_{\mathcal{T},L}$ with essential image the $\mathcal{T}$–full subcategory of $\mathcal{T}$–presentable-stable categories $\underline{\mathrm{Pr}}_{\mathcal{T},\mathrm{st},L}$.*

Proof. That $\underline{\mathrm{Sp}}_{\mathcal{T}}(-) \simeq \underline{\mathrm{Sp}}_{\mathcal{T}} \otimes (-)$ is the proposition above, which also gives the required essential image. That the functor is a $\mathcal{T}$–symmetric monoidal localisation is by the $\mathcal{T}$–idempotence of $\underline{\mathrm{Sp}}_{\mathcal{T}}$ from [Nar17, Cor. 3.28]. $\square$

2.3.3 Internal hom objects

Observation 2.3.13 ($\mathcal{T}$–right exacts on $\mathcal{T}$–stables)**.** If $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ are $\mathcal{T}$–stables, then note that the two $\mathcal{T}$–full subcategories $\underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \subseteq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \supseteq \underline{\mathrm{Fun}}^{\mathrm{rex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ agree. To wit, both mean that they are fibrewise right and left exact (since these are fibrewise stable after all); moreover, preserving finite $\mathcal{T}$–coproducts and preserving finite $\mathcal{T}$–products are equivalent since $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ were $\mathcal{T}$–semiadditive. Hence in this case we have $\underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \simeq \underline{\mathrm{Fun}}^{\mathrm{ex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \simeq \underline{\mathrm{Fun}}^{\mathrm{rex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$.

Lemma 2.3.14. *Let $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ be finite $\mathcal{T}$–complete and $\underline{\mathcal{A}}$ be finite $\mathcal{T}$–cocomplete. Then we have a canonical equivalence $\underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{Lin}}_{\mathcal{T}}(\underline{\mathcal{A}}, \underline{\mathcal{D}})) \simeq \underline{\mathrm{Lin}}_{\mathcal{T}}(\underline{\mathcal{A}}, \underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}))$.*

Proof. Note that we have the identification $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{Lin}}_{\mathcal{T}}(\underline{\mathcal{A}}, \underline{\mathcal{D}})) \simeq \underline{\mathrm{Lin}}_{\mathcal{T}}(\underline{\mathcal{A}}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}))$ since $\mathcal{T}$–limits of functor categories are computed in the target by Proposition 1.2.12. To see that we have the desired equivalence, consider the diagram

$$\begin{tikzcd}
\underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{Lin}}_{\mathcal{T}}(\underline{\mathcal{A}}, \underline{\mathcal{D}})) \arrow[r, dashed, shift left] \arrow[d, hook] & \underline{\mathrm{Lin}}_{\mathcal{T}}(\underline{\mathcal{A}}, \underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})) \arrow[l, dashed, shift left] \arrow[d, hook] \\
\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathrm{Lin}}_{\mathcal{T}}(\underline{\mathcal{A}}, \underline{\mathcal{D}})) \arrow[r, shift left, "\simeq"] & \underline{\mathrm{Lin}}_{\mathcal{T}}(\underline{\mathcal{A}}, \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})) \arrow[l, shift left]
\end{tikzcd}$$

That the bottom arrows restrict to the dashed arrows is because again by Proposition 1.2.12, $\mathcal{T}$–limits in both $\underline{\mathrm{Lin}}_{\mathcal{T}}(\underline{\mathcal{A}}, \underline{\mathcal{D}})$ and $\underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ are computed in $\underline{\mathcal{D}}$. $\square$

Corollary 2.3.15 (Internal hom object of $\mathcal{T}$–perfects)**.** *Let $\underline{\mathcal{C}}, \underline{\mathcal{D}} \in \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$. Then the $\mathcal{T}$–full subcategory $\underline{\mathrm{Fun}}^{\mathrm{ex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \subseteq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ on the $\mathcal{T}$–exact functors is also small $\mathcal{T}$–idempotent-complete-stable, that is $\underline{\mathrm{Fun}}^{\mathrm{ex}}_{\mathcal{T}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ is again an object of $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$.*

Proof. That it is small is clear. To see that it is $\mathcal{T}$–stable, just note

$$\begin{aligned}\underline{\mathrm{Fun}}^{\mathrm{ex}}_{\mathcal{T}}(\underline{\mathcal{C}},\underline{\mathcal{D}}) &\simeq \underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{C}},\underline{\mathcal{D}})\\ &\simeq \underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{C}},\underline{\mathrm{Lin}}_{\mathcal{T}}(\underline{\mathcal{S}}^{\mathrm{fin}}_{*\mathcal{T}},\underline{\mathcal{D}}))\\ &\simeq \underline{\mathrm{Lin}}_{\mathcal{T}}(\underline{\mathcal{S}}^{\mathrm{fin}}_{*\mathcal{T}},\underline{\mathrm{Fun}}^{\mathrm{lex}}_{\mathcal{T}}(\underline{\mathcal{C}},\underline{\mathcal{D}}))\\ &\simeq \underline{\mathrm{Lin}}_{\mathcal{T}}(\underline{\mathcal{S}}^{\mathrm{fin}}_{*\mathcal{T}},\underline{\mathrm{Fun}}^{\mathrm{ex}}_{\mathcal{T}}(\underline{\mathcal{C}},\underline{\mathcal{D}}))\end{aligned}$$

where the first and last equivalences are by Observation 2.3.13, the second is by Theorem 1.3.8, and the third by Lemma 2.3.14. For $\mathcal{T}$–idempotent-completeness, note that $\mathcal{T}$–colimits of $\underline{\mathrm{Fun}}^{\mathrm{ex}}_{\mathcal{T}}(\underline{\mathcal{C}},\underline{\mathcal{D}}) \simeq \underline{\mathrm{Fun}}^{\mathrm{rex}}_{\mathcal{T}}(\underline{\mathcal{C}},\underline{\mathcal{D}})$ are computed in $\underline{\mathcal{D}}$, and since being $\mathcal{T}$–idempotent-complete is just the condition of admitting certain fibrewise $\mathcal{T}$–colimits, this point is clear too. □

Proposition 2.3.16. *Let $\underline{\mathcal{C}} \in \underline{\mathrm{Cat}}^{\mathrm{perf}(\kappa)}_{\mathcal{T}}$. Then $\underline{\mathrm{Fun}}^{\mathrm{ex}}_{\mathcal{T}}(\underline{\mathrm{Sp}}^{\underline{\kappa}}_{\mathcal{T}},\underline{\mathcal{C}}) \simeq \underline{\mathcal{C}}$.*

Proof. Recall we had equivalence $\underline{\mathrm{Cat}}^{\mathrm{perf}(\kappa)}_{\mathcal{T}} \simeq \underline{\mathrm{Pr}}_{\mathcal{T},\mathrm{st},L,\kappa}$ from Theorem 2.2.16 so that $(\underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{C}})^{\underline{\kappa}} \simeq \underline{\mathcal{C}}$. Writing $\underline{\mathrm{Fun}}^{\underline{\kappa}} \subseteq \underline{\mathrm{Fun}}$ for the $\mathcal{T}$–full subcategory of parametrised functors preserving parametrised κ–compact objects. Now consider

$$\begin{aligned}\underline{\mathrm{Fun}}^{\mathrm{ex}}_{\mathcal{T}}(\underline{\mathrm{Sp}}^{\underline{\kappa}}_{\mathcal{T}},\underline{\mathcal{C}}) &\simeq \underline{\mathrm{Fun}}^{\mathrm{rex}}_{\mathcal{T}}(\underline{\mathrm{Sp}}^{\underline{\kappa}}_{\mathcal{T}},(\underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{C}})^{\underline{\kappa}})\\ &\simeq \underline{\mathrm{Fun}}^{L,\underline{\kappa}}_{\mathcal{T}}(\underline{\mathrm{Sp}}_{\mathcal{T}},\underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{C}})\\ &\simeq \underline{\mathrm{Fun}}^{L,\underline{\kappa}}_{\mathcal{T}}(\underline{\mathcal{S}}_{\mathcal{T}},\underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{C}})\\ &\simeq (\underline{\mathrm{Ind}}_{\kappa}\underline{\mathcal{C}})^{\underline{\kappa}} \simeq \underline{\mathcal{C}}\end{aligned}$$

where the second equivalence is by Proposition 1.2.38; the third equivalence is by Proposition 2.3.12; the fourth equivalence is by Theorem 1.2.24. □

Part II

Equivariant Algebraic K–theory

Chapter 3

Equivariance via parametrised theory

In this chapter, we specialise the theories developed so far to the case of equivariant homotopy theory for a finite group G. After recording some basic translations, we prove the first main result of the chapter in Theorem 3.3.4: this is an extremely general principle which says that whenever we have a G–symmetric monoidal category, Borelification (ie. forgetting all the structures and only remembering the underlying object with G–action) is always a G–symmetric monoidal process. We prove it by decategorifying a categorified formulation. In particular, whenever this functor has a right adjoint, then this is always G–lax symmetric monoidal, and so ordinary commutative algebra objects will always induce a G–commutative algebra Borel object. Consequently, we obtain in Proposition 3.3.6 a general machinery to manufacture interesting G–commutative algebras coming from commutative algebra objects endowed with G–actions, which are much easier to produce. To round out our discussion of general G–monoidal matters, we give several basic G–monoidal identifications crucial for our theory of norms on G–quadratic functors.

Finally, we study the topological Singer construction of [LR12; NS18] in the case $p = 2$ in the equivariant setting. The key result here is that upon applying $(-)^{t\Sigma_2}$, diagonalisations of G–symmetric bilinear functors are G–linear when G is odd (cf. Corollary 3.5.3). We will need this in our discussion of genuine G–hermitian K–theory for odd G in §7.1. Furthermore, this also implies, by general principles, that we obtain a refinement of the celebrated Nikolaus-Scholze Tate diagonal to the setting of genuine G–spectra in the case when G is odd and $p = 2$. We think that this has the potential of being a very interesting structure to exploit and we intend to return to this in future work.

3.1 Genuine G-category theory

Let G be a finite group. The abstract parametrised formalism treated in Part I yields the notion of G–categories by setting the base category $\mathcal{T}$ to be the opposite of the orbit category $\mathcal{O}_G^{\mathrm{op}}$ of G, which is an atomic orbital category. Recall that $\mathcal{O}_G$ is the 1–category whose objects are transitive G–sets G/H, where $H \leq G$ is a subgroup, and whose morphisms are G–equivariant maps between these. By a straightforward unwinding of definitions, we see that morphisms $G/H \to G/K$ correspond precisely to subconjugations of H into K inside G. In particular, the endomorphisms of an orbit G/H in this category are precisely given by the Weyl group $W_GH := N_GH/H$ of $H \leq G$, and so these are in fact even automorphisms. Here, $N_GH \leq G$ is the normaliser of H in G.

Therefore, a *G–category* is then an object $\mathcal{C} \in \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathrm{Cat})$, that is, the data of a category $\mathcal{C}_H$ for each $H \leq G$ together with functors generated under compositions by:

- For each inclusion $H \leq K$ of subgroups in G, a restriction functor $\mathrm{Res}_H^K : \mathcal{C}_K \to \mathcal{C}_H$,
- For each $H \leq G$, the data of self-equivalences of $\mathcal{C}_H$ by W_GH. For instance, for the orbit G/e given by the trivial subgroup, the category $\mathcal{C}_e$ is endowed with a G–action, whereas for the orbit G/G, the category $\mathcal{C}_G$ has no nontrivial self-equivalences.

Hence, one can think of a G–category $\mathcal{C}$ as an underlying category with G-action $\mathcal{C}_e$ together with its *genuine fixed points data* $(\mathcal{C}_e)^H := \mathcal{C}_H$ for each subgroup $H \leq G$.

Example 3.1.1 (Equivariant stable homotopy theory)**.** Here are two distinct and important versions of what G–spectra could mean. They should illustrate the difference between *homotopy* fixed points, which can always be recovered once we have an object with G–action, and genuine fixed points, which are *extra data* that we have to supply. In both versions, we will see a philosophical principle where the equivariant structures get "internalised" as we pass to higher fixed points.

(i) (Spectra with G–action) Purely from the datum of the trivial G–action on the category Sp, we can recover its various homotopy fixed points as $\mathrm{Sp}^{hH} \simeq \mathrm{Fun}(BH, \mathrm{Sp}) =: \mathrm{Sp}^{BH}$ for $H \leq G$. These assemble into the G–category $\underline{\mathrm{Bor}}(\mathrm{Sp})$ of *spectra with G–actions* given by $\underline{\mathrm{Bor}}(\mathrm{Sp})_H := \mathrm{Sp}^{hH} \simeq \mathrm{Sp}^{BH}$. The restriction functor $\mathrm{Res}_H^K : \mathrm{Sp}^{BK} \to \mathrm{Sp}^{BH}$ for $H \leq K$ is then just the usual restriction of a K–action to a H–action. In this case, we see that upon passing to the top fixed points, the (trivial) G–action on the whole category $\underline{\mathrm{Bor}}(\mathrm{Sp})_e = \mathrm{Sp}$ gets absorbed into the category $\underline{\mathrm{Bor}}(\mathrm{Sp})_G = \mathrm{Sp}^{BG}$, which no longer has a G–action, but whose *objects* are now endowed with G–actions.

(ii) (Genuine G–spectra) Unlike the case above where everything can be recovered just from the data of the (trivial) G–action on Sp, there is another much more highly structured setting, namely that of *genuine* G–spectra $\underline{\mathrm{Sp}}_G$. Here, we have the same underlying category with G–action, ie. $(\underline{\mathrm{Sp}}_G)_e = \mathrm{Sp}$ endowed with the trivial G–action, but now the fixed points are given by $(\underline{\mathrm{Sp}}_G)^H := \mathrm{Sp}_H$, the genuine H–spectra. In this case, the restrictions are the usual restrictions on genuine equivariant spectra.

The distinction between the two examples above highlights the fact that the data of a G–category is *much more* than just that of G–actions on a category - indeed, both the examples have *equivalent* underlying category with G–action.

This formalism also allows us to handle the notion of G–limits and G–colimits (cf. §1.2.1). In more detail, we can ask if all the functors Res^H_K for all $H \leq K \leq G$ have the property of admitting left (resp. right) adjoints. If they do, then we call these the finite G–coproducts Ind^H_K (resp. G–products Coind^H_K). Furthermore, by atomic orbitality of $\mathcal{O}^{\mathrm{op}}_G$, we get that there are canonical comparisons (cf. §1.3.1) $\mathrm{Ind}^H_K \Rightarrow \mathrm{Coind}^H_K$ and we say that a G–category $\mathcal{C}$ is *G–semiadditive* if it has the property that these canonical comparisons are equivalences. We moreover say that it is *G–stable* if it is G–semiadditive and is fibrewise stable (ie. $\mathcal{C}_H$ are stable for all $H \leq G$).

3.2 User's guide to normed G-spectra

We collect here some salient aspects of the multiplicative norms in the context of genuine G-spectra for the benefit of those who want to use these extra structures model-independently. We will however omit two important points, namely: (1) the compatibility of these norms with geometric fixed points, for which we refer the reader to [HHR16, Prop. 2.57] and [Sch20, Prop. 11.9]; (2) the complicated notion of *distributivity* which pertains to the interaction between these multiplicative norms with additive inductions, and we refer the reader to [EH19] and [QS22, §5.1] for more details on this. We take the path of deriving the following properties purely axiomatically given the formalism of G–operads.

Observation 3.2.1 (Underlying object of normed objects)**.** Let $\mathcal{C}^{\underline{\otimes}}$ be a G-symmetric monoidal category and $A \in \mathcal{C}_H$. We observe that the underlying object of the normed object $\mathrm{N}^G_H A$ is just given by $\bigotimes_{|G/H|} A$ as expected. To see this, we compute $\mathrm{Res}^G_e \mathrm{N}^G_H A$ axiomatically. We know that N^G_H and Res^G_e are given respectively by the cocartesian lift along the spans

$$\begin{tikzcd} G/H \arrow[r, equal] \arrow[d] & G/H \arrow[r] \arrow[d] & G/G \arrow[d, equal] \\ G/G \arrow[r, equal] & G/G \arrow[r, equal] & G/G \end{tikzcd} \qquad \begin{tikzcd} G/G \arrow[d, equal] & G/e \arrow[l] \arrow[r, equal] \arrow[d] & G/e \arrow[d, equal] \\ G/G & G/e \arrow[l] \arrow[r, equal] & G/e \end{tikzcd}$$

Hence composing these spans and re-expressing the terms yield that $\mathrm{Res}^G_e \mathrm{N}^G_H$ is encoded by the morphism in $\underline{\mathrm{Fin}}_{*G}$

$$\begin{tikzcd} G/H \arrow[d] & \coprod_{G/H} G/e \arrow[l] \arrow[d] \arrow[r, equal] & \coprod_{G/H} G/e \arrow[d] \arrow[r, equal] & \coprod_{G/H} G/e \arrow[d] \arrow[r] & G/e \arrow[d, equal] \\ G/G & G/e \arrow[l] \arrow[r, equal] & G/e \arrow[r, equal] & G/e \arrow[r, equal] & G/e \end{tikzcd}$$

The left span is $\prod_{G/H} \mathrm{Res}^G_e$ whereas by Recollections 1.3.18, the right span is $\bigotimes_{|G/H|}$. Hence in total we see that $\mathrm{Res}^G_e \mathrm{N}^G_H A \simeq \bigotimes_{|G/H|} \mathrm{Res}^G_e A$ as claimed.

Observation 3.2.2 (Equivariant multiplications)**.** A natural follow-up question that one might have when presented with the notion of G–commutative algebra objects is, what are the essential extra structures we are endowing on them? For $\underline{\mathcal{C}}^{\otimes}$ a G-symmetric monoidal category (for instance, $\underline{\mathrm{Sp}}^{\otimes}$), a G–commutative algebra object $A \in \mathrm{CAlg}_G(\underline{\mathcal{C}}^{\otimes})$ is then by definition a G–inert section of the cocartesian fibration $\underline{\mathcal{C}}^{\otimes} \to \underline{\mathrm{Fin}}_{*G}$. In particular, for

$$\begin{tikzcd} G/H \arrow[d] \arrow[r, equal] & G/H \arrow[d] \arrow[r, "f"] & G/G \arrow[d, equal] \\ G/G \arrow[r, equal] & G/G \arrow[r, equal] & G/G \end{tikzcd}$$

we obtain a morphism in $\mathrm{Map}^f_{\mathcal{C}_H}(\mathrm{Res}^G_H A, A) \simeq \mathrm{Map}_{\mathcal{C}_G}(\mathrm{N}^G_H \mathrm{Res}^G_H A, A)$, ie. a morphism

$$\mu^G_H : \mathrm{N}^G_H \mathrm{Res}^G_H A \to A$$

which can be viewed as the data of equivariant multiplications where we do not only have the usual n–fold multiplications $\bigotimes_n A \to A$, but also extra multiplications $\mathrm{N}^H_K \mathrm{Res}^G_K A \to \mathrm{Res}^G_H A$. Note that in the case of genuine G–spectra, as for the ordinary multiplications, these equivariant multiplication maps are maps of spectra, and so in particular, additive.

3.3 A monoidal Borelification principle

Construction 3.3.1 (Borel objects)**.** Let $\mathcal{C}$ be an ordinary category. Then we define the Borel G–category $\underline{\mathrm{Bor}}(\mathcal{C})$ associated to it to be the G–category whose value at G/H is given by $\mathrm{Fun}(BH, \mathcal{C})$ and the restriction functors are given by restriction $i^* : : \mathrm{Fun}(BK, \mathcal{C}) \to \mathrm{Fun}(BH, \mathcal{C})$ for $i \colon G/H \to G/K$ a subconjugation of H into K.

Observation 3.3.2. There is a G–adjunction $\mathrm{ev} : \widehat{\underline{\mathrm{Cat}}}_G \rightleftarrows \underline{\mathrm{Bor}}(\widehat{\mathrm{Cat}}) : \underline{\mathrm{Bor}}$ which is fibrewise given by $j^* : \widehat{\mathrm{Cat}}_H = \mathrm{Fun}(\mathcal{O}^{\mathrm{op}}_H, \widehat{\mathrm{Cat}}) \rightleftarrows \mathrm{Fun}(BH, \widehat{\mathrm{Cat}}) = (\underline{\mathrm{Bor}}(\widehat{\mathrm{Cat}}))_H : j_*$. Moreover, for $\mathcal{C} \in \widehat{\mathrm{Cat}}_H$, the adjuntion unit $\mathcal{C} \to j_* j^* \mathcal{C}$ is given levelwise by $\mathcal{C}_H \to$

$\mathcal{C}_e^{hH}$ induced by the H–equivariant map $\mathrm{Res}_e^H : \mathcal{C}_H \to \mathcal{C}_e$ present in the structure of a G–category.

Proposition 3.3.3. *Both the G–functors in the G–adjunction* $\mathrm{ev} : \underline{\widehat{\mathrm{Cat}}}_G \rightleftarrows \underline{\mathrm{Bor}}(\widehat{\mathrm{Cat}}) : \underline{\mathrm{Bor}}$ *above strongly preserve finite G–products. In particular, this G–adjunction induces the adjunction*

$$\mathrm{ev} : \mathrm{CMon}_G(\underline{\widehat{\mathrm{Cat}}}_G) \rightleftarrows \mathrm{CMon}(\mathrm{Fun}(BG, \mathrm{Cat})) : \underline{\mathrm{Bor}}$$

Proof. Let $H \leq G$ be a subgroup, $i : \mathcal{O}_H^{\mathrm{op}} \to \mathcal{O}_G^{\mathrm{op}}$ be the inclusion, and $j : BH \hookrightarrow \mathcal{O}_H^{\mathrm{op}}$ be the fully faithful inclusion. Concretely, on level G/H the adjunction is given by

$$i^* : \mathrm{Fun}(\mathcal{O}_H^{\mathrm{op}}, \widehat{\mathrm{Cat}}) \rightleftarrows \mathrm{Fun}(BH, \widehat{\mathrm{Cat}}) : i_*$$

The G–right adjoint of course strongly preserves G–products, and the statement that the left adjoint $\mathrm{ev} = i^*$ does so too translates into seeing that the square

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \widehat{\mathrm{Cat}}) \arrow[r, "j^*"] & \mathrm{Fun}(BG, \widehat{\mathrm{Cat}}) \\
\mathrm{Fun}(\mathcal{O}_H^{\mathrm{op}}, \widehat{\mathrm{Cat}}) \arrow[r, "j^*"] \arrow[u, "i_*"] & \mathrm{Fun}(BH, \widehat{\mathrm{Cat}}) \arrow[u, "i_*"']
\end{tikzcd}$$

commutes. And this is because we have equivalences (even isomorphisms!) of comma categories at $G/e \in BG \subseteq \mathcal{O}_G^{\mathrm{op}}$

$$\Big[G/e \downarrow (\mathcal{O}_H^{\mathrm{op}} \xrightarrow{i} \mathcal{O}_G^{\mathrm{op}})\Big] \simeq \coprod_{G/H} * \simeq \Big[G/e \downarrow (BH \xrightarrow{i} BG)\Big]$$

This gives us the first part of the proposition. For the second part, first observe that since both adjoints strongly preserve finite G–products, we can apply $\underline{\mathrm{CMon}}_G(-) := \underline{\mathrm{Fun}}^{\times}(\underline{\mathrm{Fin}}_{*G}, -)$ to obtain the G–adjunction

$$\mathrm{ev} : \mathrm{CMon}_G(\underline{\widehat{\mathrm{Cat}}}_G) \rightleftarrows \mathrm{CMon}_G(\underline{\mathrm{Bor}}(\widehat{\mathrm{Cat}})) : \underline{\mathrm{Bor}}$$

Hence, it would suffice to show that $\mathrm{CMon}_G(\underline{\mathrm{Bor}}(\mathcal{C})) \simeq \mathrm{CMon}(\mathcal{C})$ for any category $\mathcal{C}$ admitting finite products. By definition, we have

$$\mathrm{CMon}_G(\underline{\mathrm{Bor}}(\mathcal{C})) := \mathrm{Fun}_G^{\times}(\underline{\mathrm{Fin}}_{*G}, \underline{\mathrm{Bor}}(\mathcal{C})), \qquad \mathrm{CMon}(\mathcal{C}) := \mathrm{Fun}^{\times}(\mathrm{Fin}_*, \mathcal{C})$$

Now the adjunction $\widehat{\mathrm{Cat}}_G = \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \widehat{\mathrm{Cat}}) \rightleftarrows \mathrm{Bor}(\widehat{\mathrm{Cat}})$ is easily seen to induce the equivalence of *categories* (ie. it is a 2-adjunction)

$$\mathrm{Fun}_G(\underline{\mathrm{Fin}}_{*G}, \underline{\mathrm{Bor}}(\mathcal{C})) \xrightarrow[\simeq]{\mathrm{ev}} \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{C})$$

which clearly induces the commuting square

$$\begin{tikzcd}
\mathrm{Fun}_G(\underline{\mathrm{Fin}}_{*G}, \underline{\mathrm{Bor}}(\mathcal{C})) \arrow[r, "\mathrm{ev}", "\simeq"'] & \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{C}) \\
\mathrm{Fun}^{\underline{\times}}_G(\underline{\mathrm{Fin}}_{*G}, \underline{\mathrm{Bor}}(\mathcal{C})) \arrow[r, "\mathrm{ev}"] \arrow[u, hook] & \mathrm{Fun}^{\times}(\mathrm{Fin}_*, \mathcal{C}) \arrow[u, hook]
\end{tikzcd}$$

In particular, the bottom horizontal functor is fully faithful. To see that it is essentially surjective, we know that the inverse for the top horizontal functor is given by applying $\underline{\mathrm{Bor}}(-)$ (observe that $\underline{\mathrm{Bor}}(\mathrm{Fin}_*) \simeq \underline{\mathrm{Fin}}_{*G}$ since $\mathrm{Fun}(BG, \mathrm{Fin}_*) \simeq \mathrm{Fin}_{*G}$). Hence, if $\varphi : \mathrm{Fin}_* \to \mathcal{C}$ is a finite product-preserving functor, then $\underline{\mathrm{Bor}}(\varphi) : \underline{\mathrm{Fin}}_{*G} = \underline{\mathrm{Bor}}(\mathrm{Fin}_*) \to \underline{\mathrm{Bor}}(\mathcal{C})$ strongly preserves finite G–products because the comma categories involved in computing the right Kan extensions are just disjoint unions of points. $\square$

We distil an immediate consequence of the result above into the following principle which establishes an abstract but very important link between G–categories and their underlying category with G–action. We will have use of this in the coming subsection as well as in the hermitian K–theory setting in Part III.

Theorem 3.3.4 (Monoidal Borelification principle)**.** *Let $\underline{\mathcal{C}}^{\underline{\otimes}} \in \mathrm{CMon}_G(\widehat{\underline{\mathrm{Cat}}}_G)$ be a G–symmetric monoidal category and $\mathcal{D}^{\otimes} \in \mathrm{CMon}(\widehat{\mathrm{Cat}})$ be a symmetric monoidal category. Then:*

(i) The G–category $\underline{\mathrm{Bor}}(\mathcal{D})$ canonically refines to a G–symmetric monoidal category $\underline{\mathrm{Bor}}(\mathcal{D}^{\otimes})$. This can be concretely described as follows: for $d \in \mathrm{Fun}(BH, \mathcal{D})$ a H–object in $\underline{\mathrm{Bor}}(\mathcal{D})$, the G–object $\mathrm{N}_H^G d \in \mathrm{Fun}(BG, \mathcal{D})$ is given by $\bigotimes_{|G/H|} d$,

(ii) The unit $\underline{\mathcal{C}} \to \underline{\mathrm{Bor}}(\mathcal{C}_e)$ of the adjunction from Observation 3.3.2 canonically refines to a G–symmetric monoidal functor

$$\underline{\mathcal{C}}^{\underline{\otimes}} \to \underline{\mathrm{Bor}}(\mathcal{C}_e^{\otimes})$$

In particular, if the underlying G–functor $\underline{\mathcal{C}} \to \underline{\mathrm{Bor}}(\mathcal{C}_e)$ admits a G–right adjoint, then this canonically refines to a G–lax symmetric monoidal functor.

Remark 3.3.5. There are many interesting cases where Borelification functor $\mathcal{C} \to \underline{\mathrm{Bor}}(\mathcal{C}_e)$ has a right adjoint. For example, if $\mathcal{C}$ were G–presentable, then by the fibrewise criterion of Theorem 2.2.2 for instance, we know that $\underline{\mathrm{Bor}}(\mathcal{C}_e)$ is also G–presentable. In this case, one just has to check that the Borelification funtor strongly preserves G–colimits and then appeal to Theorem 2.2.3.

Another very useful consequence of Proposition 3.3.3 is the following which leads to ample examples of G–commutative algebras. We thank Asaf Horev for discussions leading to it, in particular, in teaching us the trick of using symmetric monoidal envelopes.

Proposition 3.3.6 (G–Borel commutative algebras)**.** *Let* $\mathcal{C}^{\otimes} \in \mathrm{CMon}(\mathrm{Cat})$ *be a symmetric monoidal category. Then* $\mathrm{CAlg}_G(\underline{\mathrm{Bor}}(\mathcal{C}^{\otimes}))^{\simeq} \simeq (\mathrm{CAlg}(\mathcal{C}^{\otimes})^{\simeq})^{hG}$.

Proof. We compute:

$$\begin{aligned}
\mathrm{CAlg}_G(\underline{\mathrm{Bor}}(\mathcal{C}^{\otimes}))^{\simeq} &\simeq \mathrm{Fun}_{\mathrm{CMon}_G(\widehat{\underline{\mathrm{Cat}}}_G)}(\underline{\mathrm{Fin}}^{\amalg}_G, \underline{\mathrm{Bor}}(\mathcal{C}^{\otimes}))^{\simeq} \\
&\simeq \mathrm{Fun}_{\mathrm{CMon}(\mathrm{Fun}(BG,\widehat{\mathrm{Cat}}))}(\mathrm{Fin}^{\amalg}, \mathcal{C}^{\otimes})^{\simeq} \\
&\simeq (\mathrm{Fun}_{\mathrm{CMon}(\widehat{\mathrm{Cat}})}(\mathrm{Fin}^{\amalg}, \mathcal{C}^{\otimes})^{\simeq})^{hG} \\
&= (\mathrm{CAlg}(\mathcal{C}^{\otimes})^{\simeq})^{hG}
\end{aligned}$$

as required, where the second equivalence is by Proposition 3.3.3. □

3.4 Basic G-symmetric monoidal identifications

Notation 3.4.1. It is useful to adopt the notation of $\underline{\mathrm{Fun}}_G^{\delta_{G/H}}(\prod_{G/H}\underline{\mathcal{C}}, \underline{\mathcal{D}})$ to mean distributive functors (cf. §1.3.4) and $\underline{\mathrm{Fun}}_G^{\delta^{\mathrm{fin}}_{G/H}}(\prod_{G/H}\underline{\mathcal{C}}, \underline{\mathcal{D}})$ to mean *finite* distributive functors (ie. replace small colimits in the definition of distributivity with those which are finite in each fibre).

Lemma 3.4.2. *Let* $\underline{\mathcal{C}}$ *be* H*–stable. Then* $\underline{\mathrm{Fun}}^*_G(\prod_{G/H}\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G) \simeq \bigotimes_{G/H}\underline{\mathrm{Fun}}^*_H(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H)$.

Proof. By Proposition 2.3.8, we already know that the top map in

$$\begin{tikzcd}
\bigotimes_{G/H}\underline{\mathrm{Fun}}_H(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H) \arrow[r, "\simeq"] & \underline{\mathrm{Fun}}_G(\prod_{G/H}\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G) \\
\bigotimes_{G/H}\underline{\mathrm{Fun}}^*_H(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H) \arrow[r] \arrow[u, hook] & \underline{\mathrm{Fun}}^*_G(\prod_{G/H}\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G) \arrow[u, hook]
\end{tikzcd}$$

is an equivalence and that the square commutes clearly. To see that every object in $\varphi \in \underline{\mathrm{Fun}}^*_G(\prod_{G/H}\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G)$ is hit by the bottom horizontal map, observe that evaluating at the level G/G yields a functor

$$\varphi_{[G/G]} : \mathcal{C}^{\mathrm{op}}_H \simeq (\prod_{G/H}\underline{\mathcal{C}}^{\mathrm{op}})_{[G/G]} \longrightarrow (\underline{\mathrm{Sp}}_G)_{[G/G]} = \mathrm{Sp}_G$$

which preserves zero objects, and hence $\mathrm{Res}^G_H \circ \varphi_{[G/G]} : \mathcal{C}^{\mathrm{op}}_H \to \mathrm{Sp}_H$ must also preserve zero objects. Thus, under $\bigotimes_{G/H}\underline{\mathrm{Fun}}_H(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H) \simeq \underline{\mathrm{Fun}}_G(\prod_{G/H}\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G)$, the preimage of φ must have been in $\bigotimes_{G/H}\underline{\mathrm{Fun}}^*_H(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H)$. □

Lemma 3.4.3. *Let* $\underline{\mathcal{C}}$ *be a* H*–stable category. Then* $(\bigotimes_{G/H}\underline{\mathcal{C}})^{\underline{\mathrm{op}}} \simeq \bigotimes_{G/H}\underline{\mathcal{C}}^{\underline{\mathrm{op}}}$.

Proof. Let $\underline{\mathcal{E}}$ be a G–stable category. Then:

$$\begin{aligned}\mathrm{Fun}^{\mathrm{ex}}_{\underline{G}}((\bigotimes_{G/H}\underline{\mathcal{C}})^{\underline{\mathrm{op}}},\underline{\mathcal{E}}) &\simeq \mathrm{Fun}^{\mathrm{ex}}_{\underline{G}}(\bigotimes_{G/H}\underline{\mathcal{C}},\underline{\mathcal{E}}^{\underline{\mathrm{op}}})\\ &\simeq \mathrm{Fun}^{\delta^{\mathrm{fin}}_{G/H}}(\prod_{G/H}\underline{\mathcal{C}},\underline{\mathcal{E}}^{\underline{\mathrm{op}}})\\ &\simeq \mathrm{Fun}^{\mathrm{ex}}_{\underline{G}}(\bigotimes_{G/H}\underline{\mathcal{C}}^{\underline{\mathrm{op}}},\underline{\mathcal{E}})\end{aligned}$$

where the second equivalence is since $\underline{\mathcal{E}}^{\underline{\mathrm{op}}}$ is still G–stable. □

Notation 3.4.4. Write $\underline{\mathrm{PSh}}^{\mathrm{st}}_{G}(\underline{\mathcal{C}})$ for $\underline{\mathrm{Fun}}_{G}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}},\underline{\mathrm{Sp}}_{G})$, the G–spectral presheaves.

Lemma 3.4.5. *Let $\underline{\mathcal{E}}$ be G–presentable and $\underline{\mathcal{C}} \in \mathrm{Cat}_{H}$. Then $y^*\colon \mathrm{Fun}^{\delta_{G/H}}_{\underline{G}}(\prod_{G/H}\underline{\mathrm{PSh}}^{\mathrm{st}}_{H}(\underline{\mathcal{C}}),\underline{\mathcal{E}}) \to \mathrm{Fun}_{\underline{G}}(\prod_{G/H}\underline{\mathcal{C}},\underline{\mathcal{E}})$ is an equivalence.*

Proof. We compute:

$$\begin{aligned}\mathrm{Fun}^{\delta_{G/H}}_{\underline{G}}\big(\prod_{G/H}\underline{\mathrm{PSh}}^{\mathrm{st}}_{H}(\underline{\mathcal{C}}),\underline{\mathcal{E}}\big) &\simeq \mathrm{Fun}^{L}_{\underline{G}}\big(\bigotimes_{G/H}\underline{\mathrm{PSh}}^{\mathrm{st}}_{H}(\underline{\mathcal{C}}),\underline{\mathcal{E}}\big)\\ &\simeq \mathrm{Fun}^{L}_{\underline{G}}\big((\bigotimes_{G/H}\underline{\mathrm{PSh}}_{H}(\underline{\mathcal{C}}))\otimes\bigotimes_{G/H}\underline{\mathrm{Sp}}_{H},\underline{\mathcal{E}}\big)\\ &\simeq \mathrm{Fun}^{L}_{\underline{G}}\big(\underline{\mathrm{PSh}}_{G}(\prod_{G/H}\underline{\mathcal{C}})\otimes\underline{\mathrm{Sp}}_{G},\underline{\mathcal{E}}\big)\\ &\simeq \mathrm{Fun}_{\underline{G}}\big(\prod_{G/H}\underline{\mathcal{C}},\underline{\mathcal{E}}\big)\end{aligned}$$

where the third equivalence is by Proposition 2.3.8 and the last by Proposition 2.3.12. □

Lemma 3.4.6. *Let $\underline{\mathcal{C}}$ be H–stable and let $i : \underline{\mathcal{C}} \hookrightarrow \underline{\mathrm{Ind}}_{H}(\underline{\mathcal{C}})$ be the inclusion. Let $\underline{\mathcal{E}}$ be a G–presentable-stable category. Then the following functor is an equivalence.*

$$\mathrm{Fun}^{\delta_{G/H}}_{\underline{G}}(\prod_{G/H}\underline{\mathrm{Ind}}_{H}\underline{\mathcal{C}},\underline{\mathcal{E}}) \xrightarrow{i^*} \mathrm{Fun}^{\delta^{\mathrm{fin}}_{G/H}}_{\underline{G}}(\prod_{G/H}\underline{\mathcal{C}},\underline{\mathcal{E}})$$

Proof. Let $L : \underline{\mathrm{PSh}}^{\mathrm{st}}_{H}(\underline{\mathcal{C}}) \rightleftarrows \underline{\mathrm{Ind}}_{H}(\underline{\mathcal{C}}) : \ell$ be the H-Bousfield localisation. Since $\underline{\mathrm{Ind}}_{H}(\underline{\mathcal{C}}) \simeq \underline{\mathrm{Fun}}^{\mathrm{lex}}_{H}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}},\underline{\mathrm{Sp}}_{H})$ by Proposition 2.1.9, in fact ℓ strongly preserves colimits. Hence we get the left vertical adjunction in the diagram

$$\begin{tikzcd}
\mathrm{Fun}^{\delta_{G/H}}_{\underline{G}}(\prod_{G/H}\underline{\mathrm{PSh}}^{\mathrm{st}}_{H}(\underline{\mathcal{C}}),\underline{\mathcal{E}}) \arrow[r, "\simeq"', "y^*"'] \arrow[d, dashed, shift right, "\ell^*"'] & \mathrm{Fun}_{\underline{G}}(\prod_{G/H}\underline{\mathcal{C}},\underline{\mathcal{E}}) \arrow[l, dashed, bend right, "y_!"'] \\
\mathrm{Fun}^{\delta_{G/H}}_{\underline{G}}(\prod_{G/H}\underline{\mathrm{Ind}}_{H}(\underline{\mathcal{C}}),\underline{\mathcal{E}}) \arrow[r, "i^*"'] \arrow[u, hook, shift right, "L^*"'] & \mathrm{Fun}^{\delta^{\mathrm{fin}}_{G/H}}_{\underline{G}}(\prod_{G/H}\underline{\mathcal{C}},\underline{\mathcal{E}}) \arrow[u, hook]
\end{tikzcd} \tag{3.1}$$

where the top horizontal is an equivalence by the preceding lemma. The solid square commutes since $L\circ y \simeq L\circ \ell\circ i \simeq i$ and because $\underline{\mathcal{C}} \hookrightarrow \underline{\mathrm{Ind}}_H\underline{\mathcal{C}}$ strongly preserves finite H–colimits, we get that $i^* : \underline{\mathrm{Fun}}_H^{\delta_{G/H}}(\prod_{G/H}\underline{\mathrm{Ind}}_H\underline{\mathcal{C}},\underline{\mathcal{E}}) \to \underline{\mathrm{Fun}}_H(\prod_{G/H}\underline{\mathcal{C}},\underline{\mathcal{E}})$ lands in $\underline{\mathrm{Fun}}_H^{\delta^{\mathrm{fin}}_{G/H}}(\prod_{G/H}\underline{\mathcal{C}},\underline{\mathcal{E}})$. Thus i^* is fully faithful. To see that it is also essentially surjective, observe that by definition $y_!$ is the G–left Kan extension along the G–fully faithful functor $y \simeq \ell\circ i$, and so we have that $i^*\ell^*y_! \simeq y^*y_! \simeq \mathrm{id}$, ie. that every object in the bottom right category is in the image of i^*. □

Corollary 3.4.7 (G–symmetric monoidality of Ind-completion)**.** *Let $\underline{\mathcal{C}}$ be a H-stable category. Then there is an equivalence $\underline{\mathrm{Ind}}_G(\bigotimes_{G/H}\underline{\mathcal{C}}) \simeq \bigotimes_{G/H}\underline{\mathrm{Ind}}_H(\underline{\mathcal{C}})$ which is compatible with the respective Yoneda embeddings. Hence, by Proposition 2.1.9 this also means that we have an equivalence $\underline{\mathrm{Fun}}^{\mathrm{ex}}_G(\bigotimes_{G/H}\underline{\mathcal{C}}^{\mathrm{op}},\underline{\mathrm{Sp}}_G) \simeq \bigotimes_{G/H}\underline{\mathrm{Fun}}^{\mathrm{ex}}_H(\underline{\mathcal{C}}^{\mathrm{op}},\underline{\mathrm{Sp}}_H)$.*

Proof. We observe the sequence of equivalences of unparametrised categories

$$\begin{aligned}\mathrm{Fun}^L_G(\underline{\mathrm{Ind}}_G(\bigotimes_{G/H}\underline{\mathcal{C}}),\underline{\mathcal{E}}) &\xrightarrow[\simeq]{i^*} \mathrm{Fun}^{\mathrm{ex}}_G(\bigotimes_{G/H}\underline{\mathcal{C}},\underline{\mathcal{E}})\\ &\xrightarrow[\simeq]{\tau^*} \mathrm{Fun}_G^{\delta^{\mathrm{fin}}_{G/H}}(\prod_{G/H}\underline{\mathcal{C}},\underline{\mathcal{E}})\\ &\xleftarrow[\simeq]{(\prod i)^*} \mathrm{Fun}_G^{\delta_{G/H}}(\prod_{G/H}\underline{\mathrm{Ind}}_H\underline{\mathcal{C}},\underline{\mathcal{E}})\\ &\xleftarrow[\simeq]{\tau^*} \mathrm{Fun}^L_G(\bigotimes_{G/H}\underline{\mathrm{Ind}}_H\underline{\mathcal{C}},\underline{\mathcal{E}})\end{aligned} \tag{3.2}$$

where the third step is by Lemma 3.4.6. Since the equivalence factors through $\bigotimes_{G/H}\underline{\mathcal{C}}$ in the first step, they must respect the Yoneda maps as claimed. □

Corollary 3.4.8. *Let $\underline{\mathcal{C}}$ be a H–stable category. Then*

$$\underline{\mathrm{map}}_{\bigotimes_{G/H}\underline{\mathcal{C}}}(-,\otimes c) \simeq \otimes_{G/H}\underline{\mathrm{map}}_{\underline{\mathcal{C}}}(-,c)$$

Proof. Let $x,c \in \prod_{G/H}\underline{\mathcal{C}}$. By plugging in $\underline{\mathcal{E}} = \underline{\mathrm{Sp}}_G$ in Eq. (3.2), we get the diagram

$$\begin{tikzcd}
\prod_{G/H}\underline{\mathcal{C}} \arrow[r, "\prod i"] \arrow[d, "\tau"'] & \prod_{G/H}\underline{\mathrm{Ind}}_H(\underline{\mathcal{C}}) \arrow[r, "\mathrm{ev}_x"] \arrow[d, "\tau"'] & \prod_{G/H}\underline{\mathrm{Sp}}_H \arrow[d, "\otimes"] \\
\bigotimes_{G/H}\underline{\mathcal{C}} \arrow[r, "\otimes i"] \arrow[dr, "i_\otimes"'] & \bigotimes_{G/H}\underline{\mathrm{Ind}}_H\underline{\mathcal{C}} \arrow[r, "\mathrm{ev}_{\otimes x}"] \arrow[d, "\simeq"] & \bigotimes_{G/H}\underline{\mathrm{Sp}}_H \simeq \underline{\mathrm{Sp}}_G \\
& \underline{\mathrm{Ind}}_G(\bigotimes_{G/H}\underline{\mathcal{C}}) \arrow[ur, "\mathrm{ev}_{\otimes x}"'] &
\end{tikzcd}$$

By starting with $c \in \prod_{G/H}\underline{\mathcal{C}}$, we get $\underline{\mathrm{map}}_{\bigotimes_{G/H}\underline{\mathcal{C}}}(\otimes x,\otimes c) \simeq \otimes_{G/H}\underline{\mathrm{map}}_{\underline{\mathcal{C}}}(x,c)$. □

3.5 Σ_2-Tate of diagonalised bilinears for odd groups

Let G be an odd group, and let $\beta : \underline{\mathcal{C}} \times \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a symmetric G–bilinear functor between G–stable categories. One can ask when the composite $\beta \circ \Delta : \mathcal{C} \xrightarrow{\Delta} \underline{\mathcal{C}} \times \underline{\mathcal{C}} \xrightarrow{\beta} \underline{\mathcal{D}}$ is G–linear. To answer this question, let us first record the following observation from group theory whose main input is an old theorem of J.S. Frame from 1941. We thank Jeroen van der Meer for his help in deriving this statement.

Proposition 3.5.1. *Let G be an odd group and $H \leq G$ a subgroup. Then the only double coset HgH which is self-inverse, ie. $HgH = Hg^{-1}H \subseteq G$, is the trivial one associated to $g = e$.*

Proof. By [Fra41, Thm. 3.2], the number of self-inverse double cosets is given by $S = \frac{1}{|G|}\sum_{g \in G} \chi_{G/H}(g^2)$ where for an element $g \in G$, $\chi_{G/H}(g)$ is the trace of the element g under the representation $\pi : G \to \mathrm{Aut}(G/H)$. Note also that this trace is equal to the trace under the linearised representation $\rho : G \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}[G/H])$. On the other hand, the map of sets given by squaring $(-)^2 : G \to G$ is surjective (if $|G| = 2n-1$, then $g^{2n-1} = e$ and so $g = (g^n)^2$) and so is a bijection. Therefore, we can even write $S = \frac{1}{|G|}\sum_{g \in G} \chi_{G/H}(g)$. Now by the standard formula for traces we have $\sum_{g \in G} \chi_{G/H}(g) = |G| \dim(\mathbb{C}[G/H]^G) = |G| \cdot 1$ and so $S = |G|/|G| = 1$ as was to be shown. □

Lemma 3.5.2. *Let G be an odd group. Let $\beta \in \underline{\mathrm{Fun}}_G^{G-\mathrm{bilin}}(\underline{\mathcal{C}} \times \underline{\mathcal{C}}, \underline{\mathcal{D}})^{h\Sigma_2}$ for $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ G–stable categories, ie. β is a symmetric G–bilinear functor. Let $X, Y \in \mathcal{C}_H$ for some $H \leq G$. Then under the double coset decompositions on the second variable as a consequence of the G–bilinearity of β*

$$\beta(\mathrm{Ind}_H^G X, \mathrm{Ind}_H^G Y) \simeq \bigoplus_{g \in H\backslash G/H} \mathrm{Ind}_{H^g \cap H}^G \beta\big(\mathrm{Res}_{H^g \cap H}^H X, \mathrm{Res}_{H^g \cap H}^{H^{g^{-1}} \cap H} \mathrm{Res}_{H^{g^{-1}} \cap H}^H Y\big)$$

$$\beta(\mathrm{Ind}_H^G Y, \mathrm{Ind}_H^G X) \simeq \bigoplus_{g \in H\backslash G/H} \mathrm{Ind}_{H^g \cap H}^G \beta\big(\mathrm{Res}_{H^g \cap H}^H Y, \mathrm{Res}_{H^g \cap H}^{H^{g^{-1}} \cap H} \mathrm{Res}_{H^{g^{-1}} \cap H}^H X\big)$$

the Σ_2–symmetry $\beta(\mathrm{Ind}_H^G X, \mathrm{Ind}_H^G Y) \xrightarrow[\simeq]{\tau} \beta(\mathrm{Ind}_H^G Y, \mathrm{Ind}_H^G X)$ swaps the HgH summand and $Hg^{-1}H$ summand of $\beta(\mathrm{Ind}_H^G X, \mathrm{Ind}_H^G Y)$ and of $\beta(\mathrm{Ind}_H^G Y, \mathrm{Ind}_H^G X)$, respectively.

Proof. Let $g \in G$ be an element. Note first that the symmetry commutes with restrictions since it comes from $\underline{\mathrm{Fun}}_G^{G-\mathrm{bilin}}(\underline{\mathcal{C}} \times \underline{\mathcal{C}}, \underline{\mathcal{D}})^{h\Sigma_2}$ where Σ_2 acts on $\underline{\mathrm{Fun}}_G^{G-\mathrm{bilin}}(\underline{\mathcal{C}} \times \underline{\mathcal{C}}, \underline{\mathcal{D}})$ via the G–functor $\underline{\mathcal{C}} \times \underline{\mathcal{C}} \xrightarrow[\simeq]{\mathrm{swap}} \underline{\mathcal{C}} \times \underline{\mathcal{C}}$. Therefore, we obtain the datum of a commuting square

$$\begin{tikzcd}
\beta(X, \mathrm{Res}^G_H \mathrm{Ind}^G_H Y) \arrow[r] \arrow[d, "\tau"', "\simeq"] & \mathrm{Ind}^H_{H^g\cap H}\beta(\mathrm{Res}^H_{H^g\cap H} X, \mathrm{Res}^{H^{g^{-1}}\cap H}_{H^g\cap H}\mathrm{Res}^H_{H^{g^{-1}}\cap H} Y) \arrow[d, "\tau"', "\simeq"] \\
\beta(\mathrm{Res}^G_H \mathrm{Ind}^G_H Y, X) \arrow[r] & \mathrm{Ind}^H_{H^g\cap H}\beta(\mathrm{Res}^{H^{g^{-1}}\cap H}_{H^g\cap H}\mathrm{Res}^H_{H^{g^{-1}}\cap H} Y, \mathrm{Res}^H_{H^g\cap H} X)
\end{tikzcd} \tag{3.3}$$

However, both double coset decompositions in the statement arise from the one on the second variable of β. Hence we will need to translate the diagram Eq. (3.3) into this form. To this end, observe tautologically that $\big(G/H^g\cap H \xrightarrow{(-)^{g^{-1}}} G/H^{g^{-1}}\cap H \to G/G\big) = \big(G/H^g\cap H \to G/G\big)$ and so $\mathrm{Ind}^G_{H^g\cap H} \simeq \mathrm{Ind}^G_{H^{g^{-1}}\cap H}\mathrm{Res}^{H^g\cap H}_{H^{g^{-1}}\cap H}$. Because of this, we have a natural equivalence $\mathrm{Ind}^G_H\beta(Y, \mathrm{Res}^G_H\mathrm{Ind}^G_H X) \simeq \mathrm{Ind}^G_H\beta(\mathrm{Res}^G_H\mathrm{Ind}^G_H Y, X)$ coming from the equivalence on the double coset decomposition summand

$$\begin{aligned}
&\mathrm{Ind}^G_{H^g\cap H}\beta\big(\mathrm{Res}^H_{H^g\cap H} Y, \mathrm{Res}^{H^{g^{-1}}\cap H}_{H^g\cap H}\mathrm{Res}^H_{H^{g^{-1}}\cap H} X\big) \\
&\simeq \mathrm{Ind}^G_{H^{g^{-1}}\cap H}\mathrm{Res}^{H^g\cap H}_{H^{g^{-1}}\cap H}\beta\big(\mathrm{Res}^H_{H^g\cap H} Y, \mathrm{Res}^{H^{g^{-1}}\cap H}_{H^g\cap H}\mathrm{Res}^H_{H^{g^{-1}}\cap H} X\big) \\
&\simeq \mathrm{Ind}^G_{H^{g^{-1}}\cap H}\beta\big(\mathrm{Res}^{H^g\cap H}_{H^{g^{-1}}\cap H}\mathrm{Res}^H_{H^g\cap H} Y, \mathrm{Res}^H_{H^{g^{-1}}\cap H} X\big)
\end{aligned}$$

Thus, combining Eq. (3.3) with this identification and using the double coset decomposition on the second variable, for $g\in G$ such that $HgH \neq Hg^{-1}H$, the symmetry τ induces

$$\begin{tikzcd}
\mathrm{Ind}^G_H\beta(X, \mathrm{Res}^G_H \mathrm{Ind}^G_H Y) \arrow[r] \arrow[d, "\tau"', "\simeq"] & \bigoplus_{k\in\{g,g^{-1}\}}\mathrm{Ind}^H_{H^g\cap H}\beta(\mathrm{Res}^H_{H^g\cap H} X, \mathrm{Res}^{H^{g^{-1}}\cap H}_{H^g\cap H}\mathrm{Res}^H_{H^{g^{-1}}\cap H} Y) \arrow[dd, "\mathrm{swap}"', "\simeq"] \\
\mathrm{Ind}^G_H\beta(\mathrm{Res}^G_H \mathrm{Ind}^G_H Y, X) \arrow[d, "\simeq"] & \\
\mathrm{Ind}^G_H\beta(Y, \mathrm{Res}^G_H \mathrm{Ind}^G_H X) \arrow[r] & \bigoplus_{k\in\{g,g^{-1}\}}\mathrm{Ind}^H_{H^g\cap H}\beta(\mathrm{Res}^{H^{g^{-1}}\cap H}_{H^g\cap H}\mathrm{Res}^H_{H^{g^{-1}}\cap H} Y, \mathrm{Res}^H_{H^g\cap H} X)
\end{tikzcd}$$

In the case when $g = e$, this square is to be interpreted as the symmetry $\mathrm{Ind}^G_H\beta(X,Y) \xrightarrow[\simeq]{\tau} \mathrm{Ind}^G_H\beta(Y,X)$. This completes the proof of the lemma. □

Corollary 3.5.3. *Let G be an odd group and let $\beta \in \underline{\mathrm{Fun}}^{G-\mathrm{bilin}}_G(\underline{\mathcal{C}}\times\underline{\mathcal{C}}, \underline{\mathcal{D}})^{h\Sigma_2}$ a symmetric G–bilinear functor between G–stable categories. Then the $(\beta\circ\Delta)^{t\Sigma_2} : \underline{\mathcal{C}} \longrightarrow \underline{\mathcal{D}}$ is G–linear.*

Proof. Setting $Y = X$ in Lemma 3.5.2, we obtain the decomposition

$$\begin{aligned}
&\beta(\mathrm{Ind}_H^G X, \mathrm{Ind}_H^G X) \\
&\simeq \mathrm{Ind}_H^G \beta(X,X) \oplus \bigoplus_{[e]\neq[g]\in[H\backslash G/H]_{\Sigma_2}} \Big[\mathrm{Ind}^G_{H^g\cap H}\beta\big(\mathrm{Res}^H_{H^g\cap H} X, \mathrm{Res}^{H^{g^{-1}}\cap H}_{H^g\cap H}\mathrm{Res}^H_{H^{g^{-1}}\cap H} X\big) \\
&\oplus \mathrm{Ind}^G_{H^g\cap H}\beta\big(\mathrm{Res}^H_{H^{g^{-1}}\cap H} X, \mathrm{Res}^{H^g\cap H}_{H^{g^{-1}}\cap H}\mathrm{Res}^H_{H^g\cap H} X\big)\Big]
\end{aligned}$$

as Σ_2–objects, where the Σ_2–action on the square bracket summands swaps the two terms. Hence, since $(-)^{t\Sigma_2}$ vanishes on Σ_2–free objects, we get $\beta(\mathrm{Ind}_H^G X, \mathrm{Ind}_H^G X)^{t\Sigma_2} \simeq \big(\mathrm{Ind}_H^G \beta(X,X)\big)^{t\Sigma_2} \simeq \mathrm{Ind}_H^G\big(\beta(X,X)^{t\Sigma_2}\big)$ as required. $\square$

3.6 The equivariant Σ_2-Tate diagonal for odd groups

Observe that $-\otimes- : \underline{\mathrm{Sp}}_G \times \underline{\mathrm{Sp}}_G \to \underline{\mathrm{Sp}}_G$ is a G–bilinear functor, and so when G is odd, Corollary 3.5.3 implies that the Singer construction $\underline{\mathrm{T}}_2(-) := ((-)^{\otimes 2})^{t\Sigma_2} : \underline{\mathrm{Sp}}_G \longrightarrow \underline{\mathrm{Sp}}_G$ is G–linear. In this subsection, we will give an alternative presentation of this result using the *distributivity* property of the tensor product.

Lemma 3.6.1. *Let $H \leq G$ be a subgroup and $f : G/H \to G/G$ the unique map. Then there is a distributivity diagram, in the sense of [EH19, Def. 2.3.1],*

$$\begin{tikzcd}
 & (G/H\times G/H)^{\amalg 2} \arrow[r, "\widetilde{\nabla}"] \arrow[dl, "\pi_1\amalg\pi_2"'] \arrow[d, "u\amalg u"] \arrow[dr, phantom, "\lrcorner"] & G/H\times G/H \arrow[d, "u"] \\
G/H^{\amalg 2} \arrow[r, "f\amalg f"'] & G/G^{\amalg 2} \arrow[r, "\nabla"'] & G/G
\end{tikzcd}$$

Moreover, this diagram is Σ_2–equivariant where $G/H^{\amalg 2}$ and $G/G^{\amalg 2}$ are given the swap action, $G/H \times G/H$ the flip action, $(G/H\times G/H)^{\amalg 2}$ the swap-flip action, and G/G the trivial action.

Proof. By [HHR16, Lem. A.36] we have the distributivity diagram

$$\begin{tikzcd}
 & G/G^{\amalg 2}\times\Gamma(G/H^{\amalg 2}\xrightarrow{f^{\amalg 2}} G/G^{\amalg 2}) \arrow[r, "\mathrm{proj}_2"] \arrow[dl, "\mathrm{ev}"'] \arrow[d, "\mathrm{proj}_1"] \arrow[dr, phantom, "\lrcorner"] & \Gamma(G/H^{\amalg 2}\xrightarrow{f^{\amalg 2}} G/G^{\amalg 2}) \arrow[d, "u"] \\
G/H^{\amalg 2} \arrow[r, "f\amalg f"'] & G/G^{\amalg 2} \arrow[r, "\nabla"'] & G/G
\end{tikzcd}$$

where $\Gamma(G/H^{\amalg 2}\xrightarrow{f^{\amalg 2}} G/G^{\amalg 2})$ is the finite G–set of sections to the map $G/H^{\amalg 2}\xrightarrow{f^{\amalg 2}} G/G^{\amalg 2}$ which also inherits a Σ_2–action from the swap action on $G/H^{\amalg 2}\xrightarrow{f^{\amalg 2}}$

$G/G^{\amalg_2}$. It is easy to see that this finite G–set with Σ_2–action is computed as $\mathrm{Map}(G/G, G/H) \times \mathrm{Map}(G/G, G/H) \cong G/H \times G/H$ with the flip Σ_2–action. Under this identification, we deduce similarly

$$G/G^{\amalg_2} \times \Gamma(G/H^{\amalg_2} \xrightarrow{f^{\amalg_2}} G/G^{\amalg_2}) \cong (G/H \times G/H)^{\amalg_2}$$

as finite G–sets with the stated Σ_2–action. And from this, the descriptions of the three maps out of it are also immediate. $\square$

Lemma 3.6.2 (Σ_2–double coset formula)**.** *Let $H \leq G$ be a subgroup. Then we have a Σ_2–decomposition of finite G–sets $G/H \times G/H \cong \coprod_{g \in H\backslash G/H} G/(H \cap {}^gH)$ where the Σ_2–action on the right hand side is given by conjugation $G/(H \cap {}^gH) \xrightarrow{\cong} G/(H \cap {}^{g^{-1}}H)$. In particular, Σ_2 acts trivially on the component $G/(H \cap {}^eH) = G/H$.*

Proof. The isomorphism of G–sets is standard and is given by the G–bijection

$$G/H \times G/H \longrightarrow \coprod_{g \in H\backslash G/H} G/(H \cap {}^gH) \quad :: \quad (gH, \widetilde{g}H) \mapsto (gH, g^{-1}\widetilde{g}H) \in G/(H \cap {}^{g^{-1}\widetilde{g}}H)$$

where we think of the coordinate $g^{-1}\widetilde{g}H$ as the index in the coproduct decomposition, and the G–action on the right hand side only acts on the gH coordinate. Under this G–bijection, we can then induce the unique Σ_2–action on the right hand side such that the G–bijection is also Σ_2–equivariant. The following diagram of Σ_2–actions

$$\begin{tikzcd}
(gH, \widetilde{g}H) \arrow[r, mapsto] \arrow[d, mapsto, "\Sigma_2"'] & (gH, g^{-1}\widetilde{g}H) \arrow[d, mapsto, "\Sigma_2"] \\
(\widetilde{g}H, gH) \arrow[r, mapsto] & (\widetilde{g}H, \widetilde{g}^{-1}gH)
\end{tikzcd}$$

then shows that Σ_2–action on $\coprod_{g \in H\backslash G/H} G/(H \cap {}^gH)$ is given by conjugation. $\square$

Proposition 3.6.3 (Equivariant Σ_2–Singer construction)**.** *Let G be a odd finite group. Then the Σ_2–Singer construction $\underline{\mathrm{T}}_2 : \underline{\mathrm{Sp}}_G \longrightarrow \underline{\mathrm{Sp}}_G$ given by $X \mapsto (X \otimes X)^{t\Sigma_2}$ is G–linear.*

Proof. By [NS18] we already know that it is linear in the ordinary sense. Hence, all that is left to show is that it preserves equivariant coproducts. For this, let $H \leq G$ be a subgroup and $X \in \mathrm{Sp}_H$. We need to show that $\mathrm{Ind}_H^G \underline{\mathrm{T}}_2 X \simeq \underline{\mathrm{T}}_2 \mathrm{Ind}_H^G X$. Now $\mathrm{Ind}_H^G X \otimes \mathrm{Ind}_H^G X$ is a Σ_2–object in Sp_G, and using the notation from Lemma 3.6.1 we can compute it as a Σ_2–object by:

$$\begin{aligned}
&\mathrm{Ind}_H^G X \otimes \mathrm{Ind}_H^G X\\
&\simeq \nabla_\otimes (f \amalg f)_\oplus (X, X)\\
&\simeq u_\oplus \widetilde{\nabla}_\otimes (\pi_1 \amalg \pi_2)^*(X, X)\\
&\simeq \bigoplus_{g \in H\backslash G/H} \mathrm{Ind}_{H\cap {}^gH}^G \Big[(\mathrm{Res}^H_{H\cap {}^gH} X) \otimes (g^* \mathrm{Res}^H_{H^g \cap H} X)\Big]\\
&\simeq \mathrm{Ind}_H^G [X \otimes X] \oplus \bigoplus_{[e]\neq[g]\in[H\backslash G/H]_{\Sigma_2}} \Bigg[\mathrm{Ind}_{H\cap {}^gH}^G \Big[(\mathrm{Res}^H_{H\cap {}^gH} X) \otimes (g^* \mathrm{Res}^H_{H^g \cap H} X)\Big]\\
&\oplus \mathrm{Ind}_{H\cap g^{-1}H}^G \Big[(\mathrm{Res}^H_{H\cap g^{-1}H} X) \otimes (g^{-1*} \mathrm{Res}^H_{H^{g^{-1}} \cap H} X)\Big]\Bigg]
\end{aligned}$$

Since G was odd, Proposition 3.5.1 gives that the Σ_2–action on the final term is the swap on $\mathrm{Ind}_H^G[X \otimes X]$ and given by the free Σ_2–swap conjugation on the terms

$$\begin{aligned}
&\Bigg[\mathrm{Ind}_{H\cap {}^gH}^G \Big[(\mathrm{Res}^H_{H\cap {}^gH} X) \otimes (g^* \mathrm{Res}^H_{H^g \cap H} X)\Big]\\
&\oplus \mathrm{Ind}_{H\cap g^{-1}H}^G \Big[(\mathrm{Res}^H_{H\cap g^{-1}H} X) \otimes (g^{-1*} \mathrm{Res}^H_{H^{g^{-1}} \cap H} X)\Big]\Bigg]
\end{aligned}$$

The second equivalence is by distributivity of $\otimes$ and Ind_H^G, and the third equivalence of Σ_2–objects is by our Σ_2–double coset decomposition Lemma 3.6.2. Therefore, since $(-)^{t\Sigma_2}$ kills Σ_2–free terms, we see that

$$\underline{\mathrm{T}}_2 \mathrm{Ind}_H^G X \simeq (\mathrm{Ind}_H^G [X \otimes X])^{t\Sigma_2} \simeq \mathrm{Ind}_H^G [X \otimes X]^{t\Sigma_2} \simeq \mathrm{Ind}_H^G \underline{\mathrm{T}}_2 2X$$

as required. □

Construction 3.6.4 (Equivariant Tate diagonal)**.** Let G be a group such that $\underline{\mathrm{T}}_2$ is G–linear, for example an odd group by Proposition 3.6.3. We know that we have an equivalence of G–categories $\underline{\Omega}^\infty : \underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathrm{Sp}}_G, \underline{\mathrm{Sp}}_G) \longrightarrow \underline{\mathrm{Fun}}^{\mathrm{lex}}(\underline{\mathrm{Sp}}_G, \underline{\mathcal{S}}_G)$. Hence since we know that $\underline{\mathrm{T}}_2$ is G–linear by Proposition 3.6.3, we have equivalences

$$\underline{\mathrm{Nat}}(\mathrm{id}, \underline{\mathrm{T}}_2) \simeq \underline{\mathrm{Nat}}(\underline{\mathrm{Map}}(-, \mathbb{S}_G), \underline{\Omega}^\infty \underline{\mathrm{T}}_2) \simeq \underline{\Omega}^\infty \underline{\mathrm{T}}_2(\mathbb{S}_G)$$

Since $\underline{\mathrm{T}}_2$ is ordinary lax symmetric monoidal, there is a unit element $1 \in \pi_0^G \underline{\Omega}^\infty \underline{\mathrm{T}}_2(\mathbb{S}_G)$ corresponding to the canonical map $\mathbb{S}_G \to \mathbb{S}_G^{h\Sigma_2} \to \mathbb{S}_G^{t\Sigma_2}$. This yields a transformation

$$\underline{\Delta}_2 : \mathrm{id} \Longrightarrow \underline{\mathrm{T}}_2$$

of functors $\underline{\mathrm{Sp}}_G \longrightarrow \underline{\mathrm{Sp}}_G$ which we call the *equivariant Tate diagonal*.

