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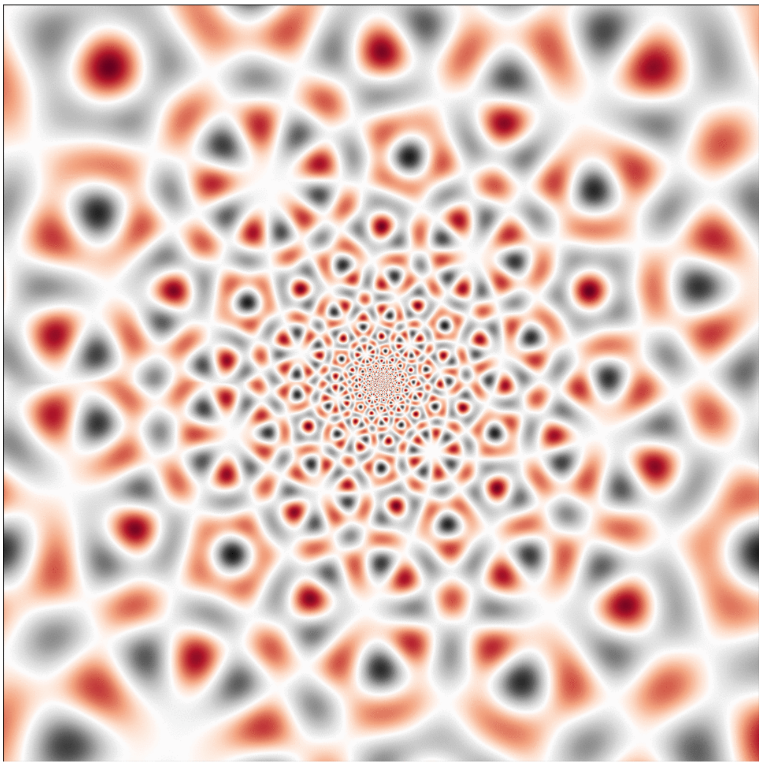

PhD thesis

Equivariant homotopy theory and higher category theory

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Abstract (English)

This thesis is divided into two parts. In the first part, we prove an equivariant generalization of the *recognition principle*: We show that for G a finite group and V a G-representation, an $\mathbb{E}_V$-algebra in G-spaces is equivalent to a V-fold loop space if and only if it is group-like. In the second part of this thesis, we show that the category of orthogonal factorization systems embeds fully faithfully into the category of double categories, a result which is used to construct an equivariant symmetric monoidal structure on the category of equivariant manifolds.

Abstract (Danish)

Denne afhandling er opdelt i to dele. I den første del beviser vi en ækvivariant generalisering af *genkendelsesprincippet*: Vi viser, at for en endelig gruppe G og en G-repræsentation V, er en $\mathbb{E}_V$-algebra i G-rum ækvivalent med et V-foldigt løkkerum, hvis og kun hvis den er grupplignende. I den anden del af afhandlingen viser vi, at kategorien af ortogonale faktorisationssystemer indlejres fuldt trofast i kategorien af dobbeltkategorier, et resultat som anvendes til at konstruere en ækvivariant symmetrisk monoidal struktur på kategorien af ækvivariante mangfoldigheder.

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I am deeply grateful to Natalie Stewart for her collaboration on our joint project, part of which is included in this thesis.

Zum Schluss will ich meiner Familie danken, Mama, Papa, Ronja, Ralf, Marion, Klaus und Edeltraud, für all die Unterstützung und Liebe in den letzten 26 Jahren.

Contents

Introduction

Equivariant homotopy theory studies the behavior of homotopy coherent structures under the additional action of a group. Such structures naturally occur in representation theory as well as in geometry and topology. In many geometric settings, it is important to organize the data in a way that preserves all the inherent coherences of the input. This thesis consists of two parts, which address two different instances of this:

- Loop spaces: given a based space X, the homotopy group $\pi_n(X)$ is defined as homotopy classes of maps from the sphere S^n to X. A more coherent way to encode this data is to remember the *loop space* $\Omega^n X = \mathrm{map}_*(S^n, X)$ of X. In fact, this loop space admits even more structure, namely the structure of an $\mathbb{E}_n$-algebra in spaces, remembering the ways to compose loops giving rise to the group structure on $\pi_n(X)$.

- The category of manifolds and its symmetric monoidal structure: the category of manifolds comes with a symmetric monoidal structure, given by disjoint union. In the equivariant world, this structure extends to a *normed* structure, also keeping track of inductions of manifolds along finite index subgroup inclusions and compatibility with the restriction functors.

In the first part of this thesis, we explore the equivariant generalization of loop spaces as $\mathbb{E}_n$-algebras.

The second part of the thesis is a purely categorical result. As an application, we discuss the construction of the aforementioned category of equivariant manifolds.

We now turn to a separate discussion of the two parts. We refer the reader to the respective chapters for a self-contained introduction to each part, also containing precise formulations of the theorems.

Part I: A genuine equivariant recognition principle for finite groups

We begin by recalling the non-equivariant case. Given a based space X, its n-fold loop space naturally admits the structure of an algebra over the *little disk operad* $\mathbb{E}_n$, which is defined using framed embeddings of n-dimensional disks. This algebra structure coherently encodes all the ways one can "compose" maps out of a sphere. On the set of path components $\pi_0(\Omega^n X) = \pi_n(X)$, this algebra structure recovers the group structure on the homotopy groups (for $n \geq 1$).

May's celebrated recognition principle provides a converse to this construction: An $\mathbb{E}_n$-algebra A in spaces is equivalent to an n-fold loop space $\Omega^n X$ if A is group-like, i.e. if the set of path components $\pi_0(A)$ is a group. If the space X is n-connective, we can even recover the space from its loop space $\Omega^n X$ as an $\mathbb{E}_n$-algebra.

In this article, we study the equivariant generalization of this question. Given a finite-dimensional G-representation V, we can define the V-fold loop space as the equivariant space of pointed maps between S^V, the one-point-compactification of V, and X. This space naturally carries the structure of an $\mathbb{E}_V$-algebra, where $\mathbb{E}_V$ is a G-operad defined using embeddings of equivariant disks framed in V.

An equivariant recognition principle asks for a list of sufficient conditions for an $\mathbb{E}_V$-algebra to be equivalent to a V-fold loop space. If $V \cong W \oplus \mathbb{R}$ contains a trivial summand, this question has been answered by May and Guillou [GM17], proving that an $\mathbb{E}_V$-algebra is a V-fold loop space if and only if all fixed points are group-like.

If V does not contain a trivial summand, we face a new difficulty: The above statement does not even make sense, as the fixed points A^G of an $\mathbb{E}_V$-algebra A do not carry a natural monoid structure.

We do however prove that the straightforward generalization of the recognition principle theorem holds: An $\mathbb{E}_V$-algebra A is a V-fold loop space if and only if A^H is group-like whenever $\dim V^H \geq 1$ (and only then it even makes sense to impose that condition).

Similarly to May and Guillou, we reduce this recognition principle to the *approximation theorem*, stating that the free group-like $\mathbb{E}_V$-algebra on a based G-space X is given by the V-fold loop space $\Omega^V \Sigma^V X$. The original non-equivariant result is due to Segal [Seg73] and the equivariant generalization in the case where V contains a trivial summand is due to Hauschild [Hau80] and Rourke and Sanderson [RS00].

Part II: On orthogonal factorization systems and double categories

We motivate the results in this part through an application: our goal is to define an equivariant version of the category of manifolds together with a normed structure, encoding compatibility between restriction, disjoint unions and inductions.

This construction will be used in forthcoming work with Natalie Stewart, generalizing equivariant factorization homology from finite groups, due to Horev [Hor19], to compact Lie groups. Such an extension requires as an input a category of equivariant manifolds with a normed structure.

Classically, a symmetric monoidal functor is encoded by a product-preserving functor

$$\mathrm{Span(Fin)} \longrightarrow \mathrm{Cat}$$

from the category of spans in finite sets to the category of categories. The forward maps are the ones encoding the multiplicative structure.

In the equivariant setting, we have to replace finite sets by disjoint unions of transitive G-spaces. In particular, in the case of compact Lie groups, this will no longer be a 1-category, but the mapping spaces come from the topological spaces of equivariant maps.

The symmetric monoidal category of manifolds is usually defined as the homotopy coherent nerve of a symmetric monoidal topologically enriched category. This seems to be hard to realize in the equivariant world: The analog of a symmetric monoidal category also incorporates *norm functors* and we do not know of a construction defining these norms as additional structure on a topologically enriched category.

One could instead try to write down a cocartesian fibration over the span category and use the straightening/unstraightening equivalence. There is also a problem with this strategy, as the span category does not admit a very natural model as a topologically enriched category.

In order to deal with this difficulty, we instead would like to work with *double categories*, categories in which there are two different types of morphisms, *horizontal* and *vertical* ones. These two different types of morphisms cannot be composed with each other, but there are *squares* witnessing compatibilities.

In the last part of this thesis, we show that *orthogonal factorization systems* embed fully faithfully into double categories. Here, an orthogonal factorization system is additional structure on a category, equipping it with two classes of morphisms, so that any morphism in the category can be factored uniquely.

We also identify certain fibrations of orthogonal factorization systems with fibrations of double categories. Using this equivalence, we construct the category of manifolds as follows:

- We write down a fibration of double categories. This can be implemented by taking homotopy coherent nerves of topologically enriched categories as the two directions are still "separated".
- We then show that this fibration of double categories comes from a fibration of orthogonal factorization systems. This enables us to pass from the world of double categories to ordinary categories. Ultimately, this will lead to the fibration which straightens to the category of manifolds.

Publication list

This thesis consists of two parts.

- An earlier version of the first part of this thesis has already appeared on the preprint server arXiv [Jur25a].
- An earlier version of the second part has already appeared on the preprint server arXiv [Jur25b]. The version in this thesis contains an appendix with an application to equivariant homotopy theory which is part of joint work with Natalie Stewart.