We now record all the important points of this subsection into the following theorem.

Theorem 3.6.5. *Let G be an odd group. Then the 2–Singer construction $\underline{\mathrm{T}}_2(-) := ((-)^{\otimes 2})^{t\Sigma_2} : \underline{\mathrm{Sp}}_G \to \underline{\mathrm{Sp}}_G$ is a G–linear functor. Moreover, there is a natural transformation of G–linear functors*

$$\mathrm{id} \xRightarrow{\underline{\Delta}_2} \underline{\mathrm{T}}_2$$

which refines the Nikolaus-Scholze Tate diagonal Δ_2 in the sense that for $X \in \mathrm{Sp}_G$, we have an identification $\mathrm{Res}^G_e \left(\underline{\Delta}_2 \colon X \Rightarrow \underline{\mathrm{T}}_2 X\right) \simeq \left(\Delta_2 \colon \mathrm{Res}^G_e X \Rightarrow \mathrm{T}_2 \mathrm{Res}^G_e X\right)$.

Chapter 4

Genuine equivariant algebraic K–theory

Building upon the notion of $\mathcal{T}$–perfect-stable categories from §2.3, we work towards introducing two natural candidates for *parametrised algebraic K–theory* in this chapter. To this end, as in the unparametrised case, we would need a good understanding of split Verdier sequences and the attendant set-theoretic considerations; these are achieved in §4.1. Taking these as ingredients, we proceed to defining two variants of parametrised algebraic K–theory in §4.2: the first one, which we term *pointwise*, is gotten by applying the functor $\mathrm{Mack}_{\mathcal{T}}(-)$ to the unparametrised K–theory functor $\mathrm{K} : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Sp}$ (more colloquially, $\mathrm{K}(\mathcal{C})^H := \mathrm{K}(\mathcal{C}^H)$); the second one, which we term *normed*, builds into the definition that it also admits the Hill-Hopkins-Ravenel norms and which, moreover, receives a natural comparison map from the first variant. In the final §4.3 of this chapter, we show that this comparison is an equivalence when G is a 2–group. This implies that in this case, the composite functor

$$\underline{\mathrm{Cat}}^{\mathrm{perf}}_G \subseteq \mathrm{Mack}_G(\mathrm{Cat}^{\mathrm{perf}}) \xrightarrow{\mathrm{K}} \mathrm{Mack}_G(\mathrm{Sp}) = \mathrm{Sp}_G$$

canonically refines to a G–lax symmetric monoidal functor (cf. Corollary 4.3.20), hence completing the algebraic K–theory program outlined in [BDG+16a] for such groups G.

4.1 Generation of split Verdier sequences

4.1.1 (Split) Verdier sequences

The notion of (split) Verdier sequences is a direct adaptation of those of [CDH+20b].

Definition 4.1.1. A sequence $\underline{\mathcal{C}} \xrightarrow{i} \underline{\mathcal{D}} \xrightarrow{p} \underline{\mathcal{E}}$ in $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$ with vanishing composite is called a *Verdier sequence* if it is both a fibre and cofibre sequence. It is moreover said to be a *split* Verdier sequence if it can be completed to $\mathcal{T}$-adjunctions

$$\begin{tikzcd} \underline{\mathcal{C}} \arrow[r, "i"] & \underline{\mathcal{D}} \arrow[l, bend right, "q"'] \arrow[l, bend left, "r"] \arrow[r, "p"] & \underline{\mathcal{E}} \arrow[l, bend right, "\ell"'] \arrow[l, bend left, "j"] \end{tikzcd}$$

where an arrow stacked above another denotes being a left adjoint.

Remark 4.1.2. Since $\mathrm{Cat}^{\mathrm{perf}}$ is semiadditive and since Mackey functors are finite product-preserving functors, we see that $\mathrm{Mack}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}}) \subseteq \mathrm{Fun}(\mathrm{Span}(\mathcal{T}), \mathrm{Cat}^{\mathrm{perf}})$ is closed under finite (co)limits, and so these are computed pointwise in $\mathrm{Mack}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$. On the other hand, sections A.1 and A.2 of [CDH+20b] give us very good control of the fibre and cofibre sequences in $\mathrm{Cat}^{\mathrm{perf}}$ in terms of (split) Verdier sequences. Hence, in conjunction with the creation of fibre and cofibre sequences under the inclusion $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \subseteq \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$ from Proposition 2.3.7, we will have a good control of the parametrised (split) Verdier sequences as defined above. We will record the consequence of this that we need in the following corollary.

Corollary 4.1.3. *Suppose we have sequences and $\mathcal{T}$-adjunctions in $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$*

$$\begin{tikzcd} \underline{\mathcal{C}} \arrow[r, bend right, "i"'] \arrow[r, phantom, "\perp"] & \underline{\mathcal{D}} \arrow[l, bend right, "q"'] \arrow[r, bend right, "p"'] \arrow[r, phantom, "\perp"] & \underline{\mathcal{E}} \arrow[l, bend right, "\ell"'] \end{tikzcd}$$

where the top and bottom composites vanish. Then the top sequence is Verdier if and only if the bottom one is. In particular, in a split Verdier sequence, all three layers of sequences are Verdier.

Proof. Since the inclusion $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \subseteq \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$ creates fibres and cofibres by Proposition 2.3.7, and since these are pointwise in $\mathrm{Mack}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$ by the remark above, we can check the Verdierness of these sequences by checking fibrewise. Suppose the top sequence is Verdier. Then by [CDH+20b, A.1.10 (iii) and A.2.1], since q was a Dwyer-Kan localisation, i must be fully faithful. Hence by [CDH+20b, A.2.5] the bottom sequence is Verdier. Applying $(-)^{\underline{\mathrm{op}}}$ everywhere, we obtain the reverse direction. $\square$

Lemma 4.1.4. *Let $f \colon W \to V$ be in $\mathcal{T}$. Then a split Verdier sequence*

$$\begin{tikzcd} \underline{\mathcal{C}} \arrow[r, hook, "i"] & \underline{\mathcal{D}} \arrow[l, bend right, "q"'] \arrow[l, bend left, "r"] \arrow[r, two heads, "p"] & \underline{\mathcal{E}} \arrow[l, bend right, hook, "\ell"'] \arrow[l, bend left, hook, "j"] \end{tikzcd}$$

in $(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}})_W$ *gives rise to one in* $(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}})_V$.

$$\begin{tikzcd} f_*\underline{\mathcal{C}} \arrow[r, hook, "i"] & f_*\underline{\mathcal{D}} \arrow[l, bend right, "q"'] \arrow[l, bend left, "r"] \arrow[r, two heads, "p"] & f_*\underline{\mathcal{E}} \arrow[l, bend right, "\ell"'] \arrow[l, bend left, "j"] \end{tikzcd}$$

Proof. We saw in Proposition 2.3.1 that $\underline{\mathrm{Pr}}_{\mathcal{T},\mathrm{st},L,\omega}$ is $\mathcal{T}$-semiadditive, and so $f_! \simeq f_*$. Hence f_* preserves (co)fibre sequences and we have bifibre sequences in the three directions above. Furthermore, Lemma 1.3.13 says that the desired three layers of sequences are all adjoints of each other, and hence they form a split Verdier sequence by Corollary 4.1.3 as required. □

4.1.2 Set-theoretic considerations

In this subsection we mimic the formulations and techniques of [CDH+, §1.1] to prepare the set-theoretic materials needed for our construction of parametrised motives in the next subsection. The goal is to obtain Corollary 4.1.11, and the reader unconcerned with such matters may wish to take this for granted and skip directly to §4.2.

First of all we will deduce the parametrised analogue of [MP87, Lem 1.7.ii] from the unparametrised version proven in [CDH+]. We will need some terminology for this.

Terminology 4.1.5. Let $\underline{\mathcal{C}}$ be a $\mathcal{T}$-cocomplete category and S be a set of objects in $\underline{\mathcal{C}}$. We say that it is *jointly conservative* if S induces a jointly conservative set of objects in each fibre of $\underline{\mathcal{C}}$, ie. for every $V \in \mathcal{T}$ and writing S_V for the set of objects of $\mathcal{C}_V$ in the set S, the functor $\prod_{x\in S_V} \mathrm{Map}_{\mathcal{C}_V}(x,-) : \mathcal{C}_V \to \prod_{x\in S_V} \mathcal{S}$ is conservative. We say that it is a *set of parametrised generators of* $\underline{\mathcal{C}}$ if the smallest $\mathcal{T}$-cocomplete subcategory of $\underline{\mathcal{C}}$ containing S is $\underline{\mathcal{C}}$ itself. In other words, every parametrised object in $\underline{\mathcal{C}}$ can be written as a parametrised colimit of objects in $\underline{\mathcal{C}}$.

Proposition 4.1.6 (Parametrised Makkai-Pitts)**.** *Let κ be a regular cardinal and $\underline{\mathcal{C}}$ a $\mathcal{T}$-cocomplete category. Let $S \subseteq \underline{\mathcal{C}}$ be a jointly conservative set of parametrised-κ-compact objects. Then S is a set of parametrised-κ-compact generators. In particular, $\underline{\mathcal{C}}$ is parametrised-κ-compactly generated.*

Proof. We want to show that for every $V \in \mathcal{T}$, any $\mathcal{T}_{/V}$–object in $\underline{\mathcal{C}}_V$ is a $\mathcal{T}_{/V}$–colimit of objects in S. By hypothesis, $\prod_{x\in S_V} \mathrm{Map}_{\mathcal{C}_V}(x,-)\colon \mathcal{C}_V \to \prod_{x\in S_V} \mathcal{S}$ is jointly conservative. Hence, by [CDH+, Prop 1.1.2], every object in $\mathcal{C}_V$ is a κ-small colimit of objects in S_V. □

Proposition 4.1.7. *The set* $\{\underline{\mathrm{Sp}}^{\omega}_{\underline{V}}, \mathrm{fun}(\Delta^1, \underline{\mathrm{Sp}}^{\omega}_{\underline{V}})\}_{V\in T}$ *is jointly conservative on* $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{T}$. *Thus,* $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{T}$ *is κ-compactly generated for all regular cardinals κ.*

Proof. Since joint conservativity is checked fibrewise, we show that $\{\underline{\mathrm{Sp}}_{\underline{V}}^{\omega}, \mathrm{fun}(\Delta^1, \underline{\mathrm{Sp}}_{\underline{V}}^{\omega})\}$ is jointly conservative on $\underline{\mathrm{Cat}}_{\underline{V}}^{\mathrm{perf}}$ for an arbitrary $V \in \mathcal{T}$. Note that $\underline{\mathrm{Sp}}_{\underline{V}}^{\omega}$ and $\mathrm{fun}(\Delta^1, \underline{\mathrm{Sp}}_{\underline{V}}^{\omega})$ corepresent the functors $\underline{\mathrm{Cat}}_{\underline{V}}^{\mathrm{perf}} \to \underline{\mathcal{S}}_{\underline{V}}$

$$\underline{\mathcal{C}} \mapsto \underline{\mathcal{C}}^{\simeq} \quad \text{and} \quad \underline{\mathcal{C}} \mapsto \mathrm{fun}(\Delta^1, \underline{\mathcal{C}})^{\simeq} \tag{4.1}$$

respectively. We only show this for the second one since the first is easier:

$$\begin{aligned}
\underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}_{\underline{V}}^{\mathrm{perf}}}(\mathrm{fun}(\Delta^1, \underline{\mathrm{Sp}}_{\underline{V}}^{\omega}), \underline{\mathcal{C}}) &\simeq \underline{\mathrm{Map}}_{\mathrm{Pr}_{V\text{-st},L,\omega}}(\mathrm{fun}(\Delta^1, \underline{\mathrm{Sp}}_{\underline{V}}), \underline{\mathrm{Ind}}_{\omega}\underline{\mathcal{C}}) \\
&\simeq \underline{\mathrm{Fun}}_{\underline{V}}^{L,\omega}(\mathrm{fun}(\Delta^1, \underline{\mathrm{Sp}}_{\underline{V}}), \underline{\mathrm{Ind}}_{\omega}\underline{\mathcal{C}})^{\simeq} \\
&\simeq \underline{\mathrm{Fun}}_{\underline{V}}^{R,\omega\text{-acc}}(\underline{\mathrm{Ind}}_{\omega}\underline{\mathcal{C}}, \mathrm{fun}(\Delta^1, \underline{\mathrm{Sp}}_{\underline{V}}))^{\simeq} \\
&\simeq \mathrm{fun}\left(\Delta^1, \underline{\mathrm{Fun}}_{\underline{V}}^{R,\omega\text{-acc}}(\underline{\mathrm{Ind}}_{\omega}\underline{\mathcal{C}}, \underline{\mathrm{Sp}}_{\underline{V}})\right)^{\simeq} \\
&\simeq \mathrm{fun}\left(\Delta^1, \underline{\mathrm{Fun}}_{\underline{V}}^{L,\omega}(\underline{\mathrm{Sp}}_{\underline{V}}, \underline{\mathrm{Ind}}_{\omega}\underline{\mathcal{C}})\right)^{\simeq} \\
&\simeq \mathrm{fun}(\Delta^1, \underline{\mathcal{C}})^{\simeq}
\end{aligned}$$

where the first equivalence is by Theorem 2.2.16; the third and fifth are by Proposition 1.2.28 and Theorem 2.2.3; the fourth by Notation 1.1.14; and the last is by Proposition 2.3.16. To see that the two functors of Eq. (4.1) are jointly conservative, suppose $\varphi : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a functor such that

$$\varphi : \underline{\mathcal{C}}^{\simeq} \xrightarrow{\simeq} \underline{\mathcal{D}}^{\simeq} \qquad \text{and} \qquad \varphi : \mathrm{fun}(\Delta^1, \underline{\mathcal{C}})^{\simeq} \xrightarrow{\simeq} \mathrm{fun}(\Delta^1, \underline{\mathcal{D}})^{\simeq}$$

are equivalences of $\mathcal{T}_{/V}$–spaces. In particular, the first equivalence implies that φ is $\mathcal{T}_{/V}$–essentially surjective. On the other hand, the fibre over $[W \to V]$ of $\mathrm{fun}(\Delta^1, \underline{\mathcal{C}})$ is $\mathrm{Fun}(\Delta^1, \mathcal{C}_W)$ and so the second equivalence together with the the formula for unparametrised mapping spaces as pullbacks $\mathrm{Fun}(\Delta^1, \mathcal{C}_W) \times_{\mathcal{C}_W^{\times 2}} \{*\}$ gives us that $\varphi : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is $\mathcal{T}_{/V}$–fully faithful. □

Notation 4.1.8. For $\mathcal{C}$ being any of $\mathrm{Cat}^{\mathrm{perf}}$, $\mathrm{Mack}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$, $\underline{\mathrm{Cofree}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$, $\underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{perf}}$, or $\underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$, we write $\mathrm{Split}(\mathcal{C})$ for the full subcategory of $\mathrm{fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ consisting of the split Verdier sequences. This is an parametrised or unparametrised category according as $\mathcal{C}$ is or not. Note that $\mathrm{Split}(\underline{\mathrm{Cofree}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})) \simeq \underline{\mathrm{Cofree}}_{\mathcal{T}}(\mathrm{Split}(\mathrm{Cat}^{\mathrm{perf}}))$ and $\mathrm{Split}(\underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})) \simeq \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Split}(\mathrm{Cat}^{\mathrm{perf}}))$ since splitness is a fibrewise notion, and $\underline{\mathrm{Cofree}}$ and $\underline{\mathrm{CMon}}$ are fibrewise constructions.

Remark 4.1.9. There is an adjunction $L : \mathrm{Fun}(\Delta^1, \mathrm{Cat}^{\mathrm{perf}}) \rightleftarrows \mathrm{Cat}^{\mathrm{perf}} : R$ where $L(\mathcal{C} \xrightarrow{f} \mathcal{D}) \simeq \mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}(\mathcal{D})$ where $\mathrm{Ar}(\mathcal{D}) := \mathrm{Fun}(\Delta^1, \mathcal{D})$ is the arrow category and $R(\mathcal{E}) \simeq (\mathrm{Ar}(\mathcal{E}) \xrightarrow{\text{target}} \mathcal{E})$. Clearly the right adjoint R preserves small colimits, and in particular all filtered colimits. Now since $\mathrm{Fun}(\Delta^1, \mathrm{Cat}^{\mathrm{perf}})$ and $\mathrm{Cat}^{\mathrm{perf}}$ are semi-additive, the left adjoint preserves finite products. Hence we can apply $\underline{\mathrm{CMon}}_{\mathcal{T}}$ to obtain a $\mathcal{T}$-adjunction

$$\underline{L}_{\mathcal{T}} : \underline{\mathrm{fun}}(\Delta^1, \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})) \rightleftarrows \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}}) : \underline{R}_{T}$$

where the $\mathcal{T}$-right adjoint preserves all fibrewise filtered colimits, and hence $\underline{L}_{\mathcal{T}}$ preserves κ-compact objects for all regular cardinals κ. This means that if $(\underline{\mathcal{C}} \xrightarrow{f} \underline{\mathcal{D}})$ is a $\mathcal{T}$-exact functor between κ-compact $\mathcal{T}$-perfect stable categories, then $\underline{\mathcal{C}} \times_{\underline{\mathcal{D}}} \mathrm{Ar}(\underline{\mathcal{D}})$ is κ-compact too. We will need this result very shortly and we refer to [CDH+] for the original treatment of this in the unparametrised setting.

Lemma 4.1.10 (Split Verdier classification)**.** $\mathrm{Split}(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}) \simeq \underline{\mathrm{fun}}(\Delta^1, \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}})$.

Proof. We will bootstrap this statement from the unparametrised statement. We know from [CDH+20b, Prop. A.2.11] that $\mathrm{Fun}(\Delta^1, \mathrm{Cat}^{\mathrm{perf}}) \simeq \mathrm{Split}(\mathrm{Cat}^{\mathrm{perf}})$: this equivalence is implemented by a functor $\mathrm{Fun}(\Delta^1, \mathrm{Cat}^{\mathrm{perf}}) \to \mathrm{Split}(\mathrm{Cat}^{\mathrm{perf}})$ which sends $(\mathcal{C} \xrightarrow{f} \mathcal{D})$ to the split Verdier sequence $(\mathcal{D} \to \mathcal{C} \times_{\mathcal{D}} \mathrm{Ar}(\mathcal{D}) \to \mathcal{C})$. Now consider

$$\begin{tikzcd}
\mathrm{Split}(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}) \arrow[d, hook] & \underline{\mathrm{fun}}(\Delta^1, \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}) \arrow[l, dashed, "\simeq"'] \arrow[d, hook] \\
\mathrm{Split}(\underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})) \arrow[d, "\simeq"'] & \underline{\mathrm{fun}}(\Delta^1, \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})) \arrow[l, dashed, "\simeq"'] \arrow[d, "\simeq"] \\
\underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Split}(\mathrm{Cat}^{\mathrm{perf}})) & \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Fun}(\Delta^1, \mathrm{Cat}^{\mathrm{perf}})) \arrow[l, "\simeq"']
\end{tikzcd}$$

where the bottom vertical equivalences are by Notation 4.1.8 and the top vertical equivalences are by Theorem 2.3.4. Then the bottom equivalence induces the middle dashed equivalence which in turn induces the top dashed equivalence as required. ∎

Corollary 4.1.11. *For any regular cardinal κ there is a small set S_κ of split Verdier sequences on κ-compact $\mathcal{T}$-perfect-stable categories such that any split Verdier sequence in $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$ can be written as a fibrewise κ-filtered colimit of sequences in S_κ.*

Proof. First note that we have

$$\mathrm{Split}(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}})^{\underline{\kappa}} \simeq \underline{\mathrm{fun}}(\Delta^1, \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}})^{\underline{\kappa}} \simeq \underline{\mathrm{fun}}(\Delta^1, (\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}})^{\underline{\kappa}}) \simeq \mathrm{Split}((\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}})^{\underline{\kappa}})$$

where the second equivalence is by [Lur09, Lem. 5.3.4.9] and the third is by Remark 4.1.9 together with Lemma 4.1.10. Now since $\mathrm{Split}(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}) \simeq \mathrm{fun}(\Delta^1, \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}})$ is κ-compactly generated for any regular cardinal κ by Proposition 4.1.7, we see that

$$\mathrm{Split}(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}) \simeq \underline{\mathrm{Ind}}_{\kappa}(\mathrm{Split}((\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}})^{\underline{\kappa}}))$$

with the $\mathcal{T}$-category $\mathrm{Split}((\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}})^{\underline{\kappa}})$ being small. $\square$

4.2 Parametrised noncommutative motives

In their ground-breaking work, Blumberg, Gepner, and Tabuada [BGT13; BGT14] showed that algebraic K–theory refines to a lax symmetric monoidal functor via the formal construction of *noncommutative motives*. The present section will carry out this general strategy by providing two parallel motivic scaffoldings: the first, which we term *pointwise*, will be the one that corepresents algebraic K–theory; the second, which we term *normed*, will admit the sought after multiplicative norms by definition. In the next §4.3, we will show that these two constructions agree in the equivariant setting when G is a finite 2–group, showing that algebraic K–theory refines to a G–lax symmetric monoidal functor in this case. The formulations and proof techniques in this section are just a straightforward mimicking of those in [CDH+].

Notation 4.2.1. Let κ be a regular cardinal. We write $\widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa}_{\mathcal{T}}$ for the smallest $\mathcal{T}$-symmetric monoidal subcategory of $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$ containing $(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}})^{\underline{\kappa}}$. In particular, since $(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}})^{\underline{\kappa}}$ is small by Proposition 4.1.7, $\widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa}_{\mathcal{T}}$ is also small. We need this slight enlargement for the technical reason that we do not know a priori that $(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}})^{\underline{\kappa}}$ inherits the $\mathcal{T}$-symmetric monoidal structure of $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$ since it is not clear that the multiplicative norms preserve parametrised-κ-compact objects. We will see why we need this technical manoeuvre in §4.2.2.

Definition 4.2.2. Let κ be a regular cardinal. Let $\mathcal{R}^{-1}_{\mathrm{pw},\kappa}$ be the collection of diagrams in $\widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa}_{\mathcal{T}} \subseteq \underline{\mathrm{PSh}}_{\mathcal{T}}(\widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa}_{\mathcal{T}})$ consisting of:

- the diagram $\underline{\mathrm{const}}_{\mathcal{T}}(\emptyset)^{\underline{\triangleright}} = \underline{*} \to \widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa}_{\mathcal{T}}$ picking the zero category (ie. the initial object),
- all split Verdier sequences.

Let $\mathcal{R}^{-1}_{\mathrm{norm},\kappa}$ be the closure of $\mathcal{R}^{-1}_{\mathrm{pw},\kappa}$ under under $f_{\otimes}$ for $f : U \to V$ a map of finite T-sets.

Definition 4.2.3. Let κ be a regular cardinal. We define:

- *unstable pointwise $\underline{\kappa}$-motives* $\underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw,un},\underline{\kappa}}$ to be $\mathcal{R}^{-1}_{\mathrm{pw},\kappa}\underline{\mathrm{PSh}}_{\mathcal{T}}(\widetilde{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\mathrm{perf},\underline{\kappa}})$,
- *unstable normed $\underline{\kappa}$-motives* $\underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{un},\underline{\kappa}}$ to be $\mathcal{R}^{-1}_{\mathrm{norm},\kappa}\underline{\mathrm{PSh}}_{\mathcal{T}}(\widetilde{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\mathrm{perf},\underline{\kappa}})$.

Remark 4.2.4. Note that $\mathcal{R}_{\mathrm{pw},\kappa}$ and $\mathcal{R}_{\mathrm{norm},\kappa}$ are small since $\widetilde{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\mathrm{perf},\underline{\kappa}}$ is, and so $\underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw,un},\underline{\kappa}}$ and $\underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{un},\underline{\kappa}}$ are $\mathcal{T}$-presentable.

Notation 4.2.5. Write $j^{\kappa}_{\mathrm{un}}\colon \widetilde{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\mathrm{perf},\underline{\kappa}} \to \underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{un},\underline{\kappa}}$ for the canonical functor. Since split Verdier sequences were already cofibre sequences in $\widetilde{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\mathrm{perf},\underline{\kappa}}$ by definition, we get from Theorem 2.2.12 that this functor is $\mathcal{T}$-fully faithful.

We now collect some basic results about these two types of motives in the next two subsections.

4.2.1 Variant 1: pointwise motives

Definition 4.2.6. Let $\underline{\mathcal{E}}$ be a $\mathcal{T}$–complete category. A $\mathcal{T}$-functor $\underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{perf}} \to \underline{\mathcal{E}}$ is said to be *additive* if it sends split Verdier sequences to fibre sequences and preserves the final objects. We write $\underline{\mathrm{Fun}}_{\mathcal{T}}^{\mathrm{add}}(\underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{perf}},\underline{\mathcal{E}}) \subseteq \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{perf}},\underline{\mathcal{E}})$ for the $\mathcal{T}$-full subcategory of such. We also similarly use the terminology of being additive when $\underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{perf}}$ is replaced with $\widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\underline{\kappa}}$.

Proposition 4.2.7 (Universal property of unstable pointwise $\underline{\kappa}$-motives)**.** *For every $\mathcal{T}$-cocomplete category $\underline{\mathcal{E}}$, $(j^{\kappa}_{\mathrm{un}})^* : \underline{\mathrm{Fun}}^{L}_{\mathcal{T}}(\underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw,un},\underline{\kappa}},\underline{\mathcal{E}}) \to \underline{\mathrm{Fun}}_{\mathcal{T}}^{\mathrm{add}}(\widetilde{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\mathrm{perf},\underline{\kappa}},\underline{\mathcal{E}})$ is an equivalence.*

Proof. This is immediate by construction and Theorem 2.2.10. □

Construction 4.2.8 (The big unstable pointwise motives)**.** Let $\kappa \leq \kappa'$ be two regular cardinals. Then the composition $\widetilde{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\mathrm{perf},\underline{\kappa}} \subseteq \widetilde{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\mathrm{perf},\underline{\kappa}'} \hookrightarrow \underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw,un},\underline{\kappa}'}$ preserves initial objects and sends split Verdier sequences to cofibre sequences. Hence by Proposition 4.2.7 we obtain a strongly $\mathcal{T}$-colimit-preserving functor $\underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{un},\underline{\kappa}} \to \underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw,un},\underline{\kappa}'}$. This is $\mathcal{T}$-fully faithful since it sends compact-generators to compact objects and is $\mathcal{T}$-fully faithful on these. We then define

$$\underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{un}} := \bigcup_{\kappa} \underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw,un},\underline{\kappa}}$$

Since we also have $\underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{perf}} \simeq \bigcup_{\kappa} \widetilde{\underline{\mathrm{Cat}}}_{\mathcal{T}}^{\mathrm{perf},\underline{\kappa}}$, we obtain a $\mathcal{T}$-fully faithful functor

$$j\colon\ : \underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathrm{perf}} \longrightarrow \underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw,un}}$$

Since the poset of regular cardinals is a large category and each of $\underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw},\mathrm{un},\underline{\kappa}}$ is large, we deduce that $\underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw},\mathrm{un}}$ is a large $\mathcal{T}$-presentable category since large unions of large sets is large. We refer to [CDH+, §1.2] for a more thorough discussion of set-theoretic considerations.

Proposition 4.2.9 (Universal property of big unstable motives, "[CDH+, Prop. 1.2.6]")**.** *For a $\mathcal{T}$–(co)complete category $\underline{\mathcal{E}}$, $(j_{\mathrm{un}})^* : \underline{\mathrm{Fun}}^L_{\mathcal{T}}(\underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw},\mathrm{un}}, \underline{\mathcal{E}}) \to \underline{\mathrm{Fun}}^{\mathrm{add}}_{\mathcal{T}}(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}, \underline{\mathcal{E}})$ is an equivalence.*

Proof. By Corollary 4.1.11 we have $\underline{\mathrm{Fun}}^{\mathcal{R}}_{\mathcal{T}}(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}, \underline{\mathcal{E}}) \simeq \lim_\kappa \underline{\mathrm{Fun}}^{\mathcal{R}}_{\mathcal{T}}(\underline{\mathrm{Cat}}^{\mathrm{perf},\underline{\kappa}}_{\mathcal{T}}, \underline{\mathcal{E}})$ obtained from the tautological equivalence $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}, \underline{\mathcal{E}}) \simeq \lim_\kappa \underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathrm{Cat}}^{\mathrm{perf},\underline{\kappa}}_{\mathcal{T}}, \underline{\mathcal{E}})$. But we also have the tautological equivalence $\underline{\mathrm{Fun}}^L_{\mathcal{T}}(\underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw},\mathrm{un}}, \underline{\mathcal{E}}) \simeq \lim_\kappa \underline{\mathrm{Fun}}^L_{\mathcal{T}}(\underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw},\mathrm{un},\underline{\kappa}}, \underline{\mathcal{E}})$. Therefore we can apply Proposition 4.2.7 to conclude. □

Construction 4.2.10 (Big stable motives)**.** Define the $\mathcal{T}$-presentable-stable category of *parametrised noncommutative motives* to be $\underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw}} := \underline{\mathrm{Sp}}_{\mathcal{T}}(\underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw},\mathrm{un}})$. This yields

$$\mathcal{Z}\colon \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \xhookrightarrow{j_{\mathrm{un}}} \underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw},\mathrm{un}} \xrightarrow{\mathrm{can}} \underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw}}$$

Just as importantly, since $\mathcal{T}$–stabilisation is a left adjoint in $\underline{\mathrm{Pr}}_{\mathcal{T},L}$, we also have $\underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw}} \simeq \bigcup_\kappa \underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw},\underline{\kappa}}$ where $\underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw},\underline{\kappa}} := \underline{\mathrm{Sp}}_{\mathcal{T}}(\underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{pw},\mathrm{un},\underline{\kappa}})$. We then obtain commuting composites

$$\begin{tikzcd}
 & & \underline{\mathrm{NMot}}_{\mathcal{T}}^{\mathrm{un},\underline{\kappa}} \arrow[dr, "\mathrm{can}"] & \\
\mathcal{Z}_\kappa\colon & \underline{\widetilde{\mathrm{Cat}}}^{\mathrm{perf},\underline{\kappa}}_{\mathcal{T}} \arrow[ur, hook, "j_{\mathrm{un}}"] \arrow[r, "\mathcal{U}_\kappa"'] & \underline{\mathrm{PSh}}^{\mathrm{st}}_{\mathcal{T}}(\underline{\widetilde{\mathrm{Cat}}}^{\mathrm{perf},\underline{\kappa}}_{\mathcal{T}}) \arrow[r, "\lambda_\kappa"'] & \underline{\mathrm{NMot}}_{\mathcal{T}}^{\underline{\kappa}}
\end{tikzcd}$$

We will use this second description to handle monoidal matters later.

Theorem 4.2.11 (Universal property of stable motives)**.** *For every $\mathcal{T}$-presentable-stable category $\underline{\mathcal{E}}$, the precomposition $\mathcal{Z}^*\colon \underline{\mathrm{Fun}}^L_{\mathcal{T}}(\underline{\mathrm{NMot}}_{\mathcal{T}}, \underline{\mathcal{E}}) \to \underline{\mathrm{Fun}}^{\mathrm{add}}_{\mathcal{T}}(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}, \underline{\mathcal{E}})$ is an equivalence.*

Proof. This is an immediate consequence of Proposition 4.2.9 and Proposition 2.3.12. □

Construction 4.2.12 (Connective algebraic K-theory)**.** Recall that ordinary (idempotent-complete) algebraic K-theory is given by the finite product-preserving functor

$$\mathcal{K} : \mathrm{Cat}^{\mathrm{perf}} \xrightarrow{\mathrm{Q}_\bullet} \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat}^{\mathrm{perf}}) \xrightarrow{(-)^{\simeq}} \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\mathrm{colim}} \mathcal{S}$$

where, using the notation from Notation 1.1.14, we have $\underline{\mathrm{Q}_n\mathcal{C}} \simeq \underline{\mathrm{fun}}_{\mathcal{T}}(\mathrm{TwAr}(\Delta^n), \underline{\mathcal{C}})$ known as Quillen's Q–construction. Since $\mathrm{CMon}(\mathcal{S}) \to \mathcal{S}$ preserves sifted colimits by [Lur17, §3.2.3], it in particular preserves geometric realisations. Hence the geometric realisation used above to define $\mathcal{K}$ acquires a canonical commutative monoid structure because we have the factorisation

$$\begin{tikzcd} \mathrm{Cat}^{\mathrm{perf}} \arrow[r, "(-)^{\simeq}"] \arrow[dr, dashed] & \mathcal{S} \\ & \mathrm{CMon}(\mathcal{S}) \arrow[u] \end{tikzcd}$$

Thus we can apply the $\mathcal{T}$-cofree Construction 1.1.16 and $\mathcal{T}$-semiadditivise to get

$$\underline{\mathcal{K}}_{\mathcal{T}} : \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}}) \longrightarrow \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathcal{S})$$

which we call the *parametrised algebraic K-theory space*. On fibres, this looks like

$$\begin{aligned} \mathrm{Mack}_{\mathcal{T}}(\mathcal{K}) : \ \mathrm{Fun}^{\times}(\mathrm{Span}(\mathcal{T}), \mathrm{Cat}^{\mathrm{perf}}) &\xrightarrow{\mathrm{Q}_\bullet} \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Fun}^{\times}(\mathrm{Span}(\mathcal{T}), \mathrm{Cat}^{\mathrm{perf}})) \\ &\xrightarrow{(-)^{\simeq}} \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Fun}^{\times}(\mathrm{Span}(\mathcal{T}), \mathrm{CMon}(\mathcal{S}))) \\ &\xrightarrow{\mathrm{colim}} \mathrm{Fun}^{\times}(\mathrm{Span}(\mathcal{T}), \mathrm{CMon}(\mathcal{S})) \end{aligned}$$

We will have use of this description soon in analysing motivic suspensions. Note also that $\underline{\mathcal{K}}_{\mathcal{T}}$ is an additive theory since we define split Verdier sequences in $\underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Cat}^{\mathrm{perf}})$ as those that are pointwise split Verdier in the usual sense. Moreover, one can deloop the algebraic K-theory space $\mathcal{K}$ to get an *algebraic K-theory spectrum* $\mathrm{K} : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Sp}$ which is the spectrum associated to the prespectrum whose n-th term is $\mathrm{colim}_{\bullet \in (\mathbf{\Delta}^{\mathrm{op}})^n}(\mathrm{Q}_\bullet\mathcal{C})^{\simeq}$ (cf. [BGT13, §7.2] or Waldhausen's original treatment [Wal85] for more details using the equivalent $\mathrm{S}_\bullet$–construction), and we write $\underline{\mathrm{K}}_{\mathcal{T}}$ for the analogous pointwise K–theory spectrum.

Lemma 4.2.13. *Let* $\underline{\mathcal{C}}, \underline{\mathcal{D}} \in \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$. *Then* $\underline{\mathrm{Fun}}^{\mathrm{ex}}_{\mathcal{T}}(\underline{\mathcal{D}}, \underline{\mathrm{Q}_n\mathcal{C}}) \simeq \underline{\mathrm{Q}_n\mathrm{Fun}}^{\mathrm{ex}}_{\mathcal{T}}(\underline{\mathcal{D}}, \underline{\mathcal{C}})$.

Proof. Since $\underline{\mathrm{Q}_n\mathcal{C}} \simeq \underline{\mathrm{fun}}_{\mathcal{T}}(\Delta^n, \underline{\mathcal{C}})$, we get from Notation 1.1.14 (1) that $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{D}}, \underline{\mathrm{Q}_n\mathcal{C}}) \simeq \underline{\mathrm{Q}_n\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{D}}, \underline{\mathcal{C}})$. But then, both $\underline{\mathrm{Q}_n\mathcal{C}}$ and $\underline{\mathrm{Fun}}_{\mathcal{T}}(\underline{\mathcal{D}}, \underline{\mathcal{C}})$ inherit $\mathcal{T}$-(co)limits from $\underline{\mathcal{C}}$ (the former by Notation 1.1.14 (2)), and so clearly we obtain the statement required. ☐

Lemma 4.2.14 (Motivic suspension, "[BGT13, §7.3], [CDH+, Prop. 1.2.9]"). *Let* $\underline{\mathcal{C}} \in \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$. *Then* $\mathrm{colim}_{\bullet \in \mathbf{\Delta}^{\mathrm{op}}} j_{\mathrm{un}}\underline{\mathrm{Q}_\bullet}(\underline{\mathcal{C}})$ *is already motivically local and moreover,*

$$\operatorname*{colim}_{\bullet \in \mathbf{\Delta}^{\mathrm{op}}} j_{\mathrm{un}}\underline{\mathrm{Q}_\bullet}(\underline{\mathcal{C}}) \simeq \Sigma j_{\mathrm{un}}(\underline{\mathcal{C}}) \in \underline{\mathrm{NMot}}^{\mathrm{un}}_{\mathcal{T}}$$

Proof. To see the first part, let $\underline{\mathcal{D}} \in \underline{\mathrm{Cat}}^{\mathrm{perf}}$. Then note that

$$\begin{aligned}\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}}(j_{\mathrm{un}}\underline{\mathcal{D}}, \operatorname*{colim}_{\bullet\in\Delta^{\mathrm{op}}} j_{\mathrm{un}}\underline{\mathrm{Q}}_{\bullet}(\underline{\mathcal{C}})) &\simeq \operatorname*{colim}_{\bullet\in\Delta^{\mathrm{op}}} \underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}}(j_{\mathrm{un}}\underline{\mathcal{D}}, j_{\mathrm{un}}\underline{\mathrm{Q}}_{\bullet}(\underline{\mathcal{C}}))\\ &\simeq \operatorname*{colim}_{\bullet\in\Delta^{\mathrm{op}}} \underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{D}}, \underline{\mathrm{Q}}_{\bullet}(\underline{\mathcal{C}}))^{\simeq}\\ &\simeq \operatorname*{colim}_{\bullet\in\Delta^{\mathrm{op}}} (\underline{\mathrm{Q}}_{\bullet}\underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{D}}, \underline{\mathcal{C}}))^{\simeq} =: \underline{\mathcal{K}}_{\mathcal{T}}(\underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{D}}, \underline{\mathcal{C}}))\end{aligned}$$

and hence, since $\underline{\mathrm{Fun}}^{\mathrm{ex}}(-, \underline{\mathcal{C}})$ preserves split Verdier sequences and since $\underline{\mathcal{K}}_{\mathcal{T}}$ is additive, we obtain that indeed $\operatorname{colim}_{\bullet\in\Delta^{\mathrm{op}}} j_{\mathrm{un}}\underline{\mathrm{Q}}_{\bullet}(\underline{\mathcal{C}})$ is motivically local as claimed.

For the second part, recall we have the simplicial split Verdier sequence

$$\underline{\mathcal{C}} \to \underline{\mathrm{Déc}_{\bullet}\mathcal{C}} \to \underline{\mathrm{Q}_{\bullet}\mathcal{C}}$$

where we have adopted the terminology décalage from [CDH+20b, Lem. 2.4.7]. The construction $\underline{\mathrm{Déc}_{\bullet}\mathcal{C}}$ is also called the simplicial path object in [BGT13, Proof of Prop. 7.17]. Now since $j_{\mathrm{un}} : \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \to \underline{\mathrm{NMot}}^{\mathrm{un}}_{\mathcal{T}}$ sends split Verdier sequences to cofibre sequences by definition of unstable motives, and cofibre sequences are stable under colimits, we can apply j_{un} to the simplicial split Verdier sequence and take geometric realisation in $\underline{\mathrm{NMot}}^{\mathrm{un}}_{\mathcal{T}}$ to get a cofibre sequence in $\underline{\mathrm{NMot}}^{\mathrm{un}}_{\mathcal{T}}$

$$j_{\mathrm{un}}(\underline{\mathcal{C}}) \to \operatorname*{colim}_{n\in\Delta^{\mathrm{op}}} j_{\mathrm{un}}\underline{\mathrm{Déc}_{\bullet}\mathcal{C}} \to \operatorname*{colim}_{\bullet\in\Delta^{\mathrm{op}}} j_{\mathrm{un}}\underline{\mathrm{Q}_{\bullet}\mathcal{C}}$$

But then we know that the middle term is always augmented over 0 and so is zero, hence the last term is a suspension of the first term, as required. □

Theorem 4.2.15 (Motivic corepresentability of K-theory, "[CDH+, Prop. 2.1.5]")**.** *Let $\underline{\mathcal{C}}, \underline{\mathcal{D}} \in \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}$. Then there is a natural equivalence*

$$\underline{\mathrm{map}}_{\underline{\mathrm{NMot}}_{\mathcal{T}}}(\mathcal{Z}\underline{\mathcal{C}}, \mathcal{Z}\underline{\mathcal{D}}) \simeq \underline{\mathrm{K}}_{\mathcal{T}}(\underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{C}}, \underline{\mathcal{D}}))$$

In particular, $\underline{\mathrm{K}}_{\mathcal{T}}$ is corepresented by $\mathcal{Z}((\underline{\mathrm{Sp}}_{\mathcal{T}})^{\underline{\omega}})$ by Proposition 2.3.16.

Proof. Firstly, note that in $\underline{\mathrm{NMot}}^{\mathrm{un}}_{\mathcal{T}}$, $\Sigma^n j_{\mathrm{un}}\mathcal{D} \simeq \operatorname{colim}_{\bullet\in(\Delta^{\mathrm{op}})^n} j_{\mathrm{un}}\underline{\mathrm{Q}}_{\bullet}\mathcal{D}$ since

$$\begin{aligned}\Sigma^n j_{\mathrm{un}}\mathcal{D} \simeq \Sigma^{n-1}\big(\operatorname*{colim}_{\bullet\in\Delta^{\mathrm{op}}} j_{\mathrm{un}}\underline{\mathrm{Q}}_{\bullet}\mathcal{D}\big) &\simeq \operatorname*{colim}_{\bullet\in\Delta^{\mathrm{op}}} \big(\Sigma^{n-1} j_{\mathrm{un}}\underline{\mathrm{Q}}_{\bullet}\mathcal{D}\big)\\ &\simeq \operatorname*{colim}_{\bullet\in\Delta^{\mathrm{op}}} \big(\Sigma^{n-2}\big(\operatorname*{colim}_{\bullet\in\Delta^{\mathrm{op}}} j_{\mathrm{un}}\underline{\mathrm{Q}}_{\bullet}\mathcal{D}\big)\big)\end{aligned}$$

and so on. Writing $\mathcal{M}$ for the motivic localisation, the left hand parametrised spectrum in the theorem statement is the one associated to the prespectrum whose n-th term, for $n \geq 1$, is

$$\begin{aligned}
\underline{\mathrm{Map}}_{\underline{\mathrm{NMot}}^{\mathrm{un}}_{\mathcal{T}}}(\mathcal{M}j_{\mathrm{un}}\underline{\mathcal{C}}, \Sigma^n j_{\mathrm{un}}\underline{\mathcal{D}}) &\simeq \underline{\mathrm{Map}}_{\underline{\mathrm{NMot}}^{\mathrm{un}}_{\mathcal{T}}}(\mathcal{M}j_{\mathrm{un}}\underline{\mathcal{C}}, \operatorname*{colim}_{\bullet\in(\Delta^{\mathrm{op}})^n} j_{\mathrm{un}}\underline{\mathrm{Q}_{\bullet}}\underline{\mathcal{D}}) \\
&\simeq \underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}}(j_{\mathrm{un}}\underline{\mathcal{C}}, \operatorname*{colim}_{\bullet\in(\Delta^{\mathrm{op}})^n} j_{\mathrm{un}}\underline{\mathrm{Q}_{\bullet}}\underline{\mathcal{D}}) \\
&\simeq \operatorname*{colim}_{\bullet\in(\Delta^{\mathrm{op}})^n} \underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_{\mathcal{T}}}(j_{\mathrm{un}}\underline{\mathcal{C}}, j_{\mathrm{un}}\underline{\mathrm{Q}_{\bullet}}\underline{\mathcal{D}}) \\
&\simeq \operatorname*{colim}_{\bullet\in(\Delta^{\mathrm{op}})^n} \underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}}}(\underline{\mathcal{C}}, \underline{\mathrm{Q}_{\bullet}}\underline{\mathcal{D}}) \\
&\simeq \operatorname*{colim}_{\bullet\in(\Delta^{\mathrm{op}})^n} \left(\underline{\mathrm{Q}_{\bullet}}\underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})\right)^{\simeq} \\
&\simeq \Omega^{\infty}\Sigma^n \underline{\mathrm{K}}_{\mathcal{T}}\left(\underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})\right)
\end{aligned}$$

where the second equivalence is since for $n \geq 1$, $\mathrm{colim}_{\bullet\in(\Delta^{\mathrm{op}})^n} j_{\mathrm{un}}\underline{\mathrm{Q}_{\bullet}}\underline{\mathcal{D}}$ is already in $\underline{\mathrm{NMot}}^{\mathrm{un}}_{\mathcal{T}}$ by Lemma 4.2.14; the fourth since j_{un} is $\mathcal{T}$-fully faithful; the fifth by Lemma 4.2.13; and the last by definition of $\underline{\mathrm{K}}_{\mathcal{T}}$. Hence both parametrised spectra in the statement have equivalent associated spectra, giving the desired conclusion. □

4.2.2 Variant 2: normed motives

Throughout this subsection, let $\mathcal{T}$ be such that for each $f\colon W \to V$, the functors $(\mathcal{T}_{/W})^{\mathrm{op}} \to (\mathcal{T}_{/V})^{\mathrm{op}}$ have finite discrete comma categories. An example for this is $\mathcal{T} = \mathcal{O}_G^{\mathrm{op}}$ for a finite group G, which will be the only case we will be interested in.

Proposition 4.2.16 (Parametrised Dwyer-Kan symmetric monoidality)**.** *Suppose $L : \underline{\mathcal{C}} \to L\underline{\mathcal{C}}$ is a $\mathcal{T}$–symmetric monoidal localisation (cf. Terminology 1.3.10). Then for any $\mathcal{T}$–symmetric monoidal category $\underline{\mathcal{D}}^{\otimes}$, the induced functor L^* : $\underline{\mathrm{Map}}^{\otimes}_{\mathcal{T}}(L\underline{\mathcal{C}}^{\otimes}, \underline{\mathcal{D}}^{\otimes}) \to \underline{\mathrm{Map}}^{\otimes, S^{-1}}_{\mathcal{T}}(\underline{\mathcal{C}}^{\otimes}, \underline{\mathcal{D}}^{\otimes})$ is an equivalence where $\underline{\mathrm{Map}}^{\otimes}_{\mathcal{T}}$ denotes the $\mathcal{T}$–symmetric monoidal functors.*

Proof. We prove this by bootstrapping from the proof of [Lur17, Prop. 4.1.7.4]. Recall from [Lur17, Cons. 4.1.7.1] that we have a category WCat whose objects are pairs $(\mathcal{C}, W)$ where $\mathcal{C}$ is a category and W is a collection of morphisms in $\mathcal{C}$ stable under composition and contains all equivalences in $\mathcal{C}$, and morphisms $f : (\mathcal{C}, W) \to (\mathcal{C}', W')$ are functors $f : \mathcal{C} \to \mathcal{C}'$ such that $f(W) \subseteq W'$. By [Lur17, Prop. 4.1.7.2] we have a Bousfield localisation

$$\mathrm{WCat} \underset{\longleftarrow}{\overset{I}{\longrightarrow}} \mathrm{Cat} \tag{4.2}$$

where both functors preserve finite products and the functor I sends $(\mathcal{C}, W)$ to the Dwyer-Kan localisation $\mathcal{C}[W^{-1}]$. Applying Construction 1.1.16 to this adjunction we get the $\mathcal{T}$–Bousfield localisation $\underline{I}_{\mathcal{T}} : \underline{\mathrm{Cofree}}_{\mathcal{T}}(\mathrm{WCat}) \rightleftarrows \underline{\mathrm{Cofree}}_{\mathcal{T}}(\mathrm{Cat}) : \underline{\mathrm{incl}}_{\mathcal{T}}$.

Moreover, since both functors in Eq. (4.2) preserve finite products, by our hypothesis on $\mathcal{T}$, the functor $\underline{I}_{\mathcal{T}}$ storngly preserves indexed products and so we even have an adjunction

$$\underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{WCat}) \underset{\mathrm{incl}}{\overset{\underline{I}_{\mathcal{T}}}{\rightleftarrows}} \underline{\mathrm{CMon}}_{\mathcal{T}}(\mathrm{Cat}) \tag{4.3}$$

Hence, since $\mathcal{T}$–symmetric monoidal categories are equivalently $\mathcal{T}$–commutative monoidss in $\underline{\mathrm{Cat}}^{\times}_{\mathcal{T}}$, for $(\mathcal{C}^{\underline{\otimes}}, S) \in \underline{\mathrm{CMon}}_{\mathcal{T}}(\underline{\mathrm{WCat}}_{\mathcal{T}})$ and $\mathcal{D}^{\underline{\otimes}} \in \underline{\mathrm{CAlg}}_{\mathcal{T}}(\underline{\mathrm{Cat}}^{\times}_{\mathcal{T}})$, the $\mathcal{T}$–adjunction Eq. (4.3) yield the equivalence $L^* : \underline{\mathrm{Map}}^{\underline{\otimes}}_{\mathcal{T}}(L\mathcal{C}^{\underline{\otimes}}, \mathcal{D}^{\underline{\otimes}}) \xrightarrow{\simeq} \underline{\mathrm{Map}}^{\underline{\otimes}, S^{-1}}_{\mathcal{T}}(\mathcal{C}^{\underline{\otimes}}, \mathcal{D}^{\underline{\otimes}})$ as desired. □

As in Construction 4.2.8, we can construct $\underline{\mathrm{NMot}}^{\mathrm{un}}_{\mathcal{T}}$:

Proposition 4.2.17 ("[CDH+, Prop. 1.2.11]"). *There is a $\mathcal{T}$-symmetric monoidal structure on $\underline{\mathrm{NMot}}^{\mathrm{un}}_{\mathcal{T}}$ such that the functor $j_{\mathrm{un}} : \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \longrightarrow \underline{\mathrm{NMot}}^{\mathrm{un}}_{\mathcal{T}}$ refines canonically to a $\mathcal{T}$-symmetric monoidal functor.*

Proof. We first argue for the case of small motives. From Corollary 2.3.10 the Yoneda embedding $\widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa}_{\mathcal{T}} \hookrightarrow \underline{\mathrm{PSh}}_{\mathcal{T}}(\widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa}_{\mathcal{T}})$ uniquely refines to a $\mathcal{T}$-symmetric monoidal functor, and so we are left to show that the $\mathcal{T}$-Bousfield localisation $\underline{\mathrm{PSh}}_{\mathcal{T}}(\widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa}_{\mathcal{T}}) \to \underline{\mathrm{NMot}}^{\mathrm{un},\underline{\kappa}}_{\mathcal{T}}$ is compatible with the $\mathcal{T}$-symmetric monoidal structure in the sense of [Lur17, Def. 2.2.1.6]. But this is an immediate consequence of our definition of $\mathcal{R}_{\mathrm{norm},\kappa}$ and [Lur17, Prop. 2.2.1.9], using the fact that we have closed up $\mathcal{R}_{\mathrm{norm},\kappa}$ under the norm operations.

Now for the case of the big motives, Proposition 2.3.9 implies that the $\mathcal{T}$-symmetric monoidal inclusion $\widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa}_{\mathcal{T}} \subseteq \widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa'}_{\mathcal{T}}$ induces a $\mathcal{T}$-symmetric monoidal refinement of $\underline{\mathrm{PSh}}_{\mathcal{T}}(\widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa}_{\mathcal{T}}) \to \underline{\mathrm{PSh}}_{\mathcal{T}}(\widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\kappa'}_{\mathcal{T}})$. On the other hand, Proposition 4.2.16 implies that this induces a $\mathcal{T}$-symmetric monoidal refinement of $\underline{\mathrm{NMot}}^{\mathrm{un},\underline{\kappa}}_{\mathcal{T}} \subseteq \underline{\mathrm{NMot}}^{\mathrm{un},\underline{\kappa}'}_{\mathcal{T}}$. Thus since filtered colimits of $\mathcal{T}$-symmetric monoidal categories are formed underlying by the obvious parametrised analogue of [Lur17, §3.2.3], we obtain a canonical $\mathcal{T}$-symmetric monoidal structure on $\underline{\mathrm{NMot}}^{\mathrm{un}}_{\mathcal{T}}$ together with a unique $\mathcal{T}$-symmetric monoidal refinement of $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \hookrightarrow \underline{\mathrm{NMot}}^{\mathrm{un}}_{\mathcal{T}}$. □

Theorem 4.2.18 (Monoidality of motives). *The $\mathcal{T}$-functor $\mathcal{Z} : \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \to \underline{\mathrm{NMot}}_{\mathcal{T}}$ canonically refines to a $\mathcal{T}$-symmetric monoidal functor.*

Proof. We already know that $j_{\mathrm{un}} : \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \to \underline{\mathrm{NMot}}^{\mathrm{un}}_{\mathcal{T}}$ is $\mathcal{T}$-symmetric monoidal by Proposition 4.2.17. Now by Lemma 1.3.11 and Proposition 2.3.12, $\underline{\mathrm{NMot}}^{\mathrm{un}}_{\mathcal{T}} \to \underline{\mathrm{NMot}}_{\mathcal{T}}$ also refines uniquely to a $\mathcal{T}$-symmetric monoidal functor. □

Unlike in the unparametrised situation where algebraic K–theory is a construction and its corepresentability in motives is a result, we now *define* normed parametrised algebraic K–theory to be that which is corepresented by the unit in motives.

Definition 4.2.19. The *normed parametrised algebraic K–theory spectrum* $\underline{\mathrm{K}}_{\mathcal{T}}$ is defined as

$$\underline{\mathrm{K}}_{\mathcal{T}} \colon \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \xrightarrow{\mathcal{Z}} \underline{\mathrm{NMot}}_{\mathcal{T}} \xrightarrow{\underline{\mathrm{map}}(\mathbb{1},-)} \underline{\mathrm{Sp}}_{\mathcal{T}}$$

Observation 4.2.20. From this definition, we can collect two immediate and important facts:

(i) The $\mathcal{T}$–functor $\underline{\mathrm{K}}_{\mathcal{T}} \colon \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \to \underline{\mathrm{Sp}}_{\mathcal{T}}$ canonically refines to a $\mathcal{T}$–lax symmtric monoidal functor because $\underline{\mathrm{map}}(\mathbb{1},-)$ canonically refines to such in general.

(ii) The $\mathcal{T}$–functors $\mathcal{Z}_{\kappa} \colon \widetilde{\underline{\mathrm{Cat}}}^{\mathrm{perf},\underline{\kappa}}_{\mathcal{T}} \to \underline{\mathrm{NMot}}^{\underline{\kappa}}_{\mathcal{T}}$ and $\mathcal{Z} \colon \underline{\mathrm{Cat}}^{\mathrm{perf}}_{\mathcal{T}} \to \underline{\mathrm{NMot}}_{\mathcal{T}}$ are additive by construction, and so by Proposition 4.2.7 and Theorem 4.2.11, there are canonical comparison maps $\Psi_{\kappa} \colon \underline{\mathrm{NMot}}^{\mathrm{pw},\underline{\kappa}}_{\mathcal{T}} \to \underline{\mathrm{NMot}}^{\underline{\kappa}}_{\mathcal{T}}$ and $\Psi \colon \underline{\mathrm{NMot}}^{\mathrm{pw}}_{\mathcal{T}} \to \underline{\mathrm{NMot}}_{\mathcal{T}}$. Furthermore, it is easy to see that the Ψ_{κ}'s assemble to induce Ψ. We do not know in general if the comparison map $\Psi : \underline{\mathrm{NMot}}^{\mathrm{pw}}_{\mathcal{T}} \to \underline{\mathrm{NMot}}_{\mathcal{T}}$ is an equivalence. However, we are able to show that it *is* so in the case of equivariant algebraic K–theory for G a 2-group, and this is the content of the next section.

4.3 Equivariant algebraic K-theory for 2–groups

In this section, we specialise the considerations of §4.2 to the case of $\mathcal{T} = \mathcal{O}^{\mathrm{op}}_G$ where G is a finite group, giving G–equivariant algebraic K–theory. The end goal is to show Theorem 4.3.19, which says that $\underline{\mathrm{K}}_G$ refines to the structure of a normed ring G–spectrum when G is a 2–group. As we will see, understanding the excision property of the algebraic K–theory *spectrum* for certain kinds of pushouts will be crucial and so we will introduce in §4.3.1 the required class of pushout diagrams. Following that, §4.3.2 and §4.3.3 will be concerned with a general analysis of some G-diagrams which will be needed in §4.3.4 to prove the main theorem. Finally, we provide a large class of examples in §4.3.5.

4.3.1 Stable additivity and right-split Verdier pushouts

Recall the notation from Construction 4.2.10

$$\begin{tikzcd} \underline{\mathrm{Cat}}^{\mathrm{perf}}_G \arrow[r, "\mathcal{U}"'] \arrow[rr, bend left, "\mathcal{Z}"] & \underline{\mathrm{PSh}}^{\mathrm{st}}_G(\underline{\mathrm{Cat}}^{\mathrm{perf}}_G) \arrow[r, "\lambda"'] & \underline{\mathrm{NMot}}_G \end{tikzcd}$$

Note that $\mathcal{U}$ is G–symmetric monoidal functor. For the sake of notational concision, we have suppressed the κ's because size issues will not be relevant.

Definition 4.3.1. A square in $\underline{\mathrm{Cat}^{\mathrm{perf}}}$

$$\begin{tikzcd}
\underline{\mathcal{A}} \arrow[r] \arrow[d, hook, shift right] \arrow[dr, phantom, "\ulcorner", very near end] & \underline{\mathcal{B}} \arrow[d, hook, shift right] \\
\underline{\mathcal{C}} \arrow[r] \arrow[u, dashed, shift right] & \underline{\mathcal{P}} \arrow[u, dashed, shift right]
\end{tikzcd}$$

is said to be a *right-split Verdier pushout* if it is a pushout diagram and the vertical arrows are right-split Verdier inclusions.

The following lemma gives the source of right-split Verdier pushouts that concern us.

Lemma 4.3.2. *Suppose we have a diagram in* $\underline{\mathrm{Cat}^{\mathrm{perf}}}$

$$\begin{tikzcd}
\underline{\mathcal{A}} \arrow[r] \arrow[d, hook, shift right] & \underline{\mathcal{B}} \\
\underline{\mathcal{C}} \arrow[u, dashed, shift right] &
\end{tikzcd}$$

such that $\underline{\mathcal{A}} \to \underline{\mathcal{B}}$ *is a biadjoint (ie. it admits adjoints on both sides and that these are equivalent). Then the pushout in* $\underline{\mathrm{Cat}^{\mathrm{perf}}}$ *is a right-split Verdier pushout.*

Proof. We work in the presentable setting by virtue of the equivalence $\underline{\mathrm{Cat}^{\mathrm{perf}}} \simeq \underline{\mathrm{Pr}}_{L,\mathrm{st},\omega}$. We compute the pushout in $\underline{\mathrm{Pr}}_{L,\mathrm{st}}$ and then check that it is already the pushout in $\underline{\mathrm{Pr}}_{L,\mathrm{st},\omega}$. Now colimits in $\underline{\mathrm{Pr}}_{L,\mathrm{st}}$ are computed as limits in $\underline{\mathrm{Pr}}_{R,\mathrm{st}}$. And so we get the solid pushout and dashed pullback square

$$\begin{tikzcd}
\underline{\mathrm{Ind}}(\underline{\mathcal{A}}) \arrow[r] \arrow[d, hook, shift right] \arrow[dr, phantom, "\ulcorner", very near end] & \underline{\mathrm{Ind}}(\underline{\mathcal{B}}) \arrow[d, shift right] \arrow[l, dashed, bend left] \\
\underline{\mathrm{Ind}}(\underline{\mathcal{C}}) \arrow[r] \arrow[u, dashed, shift right] & \underline{\mathrm{Ind}}(\underline{\mathcal{P}}) \arrow[u, dashed, shift right] \arrow[l, dashed, bend left]
\end{tikzcd}$$

Now since limits in both $\underline{\mathrm{Pr}}_{L,\mathrm{st}}$ and $\underline{\mathrm{Pr}}_{R,\mathrm{st}}$ are computed underlying, and since the top and left dashed maps are themselves left adjoints by our hypothesis, we see that the bottom and right dashed maps are also left adjoints. In particular, the solid maps $\underline{\mathrm{Ind}}(\underline{\mathcal{C}}) \to \underline{\mathrm{Ind}}(\underline{\mathcal{P}})$ and $\underline{\mathrm{Ind}}(\underline{\mathcal{B}}) \to \underline{\mathrm{Ind}}(\underline{\mathcal{P}})$ both preserve compact objects. Therefore, $\underline{\mathrm{Ind}}(\underline{\mathcal{P}})$ which is a priori a pushout in $\underline{\mathrm{Pr}}_{L,\mathrm{st}}$ is also a pushout in $\underline{\mathrm{Pr}}_{L,\mathrm{st},\omega}$. Moreover, since sections pull back to sections and since $\underline{\mathrm{Ind}}(\underline{\mathcal{A}}) \to \underline{\mathrm{Ind}}(\underline{\mathcal{C}})$ is a section of $\underline{\mathrm{Ind}}(\underline{\mathcal{C}}) \to \underline{\mathrm{Ind}}(\underline{\mathcal{A}})$, we see that $\underline{\mathrm{Ind}}(\underline{\mathcal{B}}) \to \underline{\mathrm{Ind}}(\underline{\mathcal{P}})$ is a section of $\underline{\mathrm{Ind}}(\underline{\mathcal{P}}) \to \underline{\mathrm{Ind}}(\underline{\mathcal{B}})$. This pair being adjoint to each other then automatically implies that $\underline{\mathrm{Ind}}(\underline{\mathcal{B}}) \to \underline{\mathrm{Ind}}(\underline{\mathcal{P}})$ is fully faithful, and so a right-split Verdier inclusion. $\square$

The following result is where our stability hypothesis comes in.

Lemma 4.3.3. *If we have a right-split Verdier pushout as in Definition 4.3.1, then*

$$\begin{tikzcd}
\mathcal{Z}(\underline{\mathcal{A}}) \arrow[r] \arrow[d] & \mathcal{Z}(\underline{\mathcal{B}}) \arrow[d] \\
\mathcal{Z}(\underline{\mathcal{C}}) \arrow[r] & \mathcal{Z}(\underline{\mathcal{P}})
\end{tikzcd}$$

is a pushout in the stable noncommutative motives $\underline{\mathrm{NMot}}$.