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A genuine equivariant recognition principle for finite groups

Branko Juran

For G a finite group and V a finite dimensional real G-representation, there is a G-operad $\mathbb{E}_V$ defined using embeddings of V-framed G-disks such that for any based G-space X, there is a naturally defined $\mathbb{E}_V$-algebra structure on the V-fold space $\Omega^V X$.

Given an $\mathbb{E}_V$-algebra in G-spaces and a subgroup H of G, the fixed points A^H carry the structure of an $\mathbb{E}_{\dim V^H}$-algebra in spaces. We prove that an $\mathbb{E}_V$-algebra is equivalent to a V-fold loop space if and only if A^H is group-like for all H such that $\dim V^H \geq 1$. This generalizes a result by Guillou and May by removing the assumption that V contains a trivial summand. They observed that the equivariant recognition principle follows from an equivariant version of the approximation theorem, stating that $\Omega^V \Sigma^V X$ is the free group-like $\mathbb{E}_V$-algebra on a based G-space X. This has been proven by Hauschild in the case that V contains a trivial summand and by Rourke and Sanderson in the case that X is G-connected. Our proof proceeds by showing that the equivariant approximation theorem holds for all G-representations V and all based G-spaces X.

Contents

1 Introduction

When May [May72] introduced the notion of an operad, one of the main motivations was to encode the multiplicative structure on the n-fold loop space $\Omega^n X$ of a based space X which gives rise to the multiplicative structure on the homotopy group $\pi_n(X)$ of that space. This n-fold loop space $\Omega^n X$ of a space X admits a homotopy-coherent multiplicative structure, making it an *$\mathbb{E}_n$-algebra in spaces*. Those $\mathbb{E}_n$-algebras later started appearing in other contexts, including higher algebra and the study of configuration spaces of manifolds.

One of the very first and fundamental results in the theory of those operads is May's recognition principle, classifying which of the $\mathbb{E}_n$-algebras come from n-fold loop spaces:

Theorem (May)**.** *An $\mathbb{E}_n$-algebra in spaces A is equivalent to an n-fold loop space if and only if $\pi_0(A)$ is a group.*

This was proven by May [May72] in the case that $\pi_0(A)$ is trivial (and in a different framework by Boardman and Vogt [BV73]) and Segal [Seg73] provided the missing input, the approximation theorem, to deduce the general case, as explained in [CLM76, pp. 487, footnote 21]. The hard part is the "only if"-direction, showing that it suffices that $\pi_0(A)$ is a group in order to construct a space $B^n A$ such that $A \cong \Omega^n B^n A$. The more detailed version of the above theorem does that explicitly by constructing the delooping $B^n A$ of an arbitrary $\mathbb{E}_n$-algebra A and then shows that there is a natural map $A \to \Omega^n B^n A$ which is an equivalence if and only if $\pi_0(A)$ is a group. In general this map is the so-called *group completion*, the initial map into an $\mathbb{E}_n$-algebra for which π_0 is a group.

The goal of this paper is to prove a genuine equivariant version of the above theorem, generalizing previous conditional results by May and Guillou. We will study *genuine G-spaces*, objects represented by topological spaces with an action of a finite group G. The role of the $\mathbb{E}_n$-operad is taken over by the G-operads $\mathbb{E}_V$ where V can be any n-dimensional real G-representation. Apart from applications to equivariant loop space theory, these G-operads have been used to study equivariant factorization homology [Hor19] [Zou23] [HKZ24], equivariant and real versions of topological Hochschild homology [Hor19] [Dot+21] and equivariant versions of the Hopkins-Mahowald theorem [HW20] [Lev22].

Given a G-representation V and a based G-space X, we can consider the V-fold loop space $\Omega^V X$, that is based maps from the one-point-compactification S^V of V into X. This V-fold loop space carries the structure of an $\mathbb{E}_V$-algebra. A genuine equivariant recognition principle must find a list of necessary conditions on an $\mathbb{E}_V$-algebra to be equivalent to a V-fold loop space. For a subgroup H of G, the fixed points A^H of an $\mathbb{E}_V$-algebra A carry the structure of an $\mathbb{E}_{\dim V^H}$-algebra in spaces. In particular, we only obtain a monoid structure on $\pi_0(A^H)$ if $\dim V^H \geq 1$. in the case that $A = \Omega^V X$ is a V-fold loop space, this monoid is a group. The goal of this paper is to show that this necessary condition on A to be equivalent to a V-fold loop space is also sufficient.

Before we can state the main theorem, we need the following two definitions:

Definition. Let A be an $\mathbb{E}_V$-algebra in G-spaces. We say that A is *group-like* if $\pi_0(A^H)$

is a group for all H such that $\dim V^H \geq 1$. We denote the full subcategory of group-like $\mathbb{E}_V$-algebras by $\mathrm{Alg}^{\mathrm{grp}}_{\mathbb{E}_V}(\underline{\mathcal{S}}) \subset \mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathcal{S}})$.

Given an $\mathbb{E}_V$-algebra A, we say that a map $A \to A^{\mathrm{grp}}$ to another $\mathbb{E}_V$-algebra A^{grp} exhibits A^{grp} as the group completion of A if it is an initial map to a group-like $\mathbb{E}_V$-algebra.

Definition. For V a real G-representation, a based G-space X is called V-connective if its H-fixed points X^H are $(\dim V^H - 1)$-connected for every subgroup H of G.

We will now state the genuine equivariant recognition principle in the form which also gives an explicit description of the group completion functor for non-group-like $\mathbb{E}_V$-algebras.

Theorem A (Recognition principle)**.** *There is an adjunction*

$$\Omega^V : \mathcal{S}^G_* \leftrightarrows \mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathcal{S}}) : B^V$$

between the category of based G-spaces and the category of $\mathbb{E}_V$-algebras in G-spaces such that

- *For A an $\mathbb{E}_V$-algebra in G-spaces, the unit of the adjunction*

  $$A \longrightarrow \Omega^V B^V A$$

  is an equivalence if and only if A is group-like. In general, it exhibits the target as the group completion of the source.

- *For X a based G-spaces, the counit of the adjunction*

  $$B^V \Omega^V X \longrightarrow X$$

  is an equivalence if and only if X is V-connective. In general, it exhibits the source as the V-connective cover of the target.

In particular, the above adjunction restricts to an equivalence of categories

$$\left(\mathcal{S}^G_*\right)_{\geq V} \cong \mathrm{Alg}^{\mathrm{grp}}_{\mathbb{E}_V}(\underline{\mathcal{S}})$$

between V-connective based G-spaces and group-like $\mathbb{E}_V$-algebras.

It is an easy consequence of this theorem that the free group-like $\mathbb{E}_V$-algebra on a based G-space X must be given by the V-fold loop space $\Omega^V \Sigma^V X$. Following May's strategy of using the monadicity theorem, it is actually possible to deduce the recognition principle from this special case. We therefore first prove an equivariant version of the so-called approximation theorem:

Theorem B (Approximation theorem)**.** *For X a based G-space, the natural map*

$$\mathrm{Free}^{\mathbb{E}_V} X \longrightarrow \Omega^V \Sigma^V X$$

from the free $\mathbb{E}_V$-algebra on X to the V-fold loop space of the V-fold suspension of X exhibits the target as the group completion of the source.

The non-equivariant version of the approximation theorem is the aforementioned result due to Segal [Seg73]. The equivariant version is known in the following two cases: Hauschild [Hau80] proved it in the case that V contains a trivial summand, in which the group completion can be computed by applying $\Omega B(-)$ on all fixed points. In the cited paper, he only provides details for the case $X = S^0$ but remarks that the general case works similarly. Rourke and Sanderson [RS00] gave another proof of the same special case and also proved the result for arbitrary V, provided that X is G-connected. In the latter case, the left hand side already is group-like and the map appearing in Theorem B is an equivalence. This was used by Guillou and May [GM17] to deduce Theorem A in the same cases, i.e. if V contains a trivial summand or that the unit $A \to \Omega^V B^V A$ is an equivalence if the $\mathbb{E}_V$-algebra A is connected. A version of the recognition principle for $G = C_2$ and V being the sign representation appeared in Stiennon's thesis [Sti13] and later in work by Moi [Moi20]. Both of them prove that a group-like simplicial monoid with anti-involution models an equivariant loop space.

Proof strategy

We will now elaborate on the proof strategy of the equivariant approximation theorem (Theorem B). The recognition principle (Theorem A) is a rather formal consequence of this result using the monadicity theorem.

As mentioned earlier, the main difficulty is that for the subgroups H of G for which $V^H = \{0\}$, the fixed points A^H of an $\mathbb{E}_V$-algebra A do not generally admit the structure of an $\mathbb{E}_1$-algebra. In contrast to the case where V contains a trivial summand, this means that the $\mathbb{E}_V$-group completion functor is not simply given by group completion on all fixed points. However, we use that those H-fixed points are acted on by the fixed points of the equivariant factorization homology $\left(\int_{V\setminus\{0\}} A\right)^H$.

In order to study the behavior of the group completion on those two types of fixed points separately, we define a G-operad $\mathbb{E}_V^{\mathrm{i,e}}$ which lies in between $\mathbb{E}_0$ and $\mathbb{E}_V$. For subgroups H such that $\dim V^H \geq 1$, the collection of H-fixed points of an $\mathbb{E}_V^{\mathrm{i,e}}$-algebra have the same structure as for an $\mathbb{E}_V$-algebra. However, for the H-fixed points for H such that $\dim V^H = 0$, they really only admit the structure of an $\mathbb{E}_0$-algebra in spaces without an action of the equivariant factorization homology.

Using this, we can split up the computation of the free group-like $\mathbb{E}_V$-algebra into three steps: We first compute the free $\mathbb{E}_V^{\mathrm{i,e}}$-algebra on X, then its group completion as an $\mathbb{E}_V^{\mathrm{i,e}}$-algebra and finally the free $\mathbb{E}_V$-algebra on that group completed $\mathbb{E}_V^{\mathrm{i,e}}$-algebra.

For the first step, we use the explicit formula for a free algebra for an operad to see that the H-fixed points with $\dim V^H \geq 1$ of the free $\mathbb{E}_V^{\mathrm{i,e}}$-algebra on a based G-space X are given by a certain equivariant configuration space with labels in X. Moreover, the H-fixed points for H with $\dim V^H = 0$ are just given by X^H.