Proof. First we extend the diagram with $\underline{\mathcal{E}} := \mathrm{cofib}(\underline{\mathcal{B}} \hookrightarrow \underline{\mathcal{P}})$ to obtain

$$\begin{tikzcd}
\underline{\mathcal{A}} \arrow[r] \arrow[d, hook] & \underline{\mathcal{B}} \arrow[r] \arrow[d, hook] & 0 \arrow[d] \\
\underline{\mathcal{C}} \arrow[r] \arrow[u, dashed, bend right] \arrow[ur, phantom, "\ulcorner"] & \underline{\mathcal{P}} \arrow[r] \arrow[u, dashed, bend right] \arrow[ur, phantom, "\ulcorner"] & \underline{\mathcal{E}}
\end{tikzcd}$$

Since taking cofibres of right-split Verdier inclusions give right-split Verdier sequences by Corollary 4.1.3, we get right-split Verdier sequences

$$\begin{tikzcd}
\underline{\mathcal{A}} \arrow[r, hook] & \underline{\mathcal{C}} \arrow[r, two heads] \arrow[l, dashed, bend left] & \underline{\mathcal{E}} \arrow[l, dashed, bend left]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\underline{\mathcal{B}} \arrow[r, hook] & \underline{\mathcal{P}} \arrow[r, two heads] \arrow[l, dashed, bend left] & \underline{\mathcal{E}} \arrow[l, dashed, bend left]
\end{tikzcd}$$

Hence, by [CDH+20b, Rmk. 2.7.6 (ii)], the maps $\mathcal{U}(\underline{\mathcal{C}})/\mathcal{U}(\underline{\mathcal{A}}) \to \mathcal{U}(\underline{\mathcal{E}})$ and $\mathcal{U}(\underline{\mathcal{P}})/\mathcal{U}(\underline{\mathcal{B}}) \to \mathcal{U}(\underline{\mathcal{E}})$ are λ-equivalences. Now consider the horizontal maps of vertical cofibre sequences

$$\begin{tikzcd}
\mathcal{U}(\underline{\mathcal{A}}) \arrow[r] \arrow[d, hook] & \mathcal{U}(\underline{\mathcal{B}}) \arrow[r] \arrow[d, hook] & 0 \arrow[d] \\
\mathcal{U}(\underline{\mathcal{C}}) \arrow[r] \arrow[d] & \mathcal{U}(\underline{\mathcal{P}}) \arrow[r] \arrow[d] & \mathcal{U}(\underline{\mathcal{E}}) \arrow[d, equal] \\
\mathcal{U}(\underline{\mathcal{C}})/\mathcal{U}(\underline{\mathcal{A}}) \arrow[r, dashed] \arrow[rr, bend right, "(\lambda\simeq)"'] & \mathcal{U}(\underline{\mathcal{P}})/\mathcal{U}(\underline{\mathcal{B}}) \arrow[r, "(\lambda\simeq)"] & \mathcal{U}(\underline{\mathcal{E}})
\end{tikzcd}$$

The arrows marked with $(\lambda \simeq)$ are λ-equivalences, and so the dashed map is too. On the other hand, we have this map of cofibre sequences

$$\begin{tikzcd}
\mathcal{U}(\underline{\mathcal{B}}) \arrow[r] \arrow[d, equal, "(\lambda\simeq)"] & \mathcal{U}(\underline{\mathcal{B}}) \coprod_{\mathcal{U}(\underline{\mathcal{A}})} \mathcal{U}(\underline{\mathcal{C}}) \arrow[r] \arrow[d] & \mathcal{U}(\underline{\mathcal{C}})/\mathcal{U}(\underline{\mathcal{A}}) \arrow[d, "(\lambda\simeq)"] \\
\mathcal{U}(\underline{\mathcal{B}}) \arrow[r] & \mathcal{U}(\underline{\mathcal{P}}) \arrow[r] & \mathcal{U}(\underline{\mathcal{P}})/\mathcal{U}(\underline{\mathcal{B}})
\end{tikzcd}$$

where the maps marked with $(\lambda \simeq)$ are λ-equivalences, and hence since we are in stable presheaves, the middle one is too, as was to be shown. □

4.3.2 C_2-pullbacks and –pushouts

The results in this subsection hold generally for G/H–pullbacks and –pushouts for $H \lhd G$ with $|G/H| = 2$. We phrase everything in terms of C_2 purely for notational convenience. We are grateful to Greg Arone for his indispensable suggestion to transform C_2–pushouts into ordinary pushouts of C_2–objects.

Lemma 4.3.4. *Suppose we are given a C_2-pullback diagram $(X \to Y \leftarrow X)$. Then the C_2-pullback $X \underline{\times}_Y X$ can equivalently be computed as the following ordinary pullback of C_2-objects*

$$
\begin{tikzcd}
X \underline{\times}_Y X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\eta"] \\
\mathrm{Coind}_e^{C_2} X \arrow[r] & \mathrm{Coind}_e^{C_2} \mathrm{Res}_e^{C_2} Y
\end{tikzcd}
$$

Proof. The case of C_2–spaces will imply immediately the general case of a C_2–category, and so we just show the statement in this special case. Let $\underline{J}$ be the diagram indexing the C_2-pullbacks. On underlying spaces (ie. the fibre over C_2/e) it is the usual pullback. On the fibre over C_2/C_2, we compute the right Kan extension $p_* \colon \mathrm{Fun}_{C_2}(\underline{J}, \underline{\mathcal{S}}_{C_2}) \simeq \mathrm{Fun}(\mathrm{Total}(\underline{J}), \mathcal{S}) \longrightarrow \mathrm{Fun}(\mathcal{O}_{C_2}^{\mathrm{op}}, \mathcal{S})$ along the structure projection $p \colon \mathrm{Total}(\underline{J}) \to \mathcal{O}_{C_2}^{\mathrm{op}}$ given by

$$
\begin{tikzcd}
 & & C_2/C_2 & \{g\} \arrow[l] \\
C_2/C_2 \arrow[d, mapsto] & \longrightarrow \arrow[d, mapsto, "p"'] & C_2/C_2 \arrow[u, leftrightarrow, "\simeq"'] \arrow[d, mapsto] & \{e\} \arrow[l] \arrow[u, leftrightarrow, "\simeq"'] \arrow[dl, mapsto] \\
C_2/C_2 & \longrightarrow & C_2/e \arrow[loop right, "\simeq\, C_2"] &
\end{tikzcd}
$$

We want to compute the comma category $(C_2/C_2 \downarrow p)$ in order to compute the value of the right Kan extension p_* at C_2/C_2. By inspection this is

$$
\begin{tikzcd}
 & (C_2/C_2 \downarrow \{g\}) \arrow[r, leftrightarrow, "\simeq"] \arrow[d] & (C_2/C_2 \downarrow \{e\}) \arrow[d] \\
(C_2/C_2 \downarrow C_2/C_2) \arrow[r] & (C_2/C_2 \downarrow C_2/C_2) \arrow[r, leftrightarrow, "\simeq"] & (C_2/C_2 \downarrow C_2/C_2)
\end{tikzcd}
$$

which is a pullback diagram after identification under the two equivalences. Hence for a C_2-pullback diagram $(X \to Y \leftarrow X)$, we have that the C_2-fixed point of spaces is computed as the ordinary pullback

$$(X \underline{\times}_Y X)^{C_2} \simeq X \times_{Y^e} Y^{C_2}$$

Hence the ordinary pullback in the statement of the lemma is indeed the following pullback of genuine C_2–spaces (we have used the compact notation $(A \downarrow B)$ here to mean the C_2–space $\mathcal{X}$ such that $\mathcal{X}^{C_2} \simeq A$ and $\mathcal{X}^e \simeq B$)

$$\begin{tikzcd}
((X \underline{\times}_Y X)^{C_2} \downarrow X \times_{Y^e} X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (Y^{C_2} \downarrow Y^e) \arrow[d] \\
(X \downarrow X \times X) \arrow[r] & (Y^e \downarrow Y^e \times Y^e)
\end{tikzcd}$$

Here we have also used the formula $\mathrm{Coind}_e^{C_2} X \simeq (X \downarrow X \times X)$ which can be obtained by a direct computation of the right Kan extension formula for the C_2-indexed product. □

Corollary 4.3.5 (C_2-pushout formula)**.** *Let $(B \leftarrow A \to B)$ be a C_2-pushout diagram in a C_2-category $\underline{\mathcal{C}}$. Then the C_2-pushout can be computed as the following ordinary pushout of C_2-objects*

$$\begin{tikzcd}
\mathrm{Ind}_e^{C_2} \mathrm{Res}_e^{C_2} A \arrow[r, "\varepsilon"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & A \arrow[d] \\
\mathrm{Ind}_e^{C_2} B \arrow[r] & B \underline{\amalg}_A B
\end{tikzcd}$$

Proof. We combine the lemma above together with Corollary 1.2.22 and compute. Let $Z \in \mathcal{C}_{C_2}$ be an arbitrary C_2-object. Then

$$\begin{aligned}
\underline{\mathrm{Map}}_{C_2}(B \underline{\amalg}_A B, Z) &\simeq \mathrm{Map}(B, \mathrm{Res}_e^{C_2} Z) \underline{\times}_{\underline{\mathrm{Map}}_{C_2}(A,Z)} \mathrm{Map}(B, \mathrm{Res}_e^{C_2} Z) \\
&\simeq \mathrm{Coind}_e^{C_2} \mathrm{Map}(B, \mathrm{Res}_e^{C_2} Z) \times_{\mathrm{Coind}_e^{C_2} \mathrm{Res}_e^{C_2} \underline{\mathrm{Map}}_{C_2}(A,Z)} \underline{\mathrm{Map}}_{C_2}(A, Z) \\
&\simeq \underline{\mathrm{Map}}_{C_2}(\mathrm{Ind}_e^{C_2} B \textstyle\coprod_{\mathrm{Ind}_e^{C_2} \mathrm{Res}_e^{C_2} A} A, Z)
\end{aligned}$$

as required. □

4.3.3 Norms of cofibre sequences

The aim of this subsection is to provide a decomposition result that will allow us to analyse C_2–norms of cofibre sequences. We first record the following immediate consequence of Theorem 1.2.42.

Lemma 4.3.6. *Let $\underline{L}$ and $\underline{\mathcal{C}}$ be $\mathcal{T}$-categories. Suppose $\partial : (\underline{L} \times 0)^{\trianglerighteq} \cup_{\underline{L}\times 0} (\underline{L} \times \Delta^1) \to \underline{\mathcal{C}}$ is a $\mathcal{T}$-diagram. Then we have a pushout*

$$\begin{tikzcd}
\underline{\operatorname{colim}}_{\underline{L}\times 0}\partial \arrow[r] \arrow[d] & \underline{\operatorname{colim}}_{(\underline{L}\times 0)^{\trianglerighteq}}\partial \arrow[d] \\
\underline{\operatorname{colim}}_{\underline{L}\times 1}\partial \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \underline{\operatorname{colim}}_{(\underline{L}\times 0)^{\trianglerighteq}\cup_{\underline{L}\times 0}(\underline{L}\times\Delta^1)}\partial
\end{tikzcd}$$

Notation 4.3.7. Let $H \triangleleft G$ with $|G/H| = 2$, $\underline{\mathcal{C}} \in \mathrm{Cat}_G$, and $\partial : \prod_{G/H} \underline{\mathrm{const}}_H(\Lambda^2_0) \to \underline{\mathcal{C}}$ a G–diagram. For $K \leq G$ a subgroup, since H was normal, we have that

$$G/H \times G/K = \coprod_{g \in K\backslash G/H} G/K \cap {}^gH = \coprod_{g \in K\backslash G/H} G/K \cap H$$

Since $|G/H| = 2$, we only have the following two possibilities

$$G/H \times G/K = \begin{cases} (G/K)^{\amalg 2} & \text{if } K \leq H \\ G/K \cap H & \text{if } K \not\leq H \end{cases}$$

Hence, the data of the G-diagram ∂ is determined by the data of a commuting diagram

$$\begin{tikzcd}
\Lambda^2_0 \times \Lambda^2_0 \arrow[loop left, "C_2"] \arrow[r, "\partial_H"] & \mathcal{C}_H \\
\Lambda^2_0 \arrow[u, "\Delta"] \arrow[r, "\partial_G"] & \mathcal{C}_G \arrow[u, "\mathrm{Res}^G_H"']
\end{tikzcd}$$

because the data of the diagram at other subgroups are restrictions of those from either $\mathcal{C}_H$ or $\mathcal{C}_G$. Hence we will represent the data of ∂ in the following schematic diagram

$$\begin{tikzcd}
 & & & & & & & W^G \arrow[ddd] \\
G/G & & & & Z^G \arrow[uurrr] \arrow[dlll] \arrow[ddd] & & & \\
 & A^G \arrow[ddd] & & & & & & \\
 & & & C \arrow[ddrr, dashed, leftrightarrow] & & X \arrow[ll] \arrow[rr] \arrow[dr, dashed, leftrightarrow, "C_2"] & & W^H \\
G/H & & B \arrow[ur] \arrow[dl] \arrow[dr, dashed, leftrightarrow, "C_2"'] & & Z^H \arrow[ll] \arrow[ur] \arrow[rr] \arrow[dl] & & X \arrow[ur] \arrow[dl] & \\
 & A^H & & B \arrow[ll] \arrow[rr, "C_2"] & & C & &
\end{tikzcd}$$

Lemma 4.3.8. *Let $H \lhd G$ be a normal subgroup of index 2, and suppose we have a G-diagram $\partial : \prod_{G/H} \underline{\mathrm{const}}_H(\Lambda^2_0) \longrightarrow \underline{\mathcal{C}}$ as above, where $\underline{\mathcal{C}}$ is G-finite-cocomplete and G-pointed, and where $A \simeq B \simeq C \simeq 0$. Then its G-colimit can be computed as the following fibrewise pushout*

$$\begin{tikzcd}
X \underline{\amalg}_Z X \arrow[r] \arrow[d] & W \arrow[d] \\
0 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \underline{\mathrm{colim}}\, \partial
\end{tikzcd}$$

Proof. Write $\underline{J}$ for the indexing G-category where we glue the B and A^H terms, and write $p\colon \prod_{G/H} \underline{\mathrm{const}}_H(\Lambda^2_0) \to \underline{J}$ for the G-functor which does this and $q\colon \underline{J} \to \underline{*}_G$ for the unique G-functor. Since $\underline{\mathrm{colim}}\, \partial \simeq q_! p_! \partial$, we can also compute $\underline{\mathrm{colim}}\, \partial$ by first G-left Kan extending $\partial : \prod_{G/H} \underline{\mathrm{const}}_H(\Lambda^2_0) \to \underline{\mathcal{C}}$ to a G-functor $p_!\partial : \underline{J} \to \underline{\mathcal{C}}$, and this is our first goal.

Since the functor p only changes the fibres G/K where $K \leq H$, we only need to compute the terms in question marks in the following diagram

$$\begin{tikzcd}
 & & & & & & W^G \arrow[ddd] \\
G/G & & & Z^G \arrow[urrr] \arrow[dll] \arrow[ddd] & & & \\
 & 0 \arrow[ddd] & & & & & \\
 & & ? & & X \arrow[ll] \arrow[rr] \arrow[dr, dashed, bend left, "C_2"] & & W^H \\
G/H & & & Z^H \arrow[ur] \arrow[rr] \arrow[dll] & & X \arrow[ur] \arrow[dl] & \\
 & 0 \arrow[uur] \arrow[rrr] & & & ? & &
\arrow[from=4-3, to=6-5, dashed, bend right, "C_2"']
\end{tikzcd}$$

and since this is a fibrewise H-diagram $\underline{\mathrm{const}}_H((\Lambda^2_0)^{\times 2})$, we can compute the missing terms as an ordinary left Kan extension. By a straightforward inspection, the approriate comma category diagram is

$$\begin{tikzcd}
 & & 0 & & X \arrow[ll] \\
 & 0 \arrow[ur] & & Z^H \arrow[ll] \arrow[ur] \arrow[dl] & \\
0 & & 0 \arrow[ll] & & \\
\arrow[from=2-2, to=3-1]
\end{tikzcd}$$

and hence the terms in questions marks are 0. Therefore, the G-left Kan extended diagram $p_!\partial\colon \underline{J} \to \underline{\mathcal{C}}$ is now given by

$$\begin{tikzcd}
 & & & & & & W^G \arrow[ddd] \\
G/G & & & Z^G \arrow[urrr] \arrow[dll] \arrow[ddd] & & & \\
 & 0 \arrow[ddd] & & & & & \\
 & & 0 & & X \arrow[ll] \arrow[rr] \arrow[dr, dashed, leftrightarrow, "C_2"] & & W^H \\
G/H & & & Z^H \arrow[ur] \arrow[rr] \arrow[dll] & & X \arrow[ur] \arrow[dl] & \\
 & 0 \arrow[uur] \arrow[rrr] & & & 0 \arrow[uull, dashed, leftrightarrow, bend left, "C_2"] & &
\end{tikzcd}$$

Now, if we write $\underline{L} = \prod_{G/H} \underline{\mathrm{const}}_H(\Delta^1)$ for the indexing category of $G/H = C_2$-pushouts, we have the diagram decomposition

$$\underline{J} \simeq (\underline{L} \times 0)^{\underline{\triangleright}} \cup_{\underline{L}\times 0} (\underline{L} \times \Delta^1)$$

so that $p_!\partial$ has $(\underline{L} \times 0)^{\underline{\triangleright}}$-part given by

$$\begin{tikzcd}
 & & & & W^G \arrow[dd] \\
G/G & & Z^G \arrow[urr] \arrow[dd] & & \\
 & & & X \arrow[r] \arrow[d, dashed, leftrightarrow, "C_2"] & W^H \\
G/H & & Z^H \arrow[ur] \arrow[r] & X \arrow[ur] &
\end{tikzcd}$$

and $\underline{L} \times 1$-part given by

$$\begin{tikzcd}
G/G & 0 \arrow[dd] & & \\
 & & 0 \arrow[dr, dashed, leftrightarrow, "C_2"] & \\
G/H & 0 \arrow[ur] \arrow[rr] & & 0
\end{tikzcd}$$

Thus by Lemma 4.3.6, we get the following fibrewise pushout

$$\begin{tikzcd}
X \underline{\amalg}_Z X \simeq \underline{\mathrm{colim}}_{\underline{L}\times 0} p_!\partial \arrow[r] \arrow[d] & \underline{\mathrm{colim}}_{(\underline{L}\times 0)^{\underline{\triangleright}}} p_!\partial \simeq W \arrow[d] \\
0 \simeq \underline{\mathrm{colim}}_{\underline{L}\times 1} p_!\partial \arrow[r] \arrow[ur, phantom, "\ulcorner", near end] & \underline{\mathrm{colim}}_{(\underline{L}\times 0)^{\underline{\triangleright}} \cup_{\underline{L}\times 0} (\underline{L}\times\Delta^1)} p_!\partial \simeq q_! p_! \partial \simeq \underline{\mathrm{colim}}\, \partial
\end{tikzcd}$$

as desired. □

Remark 4.3.9. The decomposition result above was inspired by [HHR16, Prop. A.43], in the case when $|G/H| = 2$. We do not yet know how to lift their point-set proof for general subgroup inclusions.

4.3.4 Pointwise K–theory is normed for 2-groups

We aim to prove that N_H^G preserves λ-equivalences, ie. if we have a split Verdier

$$\begin{tikzcd} \underline{\mathcal{C}} \arrow[r, hook] & \underline{\mathcal{D}} \arrow[l, dashed, bend right] \arrow[l, dashed, bend left] \arrow[r, two heads] & \underline{\mathcal{E}} \arrow[l, dashed, bend right, hook] \arrow[l, dashed, bend left, hook] \end{tikzcd}$$

then $\mathrm{N}_H^G(\mathcal{U}(\underline{\mathcal{D}})/\mathcal{U}(\underline{\mathcal{C}})) \to \mathcal{U}(\mathrm{N}_H^G\underline{\mathcal{E}})$ induced by the λ-equivalence $\mathcal{U}(\underline{\mathcal{D}})/\mathcal{U}(\underline{\mathcal{C}}) \to \mathcal{U}(\underline{\mathcal{E}})$ is itself a λ-equivalence. If we can show this, then we would have shown that the inclusion $\mathcal{R}_{\mathrm{pw},\kappa} \subseteq \mathcal{R}_{\mathrm{norm},\kappa}$ (cf. §4.2) is an identification, and so the comparison map $\Psi : \underline{\mathrm{NMot}}_G^{\mathrm{pw}} \to \underline{\mathrm{NMot}}_G$ from Observation 4.2.20 is an equivalence. Since size issues will not play a role in our discussions here, we will suppress any mention of κ.

Corollary 4.3.10. *Let $H \triangleleft G$ with $|G/H| = 2$. Suppose we have a pushout*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ X \arrow[r] & Y \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

in a G-symmetric monoidal G-stable category $\underline{\mathcal{C}}$. Then we have the pushout

$$\begin{tikzcd} A \otimes B \underline{\amalg}_{\mathrm{N}_H^G A} A \otimes B \arrow[r] \arrow[d] & \mathrm{N}_H^G B \arrow[d] \\ X \otimes Y \underline{\amalg}_{\mathrm{N}_H^G X} X \otimes Y \arrow[r] & \mathrm{N}_H^G Y \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

Proof. Writing C for $\mathrm{cofib}(A \to B) \simeq \mathrm{cofib}(X \to Y)$, we get from the G/H-distributivity of N_H^G together with Lemma 4.3.8 that we have the map of cofibre sequences

$$\begin{tikzcd} A \otimes B \underline{\amalg}_{\mathrm{N}_H^G A} A \otimes B \arrow[r] \arrow[d] & \mathrm{N}_H^G B \arrow[r] \arrow[d] & \mathrm{N}_H^G C \arrow[d, equal] \\ X \otimes Y \underline{\amalg}_{\mathrm{N}_H^G X} X \otimes Y \arrow[r] & \mathrm{N}_H^G Y \arrow[r] & \mathrm{N}_H^G C \end{tikzcd}$$

and so since $\underline{\mathcal{C}}$ was stable, the left square is a fibrewise pushout. $\square$

Lemma 4.3.11. *Suppose $H \triangleleft G$ with $|G/H| = 2$, and $\underline{\mathcal{A}} \xrightarrow{i} \underline{\mathcal{B}}$ is a split Verdier inclusion in $\underline{\mathrm{Cat}}_H^{\mathrm{perf}}$. Then the canonical map*

$$\mathcal{Z}(\underline{\mathcal{A}} \otimes \underline{\mathcal{B}}) \underline{\amalg}_{\mathcal{Z}(\mathrm{N}_H^G \underline{\mathcal{A}})} \mathcal{Z}(\underline{\mathcal{A}} \otimes \underline{\mathcal{B}}) \longrightarrow \mathcal{Z}(\underline{\mathcal{A}} \otimes \underline{\mathcal{B}} \underline{\amalg}_{\mathrm{N}_H^G \underline{\mathcal{A}}} \underline{\mathcal{A}} \otimes \underline{\mathcal{B}})$$

is an equivalence in $\underline{\mathrm{NMot}}_G$.

Proof. By Corollary 4.3.5 we have the pushout

$$\begin{tikzcd}
\mathrm{Ind}^G_H \mathrm{Res}^G_H \mathrm{N}^G_H \underline{\mathcal{A}} \arrow[r, "\varepsilon"] \arrow[d, shift right] & \mathrm{N}^G_H \underline{\mathcal{A}} \arrow[d, shift right] \\
\mathrm{Ind}^G_H(\underline{\mathcal{A}} \otimes \underline{\mathcal{B}}) \arrow[r] \arrow[u, dashed, shift right] & \underline{\mathcal{A}} \otimes \underline{\mathcal{B}} \amalg_{\mathrm{N}^G_H \mathcal{A}} \underline{\mathcal{A}} \otimes \underline{\mathcal{B}} \arrow[u, dashed, shift right] \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}$$

which is moreover a right-split Verdier pushout by Lemma 4.3.2. Hence by Lemma 4.3.3 we obtain the pushout square

$$\begin{tikzcd}
\mathrm{Ind}^G_H \mathrm{Res}^G_H \mathcal{Z}(\mathrm{N}^G_H \underline{\mathcal{A}}) \arrow[r, "\varepsilon"] \arrow[d] & \mathcal{Z}(\mathrm{N}^G_H \underline{\mathcal{A}}) \arrow[d] \\
\mathrm{Ind}^G_H \mathcal{Z}(\underline{\mathcal{A}} \otimes \underline{\mathcal{B}}) \arrow[r] & \mathcal{Z}(\underline{\mathcal{A}} \otimes \underline{\mathcal{B}} \amalg_{\mathrm{N}^G_H \mathcal{A}} \underline{\mathcal{A}} \otimes \underline{\mathcal{B}}) \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}$$

as desired. □

Definition 4.3.12. Let $\overline{S}$ be a collection of morphisms in a category $\underline{\mathcal{C}}$. We say that it is G-strongly saturated if the following conditions are true:

(i) (Pushout closure) Suppose we have a fibrewise pushout square in $\underline{\mathcal{C}}$

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}$$

such that the left vertical is in $\overline{S}$, then the right vertical is also in $\overline{S}$,

(ii) (G-colimit closure) The G-full subcategory $\mathrm{fun}^{\overline{S}}(\Delta^1, \underline{\mathcal{C}}) \subseteq \mathrm{fun}(\Delta^1, \underline{\mathcal{C}})$ is closed under G-colimits,

(iii) (2-out-of-3) If any two of the three morphisms in

$$\begin{tikzcd}
A \arrow[r] \arrow[rr, bend left] & B \arrow[r] & C
\end{tikzcd}$$

are in $\overline{S}$, then the third one is too.

Proposition 4.3.13 ("[Lur09, Prop. 5.5.4.15]")**.** *Let $\underline{\mathcal{C}}$ be a G-presentable category and S a set of morphisms in $\underline{\mathcal{C}}$, and $\overline{S}$ its G-strong saturation. Let $L : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be the G-Bousfield localisation at S. Then the collection of L-equivalences consists precisely of the collection $\overline{S}$.*

Proof. We will bootstrap the parametrised statement from the unparametrised version in [Lur09, Prop. 5.5.4.15]. Let T be the collection of L-equivalences. First of all, note that we have $\overline{S} \subseteq T$ since it is straightforward to check that T is a G-strongly saturated collection containing S and $\overline{S}$ is by definition the minimal such collection.

To see the reverse inclusion, let $f : X \to Y$ be an L-equivalence and now consider the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ LX \arrow[r, "\simeq", "Lf"'] & LY \end{tikzcd}$$

Now since a G-Bousfield localisation is in particular fibrewise Bousfield localisation, we can apply [Lur09, Prop. 5.5.4.15 (1)] to see that the vertical maps in the square are in $\overline{S}$. And hence by 2-out-of-3, we see that f was also in $\overline{S}$, as desired. □

Lemma 4.3.14. *Let $H \triangleleft G$ with $|G/H| = 2$ and $\underline{\mathcal{C}}$ a G-symmetric monoidal G-stable category. Suppose S is a collection of morphisms in $\underline{\mathcal{C}}$ and $\overline{S}$ its G-strong saturation. If N_H^G sends morphisms in S to morphisms in $\overline{S}$, then N_H^G also preserves all morphisms in the saturation $\overline{S}$.*

Proof. There are three operations in a G-strong saturation, namely pushout closure, colimit closure of the arrow category, and 2-out-of-3 property of compositions. We have to check that N_H^G preserves morphisms constructed under these operations. The 2-out-of-3 property is clear, and so we only have to check the first two operations. To see colimit closure of the arrow category, we need to show that if $\partial : \underline{J} \to \mathrm{fun}_H(\Delta^1, \underline{\mathcal{C}})$ is a diagram that is pointwise in the full subcategory of $\mathrm{fun}_H^{\overline{S}}(\Delta^1, \underline{\mathcal{C}})$ on those morphisms in $\overline{S}$ that are preserved by N_H^G, so that $\underline{\mathrm{colim}}_{\underline{J}}\partial \in \mathrm{fun}_H^{\overline{S}}(\Delta^1, \underline{\mathcal{C}})$, then $\mathrm{N}_H^G \underline{\mathrm{colim}}_{\underline{J}}\partial \in \mathrm{fun}_G^{\overline{S}}(\Delta^1, \underline{\mathcal{C}})$. For this, recall by G/H-distributivity that $\mathrm{N}_H^G \underline{\mathrm{colim}}_{\underline{J}}\partial$ is computed as the cone point of the G-colimit diagram

$$(\prod_{G/H} \underline{J})^{\underline{\triangleright}} \to \prod_{G/H} (\underline{J}^{\underline{\triangleright}}) \xrightarrow{\prod_{G/H}\partial} \prod_{G/H} \mathrm{fun}_H(\Delta^1, \underline{\mathcal{C}}) \xrightarrow{\mathrm{N}_H^G} \mathrm{fun}_G(\Delta^1, \underline{\mathcal{C}})$$

Now the hypothesis on ∂ ensures that, when restricted to $\prod_{G/H} \underline{J}$, this composite lands in $\mathrm{fun}_G^{\overline{S}}(\Delta^1, \underline{\mathcal{C}}) \subseteq \mathrm{fun}_G(\Delta^1, \underline{\mathcal{C}})$ and since by definition $\mathrm{fun}_G^{\overline{S}}(\Delta^1, \underline{\mathcal{C}})$ is closed under G-colimits, we obtain that the cone point $\mathrm{N}_H^G \underline{\mathrm{colim}}_{\underline{J}}\partial$ is indeed in $\mathrm{fun}_G^{\overline{S}}(\Delta^1, \underline{\mathcal{C}})$ as required.

Finally, to see pushout closure, suppose we have a pushout in $\mathcal{C}_H$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ X \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Y \end{tikzcd}$$

where the left vertical is in S (and so, by definition of saturation, the right vertical is in $\overline{S}$). Then by Corollary 4.3.10 we obtain the pushout square

$$\begin{tikzcd} A \otimes B \amalg_{\mathrm{N}_H^G A} A \otimes B \arrow[r] \arrow[d] & \mathrm{N}_H^G B \arrow[d] \\ X \otimes Y \amalg_{\mathrm{N}_H^G X} X \otimes Y \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{N}_H^G Y \end{tikzcd} \tag{4.4}$$

Hence if we can show that the left vertical map is in $\overline{S}$, then by definition, the right vertical map will be in $\overline{S}$ too. For this, by Corollary 4.3.5 we have

$$\begin{tikzcd} \mathrm{Ind}_H^G \mathrm{Res}_H^G \mathrm{N}_H^G A \arrow[r, "\varepsilon"] \arrow[d] & \mathrm{N}_H^G A \arrow[d] \\ \mathrm{Ind}_H^G (A \otimes B) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & A \otimes B \amalg_{\mathrm{N}_H^G A} A \otimes B \end{tikzcd}$$

and similarly for $X \otimes Y \amalg_{\mathrm{N}_H^G X} X \otimes Y$. Since the respective maps on the upper three terms between the ones for the pair (A, B) and the ones for the pair (X, Y) are all in $\overline{S}$ by hypothesis, so is the induced map $A \otimes B \amalg_{\mathrm{N}_H^G A} A \otimes B \to X \otimes Y \amalg_{\mathrm{N}_H^G X} X \otimes Y$. Therefore, the pushout Eq. (4.4) gives that $\mathrm{N}_H^G B \to \mathrm{N}_H^G Y$ is also in $\overline{S}$ as required. $\square$

Lemma 4.3.15. *Let $s : \Delta^0 \hookrightarrow \Delta^1$ be the source inclusion, $H \triangleleft G$ with $|G/H| = 2$, and $j : \Delta^1 \amalg_{\Delta^0}^{G/H} \Delta^1 \hookrightarrow \prod_{G/H} \Delta^1$ the inclusion. Then the functor*

$$\underline{\mathrm{Fun}}_G(\Delta^1 \amalg_{\Delta^0}^{G/H} \Delta^1, \underline{\mathrm{Sp}}_G) \to \underline{\mathrm{Fun}}_G(\prod_{G/H} \Delta^1, \underline{\mathrm{Sp}}_G)$$

induced by $\bigotimes_{G/H}(\underline{\mathrm{Sp}}_H \xrightarrow{s_!} \underline{\mathrm{Fun}}_H(\Delta^1, \underline{\mathrm{Sp}}_H))$ together with the identifications $\underline{\mathrm{Sp}}_H^{\Delta^1} \amalg_{\underline{\mathrm{Sp}}_G}^{s_!} \underline{\mathrm{Sp}}_H^{\Delta^1} \simeq \underline{\mathrm{Fun}}_G(\Delta^1 \amalg_{\Delta^0} \Delta^1, \underline{\mathrm{Sp}}_G)$ and $\bigotimes_{G/H} \underline{\mathrm{Fun}}_H(\Delta^1, \underline{\mathrm{Sp}}_H) \simeq \underline{\mathrm{Fun}}_G(\prod_{G/H} \Delta^1, \underline{\mathrm{Sp}}_G)$ is given by left Kan extension along the inclusion j, and so in particular is G-fully faithful since j is G-fully faithful.

Proof. We first consider the case of the left Kan extension $\bigotimes_{G/H}(\underline{\mathrm{Sp}}_H \xrightarrow{s_!} \underline{\mathrm{Fun}}_H(\Delta^1, \underline{\mathrm{Sp}}_H))$: here the G-symmetric monoidality of the stable Yoneda cocompletion Proposition 2.3.9

$$\underline{\mathrm{Fun}}(-, \underline{\mathrm{Sp}}) : \underline{\mathrm{Cat}}^{\times} \longrightarrow \underline{\mathrm{Pr}}_{L,\mathrm{st}}^{\otimes}$$

means that under $\bigotimes_{G/H} \underline{\mathrm{Sp}}_H \simeq \underline{\mathrm{Sp}}_G$ and $\bigotimes_{G/H} \underline{\mathrm{Fun}}_H(\Delta^1, \underline{\mathrm{Sp}}_H) \simeq \underline{\mathrm{Fun}}_G(\prod_{G/H} \Delta^1, \underline{\mathrm{Sp}}_G)$, the norm $\bigotimes_{G/H}(\underline{\mathrm{Sp}}_H \xrightarrow{s_!} \underline{\mathrm{Fun}}_H(\Delta^1, \underline{\mathrm{Sp}}_H))$ induces the G-left Kan extension $\underline{\mathrm{Sp}}_G \xrightarrow{(s \times s)_!} \underline{\mathrm{Fun}}_G(\prod_{G/H} \Delta^1, \underline{\mathrm{Sp}}_G)$. In total we get the solid $G/H = C_2$-pushout diagram in $\underline{\mathrm{Pr}}_{G,L,\mathrm{st}}$

$$\begin{tikzcd}
\underline{\mathrm{Sp}}_G \arrow[r, shift left, "(\mathrm{id}\times s)_!"] \arrow[d, shift right, "(s\times\mathrm{id})_!"'] & \underline{\mathrm{Fun}}_H(\Delta^0\times\Delta^1, \underline{\mathrm{Sp}}_H) \arrow[l, shift left, dashed, "(\mathrm{id}\times s)^*"] \arrow[d, shift right] \arrow[ddr, bend left, "(s\times\mathrm{id})_!"] & \\
\underline{\mathrm{Fun}}_H(\Delta^1\times\Delta^0, \underline{\mathrm{Sp}}_H) \arrow[u, shift right, dashed, "(s\times\mathrm{id})^*"'] \arrow[r, shift left] \arrow[ur, phantom, "\ulcorner"] \arrow[drr, bend right, "(\mathrm{id}\times s)_!"] & \underline{\mathrm{Fun}}_G(\Delta^1\underline{\amalg}^{G/H}_{\Delta^0}\Delta^1, \underline{\mathrm{Sp}}_G) \arrow[l, shift left, dashed] \arrow[u, shift right, dashed] \arrow[dr, shift left, "?"] & \\
 & & \underline{\mathrm{Fun}}_G(\prod_{G/H}\Delta^1, \underline{\mathrm{Sp}}_G) \arrow[ul, shift left, dashed, "j^*"] \arrow[uul, bend right, dashed, "(s\times\mathrm{id})^*"'] \arrow[ull, bend left, dashed, "(\mathrm{id}\times s)^*"]
\end{tikzcd}$$

and our goal is to show that the map with the question mark is the left Kan extension $j_!$. For this, note that the corresponding dashed $G/H = C_2$-pullback diagram in $\underline{\mathrm{Pr}}_{G,R,\mathrm{st}}$ clearly induces the dashed restriction functor $j^* : \underline{\mathrm{Fun}}_G(\prod_{G/H}\Delta^1, \underline{\mathrm{Sp}}_G) \to \underline{\mathrm{Fun}}_G(\Delta^1\underline{\amalg}^{G/H}_{\Delta^0}\Delta^1, \underline{\mathrm{Sp}}_G)$. Hence by uniqueness of left adjoints, we get that the map with question mark is indeed equivalent to $j_! : \underline{\mathrm{Fun}}_G(\Delta^1\underline{\amalg}^{G/H}_{\Delta^0}\Delta^1, \underline{\mathrm{Sp}}_G) \to \underline{\mathrm{Fun}}_G(\prod_{G/H}\Delta^1, \underline{\mathrm{Sp}}_G)$ as desired. $\square$

Proposition 4.3.16. *Let $H \triangleleft G$ be a normal subgroup of index 2. Then N^G_H sends the morphism $t^* : \mathcal{U}((\underline{\mathrm{Sp}}^{\underline{\omega}}_H)^{\Delta^1})/\mathcal{U}(\underline{\mathrm{Sp}}^{\underline{\omega}}_H) \to \mathcal{U}(\underline{\mathrm{Sp}}^{\underline{\omega}}_H)$ in $\mathcal{R}_{\mathrm{pw}}$ to a morphism in $\overline{\mathcal{R}}_{\mathrm{pw}}$.*

Proof. To prevent too many symbols, it will be convenient to omit the $(-)^{\underline{\omega}}$ decoration. Recall that we have the split Verdier sequence

$$\begin{tikzcd}
\underline{\mathrm{Sp}}_H \arrow[r, hook, "s_*"] \arrow[r, bend left=60, "s_!"] \arrow[r, bend right=60, dashed] & \underline{\mathrm{Sp}}_H^{\Delta^1} \arrow[l, bend right=40, "s^*"'] \arrow[l, bend left=40] \arrow[r, two heads, "t^*"] & \underline{\mathrm{Sp}}_H \arrow[l, bend right=40, "t_!"'] \arrow[l, bend left=40]
\end{tikzcd}$$

Now, we have the following commutative square, which we learnt from Achim Krause.

$$\begin{tikzcd}
\underline{\mathrm{Sp}}_H \arrow[r, "s_!"] \arrow[d, equal] & \underline{\mathrm{Sp}}_H^{\Delta^1} \arrow[d, "\mathrm{cofib}"', "\simeq"] \\
\underline{\mathrm{Sp}}_H \arrow[r, "s_*"] & \underline{\mathrm{Sp}}_H^{\Delta^1}
\end{tikzcd}$$

Hence applying N^G_H to the whole square, we get in turn the diagram

$$\begin{tikzcd}
\underline{\mathrm{Sp}}_H^{\Delta^1}\underline{\amalg}^{s_!}_{\underline{\mathrm{Sp}}_G}\underline{\mathrm{Sp}}_H^{\Delta^1} \simeq \underline{\mathrm{Fun}}_G(\Delta^1\underline{\amalg}_{\Delta^0}\Delta^1, \underline{\mathrm{Sp}}_G) \arrow[r, hook, "j_!"] \arrow[d, "\simeq"'] & \underline{\mathrm{Fun}}_G(\prod_{G/H}\Delta^1, \underline{\mathrm{Sp}}_G) \simeq \mathrm{N}^G_H(\underline{\mathrm{Sp}}_H^{\Delta^1}) \arrow[d, "\simeq"] \\
\underline{\mathrm{Sp}}_H^{\Delta^1}\underline{\amalg}^{s_*}_{\underline{\mathrm{Sp}}_G}\underline{\mathrm{Sp}}_H^{\Delta^1} \arrow[r] & \mathrm{N}^G_H(\underline{\mathrm{Sp}}_H^{\Delta^1})
\end{tikzcd}$$

where the G/H-pushout on the top left is with respect to the $s_!$ diagram and the bottom left is with respect to the s_* diagram. Since, by Lemma 4.3.15, the top arrow is $j_!$ which is G-fully faithful, so is the bottom arrow. Therefore, together with the G/H-distributivity of N_H^G, we obtain the following Verdier sequence

$$\begin{tikzcd} \underline{\mathrm{Sp}}_H^{\Delta^1} \underline{\amalg}^{s_*}_{\underline{\mathrm{Sp}}_G} \underline{\mathrm{Sp}}_H^{\Delta^1} \arrow[r, hook] & \mathrm{N}_H^G(\underline{\mathrm{Sp}}_H^{\Delta^1}) \arrow[r, two heads, "(t\times t)^*"] & \mathrm{N}_H^G\underline{\mathrm{Sp}}_H \simeq \underline{\mathrm{Sp}}_G \arrow[l, bend right, dashed, "(t\times t)_!"'] \arrow[l, bend left, dashed] \end{tikzcd}$$

which is automatically split since the right hand Verdier projection admits the dashed adjoints. Hence by definition of the motivic localisation λ, the diagonal map in

$$\begin{tikzcd} \dfrac{\mathcal{U}(\mathrm{N}_H^G(\underline{\mathrm{Sp}}_H^{\Delta^1}))}{\mathcal{U}(\underline{\mathrm{Sp}}_H^{\Delta^1})\underline{\amalg}^{s_*}_{\mathcal{U}(\underline{\mathrm{Sp}}_G)}\mathcal{U}(\underline{\mathrm{Sp}}_H^{\Delta^1})} \arrow[r] \arrow[d] & \mathcal{U}(\mathrm{N}_H^G\underline{\mathrm{Sp}}_H) \\ \dfrac{\mathcal{U}(\mathrm{N}_H^G(\underline{\mathrm{Sp}}_H^{\Delta^1}))}{\mathcal{U}(\underline{\mathrm{Sp}}_H^{\Delta^1}\underline{\amalg}^{s_*}_{\underline{\mathrm{Sp}}_G}\underline{\mathrm{Sp}}_H^{\Delta^1})} \arrow[ur] & \end{tikzcd}$$

is a morphism in $\mathcal{R}_{\mathrm{pw}}$. So to show that the top horizontal map is in $\overline{\mathcal{R}}_{\mathrm{pw}}$, it will suffice to show that the left vertical map is in $\overline{\mathcal{R}}_{\mathrm{pw}}$: this is merely the observation that we have, by definition a map of cofibre sequences in $\underline{\mathrm{PSh}}^{\mathrm{st}}(\underline{\mathrm{Cat}}^{\mathrm{perf}})$

$$\begin{tikzcd} \mathcal{U}(\underline{\mathrm{Sp}}_H^{\Delta^1})\underline{\amalg}^{s_*}_{\mathcal{U}(\underline{\mathrm{Sp}}_G)}\mathcal{U}(\underline{\mathrm{Sp}}_H^{\Delta^1}) \arrow[r] \arrow[d] & \mathcal{U}(\mathrm{N}_H^G(\underline{\mathrm{Sp}}_H^{\Delta^1})) \arrow[r] \arrow[d, equal] & \dfrac{\mathcal{U}(\mathrm{N}_H^G(\underline{\mathrm{Sp}}_H^{\Delta^1}))}{\mathcal{U}(\underline{\mathrm{Sp}}_H^{\Delta^1})\underline{\amalg}^{s_*}_{\mathcal{U}(\underline{\mathrm{Sp}}_G)}\mathcal{U}(\underline{\mathrm{Sp}}_H^{\Delta^1})} \arrow[d] \\ \mathcal{U}(\underline{\mathrm{Sp}}_H^{\Delta^1}\underline{\amalg}^{s_*}_{\underline{\mathrm{Sp}}_G}\underline{\mathrm{Sp}}_H^{\Delta^1}) \arrow[r] & \mathcal{U}(\mathrm{N}_H^G(\underline{\mathrm{Sp}}_H^{\Delta^1})) \arrow[r] & \dfrac{\mathcal{U}(\mathrm{N}_H^G(\underline{\mathrm{Sp}}_H^{\Delta^1}))}{\mathcal{U}(\underline{\mathrm{Sp}}_H^{\Delta^1}\underline{\amalg}^{s_*}_{\underline{\mathrm{Sp}}_G}\underline{\mathrm{Sp}}_H^{\Delta^1})} \end{tikzcd}$$

and the left vertical is in $\overline{\mathcal{R}}_{\mathrm{pw}}$ by Lemma 4.3.11, and hence the right vertical is in $\overline{\mathcal{R}}_{\mathrm{pw}}$ too. □

Lemma 4.3.17. *Let $H \triangleleft G$ with $|G/H| = 2$. Then N_H^G preserves morphisms in $\overline{\mathcal{R}}_{\mathrm{pw}}$.*

Proof. By Lemma 4.3.14, it suffices to show that N_H^G sends morphisms in $\mathcal{R}_{\mathrm{pw}}$ to morphisms in $\overline{\mathcal{R}}_{\mathrm{pw}}$. Now for any $\underline{\mathcal{C}} \in \mathrm{Cat}_H^{\mathrm{perf}}$, we have the following identification

$$\left(\underline{\mathcal{C}} \xrightarrow{s} \underline{\mathcal{C}}^{\Delta^1} \xrightarrow{t} \underline{\mathcal{C}}\right) \simeq \left(\underline{\mathrm{Sp}}_H^{\omega} \xrightarrow{s} (\underline{\mathrm{Sp}}_H^{\omega})^{\Delta^1} \xrightarrow{t} \underline{\mathrm{Sp}}_H^{\omega}\right) \otimes \underline{\mathcal{C}}$$

Therefore, we obtain that N_H^G sends the following morphism in $\mathcal{R}_{\mathrm{pw}}$

$$\Big(\mathcal{U}(\underline{\mathcal{C}}^{\Delta^1})/\mathcal{U}(\underline{\mathcal{C}}) \xrightarrow{t} \mathcal{U}(\underline{\mathcal{C}})\Big) \simeq \Big(\mathcal{U}((\underline{\mathrm{Sp}}_H^{\omega})^{\Delta^1})/\mathcal{U}(\underline{\mathrm{Sp}}_H^{\omega}) \to \mathcal{U}(\underline{\mathrm{Sp}}_H^{\omega})\Big) \otimes \mathcal{U}(\underline{\mathcal{C}})$$

to a morphism in $\overline{\mathcal{R}}_{\mathrm{pw}}$ by the ordinary symmetric monoidality of λ and Proposition 4.3.16. But then since the collection of morphisms in S can be taken to be of this form by [CDH+20b, Rmk. 2.7.6 (ii)], we are done. □

The final ingredient to the main theorem is the following observation in group theory.

Proposition 4.3.18. *Let p be a prime, G be a p-group, and $H \leq G$ a subgroup. Then there is a normal series $H = N_0 \lhd N_1 \lhd \cdots \lhd N_k = G$ such that the quotients $N_m/N_{m-1} \cong C_p$ for all m.*

Proof. If $H \lhd G$ is itself already normal, then this is immediate since we can just obtain this from the C_p-solvability of the p-group G/H. Suppose $H \leq G$ is a proper subgroup. We claim that we have the proper inclusion $H \lneq \mathrm{N}_H G$ into the normaliser: given this, we can now induct by taking successive normalisers and applying the statement in the case of $H \leq G$ being normal. To see the claim, consider the action of H on the left H-cosets of G. Since H fixes the coset H, this action has a point with singleton orbit, and so since everything in sight are p-groups, we get from the orbit-stabiliser that there is another left H-coset gH for some $g \in G\backslash H$. This means that for all $h \in H$, we get that $hgH = gH$, so that $g \in G\backslash H$ is a normaliser of H which is not in H, as asserted. □

Theorem 4.3.19. *Let G be a 2-group. The inclusion $\mathcal{R}_{\mathrm{pw},\kappa} \subseteq \mathcal{R}_{\mathrm{norm},\kappa}$ is an identification, and hence the comparison $\Psi : \underline{\mathrm{NMot}}_G^{\mathrm{pw}} \to \underline{\mathrm{NMot}}_G$ from Observation 4.2.20 is an equivalence.*

Proof. Let $H \leq G$ be a subgroup. We need to show that N_H^G preserves λ-equivalences, and by Proposition 4.3.13, we need to show N_H^G preserves morphisms in $\overline{\mathcal{R}}_{\mathrm{pw}}$, the G-strong saturation of $\mathcal{R}_{\mathrm{pw}} = \langle \underline{\mathcal{C}} \xrightarrow{s_*} \underline{\mathcal{C}}^{\Delta^1} \xrightarrow{t^*} \underline{\mathcal{C}} \rangle$. By Proposition 4.3.18, let $H = N_0 \lhd N_1 \lhd \cdots \lhd N_k = G$ be a C_2-normal series. Since $\mathrm{N}_H^G \simeq \mathrm{N}_{N_{k-1}}^{N_k} \circ \cdots \circ \mathrm{N}_{N_0}^{N_1}$, it would suffice to show that $\mathrm{N}_{N_{m-1}}^{N_m}$ preserves morphisms in $\overline{\mathcal{R}}_{\mathrm{pw}}$. But then $N_{m-1} \lhd N_m$ is a normal inclusion of index 2, and so this assertion is true by Lemma 4.3.17. □

Corollary 4.3.20. *Let G be a 2–group. Then $\underline{\mathrm{K}}_G^{\mathrm{pw}} \Rightarrow \underline{\mathrm{K}}_G : \underline{\mathrm{Cat}}_G^{\mathrm{perf}} \longrightarrow \underline{\mathrm{Sp}}_G$ is an equivalence. In particular, $\underline{\mathrm{K}}_G^{\mathrm{pw}}$ refines to the a G–lax symmetric monoidal structure and induces*

$$\underline{\mathrm{K}}_G : \mathrm{CAlg}_G(\underline{\mathrm{Cat}}_G^{\mathrm{perf}}) \longrightarrow \mathrm{CAlg}_G(\underline{\mathrm{Sp}}_G)$$

Corollary 4.3.21. *Let G be a 2–group and $\mathcal{C}^{\otimes} \in \mathrm{CAlg}((\mathrm{Cat}^{\mathrm{perf}})^{\otimes})$ be a small symmetric monoidal perfect-stable category. Then the collection of spectra $\{\mathrm{K}(\mathrm{Fun}(BH,\mathcal{C}))\}_{H\leq G}$ assembles canonically to a G–normed ring spectrum.*

Proof. This is an immediate combination of Corollary 4.3.20 and Example 4.3.25. □

4.3.5 Borel equivariant algebraic K–theory

Having performed a general analysis of normed equivariant algebraic K–theory, we record here a large source of examples via Theorem 3.3.4 coming from categories with G–actions.

Proposition 4.3.22. *Let G be a finite group. The functor $\mathrm{ev}_{G/e} : \underline{\mathrm{Cat}}^{\mathrm{perf}}_G \to \underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{perf}})$ canonically refines to a G–symmetric monoidal functor $\mathrm{ev}_{G/e} : (\underline{\mathrm{Cat}}^{\mathrm{perf}}_G)^{\underline{\otimes}} \to \underline{\mathrm{Bor}}((\mathrm{Cat}^{\mathrm{perf}})^{\otimes})$. Moreover, it admits a G–fully faithful right adjoint and the G–fully faithful right adjoint $\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{perf}}) \hookrightarrow \underline{\mathrm{Cat}}^{\mathrm{perf}}_G$ canonically refines to a G–lax symmetric monoidal functor.*

Proof. By Theorem 3.3.4 (ii), we are left to show that $\mathrm{ev}_{G/e}$ is the unit of the adjunction from Observation 3.3.2. As noted there, this is fibrewise induced by taking homotopy fixed points in the target of the H–equivariant map $\mathrm{Res} : \underline{\mathrm{Cat}}^{\mathrm{perf}}_H \to \mathrm{Cat}^{\mathrm{perf}}$ to yield

$$\mathrm{ev} : \underline{\mathrm{Cat}}^{\mathrm{perf}}_H \to (\mathrm{Cat}^{\mathrm{perf}})^{hH} \simeq \mathrm{Fun}(BH, \mathrm{Cat}^{\mathrm{perf}})$$

as desired. We now immediately obtain that the G–right adjoint is as claimed because fibrewise the adjunction is given by the dashed lift

$$\begin{tikzcd}
\underline{\mathrm{Cat}}^{\mathrm{perf}}_H \arrow[r, "\mathrm{ev}_{H/e}", shift left] \arrow[d, hook] & \mathrm{Fun}(BH, \mathrm{Cat}^{\mathrm{perf}}) \arrow[l, dashed, hook, shift left] \arrow[dl, hook, shift left] \\
\mathrm{Mack}_H(\mathrm{Cat}^{\mathrm{perf}}) \arrow[ur, "\mathrm{ev}_{H/e}", shift left] &
\end{tikzcd}$$

for which the diagonal adjunction is given for instance by [BGS20, §8]. □

Remark 4.3.23. Here is another way to *deduce* that the adjunction unit is G–symmetric monoidal *from* its concrete description as the evaluation. We will comment as to why we prefer the abstract approach above at the end of the remark. To wit, we know that the evaluation $\mathrm{ev}_{G/e} : \underline{\mathrm{Cat}}^{\mathrm{perf}}_G \to \underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{perf}})$ is a G-Dwyer-Kan localisation on the morphisms which are underlying equivalences, that is, it is the initial functor that sends to equivalences the morphisms $f : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ in $\mathrm{Cat}^{\mathrm{perf}}_G$ which satisfy that $\mathrm{Res}^G_e f$ is an equivalence. Since we already know that this localisation refines to a symmetric monoidal functor in the unparametrised sense, by [Lur17, Prop. 2.2.1.9] we just need to show that for all $H \leq G$, if $f : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a map of H-perfect-stable categories such that $\mathrm{Res}^G_e f$ is an equivalence, then $\mathrm{N}^G_H f : \mathrm{N}^G_H \underline{\mathcal{C}} \to \mathrm{N}^G_H \underline{\mathcal{D}}$ also satisfies that $\mathrm{Res}^G_e \mathrm{N}^G_H f$ is an equivalence. But then this is clear by Observation 3.2.1 since this is

$$\mathrm{Res}^G_e \mathrm{N}^G_H f \simeq \bigotimes_{|G/H|} \mathrm{Res}^G_e f : \bigotimes_{|G/H|} \mathrm{Res}^G_e \underline{\mathcal{C}} \longrightarrow \bigotimes_{|G/H|} \mathrm{Res}^G_e \underline{\mathcal{D}}$$

which is an equivalence by hypothesis. As mentioned above, we think that the formulation of Proposition 4.3.22 is better because here, we are using the G–symmetric monoidal structure on $\underline{\mathrm{Cat}}^{\mathrm{perf}}_G$ to *induce* one on $\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{perf}})$. A priori, we do not know that this G–symmetric monoidal structure is the one induced on $\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{perf}})$ by $(\mathrm{Cat}^{\mathrm{perf}})^{\otimes}$. It is this latter one that will provide us with a huge source of examples, as we record now.

Corollary 4.3.24. *Let G be a 2–group. Then the G–functor*

$$\underline{\mathrm{K}}_G : \underline{\mathrm{Bor}}_G(\mathrm{Cat}^{\mathrm{perf}}) \hookrightarrow \underline{\mathrm{Cat}}^{\mathrm{perf}}_G \xrightarrow{\mathcal{Z}} \underline{\mathrm{NMot}}_G \xrightarrow{\mathrm{map}(\mathbb{1}_G,-)} \underline{\mathrm{Sp}}_G \tag{4.5}$$

canonically refines to a G–lax symmetric monoidal functor $\underline{\mathrm{K}}_G \colon \underline{\mathrm{Bor}}((\mathrm{Cat}^{\mathrm{perf}})^{\otimes}) \longrightarrow \underline{\mathrm{Sp}}^{\otimes}_G$. In particular, we obtain a functor $\underline{\mathrm{K}}_G \colon \mathrm{Fun}\left(BG, \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{perf}})\right)^{\simeq} \longrightarrow \mathrm{CAlg}_G\left(\underline{\mathrm{Sp}}^{\otimes}_G\right)^{\simeq}$.

Proof. The first and last functor in Eq. (4.5) are G–lax symmetric monoidal: the first by Proposition 4.3.22 and the last by the general fact of mapping from the unit object. By Theorem 4.3.19, $\mathcal{Z}$ is G–symmetric monoidal, and hence the composite is G–lax symmetric monoidal as claimed. Applying CAlg_G and Proposition 3.3.6 gives the last statement. □

Example 4.3.25. We collect here two important sources of examples, showing that normed equivariant algebraic K–theory is in ample supply.

(i) Since $\underline{\mathrm{Sp}}^{\omega}_G \in (\underline{\mathrm{Cat}}^{\mathrm{perf}}_G)^{\underline{\otimes}}$ is the unit object, it is a G–commutative algebra, and hence $\underline{\mathrm{K}}_G(\underline{\mathrm{Sp}}^{\omega}_G)$ canonically refines a G–normed ring spectrum. In light of [BH21, Prop. 7.6] - the connection to which we do not make precise in our work - we expect that *any* G–normed ring spectrum will give rise to a G–normed ring K–theory spectrum. This would then specialise to the case above by considering the G–normed ring spectrum $\mathbb{S}_G$.

(ii) Endowed with the trivial G–action, any $\mathcal{C}^{\otimes} \in \mathrm{CAlg}((\mathrm{Cat}^{\mathrm{perf}})^{\otimes})$ gives a G–symmetric monoidal G–perfect-stable category $\underline{\mathrm{Bor}}(\mathcal{C}^{\otimes})$. Hence $\{\mathrm{K}_G(\mathrm{Fun}(BH, \mathcal{C})\}_{H \leq G}$ canonically assembles to a G–normed ring spectrum.

Part III

Equivariant Hermitian K–theory

Chapter 5

Borel equivariant Grothendieck-Witt theory

In this chapter, we study the notion of Borel equivariant GW–theory, ie. the machinery will be the Grothendieck-Witt theory introduced in [CDH+20b] whose input will be Poincaré categories equipped with G–actions. In other words, we will study the functor

$$\underline{\mathrm{GW}}\colon \mathrm{Fun}(BG, \mathrm{Cat}^{\mathrm{p}}) \xhookrightarrow{\mathrm{Bor}} \mathrm{Mack}_G(\mathrm{Cat}^{\mathrm{p}}) \xrightarrow{\mathrm{GW}} \mathrm{Mack}_G(\mathrm{Sp}) = \mathrm{Sp}_G$$

where Bor is the Borellification functor given by $(\mathcal{C}, \text{♀}) \mapsto \mathrm{Bor}(\mathcal{C}, \text{♀}) = \{(\mathcal{C}, \text{♀})^{hH}\}_{H \leq G}$. We emphasise again that by Borel equivariant, we mean that the *input* is Borel equivariant, and not the output. We will mimic the methods of §4.3 to show that Borel equivariant Grothendieck-Witt theory admits a refinement to the structure of normed ring G–spectra when G is a 2–group (Corollary 5.4.6). The general method will be similar to the K–theoretic case, but this will need to be augmented by some knowledge of the overlaying quadratic structures, and so we collect the extra ingredients in §5.1 and §5.2. Throughout the first three sections, G will be an arbitrary finite group. We restrict to the case of G being a 2–group in §5.4.

5.1 Preparatory materials

Notation 5.1.1. We write $\mathrm{Cat}^{\mathrm{p}}$ in this section to denote the Poincaré categories whose underlying category is perfect. In this way, our $\mathrm{Cat}^{\mathrm{p}}$ here will be a full subcategory of the one introduced in [CDH+20a]. It is clear that the property of being perfect is closed under all categorical operations that we will take here.

Fact 5.1.2. Denoting by $\mathrm{Split}(\mathrm{Cat}^{\mathrm{p}})$ the category of split Poincaré–Verdier sequences, we obtain from the classification result in [CDH+20b, §1.2] an equivalence $\mathrm{Split}(\mathrm{Cat}^{\mathrm{p}}) \simeq \mathrm{Fun}(\Delta^1, \mathrm{Cat}^{\mathrm{p}})$. Moreover, by [CDH+], $\mathrm{Cat}^{\mathrm{p}}$ is κ–compactly generated for every regular cardinal κ.

Lemma 5.1.3. *We have an equivalence* $\underline{\mathrm{Split}}(\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})) \simeq \mathrm{fun}(\Delta^1, \underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}}))$ *and furthermore* $\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})$ *is κ–compactly generated for every cardinal κ.*

Proof. The first claim is clear since we are just pointwise applying $\mathrm{Fun}(BH, -)$ on the equivalence from Fact 5.1.2. For the second, since $\mathrm{Cat}^{\mathrm{p}}$ is κ–compactly generated for every regular cardinal κ, and since the category of such is closed under limits, we know that $\mathrm{Fun}(BH, \mathrm{Cat}^{\mathrm{p}})$ is also κ–compactly generated for all κ, ie. $\mathrm{Ind}_{\kappa}(\mathrm{Fun}(BH, \mathrm{Cat}^{\mathrm{p}})^{\kappa})$. Now for $H \leq K \leq G$, we know that the restriction functor

$$\mathrm{Fun}(BK, \mathrm{Cat}^{\mathrm{p}}) \to \mathrm{Fun}(BH, \mathrm{Cat}^{\mathrm{p}})$$

preserves κ–compact objects since the right adjoint, given by the indexed product, is a finite limit and so commutes with κ–filtered colimits for all regular cardinals κ. Hence, we even have that $\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}}) \simeq \underline{\mathrm{Ind}}_{\kappa}(\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})^{\underline{\kappa}})$ as required. ☐

Remark 5.1.4. There is an adjunction $L : \mathrm{Fun}(\Delta^1, \mathrm{Cat}^{\mathrm{p}}) \rightleftarrows \mathrm{Cat}^{\mathrm{p}} : R$ where $L((\mathcal{C}, \Omega) \xrightarrow{f} (\mathcal{D}, \Phi)) \simeq (\mathcal{C}, \Omega) \times_{(\mathcal{D},\Phi)} \mathrm{Met}(\mathcal{D}, \Phi)$ and $R(\mathcal{E}, \Psi) \simeq (\mathrm{Met}(\mathcal{E}, \Psi) \xrightarrow{\text{target}} (\mathcal{E}, \Psi))$. The right adjoint R preserves filtered colimits since Met does. Borelifying yields the adjunction

$$\underline{L}_G : \mathrm{fun}(\Delta^1, \underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})) \rightleftarrows \underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}}) : \underline{R}_G$$

where the G-right adjoint preserves all fibrewise filtered colimits, and hence $\underline{L}_G$ preserves κ-compact objects for all regular cardinals κ. This means that if $((\mathcal{C}, \Omega) \xrightarrow{f} (\mathcal{D}, \Phi))$ is a G–equivariant Poincaré functor between equivariant κ–compact Poincaré categories, then $(\mathcal{C}, \Omega) \times_{(\mathcal{D},\Phi)} \mathrm{Met}(\mathcal{D}, \Phi)$ is equivariant κ–compact too. We will need this result shortly and we refer to [CDH+] for the original treatment of this in the unparametrised setting.

Corollary 5.1.5. *For any regular cardinal κ there is a small set S_κ of split Poincaré–Verdier sequences on κ-compact G-perfect Poincaré categories such that any split Poincaré–Verdier sequence in* $\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})$ *can be written as a fibrewise κ-filtered colimit of sequences in S_κ.*

Proof. First note that we have

$$\begin{aligned}\mathrm{Split}(\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}}))^{\underline{\kappa}} \simeq \mathrm{fun}(\Delta^1, \underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}}))^{\underline{\kappa}} &\simeq \mathrm{fun}(\Delta^1, \underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})^{\underline{\kappa}}) \\ &\simeq \mathrm{Split}(\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})^{\underline{\kappa}})\end{aligned}$$

where the second equivalence is by [Lur09, Lem. 5.3.4.9] and the third is by Remark 5.1.4 together with Fact 5.1.2. Now since $\mathrm{Split}(\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}}))$ is κ-compactly generated for any regular cardinal κ, we see that

$$\underline{\mathrm{Split}}(\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})) \simeq \underline{\mathrm{Ind}}_{\kappa}(\underline{\mathrm{Split}}(\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}}))^{\underline{\kappa}}) \simeq \underline{\mathrm{Ind}}_{\kappa}\underline{\mathrm{Split}}(\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})^{\underline{\kappa}})$$

with the G–category $\underline{\mathrm{Split}}(\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})^{\underline{\kappa}})$ being small. This is the statement to be proven. $\square$

Lemma 5.1.6. *Let* $(\mathcal{C}, ϙ) \xrightarrow{(i,\eta)} (\mathcal{D}, \Phi) \xrightarrow{(p,\theta)} (\mathcal{E}, \Psi)$ *be a split Poincaré–Verdier sequence in* $(\mathrm{Cat}^{\mathrm{p}})^{BG}$. *Then* $(\mathcal{C}, ϙ)^{hG} \xrightarrow{i} (\mathcal{D}, \Phi)^{hG} \xrightarrow{p} (\mathcal{E}, \Psi)^{hG}$ *is a split Poincaré–Verdier sequence in* $\mathrm{Cat}^{\mathrm{p}}$.

Proof. Since a split Poincaré–Verdier sequence is in particular a fibre sequence and since $(-)^{hG}$ preserves limits, it is immediately a fibre sequence in $\mathrm{Cat}^{\mathrm{p}}$. Thus, by [CDH+20b, Prop. 1.2.2], we are left to show that p admits a fully faithful right adjoint r and that

$$r^*\Phi^{hG} \Longrightarrow r^*p^*\Psi^{hG} \Longrightarrow \Psi^{hG}$$

is an equivalence. Since $(-)^{hG}$ preserves adjunctions, we know that $p : \mathcal{D}^{hG} \to \mathcal{E}^{hG}$ has a right adjoint r coming from the underlying adjunction, and since $p \circ r \simeq \mathrm{id}$ as G–equivariant endofunctors on $\mathcal{E}$, we get that $p \circ r \simeq \mathrm{id}$ as endofunctors on $\mathcal{E}^{hG}$, and hence the right adjoint is fully faithful too. That the required transformation is an equivalence is clear since $(-)^{hG}$ is applied pointwise in spectra. $\square$

We now record an arithmetic fracture in the equivariant setting. This result will not be needed anywhere in the thesis and is included merely for completeness' sake. To this end, let us first recall the following result:

Fact 5.1.7 ([CDH+20c, Prop. 2.1.12])**.** Let R be an ordinary commutative unital ring and M an invertible module with involution over R. Let S be a multiplicatively closed subset generated by an integer $\ell \in R$. Suppose the ℓ^∞-torsion in R is bounded. Then for $r \in \{s, q\}$ we have a split Poincaré–Verdier square

$$\begin{tikzcd}
(\mathcal{D}^p(R), ϙ^r_M) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (\mathcal{D}^c(R[\ell^{-1}]), ϙ^r_{M[S^{-1}]}) \arrow[d] \\
(\mathcal{D}^p(R^\wedge_\ell), ϙ^r_{M^\wedge_\ell}) \arrow[r] & (\mathcal{D}^{c'}(R^\wedge_\ell[\ell^{-1}]), ϙ^r_{M^\wedge_\ell[S^{-1}]})
\end{tikzcd}$$

where $c := \mathrm{Im}(K_0(R) \to K_0(R[\ell^{-1}]))$ and $c' := \mathrm{Im}(K_0(R^\wedge_\ell) \to K_0(R^\wedge_\ell[\ell^{-1}]))$, and the horizontal maps have fully faithful right adjoints, and hence are split Poincaré–Verdier projections.