In the second step, we use that the condition $\dim V^H \geq 1$ is equivalent to H-representation $\mathrm{res}^G_H V$ containing a trivial summand. We might therefore use the approximation theorem of Hauschild [Hau80] to deduce that the H-fixed points of the $\mathbb{E}_V^{\mathrm{i,e}}$-group completion of the free $\mathbb{E}_V^{\mathrm{i,e}}$-algebra on X are equivalent to $\left(\Omega^V \Sigma^V X\right)^H$. Moreover, the group completion does not change the other H-fixed points at all.

In the third and final step, we have to compute the free $\mathbb{E}_V$-algebra on that group completed $\mathbb{E}_V^{\mathrm{i,e}}$-algebra. This time, this will not change any H-fixed points for which $\dim V^H \geq 1$. Using that this holds for all H appearing as isotropy groups of the G-manifold $V \setminus \{0\}$ and our previous computation, we will see that the H-fixed points for $\dim V^H = 0$ are equipped with a free action of the equivariant factorization homology $\left(\int_{V\setminus\{0\}} \Omega^V \Sigma^V X\right)^H$. Finally, we will use equivariant nonabelian Poincaré duality, due to Horev, Klang and Zou, [HKZ24] to compute this factorization homology.

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2 Preliminaries

Throughout this paper, we fix a finite group G and an n-dimensional real G-representation V.

2.1 G-symmetric monoidal G-categories and G-operads

We use the theory of ∞-categories as developed by Lurie in [Lur09] and [Lur17]. We use the term *category* to refer to an ∞-category. The category of spaces (or animae, homotopy types, ...) is denoted by $\mathcal{S}$. Moreover, we use the theory of parameterized homotopy theory developed by Barwick, Glasman, Dotto, Nardin and Shah [Bar+16] and in particular parameterized operads as developed by Nardin and Shah in [Sha23] and [NS22]. In particular, we assume the reader to be familiar with G-symmetric monoidal G-categories and G-operads.

We will mostly omit the G from the notation and underline categories to indicate that they are parameterized, e.g. the G-category of (genuine) G-spaces is denoted by $\underline{\mathcal{S}}$, its based version by $\underline{\mathcal{S}}_*$. Their fixed point categories $\underline{\mathcal{S}}^G = \mathcal{S}^G$ and $\underline{\mathcal{S}}_*^G = \mathcal{S}_*^G$ then recover the categories of (genuine) G-spaces and based (genuine) G-spaces, respectively.

For $\underline{\mathcal{C}}$ a G-symmetric monoidal G-category and $K \subset H$ nested subgroups of G, we use the notation $\mathrm{Nm}_K^H \colon \underline{\mathcal{C}}^K \longrightarrow \underline{\mathcal{C}}^H$ for its norm functor.

We denote the category of G-operads by Op^G and the category of G-symmetric monoidal G-categories and G-symmetric monoidal functors by $\mathrm{Cat}^{G-\otimes}$.

The inclusion $\mathrm{Cat}^{G-\otimes} \to \mathrm{Op}^G$ admits a left adjoint, the G-envelope functor

$$\underline{\mathrm{Env}} \colon \mathrm{Op}^G \longrightarrow \mathrm{Cat}^{G-\otimes}.$$

Using that the G-envelope of the terminal G-operad is given by the G-symmetric monoidal G-category of finite G-sets $\underline{\mathrm{Fin}}$ with its cocartesian G-symmetric monoidal structure, this functor factors through the slice category $\mathrm{Cat}^{G-\otimes}{}_{/\underline{\mathrm{Fin}}}$. We will make use of the following theorem:

Theorem 2.1 (Barkan, Haugseng, Steinebrunner [BHS24, Cor. 5.2.15, Lem. 5.2.16])**.** *The G-envelope functor induces a fully faithful functor*

$$\mathrm{Op}^G \longrightarrow \mathrm{Cat}^{G-\otimes}{}_{/\underline{\mathrm{Fin}}}$$

from the category of G-operads to the slice category of G-symmetric monoidal G-categories and G-symmetric monoidal functors over the G-symmetric monoidal G-category of finite G-sets $\underline{\mathrm{Fin}}$. The essential image is spanned by the G-symmetric monoidal functors $F\colon \underline{\mathcal{C}} \to \underline{\mathrm{Fin}}$ such that the squares

$$\begin{array}{ccc} \underline{\mathcal{C}}^H \times \underline{\mathcal{C}}^H & \xrightarrow{\otimes} & \underline{\mathcal{C}}^H \\ \downarrow{\scriptstyle F^H \times F^H} & & \downarrow{\scriptstyle F^H} \\ \underline{\mathrm{Fin}}^H \times \underline{\mathrm{Fin}}^H & \xrightarrow{\sqcup} & \underline{\mathrm{Fin}}^H \end{array} \quad \textit{and} \quad \begin{array}{ccc} \underline{\mathcal{C}}^K & \xrightarrow{\mathrm{Nm}_K^H} & \underline{\mathcal{C}}^H \\ \downarrow{\scriptstyle F^K} & & \downarrow{\scriptstyle F^H} \\ \underline{\mathrm{Fin}}^K & \xrightarrow{H \times_K (-)} & \underline{\mathrm{Fin}}^H \end{array}$$

are pullbacks for all nested subgroups $K \subset H$ of G.

Remark 2.2. The above theorem in particular implies the following: Given a G-operad $\underline{\mathcal{O}}$, for a (not necessarily full) G-symmetric monoidal G-subcategory $\underline{\mathcal{C}}$ of a $\underline{\mathrm{Env}}(\underline{\mathcal{O}})$, the functor $\underline{\mathcal{C}} \subset \underline{\mathrm{Env}}(\underline{\mathcal{O}}) \to \underline{\mathrm{Fin}}^G$ is contained in the essential image of the G-envelope functor from Theorem 2.1 if the following holds true for all nested subgroups $K \subset H$:

- For all $f\colon c \to c'$ and $g\colon d \to d'$ in $\underline{\mathrm{Env}}(\underline{\mathcal{O}})^H$: If $f \otimes g\colon c \otimes d \to c' \otimes d'$ is contained in $\underline{\mathcal{C}}^H$, then so are f and g (and in particular their source and target).
- For all $f\colon c \to c'$ in $\underline{\mathrm{Env}}(\underline{\mathcal{O}})^K$: If $\mathrm{Nm}_K^H(f)\colon \mathrm{Nm}_K^H(c) \to \mathrm{Nm}_K^H(c')$ is in $\underline{\mathcal{C}}^H$, then f is in $\underline{\mathcal{C}}^K$ (and in particular its source and target)

Using the above theorem and in particular the remark, we will not work with G-operads directly but rather replace them with their G-envelope.

Given a G-symmetric monoidal G-category $\underline{\mathcal{C}}$ and a G-operad $\underline{\mathcal{O}}$, we write

$$\mathrm{Alg}_{\underline{\mathcal{O}}}(\underline{\mathcal{C}}) = \mathrm{Fun}_{\mathrm{Op}^G}(\underline{\mathcal{O}}, \underline{\mathcal{C}}) \cong \mathrm{Fun}^{G-\otimes}(\underline{\mathrm{Env}}(\underline{\mathcal{O}}), \underline{\mathcal{C}})$$

for the category of $\underline{\mathcal{O}}$-algebras in $\underline{\mathcal{C}}$. This category is the fixed point category of a G-category of algebras.

The following result gives an explicit formula for computing free algebras:

Proposition 2.3. *Let $f\colon \underline{\mathcal{O}} \to \underline{\mathcal{P}}$ be a map of G-operads. Then precomposition*

$$f^*\colon \mathrm{Alg}_{\underline{\mathcal{P}}}(\underline{\mathcal{S}}) \longrightarrow \mathrm{Alg}_{\underline{\mathcal{O}}}(\underline{\mathcal{S}})$$

admits a left adjoint, which can be computed as a left Kan extension of the induced maps out of the G-envelope, i.e. the Beck Chevalley transform of the following diagram is invertible, making the diagram commute:

$$\begin{array}{ccc} \mathrm{Alg}_{\underline{\mathcal{O}}}(\underline{\mathcal{S}}) & \xrightarrow{f_!} & \mathrm{Alg}_{\underline{\mathcal{P}}}(\underline{\mathcal{S}}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\underline{\mathrm{Env}}(\mathcal{O}), \underline{\mathcal{S}}) & \xrightarrow{\mathrm{Env}(f)_!} & \mathrm{Fun}(\underline{\mathrm{Env}}(\mathcal{P}), \underline{\mathcal{S}}) \end{array}$$

where the upper arrow is the so-called operadic left Kan extension *along f, the bottom arrow is left Kan extension along the underlying functor of $\underline{\mathrm{Env}}(f)$ and the vertical arrows are forgetful functors.*

Proof. This is a special case of [LLP25, Lem. 3.45], using that the G-category of G-spaces is distributive. □

Given a G-operad $\underline{\mathcal{O}}$, write $\underline{\mathrm{col}}(\underline{\mathcal{O}})$ for its G-category of colors. Finally, we recall the following proposition. We write

$$\mathrm{fgt}^{\underline{\mathcal{O}}} \colon \mathrm{Alg}_{\underline{\mathcal{O}}}(\underline{\mathcal{S}}) \longrightarrow \mathrm{Fun}(\underline{\mathrm{col}}(\underline{\mathcal{O}}), \underline{\mathcal{S}})$$

for the forgetful functor.

Proposition 2.4 ([NS22, Thm. 5.1.4(2), Cor. 5.1.5])**.** *For $\underline{\mathcal{O}}$ a G-operad, the forgetful functor*

$$\mathrm{fgt}^{\underline{\mathcal{O}}} \colon \mathrm{Alg}_{\underline{\mathcal{O}}}(\underline{\mathcal{S}}) \longrightarrow \mathrm{Fun}(\underline{\mathrm{col}}(\underline{\mathcal{O}}), \underline{\mathcal{S}})$$

is a conservative right adjoint preserving geometric realizations. In particular, it is monadic.

2.2 The equivariant little disk operad $\mathbb{E}_V$, V-fold loop spaces and equivariant nonabelian Poincaré duality

We will now turn towards more geometric constructions. We will recall the construction of the equivariant little disk operad $\mathbb{E}_V$ and how to realize loop spaces as algebras over those G-operads. We will then recall the statement of nonabelian Poincaré duality.

The G-symmetric monoidal G-category of n-dimensional G-manifolds from [Hor19] is denoted by $\underline{\mathrm{Man}}$. Its V-framed version is denoted by $\underline{\mathrm{Man}}_V$. The G-symmetric monoidal subcategory of finite disjoint unions of V-framed G-disks will be denoted by $\underline{\mathrm{Disk}}_V$. The G-symmetric monoidal functor $\underline{\mathrm{Disk}}_V \to \underline{\mathrm{Fin}}$ sending a disk to its G-set of equivariant path components is contained in the essential image of the sliced G-envelope functor from Theorem 2.1, so that $\underline{\mathrm{Disk}}_V$ is the G-envelope of a G-operad $\mathbb{E}_V$, see [Hor19, Prop. 3.7.4, Prop. 3.9.8]. Given another G-symmetric monoidal G-category $\underline{\mathcal{C}}$, we might therefore write

$$\mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathcal{C}}) = \mathrm{Fun}^{G-\otimes}(\underline{\mathrm{Disk}}_V, \underline{\mathcal{C}})$$

for the category of $\mathbb{E}_V$-algebras in $\underline{\mathcal{C}}$. Let us start with some general observations about these little disk operads, starting with a non-equivariant description of $\mathbb{E}_V$-algebras for V being the trivial representation. For $n \geq 0$, let triv^n denote the n-dimensional trivial representation. Let Disk_n denote the non-equivariant category of n-dimensional framed disks. Then there is a functor $\mathrm{Disk}_n \to (\underline{\mathrm{Disk}}_{\mathrm{triv}^n})^G$ into the category of G-disks framed in triv^n, equipping a disk with trivial G-action.

Lemma 2.5 ([Hor19, Lem. 7.2.1])**.** *Let $\underline{\mathcal{C}}$ be a G-symmetric monoidal G-category. The functor*

$$\begin{aligned} &\mathrm{Alg}_{\mathbb{E}_{\mathrm{triv}^n}}(\underline{\mathcal{C}}) \\ &\cong \mathrm{Fun}^{G-\otimes}(\underline{\mathrm{Disk}}_{\mathrm{triv}^n}, \underline{\mathcal{C}}) \\ &\xrightarrow{(-)^G} \mathrm{Fun}^{\otimes}\Big((\underline{\mathrm{Disk}}_{\mathrm{triv}^n})^G, \underline{\mathcal{C}}^G\Big) \\ &\longrightarrow \mathrm{Fun}^{\otimes}\big(\mathrm{Disk}_n, \underline{\mathcal{C}}^G\big) \cong \mathrm{Alg}_{\mathbb{E}_n}\big(\underline{\mathcal{C}}^G\big) \end{aligned}$$

is an equivalence.

We will implicitly use this proposition to identify based G-spaces with $\mathbb{E}_0$-algebras in G-spaces.

More generally, there is a functor $\mathrm{Disk}_{\dim V^H} \to \underline{\mathrm{Disk}}_V^H$ equipping a $\dim V^H$-dimensional non-equivariant disk with the trivial H-action and then taking products with the orthogonal complement of V^H in V. This induces a functor

$$(-)^H \colon \mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathcal{C}}) \longrightarrow \mathrm{Alg}_{\mathbb{E}_{V^H}}\big(\underline{\mathcal{C}}^H\big)$$

where we write $\mathbb{E}_{V^H}$ for the non-equivariant operad $\mathbb{E}_{\dim V^H}$ to emphasize the functoriality.

The little disk operads are functorial in representations, so that the inclusion $0 \to V$ induces a forgetful functor

$$\mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathcal{S}}) \longrightarrow \mathrm{Alg}_{\mathbb{E}_0}(\underline{\mathcal{S}}) \cong \mathcal{S}_*^G\,.$$

This functor can also be described by evaluating the strong symmetric monoidal functor $\underline{\mathrm{Disk}}_V \to \underline{\mathcal{S}}$ at $V \in \underline{\mathrm{Disk}}_V^G$, giving a G-space which admits a base point coming from the unique map from the empty disk into V. We denote the left adjoint of this functor by

$$\mathrm{Free}^{\mathbb{E}_V} : \mathcal{S}_*^G \longrightarrow \mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathcal{S}})\,,$$

omitting the $\mathbb{E}_0$ from the notation.

We will now discuss group-like algebras. Recall that for $n \geq 1$ and A an $\mathbb{E}_n$-algebra in spaces, $\pi_0(A)$ naturally admits the structure of a monoid in sets (commutative if $n \geq 2$) and that A is called *group-like* if that monoid is a group. Let us recall the following definition from the introduction:

Definition 2.6. Let A be an $\mathbb{E}_V$-algebra in $\underline{\mathcal{S}}$. We say that A is *group-like* if $\pi_0(A^H)$ is a group for all H such that $\dim V^H \geq 1$. We denote the full subcategory of group-like $\mathbb{E}_V$-algebras by $\mathrm{Alg}_{\mathbb{E}_V}^{\mathrm{grp}}(\underline{\mathcal{S}}) \subset \mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathcal{S}})$.

Finally, we note that the existence of *some* group completion functor for $\mathbb{E}_V$-algebras can be proven formally:

Proposition 2.7. *The inclusion*

$$\mathrm{Alg}^{\mathrm{grp}}_{\mathbb{E}_V}(\underline{\mathcal{S}}) \longrightarrow \mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathcal{S}})$$

admits a left adjoint.

Proof. The subcategory of group-like functors can be written as the class of local objects with respect to the set of morphisms corepresenting the shearing maps on H-fixed points, so the claim follows from [Lur09, Prop. 5.5.4.15]. □

Notation 2.8. We denote the left adjoint to the inclusion described in Proposition 2.7 by

$$\mathrm{GrpCompl}^{\mathbb{E}_V} \colon \mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathcal{S}}) \longrightarrow \mathrm{Alg}^{\mathrm{grp}}_{\mathbb{E}_V}(\underline{\mathcal{S}})$$

and call it the *group completion.*

The problem hence is not to show the existence of this functor, but to explicitly compute it. Next, we will explain how to construct an $\mathbb{E}_V$-algebra structure on a V-fold loop space.

Construction 2.9. The following construction is discussed in detail in [HKZ24, Sec. 2.3], where the reader might find more details. There is a G-symmetric monoidal functor

$$(-)^+ \colon (\underline{\mathrm{Man}})^{\sqcup} \longrightarrow ((\underline{\mathcal{S}}_*)^{\mathrm{op}})^{\vee}$$

where the target is equipped with the cartesian monoidal structure in the opposite G-category of G-spaces, i.e. the wedge product/induction. It sends a manifold to the homotopy type of its one point compactification and a map to its induced collapse map.

Construction 2.10. We will now recall the definition of the V-fold loop space functor from [HKZ24, Def. 6.2.1]. It is defined to be the composite

$$\Omega^V \colon \mathcal{S}^G_* \longrightarrow \mathrm{Fun}^{G-\otimes}\big(((\underline{\mathcal{S}}_*)^{\vee})^{\mathrm{op}}, (\underline{\mathcal{S}})^{\times}\big) \longrightarrow \mathrm{Fun}^{G-\otimes}\big(\underline{\mathrm{Disk}}_V, (\underline{\mathcal{S}})^{\times}\big)\,.$$

Here, the first functor is the Yoneda embedding. As any representable functor preserves G-limits, it indeed naturally factors through the category of G-symmetric monoidal functor from $(\underline{\mathcal{S}}_*)^{\mathrm{op}}$ and $\underline{\mathcal{S}}$ equipped with their respective cartesian symmetric monoidal structure from [Sha23, Prop. 5.12]. The second functor is the restriction along the G-symmetric forgetful functor $\underline{\mathrm{Disk}}_V \hookrightarrow \underline{\mathrm{Man}}_V \to \underline{\mathrm{Man}}$ and the functor $\underline{\mathrm{Man}} \to ((\underline{\mathcal{S}}_*)^{\vee})^{\mathrm{op}}$ from Construction 2.9.

Finally, we discuss equivariant factorization homology. We restrict our attention to factorization homology with values in the G-category of G-spaces.

Definition 2.11. We define the equivariant factorization homology as the left Kan extension along the inclusion $\underline{\mathrm{Disk}}_V \subset \underline{\mathrm{Man}}_V$:

$$\int : \mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathcal{S}}) \cong \mathrm{Fun}^{G-\otimes}(\underline{\mathrm{Disk}}_V, \underline{\mathcal{S}}) \longrightarrow \mathrm{Fun}^G(\underline{\mathrm{Disk}}_V, \underline{\mathcal{S}}) \longrightarrow \mathrm{Fun}^G(\underline{\mathrm{Man}}_V, \underline{\mathcal{S}})\,,$$

that is for A an $\mathbb{E}_V$-algebra in $\underline{\mathcal{S}}$ and M a V-framed manifold, the equivariant factorization homology of A over M is given by:

$$\int_M A \cong \mathrm{colim}\Big(\underline{\mathrm{Disk}}_{V/M} \longrightarrow \underline{\mathrm{Disk}}_V \xrightarrow{A} \underline{\mathcal{S}}\Big)$$

Remark 2.12. Using a lemma similar to Proposition 2.3 (essentially [LLP25, Thm. 3.39]), Horev shows that this Kan extension naturally admits a G-symmetric monoidal structure. We will not make use of this enhancement.