Deducing from this the following statement is then straightforward.

Proposition 5.1.8 (Equivariant arithmetic fracture)**.** *Let G be a finite group and let R be an ordinary commutative unital ring equipped with a G–action. Let M an invertible R–module with involution over R equipped with a G–action. Let S be a multiplicatively closed subset generated by an integer $\ell \in R$. Assume that the ℓ^∞-torsion in R is bounded. Then for $r \in \{s, q\}$ the square*

$$\begin{tikzcd}
(\mathcal{D}^p(R), \Upsilon^r_M)^{hG} \arrow[r] \arrow[d] & (\mathcal{D}^c(R[\ell^{-1}]), \Upsilon^r_{M[S^{-1}]})^{hG} \arrow[d] \\
(\mathcal{D}^p(R^\wedge_\ell), \Upsilon^r_{M^\wedge_\ell})^{hG} \arrow[r] & (\mathcal{D}^{c'}(R^\wedge_\ell[\ell^{-1}]), \Upsilon^r_{M^\wedge_\ell[S^{-1}]})^{hG}
\end{tikzcd}$$

is a split Poincaré–Verdier square where $c := \mathrm{Im}(K_0(R) \to K_0(R[\ell^{-1}]))$ *and* $c' := \mathrm{Im}(K_0(R^\wedge_\ell) \to K_0(R^\wedge_\ell[\ell^{-1}]))$.

Proof. By naturality, we know that all the maps in the pullback square of Fact 5.1.7 G–equivariant. Since limits commute, we still have the pullback square

$$\begin{tikzcd}
(\mathcal{D}^p(R), \Upsilon^r_M)^{hG} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathcal{D}^c(R[\ell^{-1}]), \Upsilon^r_{M[S^{-1}]})^{hG} \arrow[d] \\
(\mathcal{D}^p(R^\wedge_\ell), \Upsilon^r_{M^\wedge_\ell})^{hG} \arrow[r] & (\mathcal{D}^{c'}(R^\wedge_\ell[\ell^{-1}]), \Upsilon^r_{M^\wedge_\ell[S^{-1}]})^{hG}
\end{tikzcd}$$

Finally, by the argument in Lemma 5.1.6, the functor $(-)^{hG}$ preserves split Poincaré–Verdier projections and so the resulting pullback square is still a split Poincaré–Verdier square. □

5.2 Split Poincaré–Verdier pushouts

Lemma 5.2.1. *Suppose we have a pushout square in* $\mathrm{Cat}^{\mathrm{p}}$

$$\begin{tikzcd}
(\mathcal{C}, \Upsilon) \arrow[r, "f"] \arrow[d, hook, "i"', shift right] & (\mathcal{D}, \Phi) \arrow[d, hook, "j"] \\
(\mathcal{E}, \Psi) \arrow[r, "g"] \arrow[u, dashed, "s"', shift right] \arrow[ur, phantom, "\ulcorner", very near end] & (\mathcal{P}, \alpha)
\end{tikzcd}$$

satisfying the following list of conditions:

- *The left vertical is a split Poincaré–Verdier inclusion such that* $s : \mathcal{E} \to \mathcal{C}$ *itself has a right adjoint which is fully faithful.*
- *Both of the horizontal maps have right adjoints.*

Then $j : (\mathcal{D}, \Phi) \hookrightarrow (\mathcal{P}, \alpha)$ *is also a split Poincaré–Verdier inclusion.*

Proof. Our task is to show that the canonical transformation $\Phi \Rightarrow (j^{\mathrm{op}})^*\alpha$ is an equivalence. Let us first complete the diagram with all the data that we need to establish notations:

$$\begin{tikzcd}
(\mathcal{C},\Qoppa) \arrow[r, shift left, "f"] \arrow[d, "i"'] \arrow[d, dashed, bend right, "k"'] & (\mathcal{D},\Phi) \arrow[l, dashed, shift left, "\overline{f}"] \arrow[d, "j"'] \arrow[d, dashed, bend right, "\overline{k}"'] \\
(\mathcal{E},\Psi) \arrow[r, shift left, "g"] \arrow[u, shift right, "s"] \arrow[ur, phantom, "\ulcorner"] & (\mathcal{P},\alpha) \arrow[l, dashed, shift left, "\overline{g}"] \arrow[u, shift right, "\overline{s}"]
\end{tikzcd}$$

By the method in which pushouts are computed, we have the pullback

$$\begin{tikzcd}
\mathcal{C} & \mathcal{D} \arrow[l, "\overline{f}"'] \\
\mathcal{E} \arrow[u, "s"] & \mathcal{P} \arrow[l, "\overline{g}"] \arrow[u, "\overline{s}"'] \arrow[ul, phantom, "\ulcorner"]
\end{tikzcd}$$

where the vertical maps are Verdier projections. Hence by [CDH+20b, Lem 1.5.3, Prop. A.3.15] we see that this square is right adjointable, yielding the following equivalences

$$\left(k \circ \overline{f} \xrightarrow{\simeq} \overline{g} \circ \overline{k}\right) \quad \Longrightarrow \quad \left(f \circ s \xrightarrow{\simeq} \overline{s} \circ g\right)$$

Now recall that by definition $\alpha : \mathcal{P}^{\mathrm{op}} \to \mathrm{Sp}$ is defined as the pushout in $\mathrm{Fun}^q(\mathcal{P}^{\mathrm{op}}, \mathrm{Sp})$

$$\begin{tikzcd}
(j^{\mathrm{op}})_!(f^{\mathrm{op}})_!\Qoppa \arrow[r] \arrow[d] & (j^{\mathrm{op}})_!\Phi \arrow[d] \\
(g^{\mathrm{op}})_!\Psi \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \alpha
\end{tikzcd}$$

Applying j^* to this we get the pushout square in $\mathrm{Fun}^q(\mathcal{D}^{\mathrm{op}}, \mathrm{Sp})$

$$\begin{tikzcd}
(f^{\mathrm{op}})_!\Qoppa \simeq (j^{\mathrm{op}})^*(j^{\mathrm{op}})_!(f^{\mathrm{op}})_!\Qoppa \arrow[r] \arrow[d] & (j^{\mathrm{op}})^*(j^{\mathrm{op}})_!\Phi \simeq \Phi \arrow[d] \\
(j^{\mathrm{op}})^*(g^{\mathrm{op}})_!\Psi \arrow[r] \arrow[ur, phantom, "\ulcorner"] & (j^{\mathrm{op}})^*\alpha
\end{tikzcd} \tag{5.1}$$

Hence if we can show that the left vertical arrow is an equivalence, then we would be done. Since we have the adjunction $\overline{s}^{\mathrm{op}} \dashv j^{\mathrm{op}}$, we see that $(j^{\mathrm{op}})^* \simeq (\overline{s}^{\mathrm{op}})_!$, and similarly we have $(i^{\mathrm{op}})^* \dashv (s^{\mathrm{op}})^*$. Hence, together with our hypothesis, we have that $\Qoppa \simeq (i^{\mathrm{op}})^*\Psi \simeq (s^{\mathrm{op}})_!\Psi$. Therefore, the left vertical map in Eq. (5.1) becomes

$$(f^{\mathrm{op}})_!(s^{\mathrm{op}})_!\Psi \to (\overline{s}^{\mathrm{op}})_!(g^{\mathrm{op}})_!\Psi$$

and this is an equivalence since $f^{\mathrm{op}} \circ s^{\mathrm{op}} \simeq \overline{s}^{\mathrm{op}} \circ g^{\mathrm{op}}$ as above. □

The following is an immediate consequence of Lemma 5.2.1 since every relevant notion is pointwise.

Corollary 5.2.2. *Suppose we have a pushout square in* $\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})$

$$\begin{tikzcd} (\mathcal{C}, \Qoppa) \arrow[r, "f"] \arrow[d, hook, "i"'] & (\mathcal{D}, \Phi) \arrow[d, hook, "j"] \\ (\mathcal{E}, \Psi) \arrow[r, "g"] \arrow[u, dashed, bend right, "s"'] & (\mathcal{P}, \alpha) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

satisfying the following list of conditions:

- *The left vertical is a split Poincaré–Verdier inclusion such that* $s : \mathcal{E} \to \mathcal{C}$ *itself has a right adjoint which is fully faithful.*
- *Both of the horizontal maps have right adjoints.*

Then $j : (\mathcal{D}, \Phi) \hookrightarrow (\mathcal{P}, \alpha)$ *is also a split Poincaré–Verdier inclusion.*

Construction 5.2.3 (Temabolic structures)**.** Recall that for a Poincaré category $(\mathcal{C}, \Qoppa)$, we have the *metabolic category* $\mathrm{Met}(\mathcal{C}, \Qoppa)$ Poincaré structure on $\mathcal{C}^{\Delta^1}$ given by

$$(x \xrightarrow{f} y) \quad \mapsto \quad \mathrm{fib}(\Qoppa(y) \to \Qoppa(x))$$

On the other hand, as in Proposition 5.4.3, we have the trick equivalence

$$\mathrm{cofib} : \mathrm{Sp}_\omega^{\Delta^1} \leftrightarrows \mathrm{Sp}_\omega^{\Delta^1} : \mathrm{fib}$$

We define the *temabolic* Poincaré structure on $\mathrm{Sp}_\omega^{\Delta^1}$ as the one induced by this equivalence, that is, $\Qoppa_{\mathrm{tem}} := \Qoppa_{\mathrm{met}} \circ \mathrm{cofib}$ so that

$$\Qoppa_{\mathrm{tem}} : (x \xrightarrow{f} y) \quad \mapsto \quad \Qoppa_{\mathrm{met}}(y \to \mathrm{cofib}(f)) \quad \mapsto \quad \mathrm{fib}(\Qoppa(\mathrm{cofib}(f)) \to \Qoppa(y))$$

By construction, we then have an equivalence of Poincaré categories

$$\mathrm{cofib} : (\mathrm{Sp}_\omega^{\Delta^1}, \Qoppa_{\mathrm{tem}}) \leftrightarrows (\mathrm{Sp}_\omega^{\Delta^1}, \Qoppa_{\mathrm{met}}) : \mathrm{fib}$$

since the natural transformation $\Qoppa_{\mathrm{tem}} \Rightarrow \mathrm{cofib}^* \Qoppa_{\mathrm{met}}$ is an equivalence by definition. Furthermore, note that the linear part of temabolic structure is given by

$$\begin{aligned} L_{\Qoppa_{\mathrm{tem}}}(x \xrightarrow{f} y) &\simeq \mathrm{fib}(L_{\Qoppa}(\mathrm{cofib}(f)) \to L_{\Qoppa}(y)) \\ &\simeq \mathrm{fib}(Dx \to D\,\mathrm{fib}(f)) \simeq Dx =: D_{\mathrm{src}}(x \xrightarrow{f} y) \end{aligned}$$

The final important observation about this construction is that, if we write $s : \Delta^0 \hookrightarrow \Delta^1$ for the source inclusion, then we have the commuting square in $\mathrm{Cat}^{\mathrm{p}}$

$$\begin{tikzcd}
(\mathrm{Sp}^{\omega}, \Qoppa) \arrow[r, "s_!"] \arrow[d, equal] & (\mathrm{Sp}_{\omega}^{\Delta^1}, \Qoppa_{\mathrm{tem}}) \arrow[d, "\simeq"', "\mathrm{cofib}"] \\
(\mathrm{Sp}^{\omega}, \Qoppa) \arrow[r, "s_*"] & (\mathrm{Sp}_{\omega}^{\Delta^1}, \Qoppa_{\mathrm{met}})
\end{tikzcd}$$

Notation 5.2.4. We have the following pushout square

$$\begin{tikzcd}
\Delta^0 \amalg \Delta^0 \arrow[r, "p"] \arrow[d, hook, "s \amalg s"', shift right] \arrow[dr, phantom, "\ulcorner"] & \Delta^0 \arrow[d, hook, "h"', shift right] \arrow[ddr, bend left, hook, "d"] & \\
\Delta^1 \amalg \Delta^1 \arrow[r, "q"] \arrow[u, dashed, bend right, "r \amalg r"'] \arrow[drr, bend right, "\pi"'] & \Delta^1 \cup_{\Delta^0} \Delta^1 \arrow[u, dashed, bend right, "R"'] \arrow[dr, hook, "e"] & \\
& & \Delta^1 \times \Delta^1
\end{tikzcd}$$

Hence we get the following diagram of categories

$$\begin{tikzcd}
\mathrm{Sp}_{\omega}^{\Delta^0 \amalg \Delta^0} \arrow[r, shift right, "p_!"'] \arrow[d, shift left, "s_! \oplus s_!"] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Sp}_{\omega}^{\Delta^0} \arrow[l, shift right, "p^*"'] \arrow[d, shift left, "h_!"] \arrow[ddr, bend left, hook, "d_!"] & \\
\mathrm{Sp}_{\omega}^{\Delta^1 \amalg \Delta^1} \arrow[u, shift left, "r_! \oplus r_!"] \arrow[r, shift right, "q_!"'] \arrow[drr, bend right, "\pi_!"'] & \mathrm{Sp}_{\omega}^{\Delta^1 \cup_{\Delta^0} \Delta^1} \arrow[l, shift right, "\ell"'] \arrow[u, shift left, "R_!"] \arrow[dr, hook, "e_!"] & \\
& & \mathrm{Sp}_{\omega}^{\Delta^1 \times \Delta^1}
\end{tikzcd} \tag{5.2}$$

where here the left adjoint $\ell : \mathrm{Sp}_{\omega}^{\Delta^1 \cup_{\Delta^0} \Delta^1} \to \mathrm{Sp}_{\omega}^{\Delta^1 \amalg \Delta^1}$ is given by

$$(c \leftarrow a \to b) \mapsto (b \to b \cup_a c, c \to b \cup_a c)$$

This can be checked easily using the notion of left adjoint objects. More precisely, since we are mapping into the product $\mathrm{Sp}_{\omega}^{\Delta^1} \times \mathrm{Sp}_{\omega}^{\Delta^1}$, we can without loss of generality build this adjoint on one of the components. So suppose $(c \leftarrow a \to b) \in \mathrm{Sp}_{\omega}^{\Delta^1 \cup_{\Delta^0} \Delta^1}$ and $(x \to y) \in \mathrm{Sp}_{\omega}^{\Delta^1}$, so that $\ell(c \leftarrow a \to b) = (b \to b \cup_a c)$ and $q_!(x \to y) = (y \leftarrow x \xrightarrow{=} x)$. Then we clearly have

$$\mathrm{Map}\left((b \to b \cup_a c), (x \to y)\right) \simeq \mathrm{Map}\left((c \leftarrow a \to b), (y \leftarrow x \xrightarrow{=} x)\right)$$

by virtue of the diagram

$$\begin{tikzcd}
a \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & b \arrow[r] \arrow[d] & x \arrow[d] \\
c \arrow[r] & b \cup_a c \arrow[r] & y
\end{tikzcd}$$

One sanity check for this adjunction is that to see that, indeed, the square of left adjoints commutes. Moreover, this pushout square is clearly of the form considered in Eq. (5.2) - it is an easy check to see that it is indeed adjointable.

Lemma 5.2.5. *Endowing $\mathrm{Sp}_\omega^{\Delta^1}$ with the temabolic structure and writing $\alpha : (\mathrm{Sp}_\omega^{\Delta^1\cup_{\Delta^0}\Delta^1})^{\mathrm{op}} \to \mathrm{Sp}$ for the pushout Poincaré structure, the map $e_! : \mathrm{Sp}_\omega^{\Delta^1\cup_{\Delta^0}\Delta^1} \hookrightarrow \mathrm{Sp}_\omega^{\Delta^1\times\Delta^1}$ induces the natural transformation $\alpha \Rightarrow (e_!)^*(\Omega_{\mathrm{tem}} \otimes \Omega_{\mathrm{tem}})$ which is an equivalence, that is, it is a split Poincaré–Verdier inclusion.*

Proof. Since the pushout is taken in $\mathrm{Cat}^{\mathrm{p}}$, we already know that $e_!$ is duality-preserving, and so preserves bilinear parts. Hence, we are left to show that $L_\alpha \to (e_!)^*(L_{\Omega_{\mathrm{tem}}} \otimes L_{\Omega_{\mathrm{tem}}}) \simeq (e_!)^*(D_{\mathrm{src}} \otimes D_{\mathrm{src}})$ is an equivalence. Now, we have the pushout in $\mathrm{Fun}^{\mathrm{ex}}((\mathrm{Sp}_\omega^{\Delta^1\cup_{\Delta^0}\Delta^1})^{\mathrm{op}}, \mathrm{Sp})$

$$\begin{tikzcd}
((h_!)^{\mathrm{op}}(p_!)^{\mathrm{op}})_!(D\oplus D) \arrow[r] \arrow[d] & ((h_!)^{\mathrm{op}})_! D \arrow[d] & \\
((q_!)^{\mathrm{op}})_!(D_{\mathrm{src}}\oplus D_{\mathrm{src}}) \arrow[r] & L_\alpha \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr] & \\
& & ((e_!)^{\mathrm{op}})^*(D_{\mathrm{src}}\otimes D_{\mathrm{src}})
\end{tikzcd} \tag{5.3}$$

by definition and Notation 5.2.4, and we want to show that the diagonal map is an equivalence.

We collect all the adjunction relations we will be needing:

- $(h \dashv R) \Rightarrow (R_! \dashv R^* \simeq h_! \dashv h^*) \Rightarrow ((h_!)^{\mathrm{op}} \dashv (R_!)^{\mathrm{op}}) \Rightarrow ([(R_!)^{\mathrm{op}}]^* \dashv [(h_!)^{\mathrm{op}}]^*)$
- $((p_!)^{\mathrm{op}} \simeq (p_*)^{\mathrm{op}} \dashv (p^*)^{\mathrm{op}}) \Longrightarrow ([(p^*)^{\mathrm{op}}]^* \simeq [(p_!)^{\mathrm{op}}]_! \dashv [(p_!)^{\mathrm{op}}]^*)$
- $(\ell \dashv q_!) \Longrightarrow ((q_!)^{\mathrm{op}} \dashv \ell^{\mathrm{op}}) \Longrightarrow ([\ell^{\mathrm{op}}]^* \simeq [(q_!)^{\mathrm{op}}]_! \dashv [(q_!)^{\mathrm{op}}]^*)$

Given these, we can rewrite the pushout Eq. (5.3) as

$$\begin{tikzcd}
{[(R_!)^{\mathrm{op}}]^*[(p^*)^{\mathrm{op}}]^*(D\oplus D)} \arrow[r] \arrow[d] & {[(R_!)^{\mathrm{op}}]^* D} \arrow[d] \\
(\ell^{\mathrm{op}})^*(D_{\mathrm{src}}\oplus D_{\mathrm{src}}) \arrow[r] & L_\alpha \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Evaluating at an object $(x \to y, 0 \to 0) \in \mathrm{Sp}_\omega^{\Delta^1 \amalg \Delta^1}$ on the top three corners, we get

$$\Big([(q_!)^{\mathrm{op}}]^*[(R_!)^{\mathrm{op}}]^* D\Big)(x \to y, 0 \to 0) \simeq \Big([(R_!)^{\mathrm{op}}]^* D\Big)(y \leftarrow x \xrightarrow{=} x) \simeq Dy$$

$$\Big([(q_!)^{\mathrm{op}}]^*[(R_!)^{\mathrm{op}}]^*[(p^*)^{\mathrm{op}}]^*(D\oplus D)\Big)(x \to y, 0 \to 0) \simeq (D\oplus D)(p^{*,\mathrm{op}})(y) \simeq Dy \oplus Dy$$

$$\begin{aligned}([(q_!)^{\mathrm{op}}]^*[\ell^{\mathrm{op}}]^*(D_{\mathrm{src}}\oplus D_{\mathrm{src}}))(x\to y,0\to 0) &\simeq (D_{\mathrm{src}}\oplus D_{\mathrm{src}})(\ell^{\mathrm{op}})(y\leftarrow x\xrightarrow{=}x)\\ &\simeq (D_{\mathrm{src}}\oplus D_{\mathrm{src}})(x\to y, y\xrightarrow{=}y)\\ &\simeq Dx\oplus Dy\end{aligned}$$

Therefore we obtain that $([(q_!)^{\mathrm{op}}]^*L_\alpha)(x\to y,0\to 0)\simeq Dx$. But we know that $(e_!)^{\mathrm{op}}(q_!)^{\mathrm{op}}(x\to y,0\to 0)\simeq(\pi_!)^{\mathrm{op}}(x\to y,0\to 0)$ is given by the square in $\mathrm{Sp}_\omega^{\Delta^1\cup_{\Delta^0}\Delta^1}$

$$\begin{tikzcd} x \arrow[r, equal] \arrow[d] & x \arrow[d] \\ y \arrow[r, equal] & y \end{tikzcd} \quad = \quad (x\to y)\otimes(\mathbb{S}\xrightarrow{=}\mathbb{S})$$

and so we get that

$$[(q_!)^{\mathrm{op}}]^*[(e_!)^{\mathrm{op}}]^*(D_{\mathrm{src}}\otimes D_{\mathrm{src}})(x\to y,0\xrightarrow{=}0)\simeq Dx\otimes D\mathbb{S}\simeq Dx \tag{5.4}$$

Similarly, we could have set the first variable to zero, and so in total we have shown that $L_\alpha \Longrightarrow [(e_!)^{\mathrm{op}}]^*(D_{\mathrm{src}}\otimes D_{\mathrm{src}})$ is an equivalence upon applying the restriction $[(q_!)^{\mathrm{op}}]^*$ to $\mathrm{Sp}_\omega^{\Delta^1\amalg\Delta^1}$. On the other hand, the pushout in Eq. (5.2) is of the form considered in Lemma 5.2.1, so $h_!:\mathrm{Sp}_\omega^{\Delta^0}\hookrightarrow\mathrm{Sp}_\omega^{\Delta^1\cup_{\Delta^0}\Delta^1}$ is a Poincaré–Verdier inclusion. Thus, the transformation Eq. (5.4) is also an equivalence upon applying the restriction $[(h_!)^{\mathrm{op}}]^*$ to $\mathrm{Sp}_\omega^{\Delta^0}$.

Now by virtue of $(\mathrm{Sp}_\omega^{\Delta^1\cup_{\Delta^0}\Delta^1})^{\mathrm{op}}$ being a pushout, we have the following equivalence

$$\begin{aligned}&\mathrm{Fun}^{\mathrm{ex}}((\mathrm{Sp}_\omega^{\Delta^1\cup_{\Delta^0}\Delta^1})^{\mathrm{op}},\mathrm{Sp})\\ &\xrightarrow{\simeq}\mathrm{Fun}^{\mathrm{ex}}((\mathrm{Sp}_\omega^{\Delta^1\amalg\Delta^1})^{\mathrm{op}},\mathrm{Sp})\times_{\mathrm{Fun}^{\mathrm{ex}}((\mathrm{Sp}_\omega^{\Delta^0\amalg\Delta^0})^{\mathrm{op}},\mathrm{Sp})}\mathrm{Fun}^{\mathrm{ex}}((\mathrm{Sp}_\omega^{\Delta^0})^{\mathrm{op}},\mathrm{Sp})\end{aligned}$$

given by $[(q_!)^{\mathrm{op}}]^*\times_{[(h_!)^{\mathrm{op}}]^*}[(h_!)^{\mathrm{op}}]^*$. And hence since the morphism Eq. (5.4) in the source of this equivalence becomes an equivalence on the target, it must have been an equivalence to begin with, as was to be shown. $\square$

5.3 Borel Poincaré motives

We now imitate the strategy and techniques from §4.2 for the case of Borel equivariant GW–theory. The end goal is to show that the composite functor

$$\mathrm{Fun}(BG,\mathrm{Cat}^{\mathrm{p}})\xhookrightarrow{\mathrm{Bor}}\mathrm{Mack}_G(\mathrm{Cat}^{\mathrm{p}})\xrightarrow{\mathrm{GW}}\mathrm{Mack}_G(\mathrm{Sp})=\mathrm{Sp}_G$$

refines to a G–lax symmetric monoidal functor.

Definition 5.3.1. Let κ be a regular cardinal. The G-category of *unstable pointwise Borel $\underline{\kappa}$-motives* $\underline{\mathrm{BorMot}}_G^{\mathrm{p,un},\kappa}$ is defined to be $\mathcal{R}_{\mathrm{Bor},\kappa}^{-1}\underline{\mathrm{PSh}}_G(\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}}))$ via the construction from Theorem 2.2.10, where $\mathcal{R}_{\mathrm{Bor},\kappa}$ is the collection of diagrams in $\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})^{\underline{\kappa}}$ consisting of:

- $\underline{\mathrm{const}}_G(\varnothing)^{\underline{\triangleright}} = \underline{*} \to \underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})^{\underline{\kappa}}$ picking the zero category (ie. the initial object),
- all split Poincaré–Verdier sequences.

Remark 5.3.2. Note that $\mathcal{R}_{\mathrm{Bor},\kappa}$ is small since $\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})^{\underline{\kappa}}$ was small, and so $\underline{\mathrm{BorMot}}_G^{\mathrm{p,un},\kappa}$ is G-presentable. By the methods of Chapter 4, we see that we obtain a G–presentable $\underline{\mathrm{BorMot}}_G^{\mathrm{p,un}} := \bigcup_\kappa \underline{\mathrm{BorMot}}_G^{\mathrm{p,un},\kappa}$ and the stable version $\underline{\mathrm{BorMot}}_G^{\mathrm{p}} := \underline{\mathrm{Sp}}_G(\underline{\mathrm{BorMot}}_G^{\mathrm{p,un}})$.

Notation 5.3.3. Write $j_{\mathrm{un}}^{\kappa}\colon \underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})^{\underline{\kappa}} \to \underline{\mathrm{BorMot}}_G^{\mathrm{p,un},\kappa}$ for the canonical functor. Since split Poincaré–Verdier sequences were already cofibre sequences in $\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})^{\underline{\kappa}}$ by definition, we get from Theorem 2.2.12 that this functor is G-fully faithful. As in Chapter 4, we can get a fully faithful $j_{\mathrm{un}}\colon \underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}}) \hookrightarrow \underline{\mathrm{BorMot}}_G^{\mathrm{p,un}}$, and we also denote by $\mathcal{Z}\colon \underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}}) \to \underline{\mathrm{BorMot}}_G^{\mathrm{p}}$ the stable version. Moreover, the we will also need the similar notation $\mathcal{U}\colon \underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}}) \hookrightarrow \underline{\mathrm{PSh}}(\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}}))$ for the Yoneda embedding.

The following lemma is an immediate consequence of the fact that the hermitian Q–construction commutes with functor categories in the nonequivariant setting. This is since $\underline{\mathrm{Fun}}^{\mathrm{ex}}$ and $\underline{\mathrm{Q}}$ are just the underlying such construction together with the data of G–actions coming from the input.

Lemma 5.3.4. *Let* $(\mathcal{C},\Omega), (\mathcal{D},\Phi) \in \underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})$. *Then*

$$\underline{\mathrm{Fun}}^{\mathrm{ex}}((\mathcal{D},\Phi), \underline{\mathrm{Q}}_n(\mathcal{C},\Omega)) \simeq \underline{\mathrm{Q}}_n\underline{\mathrm{Fun}}^{\mathrm{ex}}((\mathcal{D},\Phi),(\mathcal{C},\Omega))$$

Lemma 5.3.5 (Motivic suspension, "[BGT13, §7.3], [CDH+, Prop. 1.2.9]")**.** *Let* $(\mathcal{C},\Omega) \in \underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})$. *Then* $\mathrm{colim}_{\bullet\in\Delta^{\mathrm{op}}}\, j_{\mathrm{un}}\underline{\mathrm{Q}}_\bullet(\mathcal{C},\Omega) \in \underline{\mathrm{BorMot}}_G^{\mathrm{p,un}} \subseteq \underline{\mathrm{PSh}}_G$ *and moreover,*

$$\operatorname*{colim}_{\bullet\in\Delta^{\mathrm{op}}} j_{\mathrm{un}}\underline{\mathrm{Q}}_\bullet(\mathcal{C},\Omega) \simeq \Sigma j_{\mathrm{un}}(\mathcal{C}) \in \underline{\mathrm{BorMot}}_G^{\mathrm{p,un}}$$

Proof. To see the first part, let $(\mathcal{D},\Phi) \in \underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})$. Then note that

$$\begin{aligned}
&\underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_G}(j_{\mathrm{un}}(\mathcal{D},\Phi), \operatorname*{colim}_{\bullet\in\Delta^{\mathrm{op}}} j_{\mathrm{un}}\underline{\mathrm{Q}}_\bullet(\mathcal{C},\Omega))\\
&\simeq \operatorname*{colim}_{\bullet\in\Delta^{\mathrm{op}}} \underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_G}(j_{\mathrm{un}}(\mathcal{D},\Phi), j_{\mathrm{un}}\underline{\mathrm{Q}}_\bullet(\mathcal{C},\Omega))\\
&\simeq \operatorname*{colim}_{\bullet\in\Delta^{\mathrm{op}}} \left[\mathrm{Pn}\,\mathrm{Fun}^{\mathrm{ex}}((\mathcal{D},\Phi),\underline{\mathrm{Q}}_\bullet(\mathcal{C},\Omega))\right]^{h-}\\
&\simeq \operatorname*{colim}_{\bullet\in\Delta^{\mathrm{op}}} \left[\mathrm{Pn}(\underline{\mathrm{Q}}_\bullet\,\mathrm{Fun}^{\mathrm{ex}}((\mathcal{D},\Phi),(\mathcal{C},\Omega)))\right]^{h-}\\
&\simeq \operatorname*{colim}_{\bullet\in\Delta^{\mathrm{op}}} \mathrm{Pn}\underline{\mathrm{Q}}_\bullet\left(\mathrm{Fun}^{\mathrm{ex}}((\mathcal{D},\Phi),(\mathcal{C},\Omega))^{h-}\right)\\
&=: \underline{\mathcal{GW}}_G^{\mathrm{pw}}(\mathrm{Fun}^{\mathrm{ex}}((\mathcal{D},\Phi),(\mathcal{C},\Omega))^{h-})
\end{aligned}$$

and hence, since for all $H \le G$, $\mathrm{Fun}^{\mathrm{ex}}(-,\mathcal{C})^{hH}$ preserves split Poincaré–Verdier sequences by Lemma 5.1.6 and since $\underline{\mathcal{GW}}_G^{\mathrm{pw}}$ is additive, we obtain that indeed $\operatorname{colim}_{\bullet\in\Delta^{\mathrm{op}}} j_{\mathrm{un}}\underline{\mathrm{Q}}_\bullet(\mathcal{C},\Omega)$ is motivically local as claimed.

For the second part, by [CDH+20b, Obs. 3.3.3] we have the simplicial split Poincaré–Verdier sequence $\mathrm{const}_\bullet(\mathcal{C},\Omega\Omega) \to \underline{\mathrm{Null}}_\bullet(\mathcal{C},\Omega) \to \underline{\mathrm{Q}}_\bullet(\mathcal{C},\Omega)$ in $\mathrm{Fun}(BG,\mathrm{Cat}^{\mathrm{p}})$. Now since $j_{\mathrm{un}}: \underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}}) \to \underline{\mathrm{BorMot}}_G^{\mathrm{p,un}}$ sends split Poincaré–Verdier sequences to cofibre sequences by definition of unstable motives, and cofibre sequences are stable under colimits, we can apply j_{un} to the simplicial split Poincaré–Verdier sequence and take geometric realisation in $\underline{\mathrm{BorMot}}_G^{\mathrm{p,un}}$ to get a cofibre sequence in $\underline{\mathrm{BorMot}}_G^{\mathrm{p,un}}$

$$(\mathcal{C},\Omega\Omega) \to \operatorname*{colim}_{\bullet\in\Delta^{\mathrm{op}}} \underline{\mathrm{Null}}_\bullet(\mathcal{C},\Omega) \to \operatorname*{colim}_{\bullet\in\Delta^{\mathrm{op}}} \underline{\mathrm{Q}}_\bullet(\mathcal{C},\Omega)$$

But by [CDH+20b, Lem. 3.3.1] the middle term is always augmented over 0 and so is zero, giving that the last term is a suspension of the first term as required. $\square$

Theorem 5.3.6 (Motivic corepresentability of GW, "[CDH+, Prop. 2.1.5]")**.** *Let $(\mathcal{C},\Omega)$ and $(\mathcal{D},\Phi)$ be in $\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})$. Then there is a natural equivalence in $\underline{\mathrm{Sp}}_G$*

$$\underline{\mathrm{map}}_{\underline{\mathrm{BorMot}}_G^{\mathrm{p,un}}}(\mathcal{Z}(\mathcal{C},\Omega),\mathcal{Z}(\mathcal{D},\Phi)) \simeq \underline{\mathrm{GW}}_G(\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C},\Omega),(\mathcal{D},\Phi))^{h-})$$

In particular, $\underline{\mathrm{GW}}_G$ is corepresented by $\mathcal{Z}(\mathrm{triv}_G(\mathrm{Sp}^\omega,\Omega^u))$ by Proposition 2.3.16.

Proof. Firstly, note that in $\underline{\mathrm{BorMot}}_G^{\mathrm{p,un}}$, $\Sigma^n j_{\mathrm{un}}\mathcal{D} \simeq \operatorname{colim}_{\bullet\in(\Delta^{\mathrm{op}})^n} j_{\mathrm{un}}\underline{\mathrm{Q}}_\bullet\mathcal{D}$ since

$$\Sigma^n j_{\mathrm{un}}(\mathcal{D},\Phi) \simeq \Sigma^{n-1}\big(\operatorname*{colim}_{\bullet\in\Delta^{\mathrm{op}}} j_{\mathrm{un}}\underline{\mathrm{Q}}_\bullet(\mathcal{D},\Phi)\big) \simeq \operatorname*{colim}_{\bullet\in\Delta^{\mathrm{op}}}(\Sigma^{n-1}j_{\mathrm{un}}\underline{\mathrm{Q}}_\bullet\mathcal{D}) \simeq \operatorname*{colim}_{\bullet\in\Delta^{\mathrm{op}}}(\Sigma^{n-2}\big(\operatorname*{colim}_{\bullet\in\Delta^{\mathrm{op}}} j_{\mathrm{un}}\underline{\mathrm{Q}}_\bullet\mathcal{D})\big)$$

and so on. The left hand parametrised spectrum in the theorem statement is the one associated to the prespectrum whose n-th term, for $n \geq 1$, is

$$\begin{aligned}
&\underline{\mathrm{Map}}_{\underline{\mathrm{BorMot}}_G^{\mathrm{p,un}}}(\lambda j_{\mathrm{un}}(\mathcal{C},\Omega), \Sigma^n j_{\mathrm{un}}(\mathcal{D},\Phi)) \\
&\simeq \underline{\mathrm{Map}}_{\underline{\mathrm{BorMot}}_G^{\mathrm{p,un}}}(\lambda j_{\mathrm{un}}(\mathcal{C},\Omega), \operatorname*{colim}_{\bullet\in(\Delta^{\mathrm{op}})^n} j_{\mathrm{un}}\underline{\mathrm{Q}}_\bullet(\mathcal{D},\Phi)) \\
&\simeq \underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_G}(j_{\mathrm{un}}(\mathcal{C},\Omega), \operatorname*{colim}_{\bullet\in(\Delta^{\mathrm{op}})^n} j_{\mathrm{un}}\underline{\mathrm{Q}}_\bullet(\mathcal{D},\Phi)) \\
&\simeq \operatorname*{colim}_{\bullet\in(\Delta^{\mathrm{op}})^n} \underline{\mathrm{Map}}_{\underline{\mathrm{PSh}}_G}(j_{\mathrm{un}}(\mathcal{C},\Omega), j_{\mathrm{un}}\underline{\mathrm{Q}}_\bullet(\mathcal{D},\Phi)) \\
&\simeq \operatorname*{colim}_{\bullet\in(\Delta^{\mathrm{op}})^n} \underline{\mathrm{Map}}_{\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})}((\mathcal{C},\Omega), \underline{\mathrm{Q}}_\bullet(\mathcal{D},\Phi)) \\
&\simeq \operatorname*{colim}_{\bullet\in(\Delta^{\mathrm{op}})^n} \mathrm{Pn}\underline{\mathrm{Q}}_\bullet\Big(\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C},\Omega),(\mathcal{D},\Phi))^{h-}\Big) \\
&\simeq \Omega^\infty\Sigma^n\underline{\mathrm{GW}}_G\big(\mathrm{Fun}^{\mathrm{ex}}((\mathcal{C},\Omega),(\mathcal{D},\Phi))^{h-}\big)
\end{aligned}$$

where the second equivalence is since for $n \geq 1$, $\operatorname{colim}_{\bullet\in(\Delta^{\mathrm{op}})^n} j_{\mathrm{un}}\underline{\mathrm{Q}}_\bullet(\mathcal{D},\Phi)$ is already in $\underline{\mathrm{BorMot}}_G^{\mathrm{p,un}}$ by Lemma 5.3.5; the fourth since j_{un} is G-fully faithful; the sixth by Lemma 5.3.4; and the last by definition of $\underline{\mathrm{GW}}_G$. Hence both parametrised spectra in the statement have equivalent associated spectra, giving the desired conclusion. □

5.4 The multiplicative norms

Let G now be a 2–group.

Lemma 5.4.1. *Suppose $H \triangleleft G$ with $|G/H| = 2$, and $(\mathcal{A},\Omega) \xrightarrow{i} (\mathcal{B},\Phi)$ is a split Poincaré–Verdier inclusion in $\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})_H$. Then the following map is an equivalence in $\underline{\mathrm{BorMot}}_G^{\mathrm{p}}$.*

$$\begin{aligned}
&\mathcal{Z}((\mathcal{A},\Omega)\otimes(\mathcal{B},\Phi))\underline{\amalg}_{\mathcal{Z}(\mathrm{N}_H^G(\mathcal{A},\Omega))}\mathcal{Z}((\mathcal{A},\Omega)\otimes(\mathcal{B},\Phi)) \\
&\to \mathcal{Z}((\mathcal{A},\Omega)\otimes(\mathcal{B},\Phi)\underline{\amalg}_{\mathrm{N}_H^G(\mathcal{A},\Omega)}(\mathcal{A},\Omega)\otimes(\mathcal{B},\Phi))
\end{aligned}$$

Proof. By Corollary 4.3.5 we have the pushout

$$\begin{tikzcd}
\mathrm{Ind}_H^G\mathrm{Res}_H^G\mathrm{N}_H^G(\mathcal{A},\Omega) \arrow[r, "\varepsilon"] \arrow[d, hook] & \mathrm{N}_H^G(\mathcal{A},\Omega) \arrow[d, hook] \arrow[dl, phantom, "\ulcorner"] \\
\mathrm{Ind}_H^G(\mathcal{A},\Omega)\otimes(\mathcal{B},\Phi) \arrow[r] \arrow[u, dashed, bend right] & (\mathcal{A},\Omega)\otimes(\mathcal{B},\Phi)\underline{\amalg}_{\mathrm{N}_H^G(\mathcal{A},\Omega)}(\mathcal{A},\Omega)\otimes(\mathcal{B},\Phi) \arrow[u, dashed, bend right]
\end{tikzcd}$$

which is a split Poincaré–Verdier pushout by Corollary 5.2.2. Hence by Lemma 4.3.3 we obtain the pushout square

$$\begin{tikzcd}
\mathrm{Ind}_H^G \mathrm{Res}_H^G \mathcal{Z}(\mathrm{N}_H^G(\mathcal{A},\Omega)) \arrow[rr, "\varepsilon"] \arrow[d] & & \mathcal{Z}(\mathrm{N}_H^G(\mathcal{A},\Omega)) \arrow[d] \\
\mathrm{Ind}_H^G \mathcal{Z}((\mathcal{A},\Omega)\otimes(\mathcal{B},\Phi)) \arrow[r] & \mathcal{Z}((\mathcal{A},\Omega)\otimes(\mathcal{B},\Phi)\underline{\amalg}_{\mathrm{N}_H^G A}(\mathcal{A},\Omega)\otimes(\mathcal{B},\Phi)) \arrow[ur, phantom, "\ulcorner"] &
\end{tikzcd}$$

as desired. ☐

Proposition 5.4.2 (Cofibre distributivity of Borel norms)**.** *Let $\mathcal{C}^{\otimes}$ be a pointed cocomplete symmetric monoidal category whose tensor product is bicontinuous (ie. commutes with arbitrary colimits in each variable). Suppose $X \to Y \to Z$ is a cofibre sequence in $\underline{\mathrm{Bor}}(\mathcal{C}^{\otimes})_H$ where $H \leq G$ is a subgroup of index 2. Then we obtain the following cofibre sequence in $\underline{\mathrm{Bor}}(\mathcal{C}^{\otimes})_G$*

$$X \otimes Y \underline{\amalg}^{G/H}_{X\otimes X} Y \otimes X \to \mathrm{N}_H^G Y \to \mathrm{N}_H^G Z$$

Proof. First note that $\mathrm{Res}_e^G(X \otimes Y \underline{\amalg}^{G/H}_{X\otimes X} Y \otimes X) \simeq X \otimes Y \amalg_{X\otimes X} Y \otimes X$, ie. the underlying object of G/H–pushouts are just ordinary pushouts. Secondly, since $\mathrm{fgt} : \mathrm{Fun}(BH, \mathcal{C}) \to \mathcal{C}$ preserves colimits, in particular, it reflects cofibres. Hence to check that the sequence in question is cofibre, it suffices to verify it on the underlying sequence in $\mathcal{C}$, forgetting the equivariance. Now it is a standard consequence of the bicontinuity of the tensor product. ☐

We now have all the ingredients we need to mimic the arguments in §4.3.4.

Proposition 5.4.3. *Let $H \triangleleft G$ be a normal subgroup of index 2. Then N_H^G sends the morphism $t^* : \mathcal{U}(\mathrm{Sp}_\omega^{\Delta^1}, \Omega^u_{\mathrm{met}})/\mathcal{U}(\mathrm{Sp}^\omega, \Omega\Omega^u) \to \mathcal{U}(\mathrm{Sp}^\omega, \Omega^u)$ in $\mathcal{R}_{\mathrm{Bor}}$ to a morphism in $\overline{\mathcal{R}}_{\mathrm{Bor}}$.*

Proof. Recall that we have the split Verdier sequence

$$\begin{tikzcd}
\mathrm{Sp}_\omega \arrow[r, hook, "s_*"] \arrow[r, bend left=70, "s_!"] \arrow[r, bend right=70, dashed] & \mathrm{Sp}_\omega^{\Delta^1} \arrow[l, bend right=30, "s^*"'] \arrow[l, bend left=30] \arrow[r, two heads, "t^*"] \arrow[r, bend left=70] & \mathrm{Sp}_\omega \arrow[l, bend right=30, "t_!"'] \arrow[l, bend left=30]
\end{tikzcd}$$

which underlies the split Poincaré–Verdier sequence

$$(\mathrm{Sp}^\omega, \Omega\Omega^u) \xrightarrow{s_*} \mathrm{Met}(\mathrm{Sp}^\omega, \Omega^u) \xrightarrow{t^*} (\mathrm{Sp}^\omega, \Omega^u)$$

Recall moreover that, by design of the temabolics, we have the following square

$$\begin{tikzcd}
(\mathrm{Sp}^\omega, \Omega Ϙ^u) \arrow[r, hook, "s_!"] \arrow[d, equal] & (\mathrm{Sp}_\omega^{\Delta^1}, Ϙ^u_{\mathrm{tem}}) \arrow[d, "\mathrm{cofib}"', "\simeq"] \\
(\mathrm{Sp}^\omega, \Omega Ϙ^u) \arrow[r, hook, "s_*"] & (\mathrm{Sp}_\omega^{\Delta^1}, Ϙ^u_{\mathrm{met}}) = \mathrm{Met}(\mathrm{Sp}^\omega, Ϙ^u)
\end{tikzcd}$$

Hence applying N_H^G to the whole square, we get in turn the diagram

$$\begin{tikzcd}
(\mathrm{Sp}_\omega^{\Delta^1}, \Omega Ϙ^u_{\mathrm{tem}}) \underline{\amalg}^{s_!}_{(\mathrm{Sp}^\omega, \Omega Ϙ^u)} (\mathrm{Sp}_\omega^{\Delta^1}, \Omega Ϙ^u_{\mathrm{tem}}) \arrow[r, hook, "e_!"] \arrow[d, "\simeq"] & \mathrm{N}_H^G(\mathrm{Sp}_\omega^{\Delta^1}, Ϙ_{\mathrm{tem}}) \arrow[d, "\mathrm{cofib}"', "\simeq"] \\
(\mathrm{Sp}_\omega^{\Delta^1}, \Omega Ϙ^u_{\mathrm{met}}) \underline{\amalg}^{s_*}_{(\mathrm{Sp}^\omega, \Omega Ϙ^u)} (\mathrm{Sp}_\omega^{\Delta^1}, \Omega Ϙ^u_{\mathrm{met}}) \arrow[r] & \mathrm{N}_H^G(\mathrm{Sp}_\omega^{\Delta^1}, Ϙ^u_{\mathrm{met}}) = \mathrm{N}_H^G \mathrm{Met}(\mathrm{Sp}^\omega, Ϙ^u)
\end{tikzcd}$$

where the G/H-pushout on the top left is with respect to the $s_!$ diagram and the bottom left is with respect to the s_* diagram. By Lemma 5.2.5, the top arrow $e_!$ is a split Poincaré–Verdier inclusion, hence so is the bottom arrow.

Therefore, the conclusion of the previous paragraph together with the G/H-distributivity of N_H^G Proposition 5.4.2 yields the following fibre sequence

$$\begin{tikzcd}
(\mathrm{Sp}_\omega^{\Delta^1}, \Omega Ϙ^u_{\mathrm{met}}) \underline{\amalg}^{s_*}_{(\mathrm{Sp}^\omega, \Omega Ϙ)} (\mathrm{Sp}_\omega^{\Delta^1}, \Omega Ϙ^u_{\mathrm{met}}) \arrow[r, hook] & \mathrm{N}_H^G(\mathrm{Sp}_\omega^{\Delta^1}, Ϙ^u_{\mathrm{met}}) \arrow[r, two heads, "(t\times t)^*"] & \mathrm{N}_H^G \underline{\mathrm{Sp}}_H \simeq \mathrm{N}_H^G(\mathrm{Sp}^\omega, Ϙ^u) \arrow[l, dashed, bend right, "(t\times t)_!"'] \arrow[l, dashed, bend left]
\end{tikzcd}$$

which is automatically split Poincaré–Verdier since the right hand functor admits the fully faithful dashed adjoints. Hence by definition of the motivic localisation λ, the diagonal map in the diagram

$$\begin{tikzcd}
\dfrac{\mathcal{U}(\mathrm{N}_H^G(\mathrm{Sp}_\omega^{\Delta^1}, Ϙ^u_{\mathrm{met}}))}{\mathcal{U}(\mathrm{Sp}_\omega^{\Delta^1}, \Omega Ϙ^u_{\mathrm{met}}) \underline{\amalg}^{s_*}_{\mathcal{U}(\mathrm{Sp}^\omega, \Omega Ϙ^u)} \mathcal{U}(\mathrm{Sp}_\omega^{\Delta^1}, \Omega Ϙ^u_{\mathrm{met}})} \arrow[r] \arrow[d] & \mathcal{U}(\mathrm{N}_H^G(\mathrm{Sp}^\omega, Ϙ^u)) \\
\dfrac{\mathcal{U}(\mathrm{N}_H^G(\mathrm{Sp}_\omega^{\Delta^1}, Ϙ^u_{\mathrm{met}}))}{\mathcal{U}\big((\mathrm{Sp}_\omega^{\Delta^1}, \Omega Ϙ^u_{\mathrm{met}}) \underline{\amalg}^{s_*}_{(\mathrm{Sp}^\omega, \Omega Ϙ^u)} (\mathrm{Sp}_\omega^{\Delta^1}, \Omega Ϙ^u_{\mathrm{met}})\big)} \arrow[ur]
\end{tikzcd}$$

is a morphism in $\mathcal{R}_{\mathrm{Bor}}$. So to show that the top horizontal map is in $\overline{\mathcal{R}}_{\mathrm{Bor}}$, it will suffice to show that the left vertical map is in $\overline{\mathcal{R}}_{\mathrm{Bor}}$: this is merely the observation that we have, by definition a map of cofibre sequences in $\underline{\mathrm{PSh}}_G^{\mathrm{Sp}}(\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}}))$

$$\begin{tikzcd}
\mathcal{U}(\mathrm{Sp}^{\Delta^1}_{\omega}, \Omega Ϙ^u_{\mathrm{met}}) \underline{\amalg}^{s*}_{\mathcal{U}(\mathrm{Sp}^{\omega}, \Omega Ϙ^u)} \mathcal{U}(\mathrm{Sp}^{\Delta^1}_{\omega}, \Omega Ϙ^u_{\mathrm{met}}) \arrow[r] \arrow[d] & \mathcal{U}(\mathrm{N}^G_H(\mathrm{Sp}^{\Delta^1}_{\omega}, Ϙ^u_{\mathrm{met}})) \arrow[r] \arrow[d, equal] & \frac{\mathcal{U}(\mathrm{N}^G_H(\mathrm{Sp}^{\Delta^1}_{\omega}, Ϙ^u_{\mathrm{met}}))}{\mathcal{U}(\mathrm{Sp}^{\Delta^1}_{\omega}, \Omega Ϙ^u_{\mathrm{met}}) \underline{\amalg}^{s*}_{\mathcal{U}(\mathrm{Sp}^{\omega}, \Omega Ϙ^u)} \mathcal{U}(\mathrm{Sp}^{\Delta^1}_{\omega}, \Omega Ϙ^u_{\mathrm{met}})} \arrow[d] \\
\mathcal{U}\big((\mathrm{Sp}^{\Delta^1}_{\omega}, \Omega Ϙ^u_{\mathrm{met}}) \underline{\amalg}^{s*}_{(\mathrm{Sp}^{\omega}, \Omega Ϙ^u)} (\mathrm{Sp}^{\Delta^1}_{\omega}, \Omega Ϙ^u_{\mathrm{met}})\big) \arrow[r] & \mathcal{U}(\mathrm{N}^G_H(\mathrm{Sp}^{\Delta^1}_{\omega}, Ϙ^u_{\mathrm{met}})) \arrow[r] & \frac{\mathcal{U}(\mathrm{N}^G_H(\mathrm{Sp}^{\Delta^1}_{\omega}, Ϙ^u_{\mathrm{met}}))}{\mathcal{U}\big((\mathrm{Sp}^{\Delta^1}_{\omega}, \Omega Ϙ^u_{\mathrm{met}}) \underline{\amalg}^{s*}_{(\mathrm{Sp}^{\omega}, \Omega Ϙ)} (\mathrm{Sp}^{\Delta^1}_{\omega}, \Omega Ϙ^u_{\mathrm{met}})\big)}
\end{tikzcd}$$

and the left vertical is in $\overline{\mathcal{R}}_{\mathrm{Bor}}$ by Lemma 5.4.1, and hence the right vertical is in $\overline{\mathcal{R}}_{\mathrm{Bor}}$ too. □

Lemma 5.4.4. *Let $H \triangleleft G$ be a normal subgroup of index 2. Then N^G_H preserves morphisms in $\overline{\mathcal{R}}_{\mathrm{Bor}}$.*

Proof. By Lemma 4.3.14, it suffices to show that N^G_H sends morphisms in $\mathcal{R}_{\mathrm{Bor}}$ to morphisms in $\overline{\mathcal{R}}_{\mathrm{Bor}}$. Now for any $(\mathcal{C}, Ϙ) \in \underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})_H$, by [CDH+20a, Rmk. 7.5.8] we have the following identification of the standard sequence

$$\Big((\mathcal{C}, \Omega Ϙ) \xrightarrow{s} \mathrm{Met}(\mathcal{C}, Ϙ) \xrightarrow{t} (\mathcal{C}, Ϙ)\Big) \simeq \Big((\mathrm{Sp}^{\omega}, \Omega Ϙ^u) \xrightarrow{s} \mathrm{Met}(\mathrm{Sp}^{\omega}, Ϙ^u) \xrightarrow{t} (\mathrm{Sp}^{\omega}, Ϙ^u)\Big) \otimes (\mathcal{C}, Ϙ)$$

Therefore, we obtain that N^G_H sends the following morphism in $\mathcal{R}_{\mathrm{Bor}}$

$$\Big(\mathcal{U}(\mathrm{Met}(\mathcal{C}, Ϙ))/\mathcal{U}(\mathcal{C}, \Omega Ϙ) \xrightarrow{t} \mathcal{U}(\mathcal{C}, Ϙ)\Big) \simeq \Big(\mathcal{U}(\mathrm{Met}(\mathrm{Sp}^{\omega}, Ϙ^u))/\mathcal{U}(\mathrm{Sp}^{\omega}, \Omega Ϙ^u) \to \mathcal{U}(\mathrm{Sp}^{\omega}, Ϙ^u)\Big) \otimes \mathcal{U}(\mathcal{C}, Ϙ)$$

to a morphism in $\overline{\mathcal{R}}_{\mathrm{Bor}}$ by the ordinary symmetric monoidality of λ and Proposition 5.4.3. But since the collection of morphisms in $\mathcal{R}_{\mathrm{Bor}}$ can be taken to be of this form again by [CDH+20b, Rmk. 2.7.6 (ii)], we are done. □

Theorem 5.4.5. *Let G be a 2-group. Then $\mathcal{R}_{\mathrm{Bor},\kappa}$ is closed under the multiplicative norms, and so $\lambda : \underline{\mathrm{PSh}}_G(\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})) \longrightarrow \underline{\mathrm{BorMot}}^{\mathrm{p}}_G$ canonically refines to a G–symmetric monoidal localisation.*

Proof. Exactly as in Theorem 4.3.19, replacing Lemma 4.3.17 with Lemma 5.4.4. □

Combining with Proposition 3.3.6 then yields:

Corollary 5.4.6. *Let G be a 2–group. Then $\underline{\mathrm{GW}}_G : \underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}}) \longrightarrow \underline{\mathrm{Sp}}_G$ refines to the structure of a G–lax symmetric monoidal functor. Hence, it induces the functor*

$$\underline{\mathrm{GW}}_G \colon \mathrm{Fun}(BG, \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{p}}))^{\simeq} \simeq \mathrm{CAlg}_G(\underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}}))^{\simeq} \longrightarrow \mathrm{CAlg}_G(\underline{\mathrm{Sp}}_G)^{\simeq}$$

This means that, if $(\mathcal{C}, Ϙ) \in \mathrm{Fun}(BG, \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{p}}))$ is a symmetric monoidal Poincaré category equipped with a G–action, then

$$\{\mathrm{GW}(\mathcal{C}^{hH}, Ϙ^{hH})\}_{H \leq G}$$

assembles canonically to a G–normed ring spectrum.

Chapter 6

Equivariant Goodwillie calculus

In this chapter, we translate the general setup and some results of [Dot17] into the framework of parametrised homotopy theory in preparation for the final chapter on genuine equivariant hermitian K–theory. While most, if not all, the equivariant results here are due to [Dot17], we have however chosen also to cite the various parts of [Lur17] in the statements of the results to indicate that we have used Lurie's formulations and methods for their proofs. Moreover, as already indicated in the general introduction to this thesis, while we claim no originality in this chapter, we have chosen to present many of the proofs here in the form of pure Kan extension astrology (a terminology we learnt from Shachar Carmeli in reference to the upper and lower star notations!) which might be of independent interest: the key observation here is to exploit that various adjunctions already exist at the level of the indexing posets.

After working out some cube yoga in §6.2, we record several basic observations about equivariant excisiveness in §6.3. We then prove the formula for equivariant Goodwillie approximations in the next two sections before rounding out the chapter with multilinearity matters in §6.6.

6.1 Definitions and basic constructions

Let G be a finite group throughout.

Construction 6.1.1. Let J be a finite G–set. Then the poset $\mathrm{Pos}(J)$ of subsets of J has a G–action, and so is an object in $\mathrm{Fun}(BG, \mathrm{Cat}^{(1)})$. We then write the associated G–category $\underline{\mathrm{Pos}}_G(J)$ for the image of this object under the composition $\mathrm{Fun}(BG, \mathrm{Cat}^{(1)}) \hookrightarrow \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathrm{Cat}^{(1)}) \hookrightarrow \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathrm{Cat})$. Concretely, the fibre over G/H is just $\mathrm{Pos}(J)^H$.

Notation 6.1.2. Let J be a finite G–set and $H \leq G$ a subgroup. We will be interested in the following H–subcategories:

- Write $\underline{\mathrm{Orb}}_H(J) \subseteq \underline{\mathrm{Pos}}_H(J)$ for the full subcategory spanned by the subsets of some H–orbit $o \in J/H$. Be warned that the general inclusion $\underline{\mathrm{Orb}}_H(J) \subseteq \mathrm{Res}^G_H \underline{\mathrm{Orb}}_G(J)$ is not an equivalence when $H \lneq G$ since being a transitive G–orbit is never the same as being a transitive H–orbit when $H \lneq G$.
- Let $\underline{\mathrm{Pos}}^{\varnothing}_H(J) \subseteq \underline{\mathrm{Pos}}_H(J)$ where $\varnothing$ is removed.

Definition 6.1.3 ([Dot17, Def. 2.1])**.** Let J be a finite G–set and $H \leq G$ a subgroup. Let $\underline{\mathcal{C}}$ be a H–category and $X : \underline{\mathrm{Pos}}_H(J) \to \underline{\mathcal{C}}$ be a H–diagram (such a datum is also called a J–cube). We say that it is:

(i) *G–strongly cocartesian* if it is the H–left Kan extension of $X|_{\mathrm{Res}^G_H \underline{\mathrm{Orb}}_G(J)}$.

(ii) *H–cartesian* if it is a H–limit diagram over $X|_{\underline{\mathrm{Pos}}^{\varnothing}_H(J)}$.

Remark 6.1.4. There is an asymmetry in the definition above. While being H–cartesian is the expected definition, G–strong cocartesianness always refers to being H–left Kan extended from $\mathrm{Res}^G_H \underline{\mathrm{Orb}}_G(J)$. This is due to the fact that $\mathrm{Res}^G_H \underline{\mathcal{O}}_G(J) \not\simeq \underline{\mathrm{Orb}}_H(J)$, as noted above.

Definition 6.1.5 ("[Dot17, Def. 2.10]")**.** Let J be a finite G–set, $\underline{\mathcal{C}}, \underline{\mathcal{D}} \in \mathrm{Cat}_G$ and $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ a G–functor. We say that F is J–excisive if $\mathrm{Res}^G_H F\colon \mathrm{Res}^G_H \underline{\mathcal{C}} \to \mathrm{Res}^G_H \underline{\mathcal{D}}$ sends G–strongly cocartesian $\underline{\mathrm{Pos}}_H(J)$–diagrams to H–cartesian $\underline{\mathrm{Pos}}_H(J)$–diagrams.

6.2 Basic cube yoga

Definition 6.2.1 (Face cubes, "[Lur17, Def. 6.1.1.12]")**.** Let J be a finite G–set, $H \leq G$ a subgroup, and $T \subseteq J$ a H–subset. Suppose we have a decomposition $J = U \coprod T \coprod V$ of finite H–sets. We then have a map of H–posets

$$\xi_U : \underline{\mathrm{Pos}}_H(T) \longrightarrow \underline{\mathrm{Pos}}_H(J) \quad :: \quad T_0 \mapsto (U \coprod T_0)$$

where $T_0 \subseteq T$. Given a J–cube $X : \underline{\mathrm{Pos}}_H(J) \to \underline{\mathcal{C}}$ we can precompose to get a T–cube

$$\underline{\mathrm{Pos}}_H(T) \xrightarrow{\xi_U} \underline{\mathrm{Pos}}_H(J) \xrightarrow{X} \underline{\mathcal{C}}$$

which we refer to as T*–faces* of X (note this depend on the H–decomposition of J).

Proposition 6.2.2 ("[Lur17, Prop. 6.1.1.13]")**.** *Let $\underline{\mathcal{C}}$ be a G-cocomplete category. Let J be a finite G-set and $T \subseteq J$ a G-subset, and let $X : \underline{\mathrm{Pos}}_G(J) \to \underline{\mathcal{C}}$ be a J–cube. Then:*

(1) If X is strongly G-cocartesian then every T–face of X is too.

(2) If every T-face of X is G-cartesian, then X is G-cartesian.

Proof. For (1), suppose X is G-strongly cocartesian. Choose a G-decomposition $J = U \amalg T \amalg V$, and let $Y := X \circ \xi_U \colon \underline{\mathrm{Pos}}_G(T) \to \underline{\mathcal{C}}$ be the corresponding T-face of X as constructed above. We want to show that Y is a G-left Kan extension of $Y|_{\underline{\mathrm{Orb}}_G(T)}$. Here recall that ξ_U is the H–functor

$$\xi_U : \underline{\mathrm{Pos}}_G(T) \longrightarrow \underline{\mathrm{Pos}}_G(S) \quad :: \quad T_0 \mapsto U \coprod T_0$$

For this, let $I \subseteq \underline{\mathrm{Pos}}_G(U \amalg T)$ be the subcategory on those subsets $I \subseteq J$ such that $I \cap T \in \underline{\mathrm{Orb}}_G(T)$. Note that we have adjunctions

$$h : \underline{\mathrm{Pos}}_G(U \amalg T) \rightleftarrows \underline{\mathrm{Pos}}_G(T) : \xi_U \quad \text{and} \quad \overline{h} : I \rightleftarrows \underline{\mathrm{Orb}}_G(T) : \overline{\xi}_U$$

given by $h : W \mapsto W \cap T$ and $\overline{h} : W \mapsto W \cap T$. Now consider the diagram

$$\begin{tikzcd}
\underline{\mathrm{Pos}}_G(T) \arrow[r, shift right, "\xi_U"'] \arrow[rr, bend left, "Y"] & \underline{\mathrm{Pos}}_G(U \amalg T) \arrow[l, shift right, "h"'] \arrow[r, "X"] & \mathcal{C} \\
\underline{\mathrm{Orb}}_G(T) \arrow[u, hook, "i"] \arrow[r, shift right, "\overline{\xi}_U"'] & I \arrow[u, hook, "j"'] \arrow[l, shift right, "\overline{h}"'] & \\
 & \underline{\mathrm{Orb}}_G(U \amalg T) \arrow[u, hook, "k"'] &
\end{tikzcd}$$

where both squares commute and all the vertical maps are the obvious inclusions. In particular, since we have adjunctions

$$\xi_U^* : \underline{\mathrm{Fun}}_G(\underline{\mathrm{Pos}}_G(U \amalg T), \underline{\mathcal{C}}) \rightleftarrows \underline{\mathrm{Fun}}_G(\underline{\mathrm{Pos}}_G(T), \underline{\mathcal{C}}) : h^*$$

$$\overline{\xi}_U^* : \underline{\mathrm{Fun}}_G(I, \underline{\mathcal{C}}) \rightleftarrows \underline{\mathrm{Fun}}_G(\underline{\mathrm{Orb}}_G(T), \underline{\mathcal{C}}) : \overline{h}^*$$

we get that $i_! \overline{\xi}_U^* \simeq i_! \overline{h}_! \simeq h_! j_! \simeq \xi_U^* j_!$. Moreover, by definition of G-strongly cocartesianness, we have $j_! k_! k^* j^* X \simeq X$, and so

$$j^* X \simeq j^* j_! k_! k^* j^* X \simeq k_! k^* j^* X$$

where the last equivalence is since j was fully faithful, and so $j^* j_! \simeq \mathrm{id}$ always. Therefore

$$j_! j^* X \simeq j_! k_! k^* j^* X \simeq X$$

With these observations in place, what we want to show is then that $i_! i^* Y \simeq Y$, and for this just consider the equation

$$i_! i^* Y = i_! i^* \xi_U^* X \simeq i_! \overline{\xi}_U^* j^* X \simeq \xi_U^* j_! j^* X \simeq \xi_U^* X = Y$$

where all the equivalences are by our observations above.