Construction 2.13. Let X be a based G-space and let M be a V-framed manifold, we will recall from [HKZ24] how to construct a natural map

$$\int_M \Omega^V X \longrightarrow \mathrm{map}_*(M^+, X)$$

of G-spaces. It follows from Construction 2.10 that $\Omega^V X \colon \underline{\mathrm{Disk}}_V \longrightarrow \underline{\mathcal{S}}$ can actually be extended to a functor out of $\underline{\mathrm{Man}}_V$ such that the value at M is $\mathrm{map}_*(M^+, X)$. It hence follows that there is a natural transformation

$$\Big(\underline{\mathrm{Disk}}_{V/M} \longrightarrow \underline{\mathrm{Disk}}_V \xrightarrow{\Omega^V X} \underline{\mathcal{S}}\Big) \Longrightarrow \mathrm{const}\,\mathrm{map}_*(M^+, X)\,,.$$

The natural map we wanted to construct now arises by using the universal property of parameterized colimits.

Before we state equivariant nonabelian Poincaré duality, let us recall the following definition from the introduction:

Definition 2.14. A based G-space X is called V-connective if its H-fixed points X^H are $(\dim V^H - 1)$-connected for every subgroup H of G. We write $(\mathcal{S}_*^G)_{\geq V} \subset \mathcal{S}_*^G$ for the full subcategory of V-connective spaces.

Theorem 2.15 ([HKZ24, Thm. 4.0.1], equivariant nonabelian Poincaré duality)**.** *For X a V-connective based G-space and M a V-framed manifold, the natural map from Construction 2.13*

$$\int_M \Omega^V X \longrightarrow \mathrm{map}_*(M^+, X)$$

is an equivalence of G-spaces.

3 The approximation theorem

In this section, we will prove the approximation theorem. We recommend the reader to recall the proof strategy from the introduction.

3.1 Factoring the free group-like functor

In this subsection, we will introduce the intermediate G-operads $\mathbb{E}_0 \subset \mathbb{E}_V^i \subset \mathbb{E}_V^{\mathrm{i,e}} \subset \mathbb{E}_V$, the latter one we informally described when explaining the strategy of the proof in the introduction.

Let us introduce a terminology for the subgroups of H of G for which A^H admits a monoid structure for A an $\mathbb{E}_V$-algebra.

Definition 3.1. For V a G-representation, we say that a subgroup H of G is V*-isotropy* if $\dim V^H \geq 1$.

We will now define the operad $\mathbb{E}_V^{\mathrm{i,e}} \subset \mathbb{E}_V$. The idea is that in this operad, we do not allow for any operations going from V-isotropy subgroups to subgroups which are not V-isotropy.

Definition 3.2. Let $\underline{\mathrm{Disk}}_V^{\mathrm{i,e}} \subset \underline{\mathrm{Disk}}_V$ (the "i.e" stand for isotropy, extended) denote the non-full G-symmetric monoidal subcategory spanned by all objects but on H-fixed points, but we only take those embeddings $f\colon D_1 \to D_2$ of $\mathrm{res}^G_H V$-framed H-disks such that for an H-path component $D \subset D_2$ such that $D^K \neq \emptyset$ for some non-V-isotropy $K < H$, the same is true for all path components in $f^{-1}(D)$.

We can use Remark 2.2 to see that this is an enveloping algebra of a G-operad with a map to $\mathbb{E}_V$ which we will denote by $\mathbb{E}_V^{\mathrm{i,e}}$. We therefore denote the category of $\mathbb{E}_V^{\mathrm{i,e}}$-algebras in a G-symmetric monoidal G-category $\underline{\mathcal{C}}$ by

$$\mathrm{Alg}_{\mathbb{E}_V^{\mathrm{i,e}}}(\underline{\mathcal{C}}) = \mathrm{Fun}^{G,\otimes}\left(\underline{\mathrm{Disk}}_V^{\mathrm{i,e}}, \underline{\mathcal{C}}\right).$$

It will also be helpful to have the following operad, which does not remember any fixed point data for subgroups which are not V-isotropy.

Definition 3.3. We write $\underline{\mathrm{Disk}}_V^{\mathrm{i}}$ (the "i" stands for isotropy) for the full G-symmetric monoidal subcategory of $\underline{\mathrm{Disk}}_V$ (and $\underline{\mathrm{Disk}}_V^{\mathrm{i,e}}$) spanned on H-fixed points by all H-disks for which all isotropy groups are V-isotropy. Using Remark 2.2, we see that this again is an enveloping algebra of a G-operad with a map to $\mathbb{E}_V^{\mathrm{i,e}}$ which we will denote by $\mathbb{E}_V^i$. For $\underline{\mathcal{C}}$ a G-symmetric monoidal G-category, we hence write

$$\mathrm{Alg}_{\mathbb{E}_V^i}(\underline{\mathcal{C}}) = \mathrm{Fun}^{G,\otimes}(\underline{\mathrm{Disk}}_V^{\mathrm{i}}, \underline{\mathcal{C}})$$

for the category of $\mathbb{E}_V^i$-algebras in $\underline{\mathcal{C}}$.

Finally, let us introduce the corresponding notion of group-like algebras for those operads and record that group completion exists. (Even though in this case we will construct those group completions in the proofs of Proposition 3.10 and Proposition 3.12 directly without assuming its a priori existence anyway.)

Definition 3.4. Following our conventions for the $\mathbb{E}_V$-operad, given an $\mathbb{E}_V^{\mathrm{i,e}}$-algebra A, we write A^H for the based space obtained by taking H-fixed points of the H-space which is given by evaluating the G-symmetric monoidal functor $\underline{\mathrm{Disk}}_V^{\mathrm{i,e}} \to \underline{\mathcal{S}}$ at the H-disk $\mathrm{res}^G_H V$, receiving a base point from the unique map from the empty disk into $\mathrm{res}^G_H V$. If H is V-isotropy, then these fixed points admit a natural $\mathbb{E}_{V^H}$-algebra structure and this already is well-defined for an $\mathbb{E}_V^i$-algebra. We call a $\mathbb{E}_V^{\mathrm{i,e}}$- (or $\mathbb{E}_V^i$-)algebra in spaces A *group-like* if A^H is group-like for all H which are V-isotropy and denote the corresponding subcategories of group-like algebras by $\mathrm{Alg}^{\mathrm{grp}}_{\mathbb{E}_V^{\mathrm{i,e}}}(\underline{\mathcal{S}})$ and $\mathrm{Alg}^{\mathrm{grp}}_{\mathbb{E}_V^i}(\underline{\mathcal{S}})$, respectively. The same argument as in the proof of Proposition 2.7 shows that there are left adjoints to the inclusions which we will denote by

$$\mathrm{GrpCompl}^{\mathbb{E}_V^{\mathrm{i,e}}} : \mathrm{Alg}_{\mathbb{E}_V^{\mathrm{i,e}}}(\underline{\mathcal{S}}) \longrightarrow \mathrm{Alg}^{\mathrm{grp}}_{\mathbb{E}_V^{\mathrm{i,e}}}(\underline{\mathcal{S}})$$

and

$$\mathrm{GrpCompl}^{\mathbb{E}_V^i} : \mathrm{Alg}_{\mathbb{E}_V^i}(\underline{\mathcal{S}}) \longrightarrow \mathrm{Alg}^{\mathrm{grp}}_{\mathbb{E}_V^i}(\underline{\mathcal{S}})\,.$$

Before we get into the proof, let us record the following fact, which will be useful in order to deal with the V-fold loop space functor:

Proposition 3.5. *The V-fold loop space functor*

$$\Omega^V : (\mathcal{S}^G_*)_{\geq V} \longrightarrow \mathcal{S}^G_*$$

from V-connective based G-spaces to based G-spaces is conservative and commutes with geometric realizations.

Proof. The claim about geometric realizations is proven in [CW91, Lem. 5.4].

Let us turn to the conservativity claim. If $V = W \oplus \mathbb{R}$ contains a trivial summand, we might write $\Omega^V = \Omega^W \circ \Omega$ and reduce to the same claim for W. We might hence assume that V is fixed point free.

Let $f : X \to Y$ be a map of V-connective G-spaces such that $\Omega^V f : \Omega^V X \to \Omega^V Y$ is an equivalence. By inducting on the size of the group G, we might assume that $f^H : X^H \to Y^H$ is an equivalence for all proper subgroups H of G. Now consider the cofiber sequence

$$S(V)_+ \longrightarrow S^0 \longrightarrow S^V\,.$$

Mapping into $f : X \to Y$ yields a map between fiber sequences

$$\begin{array}{ccccc} (\Omega^V X)^G & \longrightarrow & X^G & \longrightarrow & \mathrm{map}(S(V), X)^G \\ \downarrow & & \downarrow & & \downarrow \\ (\Omega^V Y)^G & \longrightarrow & Y^G & \longrightarrow & \mathrm{map}(S(V), Y)^G \end{array}$$

The right hand map is an equivalence because $S(V)$ does not have G-fixed points, and we already assumed that f is an equivalence on fixed points for all proper subgroups. The left hand map is an equivalence by assumption. Finally, $S(V)$ admits an equivariant cell structures with H-cells at most of dimension $\dim V^H - 1$. By cell induction, we conclude that any map from $S(V)$ into a V-connective space must be trivial. It follows that the base space of the two fibrations is connected and we can conclude that the map $f^G : X^G \to Y^G$ is an equivalence, as desired. □

From this we deduce:

Lemma 3.6. *The V-fold loop space functor*

$$\Omega^V \colon (S_*^G)_{\geq V} \longrightarrow \mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathcal{S}})$$

commutes with geometric realizations. The same holds when replacing $\mathbb{E}_V$ with $\mathbb{E}_V^i$.

Proof. This follows from combining Proposition 2.4 and Proposition 3.5. □

Lemma 3.7. *The H-fixed point functors*

$$(-)^H \colon \mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathcal{S}}) \to \mathrm{Alg}_{\mathbb{E}_{V^H}}(\mathcal{S})$$

commutes with geometric realizations. The same is true when replacing $\mathbb{E}_V$ with $\mathbb{E}_V^i$.

Proof. This follows immediately from Proposition 2.4, as geometric realizations in both categories can be computed after applying the forgetful functor to G-spaces or spaces, respectively. For G-spaces, the H-fixed point functor commutes with all colimits. □

3.2 Group completion for $\mathbb{E}_V^i$-algebras

As a first step, we will describe the group completion functor for $\mathbb{E}_V^i$-algebras, this step will be the main input to compute the effect of group completion on an $\mathbb{E}_V$-algebra on H-fixed points for H a V-isotropy subgroup. The following proposition is the main geometric input we are using:

Theorem 3.8 (Hauschild)**.** *Let X be a based G-space and H a V-isotropy subgroup of G. Then the natural map*

$$\mathrm{GrpCompl}^{\mathbb{E}_{V^H}}\left(\mathrm{Free}^{\mathbb{E}_V^i} X\right)^H \longrightarrow \left(\mathrm{fgt}^{\mathbb{E}_V}_{\mathbb{E}_V^i} \Omega^V \Sigma^V X\right)^H$$

is an equivalence of $\mathbb{E}_{V^H}$-algebras.

Proof. As all functors and adjunctions used in this statement are parameterized, we can assume, without loss of generality, that $H = G$ and $\mathbb{E}_V^i = \mathbb{E}_V$ because G has V-isotropy or, equivalently, V contains a trivial summand.

We will first argue, why this formula holds for $X = Y_+$ a based space obtained by adding a disjoint base point. In this case, we might use [Ste25, Rem. 2.72] to identify

$$\left(\mathrm{Free}^{\mathbb{E}_V^i} Y_+\right)^G \cong \coprod_{A \in \underline{\mathrm{Fin}}^G} (\mathbb{E}_V(A) \times \mathrm{map}(A, Y))_{h\mathrm{Aut}(A)}$$

where

$$\mathbb{E}_V(A) = \mathrm{Emb}(\oplus_A V, V) \simeq \mathrm{Conf}_A(V)\,.$$

is the space of ordered equivariant configurations $A \to V$. Modeling Y by a G-CW-complex and computing the product and the homotopy quotient, which can be computed as a quotient, as the action is free, in the category of topological spaces, we see that $\left(\mathrm{Free}^{\mathbb{E}^i_V} Y_+\right)^G$ is modeled by the space of unordered equivariant configurations in V, labeled in Y. This space is homeomorphic to the G-fixed points of the configuration space with its induced action, as defined in [RS00]. A proof of this homeomorphism for the non-labeled version can be found in [BQV23, Prop. 3.2.10] and the same arguments apply in the case with labels. The loop space $\Omega^V \Sigma^V Y_+$ can also be modeled by the space of maps of topological spaces from S^V to $\Sigma^V Y_+$. Under those identifications the result appears in [RS00]. It is originally due to Hauschild [Hau80], even though the published version only discusses the case $X = S^0$.

Finally, we can deduce the based version from the non-based version, as the bar construction for $\mathbb{E}_0$ provides a presentation of any based space as a geometric realization of spaces of the form Y_+. Moreover, all functors in question commute with geometric realizations by Lemma 3.7 and Lemma 3.6. □

Remark 3.9. We could have avoided the extra step of deducing the formula for based G-spaces from the one for based G-spaces, as computing the free objects on unbased G-spaces is enough to deduce the recognition principle Theorem A, which in turn implies the approximation theorem Theorem B also for based G-spaces. However, we decided that it is more natural to provide a proof of the approximation theorem for based G-spaces directly.

We use the above result to deduce that group completion of $\mathbb{E}^i_V$-algebras is computed pointwise:

Proposition 3.10. *Let H be a V-isotropy and let A be an $\mathbb{E}^i_V$-algebra. Then the natural map*

$$\mathrm{GrpCompl}^{\mathbb{E}_{V^H}} A^H \longrightarrow \left(\mathrm{GrpCompl}^{\mathbb{E}^i_V} A\right)^H$$

is an equivalence of $\mathbb{E}^i_V$-algebras.

Proof. We do construct the group completion functor directly. The functor

$$\Omega^V : (\mathcal{S}^G_*)_{\geq V} \longrightarrow \mathrm{Alg}_{\mathbb{E}^i_V}(\underline{\mathcal{S}})$$

commutes with limits as those are computed pointwise by Proposition 2.4 and with filtered colimits as it does with sifted colimits, which can be seen as in Lemma 3.6. (Here, we abbreviated the functor $\mathrm{fgt}^{\mathbb{E}_V}_{\mathbb{E}^i_V} \circ \Omega^V$ by Ω^V). By [Lur09, Cor. 5.5.2.9], it hence does admit a left adjoint

$$B^V : \mathrm{Alg}_{\mathbb{E}^{\mathrm{i,e}}_V}(\underline{\mathcal{S}}) \longrightarrow (\mathcal{S}^G_*)_{\geq V} .$$

We will prove that the unit

$$A \longrightarrow \Omega^V B^V A$$

is a group completion on all fixed points. This hence is a natural transformation for which the target always is group-like and an equivalence if the source is group-like. It does follow that $\Omega^V B^V$ does compute the group completion as well as that this group completion is computed pointwise.

If $A = \mathrm{Free}^{\mathbb{E}^i_V} X$ for a based G-space X, we find that $B^V \mathrm{Free}^{\mathbb{E}^i_V} X \cong \Sigma^V X$, so that the claim is precisely the content of Theorem 3.8.

We will now write a general $\mathbb{E}^i_V$-algebra A as a colimit of free algebras to reduce to this case. As we already know that the forgetful functor is monadic, we can consider the Bar construction $\mathrm{Bar}(A)\colon \Delta^{\mathrm{op}} \to \mathrm{Alg}_{\mathbb{E}^i_V}(\underline{\mathcal{S}})$ from [Lur17, Exa. 4.7.2.7] which resolves A by free $\mathbb{E}^i_V$-algebras, so that $A \cong \mathrm{colim}_{\Delta^{\mathrm{op}}} \mathrm{Bar}(A)$.

Consider the diagram of $\mathbb{E}_{V^H}$-algebras in spaces:

$$\begin{array}{ccc} A^H & \longrightarrow & \left(\Omega^V B^V A\right)^H \\ \downarrow & & \downarrow \\ \mathrm{colim}_{\Delta^{\mathrm{op}}}\left(\mathrm{Bar}(A)^H\right) & \longrightarrow & \mathrm{colim}_{\Delta^{\mathrm{op}}}\left(\left(\Omega^V B^V \mathrm{Bar}(A)\right)^H\right) \end{array}$$

where the horizontal arrows are induced by the unit of the adjunction and the vertical ones are assembly maps for the colimit. As the left adjoint B^V commutes with all colimits and Lemma 3.7 and Lemma 3.6 say that Ω^V and H-fixed points commute with geometric realizations too, we learn that the vertical morphisms are equivalences.

Now recall that $\mathrm{Bar}(A)$ is a resolution of A by free algebras, for which we already argued above using Theorem 3.8 that the unit is a group completion on H-fixed points. We therefore presented the H-fixed points of the unit as a colimit of pointwise group completions and conclude that it is a group completion, which finishes the proof. □

3.3 Group completion for $\mathbb{E}^{\mathrm{i,e}}_V$-algebras

In order to describe group completions for $\mathbb{E}^{\mathrm{i,e}}_V$-algebra, we want to use that such an algebra really just consists of the data of an $\mathbb{E}^i_V$-algebra together with a refinement of the underlying $\mathbb{E}^i_0$-algebra to an $\mathbb{E}_0$-algebra. We prove this in the special case of interest for us, for algebras in G-spaces.

Lemma 3.11. *Let $\mathcal{S}^{G,i}_*$ denote the category of based G-spaces with isotropy concentrated in V-isotropy subgroups, i.e. presheaves on the subcategory of the orbit category spanned by the corresponding orbits. The commutative diagram*

$$\begin{array}{ccc} \mathrm{Alg}_{\mathbb{E}^{\mathrm{i,e}}_V}(\underline{\mathcal{S}}) & \xrightarrow{\mathrm{fgt}^{\mathbb{E}^{\mathrm{i,e}}_V}_{\mathbb{E}^i_V}} & \mathrm{Alg}_{\mathbb{E}^i_V}(\underline{\mathcal{S}}) \\ {\scriptstyle \mathrm{fgt}^{\mathbb{E}^{\mathrm{i,e}}_V}}\downarrow & & \downarrow{\scriptstyle \mathrm{fgt}^{\mathbb{E}^i_V}} \\ \mathcal{S}^G_* & \xrightarrow{\mathrm{fgt}} & \mathcal{S}^{G,i}_* \end{array}$$

is a pullback.

Proof. We want to apply the monadicity theorem. We start by providing a description of the left adjoint to

$$\mathrm{pr}_2 \colon \mathrm{Alg}_{\mathbb{E}_V^i}(\mathcal{S}) \times_{\mathcal{S}_*^{G,i}} \mathcal{S}_*^G \longrightarrow \mathcal{S}_*^G .$$

Let $\mathrm{Free}^{\mathrm{i}} \colon \mathcal{S}_*^{G,i} \longrightarrow \mathcal{S}_*^G$ denote the fully faithful left adjoint to the forgetful functor.

Given X a based G-space, let FX be defined to be the pushout

$$\begin{array}{ccc} \mathrm{Free}^i\mathrm{fgt}^i X & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Free}^i\mathrm{fgt}^{\mathbb{E}_V^i}\mathrm{Free}^{\mathbb{E}_V^i}\mathrm{fgt}^i X & \longrightarrow & FX \end{array} \tag{1}$$

where the upper map is the counit and the lower map is Free^i applied to the unit. Then, using that fgt^i is fully faithful and preserves pushouts, one verifies that the lower map induces an equivalence $\mathrm{fgt}^i FX \cong \mathrm{fgt}^{\mathbb{E}_V^i}\mathrm{Free}^{\mathbb{E}_V^i}\mathrm{fgt}^i X$, so that $(\mathrm{Free}^{\mathbb{E}_V^i}\mathrm{fgt}^i X, FX)$ lifts to an object in $\mathrm{Alg}_{\mathbb{E}_V^i}(\mathcal{S}) \times_{\mathcal{S}_*^{G,i}} \mathcal{S}_*^G$. We claim that the right arrow in the above square exhibits this object as the left adjoint object to X under pr_2. This follows from a computation of mapping spaces in pullback categories which we leave to the reader.