For (2), suppose every T-face of X is cartesian. We want to show that $X(\emptyset)$ is the G–limit of $X|_{\underline{\mathrm{Pos}}_G^{\emptyset}(J)}$. We will proceed in two stages. Let $\mathcal{M} \subseteq \underline{\mathrm{Pos}}_G^{\emptyset}(J)$ be the subposet of all subsets that intersect nontrivially with T. The claim in the first stage is that $X|_{\underline{\mathrm{Pos}}_G^{\emptyset}(J)}$ is a G-right Kan extension of $X|_{\mathcal{M}}$. That is, considering the diagram

$$\mathcal{M} \xhookrightarrow{j} \underline{\mathrm{Pos}}_G^{\emptyset}(J) \xhookrightarrow{i} \underline{\mathrm{Pos}}_G(J) \xrightarrow{X} \underline{\mathcal{C}}$$

we want to show that the canonical map $i^*X \to j_*j^*i^*X$ is an equivalence. Since equivalences are checked pointwise, this is equivalent to showing that

$$(\xi_U^{\emptyset})^* i^* X \to (\xi_U^{\emptyset})^* j_* j^* i^* X$$

is an equivalence for all nonempty subsets $\emptyset \neq U \subseteq J$ such that $U \cap T = \emptyset$ (ie. $U \notin \mathcal{M}$), where $\xi_U^{\emptyset} : \underline{\mathrm{Pos}}_G(T) \to \underline{\mathrm{Pos}}_G^{\emptyset}(J)$ is the factorisation of $\xi_U : \underline{\mathrm{Pos}}_G(T) \to \underline{\mathrm{Pos}}_G(J)$. To see this, let us have a nonempty subset $\emptyset \neq U \subseteq J$ such that $U \cap T = \emptyset$. We want to show that $X(U)$ is computed as a G-right Kan extension of $X|_{\mathcal{M}}$. For this consider the diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[r, hook, "j"] & \underline{\mathrm{Pos}}_G^{\emptyset}(J) \arrow[dr, hook, "i"] & & \\
\mathcal{M}_{U/} \arrow[u, hook, "\overline{\ell}_U"] \arrow[r, hook, "j_U"] \arrow[ur, phantom, "\lrcorner", very near start] & \underline{\mathrm{Pos}}_G^{\emptyset}(J)_{U/} \arrow[u, hook, "\ell_U"'] \arrow[r, hook, "i_U"] \arrow[d, dashed, shift left, "\overline{\eta}_U"] & \underline{\mathrm{Pos}}_G(J) \arrow[r, "X"] & \underline{\mathcal{C}} \\
\underline{\mathrm{Pos}}_G^{\emptyset}(T) \arrow[u, hook, "\overline{\overline{\xi}}_U"] \arrow[r, hook, "k"'] & \underline{\mathrm{Pos}}_G(T) \arrow[u, hook, shift left, "\overline{\xi}_U"] \arrow[ur, "\xi_U"'] & &
\end{tikzcd}$$

enjoying the following list of properties:

- all the squares commute,
- the top square is a pullback and therefore by the pointwise formula for right Kan extensions, we have that the transformation $(\ell_U)^* j_* \Rightarrow (j_U)_*(\overline{\ell}_U)^*$ is an equivalence,
- we have adjunctions $\overline{\overline{\xi}}_U \dashv \overline{\overline{\eta}}_U$ and $\overline{\xi}_U \dashv \overline{\eta}_U$ given for example by $\overline{\xi}_U : L \mapsto L \coprod U$ and $\overline{\eta}_U : N \mapsto N \cap T$.
- $\xi_U^{\emptyset} \simeq \ell_U \circ \overline{\xi}_U$.

Now we use similar adjunction manoeuvres as before, namely note that $\overline{\xi}^*_U \simeq \overline{\eta}_{U*}$ and $\overline{\overline{\xi}}^*_U \simeq \overline{\overline{\eta}}_{U*}$. Then

$$\begin{aligned}
(\xi_U^{\emptyset})^* j_* j^* i^* X &\simeq \overline{\xi}^*_U \ell^*_U j_* j^* i^* X \\
&\simeq \overline{\xi}^*_U j_{U*} \overline{\ell}^*_U j^* i^* X \\
&\simeq \overline{\xi}^*_U j_{U*} j^*_U i^*_U X \\
&\simeq \overline{\eta}_{U*} j_{U*} j^*_U i^*_U X \\
&\simeq k_* \overline{\overline{\eta}}_{U*} j^*_U i^*_U X \\
&\simeq k_* \overline{\overline{\xi}}^*_U j^*_U i^*_U X \\
&\simeq k_* k^* \overline{\xi}^*_U i^*_U X \\
&\simeq k_* k^* \xi^*_U X \\
&\simeq \xi^*_U X \\
&\simeq (\xi^{\emptyset}_U)^* i^* X
\end{aligned}$$

where the second last equivalence is by the T–face cartesianness of X, hence the claim.

For the next stage, recall that ultimately we want to show that the canonical map $X \to i_* i^* X$ is an equivalence. But note that since the inclusion $i : \underline{\mathrm{Pos}}^{\emptyset}_H(J) \hookrightarrow \underline{\mathrm{Pos}}_H(J)$ only adds the empty subset and so if can show that $\xi^*_{\emptyset} X \to \xi^*_{\emptyset} i_* i^* X$ is an equivalence, then we would be done. In this case, we would need a similar diagram as in the previous stage adjusted by the fact that the case $U = \emptyset$ is special:

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "j"] \arrow[d, dashed, shift left, "\overline{\overline{\eta}}_{\emptyset}"] & \underline{\mathrm{Pos}}^{\emptyset}_G(J) \arrow[r, hook, "i"] & \underline{\mathrm{Pos}}_G(J) \arrow[r, "X"] \arrow[d, dashed, shift left, "\eta_{\emptyset}"] & \underline{\mathcal{C}} \\
\underline{\mathrm{Pos}}^{\emptyset}_G(T) \arrow[u, hook, shift left, "\overline{\overline{\xi}}_{\emptyset}"] \arrow[rr, hook, "k"'] & & \underline{\mathrm{Pos}}_G(T) \arrow[u, hook, shift left, "\xi_{\emptyset}"] &
\end{tikzcd}$$

For this just consider the equation

$$\begin{aligned}
\xi^*_{\emptyset} i_* i^* X &\simeq \xi^*_{\emptyset} i_* j_* j^* i^* X \\
&\simeq \eta_{\emptyset *} i_* j_* j^* i^* X \\
&\simeq k_* \overline{\overline{\eta}}_{\emptyset *} j^* i^* X \\
&\simeq k_* \overline{\overline{\xi}}^*_{\emptyset} j^* i^* X \\
&\simeq k_* k^* \xi^*_{\emptyset} X \\
&\simeq \xi^*_{\emptyset} X
\end{aligned}$$

where the first equivalence is by the first stage, and the last equivalence is by the T–face cartesianness of X using $U = \emptyset$. This completes the proof. $\square$

Lemma 6.2.3 (Pushout criterion of strong cocartesianness, "[Lur17, Prop. 6.1.1.15]"). *Let $X : \underline{\mathrm{Pos}}_G(J_+) \to \underline{\mathcal{C}}$ be an J_+-cube. Then the following are equivalent:*

(1) The J_+-cube X is G–strongly cocartesian.
(2) For every subgroup $H \leq G$ and every pair of H-invariant subsets $T, T' \subseteq J_+$ such that $T \cup T'$ has more than one G-orbit, the diagram

$$\begin{tikzcd} X(T\cap T') \arrow[r] \arrow[d] & X(T) \arrow[d] \\ X(T') \arrow[r] & X(T\cup T') \end{tikzcd}$$

is a fibrewise H-pushout.

Proof. To see (1) implies (2), note first that if $T = T \cup T'$ or $T' = T \cup T'$, then the statement is trivially true. So suppose this is not the case. We consider a couple of auxiliary H-subposets:

- $P \subseteq \underline{\mathrm{Pos}}_H(J_+)$ is given by $P = \underline{\mathrm{Pos}}_H(J_+)_{/T} \cup \underline{\mathrm{Pos}}_H(J_+)_{/T'}$, namely, the poset of subsets which are either in T or in T'.
- $P_1 := \underline{\mathrm{const}}_H(\Lambda^2_0) \simeq (\Lambda^2_0 \times \mathcal{O}^{\mathrm{op}}_H \to \mathcal{O}^{\mathrm{op}}_H)$ given by $\{T, T', T\cap T'\}$.

Now note that the inclusion $j : P_1 \subseteq P$ admits a H-left adjoint

$$\tau : P \to P_1 \quad :: \quad U \mapsto \min_{Y \in \{T, T', T\cap T'\}} (Y \supseteq U)$$

Now consider the diagram where all the squares commute

$$\begin{tikzcd} P_1^{\trianglerighteq} & P^{\trianglerighteq} \arrow[l, "\overline{\tau}"'] \arrow[r, "X"] & \underline{\mathcal{C}} \\ P_1 \arrow[u, hook, "t"] & P \arrow[l, "\tau"'] \arrow[u, hook, "w"'] \end{tikzcd}$$

with $\overline{j} : P_1^{\trianglerighteq} \hookrightarrow P^{\trianglerighteq}$ and $j : P_1 \hookrightarrow P$,

where $P_1^{\trianglerighteq}$ and $P^{\trianglerighteq}$ are thought of as the subcategories where $T \cup T'$ has been added. What we want to show is that $t_!t^*\overline{j}^*X \to \overline{j}^*X$ is an equivalence. And for this, just consider

$$\begin{aligned} t_!t^*\overline{j}^*X &\simeq t_!j^*w^*X \\ &\simeq t_!\tau_!w^*X \\ &\simeq \overline{\tau}_!w_!w^*X \\ &\simeq \overline{\tau}_!X \simeq \overline{j}^*X \end{aligned}$$

where the fourth equivalence is since $w_! w^* X \simeq X$ as X was G-strongly cocartesian. This completes this part.

To see (2) implies (1), writing $k : \underline{\mathrm{Orb}}_G(J_+) \hookrightarrow \underline{\mathrm{Pos}}_G(J_+)$ for the G-inclusion, we need to show that $k_! k^* X \to X$ is an equivalence. Since equivalences are tested pointwise, we are reduced to show the following: for each $H \leq G$ and $T \in \underline{\mathrm{Pos}}_H(J_+)$, writing $T : \underline{*} \to \underline{\mathrm{Pos}}_H(J_+)$ for the H–object T, we have that

$$T^* k_! k^* X \to T^* X$$

is an equivalence. If T were already in $\mathrm{Res}^G_H \underline{\mathrm{Orb}}_G(J_+)$ then since $k^* k_! k^* X \simeq k^* X$ (because k was G-fully faithful), the statement is true. We now prove the general case by induction on the size of T. Since T intersects non-trivially with more than one G-orbit, we can write $T = T' \cup T''$ where $0 < |T'|, |T''| < |T|$. Because (1) implies (2), we get that $k_! k^* X$ - which is G–strongly cocartesian by definition - satisfies the pushout property in (2). Moreover, by hypothesis, X does too. Now the induction hypotheses give that $k_! k^* X \to X$ is an equivalence on $T' \cap T'', T', T''$, and so by the pushout properties of both $k_! k^* X$ and X, it must be an equivalence on $T = T' \cup T''$ also, as required. □

6.3 Basic results on equivariant excisiveness

Equipped with the cube yoga of the previous section, we are now ready to deduce some basic theory on equivariant excisiveness.

Corollary 6.3.1. *Let J be a finite G–set and $K \subseteq J$ be a G–invariant subset. Then every K-excisive G–functor is also J-excisive.*

Proof. Let $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a K-excisive G–functor. Write a G–decomposition $J = I \coprod K$. Let $X : \underline{\mathrm{Pos}}_H(J_+) \to \underline{\mathcal{C}}$ be G–strongly cocartesian. By Proposition 6.2.2 (1), all K-faces are also G–strongly cocartesian. Since F was K-excisive, all the K-faces of FX are K-cartesian, and hence FX is G-cartesian by Proposition 6.2.2 (2). Hence, FX is G–cartesian. □

The proof of the following result involves some manoeuvring with cofinality arguments which we will not reproduce here. We refer the reader to [Dot17] directly, and especially the arguments in [DM16, Props. A.1 - A.3] where this is proved. All the cofinality arguments involve only 1–categories since they are about the cofinality of $\underline{\mathrm{Pos}}_G(K_+) \to \underline{\mathrm{Pos}}_G(J_+)$ when $K \to J$ is a surjective map of finite G–sets.

Lemma 6.3.2 ([Dot17, Prop. 2.15])**.** *Let $p : K \to J$ be a map of finite G–sets inducing an isomorphism on G–orbits. Then any K-excisive G–functor $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is also J-excisive. In particular, any K-excisive G–functor is $|K/G|$-excisive, in the sense of ordinary Goodwillie calculus.*

The following is now an immediate consequence of Corollary 6.3.1 and Lemma 6.3.2.

Theorem 6.3.3 ([Dot17, Cor. 2.16]). *Let $p : K \to J$ be a map of finite G–sets inducing an injection on G–orbits. Then any K-excisive G–functor $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is also J-excisive.*

In the remainder of this section, we will work towards showing that the notion of equivariant excisiveness of free G–sets is compatible with restrictions. This is a special case of [Dot17, Prop. 2.34]. First of all, we record the following lemma whose proof is immediate.

Lemma 6.3.4. *Let $H \leq G$ be a subgroup. Then there is a H–adjunction*

$$i : \underline{\mathrm{Pos}}_H(nH_+) \rightleftarrows \underline{\mathrm{Pos}}_H(\mathrm{Res}^G_H nG_+) = \mathrm{Res}^G_H \underline{\mathrm{Pos}}_G(nG_+) : \pi$$

where i is the inclusion induced by the H–equivariant inclusion of H–sets $H \hookrightarrow G \cong \coprod_{G/H} H$, and $\pi(S) := i(nH_+) \cap S \subseteq nH_+$. Moreover, this is a retraction, ie. $\pi \circ i = \mathrm{id}$.

Lemma 6.3.5. *Let $\underline{\mathcal{C}}$ be a G–complete category, and let $i : \underline{I} \rightleftarrows \underline{K} : \pi$ be a G–adjunction. Then $i^* : \underline{\mathcal{C}}^{\underline{K}^{\triangleleft}} \to \underline{\mathcal{C}}^{\underline{I}^{\triangleleft}}$ sends $\underline{K}$–cartesian diagrams to $\underline{I}$–cartesian diagrams.*

Proof. The given adjunction extends to the following commuting squares of adjunctions

$$\begin{tikzcd} \underline{I} \arrow[r, shift left, "i"] \arrow[d, hook, "j"'] & \underline{K} \arrow[l, shift left, "\pi"] \arrow[d, hook, "k"] \\ \underline{I}^{\triangleleft} \arrow[r, shift left, "\bar{i}"] & \underline{K}^{\triangleleft} \arrow[l, shift left, "\bar{\pi}"] \end{tikzcd}$$

It would now suffice to show that the following square on the left commutes

$$\begin{tikzcd} \underline{\mathcal{C}}^{\underline{I}} \arrow[d, "j_*"'] & \underline{\mathcal{C}}^{\underline{K}} \arrow[l, "i^*"'] \arrow[d, "k_*"] \\ \underline{\mathcal{C}}^{\underline{I}^{\triangleleft}} & \underline{\mathcal{C}}^{\underline{K}^{\triangleleft}} \arrow[l, "\bar{i}^*"'] \end{tikzcd} \qquad \begin{tikzcd} \underline{\mathcal{C}}^{\underline{I}} \arrow[r, "\pi^*"] & \underline{\mathcal{C}}^{\underline{K}} \\ \underline{\mathcal{C}}^{\underline{I}^{\triangleleft}} \arrow[u, "j^*"] \arrow[r, "\bar{\pi}^*"'] & \underline{\mathcal{C}}^{\underline{K}^{\triangleleft}} \arrow[u, "k^*"'] \end{tikzcd}$$

Taking left adjoints, we obtain the right hand square which obviously commutes. This completes the proof. □

Proposition 6.3.6. *Let $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be an nG–excisive functor between $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ which strongly admit finite G–(co)limits. Then $\mathrm{Res}^G_H F : \mathrm{Res}^G_H \underline{\mathcal{C}} \to \mathrm{Res}^G_H \underline{\mathcal{D}}$ is nH–excisive.*

Proof. First note that the map i from Lemma 6.3.4 restricts to give the first of the following pair of commuting squares, which then in turn induces the second one

$$\begin{tikzcd}
\underline{\mathrm{Orb}}_H(nH_+) \arrow[r, "i_{\mathcal{O}}"] \arrow[d, hook, "u"'] & \mathrm{Res}^G_H \underline{\mathrm{Orb}}_G(nG_+) \arrow[d, hook, "v"] \\
\underline{\mathrm{Pos}}_H(nH_+) \arrow[r, "i"] & \mathrm{Res}^G_H \underline{\mathrm{Pos}}_G(nG_+)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Res}^G_H \underline{\mathcal{C}}^{\underline{\mathrm{Orb}}_H(nH_+)} \arrow[r, "(i_{\mathcal{O}})_!"] \arrow[d, "u_!"'] & \mathrm{Res}^G_H \underline{\mathcal{C}}^{\mathrm{Res}^G_H \underline{\mathrm{Orb}}_G(nG_+)} \arrow[d, "v_!"] \\
\mathrm{Res}^G_H \underline{\mathcal{C}}^{\underline{\mathrm{Pos}}_H(nH_+)} \arrow[r, "i_! \simeq \pi^*"'] & \mathrm{Res}^G_H \underline{\mathcal{C}}^{\mathrm{Res}^G_H \underline{\mathrm{Pos}}_G(nG_+)}
\end{tikzcd}$$

Hence, the functor π^* sends H–strongly cocartesian cubes to G–strongly cocartesian ones.

Now to prove the statement of the proposition, consider the diagram

$$\begin{tikzcd}
\mathrm{Fun}_H\left(\underline{\mathrm{Pos}}_H(nH_+), \mathrm{Res}^G_H \underline{\mathcal{C}}\right) \arrow[r, "(\mathrm{Res}^G_H F)_*"] \arrow[d, "\pi^*"'] & \mathrm{Fun}_H\left(\underline{\mathrm{Pos}}_H(nH_+), \mathrm{Res}^G_H \underline{\mathcal{D}}\right) \\
\mathrm{Fun}_H\left(\underline{\mathrm{Pos}}_H(\mathrm{Res}^G_H nG_+), \mathrm{Res}^G_H \underline{\mathcal{C}}\right) \arrow[r, "(\mathrm{Res}^G_H F)_*"] & \mathrm{Fun}_H\left(\underline{\mathrm{Pos}}_H(\mathrm{Res}^G_H nG_+), \mathrm{Res}^G_H \underline{\mathcal{D}}\right) \arrow[u, "i^*"']
\end{tikzcd}$$

which commutes since $\pi \circ i = \mathrm{id}$ from Lemma 6.3.4. Suppose we start with an X on the top left which is a H–strongly cocartesian $\underline{\mathrm{Pos}}_H(nH_+)$–diagram. By the paragraph above, its image $\pi^* X$ is a G–strongly cocartesian $\underline{\mathrm{Pos}}_H(\mathrm{Res}^G_H nG_+)$–diagram. Hence, by nG_+–excisiveness of F, its further image $(\mathrm{Res}^G_H F)_* \pi^* X$ on the bottom right is H–cartesian. But then since the functor i^* strongly preserves H–cartesian diagrams Lemma 6.3.5, we see that the final image $(\mathrm{Res}^G_H)_* X$ on the top right corner is a H–cartesian $\underline{\mathrm{Pos}}_H(nH_+)$-diagram, as required. □

6.4 Equivariant Goodwillie approximations

Definition 6.4.1. Let $\underline{\mathcal{C}}$ be a G-category. We say that it is *G-differentiable* if:

- It is G-finite-complete,
- It admits fibrewise filtered colimits,
- The fibrewise sequential colimits are G-left exact functors, that is, they strongly preserve G-finite limits.

Definition 6.4.2. Let $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ be G-categories admitting G-final objects and $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ a G-functor. We say that it is *G-reduced* if it preserves G-final objects.

Construction 6.4.3 (Goodwillie approximations, "[Lur17, Cons. 6.1.1.18, 22, 27]"). Let J be a finite G–set. There will be three important constructions: that of S–cones, of T_{J_+}, and Goodwillie's approximations P_{J_+}. Let $\underline{\mathcal{C}}$ be G-finite-cocomplete and have G-final objects.

(a) (S–cones) Let $\underline{\mathrm{Fin}}^{\mathrm{inj}}_{G\leq n} \subseteq \underline{\mathrm{Fin}}^{\mathrm{inj}}_{G}$ denote the full subcategory of finite G–sets with $\leq n$ G-orbits and injective maps. This is subcategory that really depends on G and so for a proper subgroup $H \lneq G$, $(\underline{\mathrm{Fin}}^{\mathrm{inj}}_{G\leq n})_H$ contains finite H–sets which have more than n H–orbits in general. Now for $X \in \underline{\mathcal{C}}$, we obtain a diagram $F_X \in \underline{\mathrm{Fun}}_G(\mathrm{Fin}^{\mathrm{inj}}_G, \underline{\mathcal{C}})$ by

$$X \overset{\mathrm{RKE}}{\longmapsto} F_X|_{\underline{\mathrm{Fin}}^{\mathrm{inj}}_{G\leq 1}} \overset{\mathrm{LKE}}{\longmapsto} F_X$$

Let us temporarily denote the full subcategory of $\underline{\mathrm{Fun}}_G(\underline{\mathrm{Fin}}^{\mathrm{inj}}_G, \underline{\mathcal{C}})$ for these things by $\underline{\mathrm{Fun}}^{\mathrm{cone}}_G(\underline{\mathrm{Fin}}^{\mathrm{inj}}_G, \underline{\mathcal{C}})$. Then since everything was obtained by Kan extensions, the evaluation $\mathrm{ev}_\varnothing : \underline{\mathrm{Fun}}^{\mathrm{cone}}_G(\underline{\mathrm{Fin}}^{\mathrm{inj}}_G, \underline{\mathcal{C}}) \to \underline{\mathcal{C}}$ is an equivalence. Choosing an inverse and currying we obtain the following functor, where we call $C_S(X)$ the S–cones of X.

$$\underline{\mathcal{C}} \times \underline{\mathrm{Fin}}^{\mathrm{inj}}_G \longrightarrow \underline{\mathcal{C}} \quad :: \quad (X, S) \mapsto C_S(X) := F_X(S)$$

(b) Let $\underline{\mathcal{D}}$ have finite limits and $F \colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a G–functor. We define

$$T_{J_+}F \colon \underline{\mathcal{C}} \to \underline{\mathcal{D}} \quad :: \quad X \mapsto \underline{\lim}_{S \in \underline{\mathrm{Pos}}^{\varnothing}_G(J_+)} F(C_S(X))$$

The canonical map $F(X) = F(C_\varnothing(X)) \to \underline{\lim}_{S\in\underline{\mathrm{Pos}}^{\varnothing}_G(J_+)} F(C_S(X))$ determines a transformation $\theta : F \implies T_{J_+}F$. Now $C_{(-)}(X)$ is by construction a G–strongly cocartesian J_+-cube, and so if F were J_+-excisive we would get $\underline{\lim}_{S\in\underline{\mathrm{Pos}}^{\varnothing}_G(J_+)} F(C_S(X)) \simeq F(C_\varnothing(X)) \simeq F(X)$, so that $\theta : F \Rightarrow T_{J_+}F$ is an equivalence.

(c) (Goodwillie approximations) Suppose $\underline{\mathcal{D}}$ is differentiable and $F \colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a G–functor. For each finite G–set J_+ denote by the filtered colimit

$$P_{J_+}F := \mathrm{colim}\left(F \xrightarrow{\theta_F} T_{J_+}F \xrightarrow{\theta_{T_{J_+}F}} T_{J_+}T_{J_+}F \to \cdots\right)$$

This is what we call *equivariant Goodwillie approximation*. Note that this is G–left exact by G–differentiability of $\underline{\mathcal{D}}$.

Observation 6.4.4. Here are some easy but important observations about these:

(a) $T_{1_+}F \colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is constant with value $F(*)$. If F is reduced then $T_F \simeq \Omega_{\underline{\mathcal{D}}} \circ F \circ \Sigma_{\underline{\mathcal{C}}}$

(b) Suppose we are given functors $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ and $F'\colon \underline{\mathcal{D}} \to \underline{\mathcal{E}}$ with F strongly preserving finite G–colimits and G–final objects. Then we have equivalences

$$\big(T_{J_+}(F' \circ F)\big)(-) \simeq (T_{J_+}F') \circ F \quad \text{and} \quad P_{J_+}(F' \circ F) \simeq (P_{J_+}F') \circ F$$

The second one follows immediately from the first identification. For the first one, note $T_{J_+}(F' \circ F)(-) \simeq \underline{\lim}_{S \in \underline{\mathrm{Pos}}_G^{\varnothing}(J_+)}(F' \circ F)(C_S(-)) \simeq \underline{\lim}_{S \in \underline{\mathrm{Pos}}_G^{\varnothing}(J_+)} F'(C_S F(-)) \simeq \big((T_{J_+}F') \circ F\big)(-)$.

(c) Suppose we were given functors $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ and $F'\colon \underline{\mathcal{D}} \to \underline{\mathcal{E}}$ such that F' strongly preserves finite G–limits and fibrewise sequential colimits. Then we have

$$P_{J_+}(F' \circ F) \simeq F' \circ P_{J_+}F$$

(d) If $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is reduced, then $T_{1_+}F \simeq \mathrm{colim}_m\, \Omega^m_{\underline{\mathcal{D}}} \circ F \circ \Sigma^m_{\underline{\mathcal{C}}}$.

We will now work towards proving that the Goodwillie approximations are the universal excisive approximations, and for this, we will need some preparatory lemmas.

Notation 6.4.5. Let $i : \underline{\mathrm{Pos}}_G^{\varnothing}(J) \hookrightarrow \underline{\mathrm{Pos}}_G(J)$ be the inclusion. Write $\underline{\mathrm{Fun}}_G^{\mathrm{cart}}(\underline{\mathrm{Pos}}_G(J), \underline{\mathcal{D}})$ for the full subcategory consisting of the image of $i_* : \underline{\mathrm{Fun}}_G(\underline{\mathrm{Pos}}_G^{\varnothing}(J), \underline{\mathcal{D}}) \to \underline{\mathrm{Fun}}_G(\underline{\mathrm{Pos}}_G(J), \underline{\mathcal{D}})$.

Lemma 6.4.6 ("Rezk, [Lur17, Lem. 6.1.1.26]")**.** *Let $\underline{\mathcal{C}}$ be G-finite-cocomplete and have G-final objects, and $\underline{\mathcal{D}}$ be G-finite-complete. Let $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a G–functor. Let J be a finite G–set. Suppose $X : \underline{\mathrm{Pos}}_G(J_+) \to \underline{\mathcal{C}}$ is a G–strongly cocartesian J_+–cube. Then the canonical map $\theta_F : F(X) \to (T_{J_+}F)(X)$ constructed above factors through a G–cartesian J_+–cube of $\underline{\mathcal{D}}$.*

Proof. We will need four auxiliary points:

(a) Let $\zeta\colon \underline{\mathcal{C}} \to \underline{\mathrm{Fun}}_G(\underline{\mathrm{Pos}}_G(J_+), \underline{\mathcal{C}})$ be given by $\zeta : c \mapsto \big(I \mapsto C_I(c)\big)$. Observe that for any $c \in \underline{\mathcal{C}}$, $\zeta(c)|_{\underline{\mathrm{Orb}}_G(J_+)}$ is a G–right Kan extension of $\zeta(c)|_{\underline{\mathrm{Pos}}^{\leq 0}(J_+)}$ by construction.

(b) Let $X_I : \underline{\mathrm{Pos}}_G(J_+) \to \underline{\mathcal{C}}$ be the functor $S \mapsto X(I \cup S)$. That is, we are squashing X so that it lies on things above I: more formally, let $\alpha_I : \underline{\mathrm{Pos}}_G(J_+) \to \underline{\mathrm{Pos}}_G(J_+)$ be the non-G-fully faithful functor $S \mapsto (I \cup S)$. Then define $X_I := \alpha_I^* X$. Note that since X was G–strongly cocartesian, X_I is also G-strongly cocartesian by applying the criterion Lemma 6.2.3. To wit, if $T, T' \subseteq J_+$ are H-subsets such that $T \cup T'$ has more than one G-orbit, then so does $T \cup T' \cup I$ a fortiori, and hence

$$\begin{tikzcd}
X((T\cup I)\cap(T'\cup I)) = X((T\cap T')\cup I) \arrow[r] \arrow[d] & X(T\cup I) \arrow[d] \\
X(T'\cup I) \arrow[r] & X(T\cup T'\cup I) = X((T\cup I)\cup(T'\cup I))
\end{tikzcd}$$

is a fibrewise H-pushout since X was G–strongly cocartesian, as required.

(c) We denote the functors

$$\begin{aligned} \tau : \underline{\mathrm{Pos}}_G(J_+) \times \underline{\mathrm{Pos}}_G(J_+) \to \underline{\mathrm{Pos}}_G(J_+) \quad &:: \quad (S', S'') \mapsto (S' \cup S'') \\ j : \underline{\mathrm{Pos}}_G^{\varnothing}(J_+) \hookrightarrow \underline{\mathrm{Pos}}_G(J_+) \quad &\quad \text{the canonical inclusion} \end{aligned}$$

We can then consider the composition

$$\begin{aligned} \mathrm{Fun}_G(\underline{\mathrm{Pos}}_G(J_+), \underline{\mathcal{C}}) &\xrightarrow{F_*} \underline{\mathrm{Fun}}_G(\underline{\mathrm{Pos}}_G(J_+), \underline{\mathcal{D}}) \\ &\xrightarrow{\tau^*} \mathrm{Fun}_G(\underline{\mathrm{Pos}}_G(J_+) \times \underline{\mathrm{Pos}}_G(J_+), \underline{\mathcal{D}}) \\ &\xrightarrow{(j\times 1)^*} \mathrm{Fun}_G(\underline{\mathrm{Pos}}_G^{\varnothing}(J_+) \times \underline{\mathrm{Pos}}_G(J_+), \underline{\mathcal{D}}) \end{aligned}$$

We claim that for $X \in \mathrm{Fun}_G(\underline{\mathrm{Pos}}_G(J_+), \underline{\mathcal{C}})$ is G–strongly cocartesian, it lands in

$$\begin{aligned} &\mathrm{Fun}_G\big(\underline{\mathrm{Pos}}_G^{\varnothing}(J_+), \mathrm{Fun}_G^{\mathrm{cart}}(\underline{\mathrm{Pos}}_G(J_+), \underline{\mathcal{D}})\big) \\ \subseteq\ &\mathrm{Fun}_G\big(\underline{\mathrm{Pos}}_G^{\varnothing}(J_+), \mathrm{Fun}_G(\underline{\mathrm{Pos}}_G(J_+), \underline{\mathcal{D}})\big) \end{aligned}$$

under this composition. We need to show that for any $S' \in (\underline{\mathrm{Pos}}_G^{\varnothing}(J_+))_G = \mathrm{Pos}^{\varnothing}(J_+)^G$, the associated G-diagram

$$Z : \underline{\mathrm{Pos}}_G(S) \to \underline{\mathcal{D}} \quad :: \quad T \mapsto FX_{S'}(T) = FX(T \cup S')$$

is G-cartesian. For this, note that all the S'-faces of Z are constant since for any G-invariant decomposition $S = U \coprod S' \coprod V$, the G-diagram $Z \circ \xi_U : \underline{\mathrm{Pos}}_G(S') \to \underline{\mathrm{Pos}}_G(S) \to \underline{\mathcal{D}}$ given by

$$S'' \mapsto U \amalg S'' \mapsto FX((U \amalg S'') \cup S') \simeq FX(U \amalg S')$$

is constant. Invoking Proposition 6.2.2 (2), we see that Z is G-cartesian, as required.

(d) Let $p : \underline{\mathrm{Pos}}_G^{\varnothing}(S) \to \underline{*}$ be the unique map. Postcomposing the composition above further with the limit $p_* : \underline{\mathrm{Fun}}_G(\underline{\mathrm{Pos}}_G^{\varnothing}(S), \underline{\mathrm{Fun}}_G(\underline{\mathrm{Pos}}_G(S), \underline{\mathcal{D}})) \to \underline{\mathrm{Fun}}_G(\underline{\mathrm{Pos}}_G(S), \underline{\mathcal{D}})$, we can define $Y \in \underline{\mathrm{Fun}}_G(\underline{\mathrm{Pos}}_G(S), \underline{\mathcal{D}})$ to be $Y := p_*(j \times 1)^* \tau^* FX$ so that concretely for example, for $I \subseteq J_+$ a G–invariant subset we have

$$Y(I) := \underline{\lim}_{S' \in \underline{\mathrm{Pos}}_G^{\varnothing}(J_+)} F(X_I(S')) = \underline{\lim}_{S' \in \underline{\mathrm{Pos}}_G^{\varnothing}(J_+)} F(X(I \cup S'))$$

Since cartesianness is preserved under taking G–limits, by point (c), Y is G–cartesian.

Given these, since $\zeta(X(I))|_{\underline{\mathrm{Orb}}_G(J_+)}$ is a G–right Kan extension by (a) and X_I is G–left Kan extended from $\underline{\mathrm{Orb}}_G(J_+)$ by (b), we get that the identity map $X_I(\varnothing) \to \zeta(X(I))(\varnothing)$ admits a unique extension to a map of J_+–cubes $\eta : X_I \to \zeta(X(I))$ which when evaluated at $S \in \underline{\mathrm{Pos}}_G(J_+)$ is

$$\Big(X_I \to \xi(X(I))\Big)(S) = \Big(X(I \cup S) \to C_S(X(I))\Big)$$

depending functorially on I. We then observe that this map η provides the factorisation of $\theta : F(X) \to (T_{J_+}F)(X)$ given by $F(X) \to Y \xrightarrow{\eta} (T_{J_+}F)(X)$ as sought. □

Lemma 6.4.7 (Goodwillie approximations are excisive, "[Lur17, Lem. 6.1.1.33]"**)**. *Let $\underline{\mathcal{C}}$ be G-finite-cocomplete and admit G-final objects, and J a finite G-set. Suppose $\underline{\mathcal{D}}$ is G-differentiable and $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a functor. Then $P_{J_+}F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is J_+-excisive.*

Proof. Let $X : \underline{\mathrm{Pos}}_G(J_+) \to \underline{\mathcal{C}}$ be a G–strongly cocartesian J_+-cube. We want to show that $(P_{J_+}F)(X)$ is G-cartesian. Now by definition, we have that

$$(P_{J_+}F)(X) := \mathrm{colim}\,\big(F \to (T_{J_+}F)(X) \to (T^2_{J_+}F)(X) \to \cdots\big)$$

By the Rezk Lemma 6.4.6 we have a G–cartesian factorisation

$$(T^k_{J_+}F)(X) \to Y_k \to (T^{k+1}_{J_+}F)(X)$$

so that we could alternatively have gotten $(P_{J_+}F)(X)$ by a sequential colimit of

$$Y_0 \to Y_1 \to Y_2 \to \cdots$$

Since each Y_i is G-cartesian in $\underline{\mathcal{D}}$ and finite G–limits commute with sequential colimits in $\underline{\mathcal{D}}$ by G–differentiability, $(P_{J_+}F)(X) \simeq \mathrm{colim}_k Y_k$ is G–cartesian as wanted. □

Lemma 6.4.8 (Idempotence of Goodwillie approximations, "[Lur17, Lem. 6.1.1.35]"**)**. *Let $\underline{\mathcal{C}}$ be G-finite-cocomplete and G–pointed. Suppose $\underline{\mathcal{D}}$ is G-differentiable and $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a functor. Let $\theta : F \to T_{J_+}F$ be the canonical comparison. Then $P_{J_+}\theta : P_{J_+}F \to P_{J_+}T_{J_+}F$ is an equivalence. Therefore, the canonical transformation $P_{J_+}F \to P_{J_+}P_{J_+}F$ is an equivalence.*

Proof. Recall from the construction that P_{J_+} strongly commutes with finite G–limits (since T_{J_+} is just a G–limit construction, and by hypothesis fibrewise sequential colimits strongly preserve finite G-limits in $\underline{\mathcal{D}}$) and so we have $P_{J_+}T_{J_+}F \to \underline{\lim}_{S\in\underline{\mathrm{Pos}}^{\varnothing}_G(J_+)} P_{J_+}(F \circ C_S)$ is an equivalence. Now $C_S\colon \underline{\mathcal{C}} \to \underline{\mathcal{C}}$ strongly preserves G–colimits since $\underline{\mathcal{C}}$ ss G–pointed and so G–colimits of the G–point is again the G–point. Hence, by Observation 6.4.4 we get that $P_{J_+}T_{J_+}F \simeq \underline{\lim}_{S\in\underline{\mathrm{Pos}}^{\varnothing}_G(J_+)}(P_{J_+}F) \circ C_S$. But then $P_{J_+}F$ was J_+-excisive by Lemma 6.4.7, and so $\underline{\lim}_{S\in\underline{\mathrm{Pos}}^{\varnothing}_G(J_+)}(P_{J_+}F) \circ C_S \simeq P_{J_+}F$ by Construction 6.4.3 (b), and hence

$$P_{J_+}F \to P_{J_+}T_{J_+}F \simeq \underline{\lim}_{S\in\underline{\mathrm{Pos}}^{\varnothing}_G(J_+)}(P_{J_+}F) \circ C_S$$

is an equivalence as required. The last statement is just because $P_{J_+}T^k_{J_+}F \to P_{J_+}T^{k+1}_{J_+}F$ is an equivalence by the first part, and so the structure maps in the fibrewise sequential colimit defining $P_{J_+}P_{J_+}F$ are all equivalences. □

Notation 6.4.9. For a finite G–set J, we write $\mathrm{Exc}_G^{J_+} \subseteq \mathrm{Fun}_G$ for the full subcategory of J_+–excisive G–functors.

We are at last ready to state and prove the following.

Theorem 6.4.10 ("[Lur17, Thm. 6.1.1.10]")**.** *Let $\underline{\mathcal{C}}$ be G-finite-cocomplete and G–pointed. Suppose $\underline{\mathcal{D}}$ is G-differentiable. Then the inclusion $\mathrm{Exc}_G^{J_+}(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \subseteq \mathrm{Fun}_G(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ admits a G–left exact left adjoint $P_{J_+} : \mathrm{Fun}_G(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \to \mathrm{Exc}_G^{J_+}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$.*

Proof. We already have G–left exactness by G–differentiability of $\underline{\mathcal{D}}$ and also that the image of P_{J_+} lands in $\mathrm{Exc}_G^{J_+}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ by Lemma 6.4.7. On the other hand, if F were J-excisive, then by Lemma 6.4.8 we see that $F \to T_{J_+}F$ is an equivalence and so $F \to P_{J_+}F$ is an equivalence, and hence $\mathrm{Exc}_G^{J_+}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ consists precisely of the image of P_{J_+}. To see that it is a Bousfield localisation we just need to show that

$$P_{J_+}\theta_F, \theta_{P_{J_+}F} : P_{J_+}F \to P_{J_+}P_{J_+}F$$

are equivalences. The case of $P_{J_+}\theta_F$ is covered already by Lemma 6.4.8, whereas that of $\theta_{P_{J_+}F}$ is also done since $P_{J_+}F$ was J_+-excisive. □

6.5 Restriction-compatibility of Goodwillie approximations

Recollections 6.5.1. Suppose we have a diagram. Consider the commuting diagram

$$\begin{tikzcd} I \arrow[r, "i"] \arrow[dr, "p"'] & J \arrow[d, "q"] \\ & * \end{tikzcd}$$

and let $\underline{\mathcal{C}}$ be a G–category with the requisite G–limits. Then there is a canonical transformation

$$q_* \Longrightarrow p_*i^*$$

coming from $\mathrm{Nat}(q_*, p_*i^*) \simeq \mathrm{Nat}(p^*q_*, i^*) \simeq \mathrm{Nat}(i^*q^*q_*, i^*)$ and using the image of the $(q^* \dashv q_*)$-counit under the functor i^*. By applying the obvious units and counits, this transformation adjoints to

$$i_!p^* \Longrightarrow q^*$$

Construction 6.5.2. We first need to construct canonical transformations under $\mathrm{Res}^G_H F$

$$\zeta_F : \mathrm{Res}^G_H T_{nG_+} F \Longrightarrow T_{nH_+} \mathrm{Res}^G_H F$$

which are natural in $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ (cf. Construction 6.4.3 for the construction of T_{J_+}). For this, we collect some notations and constructions that we have seen above:

- For any G–category $\underline{\mathcal{E}}$, we write
$$\widetilde{C} : \underline{\mathcal{E}} \to \underline{\mathcal{E}}^{\underline{\mathrm{Pos}}^{\varnothing}_H(\mathrm{Res}^G_H nG_+)} \qquad\qquad C : \underline{\mathcal{E}} \to \underline{\mathcal{E}}^{\underline{\mathrm{Pos}}^{\varnothing}_H(nH_+)}$$
for the cone constructions of Construction 6.4.3 (a).
- Recall from Lemma 6.3.4 that we had the commuting diagram
$$\begin{tikzcd} \underline{\mathrm{Pos}}^{\varnothing}_H(nH_+) \arrow[r, hook, "i"] \arrow[dr, "p"'] & \underline{\mathrm{Pos}}^{\varnothing}_H(\mathrm{Res}^G_H nG_+) \arrow[d, "q"] \\ & \underline{*} \end{tikzcd}$$
Moreover, since this functor i is H–fully faithful, the functor
$$i^* : \underline{\mathcal{C}}^{\underline{\mathrm{Pos}}^{\varnothing}_H(\mathrm{Res}^G_H nG_+)} \to \underline{\mathcal{C}}^{\underline{\mathrm{Pos}}^{\varnothing}_H(nH_+)}$$
strongly preserves left and right H–Kan extensions. Therefore, since the functors $\widetilde{C}$ and C were constructed as a combination of left and right H–Kan extensions, we see that the following triangle commutes.
$$\begin{tikzcd} \underline{\mathcal{C}} \arrow[r, "\widetilde{C}"] \arrow[dr, "C"'] & \underline{\mathcal{C}}^{\underline{\mathrm{Pos}}^{\varnothing}_H(\mathrm{Res}^G_H nG_+)} \arrow[d, "i^*"] \\ & \underline{\mathcal{C}}^{\underline{\mathrm{Pos}}^{\varnothing}_H(nH_+)} \end{tikzcd}$$

With these notations set, we observe that

$$\mathrm{Res}^G_H T_{nG_+} F \simeq q_*\Big((\mathrm{Res}^G_H F) \circ \widetilde{C}\Big)$$

$$T_{nH_+} \mathrm{Res}^G_H F \simeq p_*\Big((\mathrm{Res}^G_H F) \circ C\Big) \simeq p_* \widetilde{i^*}\Big((\mathrm{Res}^G_H F) \circ \widetilde{C}\Big)$$

Hence we obtain a natural transformation

$$\zeta_F : \mathrm{Res}^G_H T_{nG_+} F \simeq q_*\Big((\mathrm{Res}^G_H F) \circ \widetilde{C}\Big) \Rightarrow p_* i^*\Big((\mathrm{Res}^G_H F) \circ \widetilde{C}\Big) \simeq T_{nH_+} \mathrm{Res}^G_H F$$

by the construction Recollections 6.5.1. Furthermore, this clearly is a natural transformation under $\mathrm{Res}^G_H F$. By naturality of all the transformations constructed, we get

$$\xi_F : \mathrm{Res}^G_H P_{nG_+} F \Longrightarrow P_{nH_+} \mathrm{Res}^G_H F$$

We reproduce Dotto's proof [Dot17, Thm. 2.31] of the following for convenience:

Theorem 6.5.3. *The map* $\xi_F : \operatorname{Res}^G_H P_{nG_+} F \Rightarrow P_{nH_+} \operatorname{Res}^G_H F$ *is an equivalence.*

Proof. In order to show that it is an equivalence, we first claim that the diagram

$$\begin{tikzcd}
\operatorname{Res}^G_H P_{nG_+} F \arrow[r, "\xi_F"] \arrow[dr, bend right, "\rho_F"'] & P_{nH_+} \operatorname{Res}^G_H F \arrow[d, "P_{nH_+} \operatorname{Res}^G_H \rho_F"] \arrow[dr, bend left, "P_{nH_+}\rho_F"] & \\
& P_{nH_+} \operatorname{Res}^G_H P_{nG_+} F \arrow[r, "P_{nH_+}\xi_F"'] & P_{nH_+} P_{nH_+} \operatorname{Res}^G_H F
\end{tikzcd} \tag{6.1}$$

commutes: the right triangle commutes since ζ_F, and hence ξ_F, was constructed as a transformation under $\operatorname{Res}^G_H F$; the left triangle commutes is by considering the diagram

$$\begin{tikzcd}
\operatorname{Res}^G_H P_{nG_+} F \arrow[r, "\operatorname{Res}^G_H \rho_{P_{nG_+} F}", "\simeq"'] \arrow[dd, "\rho_F"] & \operatorname{Res}^G_H P_{nG_+} P_{nG_+} F \arrow[dd, "\xi_{P_{nG_+} F}"] & \operatorname{Res}^G_H P_{nG_+} F \arrow[l, "\operatorname{Res}^G_H P_{nG_+}\rho_F"', "\simeq"] \arrow[d, "\xi_F"] \\
& & P_{nH_+} \operatorname{Res}^G_H F \arrow[d, "P_{nH_+} \operatorname{Res}^G_H \rho_F"] \\
P_{nH_+} \operatorname{Res}^G_H P_{nG_+} F \arrow[r, equal] & P_{nH_+} \operatorname{Res}^G_H P_{nG_+} F \arrow[r, equal] & P_{nH_+} \operatorname{Res}^G_H P_{nG_+} F
\end{tikzcd} \tag{6.2}$$

where the right square commutes by naturality of ξ, and the left square commutes by construction of ρ and ξ: to wit, ξ is induced by the horizontal maps upon taking directed colimits on both sides

$$\begin{tikzcd}
\operatorname{Res}^G_H E \arrow[r, equal] \arrow[d, "\operatorname{Res}^G_H \theta_E"'] & \operatorname{Res}^G_H E \arrow[d, "\theta_{\operatorname{Res}^G_H E}"] \\
\operatorname{Res}^G_H T_{nG_+} E \arrow[r, "\zeta_E"] \arrow[d, "\operatorname{Res}^G_H \theta_{T_{nG_+} E}"'] & T_{nH_+} \operatorname{Res}^G_H E \arrow[d, "\theta_{T_{nH_+} \operatorname{Res}^G_H E}"] \\
\operatorname{Res}^G_H T_{nG_+} T_{nG_+} E \arrow[r, "\zeta\circ\zeta"] \arrow[d] & T_{nH_+} T_{nH_+} \operatorname{Res}^G_H E \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

On the other hand, the ρ maps are induced by the top two vertical maps, and hence passing to colimits and setting $E = P_{nG_+} F$, we see indeed that $\xi_{P_{nG_+} F} \circ \operatorname{Res}^G_H \rho_{P_{nG_+} F} \simeq \rho_F$. Therefore, the diagram Eq. (6.2) commutes, and hence the left triangle of Eq. (6.1) commutes.

By considering the usual 2-out-of-6 principle, if we can show that the curved maps in Eq. (6.1) are equivalences, then we would have shown that all maps in sight are equivalences, and in particular, so is ξ_F as required. That $P_{nH_+}\rho_F$ is an equivalence is clear by Lemma 6.4.8, and that ρ_F is an equivalence is because $\mathrm{Res}^G_H P_{nG_+}F$ is already nH_+–excisive by Proposition 6.3.6. □

6.6 Multilinearity

Construction 6.6.1 (Reductions, [Lur17, Cons. 6.1.3.15])**.** Let $\underline{\mathcal{C}}_1, \ldots, \underline{\mathcal{C}}_m$ admit G–final objects $*_i$ and $\underline{\mathcal{D}}$ is G–pointed and G–finite-complete. By finality, for each $1 \leq i \leq m$, we have natural transformations $\alpha_i : \mathrm{id}_{\underline{\mathcal{C}}_i} \implies *_i$, so that we get a functor

$$\overline{F} : \underline{\mathcal{C}}_1 \times \cdots \times \underline{\mathcal{C}}_m \times \mathrm{Pos}(S) \xrightarrow{\prod \alpha_i} \underline{\mathcal{C}}_1 \times \cdots \times \underline{\mathcal{C}}_m \xrightarrow{F} \underline{\mathcal{D}}$$

given by

$$(X_1, \ldots, X_m, T) \mapsto F(X'_1, \ldots, X'_m)$$

$$X'_i := \begin{cases} X_i & \text{if } i \notin T \\ *_i & \text{if } i \in T \end{cases}$$

We let $F^T := F|_T$. Hence the functor $\overline{F}$ restricts to a natural transformation $\beta : F = F^{\emptyset} \implies \lim_{\emptyset \neq T \subseteq S} F^T$ where $S = \{1, \ldots, m\}$, and we define the *reduction of* F to be

$$F^{\mathrm{red}} := \mathrm{fib}(\beta : F \implies \lim_{\emptyset \neq T \subseteq S} F^T)$$

It is then elementary to check (cf. [Lur17, Prop. 6.1.3.17]) that F^{red} is multi-reduced and provides a right adjoint to the inclusion of multi-reduced functors into all functors.

Construction 6.6.2 (Cross-effects, "[Lur17, Cons. 6.1.3.20]", [Dot17, before Prop. 3.23])**.** Let $\underline{\mathcal{C}}$ be G–finite-cocomplete and has a G–final object, and $\underline{\mathcal{D}}$ is G–pointed and G–finite-complete. We denote the n–addition functor by

$$q_n : \underline{\mathcal{C}}^{\times n} \longrightarrow \underline{\mathcal{C}} \quad :: \quad (X_1, \ldots, X_n) \mapsto \coprod_i X_i$$

Then for every functor $F \colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ we define the *n-th cross-effect* to be $\mathrm{cr}_n F := (F \circ q_n)^{\mathrm{red}}$. That is, equivariant cross-effects are defined just to be a fibrewise construction.

Proposition 6.6.3 (Multilinear excision, [Dot17, Prop. 3.23], "[Lur17, Prop. 6.1.3.22]")**.** *Let $\underline{\mathcal{C}}$ be G–finite-cocomplete and $\underline{\mathcal{D}}$ be G–finite-complete. Let $F \colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be an nG_+-excisive functor. For each $m \leq n+1$ the cross-effect $\mathrm{cr}_m(F) : \underline{\mathcal{C}}^{\times m} \to \underline{\mathcal{D}}$ is $((n-m+1)G_+, \ldots, (n-m+1)G_+)$-excisive. In particular, $\mathrm{cr}_{n+1}F \simeq 0$.*

Proof. We perform induction over n. When $m = 0$, this is vacuous. For $m \geq 1$, note that reduction is a limit construction in the target category, and excision is a limit condition on the target category, and so reduction and taking limits do not worsen excision, ie. reductions or limits of J_+–excisive functors are still J_+–excisive. Now $\mathrm{cr}_1(F) = \mathrm{fib}(F \implies F_0)$ is given by the fibre of a natural transformation $F \implies F_0$ where F_0 is constant. So since F and F_0 are both nG_+–excisive, so is $\mathrm{cr}_1(F)$. Let us therefore assume that $m \geq 2$. Fixing $X_2, \ldots, X_m \in \underline{\mathcal{C}}$ we need to show that the functor $\underline{\mathcal{C}} \to \underline{\mathcal{D}}$ given by $X_1 \mapsto \mathrm{cr}_m(F)(X_1, \ldots, X_m)$ is $(n - m + 1)G_+$-excisive. Let $*$ be a G–final object in $\underline{\mathcal{C}}$. Define the functor

$$\widetilde{F} \coloneqq \mathrm{fib}\left(F(- \amalg X_m) \Rightarrow F(- \amalg *)\right) : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$$

By splitting away the X_m part in the definition of $\mathrm{cr}_m(F)$ we see that

$$\mathrm{cr}_m(F)(X_1, \ldots, X_m) \simeq \mathrm{cr}_{m-1}(\widetilde{F})(X_1, \ldots, X_{m-1})$$

and so it suffices to show that $\mathrm{cr}_{m-1}(\widetilde{F})$ is $((n - m + 1)G_+, \ldots, (n - m + 1)G_+)$–excisive. By induction on n, we in turn need to show that $\widetilde{F}$ is $(n - 1)G_+$-excisive.

To this end let $S = (n-1)G$ and let $Y : \mathrm{Pos}(S_+) \to \underline{\mathcal{C}}$ be a G–strongly cocartesian S_+–cube. Let $Y_* : \mathrm{Pos}((S_+)_*) \to \underline{\mathcal{C}}$ be defined as

$$Y_*(T) \coloneqq \begin{cases} Y(T) \amalg X_m & \text{if } * \notin T \\ Y(T \backslash \{*\}) \amalg * & \text{if } * \in T \end{cases}$$

Now Y_* is a G–strongly cocartesian $(S_+)_*$-cube since for every $T \subseteq S_+$, $Y_*(T) = Y(T) \amalg X_m$ and Y was a G–strongly cocartesian S_+-cube by hypothesis, whereas $Y_*(T_*) = Y(T) \amalg *$ can be checked using the pushout criterion Lemma 6.2.3. Hence since F was $(S \amalg *)_+$-excisive by Lemma 6.3.2, $F \circ (Y_*) : \mathrm{Pos}((S_+)_*) \to \underline{\mathcal{D}}$ is a cartesian $(S_+)_*$-cube in $\underline{\mathcal{D}}$, and so the diagram

$$\begin{tikzcd} F(Y(\emptyset) \amalg X_m) \arrow[r] \arrow[d] & \lim_{\emptyset \neq T \subseteq S} F(Y(T) \amalg X_m) \arrow[d] \\ F(Y(\emptyset) \amalg *) \arrow[r] & \lim_{\emptyset \neq T \subseteq S} F(Y(T) \amalg *) \end{tikzcd}$$

is a pullback. Therefore the vertical fibres, which are $\widetilde{F}$, are equivalent, whence $\widetilde{F}(Y) : \underline{\mathrm{Pos}}_G(S_+) \to \underline{\mathcal{D}}$ is a cartesian S_+–cube in $\underline{\mathcal{D}}$, as was to be shown. □

We have not succeeded in using the methods of [Lur17, Cor. 6.1.3.5] to prove the following result. As such, having nothing to add to Dotto's proof, we quote the followingwithout proof.

Proposition 6.6.4 (Additivity of free equivariant excision, [Dot17, Prop. 3.19]). *Let $\underline{\mathcal{C}}$ be G–finite-cocomplete and $\underline{\mathcal{E}}$ be G–finite-complete. Let $F : \underline{\mathcal{C}}^{\times r} \to \underline{\mathcal{E}}$ be $(n_1 G_+, \ldots, n_r G_+)$-excisive. If we write $n \coloneqq n_1 + \cdots + n_r$, then $F\Delta$ is nG_+-excisive.*

Chapter 7

Genuine equivariant hermitian K-theory

In this chapter, we explore a generalisation of the theory developed in [CDH+20a; CDH+20b; CDH+20c; CDH+] through a notion of genuine equivariant Poincaré categories. Here, we will use Dotto's equivariant Goodwillie calculus [Dot17] to formulate the concept of G–quadratic functors. As we will see in §7.1.5, we could of course bypass Goodwillie calculus altogether and just *define* G–quadraticity directly via the desired stable recollement. Nevertheless, we think that it is conceptually satisfying to have the more general notion supporting the hermitian theory.

Throughout §7.1-§7.3, G will be supposed to be an odd group. As we will see in Example 7.1.4, we need this assumption in order to ensure that G–bilinear forms is a source of G–quadratic forms, which in the nonequivariant case is a crucial part of the theory developed in [CDH+20a]. The essential reason for this is one that we have seen in Corollary 3.5.3.

In the final §7.4.2, of this final chapter, we indicate potential applications of the general approach of "genuinising" equivariant hermitian K–theory. In §7.4.1, we consider the case of $G = C_2$, in which case we will need to further assume that 2 is inverted. To avoid confusions between the C_2–action coming from the hermitian dualities and the C_2–equivariance coming from the case $G = C_2$, we have opted to denote the hermitian duality equivariance by Σ_2; this notation is supposed to evoke an impression of the swap action. In §7.4.2, we return to the original motivation for this entire thesis where we sketch the potential use of the Hill-Hopkins-Ravenel norms on equivariant L–theory to prove descent results analogous to the ones in [CMN+20] via the methods of [Gre93].

7.1 Foundations

As mentioned above, the group G will always be assumed to be odd throughout this section.

7.1.1 Quadratics, bilinears, linears

Definition 7.1.1. Let $\underline{\mathcal{C}}, \underline{\mathcal{D}}, \underline{\mathcal{E}}$ be G-stable categories.

- A G-functor $\beta : \underline{\mathcal{C}} \underline{\times} \underline{\mathcal{C}} \to \underline{\mathrm{Sp}}_G$ is *bireduced* if $\beta(x, y) \simeq 0$ when $x \simeq 0$ or $y \simeq 0$.
- A G-functor $b : \underline{\mathcal{C}} \underline{\times} \underline{\mathcal{D}} \to \underline{\mathcal{E}}$ is *G-bilinear* if it is G-exact in each variable.

Definition 7.1.2. Let $\underline{\mathcal{C}}$ be G-stable and $\underline{\Omega} : \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Sp}}_G$ a G-functor. We say that $\underline{\Omega}$ is:

- *G-quadratic* if it is reduced, $2G_+$–excisive, and the canonical transformation $P_{1_+}\underline{\Omega} \Rightarrow P_{G_+}\underline{\Omega}$ is an equivalence (cf. Chapter 6 for the meanings of these terms),
- *G-linear* if it is reduced and G_+-excisive (in other words, G-exact).

Remark 7.1.3. The extra condition $P_{1_+}\underline{\Omega} \Rightarrow P_{G_+}\underline{\Omega}$ in the definition of G–quadratic functors is to guarantee the quadratic stable recollement (cf. Theorem 7.1.24).

Example 7.1.4. Here are the two most important classes of G–quadratic functors. The reason for this will be made precise in Theorem 7.1.24.

(i) By Theorem 6.3.3 we see that being G–linear implies being G–quadratic.
(ii) For a G–bilinear $\beta : \underline{\mathcal{C}}^{\mathrm{op}} \underline{\times} \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Sp}}_G$, the functor $(\beta^{\Delta})^{h\Sigma_2}(x) := \beta(x, x)^{h\Sigma_2}$ is a reduced $2G_+$–excisive functor by Proposition 6.6.4. Moreover, we will see in Construction 7.1.13 that the linearisation is computed as $P_{1_+}(\beta^{\Delta})^{h\Sigma_2} = \mathrm{cofib}\left((\beta^{\Delta})_{h\Sigma_2} \Rightarrow (\beta^{\Delta})^{h\Sigma_2}\right) \simeq (\beta^{\Delta})^{t\Sigma_2}$ and this is G–exact by Corollary 3.5.3, since G was odd.

Let us now establish some useful notations analogous to those of [CDH+20a].

Notation 7.1.5. Let $\underline{\mathcal{C}}$ be G-stable. Then we denote by

- $\underline{\mathrm{Fun}}^*(\underline{\mathcal{C}}^{\mathrm{op}} \underline{\times} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G)$ the G-category of reduced G-functors. Note that this has a canonical Σ_2-action given by swapping the two copies of $\underline{\mathcal{C}}^{\mathrm{op}}$,
- $\underline{\mathrm{BiRed}}(\underline{\mathcal{C}}) \subseteq \underline{\mathrm{Fun}}^*(\underline{\mathcal{C}}^{\mathrm{op}} \underline{\times} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G)$ the G-full subcategory of bireduced functors,
- $\underline{\mathrm{Fun}}^{\mathrm{b}}(\underline{\mathcal{C}}) \subseteq \underline{\mathrm{BiRed}}(\underline{\mathcal{C}})$ the G-full subcategory of G-bilinear functors. This is a well-defined G-category since restrictions send G-bilinear G-functors to H-bilinear H-functors for all $H \leq G$ by virtue of Theorem 6.5.3. This also clearly inherits the Σ_2-action.

- $\underline{\mathrm{Fun}}^{\mathrm{s}}(\mathcal{C}) := \underline{\mathrm{Fun}}^{\mathrm{b}}(\mathcal{C})^{h\Sigma_2}$ which we call the G-category of *G-symmetric bilinear functors*,
- $\underline{\mathrm{Fun}}^{\mathrm{q}}(\mathcal{C}) \subseteq \underline{\mathrm{Fun}}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G)$ the G-full subcategory of G-quadratic functors. Again, this is well-defined, ie. restrictions of G-quadratics are H-quadratic, by virtue of Proposition 6.3.6 and Theorem 6.5.3.
- $\underline{\mathrm{Fun}}^{\mathrm{ex}}(\mathcal{C}) := \underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G)$ is a G-full subcategory of $\underline{\mathrm{Fun}}^{\mathrm{q}}(\mathcal{C})$.

Remark 7.1.6 (G-quadratic (co)completeness)**.** Observe that since being G-cartesian in the G-stable category $\underline{\mathrm{Sp}}_G$ is preserved under arbitrary G-(co)limits, the G-full subcategory $\underline{\mathrm{Fun}}^{\mathrm{q}}(\mathcal{C}) \subseteq \underline{\mathrm{Fun}}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G)$ is closed under arbitrary G-(co)limits, and so is in particular G-stable also.

Construction 7.1.7 (Bireduction)**.** Note that we have a retraction

$$\beta(x,0) \oplus \beta(0,y) \to \beta(x,y) \to \beta(x,0) \oplus \beta(0,y)$$

and so taking the cofibre of the first map (or equivalently fibre of the second map) gives a bireduced form which we denote by $\beta(-,-)^{\mathrm{red}}$. Note that this commutes with restriction along pairs of reduced functors. This also commutes with the flip functor and so the bireduction refines to a functor

$$(-)^{\mathrm{red}} : \underline{\mathrm{Fun}}^*(\underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G)^{h\Sigma_2} \to \underline{\mathrm{BiRed}}(\underline{\mathcal{C}})^{h\Sigma_2}$$

Since all the G–categories in sight are G–stable, taking bireduction strongly commutes with arbitrary G-(co)limits: in fact, it participates in a biadjunction (cf. Lemma 7.1.17).

Construction 7.1.8 (Cross-effects)**.** We specialise the discussion from Construction 6.6.2 in the setting of equivariant hermitian K–theory. Let $\underline{\mathcal{C}}$ be G–stable and $\underline{\Omega} : \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Sp}}_G$ be a reduced functor. We define the *cross-effect* (or *polarisation*) to be

$$B_{\underline{\Omega}} := \underline{\Omega}(- \oplus -)^{\mathrm{red}} : \underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Sp}}_G$$

yielding a functor

$$B_{(-)} : \underline{\mathrm{Fun}}^*(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G) \to \underline{\mathrm{BiRed}}(\mathcal{C})$$

This functor commutes with restrictions along direct sum preserving reduced functors, ie. for $f : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$, we have $(f \times f)^* B^{\underline{\mathcal{D}}}_{\underline{\Omega}} \simeq B^{\underline{\mathcal{C}}}_{f^*\underline{\Omega}}$. We can also define

$$B^{\Delta}_{(-)} := \Delta^* \circ B_{(-)} : \underline{\mathrm{Fun}}^*(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G) \to \underline{\mathrm{BiRed}}(\underline{\mathcal{C}}) \to \underline{\mathrm{Fun}}^*(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G)$$

Since B^{red} is a retract of B, we obtain the following canonical natural transformations

$$(B^{\Delta}_{\underline{\Omega}} \Rightarrow \underline{\Omega}) = \Big(X \mapsto \big(\underline{\Omega}(X \oplus X)^{\mathrm{red}} \to \underline{\Omega}(X \oplus X) \xrightarrow{\Delta^*} \underline{\Omega}(X)\big)\Big)$$

$$(\underline{\Omega} \Rightarrow B^{\Delta}_{\underline{\Omega}}) = \Big(X \mapsto \big(\underline{\Omega}(X) \xrightarrow{\nabla^*} \underline{\Omega}(X \oplus X) \to \underline{\Omega}(X \oplus X)^{\mathrm{red}}\big)\Big)$$

Lemma 7.1.9 (Σ_2-equivariance)**.** *We collect all the available Σ_2-equivariance here.*

(i) The cross effect $B_{\underline{\mathcal{Q}}}$ is symmetric, ie. it is in the image of the forgetful functor

$$\underline{\mathrm{Fun}}^*(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}\times\underline{\mathcal{C}}^{\underline{\mathrm{op}}},\underline{\mathrm{Sp}}_G)^{h\Sigma_2}\to\underline{\mathrm{Fun}}^*(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}\times\underline{\mathcal{C}}^{\underline{\mathrm{op}}},\underline{\mathrm{Sp}}_G)$$

(ii) The functor $\Delta^*:\underline{\mathrm{Fun}}^*(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}\times\underline{\mathcal{C}}^{\underline{\mathrm{op}}},\underline{\mathrm{Sp}}_G)\to\underline{\mathrm{Fun}}^*(\underline{\mathcal{C}}^{\underline{\mathrm{op}}},\underline{\mathrm{Sp}}_G)$ is Σ_2-equivariant.

(iii) And so the natural transformations constructed above descend to transformations

$$(B^{\Delta}_{\underline{\mathcal{Q}}})_{h\Sigma_2}\Rightarrow\underline{\mathcal{Q}}\Rightarrow(B^{\Delta}_{\underline{\mathcal{Q}}})^{h\Sigma_2}$$

Proof. Since the bireduction functor was Σ_2-equivariant, we just have to show that $\underline{\mathcal{Q}}\circ\oplus\in\underline{\mathrm{Fun}}^*(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}\times\underline{\mathcal{C}}^{\underline{\mathrm{op}}},\underline{\mathrm{Sp}}_G)$ lives in the image from $\underline{\mathrm{Fun}}^*(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}\times\underline{\mathcal{C}}^{\underline{\mathrm{op}}},\underline{\mathrm{Sp}}_G)^{h\Sigma_2}$. And for this it suffices to note that $\oplus:\underline{\mathcal{C}}^{\underline{\mathrm{op}}}\times\underline{\mathcal{C}}^{\underline{\mathrm{op}}}\to\underline{\mathcal{C}}$ inherits a Σ_2-equivariant structure from the cartesian symmetric monoidal structure on $\underline{\mathcal{C}}^{\underline{\mathrm{op}}}$. $\square$

Observation 7.1.10. For a G–bilinear $\beta:\underline{\mathcal{C}}^{\underline{\mathrm{op}}}\times\underline{\mathcal{C}}^{\underline{\mathrm{op}}}\to\underline{\mathrm{Sp}}_G$, unwinding the bireduction process, the symmetric bilinear part associated to $\beta^{\Delta}:\underline{\mathcal{C}}^{\underline{\mathrm{op}}}\to\underline{\mathrm{Sp}}_G$ is given by $(x,y)\mapsto\beta(x,y)\oplus\beta(y,x)$.

Construction 7.1.11 (Associated quadratics)**.** For $\beta\in\underline{\mathrm{Fun}}^{\mathrm{s}}(\underline{\mathcal{C}})$ we can define

$$\underline{\mathcal{Q}}^q_{\beta}(x):=\beta^{\Delta}_{h\Sigma_2}(x)=\beta(x,x)_{h\Sigma_2}\qquad\underline{\mathcal{Q}}^s_{\beta}(x):=(\beta^{\Delta})^{h\Sigma_2}(x)=\beta(x,x)^{h\Sigma_2}$$

and these are both G–quadratic: this is because β^{Δ} is G–quadratic by Example 7.1.4 and $\underline{\mathrm{Fun}}^{\mathrm{q}}(\underline{\mathcal{C}})$ is closed under G–(co)limits in $\underline{\mathrm{Fun}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}},\underline{\mathrm{Sp}}_G)$ by Remark 7.1.6. Note that

$$B_{\mathcal{Q}^q_B}\simeq B\simeq B_{\mathcal{Q}^s_B}$$

since taking cross-effects commute with (co)limits and so by Observation 7.1.10 we get

$$\begin{aligned}B_{\underline{\mathcal{Q}}^q_{\beta}}(x,y)\simeq(\beta(x,y)\oplus\beta(y,x))_{h\Sigma_2}\simeq\beta(x,y)&\\ \simeq(\beta(x,y)\oplus\beta(y,x))^{h\Sigma_2}&\simeq B_{\underline{\mathcal{Q}}^s_{\beta}}(x,y)\end{aligned}$$

Definition 7.1.12 (Bilinear parts)**.** By Lemma 7.1.9 and Proposition 7.1.15 we see that $B_{(-)}:\underline{\mathrm{Fun}}^{\mathrm{q}}(\underline{\mathcal{C}})\to\underline{\mathrm{Fun}}^*(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}\times\underline{\mathcal{C}}^{\underline{\mathrm{op}}},\underline{\mathrm{Sp}}_G)$ lifts to $B_{(-)}:\underline{\mathrm{Fun}}^{\mathrm{q}}(\underline{\mathcal{C}})\to\underline{\mathrm{Fun}}^{\mathrm{s}}(\underline{\mathcal{C}})$. We call $B_{\underline{\mathcal{Q}}}$ the *symmetric bilinear part* of $\underline{\mathcal{Q}}$, and the underlying bilinear functor $B_{\underline{\mathcal{Q}}}$ as the *bilinear part* of $\underline{\mathcal{Q}}$.

Construction 7.1.13 ((Co)linear parts and (co)homogeneity)**.** Let $\underline{\mathcal{Q}}:\underline{\mathcal{C}}^{\underline{\mathrm{op}}}\to\underline{\mathrm{Sp}}_G$ be G–quadratic. We define the linear (resp. colinear) part to be the cofibre (resp. fibre)

$$(B^{\Delta}_{\underline{\mathcal{Q}}})_{h\Sigma_2}\Rightarrow\underline{\mathcal{Q}}\Rightarrow L_{\underline{\mathcal{Q}}}\quad\text{or}\quad cL_{\underline{\mathcal{Q}}}\Rightarrow\underline{\mathcal{Q}}\Rightarrow(B^{\Delta}_{\underline{\mathcal{Q}}})^{h\Sigma_2}$$

These will be shown to be G–exact in Proposition 7.1.15, justifying the names. If $L_{\underline{\mathfrak{Q}}} \simeq *$ (resp. $cL_{\underline{\mathfrak{Q}}} \simeq *$) then we say that $\underline{\mathfrak{Q}}$ is homogeneous (resp. cohomogeneous). By Construction 7.1.8 these constructions commute with restrictions along G–exact functors.

7.1.2 Recognition criteria

We now come to one of the most important basic results that will be the bread-and-butter of this story.

Fact 7.1.14. Let $\underline{\mathfrak{Q}} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be an ordinary, nonequivariant quadratic functor such that $B_{\underline{\mathfrak{Q}}} \simeq 0$. Then it is in fact linear.

Proposition 7.1.15 (Characterisations of G-quadratics, "[CDH+20a, Prop. 1.1.13]")**.** *Let $\underline{\mathfrak{Q}} : \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Sp}}_G$ be a G-functor. Then the following are equivalent:.*

(i) $\underline{\mathfrak{Q}}$ is G–quadratic,

(ii) $B_{\underline{\mathfrak{Q}}}$ is G-bilinear and fib $\left(\underline{\mathfrak{Q}} \Rightarrow \underline{\mathfrak{Q}}^{s}_{B_{\underline{\mathfrak{Q}}}}\right) : \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Sp}}_G$ *is G-exact,*

(iii) $B_{\underline{\mathfrak{Q}}}$ is G-bilinear and cofib $\left(\underline{\mathfrak{Q}}^{q}_{B_{\underline{\mathfrak{Q}}}} \Rightarrow \underline{\mathfrak{Q}}\right) : \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Sp}}_G$ *is G-exact.*

Proof. As pointed out in Remark 7.1.6, since $\underline{\mathrm{Sp}}_G$ is G–stable the property of being reduced and $2G_+$-excisive is closed under G–limits and G–colimits. Hence (ii) and (iii) implies (i) since G-exact functors and diagonal restrictions of G-bilinears are reduced, $2G_+$-excisive, and satisfies $P_{1_+} \Rightarrow P_{G_+}$ being an equivalence Example 7.1.4. For the reverse implications, we invoke Proposition 6.6.3 to get that the cross-effect is G-bilinear. On the other hand, applying cross-effects, noting that it preserves (co)fibres and by Construction 7.1.11, we see that the cross-effect on $F :=$ fib $\left(\underline{\mathfrak{Q}} \Rightarrow \underline{\mathfrak{Q}}^{s}_{B_{\underline{\mathfrak{Q}}}}\right)$ and $C :=$ cofib $\left(\underline{\mathfrak{Q}}^{q}_{B_{\underline{\mathfrak{Q}}}} \Rightarrow \underline{\mathfrak{Q}}\right)$ are trivial, and so we see that the fibre and cofibre are reduced $2G_+$-excisive with trivial cross-effects. This means that $F \Rightarrow P_{1_+}F$ and $C \Rightarrow P_{1_+}C$ are equivalences. But then because $P_{1_+}\underline{\mathfrak{Q}} \Rightarrow P_{G_+}\underline{\mathfrak{Q}}$ is an equivalence, the same also holds for F and C. Hence $F \Rightarrow P_{G_+}F$ and $C \Rightarrow P_{G_+}C$ are equivalences, as required. $\square$

Proposition 7.1.16 (Characterisations of (co)homogeneity, "[CDH+20a, Lem. 1.3.1]")**.** *Let $\underline{\mathfrak{Q}} : \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Sp}}_G$ be a G-quadratic functor. Then the following conditions are equivalent for being G-homogeneous.*

(i) The map $\underline{\mathfrak{Q}}^{q}_{B_{\underline{\mathfrak{Q}}}} \Rightarrow \underline{\mathfrak{Q}}$ is an equivalence.

(ii) $\underline{\mathfrak{Q}}$ is equivalent to a quadratic functor of form $\underline{\mathfrak{Q}}^{q}_{\beta}$ for $\beta \in \underline{\mathrm{Fun}}^{\mathrm{s}}(\underline{\mathcal{C}})$

(iii) The G-spectrum $\underline{\mathrm{Nat}}(\underline{\mathfrak{Q}}, \lambda)$ is trivial for any G-linear $\lambda : \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Sp}}_G$.

Dually we have the characterisations for G-cohomogeneous functors.

Proof. That (i) implies (ii) is immediate. That (ii) implies (iii) is an immediate consequence of Proposition 7.1.20, since $L_{\underline{\Qoppa}} \simeq 0$ by definition of the G–linear part. That (iii) implies (i) is again a consequence of Proposition 7.1.20 since by definition (i) is saying precisely that $L_{\underline{\Qoppa}} \simeq 0$. □

7.1.3 Adjunctions in the small

Lemma 7.1.17 (Bireduction adjunction, "[CDH+20a, Lem. 1.1.3]")**.** *We have a biadjunction*

$$\begin{tikzcd} \underline{\mathrm{BiRed}}(\underline{\mathcal{C}}) \arrow[r, hook] & \underline{\mathrm{Fun}}^*(\underline{\mathcal{C}}^{\underline{\mathrm{op}}} \underline{\times} \underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathrm{Sp}}_G) \arrow[l, bend right=40, "(-)^{\mathrm{red}}"'] \arrow[l, bend left=40, "(-)^{\mathrm{red}}"] \end{tikzcd}$$

Proof. Immediate from the retraction and inclusion maps. □

Corollary 7.1.18 (Cross-effect adjunction, "[CDH+20a, Rmk. 1.1.8]")**.** *The biadjunction $\Delta : \underline{\mathcal{C}}^{\underline{\mathrm{op}}} \rightleftarrows \underline{\mathcal{C}}^{\underline{\mathrm{op}}} \underline{\times} \underline{\mathcal{C}}^{\underline{\mathrm{op}}} : \oplus$ together with the bireduction biadjunction induces a biadjunction*

$$\begin{tikzcd} \underline{\mathrm{Fun}}^*(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathrm{Sp}}_G) \arrow[r, shift left, "\oplus^*"] & \underline{\mathrm{Fun}}^*(\underline{\mathcal{C}}^{\underline{\mathrm{op}}} \underline{\times} \underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathrm{Sp}}_G) \arrow[l, shift left, "\Delta^*"] \arrow[r, shift left, "(-)^{\mathrm{red}}"] & \underline{\mathrm{BiRed}}(\underline{\mathcal{C}}) \arrow[l, shift left, hook] \end{tikzcd}$$

where the top composite is precisely $B_{(-)}$ by definition. The diagonal $\Delta_x : x \to x \oplus x$ and fold $\nabla_x : x \oplus x \to x$ induce the counit and unit

$$B^{\Delta}_{\underline{\Qoppa}} \Rightarrow \underline{\Qoppa} \Rightarrow B^{\Delta}_{\underline{\Qoppa}}$$

respectively.

Corollary 7.1.19 (Quadratic-bilinear biadjunction, "[CDH+20a, Rmk. 1.1.18]")**.** *We have a biadjunction*

$$\begin{tikzcd} \underline{\mathrm{Fun}}^{\mathrm{q}}(\underline{\mathcal{C}}) \arrow[r, "B_{(-)}"] & \underline{\mathrm{Fun}}^{\mathrm{b}}(\underline{\mathcal{C}}) \arrow[l, bend right=50, "\Delta^*"'] \arrow[l, bend left=50, "\Delta^*"] \end{tikzcd}$$

with unit and counit given by the natural maps

$$B_{\underline{\Qoppa}}(x,x) \to \underline{\Qoppa}(x) \to B_{\underline{\Qoppa}}(x,x)$$

Proof. This is just by applying Corollary 7.1.18: consider the diagram

$$
\begin{tikzcd}
\underline{\mathrm{Fun}}_*(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathrm{Sp}}_G) \arrow[r, "B_{(-)}"] & \underline{\mathrm{BiRed}}(\underline{\mathcal{C}}) \arrow[l, bend right, "\Delta^*"'] \arrow[l, bend left, "\Delta^*"] \\
\underline{\mathrm{Fun}}^{\underline{\mathrm{q}}}(\underline{\mathcal{C}}) \arrow[u, hook] \arrow[r, "B_{(-)}"] & \underline{\mathrm{Fun}}^{\underline{\mathrm{b}}}(\underline{\mathcal{C}}) \arrow[u, hook] \arrow[l, bend right, "\Delta^*"'] \arrow[l, bend left, "\Delta^*"]
\end{tikzcd}
$$

where the $B_{(-)}$ square commutes by Proposition 7.1.15 and the Δ^* squares commute by Observation 7.1.10. □

Proposition 7.1.20 (Quadratic-(co)linear adjunctions, "[CDH+20a, Lem. 1.1.24]")**.** *The natural transformations $\underline{\Qoppa} \Rightarrow L_{\underline{\Qoppa}}$ and $cL_{\underline{\Qoppa}} \Rightarrow \underline{\Qoppa}$ exhibits the unit (resp. counit) of the adjunctions*

$$
\begin{tikzcd}
\underline{\mathrm{Fun}}^{\underline{\mathrm{q}}}(\underline{\mathcal{C}}) \arrow[r, bend left, "L_{(-)}"] \arrow[r, bend right, "cL_{(-)}"'] & \underline{\mathrm{Fun}}^{\underline{\mathrm{ex}}}(\underline{\mathcal{C}}) \arrow[l, hook']
\end{tikzcd}
$$

Proof. We show the linear part. We just need to show that the mapping G-spectrum from the fibre $(B^{\Delta}_{\underline{\Qoppa}})_{h\Sigma_2}$ of $\underline{\Qoppa} \Rightarrow L_{\underline{\Qoppa}}$ to any G-exact functor is zero. So let f be a G-exact functor.

$$\underline{\mathrm{map}}((B^{\Delta}_{\underline{\Qoppa}})_{h\Sigma_2}, f) \simeq \underline{\mathrm{map}}(\Delta^* B_{\underline{\Qoppa}}, f)^{h\Sigma_2} \simeq \underline{\mathrm{map}}(B_{\underline{\Qoppa}}, B_f)^{h\Sigma_2} \simeq 0$$

where the second equivalence is by Corollary 7.1.19 and $B_f \simeq 0$ for f G-exact by Proposition 6.6.3. □

Corollary 7.1.21 (Quadratic-symmetric bilinear adjunction, "[CDH+20a, Cor. 1.3.3]")**.** *We have an adjunction*

$$
\begin{tikzcd}
\underline{\mathrm{Fun}}^{\underline{\mathrm{q}}}(\underline{\mathcal{C}}) \arrow[r, "B_{(-)}"] & \underline{\mathrm{Fun}}^{\underline{\mathrm{s}}}(\underline{\mathcal{C}}) \arrow[l, bend right, "\Qoppa^q_{(-)}"'] \arrow[l, bend left, "\Qoppa^s_{(-)}"]
\end{tikzcd}
$$

where both $\Qoppa^q$ and $\Qoppa^s$ are G-fully faithful, and their essential images are precisely the G-homogeneous and G-cohomogeneous functors, respectively.

Proof. We will argue in the homogeneous case, and the other will then be similar. We will show two things in turn: (a) that $\Upsilon^q_{(-)} : \underline{\mathrm{Fun}}^{\mathrm{s}}(\underline{\mathcal{C}}) \to \underline{\mathrm{Fun}}^{\mathrm{q}}(\underline{\mathcal{C}})$ is G-fully faithful with the prescribed essential image; (b) that we have an adjunction $\Upsilon^q_{(-)} \dashv B_{(-)}$. To see (a), we factor it as

$$\Upsilon^q_{(-)} : \underline{\mathrm{Fun}}^{\mathrm{s}}(\underline{\mathcal{C}}) \xrightarrow{\varphi} \underline{\mathrm{Fun}}^{\mathrm{hom}}(\underline{\mathcal{C}}) \subseteq \underline{\mathrm{Fun}}^{\mathrm{q}}(\underline{\mathcal{C}})$$

where $\underline{\mathrm{Fun}}^{\mathrm{hom}}(\underline{\mathcal{C}})$ is the G-full subcategory spanned by G-homogeneous quadratics. We have this factorisation by the characterisation of G-homogeneity Proposition 7.1.16. On the other hand, the formation of cross-effects

$$\psi : \underline{\mathrm{Fun}}^{\mathrm{hom}}(\underline{\mathcal{C}}) \subseteq \underline{\mathrm{Fun}}^{\mathrm{q}}(\underline{\mathcal{C}}) \xrightarrow{B_{(-)}} \underline{\mathrm{Fun}}^{\mathrm{s}}(\underline{\mathcal{C}})$$

gives a right inverse $\varphi \circ \psi \simeq \mathrm{id}$ by Proposition 7.1.16, whereas Construction 7.1.11 gives that $\psi \circ \varphi \simeq \mathrm{id}$, as required.

Finally to see (b), standard adjunction yoga says that we need to show that the natural comparison $\varepsilon : \Upsilon^q_{B_{\underline{\Upsilon}}} \Rightarrow \underline{\Upsilon}$ induces an equivalence

$$\underline{\mathrm{Nat}}_s(\beta, B_{\underline{\Upsilon}}) \xrightarrow{\Upsilon^q_{(-)}} \underline{\mathrm{Nat}}_q(\Upsilon^q_\beta, \Upsilon^q_{B_{\underline{\Upsilon}}}) \xrightarrow{\varepsilon_*} \underline{\mathrm{Nat}}_q(\Upsilon^q_\beta, \underline{\Upsilon})$$

for all $\beta \in \underline{\mathrm{Fun}}^{\mathrm{s}}(\underline{\mathcal{C}})$. Now the first map is an equivalence by (a). On the other hand, the second map is also an equivalence since $\mathrm{cofib}(\Upsilon^q_{B_{\underline{\Upsilon}}} \Rightarrow \underline{\Upsilon}) \simeq L_{\underline{\Upsilon}}$, and $\underline{\mathrm{Nat}}_q(\Upsilon^q_\beta, L_{\underline{\Upsilon}}) \simeq 0$ by Proposition 7.1.16. $\square$

7.1.4 The quadratic stable recollement

The notion of equivariant stable recollement that we need will be a fibrewise one, following [CDH+20b, Prop. A.2.10].

Definition 7.1.22. Let $\underline{\mathcal{C}} \xrightarrow{f} \underline{\mathcal{D}} \xrightarrow{p} \underline{\mathcal{E}}$ be functors between G-stable categories with trivial composite. Then we say that it is a *stable recollement* if the following conditions hold:

(i) It is a fibre sequence (in particular, this means that f is fully faithful),

(ii) f admits a G–left adjoint (that is, it participates in a G–Bousfield localisation)

(iii) p admits a fully faithful G–right adjoint (that is, it is a G–Bousfield localisation).

Remark 7.1.23. By [CDH+20b, Lem. A.2.5], a stable recollement in fact always complete automatically to a split Verdier sequence (cf. §4.1.1)

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "f"] & \mathcal{D} \arrow[l, bend right, "\overline{g}"'] \arrow[l, bend left, dashed, "g"] \arrow[r, "p"] & \mathcal{E} \arrow[l, bend right, dashed, "\overline{q}"'] \arrow[l, bend left, "q"]
\end{tikzcd}$$

In this way, we have the slogan "stable recollement = split Verdier sequences".

We now state the main theorem of this section which organises G–quadraticity, G–linearity, and G–symmetric bilinearity in a stable recollement analogous to [CDH+20a, Cor. 1.3.12].

Theorem 7.1.24 (Quadratic stable recollement)**.** *We have the stable recollement*

$$\begin{tikzcd}
\underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{C}}) \arrow[r, hook] & \underline{\mathrm{Fun}}^{\mathrm{q}}(\underline{\mathcal{C}}) \arrow[l, bend right, "L_{(-)}"'] \arrow[l, bend left, "cL_{(-)}"] \arrow[r, "B_{(-)}"] & \underline{\mathrm{Fun}}^{\mathrm{s}}(\underline{\mathcal{C}}) \arrow[l, bend right, "\text{Ϙ}^{q}_{(-)}"'] \arrow[l, bend left, "\text{Ϙ}^{s}_{(-)}"]
\end{tikzcd}$$

In particular by standard recollement fractures we have the cartesian square for any $\underline{\text{Ϙ}} \in \underline{\mathrm{Fun}}^{\mathrm{q}}(\underline{\mathcal{C}})$

$$\begin{tikzcd}
\underline{\text{Ϙ}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L_{\underline{\text{Ϙ}}} \arrow[d] \\
\text{Ϙ}^{s}_{B_{\underline{\text{Ϙ}}}} \arrow[r] & L_{\text{Ϙ}^{s}_{B_{\underline{\text{Ϙ}}}}}
\end{tikzcd}$$

where the right vertical is the linearisation of the left. Moreover the bottom map is equivalent to

$$B_{\underline{\text{Ϙ}}}(X,X)^{h\Sigma_2} \to B_{\underline{\text{Ϙ}}}(X,X)^{t\Sigma_2}$$

the usual Tate map.

Proof. To see the stable recollement, we need to check the axioms of Definition 7.1.22: Proposition 7.1.20 gives axiom (ii); Corollary 7.1.21 gives axiom (iii); and for axiom (i), suppose $\text{Ϙ} \in \underline{\mathrm{Fun}}^{\mathrm{q}}(\underline{\mathcal{C}})$ such that $B_{\underline{\text{Ϙ}}} \simeq 0$. Then by Fact 7.1.14, we have that the canonical transformation $\underline{\text{Ϙ}} \Rightarrow P_{1_+}\underline{\text{Ϙ}}$ is an equivalence. But then by definition of G-quadratic functors, this means that the composite transformation $\underline{\text{Ϙ}} \Rightarrow P_{1_+}\underline{\text{Ϙ}} \Rightarrow P_{G_+}\underline{\text{Ϙ}}$ is an equivalence, hence implying that the middle sequence is a fibre sequence as required.

We now prove the last assertion. By general principles of stable recollement we know that $L_{\text{Ϙ}^{s}_{B_{\underline{\text{Ϙ}}}}}$ is computed as the cofibre of the adjunction counit

$$\text{Ϙ}^{q}_{B_{\underline{\text{Ϙ}}}} \simeq \text{Ϙ}^{q}_{B_{\text{Ϙ}^{s}_{B_{\underline{\text{Ϙ}}}}}} \Rightarrow \text{Ϙ}^{s}_{B_{\underline{\text{Ϙ}}}}$$

Now, by unwinding adjunctions, we have

$$B_{(-)} : \underline{\mathrm{Nat}}^{\mathrm{q}}(Ϙ^{q}_{B_{\underline{Ϙ}}}, Ϙ^{s}_{B_{\underline{Ϙ}}}) \xrightarrow{\simeq} \underline{\mathrm{Nat}}^{\mathrm{s}}(B_{\underline{Ϙ}}, B_{\underline{Ϙ}})$$

hence since $B_{(-)}$ strongly preserve all G-limits and G-colimits, it preserves norm maps and so to show that the natural transformation in question is given by the Σ_2–norm map, it suffices to show that it has the same image as the norm map under $B_{(-)}$. By an easy unwinding of adjunctions, the image of the map of interest under the functor $B_{(-)}$ is the identity natural transformation $B \Rightarrow B$. On the other hand, applying $B_{(-)}$ to the norm $(B^{\Delta}_{\underline{Ϙ}})_{h\Sigma_2} \Rightarrow (B^{\Delta}_{\underline{Ϙ}})^{h\Sigma_2}$ gives

$$(B \oplus B)_{h\Sigma_2} \to (B \oplus B)^{h\Sigma_2}$$

which *is* the identity on B by the general theory on norms. This completes the proof. $\square$

Remark 7.1.25. To sum up the situation, we have the cofibre sequences in $\underline{\mathrm{Sp}}_G$

$$\begin{tikzcd}
 & cL_{B_{\underline{Ϙ}}}(x) \arrow[r, equal] \arrow[d, "\simeq"', "\text{linear}"] & cL_{\underline{Ϙ}}(x) \arrow[d] \\
B_{B_{\underline{Ϙ}}}(x,x)_{h\Sigma_2} \arrow[r, "\simeq", "\text{homogeneous}"'] \arrow[d, equal] & B_{\underline{Ϙ}}(x) \arrow[r] \arrow[d, "\simeq"', "\text{cohomogeneous}"] & L_{B_{\underline{Ϙ}}}(x) \arrow[d] \\
B_{B_{\underline{Ϙ}}}(x,x)_{h\Sigma_2} \arrow[r, "\text{norm}"'] & B_{B_{\underline{Ϙ}}}(x,x)^{h\Sigma_2} \arrow[r] & B_{B_{\underline{Ϙ}}}(x,x)^{t\Sigma_2}
\end{tikzcd}$$

7.1.5 G-quadraticity vs reduced $2G_+$-excisive

In this subsection, the main goal is to show that we have a G–Bousfield localisation

$$\underline{\mathrm{Fun}}^{*}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathrm{Sp}}_G) \rightleftarrows \underline{\mathrm{Fun}}^{\mathrm{q}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathrm{Sp}}_G)$$

which we will have use of in §7.2. We will also use this to explain the distinction between the weaker condition of being reduced $2G_+$–excisive and the stronger one of G–quadraticity.

By the stable recollement categorical decomposition, we have the following pullback of G–categories and the induced dashed functor F to the pullback $\underline{\mathrm{Fun}}^{\mathrm{q}}(\underline{\mathcal{C}})$

$$\begin{tikzcd}
\underline{\mathrm{Fun}}^*(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}) \arrow[dr, dashed, "F"] \arrow[ddr, bend right, "B"'] \arrow[drr, bend left, "L \Rightarrow L_{\underline{\text{Ϙ}}^s_B}"] & & \\
& \underline{\mathrm{Fun}}^{\mathrm{q}}(\underline{\mathcal{C}}) \arrow[r, "L \Rightarrow L_{\underline{\text{Ϙ}}^s_B}"] \arrow[d, "B"'] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{C}})^{\Delta^1} \arrow[d, "\mathrm{tgt}"] \\
& \underline{\mathrm{Fun}}^{\mathrm{s}}(\underline{\mathcal{C}}) \arrow[r, "L_{\underline{\text{Ϙ}}^s}"'] & \underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{C}})
\end{tikzcd}$$

We show that the functor $F : \underline{\mathrm{Fun}}^*(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}) \to \underline{\mathrm{Fun}}^{\mathrm{q}}(\mathcal{C})$ is a left adjoint to the inclusion. For this, we will need the following standard result:

Lemma 7.1.26. *Suppose we have a pullback of categories*

$$\begin{tikzcd}
\mathcal{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}^{\Delta^1} \arrow[d, "\mathrm{tgt}"] \\
\mathcal{E} \arrow[r, "\varphi"'] & \mathcal{C}
\end{tikzcd}$$

Then we have a natural equivalence of mapping spaces

$$\begin{aligned}
&\mathrm{Map}_{\mathcal{D}}\Big((e, c \to d, \varphi(e) \simeq d), (e', c' \to d', \varphi(e') \simeq d')\Big) \\
\simeq\ &\mathrm{Map}_{\mathcal{E}}(e, e') \times_{\mathrm{Map}_{\mathcal{C}}(c,d')} \mathrm{Map}_{\mathcal{C}}(c, c')
\end{aligned}$$

Proof. For this just observe the following sequence of equivalences

$$\begin{aligned}
&\mathrm{Map}_{\mathcal{D}}\Big((e, c \to d, \varphi(e) \simeq d), (e', c' \to d', \varphi(e') \simeq d')\Big) \\
\simeq\ &\mathrm{Map}_{\mathcal{E}}(e, e') \times_{\mathrm{Map}_{\mathcal{C}}(d,d')} \mathrm{Map}_{\mathcal{C}^{\Delta^1}}\Big((c \to d), (c' \to d')\Big) \\
\simeq\ &\mathrm{Map}_{\mathcal{E}}(e, e') \times_{\mathrm{Map}_{\mathcal{C}}(d,d')} \left(\mathrm{Map}_{\mathcal{C}}(d, d') \times_{\mathrm{Map}_{\mathcal{C}}(c,d')} \mathrm{Map}_{\mathcal{C}}(c, c')\right) \\
\simeq\ &\mathrm{Map}_{\mathcal{E}}(e, e') \times_{\mathrm{Map}_{\mathcal{C}}(c,d')} \mathrm{Map}_{\mathcal{C}}(c, c')
\end{aligned}$$

as claimed. $\square$

Proposition 7.1.27. *There is a G–adjunction*

$$\underline{\mathrm{Fun}}^*(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}) \underset{}{\overset{F}{\rightleftarrows}} \underline{\mathrm{Fun}}^{\mathrm{q}}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}})$$

where F is the functor considered above. Explicitly, it is given by

$$
\underline{\Qoppa} \quad \mapsto \quad
\begin{tikzcd}
F_{\underline{\Qoppa}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L_{\underline{\Qoppa}} \arrow[d] \\
(B^{\Delta}_{\underline{\Qoppa}})^{h\Sigma_2} \arrow[r] & (B^{\Delta}_{\underline{\Qoppa}})^{t\Sigma_2}
\end{tikzcd}
$$

Proof. Let $\underline{\Psi} \in \underline{\mathrm{Fun}}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}})$ and $\underline{\Qoppa} \in \underline{\mathrm{Fun}}^{\mathrm{q}}(\underline{\mathcal{C}}) = \underline{\mathrm{Fun}}^{\mathrm{q}}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}})$. We need to show that

$$
\underline{\mathrm{Map}}_{\underline{\mathrm{Fun}}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}})}(\underline{\Psi}, \underline{\Qoppa}) \xrightarrow{F} \underline{\mathrm{Map}}_{\underline{\mathrm{Fun}}^{\mathrm{q}}(\underline{\mathcal{C}})}(F\underline{\Psi}, F\underline{\Qoppa}) \simeq \underline{\mathrm{Map}}_{\underline{\mathrm{Fun}}^{\mathrm{q}}(\underline{\mathcal{C}})}(F\underline{\Psi}, \underline{\Qoppa}) \tag{7.1}
$$

is an equivalence. Now since it is clear that the inclusion functor strongly preserves finite G–limits, and since $\underline{\Qoppa}$ can be expressed as a pullback by Theorem 7.1.24, we see that the target is computed as

$$
\begin{tikzcd}
\underline{\mathrm{Map}}_{\underline{\mathrm{Fun}}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}})}(\underline{\Psi}, \underline{\Qoppa}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Map}}_{\underline{\mathrm{Fun}}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}})}(\underline{\Psi}, L_{\underline{\Qoppa}}) \arrow[d] \\
\underline{\mathrm{Map}}_{\underline{\mathrm{Fun}}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}})}(\underline{\Psi}, (B^{\Delta}_{\underline{\Qoppa}})^{h\Sigma_2}) \arrow[r] & \underline{\mathrm{Map}}_{\underline{\mathrm{Fun}}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}})}(\underline{\Psi}, (B^{\Delta}_{\underline{\Qoppa}})^{t\Sigma_2})\underline{\mathrm{Map}}_{\underline{\mathrm{Fun}}^{\mathrm{s}}(\underline{\mathcal{C}})}
\end{tikzcd}
$$

which in turn is given by

$$
\begin{tikzcd}
\underline{\mathrm{Map}}_{\underline{\mathrm{Fun}}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}})}(\underline{\Psi}, \underline{\Qoppa}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Map}}_{\underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{C}})}(L_{\underline{\Psi}}, L_{\underline{\Qoppa}}) \arrow[d] \\
(B_{\underline{\Psi}}, B_{\underline{\Qoppa}}) \arrow[r] & \underline{\mathrm{Map}}_{\underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{C}})}(L_{\underline{\Psi}}, L_{\underline{\Qoppa}^{s}_{B_{\underline{\Qoppa}}}})
\end{tikzcd}
$$

where we have also used the natural equivalence $(B^{\Delta}_{\underline{\mathcal{C}}})^{t\Sigma_2} \simeq L_{\underline{\mathcal{C}}^{s}_{B_{\underline{\mathcal{C}}}}}$ for the bottom right identification. On the other hand, by the formula for mapping spaces in stable recollements Lemma 7.1.26, we see that the target in Eq. (7.1) is computed as the pullback square on the right. Moreover, since the functor F was constructed by universal property of pullbacks, the map F on mapping spaces in Eq. (7.1) is indeed the one implementing this identification, and so it is an equivalence, as was to be shown. $\square$

Remark 7.1.28. To summarise the all the available adjunctions vis-a-vis G–quadraticity vs reduced $2G_+$–excisiveness, we have

$$
\underline{\mathrm{Fun}}^{*}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}) \underset{\longleftarrow}{\xrightarrow{P_{2G_+}}} \underline{\mathrm{Fun}}^{2G-\mathrm{exc},*}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G) \underset{\longleftarrow}{\xrightarrow{F}} \underline{\mathrm{Fun}}^{\mathrm{q}}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}})
$$

Hence the difference between being merely reduced $2G_+$–excisive and being G–quadratic is precisely that the latter are the ones are those that admit the fracture square decomposition as in Proposition 7.1.27.

7.1.6 Hermitian and Poincaré structures

Definition 7.1.29. A G–*hermitian* category is a pair $(\underline{\mathcal{C}}, \underline{\Omega})$ where $\underline{\mathcal{C}}$ is small G–perfect-stable and $\underline{\Omega}$ is G–quadratic.

Construction 7.1.30. These can be organised into a large G–category $\underline{\mathrm{Cat}}^{\mathrm{h}}_G$ given by unstraightening $(\underline{\mathrm{Cat}}^{\mathrm{perf}}_G)^{\underline{\mathrm{op}}} \to \widehat{\underline{\mathrm{Cat}}}_G \;::\; \underline{\mathcal{C}} \mapsto \underline{\mathrm{Fun}}^{\underline{\mathrm{q}}}(\underline{\mathcal{C}})$ using Theorem 1.1.18. Unwinding definitions, we see that a *G-hermitian functor* $(\underline{\mathcal{C}}, \underline{\Omega}) \to (\underline{\mathcal{C}}', \underline{\Omega}')$ consists of a G–linear functor $f : \underline{\mathcal{C}} \to \underline{\mathcal{C}}'$ and a natural transformation $\eta : \underline{\Omega} \Rightarrow f^*\underline{\Omega}'$.

We now explore some categorified notions of non-degeneracies that will lead to the notion of G–Poincaré categories.

Construction 7.1.31 (The duality functor)**.** Let $\beta \in \underline{\mathrm{Fun}}^{\underline{\mathrm{b}}}(\underline{\mathcal{C}})$ be G–bilinear. Suppose the following curried functor lands in the representables

$$\underline{\mathcal{C}}^{\underline{\mathrm{op}}} \to \underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathrm{Sp}}_G) \quad :: \quad y \mapsto \beta(-, y)$$

Then this functor can then be lifted to a functor

$$D^R_\beta : \underline{\mathcal{C}}^{\underline{\mathrm{op}}} \to \underline{\mathcal{C}}$$

so that we have $\beta(x, y) \simeq \underline{\mathrm{map}}_{\underline{\mathcal{C}}}(x, D^R_\beta y)$. Similarly, we may also be in a situation where $\beta(x, -) \simeq \underline{\mathrm{map}}_{\underline{\mathcal{C}}}(-, D^L_\beta x)$. Clearly if β were symmetric then it is right non-degenerate if and only if it is left non-degenerate.

Definition 7.1.32. If β were non-degenerate symmetric, then writing $D : \underline{\mathcal{C}}^{\underline{\mathrm{op}}} \to \underline{\mathcal{C}}$ and $D^{\underline{\mathrm{op}}}$ for the opposite, we see that

$$\underline{\mathrm{map}}_{\underline{\mathcal{C}}}(x, Dy) \simeq \beta(x, y) \simeq \beta(y, x) \simeq \underline{\mathrm{map}}_{\underline{\mathcal{C}}}(y, Dx) \simeq \underline{\mathrm{map}}_{\underline{\mathcal{C}}^{\underline{\mathrm{op}}}}(D^{\underline{\mathrm{op}}} x, y)$$

and so D^{op} *is* the left adjoint to D. We define the *duality evaluation* to be the unit map

$$\mathrm{ev} : id \Rightarrow DD^{\underline{\mathrm{op}}} : \underline{\mathcal{C}} \to \underline{\mathcal{C}}$$

A symmetric G–bilinear functor is called *perfect* if ev is an equivalence, and this implies $D_\beta : \underline{\mathcal{C}}, \underline{\Omega}^{\underline{\mathrm{op}}} \to \underline{\mathcal{C}}$ is an equivalence.

Definition 7.1.33. We say that a bilinear functor β is right (resp. left) non-degenerate if $\beta(-, y)$ (resp. $\beta(x, -)$) are representables. If it is both left and right non-degenerate, we say it is *non-degenerate*. In this case the resulting dualities are of course adjoint to each other as $D^L_\beta : \underline{\mathcal{C}} \rightleftarrows \underline{\mathcal{C}}^{\underline{\mathrm{op}}} : D^R_\beta$. We say that a quadratic functor $\underline{\Omega} : \underline{\mathcal{C}}^{\underline{\mathrm{op}}} \to \underline{\mathrm{Sp}}_G$ is *non-degenerate* if the G–bilinear part $B_{\underline{\Omega}}$ is non-degenerate. We denote by

$$\underline{\mathrm{Fun}}^{\mathrm{nb}}(\underline{\mathcal{C}}) \subseteq \underline{\mathrm{Fun}}^{\underline{\mathrm{b}}}(\underline{\mathcal{C}}) \qquad \underline{\mathrm{Fun}}^{\mathrm{ns}}(\underline{\mathcal{C}}) \subseteq \underline{\mathrm{Fun}}^{\underline{\mathrm{s}}}(\underline{\mathcal{C}}) \qquad \underline{\mathrm{Fun}}^{\mathrm{nq}}(\underline{\mathcal{C}}) \subseteq \underline{\mathrm{Fun}}^{\underline{\mathrm{q}}}(\underline{\mathcal{C}})$$

for the full subcategories spanned by non-degenerates.

Here is a basic result analogous to [CDH+20a, Lem. 1.2.4].

Lemma 7.1.34. *Let* $(\underline{\mathcal{C}}, \underline{\mathrm{Ϙ}}), (\underline{\mathcal{C}}', \underline{\mathrm{Ϙ}}')$ *be two non-degenerate G–hermitian categories with associated dualities* $D_{\underline{\mathrm{Ϙ}}} : \underline{\mathcal{C}}^{\underline{\mathrm{op}}} \to \underline{\mathcal{C}}$ *and* $D_{\underline{\mathrm{Ϙ}}'} : \underline{\mathcal{C}}^{\underline{\mathrm{op}}} \to \underline{\mathcal{C}}'$. *Let* $f, g : \underline{\mathcal{C}} \to \underline{\mathcal{C}}'$ *be G–exact functors. Then there is a natural equivalence*

$$\underline{\mathrm{nat}}^{\underline{\mathrm{b}}}(B_{\underline{\mathrm{Ϙ}}}, (f \times g)^* B_{\underline{\mathrm{Ϙ}}'}) \simeq \underline{\mathrm{nat}}^{\underline{\mathrm{ex}}}(f D_{\underline{\mathrm{Ϙ}}}, D_{\underline{\mathrm{Ϙ}}'} g^{\underline{\mathrm{op}}})$$

Proof. We have $\underline{\mathrm{nat}}_{\underline{\mathcal{C}},\underline{\mathcal{C}}}(B_{\underline{\mathrm{Ϙ}}}, (f \times g)^* B_{\underline{\mathrm{Ϙ}}'}) \simeq \underline{\mathrm{nat}}_{\underline{\mathcal{C}}',\underline{\mathcal{C}}}((f \times 1)_! B_{\underline{\mathrm{Ϙ}}}, (1 \times g)^* B_{\underline{\mathrm{Ϙ}}'})$. And by hypothesis, for fixed $y \in \underline{\mathcal{C}}$ we have $B_{\underline{\mathrm{Ϙ}}}(-, y) \simeq \underline{\mathrm{map}}_{\underline{\mathcal{C}}}(-, Dy)$. But by easy adjunction yoga we see that left Kan extensions commute with representables and so we have

$$(f \times 1)_! \underline{\mathrm{map}}_{\underline{\mathrm{Ϙ}}}(-, Dy) \simeq \underline{\mathrm{map}}_{\underline{\mathrm{Ϙ}}'}(-, f Dy)$$

Hence in total we have

$$\begin{aligned}
&\underline{\mathrm{nat}}_{\underline{\mathcal{C}},\underline{\mathcal{C}}}(B_{\underline{\mathrm{Ϙ}}}, (f \times g)^* B_{\underline{\mathrm{Ϙ}}'}) \\
\simeq\ & \underline{\mathrm{nat}}_{\mathrm{Fun}((\underline{\mathcal{C}}'^{\underline{\mathrm{op}}} \times \underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathrm{Sp}}_G)}((f \times 1)_! B_{\underline{\mathrm{Ϙ}}}, (1 \times g)^* B_{\underline{\mathrm{Ϙ}}'}) \\
\simeq\ & \underline{\mathrm{nat}}_{\mathrm{Fun}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \mathrm{Fun}(\underline{\mathcal{C}}'^{\underline{\mathrm{op}}}, \underline{\mathrm{Sp}}_G))}((f \times 1)_! B_{\underline{\mathrm{Ϙ}}}, (1 \times g)^* B_{\underline{\mathrm{Ϙ}}'}) \\
\simeq\ & \underline{\lim}_{(x \to y) \in \underline{\mathrm{TwAr}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}})} \underline{\mathrm{nat}}_{\mathrm{Fun}(\underline{\mathcal{C}}'^{\underline{\mathrm{op}}}, \underline{\mathrm{Sp}}_G)}((f \times 1)_! B_{\underline{\mathrm{Ϙ}}}(-, x), (1 \times g)^* B_{\underline{\mathrm{Ϙ}}'}(-, y)) \\
\simeq\ & \underline{\lim}_{(x \to y) \in \underline{\mathrm{TwAr}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}})} \underline{\mathrm{nat}}_{\mathrm{Fun}(\underline{\mathcal{C}}'^{\underline{\mathrm{op}}}, \underline{\mathrm{Sp}}_G)}(\underline{\mathrm{map}}_{\underline{\mathcal{C}}'}(-, f Dx), \underline{\mathrm{map}}_{\underline{\mathcal{C}}'}(-, D' gy)) \\
\simeq\ & \underline{\lim}_{(x \to y) \in \underline{\mathrm{TwAr}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}})} \underline{\mathrm{map}}_{\underline{\mathcal{C}}'}(f Dx, D' gy) \\
\simeq\ & \underline{\mathrm{nat}}_{\underline{\mathcal{C}}}(f D, D' g^{\underline{\mathrm{op}}})
\end{aligned}$$

as required. Here we have used the parametrised twisted arrow formula for natural transformations from Proposition 1.2.31. □

This allows us to frame the following important definitions.

Definition 7.1.35 (Duality preservation)**.** Given a G–hermitian functor $(f, \eta) : (\underline{\mathcal{C}}, \underline{\mathrm{Ϙ}}) \to (\underline{\mathcal{C}}', \underline{\mathrm{Ϙ}}')$, since [CDH+20a, Rmk. 1.1.6] says $(f \times f)^* B_{\underline{\mathrm{Ϙ}}'} \simeq B_{f^* \underline{\mathrm{Ϙ}}'}$, we get a transformation

$$\beta_\eta : B_{\underline{\mathrm{Ϙ}}} \Rightarrow (f \times f)^* B_{\underline{\mathrm{Ϙ}}'}$$

We then denote by

$$\tau_\eta : f D_{\underline{\mathrm{Ϙ}}} \Rightarrow D_{\underline{\mathrm{Ϙ}}'} f^{\underline{\mathrm{op}}}$$

the transformation corresponding to the data B_η by Lemma 7.1.34 and the equivalence $(f \times f)^* B_{\underline{\mathrm{Ϙ}}'} \simeq B_{f^* \underline{\mathrm{Ϙ}}'}$. We say that a G–hermitian functor is *duality-preserving* if this τ_η is an equivalence.

Remark 7.1.36. Note that all these non-degeneracy conditions depend only on the (symmetric) bilinear part of a quadratic functor.

Definition 7.1.37. A G–hermitian category $(\underline{\mathcal{C}}, \underline{\Omega})$ is called *G–Poincaré* if it satisfies the property of $B_{\underline{\Omega}}$ being perfect. We let $\underline{\mathrm{Cat}}_G^{\mathrm{p}} \subset \underline{\mathrm{Cat}}_G^{\mathrm{h}}$ denote the non-full subcategory spanned by G–Poincaré categories and duality-preserving functors.

Observation 7.1.38. The constructions Ω^q_β and Ω^s_β of Construction 7.1.11 given a G–symmetric bilinear β are G–Poincaré if and only if β is perfect.

Observation 7.1.39 (Equivariant shifts)**.** Let V be a finite dimensional G–representation. Note that we have following easy identifications

(i) $B_{\Omega^V \underline{\Omega}} \simeq \Omega^V B_{\underline{\Omega}}$

(ii) $L_{\Omega^V \underline{\Omega}} \simeq \Omega^V L_{\underline{\Omega}}$

(iii) $D_{\Omega^V \underline{\Omega}} \simeq \Omega^V D_{\underline{\Omega}}$

In particular, $\underline{\Omega}$ is non-degenerate or perfect if and only if $\Omega^V \underline{\Omega}$ is. Hence a G–hermitian category $(\underline{\mathcal{C}}, \underline{\Omega})$ is G–Poincaré if and only if $(\underline{\mathcal{C}}, \Omega^V \underline{\Omega})$ is for all G–representations V.

Lemma 7.1.40 (Quadraticity is connective, "[CDH+20a, Lem. 1.1.25]")**.** *Let $(\underline{\mathcal{C}}, \underline{\Omega})$ be a G–hermitian category. Suppose $\underline{\Omega}$ is pointwise coconnective. Then $\underline{\Omega} \simeq 0$. Hence, equivalences between quadratic functors can be detected after applying $\underline{\Omega}^\infty$ and we can as well just consider connective covers of G–quadratic functors.*

Proof. If $\underline{\Omega}$ were G–exact, then $\underline{\Omega} \simeq 0$ since for $n \in \mathbb{Z}$ and for $H \leq G$, we get $\pi_n^H \underline{\Omega}(x) = \pi_1^H \underline{\Omega}(\Sigma^{n-1} x)$ where x is an arbitrary H–object. For general $\underline{\Omega}$, recall from Construction 7.1.7 that $B_{\underline{\Omega}}(x, y)$ is a direct summand of $\underline{\Omega}(x \oplus y)$ and so is also coconnective. Hence for each H–object x we have a G–exact functor $B_{\underline{\Omega}}(x, -) : \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Sp}}_G$ which is coconnective, and thus so also zero by the argument above. But then by Proposition 7.1.15, $\underline{\Omega}$ is G–exact, and so also zero. □

Definition 7.1.41. Let $(\underline{\mathcal{C}}, \underline{\Omega})$ be a G–hermitian category and $x \in \underline{\mathcal{C}}$.

(i) A *G–hermitian form on x* is defined to be a point $q \in \underline{\Omega}^\infty \underline{\Omega}(x)$. We can then get the *$G$–category* $\underline{\mathrm{He}}(\underline{\mathcal{C}}, \underline{\Omega})$ *of G–hermitian objects in* $(\underline{\mathcal{C}}, \underline{\Omega})$ to be the unstraightening of $\underline{\Omega}^\infty \underline{\Omega} : \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathcal{S}}_G$. We define $\underline{\mathrm{Fm}}(\underline{\mathcal{C}}, \underline{\Omega}) := \underline{\mathrm{He}}(\underline{\mathcal{C}}, \underline{\Omega})^{\simeq}$, the G–space of hermitian objects.

(ii) If $\underline{\Omega}$ were non-degenerate, then $\underline{\Omega}^\infty B_{\underline{\Omega}}(x, x) \simeq \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(x, D_{\underline{\Omega}} x)$. In this case, a G–hermitian object (x, q) determines $q_\# : x \to D_{\underline{\Omega}} x$. We say that a G–hermitian form is *G–Poincaré* if $q_\#$ is an equivalence. Let $\underline{\mathrm{Pn}}(\underline{\mathcal{C}}, \underline{\Omega}) \subseteq \underline{\mathrm{Fm}}(\underline{\mathcal{C}}, \underline{\Omega}) \in \underline{\mathcal{S}}_G$ denote the full G–subgroupoid of G–Poincaré objects.

Remark 7.1.42. As in [CDH+20a, §2.1], we can upgrade the above to G–functors

$$\underline{\mathrm{He}} : \underline{\mathrm{Cat}}^{\mathrm{h}}_G \longrightarrow \underline{\mathrm{Cat}}_G \qquad \underline{\mathrm{Fm}} : \underline{\mathrm{Cat}}^{\mathrm{h}}_G \longrightarrow \underline{\mathcal{S}}_G \qquad \underline{\mathrm{Pn}} : \underline{\mathrm{Cat}}^{\mathrm{p}}_G \longrightarrow \underline{\mathcal{S}}_G$$

Unwinding definitions, for $(\underline{\mathcal{C}}, \underline{\Omega}) \in \underline{\mathrm{Cat}}^{\mathrm{p}}_G$ and $H \leq G$, we obtain the identification

$$\underline{\mathrm{Pn}}(\underline{\mathcal{C}}, \underline{\Omega})^H \simeq \mathrm{Pn}\big(\mathcal{C}_H, \underline{\Omega}^H : \mathcal{C}_H^{\mathrm{op}} \xrightarrow{\Omega_H} \mathrm{Sp}_H \xrightarrow{(-)^H} \mathrm{Sp}\big) \in \mathcal{S}$$

where Pn is the space of Poincaré forms in [CDH+20a].

7.1.7 Universal Poincaré category

Definition 7.1.43. We define the *universal G–Poincaré category* $(\underline{\mathrm{Sp}}^{\omega}_G, \underline{\Omega}^{\mathrm{u}})$ as the one obtained as the pullback square in $\underline{\mathrm{Sp}}_G$

$$\begin{tikzcd}
\underline{\Omega}^{\mathrm{u}}(x) \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & \underline{\mathrm{map}}(x, \mathbb{S}_G) \arrow[d, "\Delta_2"] \\
\big[\underline{\mathrm{map}}(x, \mathbb{S}_G) \otimes \underline{\mathrm{map}}(x, \mathbb{S}_G)\big]^{h\Sigma_2} \arrow[rr] & & \big[\underline{\mathrm{map}}(x, \mathbb{S}_G) \otimes \underline{\mathrm{map}}(x, \mathbb{S}_G)\big]^{t\Sigma_2}
\end{tikzcd}$$

where the map Δ_2 is the Σ_2–Tate diagonal for odd groups G constructed in Construction 3.6.4.

Construction 7.1.44 (Universal G–Poincaré form)**.** By definition, applying the universal G–Poincaré structure to $\mathbb{S}_G$ we obtain the pullback square

$$\begin{tikzcd}
\underline{\Omega}^{\mathrm{u}}(\mathbb{S}_G) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{S}_G \arrow[d, "\Delta_2"] \\
\mathbb{S}_G^{h\Sigma_2} \arrow[r] & \mathbb{S}_G^{t\Sigma_2}
\end{tikzcd}$$

and so we obtain a canonical

$$q^{\mathrm{u}} : \mathbb{S}_G \to \underline{\Omega}^{\mathrm{u}}(\mathbb{S}_G)$$

since the Tate diagonal on the sphere spectrum is given by the usual Tate map $\mathbb{S}_G \to \mathbb{S}_G^{t\Sigma_2}$. By construction we get that

$$q^{\mathrm{u}}_{\#} : \mathbb{S}_G \to D\mathbb{S}_G = \mathbb{S}_G$$

is homotopic to the identity because it is the map induced by

$$\mathbb{S}_G \xrightarrow{\mathrm{can}} \mathbb{S}_G^{h\Sigma_2} \simeq \underline{\mathrm{map}}(\mathbb{S}_G, D\mathbb{S}_G)^{h\Sigma_2}$$

In particular q^{u} gives a Poincaré object $(\mathbb{S}_G, q^{\mathrm{u}})$.

The terminology of universal G–Poincaré categories is justified by the following analogue of [CDH+20a, Lem. 4.1.1].

Lemma 7.1.45. *For every quadratic $\underline{Ϙ} : (\underline{\mathrm{Sp}}^{\omega}_{G})^{\underline{\mathrm{op}}} \to \underline{\mathrm{Sp}}_G$ the map*

$$\underline{\mathrm{nat}}^{\mathrm{q}}(\underline{Ϙ}^{\mathrm{u}}, \underline{Ϙ}) \to \underline{\mathrm{map}}(\underline{Ϙ}^{\mathrm{u}}(\mathbb{S}_G), \underline{Ϙ}(\mathbb{S}_G)) \xrightarrow{(q^{\mathrm{u}})^*} \underline{\mathrm{map}}(\mathbb{S}_G, \underline{Ϙ}(\mathbb{S}_G)) = \underline{Ϙ}(\mathbb{S}_G)$$

is an equivalence.

Proof. Since the $\underline{Ϙ}$'s satisfying that the comparison is an equivalence is closed under limits, we can use the G–quadratic stable recollement Theorem 7.1.24 to show it separately for G–exact functors and those of the form $\underline{Ϙ} = (\Delta^*\beta)^{h\Sigma_2}$.

For G–exact $\underline{Ϙ}$'s we use the adjunction $L : \underline{\mathrm{Fun}}^{\mathrm{q}}(\underline{\mathcal{C}}) \rightleftarrows \underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{C}}) : \mathrm{incl}$. Also recall that the linearisation of $Ϙ^{\mathrm{u}}$ is the G–Spanier-Whitehead dualisation D. Then, indeed

$$\underline{\mathrm{nat}}^{\mathrm{q}}(\underline{Ϙ}^{\mathrm{u}}, \underline{Ϙ}) \simeq \underline{\mathrm{nat}}^{\mathrm{ex}}(\underline{\mathrm{map}}(-, \mathbb{S}_G), \underline{Ϙ}) \simeq \underline{Ϙ}(\mathbb{S}_G)$$

For $\underline{Ϙ} = (\Delta^*\beta)^{h\Sigma_2}$, we use the adjunction $B : \underline{\mathrm{Fun}}^{\mathrm{q}}(\underline{\mathcal{C}}) \rightleftarrows \underline{\mathrm{Fun}}^{\mathrm{b}}(\underline{\mathcal{C}}) : \Delta^*$. Thus

$$\begin{aligned} \underline{\mathrm{nat}}^{\mathrm{q}}(Ϙ^{\mathrm{u}}, (\Delta^*\beta)^{h\Sigma_2}) &\simeq \underline{\mathrm{nat}}^{\mathrm{q}}(Ϙ^{\mathrm{u}}, \Delta^*\beta)^{h\Sigma_2} \\ &\simeq \underline{\mathrm{nat}}^{\mathrm{b}}\big(\underline{\mathrm{map}}(-, \mathbb{S}_G) \otimes \underline{\mathrm{map}}(-, \mathbb{S}_G), \beta\big)^{h\Sigma_2} \\ &\simeq \beta(\mathbb{S}_G, \mathbb{S}_G)^{h\Sigma_2} \\ &= (\Delta^*\beta)^{h\Sigma_2}(\mathbb{S}_G) \end{aligned}$$

as was to be shown. □

Lemma 7.1.46. *The functor $(-)^{\simeq} : \underline{\mathrm{Cat}}^{\mathrm{perf}}_{G} \to \underline{\mathcal{S}}_G$ is corepresented by $\underline{\mathrm{Sp}}^{\omega}_{G}$.*

Proof. This is an immediate consequence of Proposition 2.3.16. □

We are now in position to obtain the following analogue of [CDH+20a, Prop. 4.1.3] by an immediate mimicking of the proof there.

Theorem 7.1.47 (Universality)**.** *The data $(\underline{\mathrm{Sp}}^{\omega}_{G}, \underline{Ϙ}^{\mathrm{u}})$ and $(\underline{\mathrm{Sp}}^{\omega}_{G}, \underline{Ϙ}^{\mathrm{u}}, \mathbb{S}_G, q^{\mathrm{u}})$ corepresent the functors $\underline{\mathrm{Fm}}$ and $\underline{\mathrm{Pn}}$ respectively.*

Proof. We first work towards the case of G–hermitian forms. We want to show that

$$\underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}^{\mathrm{h}}_{G}}((\underline{\mathrm{Sp}}^{\omega}_{G}, \underline{Ϙ}^{\mathrm{u}}), (\underline{\mathcal{C}}, \underline{Ϙ})) \to \underline{\mathrm{Fm}}(\underline{\mathcal{C}}, \underline{Ϙ}) \quad :: \quad (g, \eta) \mapsto (g(\mathbb{S}_G), \eta_{\mathbb{S}_G} \circ q^{\mathrm{u}})$$

is a natural equivalence. Here recall that $\eta : \underline{Ϙ}^{\mathrm{u}} \Rightarrow g^*\underline{Ϙ}$ is a natural transformation. To do this we separate the structures in a fibre sequence of G-spaces:

$$\begin{tikzcd} \underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}^{\mathrm{h}}_{G}}((\underline{\mathrm{Sp}}^{\omega}_{G}, \underline{Ϙ}^{\mathrm{u}}), (\underline{\mathcal{C}}, \underline{Ϙ})) \arrow[r] \arrow[d] & \underline{\mathrm{Fm}}(\underline{\mathcal{C}}, \underline{Ϙ}) \arrow[d] \\ \underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}^{\mathrm{perf}}_{G}}(\underline{\mathrm{Sp}}^{\omega}_{G}, \underline{\mathcal{C}}) \arrow[r] & \underline{\mathcal{C}}^{\simeq} \end{tikzcd} \tag{7.2}$$

where the bottom map is an equivalence by the previous lemma. Now the induced map on fibres over $g \in \underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}^{\mathrm{perf}}_G}(\underline{\mathrm{Sp}}^{\omega}_G, \underline{\mathcal{C}})$ is $\underline{\mathrm{Nat}}^{\mathrm{q}}(\underline{\Qoppa}^{\mathrm{u}}, g^*\underline{\Qoppa}) \to \underline{\Omega}^{\infty}\underline{\Qoppa}(g(\mathbb{S}_G))$ and this is an equivalence by Lemma 7.1.45. Hence the top horizontal is an equivalence.

For the Poincaré structure, we know that $\underline{\mathrm{Cat}}^{\mathrm{p}}_G \subset \underline{\mathrm{Cat}}^{\mathrm{h}}_G$ is a non-full subcategory by definition consisting of G–Poincaré categories and duality-preserving functors, and so the fibres on both sides of Eq. (7.2) are the subcomponents of the fibres in the G–hermitian case cut out by certain non-degeneracy properties. And so to show an equivalence on the fibres it is enough to show that the transformations on the source satisfies the G–Poincaré conditions if and only if the G–hermitian forms on the target are G–Poincaré. Now for $(\underline{\mathcal{C}}, \underline{\Qoppa})$ a G-Poincaré category and $g \in \underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}^{\mathrm{perf}}_G}(\underline{\mathrm{Sp}}^{\omega}_G, \underline{\mathcal{C}})$ we have an induced map

$$\underline{\mathrm{Nat}}^{\mathrm{b}}(B_{\underline{\Qoppa}^{\mathrm{u}}}, (g \times g)^* B_{\underline{\Qoppa}}) \simeq \underline{\mathrm{Nat}}^{\mathrm{ex}}(g D_{\underline{\Qoppa}^{\mathrm{u}}}, D_{\underline{\Qoppa}} g^{\mathrm{op}}) \xrightarrow{\mathrm{ev}_{\mathbb{S}_G}} \underline{\mathrm{Map}}_{\underline{\mathcal{C}}}(g(\mathbb{S}_G), D_{\underline{\Qoppa}} g(\mathbb{S}_G))$$

where the first equivalence is by Lemma 7.1.34. This map can be described concretely as

$$\Big(\tau : g D_{\underline{\Qoppa}^{\mathrm{u}}} \Rightarrow D_{\underline{\Qoppa}} g^{\mathrm{op}}\Big) \mapsto \Big(g(\mathbb{S}_G) \xrightarrow{g(q^{\mathrm{u}}_{\#})} g(D\mathbb{S}_G) \xrightarrow{\tau} Dg(\mathbb{S}_G)\Big)$$

Since $q^u_{\#}$ is an equivalence, τ being an equivalence implies the target is also equivalence. On the other hand, if the target is an equivalence, then τ is also an equivalence since all compact G-spectra are built from $\mathbb{S}_G$ by finite G-colimits and retracts, the orbits $\underline{\Sigma}^{\infty} G/H_+$ being given by the indexed biproduct $\mathrm{Ind}^G_H \mathrm{Res}^G_H \mathbb{S}_G$. □

7.1.8 Hyperbolic categories

Construction 7.1.48 (Hyperbolic categories)**.** Let $\underline{\mathcal{C}} \in \underline{\mathrm{Cat}}^{\mathrm{perf}}_G$. We define the hyperbolic category $\underline{\mathrm{Hyp}}(\underline{\mathcal{C}})$ associated to it to be the G–hermitian category whose underlying G–stable category is $\underline{\mathcal{C}} \oplus \underline{\mathcal{C}}^{\mathrm{op}}$ and whose G–quadratic datum is given by

$$\underline{\Qoppa}_{\underline{\mathrm{Hyp}}} : \underline{\mathcal{C}} \oplus \underline{\mathcal{C}}^{\mathrm{op}} \longrightarrow \underline{\mathrm{Sp}}_G \quad :: \quad (x, y) \mapsto \underline{\mathrm{map}}_{\underline{\mathcal{C}}}(x, y)$$

This is easily checked even to be a G–Poincaré category with duality $(x, y) \mapsto (y, x)$.

Instead of arguing via the more sophisticated and more refined pairing constructions as done in [CDH+20a, §7.3] (which gives a stronger conclusion involving also $\underline{\mathrm{Cat}}^{\mathrm{h}}_G$), we have opted to sketch the proof using just elementary methods.