Let us now apply the monadicity theorem to the following diagram:

$$\begin{array}{ccc} \mathrm{Alg}_{\mathbb{E}_V^{\mathrm{i,e}}}(\underline{\mathcal{S}}) & \xrightarrow{\left(\mathrm{fgt}^{\mathbb{E}_V^{\mathrm{i,e}}}_{\mathbb{E}_V^i},\, \mathrm{fgt}^{\mathbb{E}_V^{\mathrm{i,e}}}_{\mathbb{E}_0}\right)} & \mathrm{Alg}_{\mathbb{E}_V^i}(\underline{\mathcal{S}}) \times_{\mathcal{S}_*^{G,i}} \mathcal{S}_*^G \\ & {\scriptstyle \mathrm{fgt}^{\mathbb{E}_V^{\mathrm{i,e}}}} \searrow \quad \swarrow {\scriptstyle \mathrm{fgt}\circ\mathrm{pr}_2} & \\ & \mathcal{S}^G & \end{array}$$

Both functors are monadic by Proposition 2.4, as the right hand side is the category of algebras over the pushout $\mathbb{E}_0 \leftarrow \mathbb{E}_0^i \rightarrow \mathbb{E}_V^i$.

Now we need to show that the top map preserves free objects. Let

$$\mathrm{Free} \colon \mathcal{S}_*^G \longrightarrow \mathrm{Alg}_{\mathbb{E}_V^i}(\underline{\mathcal{S}}) \times_{\mathcal{S}_*^{G,i}} \mathcal{S}_*^G$$

denote the adjoint of pr_2 which we constructed above. It is enough to show that for any based G-space X, the natural map

$$\mathrm{Free}(X) \longrightarrow \left(\mathrm{fgt}^{\mathbb{E}_V^{\mathrm{i,e}}}_{\mathbb{E}_V^i}, \mathrm{fgt}^{\mathbb{E}_V^{\mathrm{i,e}}}\right)\left(\mathrm{Free}^{\mathbb{E}_V^{\mathrm{i,e}}} X\right) \tag{2}$$

is an equivalence. This can be checked on H-fixed points for H all subgroups of G. As all the functors in question come from parameterized adjunctions, we might moreover assume that $H = G$. If G itself is V-isotropy, the statement becomes trivial as $\mathbb{E}_V^i = \mathbb{E}_V^{\mathrm{i,e}}$.

So, we are left with the case where G is not V-isotropy, i.e. $V^G = \{0\}$. Here, we use that the left arrow in (1) becomes an equivalence after applying G-fixed points, as Free^i just inserts the initial object $*$ on all fixed points which are not V-isotropy. We conclude that $(\mathrm{Free}(X))^G \simeq X^G$ and unwinding definitions, we are left to show that

$$X^G \longrightarrow \left(\mathrm{Free}^{\mathbb{E}_V^{\mathrm{i,e}}} X\right)^G$$

is an equivalence. This follows from the formula for the free algebra from Proposition 2.3, using that in the category $\underline{\mathrm{Disk}}_V^{\mathrm{i,e}}$, the only G-disks which embed into V, are the disk itself and the empty disk. □

Now we are ready to prove that group completion for $\mathbb{E}_V^{\mathrm{i,e}}$-algebras coincides with group completion on $\mathbb{E}_V^{i}$-algebras on H-fixed points for H a V-isotropy subgroup and just does not change anything on all other fixed points:

Proposition 3.12. *Let A be a $\mathbb{E}_V^{\mathrm{i,e}}$-algebra and H a subgroup of G.*

- *If H is V-isotropy, then the natural map*

$$\mathrm{GrpCompl}^{\mathbb{E}_{V^H}} A^H \longrightarrow \left(\mathrm{GrpCompl}^{\mathbb{E}_V^{\mathrm{i,e}}} A\right)^H$$

  is an equivalence of $\mathbb{E}_{V^H}$-algebras.

- *If H is not V-isotropy, then the unit on H-fixed points*

$$A^H \longrightarrow \left(\mathrm{GrpCompl}^{\mathbb{E}_V^{\mathrm{i,e}}} A\right)^H$$

  is an equivalence of based spaces.

Proof. We will construct the group completion explicitly and check that it has the desired properties. For this we use the equivalence

$$\mathrm{Alg}_{\mathbb{E}_V^{\mathrm{i,e}}}(\underline{\mathcal{S}}) \cong \mathrm{Alg}_{\mathbb{E}_V^{i}}(\underline{\mathcal{S}}) \times_{\mathcal{S}_*^{G,i}} \mathcal{S}_*^G$$

from Lemma 3.11 which restricts to an equivalence

$$\mathrm{Alg}^{\mathrm{grp}}_{\mathbb{E}_V^{\mathrm{i,e}}}(\underline{\mathcal{S}}) \cong \mathrm{Alg}^{\mathrm{grp}}_{\mathbb{E}_V^{i}}(\underline{\mathcal{S}}) \times_{\mathcal{S}_*^{G,i}} \mathcal{S}_*^G\,.$$

Given

$$(A, \varphi\colon \mathrm{fgt}^{\mathbb{E}_V^i} A \cong \mathrm{fgt}^i X, X) \in \mathrm{Alg}_{\mathbb{E}_V^{i}}(\underline{\mathcal{S}}) \times_{\mathcal{S}_*^{G,i}} \mathcal{S}_*^G\,,$$

we construct its group completion as follows: Its underlying $\mathbb{E}_V^i$-algebra is given by $\mathrm{GrpCompl}^{\mathbb{E}_V^i} A$. Its underlying G-space is defined to be the pushout P of

$$\begin{array}{ccc} \mathrm{Free}^i(\mathrm{fgt}^{\mathbb{E}_V^i} A) & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ \mathrm{Free}^i(\mathrm{fgt}^{\mathbb{E}_V^i} \mathrm{GrpCompl}^{\mathbb{E}_V^i} A) & \longrightarrow & P \end{array}$$

Here $\mathrm{Free}^i\colon \mathcal{S}_*^{G,i} \to \mathcal{S}_*^G$ denotes the left adjoint to the forgetful functor. The left map is the unit of the group completion adjunction. The upper map is the counit of the forgetful free adjunction, using the equivalence $\varphi\colon \mathrm{fgt}^{\mathbb{E}_V^i} A \cong \mathrm{fgt}^i X$.

Moreover, the upper map becomes an equivalence after applying $\mathrm{fgt}^i\colon \mathcal{S}_*^G \to \mathcal{S}_*^{G,i}$, so that the lower map gives an equivalence

$$\bar{\varphi}\colon \mathrm{fgt}^{\mathbb{E}_V^i}\mathrm{GrpCompl}^{\mathbb{E}_V^i}A \cong \mathrm{fgt}^i\mathrm{Free}^i\mathrm{fgt}^{\mathbb{E}_V^i}\mathrm{GrpCompl}^{\mathbb{E}_V^i}A \cong \mathrm{fgt}^i P\,,$$

so that we obtain an element

$$(\mathrm{GrpCompl}^{\mathbb{E}_V^i}A, \bar{\varphi}, P) \in \mathrm{Alg}_{\mathbb{E}_V^i}(\underline{\mathcal{S}}) \times_{\mathcal{S}_*^{G,i}} \mathcal{S}_*^G\,.$$

Now we have maps $X \to P$ and $A \to \mathrm{GrpCompl}^{\mathbb{E}_V^i}A$ and by construction, a homotopy witnessing compatibility of the equivalences φ and $\bar{\varphi}$ after applying fgt^i or $\mathrm{fgt}^{\mathbb{E}_V^i}$, respectively, i.e. we constructed a map

$$(A, \varphi, X) \longrightarrow (\mathrm{GrpCompl}^{\mathbb{E}_V^i}A, \bar{\varphi}, P)\,.$$

Moreover, all this constructions can be made functorially, so that we constructed a natural transformation from any element in $\mathrm{Alg}_{\mathbb{E}_V^i}(\underline{\mathcal{S}}) \times_{\mathcal{S}_*^{G,i}} \mathcal{S}_*^G$ into a group-like element which is an equivalence if the source was group-like. It follows that this natural transformation exhibits the target as the group completion of the source. It moreover follows that this group completion is an equivalence on fixed points which are not V-isotropy and is given by group completion of the underlying $\mathbb{E}_V^i$-algebra on the fixed points which are V-isotropy. Finally, it follows that this $\mathbb{E}_V^{\mathrm{i,e}}$-group completion is given by (non-equivariant) group completion on those fixed points by Proposition 3.10. □

3.4 The free $\mathbb{E}_V$-algebra on an $\mathbb{E}_V^{\mathrm{i,e}}$-algebra

In this section, we will compute the free $\mathbb{E}_V$-algebra on an $\mathbb{E}_V^{\mathrm{i,e}}$-algebra A. The main observation is that the H-fixed points A^H for H not V-isotropy are acted on by the H-fixed points of the equivariant factorization homology $\int_{V\setminus\{0\}} A$ and that this is somehow "the only additional structure", as the H-fixed points of the free $\mathbb{E}_V$-algebra on A will be the free $\left(\int_{V\setminus\{0\}} A\right)^H$-space on A^H.

Construction 3.13. As already explained in [Lev22, Cor. 2.2], for A an $\mathbb{E}_V$-algebra and H a subgroup which is not V-isotropy, there is a natural $\left(\int_{V\setminus\{0\}} A\right)$-action on A. It can be constructed as follows: The natural $\mathbb{E}_1$-structure on $\mathbb{R}$ in Man (one-dimensional manifolds, in that case), gives rise to an $\mathbb{E}_1$-structure on $V \setminus \{0\} \cong \mathbb{R} \times S(V)$ in $\underline{\mathrm{Man}}_V^G$ (even though this product is not a product as framed manifolds, the necessary embeddings are framed). Moreover, the V is a module over $V \setminus \{0\}$. Applying factorization homology then yields the action.

Proposition 3.14. *Let A be an $\mathbb{E}_V^{\mathrm{i,e}}$-algebra in G-spaces and let H be a subgroup of G.*

- *If H is V-isotropy, then the induced map on H-fixed points of the unit*

$$A^H \longrightarrow \left(\mathrm{Free}_{\mathbb{E}_V^{\mathrm{i,e}}}^{\mathbb{E}_V} A\right)^H$$

is an equivalence of $\mathbb{E}_{V^H}$-algebras.