Proposition 7.1.49. *There is a G–adjunction*

$$\begin{tikzcd} \underline{\mathrm{Cat}}^{\mathrm{p}}_G \arrow[r, "\mathrm{fgt}"] & \underline{\mathrm{Cat}}^{\mathrm{perf}}_G \arrow[l, bend right, "\underline{\mathrm{Hyp}}"'] \arrow[l, bend left, "\underline{\mathrm{Hyp}}"] \end{tikzcd}$$

Proof sketch: Let $(\underline{\mathcal{C}},\underline{Ϙ}) \in \underline{\mathrm{Cat}}^{\mathrm{p}}_G$ and $\mathcal{D} \in \underline{\mathrm{Cat}}^{\mathrm{perf}}_G$. We claim that $\underline{\mathrm{Hyp}}$ being right adjoint to fgt is witnessed by the unit and counit maps

$$\begin{aligned}(\mathrm{fgt},\eta)\colon (\underline{\mathcal{C}},\underline{Ϙ}) \to \underline{\mathrm{Hyp}}(\underline{\mathcal{C}}) \quad :: \quad \eta_x\colon \underline{Ϙ}(x) \Rightarrow \underline{Ϙ}_{\mathrm{Hyp}}\mathrm{fgt}(x) &:= \underline{Ϙ}_{\mathrm{Hyp}}(x, D_{Ϙ}x)\\ &\simeq \underline{\mathrm{map}}_{\mathcal{C}}(x,x)\end{aligned}$$

$$\pi\colon \mathrm{fgt}\circ\underline{\mathrm{Hyp}}(\underline{\mathcal{D}}) \simeq \underline{\mathcal{D}}\oplus\underline{\mathcal{D}}^{\mathrm{op}} \longrightarrow \underline{\mathcal{D}} \quad :: \quad (d,d') \mapsto d$$

and that $\underline{\mathrm{Hyp}}$ being left adjoint to fgt is witnessed by the unit and counit maps

$$(\mathrm{hyp},\varepsilon)\colon \underline{\mathrm{Hyp}}(\underline{\mathcal{C}}) \to (\underline{\mathcal{C}},\underline{Ϙ}) \quad :: \quad \varepsilon_{(x,y)}\colon \underline{Ϙ}_{\mathrm{Hyp}}(x,y) = \underline{\mathrm{map}}_{\mathcal{C}}(x,y) \Rightarrow \underline{Ϙ}(\mathrm{hyp}(x,y)) := \underline{Ϙ}(x\oplus D_{Ϙ}y)$$

$$i : \underline{\mathcal{D}} \longrightarrow \mathrm{fgt}\circ\underline{\mathrm{Hyp}}(\underline{\mathcal{D}}) \simeq \underline{\mathcal{D}}\oplus\underline{\mathcal{D}}^{\mathrm{op}} \quad :: \quad d \mapsto (d,0)$$

We only show the case of being left adjoint, the other one being similar. In this case, we want to show that the composite

$$\underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}^{\mathrm{p}}}(\underline{\mathrm{Hyp}}(\underline{\mathcal{D}}),(\underline{\mathcal{C}},\underline{Ϙ})) \xrightarrow{\mathrm{fgt}} \underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}^{\mathrm{perf}}}(\underline{\mathcal{D}}\oplus\underline{\mathcal{D}}^{\mathrm{op}},\underline{\mathcal{C}}) \xrightarrow{i^*} \underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}^{\mathrm{perf}}}(\underline{\mathcal{D}},\underline{\mathcal{C}})$$

is an equivalence. To this end, we need to show that the two triangles in

$$\begin{tikzcd}
\underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}^{\mathrm{p}}}(\underline{\mathrm{Hyp}}(\underline{\mathcal{D}}),(\underline{\mathcal{C}},\underline{Ϙ})) \arrow[r, "\mathrm{fgt}"] \arrow[dd, equal] & \underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}^{\mathrm{perf}}}(\underline{\mathcal{D}}\oplus\underline{\mathcal{D}}^{\mathrm{op}},\underline{\mathcal{C}}) \arrow[r, "i^*"] & \underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}^{\mathrm{perf}}}(\underline{\mathcal{D}},\underline{\mathcal{C}}) \arrow[dl, "\underline{\mathrm{Hyp}}"] \arrow[dd, equal] \\
& \underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}^{\mathrm{p}}}(\underline{\mathrm{Hyp}}(\underline{\mathcal{D}}),\underline{\mathrm{Hyp}}(\underline{\mathcal{C}})) \arrow[dl, "\mathrm{hyp}"'] & \\
\underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}^{\mathrm{p}}}(\underline{\mathrm{Hyp}}(\underline{\mathcal{D}}),(\underline{\mathcal{C}},\underline{Ϙ})) \arrow[r, "\mathrm{fgt}"] & \underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}^{\mathrm{perf}}}(\underline{\mathcal{D}}\oplus\underline{\mathcal{D}}^{\mathrm{op}},\underline{\mathcal{C}}) \arrow[r, "i^*"] & \underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}^{\mathrm{perf}}}(\underline{\mathcal{D}},\underline{\mathcal{C}})
\end{tikzcd}$$

commute. We only show that the top triangle commutes. So let $(f,\eta) : \underline{\mathrm{Hyp}}(\underline{\mathcal{D}}) \to (\underline{\mathcal{C}},\underline{Ϙ})$ be a G–Poincaré functor. Then the composition sends it to

$$\begin{aligned}\left(\underline{\mathrm{Hyp}}(\underline{\mathcal{D}}) \xrightarrow{(f,\eta)} (\underline{\mathcal{C}},\underline{Ϙ})\right) &\mapsto \left(\underline{\mathcal{D}}\oplus\underline{\mathcal{D}}^{\mathrm{op}} \xrightarrow{f} \underline{\mathcal{C}}\right)\\
&\mapsto \left(\underline{\mathcal{D}} \xrightarrow{i} \underline{\mathcal{D}}\oplus\underline{\mathcal{D}}^{\mathrm{op}} \xrightarrow{f} \underline{\mathcal{C}}\right)\\
&\mapsto \left(\underline{\mathcal{D}}\oplus\underline{\mathcal{D}}^{\mathrm{op}} \xrightarrow{\begin{pmatrix} fi & 0\\ 0 & (fi)^{\mathrm{op}}\end{pmatrix}} \underline{\mathcal{C}}\oplus\underline{\mathcal{C}}^{\mathrm{op}}\right)\\
&\mapsto \left(\underline{\mathcal{D}}\oplus\underline{\mathcal{D}}^{\mathrm{op}} \xrightarrow{\begin{pmatrix} fi & 0\\ 0 & (fi)^{\mathrm{op}}\end{pmatrix}} \underline{\mathcal{C}}\oplus\underline{\mathcal{C}}^{\mathrm{op}} \xrightarrow{\mathrm{hyp}} \underline{\mathcal{C}}\right)\end{aligned}$$

where the Poincaré structure in the last term is given, for $(d,d') \in \underline{\mathcal{D}}^{\mathrm{op}}\oplus\underline{\mathcal{D}}$, by

$$\begin{aligned}\underline{\mathrm{map}}_{\mathcal{D}}(d,d') \xrightarrow{f} \underline{\mathrm{map}}_{\mathcal{C}}(f(d,0),f(d',0)) &\simeq B_{Ϙ}(f(d,0),D_{Ϙ}f(d',0))\\ &\xrightarrow{\mathrm{can}} \underline{Ϙ}(f(d,0),D_{Ϙ}f(d',0))\end{aligned}$$

We claim that this is naturally equivalent to (f,η), ie. the square

$$\begin{tikzcd} \underline{\mathrm{map}}_{\mathcal{D}}(d,d') \arrow[r, "f"] \arrow[d, "\eta_{(d,d')}"'] & B_{\underline{\Omega}}(f(d,0), D_{\underline{\Omega}}f(d',0)) \arrow[d, "\mathrm{can}"] \\ \underline{\Omega}(f(d,d')) \arrow[r, "\simeq"'] & \underline{\Omega}(f(d,0)\oplus D_{\underline{\Omega}}f(d',0)) \end{tikzcd}$$

commutes canonically. Here, we have used that $f(d,d') \simeq f(d,0)\oplus D_{\Omega}f(d',0)$ since $D_{\Omega}f(d',0) \simeq fD_{\mathrm{Hyp}}(d',0) \simeq f(0,d')$ by Poincaréness of the functor (f,η). In any case, to see that the square indeed commutes, we write suggestively $\underline{\mathrm{map}}_{\mathcal{D}}(d,d') = \underline{\Omega}_{\mathrm{Hyp}}((d,0)\oplus(0,d'))$, and note that $B_{\underline{\Omega}_{\mathrm{Hyp}}}((d,0),D_{\mathrm{Hyp}}(d',0)) \xrightarrow{\mathrm{can}} \underline{\Omega}_{\mathrm{Hyp}}((d,0)\oplus(0,d'))$ is an equivalence. Then the required square is just a consequence of the following commuting square

$$\begin{tikzcd} \underline{\Omega}_{\mathrm{Hyp}}((d,0)\oplus(0,d')) \arrow[d, "\eta"] & B_{\underline{\Omega}_{\mathrm{Hyp}}}((d,0),D_{\mathrm{Hyp}}(d',0)) \arrow[l, "\mathrm{can}"', "\simeq"] \arrow[d, "f"] \\ \underline{\Omega}(f(d,0)\oplus f(0,d')) & B_{\underline{\Omega}}(f(d,0), fD_{\mathrm{Hyp}}(d',0)) \arrow[l, "\mathrm{can}"] \end{tikzcd}$$

coming from the naturality of η. □

7.1.9 Metabolic categories and the algebraic Thom isomorphism

Construction 7.1.50 (Metabolic categories)**.** Let $(\underline{\mathcal{C}},\underline{\Omega}) \in \underline{\mathrm{Cat}}^{\mathrm{h}}_G$. We define the *metabolic category* $\underline{\mathrm{Met}}(\underline{\mathcal{C}},\underline{\Omega})$ associated to it to be the G–hermitian category whose underlying G–stable category is given by $\underline{\mathcal{C}}^{\Delta^1}$ and whose G–quadratic datum is

$$\underline{\Omega}_{\mathrm{Met}}\colon (\underline{\mathcal{C}}^{\Delta^1})^{\underline{\mathrm{op}}} \longrightarrow \underline{\mathrm{Sp}}_G \quad :: \quad (x \xrightarrow{f} y) \mapsto \mathrm{fib}\left(\underline{\Omega}(y) \xrightarrow{f^*} \underline{\Omega}(x)\right)$$

It is not hard to check that $\underline{\mathrm{Met}}(\underline{\mathcal{C}},\underline{\Omega})$ is G–Poincaré when $(\underline{\mathcal{C}},\underline{\Omega})$ is.

Construction 7.1.51. Let $(\underline{\mathcal{C}},\underline{\Omega}) \in \underline{\mathrm{Cat}}^{\mathrm{h}}_G$. We denote by $\underline{\mathrm{Ar}}(\underline{\mathcal{C}},\underline{\Omega})$ the G–hermitian category with underlying G–category $\underline{\mathrm{Ar}}(\underline{\mathcal{C}})$ and whose G–quadratic datum is given by

$$\begin{tikzcd} \underline{\Omega}_{\mathrm{Ar}}(z \xrightarrow{f} w) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \underline{\Omega}(z) \arrow[d] \\ B_{\underline{\Omega}}(z,w) \arrow[r] & B_{\underline{\Omega}}(z,z) \end{tikzcd}$$

One then checks, as in [CDH+20a, Rmk. 2.4.2], that $\underline{\mathrm{Ar}}(\underline{\mathcal{C}},\underline{\Omega})$ is G–Poincaré if $(\underline{\mathcal{C}},\underline{\Omega})$ were.

The following is a direct proof of the genuine equivariant analogue of the algebraic Thom isomorphism in [CDH+20a, Cor. 2.4.6] without invoking the pairing construction from [CDH+20a, §7.3]. The possibility of such a proof was indicated in the paragraph following [CDH+20a, Prop. 2.4.3].

Proposition 7.1.52 (Algebraic Thom isomorphism)**.** *Let* $(\underline{\mathcal{C}}, \underline{\Omega}) \in \underline{\mathrm{Cat}}^{\mathrm{p}}_G$. *Then the association* $[w \to x] \mapsto \mathrm{fib}(w \to x)$ *induces an equivalence*

$$\underline{\mathrm{Pn}}\big(\underline{\mathrm{Met}}(\underline{\mathcal{C}}, \underline{\Omega})\big) \to \underline{\mathrm{Fm}}(\underline{\mathcal{C}}, \Omega\underline{\Omega})$$

Proof. As in [CDH+20a, §2.4], we show instead that

$$\underline{\mathrm{Pn}}\big(\underline{\mathrm{Ar}}(\underline{\mathcal{C}}, \underline{\Omega})\big) \xrightarrow{s} \underline{\mathrm{Fm}}(\underline{\mathcal{C}}, \underline{\Omega})$$

is an equivalence. Recall from above that an object on the source is a tuple $(z \xrightarrow{f} w, q, g, \eta)$ where $q \in \underline{\Omega}^{\infty}\underline{\Omega}(z)$, $g : w \to Dz$ is an equivalence, and η is an equivalence $q_\# \simeq g \circ f$.

To this end, we show that $\underline{\mathrm{Pn}}\big(\underline{\mathrm{Ar}}(\underline{\mathcal{C}}, \underline{\Omega})\big) \xrightarrow{s} \underline{\mathrm{Fm}}(\underline{\mathcal{C}}, \underline{\Omega})$ has a G–right adjoint i given by

$$i : (a, p) \mapsto (a \xrightarrow{p_\#} Da, p, \mathrm{id}, \mathrm{id})$$

Since both sides are G–spaces, the adjunction would imply that they are mutual inverses. Before that, observe that for $(z \xrightarrow{f} w, q, g, \eta) \in \underline{\mathrm{Pn}}\big(\underline{\mathrm{Ar}}(\underline{\mathcal{C}}, \underline{\Omega})\big)$, by virtue of this being a Poincaré form, the canonical vertical map

$$\begin{tikzcd} z \arrow[r, "f"] \arrow[d, equal] & w \arrow[d, "g", "\simeq"'] \\ z \arrow[r, "q_\#"'] & Dz \end{tikzcd}$$

is an equivalence and so there is a canonical equivalence $u : (z \xrightarrow{f} w, q, g, \eta) \to (z \xrightarrow{q_\#} Dz, q, \mathrm{id}, \mathrm{id})$.

Now to construct the G–right adjoint. Since adjunctions can be constructed objectwise by Proposition 1.2.26, let $(a, p) \in \underline{\mathrm{He}}(\underline{\mathcal{C}}, \underline{\Omega})$. We need to show that the map

$$\underline{\mathrm{Map}}_{\underline{\mathrm{Pn}}\left(\underline{\mathrm{Ar}}(\underline{\mathcal{C}}, \underline{\Omega})\right)}\big((z \xrightarrow{f} w, q, g, \eta), (a \xrightarrow{p_\#} Da, p, \mathrm{id}, \mathrm{id})\big) \xrightarrow{s} \underline{\mathrm{Map}}_{\underline{\mathrm{Fm}}(\underline{\mathcal{C}}, \underline{\Omega})}\big((z, q), (a, p)\big)$$

is an equivalence. In fact, since s sends the canonical map $u : (z \xrightarrow{f} w, q, g, \eta) \to (z \xrightarrow{q_\#} Dz, q, \mathrm{id}, \mathrm{id})$ above to an equivalence, it suffices to show that the following map is an equivalence

$$\underline{\mathrm{Map}}_{\underline{\mathrm{Pn}}\left(\underline{\mathrm{Ar}}(\underline{\mathcal{C}}, \underline{\Omega})\right)}\big((z \xrightarrow{q_\#} Dz, q, \mathrm{id}, \mathrm{id}), (a \xrightarrow{p_\#} Da, p, \mathrm{id}, \mathrm{id})\big)$$
$$\xrightarrow{s} \underline{\mathrm{Map}}_{\underline{\mathrm{Fm}}(\underline{\mathcal{C}}, \underline{\Omega})}\big((z, q), (a, p)\big)$$

For this, note that the two maps of G–spaces

$$\underline{\mathrm{Fm}}(\underline{\mathcal{C}}, \underline{\Qoppa}) \to \underline{\mathcal{C}}^{\simeq} \qquad \text{and} \qquad \underline{\mathrm{Pn}}(\underline{\mathrm{Ar}}(\underline{\mathcal{C}}, \underline{\Qoppa})) \xrightarrow{s} \underline{\mathrm{Fm}}(\underline{\mathcal{C}}, \underline{\Qoppa}) \to \underline{\mathcal{C}}^{\simeq}$$

are cocartesian fibrations. Hence, it will suffice to show that, for a fixed equivalence $\varphi : a \xrightarrow{\simeq} z$ in $\underline{\mathcal{C}}^{\simeq}$, the map of G–spaces

$$\begin{tikzcd}
\underline{\mathrm{Map}}^{\varphi}_{\underline{\mathrm{Pn}}(\underline{\mathrm{Ar}}(\underline{\mathcal{C}}, \underline{\Qoppa}))}((z \xrightarrow{q_\#} Dz, q, \mathrm{id}, \mathrm{id}), (a \xrightarrow{p_\#} Da, p, \mathrm{id}, \mathrm{id})) \arrow[r, "s"] \arrow[d, "\simeq"'] & \underline{\mathrm{Map}}^{\varphi}_{\underline{\mathrm{Fm}}(\underline{\mathcal{C}}, \underline{\Qoppa})}((z, q), (a, p)) \arrow[d, "\simeq"] \\
\underline{\mathrm{Map}}_{\underline{\mathrm{Pn}}(\underline{\mathrm{Ar}}(\underline{\mathcal{C}}, \underline{\Qoppa}))_a}((a \xrightarrow{\varphi(q)_\#} Da, \varphi(q), \mathrm{id}, \mathrm{id}), (a \xrightarrow{p_\#} Da, p, \mathrm{id}, \mathrm{id})) \arrow[r, "s"] & \underline{\mathrm{Map}}_{\underline{\mathrm{Fm}}(\underline{\mathcal{C}}, \underline{\Qoppa})_a}((a, \varphi(q)), (a, p))
\end{tikzcd}$$

If $\varphi(q) \neq p \in \pi_0^G \underline{\Omega}^\infty \Qoppa(a)$, then both sides are empty. If on the other hand, $\varphi(q) = p$, then both sides are equivalent to $\Omega\underline{\Omega}^\infty \Qoppa(a)$, considered as the loop space at the point $\varphi(q) - 0 = 0 \in \pi_0^G \underline{\Omega}^\infty \Qoppa(a)$, and so it is indeed an equivalence as required. $\square$

7.2 G-norms of hermitian categories

Having set up the foundations, we are now ready to deal with the subtler G–symmetric monoidal structures. As is apparent by now, the pervasive theme in this thesis is that these structures are qualitatively harder to work with than their nonequivariant counterparts owing to the fact that we have no good way to decompose G–cubes to perform currying arguments. In this section, we will see this problem again, and therefore we have had to produce proofs which are entirely different from the ones in [CDH+20a, Prop. 5.1.3]. These proofs proceed by purely monoidal principles and so we think that they provide better reasons why these statements are true. We are very much indebted to Maxime Ramzi who provided a key observation that the external norm $\boxtimes$ *is* given by the universal functor to the norm of categories.

7.2.1 Norm constructions and formulas

Construction 7.2.1. Let $\underline{\Qoppa} : \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Sp}}_H$ be a reduced $2H$-excisive functor. Then we can define the reduced $2G_+$-excisive functor $\bigotimes_H^G \underline{\Qoppa} : \bigotimes_H^G \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Sp}}_G$ as the diagonal functor in the diagram

$$\begin{tikzcd}
\prod_{G/H} \underline{\mathcal{C}}^{\mathrm{op}} \arrow[r, "\prod_{G/H} \underline{\Qoppa}"] \arrow[d, "\tau"'] \arrow[rr, bend left, "\boxtimes_{G/H} \underline{\Qoppa}"] & \prod_{G/H} \underline{\mathrm{Sp}}_H \arrow[r, "\otimes"] & \underline{\mathrm{Sp}}_G \\
\bigotimes_{G/H} \underline{\mathcal{C}}^{\mathrm{op}} \arrow[rru, "F\tau_! \boxtimes_{G/H} \underline{\Qoppa}"'] & &
\end{tikzcd}$$

where $F : \underline{\mathrm{Fun}}^*(\bigotimes_{G/H}\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G) \to \underline{\mathrm{Fun}}^{\mathrm{q}}(\bigotimes_{G/H}\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G)$ is the G–quadratisation of Proposition 7.1.27. If we have a H-bilinear functor $\beta : \underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Sp}}_H$, then we can define G-bilinear functor $\bigotimes_H^G \beta : \bigotimes_H^G \underline{\mathcal{C}}^{\mathrm{op}} \times \bigotimes_H^G \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Sp}}_G$ as the diagonal functor in the diagram

$$\begin{tikzcd}
\prod_{G/H}\underline{\mathcal{C}}^{\mathrm{op}} \times \prod_{G/H}\underline{\mathcal{C}}^{\mathrm{op}} \arrow[r, "\prod_{G/H}\beta"] \arrow[d, "\tau\times\tau"'] & \prod_{G/H}\underline{\mathrm{Sp}}_H \arrow[r, "\otimes"] & \underline{\mathrm{Sp}}_G \\
\bigotimes_{G/H}\underline{\mathcal{C}}^{\mathrm{op}} \times \bigotimes_{G/H}\underline{\mathcal{C}}^{\mathrm{op}} \arrow[urr, "p_{G_+,G_+}(\tau\times\tau)_!\boxtimes_{G/H}\beta"'] & &
\end{tikzcd}$$

The aim of this subsection is to show that

$$B_{\otimes_H^G \underline{\mathrm{Q}}} \simeq \otimes_H^G B_{\underline{\mathrm{Q}}} \qquad \text{and} \qquad L_{\otimes_H^G \underline{\mathrm{Q}}} \simeq \otimes_H^G L_{\underline{\mathrm{Q}}} \qquad \text{and} \qquad D_{\otimes_H^G \underline{\mathrm{Q}}} \simeq \otimes_H^G D_{\underline{\mathrm{Q}}}$$

Notation 7.2.2. We recall and establish some notations about distributivity. Let $\underline{\mathcal{C}}$ be a H-cocomplete category and $\underline{\mathcal{D}}$ a H–finite-cocomplete category. Write $\mathrm{Fun}_G^{\delta_{G/H}}(\prod_{G/H}\underline{\mathcal{C}}, \underline{\mathrm{Sp}}_G) \subseteq \mathrm{Fun}_G(\prod_{G/H}\underline{\mathcal{C}}, \underline{\mathrm{Sp}}_G)$ for the G–full subcategory of G/H–distributive functors in the sense of §1.3.4, and write $\mathrm{Fun}_G^{\prod_{G/H}}(\prod_{G/H}\underline{\mathcal{D}}, \underline{\mathrm{Sp}}_G) \subseteq \mathrm{Fun}_G(\prod_{G/H}\underline{\mathcal{D}}, \underline{\mathrm{Sp}}_G)$ for the G–full subcategory of G/H–finite-distributive functors, ie. those which are G/H–distributive against finite H–colimits.

Proposition 7.2.3. *Let $\underline{\mathrm{Q}} : \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Sp}}_H$ be a H–quadratic functor. Then $B_{\otimes_H^G \underline{\mathrm{Q}}} \simeq \bigotimes_H^G B_{\underline{\mathrm{Q}}}$.*

Proof. Recall that we have a H-biadjunction

$$B : \underline{\mathrm{Fun}}_H^*(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H) \rightleftarrows \underline{\mathrm{Fun}}_H^{*,*}(\underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H) : \Delta^*$$

Hence by Lemma 1.3.13 we get a G-biadjunction

$$\prod_{G/H} B : \prod_{G/H} \underline{\mathrm{Fun}}_H^*(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H) \rightleftarrows \prod_{G/H} \underline{\mathrm{Fun}}_H^{*,*}(\underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H) : \prod_{G/H} \Delta^*$$

and hence by Lemma 3.4.2 and Lemma 1.3.17, we get a G-adjunction

$$\begin{tikzcd}
\bigotimes_{G/H}\underline{\mathrm{Fun}}_H^*(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H) \arrow[r, shift left, "\otimes_H^G B"] \arrow[d, equal] & \bigotimes_{G/H}\underline{\mathrm{Fun}}_H^{*,*}(\underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H) \arrow[l, shift left, "\otimes_H^G \Delta^*"] \arrow[d, equal] \\
\underline{\mathrm{Fun}}_G^*(\prod_{G/H}\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G) \arrow[r, shift left, "\overline{B}"] & \underline{\mathrm{Fun}}_G^{*,*}(\prod_{G/H}\underline{\mathcal{C}}^{\mathrm{op}} \times \prod_{G/H}\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G) \arrow[l, shift left, "\psi"]
\end{tikzcd}$$

which moreover makes the following squares commute

$$\begin{tikzcd}
\prod_{G/H} \underline{\mathrm{Fun}}^*_H(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H) \arrow[r, shift left, "\prod_H^G B"] \arrow[d, "\boxtimes"'] & \prod_{G/H} \underline{\mathrm{Fun}}^{*,*}_H(\underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H) \arrow[l, shift left, "\prod_H^G \Delta^*"] \arrow[d, "\boxtimes"] \\
\underline{\mathrm{Fun}}^*_G(\prod_{G/H} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G) \arrow[r, shift left, "\overline{B}"] & \underline{\mathrm{Fun}}^{*,*}_G(\prod_{G/H} \underline{\mathcal{C}}^{\mathrm{op}} \times \prod_{G/H} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G) \arrow[l, shift left, "\psi"]
\end{tikzcd}$$

We first claim that $\psi \simeq \Delta^*$. To see this, note that we have the commuting diagram

$$\begin{tikzcd}
\prod_{G/H} \underline{\mathrm{Fun}}^*_H(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H) \arrow[d, "\boxtimes"'] & \prod_{G/H} \underline{\mathrm{Fun}}^{*,*}_H(\underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H) \arrow[l, "\prod_H^G \Delta^*"] \arrow[d, "\boxtimes"] \\
\underline{\mathrm{Fun}}^*_G(\prod_{G/H} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G) & \underline{\mathrm{Fun}}^{*,*}_G(\prod_{G/H} \underline{\mathcal{C}}^{\mathrm{op}} \times \prod_{G/H} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G) \arrow[l, "\Delta^*"]
\end{tikzcd}$$

and so by universality of the map

$$\prod_{G/H} \underline{\mathrm{Fun}}^{*,*}_H(\underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H) \xrightarrow{\boxtimes} \underline{\mathrm{Fun}}^{*,*}_G(\prod_{G/H} \underline{\mathcal{C}}^{\mathrm{op}} \times \prod_{G/H} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G) \simeq \bigotimes_{G/H} \underline{\mathrm{Fun}}^{*,*}_H(\underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H)$$

we get that $\psi \simeq \Delta^*$ as required.

All in all, we have the commuting upper solid square and dashed adjunctions

$$\begin{tikzcd}
\prod_{G/H} \underline{\mathrm{Fun}}_H(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H) \arrow[r, "\prod_H^G B"] \arrow[d, "\boxtimes"'] & \prod_{G/H} \underline{\mathrm{Fun}}^{*,*}_H(\underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H) \arrow[d, "\boxtimes"] \\
\underline{\mathrm{Fun}}^*_G(\prod_{G/H} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G) \arrow[r, shift left, "\overline{B}"] \arrow[d, shift right, "\tau_!"'] & \underline{\mathrm{Fun}}^{*,*}_G(\prod_{G/H} \underline{\mathcal{C}}^{\mathrm{op}} \times \prod_{G/H} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G) \arrow[l, shift left, dashed, "\Delta^*"] \arrow[d, shift right, "(\tau\times\tau)_!"'] \\
\underline{\mathrm{Fun}}^*_G(\bigotimes_{G/H} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G) \arrow[u, shift right, dashed, "\tau^*"'] \arrow[d, shift right, "F"'] & \underline{\mathrm{Fun}}^{*,*}_G(\bigotimes_{G/H} \underline{\mathcal{C}}^{\mathrm{op}} \times \bigotimes_{G/H} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G) \arrow[u, shift right, dashed, "(\tau\times\tau)^*"'] \arrow[d, shift right, "P_{G_+,G_+}"'] \\
\underline{\mathrm{Fun}}^{\mathrm{q}}_G(\bigotimes_{G/H} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G) \arrow[u, shift right, dashed, hook] \arrow[r, shift left, "B"] & \underline{\mathrm{Fun}}^{G\text{-bilin}}_G(\bigotimes_{G/H} \underline{\mathcal{C}}^{\mathrm{op}} \times \bigotimes_{G/H} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G) \arrow[l, shift left, dashed, "\Delta^*"] \arrow[u, shift right, dashed, hook]
\end{tikzcd}$$

The dashed square clearly commutes, and so the bottom solid rectangle also commutes, and hence we get $\otimes_{G/H} B_{\underline{\mathrm{Q}}} \simeq B_{\otimes_{G/H}\underline{\mathrm{Q}}}$ as required. $\square$

Proposition 7.2.4. *Let $\underline{\mathrm{Q}} : \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Sp}}_H$ be a H–quadratic functor. Then $L_{\otimes_H^G \underline{\mathrm{Q}}} \simeq \bigotimes_H^G L_{\underline{\mathrm{Q}}}$.*

Proof. First of all note that we have the accessible H-Bousfield localisation

$$\underline{\mathrm{Fun}}_H(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H) \underset{\longleftarrow}{\xrightarrow{L}} \underline{\mathrm{Fun}}^{\mathrm{ex}}_H(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H)$$

And so since the domain is H-presentable, so is the codomain by Theorem 2.2.2 (5). On the other hand, it is clear that the codomain is closed under arbitrary H-colimits in the source, and so by the adjoint functor theorem Theorem 2.2.3, the inclusion also admits a right adjoint. Hence by Lemma 1.3.17 we get the commuting diagrams of adjunctions

$$\begin{tikzcd}
\prod_{G/H} \underline{\mathrm{Fun}}_H(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H) \arrow[r, shift left, "\prod_{G/H} L"] \arrow[d, "\boxtimes"'] & \prod_{G/H} \underline{\mathrm{Fun}}^{\mathrm{ex}}_H(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H) \arrow[l, shift left, hook'] \arrow[d, "\boxtimes"] \\
\bigotimes_{G/H} \underline{\mathrm{Fun}}_H(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H) \arrow[r, shift left, "\otimes_{G/H} L"] \arrow[d, "\simeq"'] & \bigotimes_{G/H} \underline{\mathrm{Fun}}^{\mathrm{ex}}_H(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H) \arrow[l, shift left, hook'] \arrow[d, "\simeq"] \\
\underline{\mathrm{Fun}}_G(\prod_{G/H} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G) \arrow[r, shift left] \arrow[d, shift right, "\tau_!"'] & \underline{\mathrm{Fun}}^{\mathrm{ex}}_G(\bigotimes_{G/H} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G) \arrow[l, shift left, dashed, hook'] \arrow[d, equal] \\
\underline{\mathrm{Fun}}_G(\bigotimes_{G/H} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G) \arrow[r, shift left, "P_{G_+}"] \arrow[u, shift right, dashed, "\tau^*"'] & \underline{\mathrm{Fun}}^{\mathrm{ex}}_G(\bigotimes_{G/H} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G) \arrow[l, shift left, dashed, hook']
\end{tikzcd}$$

where the left middle vertical equivalence is by the G–symmetric monoidality of G–Yoneda cocompletion §1.3.2 and the right middle vertical equivalence is by virtue of Corollary 3.4.7. The dashed triangle clearly commutes and so by taking left adjoints the bottom solid square commutes too. Hence in total we get $\otimes_{G/H} L_{\underline{\mathrm{Q}}} \simeq L_{\otimes_{G/H}\underline{\mathrm{Q}}}$ as desired. ☐

Proposition 7.2.5. *Let $\underline{\mathrm{Q}} : \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Sp}}_H$ be a H–quadratic functor. Then $D_{\otimes^G_H \underline{\mathrm{Q}}} \simeq \bigotimes^G_H D_{\underline{\mathrm{Q}}}$.*

Proof. To see that $D_{\otimes^G_H \underline{\mathrm{Q}}} \simeq \otimes^G_H D_{\underline{\mathrm{Q}}}$, observe the sequence of equivalences

$$B_{\otimes^G_H \underline{\mathrm{Q}}}(-,-) \simeq \otimes^G_H B_{\underline{\mathrm{Q}}}(-,-) \simeq \otimes^G_H \underline{\mathrm{map}}_{\underline{\mathcal{C}}}(-, D_{\underline{\mathrm{Q}}}-) \simeq \mathrm{map}_{\otimes^G_H \underline{\mathcal{C}}}(-, \otimes^G_H D_{\underline{\mathrm{Q}}}-)$$

where the first equivalence is by Proposition 7.2.3 and the third by Corollary 3.4.8. ☐

7.2.2 Transitivity of norms on quadratic structures

Let $K \leq H \leq G$ be subgroup inclusions and $\underline{\mathrm{Q}} : \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Sp}}_K$ be a K-quadratic functor. In order to mimic the proof of [CDH+20a, Thm. 5.2.7] for showing that $\underline{\mathrm{Cat}}^{\mathrm{p}}_G$ refines to a G–symmetric monoidal structure in §7.3.1, we need to show that the canonical comparison

$$\bigotimes_{G/H} \bigotimes_{H/K} \underline{\mathrm{Q}} \to \bigotimes_{G/K} \underline{\mathrm{Q}}$$

is an equivalence. But equivalences of G-quadratic functors are detected jointly by the linear and the bilinear part by the recollement Theorem 7.1.24, and since by the above sections we have the equivalences

$$B_{\otimes_{G/H}\otimes_{H/K}\underline{\Omega}} \simeq \bigotimes_{G/H}\bigotimes_{H/K} B_{\underline{\Omega}} \qquad L_{\otimes_{G/H}\otimes_{H/K}\underline{\Omega}} \simeq \bigotimes_{G/H}\bigotimes_{H/K} L_{\underline{\Omega}}$$

it suffices to show that for a K-bilinear functor $\beta : \underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Sp}}_K$ and a K-linear functor $\ell : \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Sp}}_K$, we have the equivalences

$$\bigotimes_{G/H}\bigotimes_{H/K} \beta \simeq \bigotimes_{G/K} \beta \qquad \bigotimes_{G/H}\bigotimes_{H/K} \ell \simeq \bigotimes_{G/K} \ell \tag{7.3}$$

Given these, we would then have shown that the following maps

$$B_{\otimes_{G/H}\otimes_{H/K}\underline{\Omega}} \to B_{\otimes_{G/K}\underline{\Omega}} \qquad L_{\otimes_{G/H}\otimes_{H/K}\underline{\Omega}} \to L_{\otimes_{G/K}\underline{\Omega}}$$

are equivalences, as desired.

Proposition 7.2.6. *Let $K \leq H \leq G$ be subgroup inclusions and $\underline{\Omega} : \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Sp}}_K$ be a K-quadratic functor. Then the canonical comparison $\otimes_{G/H}\otimes_{H/K}\underline{\Omega} \to \otimes_{G/K}\underline{\Omega}$ is an equivalence.*

Proof. As commented above, it suffices to show that for a K-bilinear functor $\beta : \underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Sp}}_K$ and a K-linear functor $\ell : \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathrm{Sp}}_K$, we have the equivalences in Eq. (7.3). For the case of ℓ, we note that $\ell \mapsto \otimes_{G/H}\otimes_{H/K}\ell$ is implemented by the functor

$$\begin{aligned}
\prod_{G/H}\prod_{H/K} \underline{\mathrm{Fun}}^{\mathrm{ex}}_K(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_K) &\xrightarrow{\prod_{G/H}\boxtimes_{H/K}} \prod_{G/H}\bigotimes_{H/K} \underline{\mathrm{Fun}}^{\mathrm{ex}}_K(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_K) \\
&\simeq \prod_{G/H} \underline{\mathrm{Fun}}^{\mathrm{ex}}_H(\bigotimes_{H/K} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H) \\
&\xrightarrow{\boxtimes_{G/H}} \bigotimes_{G/H} \underline{\mathrm{Fun}}^{\mathrm{ex}}_H(\bigotimes_{H/K} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_H) \\
&\simeq \underline{\mathrm{Fun}}^{\mathrm{ex}}_G(\bigotimes_{G/H}\bigotimes_{H/K} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G)
\end{aligned}$$

where the two equivalences are by virtue of the G-symmetric monoidality of $\underline{\mathrm{Ind}}$ from Corollary 3.4.7. But then this whole composite is also the universal G/K-distributive functor

$$\begin{aligned}
&\prod_{G/H}\prod_{H/K} \underline{\mathrm{Fun}}^{\mathrm{ex}}_K(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_K) \simeq \prod_{G/K} \underline{\mathrm{Fun}}^{\mathrm{ex}}_K(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_K) \\
&\longrightarrow \bigotimes_{G/K} \underline{\mathrm{Fun}}^{\mathrm{ex}}_K(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_K) \simeq \underline{\mathrm{Fun}}^{\mathrm{ex}}_G(\bigotimes_{G/K} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_G)
\end{aligned}$$

which corresponds to $\ell \mapsto \bigotimes_{G/K} \ell$. Thus we have $\bigotimes_{G/H} \bigotimes_{H/K} \ell \simeq \bigotimes_{G/K} \ell$ as required. For the claim Eq. (7.3) for β, we use the equivalence

$$\underline{\mathrm{Fun}}^{\mathrm{b}}_{K}(\underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_{K}) \simeq \underline{\mathrm{Fun}}^{\mathrm{ex}}_{K}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_{K}) \otimes \underline{\mathrm{Fun}}^{\mathrm{ex}}_{K}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_{K})$$

We can now carry out an argument similar to the above. To wit, we know that $\beta \mapsto \bigotimes_{G/H} \bigotimes_{H/K} \beta$ corresponds to the composite

$$\begin{aligned}
&\prod_{G/H} \prod_{H/K} \underline{\mathrm{Fun}}^{\mathrm{b}}_{K}(\underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_{K}) \\
&\xrightarrow{\prod_{G/H} \boxtimes_{H/K}} \prod_{G/H} \bigotimes_{H/K} \underline{\mathrm{Fun}}^{\mathrm{b}}_{K}(\underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_{K}) \\
&\simeq \prod_{G/H} \Big(\underline{\mathrm{Fun}}^{\mathrm{ex}}_{H}(\bigotimes_{H/K} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_{H}) \otimes \underline{\mathrm{Fun}}^{\mathrm{ex}}_{H}(\bigotimes_{H/K} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_{H})\Big) \\
&\xrightarrow{\boxtimes_{G/H}} \bigotimes_{G/H} \Big(\underline{\mathrm{Fun}}^{\mathrm{ex}}_{H}(\bigotimes_{H/K} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_{H}) \otimes \underline{\mathrm{Fun}}^{\mathrm{ex}}_{H}(\bigotimes_{H/K} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_{H})\Big) \\
&\simeq \underline{\mathrm{Fun}}^{\mathrm{ex}}_{G}(\bigotimes_{G/H} \bigotimes_{H/K} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_{G}) \otimes \underline{\mathrm{Fun}}^{\mathrm{ex}}_{G}(\bigotimes_{G/H} \bigotimes_{H/K} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_{G}) \\
&\simeq \underline{\mathrm{Fun}}^{\mathrm{b}}_{G}(\bigotimes_{G/K} \underline{\mathcal{C}}^{\mathrm{op}} \times \bigotimes_{G/K} \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_{G})
\end{aligned}$$

Similarly as in the argument above, this composite is also clearly the universal G/K-distributive functor which corresponds to $\beta \mapsto \bigotimes_{G/K} \beta$, and so $\bigotimes_{G/H} \bigotimes_{H/K} \beta \simeq \bigotimes_{G/K} \beta$ as desired. $\square$

7.3 Category of G-Poincaré categories

With the basic theory in place, we now assemble these ingredients to prove various structural results about the whole G–category $\underline{\mathrm{Cat}}^{\mathrm{p}}_{G}$.

7.3.1 Symmetric monoidality of G-Poincaré categories

Construction 7.3.1 (Equivariant lax arrows)**.** We will construct a G–symmetric monoidal structure on $\underline{\mathrm{Cat}}^{\mathrm{p}}_{G}$ analogously to [CDH+20a, §5.2]. For this, recall their notation of the full subcategory $\widehat{\mathrm{LaxAr}} \subseteq \widehat{\mathrm{Cat}}_{/\Delta^1}$ spanned by the cartesian fibrations over Δ^1. Note that this is a strictly larger category than $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \widehat{\mathrm{Cat}})$ since the maps between these cartesian fibrations do not in general preserve cartesian morphisms. Concretely, for $M, N \in \widehat{\mathrm{LaxAr}}$, a morphism $f : M \to N$ is given by the following lax commuting diagram

$$\begin{tikzcd} M_1 \arrow[r, "f_1"] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & N_1 \arrow[d] \\ M_0 \arrow[r, "f_0"'] & N_0 \end{tikzcd}$$

where M_i, N_i are the fibres over $i \in \{0,1\}$.

Now we define the equivariant version as the full subcategory

$$\begin{aligned} \widehat{\mathrm{LaxAr}}_G := \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \widehat{\mathrm{LaxAr}}) &\subseteq \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \widehat{\mathrm{Cat}}_{/\Delta^1}) \\ &\simeq \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \widehat{\mathrm{Cat}})_{/\underline{\Delta}^1} = (\widehat{\mathrm{Cat}}_G)_{/\underline{\Delta}^1} \end{aligned}$$

where here $\underline{\Delta}^1$ denotes the G–category whose fibres are Δ^1. This clearly assembles to a G–category $\widehat{\underline{\mathrm{LaxAr}}}_G$. Explicitly, an object here is given by morphisms of G–categories $\underline{\mathcal{C}} \to \underline{\Delta}^1$ which has the property of being cartesian when evaluated over each orbit G/H. Since $\mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \widehat{\mathrm{LaxAr}}) \subseteq \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \widehat{\mathrm{Cat}}_{/\Delta^1})$ clearly inherits products, we get that

$$\widehat{\underline{\mathrm{LaxAr}}}_G \subseteq (\widehat{\underline{\mathrm{Cat}}}_G)_{/\underline{\Delta}^1}$$

inherits the indexed products from the right hand side. Since $\mathrm{ev}_0 : \widehat{\mathrm{LaxAr}} \to \widehat{\mathrm{Cat}}$ preserves products, we see that $\mathrm{ev}_0 : \widehat{\underline{\mathrm{LaxAr}}}_G \longrightarrow \widehat{\underline{\mathrm{Cat}}}_G$ strongly preserves indexed products. Moreover, if we denote by $\underline{\mathrm{LaxAr}}_G \subseteq \widehat{\underline{\mathrm{LaxAr}}}_G$ for the G–full subcategory of those objects whose fibre over 1 is small, then clearly this inclusion creates indexed products. So, in total we obtain that

$$\mathrm{ev}_0 : \underline{\mathrm{LaxAr}}_G \longrightarrow \widehat{\underline{\mathrm{Cat}}}_G \quad :: \quad [\mathcal{C} \to \underline{\Delta}^1] \mapsto \underline{\mathcal{C}}_0$$

preserves indexed products, and so is a G–symmetric monoidal functor when both sides are equipped with the G–cartesian symmetric monoidal structure. Hence, if $\underline{\mathcal{D}} \in \underline{\mathrm{CMon}}_G(\widehat{\underline{\mathrm{Cat}}}_G)$ is a G–symmetric monoidal category, the fibre $(\underline{\mathrm{Cat}}_G)_{//\underline{\mathcal{D}}}$ of ev_0 over $\mathcal{D}$ inherits a G–symmetric monoidal structure which we denote by $(\underline{\mathrm{Cat}}_G)^{\otimes}_{//\underline{\mathcal{D}}}$ and importantly, the G–functor

$$(\underline{\mathrm{Cat}}_G)^{\otimes}_{//\underline{\mathcal{D}}} \to \underline{\mathrm{LaxAr}}^{\times}_G$$

is G–symmetric monoidal. In particular, since the triangle

$$\begin{tikzcd} (\underline{\mathrm{Cat}}_G)_{//\underline{\mathcal{D}}} \arrow[r] \arrow[dr, "\mathrm{ev}_1"'] & \underline{\mathrm{LaxAr}}_G \arrow[d, "\mathrm{ev}_1"] \\ & \underline{\mathrm{Cat}}_G \end{tikzcd}$$

commutes, where the vertical functor is G–symmetric monoidal with respect to the G–cartesian symmetric monoidal structures, we see that the evaluation ev_1 refines to a G–symmetric monoidal functor

$$\mathrm{ev}_1 : (\underline{\mathrm{Cat}}_G)^{\otimes}_{//\underline{\mathcal{D}}} \longrightarrow \underline{\mathrm{Cat}}^{\times}_G$$

Construction 7.3.2 (G–symmetric monoidal structure on G–Poincaré categories)**.** We will now set $\underline{\mathcal{D}} = \underline{\mathrm{Sp}}_G$ from the above construction. We now define $(\underline{\mathrm{Cat}}_G)^{\otimes}_{\underline{\mathrm{op}}//\underline{\mathcal{D}}}$ to be the pullback in G–symmetric monoidal categories $\underline{\mathrm{CMon}}_G(\underline{\mathrm{Cat}}_G)$

$$\begin{tikzcd}
(\underline{\mathrm{Cat}}_G)^{\otimes}_{\underline{\mathrm{op}}//\underline{\mathcal{D}}} \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (\underline{\mathrm{Cat}}_G)^{\otimes}_{//\underline{\mathcal{D}}} \arrow[d, "\mathrm{ev}_1"] \\
\underline{\mathrm{Cat}}^{\times}_G \arrow[r, "\simeq", "(-)^{\underline{\mathrm{op}}}"'] & \underline{\mathrm{Cat}}^{\times}_G
\end{tikzcd}$$

This makes sense since the bottom functor is an equivalence, and so is G–symmetric monoidal, and the vertical ev_1 functor is also G–symmetric monoidal by the last part of Construction 7.3.1. Concretely, for $H \leq G$, H–objects of this G–symmetric monoidal category are given by pairs $(\underline{\mathcal{C}}, \underline{\Omega})$ of a H–category $\underline{\mathcal{C}}$ and a H–functor $\underline{\Omega} : \underline{\mathcal{C}}^{\underline{\mathrm{op}}} \to \underline{\mathrm{Sp}}_H$. For a H–object $(\underline{\mathcal{C}}, \underline{\Omega})$ and G–object $(\underline{\mathcal{D}}, \underline{\Phi})$, a morphism $(f, \eta) : (\underline{\mathcal{C}}, \underline{\Omega}) \to (\underline{\mathcal{D}}, \underline{\Phi})$ consists of a G–functor $f : \prod_{G/H} \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ and a natural transformation $\eta : \boxtimes_{G/H}\underline{\Omega} \Rightarrow \underline{\Phi} \circ f^{\underline{\mathrm{op}}}$ where $\boxtimes_{G/H}\underline{\Omega}$ is defined as the composite

$$\boxtimes_{G/H}\underline{\Omega} : \prod_{G/H} \underline{\mathcal{C}}^{\underline{\mathrm{op}}} \xrightarrow{\prod_{G/H}\underline{\Omega}} \prod_{G/H} \underline{\mathrm{Sp}}_H \xrightarrow{\otimes} \underline{\mathrm{Sp}}_G$$

We now define the G–operad $(\underline{\mathrm{Cat}}^{\mathrm{h}}_G)^{\otimes}$ as the G–suboperad of $(\underline{\mathrm{Cat}}_G)^{\otimes}_{\underline{\mathrm{op}}//\underline{\mathcal{D}}}$ spanned by, for every $H \leq G$, pairs $(\underline{\mathcal{C}}, \underline{\Omega})$ where $\underline{\mathcal{C}}$ is H–perfect-stable and $\underline{\Omega} : \underline{\mathcal{C}}^{\underline{\mathrm{op}}} \to \underline{\mathrm{Sp}}_H$ is H–quadratic, and for $(\underline{\mathcal{C}}, \underline{\Omega})$ a H–object and $(\underline{\mathcal{D}}, \underline{\Phi})$ a G–object, a morphism $(f, \eta) : (\underline{\mathcal{C}}, \underline{\Omega}) \to (\underline{\mathcal{D}}, \underline{\Phi})$ is one where $f : \prod_{G/H} \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is G/H–finite distributive, ie. G/H–distributive for finite H–colimits in $\underline{\mathcal{C}}$. By definition, the G–lax symmetric monoidal composite $(\underline{\mathrm{Cat}}^{\mathrm{h}}_G)^{\otimes} \to (\underline{\mathrm{Cat}}_G)^{\otimes}_{\underline{\mathrm{op}}//\underline{\mathcal{D}}} \xrightarrow{\mathrm{ev}_1} \underline{\mathrm{Cat}}^{\times}_G$ factors through the G–suboperad

$$(\underline{\mathrm{Cat}}^{\mathrm{h}}_G)^{\otimes} \to \underline{\mathrm{Cat}}^{\mathrm{perf}}_G \subseteq \underline{\mathrm{Cat}}^{\times}_G$$

We now have all the ingredients to state and prove the genuine equivariant analogue of [CDH+20a, Thm. 5.2.8].

Theorem 7.3.3.

(i) The G–operad $(\underline{\mathrm{Cat}}^{\mathrm{h}}_G)^{\otimes}$ is G–symmetric monoidal with G–tensor product from §7.2.

(ii) *The map of G–operads $(\underline{\mathrm{Cat}}^{\mathrm{h}}_G)^{\underline{\otimes}} \to \underline{\mathrm{Cat}}^{\mathrm{perf}}_G$ is G–symmetric monoidal.*

(iii) *The G–symmetric monoidal structure on $(\underline{\mathrm{Cat}}^{\mathrm{h}}_G)^{\underline{\otimes}}$ restricts to a G–symmetric monoidal structure on the G–subcategory $\underline{\mathrm{Cat}}^{\mathrm{p}}_G$, and so in particular the resulting non-full inclusion $(\underline{\mathrm{Cat}}^{\mathrm{p}}_G)^{\underline{\otimes}} \subset (\underline{\mathrm{Cat}}^{\mathrm{h}}_G)^{\underline{\otimes}}$ is G–symmetric monoidal.*

Proof. For part (i), observe that the construction of the G–norms on hermitian categories in §7.2.1 means that the G–operad $(\underline{\mathrm{Cat}}^{\mathrm{h}}_G)^{\underline{\otimes}}$ has the property of being a locally cocartesian fibration over $\underline{\mathrm{Fin}}_{*G}$, and so it suffices to show that compositions of local cocartesian lifts are local cocartesian: this is precisely supplied by the transitivity of the norm constructions from §7.2.2. Part (ii) is immediate by construction of the underlying G–stable category of the norm of hermitian categories. Finally, for (iii), since $\underline{\mathrm{Cat}}^{\mathrm{p}}_G \subset \underline{\mathrm{Cat}}^{\mathrm{h}}_G$ contains all the equivalences in the bigger category, we only have to show that G–norms of Poincaré categories are again Poincaré and that the tensor unit $(\underline{\mathrm{Sp}}^{\omega}_G, \underline{\Omega}^{\mathrm{u}})$ of $(\underline{\mathrm{Cat}}^{\mathrm{h}}_G)^{\underline{\otimes}}$ is G–Poincaré. The latter fact is clear, and the former is given by Proposition 7.2.5. $\square$

7.3.2 Borelification principle for G-Poincaré categories

By Theorem 3.3.4 we know that we have G–functor

$$\mathrm{ev}_e : \underline{\mathrm{Cat}}^{\mathrm{p}}_G \longrightarrow \underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})$$

which refines to a G–symmetric monoidal functor. We now give a concrete description of the G–right adjoint of this functor, which we denote by i_*.

Lemma 7.3.4. *The right adjoint $i_*(\mathcal{D}, \Phi)$ is computed as follows: for $H \leq G$ a subgroup, we have*

$$\begin{tikzcd} \mathcal{D}^{hG} \arrow[r, "\Phi^{hG}"] \arrow[d, "\mathrm{Res}"'] & \mathrm{Fun}(BG, \mathrm{Sp}) \arrow[r, hook] \arrow[d, "\mathrm{Res}"] & \mathrm{Sp}_G \arrow[d, "\mathrm{Res}"] \\ \mathcal{D}^{hH} \arrow[r, "\Phi^{hH}"'] & \mathrm{Fun}(BH, \mathrm{Sp}) \arrow[r, hook] & \mathrm{Sp}_H \end{tikzcd}$$

We denote this construction by $\underline{\mathrm{Bor}}(\mathcal{D}, \Phi)$.

Proof. Let $(\underline{\mathcal{C}}, \underline{\Omega}) \in \mathrm{Cat}^{\mathrm{p}}_G$. We need to show that the map

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{p}}_G}((\underline{\mathcal{C}}, \underline{\Omega}), \underline{\mathrm{Bor}}(\mathcal{D}, \Phi)) \xrightarrow{\mathrm{ev}_e} \mathrm{Map}_{\mathrm{Cat}^{\mathrm{p}}}((\mathcal{C}_e, \Omega_e), (\mathcal{D}, \Phi))^{hG}$$

is an equivalence. For this, observe first that this sits in the middle of a map of fibre sequences with the map on bases given by

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{perf}}_G}(\underline{\mathcal{C}}, \underline{\mathrm{Bor}}(\mathcal{D})) \xrightarrow{\mathrm{ev}_e} \mathrm{Map}_{\mathrm{Cat}^{\mathrm{p}}}(\mathcal{C}_e, \mathcal{D})^{hG}$$

and on the fibre over $f \in \mathrm{Map}_{\underline{\mathrm{Cat}}^{\mathrm{perf}}_G}(\underline{\mathcal{C}}, \underline{\mathrm{Bor}}(\mathcal{D}))$, the map

$$\mathrm{Nat}^{\underline{q}}(\underline{\Omega}, f^*\Phi_{\underline{\mathrm{Bor}}}) \xrightarrow{\mathrm{ev}_e} \mathrm{Nat}^{q}(\Omega_e, f_e^*\Phi)^{hG}$$

Hence to show that the required map is an equivalence, it suffices to show that the map on bases and on fibres are equivalences. Now the left hand side consists of the following data:

(i) A G–exact functor $f : \underline{\mathcal{C}} \to \underline{\mathrm{Bor}}(\mathcal{D})$ which by adjunction on the underlying G–category $\underline{\mathrm{Cat}}^{\mathrm{perf}}_G$, is the same data as a G–equivariant exact functor $f_e : \mathcal{C}_e \to \mathcal{D}$.

(ii) A natural transformation $\underline{\Omega} \Rightarrow f^*\Phi_{\underline{\mathrm{Bor}}}$, ie. a morphism in $\underline{\mathrm{Fun}}^{\underline{q}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathrm{Sp}}_G)$. Now recall that we had a G–Bousfield localisation $\mathrm{ev}_e : \underline{\mathrm{Sp}}_G \rightleftarrows \underline{\mathrm{Bor}}(\mathrm{Sp}) : i_*$, and $f_*\Phi_{\underline{\mathrm{Bor}}}$ is in the image of i_*. Hence we get the equivalence

$$\mathrm{Nat}(\underline{\Omega}, f^*\Phi_{\mathrm{Bor}}) \simeq \mathrm{Nat}(\Omega_e, f_e^*\Phi)$$

(iii) The transformation in (ii) induces a commutation of the square

$$\begin{tikzcd}
\underline{\mathcal{C}}^{\underline{\mathrm{op}}} \arrow[r, "f"] \arrow[d, "D_{\mathcal{C}}", "\simeq"'] & \underline{\mathrm{Bor}}(\mathcal{D})^{\underline{\mathrm{op}}} \arrow[d, "D_{\mathcal{D}}", "\simeq"'] \\
\underline{\mathcal{C}} \arrow[r, "f"] & \underline{\mathrm{Bor}}(\mathcal{D})
\end{tikzcd}$$

But by adjunction, this is equivalent to the commutation of the square

$$\begin{tikzcd}
\mathcal{C}_e^{\mathrm{op}} \arrow[r, "f_e"] \arrow[d, "D_{\mathcal{C}}", "\simeq"'] & \mathcal{D}^{\mathrm{op}} \arrow[d, "D_{\mathcal{D}}", "\simeq"'] \\
\mathcal{C}_e \arrow[r, "f_e"] & \mathcal{D}
\end{tikzcd}$$

Hence it is easy to see that we have an equivalence on the bases, and the conditions imposed on both sides on the fibres are equivalent, and so indeed we get the desired equivalence. □

As a consequence, we see that $\mathrm{ev}_e \circ \underline{\mathrm{Bor}}(\mathcal{C}, \Omega) \to (\mathcal{C}, \Omega)$ is an equivalence, and so this is a G–Bousfield localisation. Moreover, using that ev refines to a G–symmetric monoidal functor from Theorem 3.3.4 and that Proposition 3.3.6 implies that any symmetric monoidal Poincaré category $(\mathcal{C}, \Omega)$ equipped with a G–action induces a G–symmetric monoidal category $\underline{\mathrm{Bor}}(\mathcal{C}, \Omega)$, we immediately obtain:

Corollary 7.3.5. *There is a G–Bousfield localisation*

$$\underline{\mathrm{Cat}}^{\mathrm{p}}_G \underset{\underline{\mathrm{Bor}}}{\overset{\mathrm{ev}_e}{\rightleftarrows}} \underline{\mathrm{Bor}}(\mathrm{Cat}^{\mathrm{p}})$$

where ev_e refines to a G–symmetric monoidal functor, and so $\underline{\mathrm{Bor}}$ refines to a G–lax symmetric monoidal functor. Thus, for any $(\mathcal{C}, \Omega) \in \mathrm{Fun}(BG, \mathrm{CAlg}((\mathrm{Cat}^{\mathrm{p}})^{\otimes})) \simeq \mathrm{CAlg}_G(\underline{\mathrm{Bor}}((\mathrm{Cat}^{\mathrm{p}})^{\otimes}))$,

$$\{G/H \mapsto (\mathcal{C}, \Omega)^{hH}\}_{H \leq G}$$

assembles to a G–symmetric monoidal G–Poincaré category.

7.3.3 G-presentable-semiadditivity in the large

Construction 7.3.6 (Box sums of hermitian structures)**.** Let $(\underline{\mathcal{C}}_H, \underline{\Omega}_H)$ be a H–hermitian category. First of all, note that there is a H–biadjunction

$$\pi : \mathrm{Res}^G_H \bigoplus_{G/H} \underline{\mathcal{C}}_H \rightleftarrows \underline{\mathcal{C}}_H : i$$

where π projects onto the $e \in H\backslash G/H$ summand in the double coset decomposition of $\mathrm{Res}^G_H \bigoplus_{G/H}$ and i the inclusion of this summand. Now consider the composite G–functor

$$\begin{aligned}\prod_{G/H} \underline{\mathrm{Fun}}^{\mathrm{q}}_H(\underline{\mathcal{C}}^{\mathrm{op}}_H, \underline{\mathrm{Sp}}_H) &\xrightarrow{\pi^*} \prod_{G/H} \underline{\mathrm{Fun}}^{\mathrm{q}}_H(\mathrm{Res}^G_H \bigoplus_{G/H} \underline{\mathcal{C}}^{\mathrm{op}}_H, \underline{\mathrm{Sp}}_H) \\ &\xrightarrow{\mathrm{Ind}^G_H} \underline{\mathrm{Fun}}^{\mathrm{q}}_G(\bigoplus_{G/H} \underline{\mathcal{C}}^{\mathrm{op}}_H, \underline{\mathrm{Sp}}_G)\end{aligned}$$

where Ind^G_H here is the one corresponding to $\underline{\mathrm{Fun}}^{\mathrm{q}}_G(\bigoplus_{G/H} \underline{\mathcal{C}}^{\mathrm{op}}_H, \underline{\mathrm{Sp}}_G)$ admitting indexed biproducts. Upon taking the fibre over G/G, we obtain

$$\begin{aligned}\mathrm{Fun}^{\mathrm{q}}_H(\underline{\mathcal{C}}^{\mathrm{op}}_H, \underline{\mathrm{Sp}}_H) &\xrightarrow{\pi^*} \mathrm{Fun}^{\mathrm{q}}_H(\mathrm{Res}^G_H \bigoplus_{G/H} \underline{\mathcal{C}}^{\mathrm{op}}_H, \underline{\mathrm{Sp}}_H) \\ &\xrightarrow{\mathrm{Ind}^G_H} \mathrm{Fun}^{\mathrm{q}}_G(\bigoplus_{G/H} \underline{\mathcal{C}}^{\mathrm{op}}_H, \underline{\mathrm{Sp}}_G)\end{aligned}$$

which denote as $\underline{\Omega}_H \mapsto \boxplus_{G/H} \underline{\Omega}_H := \mathrm{Ind}^G_H \pi^* \underline{\Omega}_H$.

Lemma 7.3.7. *Let $(\underline{\mathcal{C}}_H, \underline{\Omega}_H)$ be a H–hermitian category. Then $B_{\boxplus_{G/H}\underline{\Omega}_H} \simeq \boxplus_{G/H} B_{\underline{\Omega}_H}$. In particular, if $(\underline{\mathcal{C}}_H, \underline{\Omega}_H)$ were a H–Poincaré category, then $(\bigoplus_{G/H} \underline{\mathcal{C}}_H, \boxplus_{G/H} \underline{\Omega}_H)$ is a G–Poincaré category with duality $D_{\boxplus_{G/H}\underline{\Omega}_H} \simeq \oplus_{G/H} D_{\underline{\Omega}_H}$.*

Proof. For the first statement, consider

$$B_{\boxplus_{G/H}\underline{Ϙ}_H} = B_{\mathrm{Ind}_H^G \pi^* \underline{Ϙ}_H} \simeq \mathrm{Ind}_H^G \pi^* B_{\underline{Ϙ}_H} = \boxplus_{G/H} B_{\underline{Ϙ}_H}$$

since the bilinear part construction B is a finite fibrewise colimit, and so commutes with everything in sight.

For the second statement, we argue first that

$$\boxplus_{G/H}\underline{\mathrm{map}}_{\underline{\mathcal{C}}_H}(-,-) \simeq \underline{\mathrm{map}}_{\bigoplus_{G/H}\underline{\mathcal{C}}_H}(-,-) \tag{7.4}$$

Consider the composite which defines $\boxplus_{G/H}$

$$\begin{aligned}\mathrm{Fun}_H(\underline{\mathcal{C}}_H^{\mathrm{op}} \times \underline{\mathcal{C}}_H, \underline{\mathrm{Sp}}_H) &\xrightarrow{\pi^*} \mathrm{Fun}_H(\mathrm{Res}_H^G \bigoplus_{G/H} (\underline{\mathcal{C}}_H^{\mathrm{op}} \times \underline{\mathcal{C}}_H), \underline{\mathrm{Sp}}_H) \\ &\xrightarrow{\mathrm{Ind}_H^G} \mathrm{Fun}_G(\bigoplus_{G/H} \underline{\mathcal{C}}_H, \underline{\mathrm{Sp}}_G)\end{aligned}$$

Recall that the H–mapping space functor $\underline{\mathrm{Map}}_{\underline{\mathcal{C}}_H}(-,-) : \underline{\mathcal{C}}_H^{\mathrm{op}} \times \underline{\mathcal{C}}_H \to \underline{\mathcal{S}}_H$ unstraightens to the parametrised twisted arrow category, which preserves indexed products, and hence

$$\prod_{G/H}(\underline{\mathcal{C}}_H^{\mathrm{op}} \times \underline{\mathcal{C}}_H) \xrightarrow{\prod_{G/H}\underline{\mathrm{Map}}_H(-,-)} \prod_{G/H}\underline{\mathcal{S}}_H \xrightarrow{\prod_{G/H}} \underline{\mathcal{S}}_G$$

is equivalent to $\underline{\mathrm{Map}}_{\prod_{G/H}\underline{\mathcal{C}}_H}(-,-)$. Now using that $\pi : \mathrm{Res}_H^G \bigoplus_{G/H} \Rightarrow \mathrm{id}$ is the adjunction counit and unwinding definitions, this composite is also $\boxplus_{G/H}\underline{\mathrm{Map}}_{\underline{\mathcal{C}}_H}(-,-)$, and hence we obtain indeed that Eq. (7.4) is true.

Given this, we then see that

$$\begin{aligned} B_{\boxplus_{G/H}\underline{Ϙ}_H} &\simeq \boxplus_{G/H} B_{\underline{Ϙ}_H} \\ &\simeq \boxplus_{G/H}\underline{\mathrm{map}}_{\underline{\mathcal{C}}_H}(-, D_{\underline{Ϙ}_H}-) \\ &\simeq \underline{\mathrm{map}}_{\bigoplus_{G/H}\underline{\mathcal{C}}_H}(-, \oplus_{G/H} D_{\underline{Ϙ}_H}-)\end{aligned}$$

and so since $D_{\underline{Ϙ}_H}$ was an equivalence, so too is $D_{\boxplus_{G/H}\underline{Ϙ}_H} \simeq \oplus_{G/H} D_{\underline{Ϙ}_H}$. In other words, $(\bigoplus_{G/H}\underline{\mathcal{C}}_H, \boxplus_{G/H}\underline{Ϙ}_H)$ is a G–Poincaré category with the prescribed duality. □

Lemma 7.3.8. *The G–categories $\underline{\mathrm{Cat}}_G^{\mathrm{p}}$ and $\underline{\mathrm{Cat}}_G^{\mathrm{h}}$ are G–semiadditive. In fact, the (non-full) inclusion $\underline{\mathrm{Cat}}_G^{\mathrm{p}} \subset \underline{\mathrm{Cat}}_G^{\mathrm{h}}$ creates these indexed biproducts.*

Proof. We first show that $\underline{\mathrm{Cat}}_G^{\mathrm{h}}$ is G–semiadditive. Let $(\underline{\mathcal{C}}_H, \underline{Ϙ}_H)$ be a H–hermitian category. We claim that $(\bigoplus_{G/H}\underline{\mathcal{C}}_H, \boxplus_{G/H}\underline{Ϙ}_H)$ is both the G/H–product and –coproduct in $\underline{\mathrm{Cat}}_G^{\mathrm{h}}$.

To see that it is a G/H–coproduct, let $(\underline{\mathcal{D}}_G, \underline{\Phi}_G)$ be a G–hermitian category. We need to show that the canonical comparison

$$\underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}^{\mathrm{h}}_G}\big((\bigoplus_{G/H}\underline{\mathcal{C}}_H, \boxplus_{G/H}\underline{\Omega}_H), (\underline{\mathcal{D}}_G, \underline{\Phi}_G)\big) \longrightarrow \prod_{G/H}\underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}^{\mathrm{h}}_H}\big((\underline{\mathcal{C}}_H, \underline{\Omega}_H), (\underline{\mathcal{D}}_H, \underline{\Phi}_H)\big) \tag{7.5}$$

induced by the summand inclusion $(\underline{\mathcal{C}}_H, \underline{\Omega}_H) \hookrightarrow (\mathrm{Res}^G_H \bigoplus_{G/H}\underline{\mathcal{C}}_H, \mathrm{Res}^G_H \boxplus_{G/H}\underline{\Omega}_H)$ is an equivalence. For this, note that for a G–exact functor $f : \bigoplus_{G/H}\underline{\mathcal{C}}_H \to \underline{\mathcal{D}}_G$, we have the vertical map of fibre sequences of G–spaces

$$\begin{tikzcd}
\underline{\mathrm{Nat}}_G(\boxplus_{G/H}\underline{\Omega}_H, f^*\underline{\Phi}_G) \arrow[r] \arrow[d] & \underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}^{\mathrm{h}}_G}((\bigoplus_{G/H}\underline{\mathcal{C}}_H, \boxplus_{G/H}\underline{\Omega}_H), (\underline{\mathcal{D}}_G, \underline{\Phi}_G)) \arrow[r] \arrow[d] & \underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}^{\mathrm{perf}}_G}(\bigoplus_{G/H}\underline{\mathcal{C}}_H, \underline{\mathcal{D}}_G) \arrow[d, "\simeq"] \\
\prod_{G/H}\underline{\mathrm{Nat}}_H(\underline{\Omega}_H, i^*f^*\underline{\Phi}_G) \arrow[r] & \prod_{G/H}\underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}^{\mathrm{h}}_H}(\mathrm{Res}^G_H(\bigoplus_{G/H}\underline{\mathcal{C}}_H, \boxplus_{G/H}\underline{\Omega}_H), (\underline{\mathcal{D}}_H, \underline{\Phi}_H)) \arrow[r] & \prod_{G/H}\underline{\mathrm{Map}}_{\underline{\mathrm{Cat}}^{\mathrm{perf}}_H}(\underline{\mathcal{C}}_H, \underline{\mathcal{D}}_H)
\end{tikzcd}$$

where the middle vertical is the map that we want to show is an equivalence. Hence, we need to show that the left vertical map is an equivalence, and for this we consider the computation (using the notation from Construction 7.3.6)

$$\begin{aligned}
\underline{\mathrm{Nat}}_G(\boxplus_{G/H}\underline{\Omega}_H, f^*\underline{\Phi}_G) &= \underline{\mathrm{Nat}}_G(\mathrm{Ind}^G_H\pi^*\underline{\Omega}_H, f^*\underline{\Phi}_G) \\
&\simeq \prod_{G/H}\underline{\mathrm{Nat}}_H(\pi^*\underline{\Omega}_H, f^*\underline{\Phi}_H) \\
&\simeq \prod_{G/H}\underline{\mathrm{Nat}}_H(\underline{\Omega}_H, i^*f^*\underline{\Phi}_H)
\end{aligned}$$

which completes the proof for the G/H–coproduct case. To see the case of G/H–products, we perform a similar computation, but this time showing that

$$\underline{\mathrm{Nat}}_G(\underline{\Phi}_G, h^*\boxplus_{G/H}\underline{\Omega}_H) \longrightarrow \prod_{G/H}\underline{\mathrm{Nat}}_H(\underline{\Phi}_H, (\pi\circ h)^*\underline{\Omega}_H)$$

is an equivalence, for a G–exact map $h : \underline{\mathcal{D}}_G \to \bigoplus_{G/H}\underline{\mathcal{C}}_H$.

We now work towards the case of $\underline{\mathrm{Cat}}^{\mathrm{p}}_G$. By Lemma 7.3.7 we already know that if $(\underline{\mathcal{C}}_H, \underline{\Omega}_H)$ is a H–Poincaré category, then $(\bigoplus_{G/H}\underline{\mathcal{C}}_H, \boxplus_{G/H}\underline{\Omega}_H)$ is a G–Poincaré category with duality $\oplus_{G/H}D_{\underline{\Omega}_H} : \bigoplus_{G/H}\underline{\mathcal{C}}^{\mathrm{op}}_H \xrightarrow{\simeq} \bigoplus_{G/H}\underline{\mathcal{C}}_H$. We are therefore left to show the following: let $(\underline{\mathcal{D}}_G, \underline{\Phi}_G)$ be a G–Poincaré category and $(f, \eta) : (\underline{\mathcal{C}}_H, \underline{\Omega}_H) \to (\underline{\mathcal{D}}_H, \underline{\Phi}_H)$ a H–Poincaré functor. Then the corresponding G–hermitian functor

$$(\overline{f}, \overline{\eta}) : (\bigoplus_{G/H}\underline{\mathcal{C}}_H, \boxplus_{G/H}\underline{\Omega}_H) \to (\underline{\mathcal{D}}_G, \underline{\Phi}_G)$$

coming from the equivalence Eq. (7.5) is in fact a G–Poincaré functor, ie. it preserves the duality. Note that concretely, $\overline{f}$ is given by the composite $\overline{f} : \bigoplus_{G/H}\underline{\mathcal{C}}_H \xrightarrow{\oplus_{G/H}f} \bigoplus_{G/H}\underline{\mathcal{D}}_H \xrightarrow{\mathrm{Ind}^G_H} \underline{\mathcal{D}}_G$ and so since we have a commuting diagram

$$\begin{tikzcd}
\overline{f}^{\mathrm{op}} : \bigoplus_{G/H} \underline{\mathcal{C}}_H^{\mathrm{op}} \arrow[r, "\oplus_{G/H} f^{\mathrm{op}}"] \arrow[d, "\simeq"', "\oplus_{G/H} D_{\Qoppa_H}"] & \bigoplus_{G/H} \underline{\mathcal{D}}_H^{\mathrm{op}} \arrow[r, "\mathrm{Ind}_H^G"] \arrow[d, "\simeq"', "\oplus_{G/H} D_{\Phi_H}"] & \underline{\mathcal{D}}_G^{\mathrm{op}} \arrow[d, "\simeq"', "D_{\Phi_G}"] \\
\overline{f} : \bigoplus_{G/H} \underline{\mathcal{C}}_H \arrow[r, "\oplus_{G/H} f"'] & \bigoplus_{G/H} \underline{\mathcal{D}}_H \arrow[r, "\mathrm{Ind}_H^G"'] & \underline{\mathcal{D}}_G
\end{tikzcd}$$

we see that indeed $(\overline{f}, \overline{\eta})$ is duality-preserving. Similarly, one can show that a H–Poincaré functor $(h, \xi) : (\underline{\mathcal{D}}_H, \underline{\Phi}_H) \to (\underline{\mathcal{C}}_H, \underline{\Qoppa}_H)$ induces a G–Poincaré functor $(\overline{h}, \overline{\xi}) : (\underline{\mathcal{D}}_G, \underline{\Phi}_G) \to (\bigoplus_{G/H} \underline{\mathcal{C}}_H, \boxplus_{G/H} \underline{\Qoppa}_H)$. All in all, these show that $(\bigoplus_{G/H} \underline{\mathcal{C}}_H, \boxplus_{G/H} \underline{\Qoppa}_H)$ satisfies the universal property of the indexed biproduct in $\underline{\mathrm{Cat}}^{\mathrm{p}}_G$. □

Proposition 7.3.9. *The G–categories $\underline{\mathrm{Cat}}^{\mathrm{h}}_G$ and $\underline{\mathrm{Cat}}^{\mathrm{p}}_G$ are G–semiadditve, G–cocomplete, and G–complete. In fact, both functors in $\underline{\mathrm{Cat}}^{\mathrm{p}}_G \to \underline{\mathrm{Cat}}^{\mathrm{h}}_G \to \underline{\mathrm{Cat}}^{\mathrm{perf}}_G$ strongly preserve G–colimits and G–limits.*

Proof. The same proof as for [CDH+20a, Prop. 6.1.2, 6.1.4] shows that $\underline{\mathrm{Cat}}^{\mathrm{h}}_G$ has fibrewise G–colimits and –limits, and that the forgetful functor $\mathrm{fgt} : \underline{\mathrm{Cat}}^{\mathrm{h}}_G \to \underline{\mathrm{Cat}}^{\mathrm{perf}}_G$ strongly preserves these, and that a similar statement holds for $\underline{\mathrm{Cat}}^{\mathrm{p}}_G$ and the inclusion functor $\underline{\mathrm{Cat}}^{\mathrm{p}}_G \hookrightarrow \underline{\mathrm{Cat}}^{\mathrm{h}}_G$. On the other hand, we know by Lemma 7.3.8 that $\underline{\mathrm{Cat}}^{\mathrm{p}}_G$ and $\underline{\mathrm{Cat}}^{\mathrm{h}}_G$ are G–semiadditive and that the inclusion $\underline{\mathrm{Cat}}^{\mathrm{p}}_G \hookrightarrow \underline{\mathrm{Cat}}^{\mathrm{h}}_G$ strongly preserves these indexed biproducts. Therefore, all in all, we obtain by Theorem 1.2.9 that both $\underline{\mathrm{Cat}}^{\mathrm{p}}_G$ and $\underline{\mathrm{Cat}}^{\mathrm{h}}_G$ are G–cocomplete and G–complete, and that both functors in $\underline{\mathrm{Cat}}^{\mathrm{p}}_G \to \underline{\mathrm{Cat}}^{\mathrm{h}}_G \to \underline{\mathrm{Cat}}^{\mathrm{perf}}_G$ strongly preserve G–colimits and –limits, as desired. □

Lemma 7.3.10. *The three G–Poincaré categories*

$$\underline{\mathrm{Hyp}}(\underline{\mathrm{Sp}}^{\omega}_G),\ \underline{\mathrm{Hyp}}(\mathrm{Ar}(\underline{\mathrm{Sp}}^{\omega}_G)),\ \textit{and}\ \underline{\mathrm{Met}}(\underline{\mathrm{Sp}}^{\omega}_G, \underline{\Qoppa}^{\mathrm{u}})$$

form a set of G–ω–compact generators for $\underline{\mathrm{Cat}}^{\mathrm{p}}_G$.

Proof. Note that the G–Poincaré categories corepresent respectively the G–functors

$$\Big((\underline{\mathcal{C}}, \underline{\Qoppa}) \mapsto \underline{\mathcal{C}}^{\simeq}\Big),\ \Big((\underline{\mathcal{C}}, \underline{\Qoppa}) \mapsto \mathrm{Ar}(\underline{\mathcal{C}})^{\simeq}\Big),\ \text{and}\ \Big((\underline{\mathcal{C}}, \underline{\Qoppa}) \mapsto \underline{\mathrm{Fm}}(\underline{\mathcal{C}}, \underline{\Qoppa})\Big)$$

the first two by the adjunction Proposition 7.1.49 together with Lemma 7.1.46, and the third is by Proposition 7.1.52. Since each of these functors clearly strongly preserves fibrewise ω–filtered colimits, the three G–Poincaré categories are G–ω–compact. Hence, by Proposition 4.1.6, it will now suffice to show that they are jointly conservative. For this, suppose we have a G–Poincaré functor

$$(f, \eta) : (\underline{\mathcal{C}}, \underline{\Qoppa}) \to (\underline{\mathcal{D}}, \underline{\Phi})$$

which induces equivalences

$$\underline{\mathcal{C}}^{\cong} \xrightarrow{\simeq} \underline{\mathcal{D}}^{\cong} \qquad \mathrm{Ar}(\underline{\mathcal{C}})^{\cong} \xrightarrow{\simeq} \mathrm{Ar}(\underline{\mathcal{D}})^{\cong} \qquad \underline{\mathrm{Fm}}(\underline{\mathcal{C}}, \underline{\Omega}) \xrightarrow{\simeq} \underline{\mathrm{Fm}}(\underline{\mathcal{D}}, \underline{\Phi})$$

The first two then implies that the underlying G–functor $f : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is an equivalence. To obtain that it is also an equivalence of G–Poincaré categories, we need to show that for all $x \in \underline{\mathcal{C}}$, the morphism

$$\eta_x : \underline{\Omega}(x) \Longrightarrow \underline{\Phi} f(x)$$

is an equivalence. By Lemma 7.1.40, we can test this by applying $\underline{\Omega}^{\infty}$. In this case, this map is just the map induced on the vertical fibres of

$$\begin{tikzcd}
\underline{\mathrm{Fm}}(\underline{\mathcal{C}}, \underline{\Omega}) \arrow[r, "\simeq"] \arrow[d] & \underline{\mathrm{Fm}}(\underline{\mathcal{D}}, \underline{\Phi}) \arrow[d] \\
\underline{\mathcal{C}}^{\cong} \arrow[r, "\simeq"] & \underline{\mathcal{D}}^{\cong}
\end{tikzcd}$$

over $x \in \underline{\mathcal{C}}$ and $f(x) \in \underline{\mathcal{D}}$ respectively, and is therefore an equivalence, as desired. $\square$

We now come to the main theorem of this subsection.

Theorem 7.3.11. *The G–category $\underline{\mathrm{Cat}}^{\mathrm{p}}_G$ is G–presentable-semiadditive.*

Proof. We already know by Proposition 7.3.9 that it is G–semiadditive. Since $\underline{\mathrm{Cat}}^{\mathrm{p}}_G$ is G–cocomplete by Proposition 7.3.9, we can combine Proposition 4.1.6 and Lemma 7.3.10 to yield that the canonical comparison $\underline{\mathrm{Ind}}_{\omega}((\underline{\mathrm{Cat}}^{\mathrm{p}}_G)^{\underline{\omega}}) \to \underline{\mathrm{Cat}}^{\mathrm{p}}_G$ is an equivalence. Hence, by Theorem 2.2.2 we get the G–presentability of $\underline{\mathrm{Cat}}^{\mathrm{p}}_G$. $\square$

7.4 Potential applications: equivariant periodicities

We now give two advertisements as to the potential usefulness of the genuine equivariant point of view for hermitian K–theory. Both of these are centred around obtaining equivariant periodicities for L–theory, proceeding however via very different means. In §7.4.1 we see that upon inverting 2, the theory of the first three sections in this chapter goes through when G is even. Our hope is that this can be a good method of manufacturing equivariantly periodic genuine G–spectra. From this, we show how to obtain equivariantly periodic L–theory for $G = C_2$. In §7.4.2, we speculate on a strategy to prove descent theorems for equivariant L–theory given a refinement of it to a G–spectrum with norms.

7.4.1 C_2-Ranicki periodicity

So far the theory was only developed for when G is an odd group: this was because $(\beta \circ \Delta)^{t\Sigma_2}$, for β a G–bilinear functor into $\underline{\mathrm{Sp}}_G$ can fail to be G–quadratic when G contains C_2. However, we might still be interested when G is an even group. In this subsection, we explore that possibility and the group G is fixed to be the cyclic group C_2 of order 2. Importantly, our standing assumption in this subsection is that $\underline{\Omega}$ factors through the full subcategory $\underline{\mathrm{Sp}}_{C_2}[\frac{1}{2}] \subseteq \underline{\mathrm{Sp}}_{C_2}$ of 2–inverted C_2–spectra in order to guarantee that $(\beta \circ \Delta)^{t\Sigma_2} \simeq 0$ and so is C_2–quadratic. The theory of §7.1 can then be carried out similarly. In this case however, instead of an interesting stable recollement as in Theorem 7.1.24, we in fact even have a splitting

$$\underline{\mathrm{Fun}}^{\mathrm{q}}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_{C_2}[\frac{1}{2}]) \simeq \underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_{C_2}[\frac{1}{2}]) \times \underline{\mathrm{Fun}}^{\mathrm{s}}(\underline{\mathcal{C}}^{\mathrm{op}} \times \underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathrm{Sp}}_{C_2}[\frac{1}{2}])$$

This is because there is no interesting gluing via the Σ_2–Tate construction, which in this case, is always trivial since 2 was inverted.

Notation 7.4.1. We record here two conventions that we use just in this exploratory subsection.

- Throughout this subsection, in order not to overload the notations, 2 being inverted is always implicit and we will omit writing $[\frac{1}{2}]$ for the receptacle of the C_2–quadratic structures. In other words, $\underline{\mathrm{Sp}}_{C_2}$ will always stand for $\underline{\mathrm{Sp}}_{C_2}[\frac{1}{2}]$ and Sp will mean $\mathrm{Sp}[\frac{1}{2}]$. Thus, for example $\underline{\mathrm{Cat}}^{\mathrm{p}}_{C_2}$ would be defined as a non-full subcategory of $\underline{\mathrm{Cat}}^{\mathrm{h}}_{C_2}$ which is in turn defined as the unstraightening of

$$(\underline{\mathrm{Cat}}^{\mathrm{perf}}_{C_2})^{\underline{\mathrm{op}}} \longrightarrow \widehat{\underline{\mathrm{Cat}}}_{C_2} \quad :: \quad \underline{\mathcal{C}} \mapsto \underline{\mathrm{Fun}}^{\mathrm{q}}(\underline{\mathcal{C}}^{\underline{\mathrm{op}}}, \underline{\mathrm{Sp}}_{C_2}[\frac{1}{2}])$$

- We emphasise here that there are two different C_2-actions, one coming from the hermitianness and the other from the equivariant direction. Because of this, we will denote the C_2 group coming from hermitian structures with Σ_2 instead, as we have been doing from the beginning of this chapter.

Construction 7.4.2 (Pointwise L–theory)**.** For the purposes of this subsection, by (equivariant) L–theory here, we mean *pointwise* L–theory. Namely, to a $(\underline{\mathcal{C}}, \underline{\Omega}) \in \underline{\mathrm{Cat}}^{\mathrm{p}}_{G}$, we can associate a G–spectrum $\underline{\mathrm{L}}(\underline{\mathcal{C}}, \underline{\Omega})$ whose genuine H–fixed point for $H \leq G$ is given by

$$\underline{\mathrm{L}}(\underline{\mathcal{C}}, \underline{\Omega}) := \mathrm{L}(\mathcal{C}_H, \mathcal{C}_H^{\mathrm{op}} \xrightarrow{\Omega_H} \mathrm{Sp}_H \xrightarrow{(-)^H} \mathrm{Sp})$$

where L here is the one constructed in [CDH+20b]. In other words, we perform the ad-construction of [CDH+20b] levelwise on each genuine fixed point of the G–space $\underline{\mathrm{Pn}}(\underline{\mathcal{C}}, \underline{\Omega})$. Importantly, this inherits the excision against split Poincaré–Verdier sequences satisfied by ordinary L–theory as well as the fact that $\underline{\mathrm{L}}(\underline{\mathrm{Hyp}}(\underline{\mathcal{C}})) \simeq 0$ since $\underline{\mathrm{Hyp}}(\underline{\mathcal{C}})$ is levelwise hyperbolic in the ordinary sense.

Recollections 7.4.3. A C_2-Poincaré category $(\underline{\mathcal{C}}, \underline{\Omega}) \in \underline{\mathrm{Cat}}^{\mathrm{p}}_{C_2}$ is the datum of a C_2-category $\underline{\mathcal{C}}$ together with a 2–inverted C_2-quadratic functor $\underline{\Omega} : \underline{\mathcal{C}}^{\underline{\mathrm{op}}} \to \underline{\mathrm{Sp}}_{C_2}$, ie. a diagram

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}}_{C_2} \arrow[r, "\Omega_{C_2}"] \arrow[d, "\mathrm{Res}"] & \mathrm{Sp}^{C_2} \arrow[d, "\mathrm{Res}"] \\ \mathcal{C}^{\mathrm{op}}_e \arrow[r, "\Omega_e"] & \mathrm{Sp} \end{tikzcd}$$

and the C_2-functor $\underline{\Omega}$ sits in a recollement square

$$\begin{tikzcd} \underline{\Omega} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L_{\underline{\Omega}} \arrow[d] \\ (B^{\Delta}_{\underline{\Omega}})^{h\Sigma_2} \arrow[r] & (B^{\Delta}_{\underline{\Omega}})^{t\Sigma_2} \simeq 0 \end{tikzcd}$$

where the Tate term is zero since 2 was inverted. Hence, we even have the splitting $\underline{\Omega} \simeq L_{\underline{\Omega}} \times (B^{\Delta}_{\underline{\Omega}})^{h\Sigma_2}$. However, this special property will not be used anywhere here.

Construction 7.4.4 (C_2-spans)**.** Let $(\underline{\mathcal{C}}, \underline{\Omega}) \in \underline{\mathrm{Cat}}^{\mathrm{p}}_{C_2}$. Write $\underline{Q}_1(\underline{\mathcal{C}}, \underline{\Omega})$ for the C_2–Poincaré category whose underlying C_2-category is $\underline{\mathrm{Fun}}(\underline{\Lambda}^2_0, \underline{\mathcal{C}})$ and the C_2–Poincaré structure given by the C_2-pullback

$$(y \leftarrow x \to y) \mapsto \underline{\Omega}^{\Delta}(y \leftarrow x \to y) := \underline{\Omega}(y) \underline{\times}_{\underline{\Omega}(x)} \underline{\Omega}(y)$$

Since the cross-effect functor commutes with all parametrised limits and colimits, we get that $B_{\underline{\Omega}^{\Delta}} \simeq \underline{\lim}_{\underline{\Lambda}} B_{\underline{\Omega}}$, and so the duality is the expected one involving C_2-pullbacks. Note that we have a C_2-self-equivalence

$$\begin{tikzcd} \mathrm{Fun}_{C_2}(\underline{\Lambda}^2_0, \underline{\mathcal{C}}) \arrow[r, "\mathrm{cofib}", "\simeq"'] \arrow[d, "\mathrm{Res}"] & \mathrm{Fun}_{C_2}(\underline{\Lambda}^2_0, \underline{\mathcal{C}}) \arrow[d, "\mathrm{Res}"] \\ \mathrm{Fun}(\Lambda^2_0, \mathcal{C}_e) \arrow[r, "\mathrm{cofib} \cup \mathrm{cofib}", "\simeq"'] & \mathrm{Fun}(\Lambda^2_0, \mathcal{C}_e) \end{tikzcd}$$

given by

$$\underline{\mathrm{cofib}}(a \xleftarrow{f} w \xrightarrow{f} a) = \Big(\mathrm{cofib}(f) \leftarrow a \underline{\amalg}_w a \to \mathrm{cofib}(f)\Big)$$

with inverse given similarly by $(\underline{\mathrm{fib}}, \mathrm{fib} \times \mathrm{fib})$.

Construction 7.4.5 (The C_2-bordism span error term)**.** Let $s : \Delta^0 \hookrightarrow \underline{\Lambda}^2_0$ be the inclusion of the join in the C_2-span and $j : \underline{\amalg}_{C_2} * \hookrightarrow \underline{\Lambda}^2_0$ be the inclusion of the C_2-points. Let $(\underline{\mathcal{C}}, \underline{\Omega}) \in \underline{\mathrm{Cat}}^{\mathrm{p}}_{C_2}$ be a C_2-Poincaré category. Then first note that the fully faithful left Kan extension $s_! : \underline{\mathcal{C}} \hookrightarrow \underset{\sim}{Q}_1(\underline{\mathcal{C}})$ is given by the constant diagram, the fully faithful right Kan extension s_* is given by extension-by-zero, the fully faithful left Kan extension $j_!$ is given by extension-by-zero, and the fully faithful right adjoint j_* is given by $x \mapsto (\mathrm{Res}^{C_2} x \leftarrow \mathrm{Ind}^{C_2} \mathrm{Res}^{C_2} x \rightarrow \mathrm{Res}^{C_2} x)$. Note that $s_!$ is duality-preserving since

$$\begin{tikzcd} x \arrow[r, mapsto, "s_!"] \arrow[d, mapsto, "D"'] & (x \xleftarrow{=} x \xrightarrow{=} x) \arrow[d, mapsto, "D^{\underline{\Delta}}"] \\ Dx \arrow[r, mapsto, "s_!"] & (Dx \xleftarrow{=} Dx \xrightarrow{=} Dx) \end{tikzcd}$$

and hence is a C_2-Poincaré functor. Be warned however that s_* is *not* a C_2-Poincaré functor since

$$D^{\underline{\Delta}}(s_*) = 0\underline{\times}_{Dx}0 \simeq \Omega^{\sigma} Dx$$

Observe that the C_2-self equivalence above interchanges $s_!$ and s_* since

$$\begin{tikzcd} \underline{\mathcal{C}} \arrow[r, hook, "s_!"] \arrow[d, equal] & \underset{\sim}{Q}_1(\underline{\mathcal{C}}) \arrow[d, shift right, "\underline{\mathrm{cofib}}"'] \arrow[d, phantom, "\simeq"] \\ \underline{\mathcal{C}} \arrow[r, hook, "s_*"] & \underset{\sim}{Q}_1(\underline{\mathcal{C}}) \arrow[u, shift right, "\underline{\mathrm{fib}}"'] \end{tikzcd}$$

Now observe that we have a bifibre sequence in $\underline{\mathrm{Cat}}^{\mathrm{perf}}_{C_2}$

$$\underline{\mathcal{C}} \overset{s^*}{\twoheadleftarrow} \underset{\sim}{Q}_1(\underline{\mathcal{C}}) \overset{j_!}{\hookleftarrow} \mathrm{Ind}^{C_2} \mathrm{Res}^{C_2} \underline{\mathcal{C}}$$

Writing p and ℓ for the left adjoints $\ell \dashv p \dashv j_!$, we have the solid commutative square

$$\begin{tikzcd} \underset{\sim}{Q}_1(\underline{\mathcal{C}}) \arrow[r, dashed, shift left, "p"] \arrow[d, "\underline{\mathrm{cofib}}"', "\simeq"] & \mathrm{Ind}^{C_2} \mathrm{Res}^{C_2} \underline{\mathcal{C}} \arrow[l, hook', shift left, "j_!"] \arrow[l, dashed, bend right, "\ell"'] \arrow[d, equal] \\ \underset{\sim}{Q}_1(\underline{\mathcal{C}}) \arrow[r, dashed, shift left, "j^*"] & \mathrm{Ind}^{C_2} \mathrm{Res}^{C_2} \underline{\mathcal{C}} \arrow[l, hook', shift left, "j_*"] \arrow[l, dashed, bend right, "j_!"'] \end{tikzcd} \tag{7.6}$$

and hence we can compute p as $j^* \circ \underline{\mathrm{cofib}}$ to give

$$p : (a \xleftarrow{f} w \xrightarrow{f} a) \mapsto \mathrm{cofib}(f)$$

and compute ℓ as $\underline{\mathrm{fib}} \circ j_!$ to give

$$\ell : x \mapsto \big(\Omega x \leftarrow \mathrm{Ind}^{C_2} \Omega x \rightarrow \Omega x\big)$$

From this we can collect some basic information:

- Taking the cofibre in $\underline{\mathrm{Cat}}^{\mathrm{p}}_{C_2}$ we get a split Poincaré–Verdier sequence

$$(\underline{\mathcal{C}}, \underline{Ϙ}) \xhookrightarrow{s_!} (\underline{Q}_1(\underline{\mathcal{C}}), \underline{Ϙ}^{\Delta}) \xrightarrow{p} \Big(\mathrm{Ind}^{C_2}\mathrm{Res}^{C_2}\underline{\mathcal{C}}, (p^{\mathrm{op}})_!(\underline{Ϙ}^{\Delta})\Big)$$

where we know that $\underline{\Psi} := (p^{\mathrm{op}})_!(\underline{Ϙ}^{\Delta})$ is computed as $(\ell^{\mathrm{op}})^*(\underline{Ϙ}^{\Delta})$. Therefore, we get that for $x \in \mathrm{Ind}^{C_2}\mathrm{Res}^{C_2}\underline{\mathcal{C}}$, $\underline{\Psi}(x) := (p^{\mathrm{op}})_!(\underline{Ϙ}^{\Delta})(x) \in \mathrm{Sp}^{C_2}$ is computed as the C_2–pullback

$$\underline{\Psi}(x) := Ϙ_e(\Omega x)\underline{\times}_{Ϙ_{C_2}(\mathrm{Ind}^{C_2}\Omega x)}Ϙ_e(\Omega x)$$

Similarly, for $(a,b) \in \mathrm{Res}^{C_2}\mathrm{Ind}^{C_2}\mathrm{Res}^{C_2}\underline{\mathcal{C}}$, we have that

$$\ell : (a,b) \mapsto \big(\Omega a \leftarrow \Omega a \oplus \Omega b \rightarrow \Omega b\big)$$

and so $\Psi = (p_!^{\mathrm{op}})(\underline{Ϙ}^{\Delta})(x,y)$ is computed as

$$\Psi_e(a,b) \simeq Ϙ_e(\Omega a) \times_{Ϙ_e(\Omega a \oplus \Omega b)} Ϙ_e(\Omega b) \simeq \Omega B_{Ϙ_e}(\Omega a, \Omega b) \simeq \Sigma B_{Ϙ_e}(a,b)$$

- Moreover, it is formally straightforward to see that the duality must be given by $p \circ D_{\underline{Ϙ}^{\Delta}} \circ \ell^{\mathrm{op}}$, and so the effect of the duality is given by on the C_2-fixed points as

$$\begin{aligned} x &\overset{\ell}{\mapsto} (\Omega x \leftarrow \mathrm{Ind}^{C_2}\Omega x \rightarrow \Omega x) \\ &\xmapsto{D_{\underline{Ϙ}^{\Delta}}} (D_e\Omega y \leftarrow D_e\Omega y \times_{\mathrm{Ind}^{C_2}\Omega x} D_e\Omega x \rightarrow D_e\Omega x) \\ &\overset{p}{\mapsto} D_e\Omega x \simeq \Sigma D_e x \end{aligned}$$

and similarly, on the underlying thing,

$$\begin{aligned} (x,y) &\overset{\ell}{\mapsto} (\Omega y \leftarrow \Omega y \oplus \Omega x \rightarrow \Omega x) \\ &\xmapsto{D_{\underline{Ϙ}^{\Lambda}}} (D_e\Omega y \leftarrow D_e\Omega y \times_{D_e\Omega y \oplus D_e\Omega x} D_e\Omega x \rightarrow D_e\Omega x) \\ &\overset{p}{\mapsto} (D_e\Omega y, D_e\Omega x) \simeq \Sigma(D_e y, D_e x) \end{aligned}$$

Lemma 7.4.6. *The C_2–quadratic structure $\underline{\Psi}$ is homogeneous, ie. the canonical map $(B^{\Delta}_{\underline{\Psi}})_{h\Sigma_2} \rightarrow \underline{\Psi}$ is an equivalence.*

Proof. We just have to show that the linear part vanishes. The description of Ψ_e makes this clear on the underlying functor, and for the full C_2-quadratic functor, just note

$$L_{\underline{\Psi}} = L_{Ϙ_e(x)\underline{\times}_{Ϙ_{C_2}(\mathrm{Ind}^{C_2}x)}Ϙ_e(x)} \simeq L_{Ϙ_e(x)}\underline{\times}_{L_{Ϙ_{C_2}(\mathrm{Ind}^{C_2}x)}} L_{Ϙ_e(x)} = 0$$

because the C_2-square

$$\begin{tikzcd} \mathrm{Ind}^{C_2} x \arrow[r] \arrow[d] & x \arrow[d] \\ x \arrow[r] & 0 \end{tikzcd}$$

is a C_2-pushout-pullback in $\underline{\mathcal{C}}$ and $L_{\underline{\Omega}}$ preserves this. □

Notation 7.4.7. Following Glasman [Gla17], we denote by $\Xi : \mathrm{Sp} \hookrightarrow \mathrm{Sp}_{C_2}$ the fully faithful right adjoint of the C_2-geometric fixed point functor Φ^{C_2}. Concretely, Ξ takes a spectrum X to the C_2-Mackey functor with trivial underlying spectrum and top fixed point X.

Lemma 7.4.8. *The C_2–Poincaré category $(\mathrm{Ind}^{C_2}\mathrm{Res}^{C_2}\underline{\mathcal{C}}, \underline{\Psi})$ constructed above satisfies*

$$\underline{\mathrm{L}}(\mathrm{Ind}_e^{C_2}\mathrm{Res}_e^{C_2}\underline{\mathcal{C}}, \underline{\Psi}) \simeq \Xi\mathrm{L}(\mathcal{C}_e, \Sigma\Omega_e^q) \in \mathrm{Sp}_{C_2}$$

Proof. We need to show two points, namely, that

$$\underline{\mathrm{L}}(\mathrm{Ind}_e^{C_2}\mathrm{Res}_e^{C_2}\underline{\mathcal{C}}, \underline{\Psi})^{C_2} := \mathrm{L}(\mathcal{C}_e, (\Psi_{C_2})^{C_2}) \simeq \mathrm{L}(\mathcal{C}_e, \Sigma\Omega_e^q) \in \mathrm{Sp} \tag{7.7}$$

and that

$$\underline{\mathrm{L}}(\mathrm{Ind}_e^{C_2}\mathrm{Res}_e^{C_2}\underline{\mathcal{C}}, \underline{\Psi})^e := \mathrm{L}(\mathcal{C}_e \oplus \mathcal{C}_e, \Psi_e) \simeq 0 \in \mathrm{Sp} \tag{7.8}$$

Firstly, by Lemma 7.4.6 we have $\underline{\Psi} \simeq (B^{\Delta}_{\underline{\Psi}})_{h\Sigma_2}$ and so we need to analyse $B^{\Delta}_{\underline{\Psi}}$ together with its Σ_2–action. By representability of C_2–Poincaré structures, we know that for $a, x \in \mathcal{C}_e = (\mathrm{Ind}_e^{C_2}\mathrm{Res}_e^{C_2}\underline{\mathcal{C}})^{C_2}$, we have

$$B_{\Psi_{C_2}}(a,x) \simeq \underline{\mathrm{map}}_{\mathrm{Ind}_e^{C_2}\mathrm{Res}_e^{C_2}\underline{\mathcal{C}}}(a, \Sigma D_e x) \simeq \mathrm{Ind}_e^{C_2}\Sigma\,\mathrm{map}_{\mathcal{C}_e}(a, D_e x) \simeq \mathrm{Ind}_e^{C_2}\Sigma B_{\Omega_e}(a,x)$$

Thus setting $a = x$, we obtain

$$(\Psi_{C_2})^{C_2}(x) \simeq \left[B^{\Delta}_{\Psi_{C_2}}(x)^{C_2}\right]_{h\Sigma_2} \simeq \left[(\mathrm{Ind}^{C_2}\Sigma B_{\Omega_e}(x,x))^{C_2}\right]_{h\Sigma_2} \simeq \Sigma B_{\Omega_e}(x,x)_{h\Sigma_2} \in \mathrm{Sp}$$

which gives $(\Psi_{C_2})^{C_2} \simeq \Sigma\Omega_e^q$, and hence Eq. (7.7).

For Eq. (7.8), letting $a, b, x, y \in \mathcal{C}_e$, we have

$$\begin{aligned} B_{\Psi_e}((a,b),(x,y)) &\simeq B_{\Sigma B_{\Omega_e}}((a,b),(x,y)) \\ &\simeq \Sigma\,\mathrm{fib}\left(B_{\Omega_e}(a\oplus x, b\oplus y) \to B_{\Omega_e}(a,b) \oplus B_{\Omega_e}(x,y)\right) \\ &\simeq \Sigma B_{\Omega_e}(a,y) \oplus \Sigma B_{\Omega_e}(x,b) \end{aligned}$$

and hence by setting $(a,b) = (x,y)$, we get

$$\Psi_e(x,y) \simeq \left[B^{\Delta}_{\Psi_e}(x,y)\right]_{h\Sigma_2} \simeq \left[\Sigma B_{\Omega_e}(x,y) \oplus \Sigma B_{\Omega_e}(x,y)\right]_{h\Sigma_2} \simeq \Sigma B_{\Omega_e}(x,y) \in \mathrm{Sp}$$

But it is standard nonequivariant theory that $(\mathcal{C}_e \oplus \mathcal{C}_e, B_{\Omega_e}) \simeq \mathrm{Hyp}(\mathcal{C}_e)$ via the duality $\mathrm{id}_{\mathcal{C}_e} \oplus D_e$, and since L–theory vanishes on hyperbolics, we get $\mathrm{L}(\mathcal{C}_e \oplus \mathcal{C}_e, \Psi_e) \simeq 0$. □

Proposition 7.4.9. *Let $(\underline{\mathcal{C}}, \underline{\Omega}) \in \underline{\mathrm{Cat}}^{\mathrm{p}}_{C_2}$. Then there is a split Poincaré–Verdier sequence*

$$(\underline{\mathcal{C}}, \Omega^{\sigma}\underline{\Omega}) \xrightarrow{s_*} (\underline{\mathcal{C}}, \underline{\Omega})^{\Lambda} \xrightarrow{j^*} \mathrm{Ind}^{C_2}\mathrm{Res}^{C_2}(\underline{\mathcal{C}}, \underline{\Omega})$$

Proof. We know that $s_* : \underline{\mathcal{C}} \hookrightarrow \underline{\mathcal{C}}^{\Lambda}$ is a split Verdier inclusion. To see that it is even a split Poincaré–Verdier inclusion, we need to show that $\Omega^{\sigma}\underline{\Omega} \Rightarrow (s_*)^*(\underline{\Omega}^{\Lambda})$ is an equivalence. Let $x \in \underline{\mathcal{C}}$. So

$$(s_*)^*(\underline{\Omega}^{\Lambda})(x) \simeq (\underline{\Omega}^{\Lambda})(0 \leftarrow x \rightarrow 0) \simeq \Omega^{\sigma}\underline{\Omega}(x)$$

By the square Eq. (7.6), we know that the cofibre map $\underline{\mathcal{C}}^{\Lambda} \rightarrow \mathrm{Ind}^{C_2}\mathrm{Res}^{C_2}\underline{\mathcal{C}}$ is given by j^*, ie. evaluation at the two feet of a C_2-span. By the same token, the pushforward quadratic structure is given by $(j_!^{\mathrm{op}})^*(\underline{\Omega}^{\Lambda})$, ie.

$$x \mapsto \underline{\Omega}^{\Lambda}(x \leftarrow 0 \rightarrow x) \simeq \mathrm{Ind}^{C_2}\Omega_e(x)$$

Hence we obtain that, indeed, the cofibre term is $\mathrm{Ind}^{C_2}\mathrm{Res}^{C_2}(\underline{\mathcal{C}}, \underline{\Omega})$. □

Lemma 7.4.10. *Let $(\underline{\mathcal{C}}, \underline{\Omega})$ be a C_2–Poincaré category. There is a failure exact sequence*

$$\Omega^{\sigma}\underline{\mathrm{L}}(\underline{\mathcal{C}}, \underline{\Omega}) \rightarrow \underline{\mathrm{L}}(\underline{\mathcal{C}}, \Omega^{\sigma}\underline{\Omega}) \rightarrow \Xi\mathrm{L}(\mathcal{C}_e, \Sigma\Omega_e^{q})$$

Proof. Consider the diagram of fibre sequences of genuine C_2–spectra

$$\begin{tikzcd}
\Omega^{\sigma}\underline{\mathrm{L}}(\underline{\mathcal{C}}, \underline{\Omega}) \arrow[r] \arrow[d] & \underline{\mathrm{L}}(\underline{\mathcal{C}}, \underline{\Omega}) \arrow[r] \arrow[d, "s_!"] & \mathrm{Ind}^{C_2}\mathrm{Res}^{C_2}\underline{\mathrm{L}}(\underline{\mathcal{C}}, \underline{\Omega}) \arrow[d, equal] \\
\underline{\mathrm{L}}(\underline{\mathcal{C}}, \Omega^{\sigma}\underline{\Omega}) \arrow[r, "s_*"] \arrow[d] & \underline{\mathrm{L}}(\underline{\mathcal{C}}^{\Lambda}, \underline{\Omega}^{\Lambda}) \arrow[r, "j^*"] \arrow[d] & \mathrm{Ind}^{C_2}\mathrm{Res}^{C_2}\underline{\mathrm{L}}(\underline{\mathcal{C}}, \underline{\Omega}) \arrow[d] \\
\Xi\mathrm{L}(\mathcal{C}_e, \Sigma\Omega_e^{q}) \arrow[r, equal] & \Xi\mathrm{L}(\mathcal{C}_e, \Sigma\Omega_e^{q}) \arrow[r] & 0
\end{tikzcd}$$

where here the top horizontal sequence is by the defining sequence of sign loops, the middle horizontal sequence is by Proposition 7.4.9, and the middle vertical sequence is by Construction 7.4.5 and Lemma 7.4.8. □

Theorem 7.4.11. *Suppose we have a C_2-Poincaré category $(\underline{\mathcal{C}}, \underline{\Omega})$ such that we have an equivalence $(\mathcal{C}_e, \Omega_e^{q}) \simeq (\mathcal{C}_e, \Sigma^2\Omega_e^{q})$ on the underlying Poincaré category, for example, $(\underline{\mathrm{Mod}}_{\mathrm{Sp}_{C_2}[\frac{1}{2}]}(KU_{C_2}[\frac{1}{2}])^{\omega}, \underline{\Omega}^{s})$ via the nonequivariant Bott periodicity. Then there is a natural equivalence*

$$\Omega^{2\sigma}\underline{\mathrm{L}}(\underline{\mathcal{C}}, \underline{\Omega}) \simeq \underline{\mathrm{L}}(\underline{\mathcal{C}}, \Omega^{2\sigma}\underline{\Omega})$$

Proof. Observe that we have a diagram of fibre sequences of genuine C_2–spectra

$$\begin{tikzcd}
\Omega^{2\sigma}\underline{\mathrm{L}}(\underline{\mathcal{C}}, \underline{\Qoppa}) \arrow[r] \arrow[d] & \Omega^{\sigma}\underline{\mathrm{L}}(\underline{\mathcal{C}}, \Omega^{\sigma}\underline{\Qoppa}) \arrow[r] \arrow[d] & \Omega^{\sigma}\Xi\mathrm{L}(\mathcal{C}_e, \Sigma\Qoppa^q_e) \arrow[d] \\
\underline{\mathrm{L}}(\underline{\mathcal{C}}, \Omega^{2\sigma}\underline{\Qoppa}) \arrow[r, equal] \arrow[d] & \underline{\mathrm{L}}(\underline{\mathcal{C}}, \Omega^{2\sigma}\underline{\Qoppa}) \arrow[r] \arrow[d] & 0 \arrow[d] \\
E \arrow[r] \arrow[d, equal] & \Xi\mathrm{L}(\mathcal{C}_e, \Sigma\Omega\Qoppa^q_e) \arrow[r] \arrow[d, equal] & \Sigma\Omega^{\sigma}\Xi\mathrm{L}(\mathcal{C}_e, \Sigma\Qoppa^q_e) \arrow[d, equal] \\
E \arrow[r] & \Xi\mathrm{L}(\mathcal{C}_e, \Qoppa^q_e) \arrow[r] & \Omega^{\sigma}\Xi\mathrm{L}(\mathcal{C}_e, \Sigma^2\Qoppa^q_e)
\end{tikzcd}$$

Now note that for genuine C_2–spectra of the form ΞX, ie. those whose underlying spectrum is trivial, we have that the canonical comparison $\Omega^{\sigma}\Xi X \to \Xi X$ induced by $S^0 \to S^{\sigma}$ is an equivalence since both have trivial underlying spectrum and have equivalent C_2-geometric fixed points since $\Phi^{C_2}(S^{\sigma}) \simeq S^0$. Hence the bottom sequence looks like

$$E \longrightarrow \Xi\mathrm{L}(\mathcal{C}_e, \Qoppa^q_e) \longrightarrow \Xi\mathrm{L}(\mathcal{C}_e, \Sigma^2\Qoppa^q_e) \tag{7.9}$$

and therefore by our hypothesis, $E \simeq 0$. □

7.4.2 Speculations: descent via multiplicative norms

The speculative materials in this short subsection was the original motivation for our study on genuine equivariant hermitian K–theory. The main idea is a simple combination of methods from [Gre93] and [CMN+20] to prove completion theorems for equivariant L–theory. The reason it is still speculative at the moment is that in spite of our protracted investigations above, we have not yet obtained the crucial ingredient that L–theory admits the Hill-Hopkins-Ravenel norms for odd groups. Needless to say, this is a direction of further work that we very much intend to return to in the future.

Construction 7.4.12. Suppose that $I := \ker\left(L^s_0(G,R) \to L^s_0(e,R)\right)$ is finitely generated ideal, and write this as $I = (a_1, \ldots, a_d)$. Then we follow Greenlees-May and define

$$K(a_i) := \mathrm{fib}\left(L^s(G,R) \to L^s(G,R)[a_i^{-1}]\right)$$

and

$$K(I) := \bigotimes_{i=1}^{r} K(a_i) \in \mathrm{Mod}_{\mathrm{Sp}^G}(L^s(G,R)) \longrightarrow L^s(G,R)$$

Here the tensor is over $L^s(G,R)$ since we are working in this module category. It turns out moreover that this $K(I)$ only depends on the radical of I. Now note that

$K(I) \to L^s(G,R)$ is an underlying equivalence since $a_i : L^s(G,R) \to L^s(G,R)$ are underlying nullhomotopic, and so we have equivalence

$$EG_+ \otimes K(I) \xrightarrow[\text{can}]{\simeq} EG_+ \otimes L^s(G,R)$$

and hence obtain the following diagram

$$\begin{tikzcd} c:\ EG_+ \otimes L^s(G,R) \arrow[r, "\text{can}^{-1}"] \arrow[dr] & EG_+ \otimes K(I) \arrow[r] & K(I) \arrow[dl] \\ & L^s(G,R) & \end{tikzcd}$$

Proposition 7.4.13 ([BHV18, Prop. 3.19])**.** *There are spectral sequences*

$$E_2^{h,k} = H_I^{-h}(L_*^s(G,R); M_*^G))_k \Longrightarrow \pi_{h+k}^G(M \otimes K(I)) \qquad d^r : E_{h,k}^r \to E_{h-r,k+r-1}^r$$

$$E_{h,k}^2 = (H_h^I(L_*^s(G,R); M_*^G))_k \Longrightarrow \pi_{h+k}\Lambda^I M \qquad d^r : E_{h,k}^r \to E_{h-r,k+r-1}^r$$

for any $M \in \mathrm{Mod}_{\mathrm{Sp}^G}(L^s(G,R))$*. Moreover,* H_I^* *is an* I*-torsion module.*

Lemma 7.4.14 (I-adic ambidexterity via finiteness)**.** *Let* $M \in \mathrm{Mod}_{\mathrm{Sp}^G}(R_G)$ *where both* M *and* R_G *have finite equivariant homotopy groups. Let* $I \leq \pi_0^G R_G$ *be an ideal. Then*

$$K(I) \otimes F(K(I), M) \to F(K(I), M)$$

is an equivalence. In other words, the I*-Tate construction* $L_I \Lambda^I M$ *is contractible.*

Proof. By [GM92, Thm. 1.9] we know that

$$H_i^I(\pi_t^G M)\& = \begin{cases} (\pi_t^G M)_I^\wedge & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

On the other hand, since $\pi_t^G M$ is finite we have that $(\pi_t^G M)_I^\wedge \cong (\pi_t^G M)/I^n$ for some $n \gg 0$. Hence in particular it is I-primary torsion. Now if $I = (a_1, \ldots, a_d)$ then by construction and hence since homotopy groups commute with filtered colimits we get

$$\pi_t^G(L_I \Lambda^I M) \cong (\pi_t^G M)_I^\wedge[(a_1 \cdots a_d)^{-1}] \cong ((\pi_t^G M)/I^n)[(a_1 \cdots a_d)^{-1}] = 0$$

as required. □

Lemma 7.4.15 (Greenlees' argument)**.** *Suppose* R_G *satisfies equivariant periodicity with respect to the reduced regular representation and let* $I = I_G = \ker(R_0^G \to R_0^e)$*. Then we have that* $\widetilde{E\mathcal{P}} \otimes K(I) \simeq 0$.

Proof. We know that $\widetilde{E\mathcal{P}} \simeq S^{\infty\overline{\rho}_G}$. Also, for any $M \in \mathrm{Mod}_{\mathrm{Sp}^G}(R_G)$ we have

$$\lambda : \pi_n^G M \xrightarrow{e_*} \pi_n^G(M \otimes S^{\overline{\rho}_G}) \xrightarrow[\beta]{\cong} \pi_n^G M$$

where $e : S^0 \to S^{\overline{\rho}_G}$ is the Euler map. And hence in the presence of the Bott isomorphism $\pi_n^G(\widetilde{E\mathcal{P}} \otimes M)$ is computed by inverting some class $\lambda \in \pi_0^G R_G$ which is concretely given as the composite

$$\lambda : R_G \xrightarrow{e} R_G \otimes S^{\overline{\rho}_G} \xrightarrow{\beta} R_G$$

Now note that, since the Euler map $e : S^0 \to S^{\overline{\rho}_G}$ is nullhomotopic nonequivariantly (ie. $\mathrm{Res}_e^G e \simeq 0$), we get that $\lambda \in I$. On the other hand, by the spectral sequence above and that localisation is exact we get that

$$E_2^{h,k} = H_I^{-h}(R_*^G; M_*^G))_k[\lambda^{-1}] \Longrightarrow \pi_{h+k}^G(\widetilde{E\mathcal{P}} \otimes K(I))$$

But then H_I^* is an I-torsion module and so everything vanishes and we are done. $\square$

We are now ready to state the main theorem on nilpotent descent in equivariant L-theory.

Theorem 7.4.16 (Nilpotent descent for equivariant L-theory)**.** *Let G be any finite group. Suppose that*

(i) $I := I_G = \ker\big(L_0^s(G,R) \to L_0^s(e,R)\big)$ is finitely generated
(ii) The map $K(I) \otimes F(K(I), L^s(G,R)) \to F(K(I), L^s(G,R))$ is an equivalence
(iii) $\mathrm{Res}_H^G I_G$ and $I_H \leq L_0^s(H,R)$ have the same radicals for all $H \leq G$.

For example, by I-adic ambidexterity Lemma 7.4.14 above, the first two conditions are satisfied when $L_n^s(G,R)$ are all finite for all G finite groups. Then we have that:

(i) $L^s(G,R)_I^\wedge = F\big(K(I), L^s(G,R)\big)$ is $\{e\}$-nilpotent, and thus so are all its modules in Sp^G.
(ii) For any $(\mathcal{C},\Omega) \in \mathrm{Fun}(BG, \mathrm{Cat}^p_{(R,\Omega^s)})$ the natural maps

$$\begin{aligned}\mathrm{coBor}\big(L(\mathrm{coBor}(\mathcal{C},\Omega))_I^\wedge\big) &\to L(\mathrm{coBor}(\mathcal{C},\Omega))_I^\wedge \\ &\to L(\mathrm{Bor}(\mathcal{C},\Omega))_I^\wedge \to \mathrm{Bor}\big(L(\mathrm{Bor}(\mathcal{C},\Omega))_I^\wedge\big)\end{aligned}$$

are all equivalences and hence in particular upon taking genuine G-fixed points we get that the sequence

$$L(\mathcal{C},\Omega)_{hG} \to (L((\mathcal{C},\Omega)_{hG})_I^\wedge \to L((\mathcal{C},\Omega)^{hG})_I^\wedge \to L(\mathcal{C},\Omega)^{hG}$$

are all equivalences.

Proof. By [MNN17, Thm. 6.41] we need to show that $\Phi^H \operatorname{Res}^G_H \left(L^s(G,R)^\wedge_I\right) \simeq 0$ for all $0 \neq H \leq G$. Now we know that $K(I)$ (and therefore I-completion) only depends on the radical of I and so by hypothesis (c) we get that $\operatorname{Res}^G_H \left(L^s(G,R)^\wedge_I\right) \simeq L^s(H,R)^\wedge_I$, and so we might as well just show it for $H = G$. In other words, we want to show that $\widetilde{E\mathcal{P}} \otimes F(K(I), L^s(G,R)) \simeq 0$. But by hypothesis (b) we have that this is equivalent to $\widetilde{E\mathcal{P}} \otimes K(I) \otimes F(K(I), L^s(G,R))$ and we know $\widetilde{E\mathcal{P}} \otimes K(I)$ vanishes by the Greenlees argument Lemma 7.4.15. The second part is just because everything in sight is $\{e\}$-nilpotent since $L^s(G,R)^\wedge_I$ is, and so in particular Borel-complete. But then on Borel-complete objects, equivalences can be checked on the underlying objects, for which everything is clear. □

Lemma 7.4.17 (Norm radical lemma)**.** *For abelian groups G, the hypothesis $\sqrt{\operatorname{Res}^G_H I_G} = \sqrt{I_H}$ is automatic.*

Proof. We always have $\sqrt{\operatorname{Res}^G_H I_G} \subseteq \sqrt{I_H}$. To see the opposite, let $a \in W(H,R)$ such that $a^n \in I_H$. Then using the Evens norm we get

$$\operatorname{Res}^G_H N^G_H(a^n) = \prod_{H\backslash G/H} N^H_{H^g \cap H} \operatorname{Res}^{H^g}_{H^g \cap H} g_*(a^n)$$

Since all our groups were abelian, there are no interesting conjugations and so the right hand side looks like $a^m := \prod_{H\backslash G/H} a^n$, and hence $a^m \in \operatorname{Res}^G_H I_G$ as required. □

We learnt of the following simple observation from [CP84, Lem. V.2.1] which was stated without proof. We thank Jesper Grodal for the key insight in the argument.

Lemma 7.4.18. *If G is a p-group and F is a finite field f characteristic p then the inclusion and forgetful map induces an isomorphism $W(G,F) \cong W(F)$.*

Proof. The key point is that elements in $W(G,F)$ are all semisimple things: to see this, let (M,q) be a form with G-isometric action. Suppose $(N,q) \leq (M,q)$ is a nontrivial proper G-invariant subspace. Then the complement is also G-invariant since if $g \in G$ and $w \in V^\perp$, then for all $v \in N$

$$q(v, gw) = q(g^{-1}v, w) = 0$$

Hence every form is isometric to a sum of simples. But then in this modular case, we know that the only simple modules are F with the trivial action. □

We now collect interesting examples one can glean from the theorem based on classical knowledge of Witt rings.

Corollary 7.4.19. *Let k be a finite field of characteristic p where p is an odd prime.*

(i) *If G is a p-group, then all the maps*

$$L(\mathcal{C},\Omega)_{hG} \to L((\mathcal{C},\Omega)_{hG}) \to L((\mathcal{C},\Omega)^{hG}) \to L(\mathcal{C},\Omega)^{hG}$$

are equivalences for all $(\mathcal{C},\Omega) \in \mathrm{Fun}(BG, \mathrm{Mod}_{\mathrm{Cat}^p}(\mathrm{Mod}_k^\omega, \Omega_k^s))$. *In particular, this will be true for any Galois action on k-algebras with Galois group G.*

(ii) *If G is an abelian group, then all the maps*

$$L(\mathcal{C},\Omega)_{hG} \to L((\mathcal{C},\Omega)_{hG})_I^\wedge \to L((\mathcal{C},\Omega)^{hG})_I^\wedge \to L(\mathcal{C},\Omega)^{hG}$$

are equivalences for all $(\mathcal{C},\Omega) \in \mathrm{Fun}(BG, \mathrm{Mod}_{\mathrm{Cat}^p}(\mathrm{Mod}_k^\omega, \Omega_k^s))$. *In particular, this will be true for any Galois action on k-algebras with Galois group G.*

Proof. We just need to see that in either case, the hypotheses of the theorem are satisfied. For (i), we know from Lemma 7.4.18 that the augmentation ideals are all zero, and so in particular we do not need the completions in the statement. Furthermore, by standard calculation, the Witt groups of finite fields are finite. Hence all the hypotheses are satisfied. As for (ii), use the norm radical Lemma 7.4.17 to get the hypothesis about augmentation ideals, and [ACH77] gives finiteness of all the Witt groups in sight. □

Concluding words

轻轻的我走了，正如我轻轻的来，
我轻轻的招手，作别西天的云彩。

徐志摩

There is a red thread running through a large swath of our work above, namely the distinguished role that the prime 2 plays. As a convenient point of reference, here are all the instances of this in this thesis:

(i) In proving that pointwise K–theory is equivalent to genuine equivariant K–theory when G is a 2–group (cf. Theorem 4.3.19), we used that the universal property of algebraic K–theory is articulated in terms of certain pushout squares, ie. diagrams indexed over $(\Delta^1)^{\times 2}$.

(ii) In showing that $(-)^{t\Sigma_2}$ of a G–bilinear functor is G–linear when G is an odd group (cf. Corollary 3.5.3), we have used a double coset counting argument which showed that when G is odd, the only self-inverse double coset is the trivial one. This observation was important to us for two reasons: (a) it allowed us to refine the Nikolaus-Scholze Σ_2–Tate diagonal to genuine G–spectra when G is odd, and we used this as an input to construct the universal G–Poincaré category in §7.1.7; (b) in the setting of genuine equivariant hermitian K–theory, we also used the Tate linearity above to ensure that the linear approximations to G–quadratic functors are automatically G–linear (cf. Example 7.1.4).

(iii) For $G = C_2$, we formulated a genuine equivariant periodicity for L–theory with respect to the sign loops Ω^σ in Theorem 7.4.11, where we exploited the well-known special property of the group C_2 in admitting a cofibre sequence of C_2–spaces $C_{2+} \to S^0 \to S^\sigma$.

Many natural questions suggest themselves from our investigations above, and we conclude this thesis by recording some of these. For one, we think that it is very desirable to have a more precise understanding of the difference between pointwise equivariant algebraic K–theory and genuine equivariant algebraic K–theory. In Corollary 4.3.20, we proved that these are the same when G is a 2–group, but we do not presently know if they should be equivalent for general groups. As explained above, the essential point is in whether or not algebraic K–theory satisfies descent against specific kinds of cubes. Therefore, as a first step, it would be beneficial to probe the descent of algebraic K–theory for $(\mathcal{C} \hookrightarrow \mathcal{C}^{\Delta^1})^{\otimes 3}$ for suitable $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$ with sufficiently computable $\mathrm{K}(\mathcal{C})$.

While we are less sure of what to expect in the genuine equivariant algebraic K–theory case above, we expect that *Borel equivariant* algebraic K–theory and Grothendieck–Witt theory should admit the norms for *all* finite groups G and not just for 2–groups. As in Chapter 5, Borel equivariance here pertains for example to G–spectral Mackey functors such as $\{\mathrm{K}(\mathcal{C}^{hH})\}_{H\leq G}$ for some $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$.

Besides that, we think that our work on genuine equivariant hermitian K–theory has only scratched the surface to a potentially deeper story. For instance, while the case of odd G worked smoothly, for that of even G we had to further impose the inversion of the prime 2 at the cost of precluding any interesting "Tate-gluing" data. We have recently learnt from Jay Shah about his joint work with J.D. Quigley on parametrised Tate constructions [SQ22] which seems to give a good candidate for the correct Tate constructions and this is something we would very much like to explore in future work. Our belief is that the interface between the parity of $|G|$ and the distinctly Σ_2–equivariant behaviour of Poincaré categories should only be a wrinkle, but not a tear, in the fabric of hermitian K–theory, as it were.

Furthermore, the manoeuvre of algebraic surgery was a technique of central import in [CDH+20c] as it allowed one to reduce many questions about L–theory to a calculation in classical algebra. We expect that elaborating this aspect of equivariant L–theory might be a wellspring of much interesting work. Indeed, one of our initial hopes was that one might be able to use algebraic surgery to prove new induction theorems for equivariant L–theory in the sense of [Dre75].

Finally, apart from finding concrete cases where the nilpotence descent approach in the speculative §7.4.2 might prove fruitful, another direction of work vis-à-vis L–theory could be in studying equivariant L–theory coming from Galois extensions of ring spectra. Since such Galois extensions were of fundamental importance in classical Witt theory, we expect that it could be just as interesting, if not more so, in the setting of higher algebra.

Bibliography

[ACH77] J.P. Alexander, P.E. Conner, and G.C. Hamrick. "Odd Order Group Actions and Witt Classification of Innerproducts". In: Lecture Notes in Mathematics 625 (1977).

[BDG+16a] C. Barwick, E. Dotto, S. Glasman, D. Nardin, and J. Shah. "Parametrized Higher Category Theory and Higher Algebra: A General Introduction". In: arXiv:1608.03654 (2016).

[BDG+16b] C. Barwick, E. Dotto, S. Glasman, D. Nardin, and J. Shah. "Parametrized Higher Category Theory and Higher Algebra: Exposé I – Elements of Parametrized Higher Category Theory". In: arXiv:1608.03657 (2016).

[BGN14] C. Barwick, S. Glasman, and D. Nardin. "Dualizing Cartesian and Cocartesian Fibrations". In: arXiv:1409.2165 (2014).

[BGS20] C. Barwick, S. Glasman, and J. Shah. "Spectral Mackey Functors and Equivariant Algebraic K-theory (II)". In: Tunis. J. Math. 2 (2020), pp. 97–146.

[BGT13] A. Blumberg, D. Gepner, and G. Tabuada. "A Universal Characterization of Higher Algebraic K-Theory". In: Geom. Top. 17.2 (2013), pp. 733–838.

[BGT14] A. Blumberg, D. Gepner, and G. Tabuada. "Uniqueness of the Multiplicative Cyclotomic Trace". In: Adv. in Math. 260 (2014), pp. 191–232.

[BH21] Tom Bachmann and Marc Hoyois. "Norms in Motivic Homotopy Theory". In: Astérisque 425 (2021), p. 207.

[BHV18] T. Barthel, D. Heard, and G. Valenzuela. "Local Duality in Algebra and Topology". In: Adv. in Math. 335 (2018), pp. 563–663.

[BMM+21] G. Bertrand, P. May, M. Merling, and A. Osorno. "Multiplicative Equivariant K-theory and the Barratt-Priddy-Quillen theorem". In: arXiv:2102.13246 (2021).

[CDH+] B. Calment, E. Dotto, Y. Harpaz, F. Hebestreit, M. Land, K. Moi, D. Nardin, T. Nikolaus, and W. Steimle. “Hermitian K-Theory for Stable ∞-Categories IV: Poincaré Motives and Karoubi-Grothendieck-Witt Groups”. *in preparation*.

[CDH+20a] B. Calment, E. Dotto, Y. Harpaz, F. Hebestreit, M. Land, K. Moi, D. Nardin, T. Nikolaus, and W. Steimle. “Hermitian K-Theory for Stable ∞-Categories I: Foundations”. In: (2020).

[CDH+20b] B. Calment, E. Dotto, Y. Harpaz, F. Hebestreit, M. Land, K. Moi, D. Nardin, T. Nikolaus, and W. Steimle. “Hermitian K-Theory for Stable ∞-Categories II: Cobordism Categories and Additivity”. In: (2020).

[CDH+20c] B. Calment, E. Dotto, Y. Harpaz, F. Hebestreit, M. Land, K. Moi, D. Nardin, T. Nikolaus, and W. Steimle. “Hermitian K-Theory for Stable ∞-Categories III: Grothendieck-Witt Groups of Rings”. In: (2020).

[CMN+20] D. Clausen, A. Mathew, N. Naumann, and J. Noel. “Descent and Vanishing in Chromatic Algebraic K-Theory via Group Actions”. In: arXiv:2011.08233 (2020).

[CP84] P.E. Conner and R. Perlis. “A Survey of Trace Forms of Algebraic Number Fields”. In: Series in Pure Mathematics Vol. 2 (1984).

[DM16] E. Dotto and K. Moi. “Homotopy Theory of G-diagrams and Equivariant Excision”. In: Alg. Geom. Top. 16 (2016), pp. 325–395.

[Dot17] E. Dotto. “Higher Equivariant Excision”. In: Adv. Math. 309 (2017), pp. 1–96.

[Dre75] A. Dress. “Induction and Structure Theorems for Orthogonal Representations of Finite Groups”. In: Ann. of Math. 102 (1975), pp. 291–325.

[EH19] E. Elmanto and R. Haugseng. “On distributivity in higher algebra I: The universal property of bispans”. In: arXiv:2010.15722 (2019).

[Fra41] J.S. Frame. “The Double Cosets of a Finite Group”. In: Bull. Amer. Math. Soc. 49 (1941), pp. 458–467.

[GGN15] D. Gepner, M. Groth, and T. Nikolaus. “Universality of Multiplicative Infinite Loop Space Machines”. In: Alg. Geom. Top. 15 (2015), pp. 3107–3153.

[GHN17] D. Gepner, R. Haugseng, and T. Nikolaus. “Lax Colimits and Free Fibrations in ∞-Categories”. In: Doc. Math. 22 (2017), pp. 1225–1266.

[Gla17] S. Glasman. “Stratified Categories, Geometric Fixed Points and a Generalized Arone-Ching Theorem”. In: arXiv:1507.01976 (2017).

[GM92] J.P.C. Greenlees and J.P. May. “Derived Functors of I-adic Completion and Local Homology”. In: J. of Algebra 149 (1992), pp. 438–453.

[GM97] J.P.C. Greenlees and J.P. May. “Localization and Completion Theorems for MU–module Spectra”. In: Ann. of Math. 146 (1997), pp. 509–544.

[Gre93] J.P.C. Greenlees. “K-homology of Universal Spaces and Local Cohomology of the Representation Ring”. In: Topology 32 (1993), pp. 295–308.

[HHR16] M. Hill, M. Hopkins, and D. Ravenel. “On the Non-Existence of Elements of Kervaire Invariant One”. In: Ann. of Math. 184 (2016), pp. 1–262.

[Hil22a] K. Hilman. “Parametrised Noncommutative Motives and Equivariant Algebraic K-theory”. In: arXiv:2202.02591 (2022).

[Hil22b] K. Hilman. “Parametrised Presentability over Orbital Categories”. In: arXiv:2202.02594 (2022).

[Hin16] V. Hinich. “Dwyer-Kan Localization Revisited”. In: Homology Homotopy Appl. 18.1 (2016), pp. 27–48.

[HW21] F. Hebestreit and F. Wagner. “Lecture Notes for Algebraic and Hermitian K-Theory”. In: author’s webpage (2021).

[Lei22] M. Leip. “On the Hochschild Homology of Hypersurfaces as a Mixed Complex”. In: PhD Thesis, University of Copenhagen (2022).

[Len21] T. Lenz. “G-Global Homotopy Theory and Algebraic K-Theory”. In: arXiv:2012.12676 (2021).

[LR12] S. Lunøe-Nielsen and J. Rognes. “The Topological Singer Construction”. In: Doc. Math. 17 (2012), pp. 861–909.

[Lur09] J. Lurie. “Higher Topos Theory”. In: Ann. of Math. Studies 170 (2009), pp. xviii+925.

[Lur11] J. Lurie. “Lecture Notes on Algebraic L–Theory and Surgery”. In: author’s webpage (2011).

[Lur17] J. Lurie. “Higher Algebra”. In: author’s webpage (2017).

[Mer17] M. Merling. “Equivariant Algebraic K-theory of G-rings”. In: Math. Zeit. 285 (2017), pp. 1205–1248.

[MNN17] A. Mathew, N. Naumann, and J. Noel. “Nilpotence and Descent in Equivariant Stable Homotopy Theory”. In: Adv. in Math. 305 (2017), pp. 994–1084.

[MP87] M. Makkai and A. M. Pitts. “Some Results on Locally Finitely Presentable Categories”. In: Trans. Amer. Math. Soc. 299.2 (1987), pp. 473–496.

[Nar17] D. Nardin. “Stability and Distributivity over Orbital ∞-categories”. In: PhD Thesis, MIT (2017).

[NS18] T. Nikolaus and P. Scholze. “On Topological Cyclic Homology”. In: Acta Math. 221.2 (2018), pp. 203–409.

[NS22] D. Nardin and J. Shah. “Parametrized and Equivariant Higher Algebra”. In: arXiv:2203.00072 (2022).

[QS22] J.D. Quigley and J. Shah. “On the Parametrised Tate Construction”. In: arXiv:2110.07707 (2022).

[RV19] E. Riehl and D. Verity. “Infinity Category Theory from Scratch”. In: arXiv:1608.05314 (2019).

[RV22] E. Riehl and D. Verity. “Elements of ∞-Category Theory”. In: Cambridge Studies in Advanced Mathematics 194 (2022), pp. xix+759.

[Sch19] S. Schwede. “Global Algebraic K-theory”. In: arXiv:1912.08872 (2019).

[Sch20] S. Schwede. “Lecture Notes on Equivariant Stable Homotopy Theory”. In: author’s webpage (2020).

[Sha22a] J. Shah. “Parametrized Higher Category Theory”. In: to appear in Alg. Geom. Top. (2022).

[Sha22b] J. Shah. “Parametrized Higher Category Theory II: Universal Constructions”. In: arXiv:2109.11954 (2022).

[SQ22] J. Shah and J.D. Quigley. “On the Parametrized Tate Construction”. In: 2110.07707 (2022).

[Wal85] F. Waldhausen. “Algebraic K-Theory of Spaces”. In: Lecture Notes in Math. 1126 (1985), pp. 318–419.

Index of symbols