- *If H is not V-isotropy, then by Construction 3.13,* $\left(\mathrm{Free}^{\mathbb{E}_V}_{\mathbb{E}_V^{\mathrm{i,e}}} A\right)^H$ *has a natural* $\left(\int_{\mathrm{res}^G_H V\setminus\{0\}} A\right)^H$*-action, so that the unit uniquely extends to a* $\left(\int_{\mathrm{res}^G_H V\setminus\{0\}} A\right)^H$*-equivariant map*

$$\left(\int_{\mathrm{res}^G_H V\setminus\{0\}} A\right)^H \times A^H \longrightarrow \left(\mathrm{Free}^{\mathbb{E}_V}_{\mathbb{E}_V^{\mathrm{i,e}}} A\right)^H.$$

out of the free $\left(\int_{\mathrm{res}^G_H V\setminus\{0\}} A\right)^H$*-space on A^H. This map is an equivalence.*

Proof. As all the adjunctions in question are parameterized, we might assume that $H = G$. The first part then becomes trivial as $\mathbb{E}_V^i = \mathbb{E}_V^{\mathrm{i,e}}$ in case that G is V-isotropy.

We might therefore assume that G is not V-isotropy. Consider the functor

$$-\sqcup(V \xrightarrow{\mathrm{id}} V)\colon \left(\underline{\mathrm{Disk}}^{\mathrm{i}}_V \times_{\underline{\mathrm{Disk}}_V} \underline{\mathrm{Disk}}_{V/V}\right)^G \longrightarrow \left(\underline{\mathrm{Disk}}^{\mathrm{i,e}}_V \times_{\underline{\mathrm{Disk}}_V} \underline{\mathrm{Disk}}_{V/V}\right)^G$$

adding one copy of V with the identity structure map to V. We claim that the functor is cofinal. Indeed, using that in $(\underline{\mathrm{Disk}}^{\mathrm{i,e}}_V)^G$ only the empty disk and V map into V, one sees that the functor is fully faithful. Using the same observation, one can also check that every object in the target has an initial morphism to one from the source (adding a disk if necessary), so that the functor is even a right adjoint inclusion.

Now we apply Proposition 2.3 to find:

$$\begin{aligned}
&\left(\mathrm{Free}^{\mathbb{E}_V}_{\mathbb{E}_V^{\mathrm{i,e}}} A\right)(V)^G \\
&\cong \mathrm{colim}\left(\underline{\mathrm{Disk}}^{\mathrm{i,e}}_V \times_{\underline{\mathrm{Disk}}_V} \underline{\mathrm{Disk}}_{V/V} \to \underline{\mathrm{Disk}}^{\mathrm{i,e}}_V \xrightarrow{A} \underline{\mathcal{S}}\right)^G \\
&\cong \mathrm{colim}\left(\left(\underline{\mathrm{Disk}}^{\mathrm{i,e}}_V \times_{\underline{\mathrm{Disk}}_V} \underline{\mathrm{Disk}}_{V/V}\right)^G \to \left(\underline{\mathrm{Disk}}^{\mathrm{i,e}}_V\right)^G \xrightarrow{A^G} \underline{\mathcal{S}}^G \xrightarrow{(-)^G} \mathcal{S}\right) \\
&\cong \mathrm{colim}\left(\left(\underline{\mathrm{Disk}}^{\mathrm{i}}_V \times_{\underline{\mathrm{Disk}}_V} \underline{\mathrm{Disk}}_{V/V}\right)^G \xrightarrow{\sqcup V} \left(\underline{\mathrm{Disk}}^{\mathrm{i,e}}_V \times_{\underline{\mathrm{Disk}}_V} \underline{\mathrm{Disk}}_{V}\right)^G \to (\underline{\mathrm{Disk}}^{\mathrm{i,e}}_V)^G \xrightarrow{A^G} \underline{\mathcal{S}}^G \xrightarrow{(-)^G} \mathcal{S}\right) \\
&\cong \mathrm{colim}\left(\left(\underline{\mathrm{Disk}}^{\mathrm{i}}_V \times_{\underline{\mathrm{Disk}}_V} \underline{\mathrm{Disk}}_{V/V}\right)^G \to \left(\underline{\mathrm{Disk}}^{\mathrm{i}}_V\right)^G \xrightarrow{A^G} \underline{\mathcal{S}}^G \xrightarrow{(-)^G} \mathcal{S} \xrightarrow{\times A(V)^G} \mathcal{S}\right) \\
&\cong \mathrm{colim}\left(\left(\underline{\mathrm{Disk}}^{\mathrm{i}}_V \times_{\underline{\mathrm{Disk}}_V} \underline{\mathrm{Disk}}_{V/V}\right)^G \to \left(\underline{\mathrm{Disk}}^{\mathrm{i}}_V\right)^G \xrightarrow{A^G} \underline{\mathcal{S}} \xrightarrow{(-)^G}\right) \times A(V)^G \\
&\cong \left(\int_{V\setminus\{0\}} A\right)^G \times A(V)^G
\end{aligned}$$

Here, we also used the fact [Sha23, Prop. 5.5] that the G-fixed points of a G-colimit of a functor $\underline{\mathcal{C}} \to \underline{\mathcal{S}}$ is naturally equivalent to the colimit of $\underline{\mathcal{C}}^G \to \underline{\mathcal{S}}^G \to \mathcal{S}$ where the last functor is taking G-fixed points.

For the last isomorphism, we also used that the isotropy groups of $V \setminus \{0\}$ are V-isotropy and, moreover, any embedding of a V-isotropy disks into V automatically lands in $V \setminus \{0\}$. It follows that the inclusion $V \setminus \{0\} \hookrightarrow V$ induces an equivalence

$$\left(\underline{\mathrm{Disk}}_{V/V\setminus\{0\}}\right)^G \simeq \left(\underline{\mathrm{Disk}}^{\mathrm{i}}_V \times_{\underline{\mathrm{Disk}}_V} \underline{\mathrm{Disk}}_{V/V}\right)^G.$$

The colimit over the left hand side is, again by [Sha23, Prop. 5.5], the fixed points of the equivariant factorization homology. The inclusion $V \setminus \{0\} \to V$ is also the one used to define the action of the factorization homology on the fixed points in Construction 3.13, so the isomorphism is the one induced by the action. □

3.5 Assembling the argument

We are now ready to prove the approximation theorem by computing the free group-like $\mathbb{E}_V$-algebra on a based space X. As explained in the proof strategy, we will first compute the free $\mathbb{E}_V^{\mathrm{i,e}}$-algebra on X, then determine its group completion using Proposition 3.12 and the result of Hauschild from Theorem 3.8. Then, we will compute the free $\mathbb{E}_V$-algebra on this free group-like $\mathbb{E}_V^{\mathrm{i,e}}$-algebra using Proposition 3.14 and equivariant nonabelian Poincaré duality, Theorem 2.15.

Theorem 3.15 (Approximation theorem)**.** *For X a based G-space, the natural map*

$$\mathrm{Free}^{\mathbb{E}_V} X \longrightarrow \Omega^V \Sigma^V X$$

from the free $\mathbb{E}_V$-algebra on X to the V-fold loop space of the V-fold suspension of X exhibits the source as the group completion of the target.

Proof. Consider the following commutative diagram of forgetful functors and inclusions of subcategories:

$$\begin{array}{ccccc} \mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathcal{S}}) & \longrightarrow & \mathrm{Alg}_{\mathbb{E}_V^{\mathrm{i,e}}}(\underline{\mathcal{S}}) & \longrightarrow & \mathrm{Alg}_{\mathbb{E}_0}(\underline{\mathcal{S}}) \\ \uparrow & & \uparrow & & \\ \mathrm{Alg}^{\mathrm{grp}}_{\mathbb{E}_V}(\underline{\mathcal{S}}) & \longrightarrow & \mathrm{Alg}^{\mathrm{grp}}_{\mathbb{E}_V^{\mathrm{i,e}}}(\underline{\mathcal{S}}) & & \end{array}$$

It follows from the description in Proposition 3.14 that the free $\mathbb{E}_V$-algebra on a group-like $\mathbb{E}_V^{\mathrm{i,e}}$-algebra is automatically group-like. We conclude that the adjunction between forgetful and free functor of $\mathbb{E}_V$ and $\mathbb{E}_V^{\mathrm{i,e}}$ restricts to an adjunction on the subcategories of the respective group-like objects. Therefore, passing to left adjoints in the above diagram yields a natural equivalence

$$\mathrm{GrpCompl}^{\mathbb{E}_V} \mathrm{Free}^{\mathbb{E}_V} X \cong \mathrm{Free}^{\mathbb{E}_V}_{\mathbb{E}_V^{\mathrm{i,e}}} \mathrm{GrpCompl}^{\mathbb{E}_V^{\mathrm{i,e}}} \mathrm{Free}^{\mathbb{E}_V^{\mathrm{i,e}}} X\,.$$

We can hence rephrase the theorem, asking whether the natural map

$$\mathrm{Free}^{\mathbb{E}_V}_{\mathbb{E}_V^{\mathrm{i,e}}} \mathrm{GrpCompl}^{\mathbb{E}_V^{\mathrm{i,e}}} \mathrm{Free}^{\mathbb{E}_V^{\mathrm{i,e}}} X \longrightarrow \Omega^V \Sigma^V X \tag{3}$$

is an equivalence which is true if it is an equivalence on all fixed points.

We start by showing that it is an equivalence on H-fixed points for H being V-isotropy. Recall from Theorem 3.8 that the natural map

$$\mathrm{Free}^{\mathbb{E}_V^{i}} X \longrightarrow \mathrm{fgt}^{\mathbb{E}_V}_{\mathbb{E}_V^{i}} \Omega^V \Sigma^V X$$

is a group completion on H-fixed points for H being V-isotropy. As $(\underline{\mathrm{Disk}}_V^{\mathrm{i}})^H \hookrightarrow (\underline{\mathrm{Disk}}_V^{\mathrm{i,e}})^H$ is an equivalence for all V-isotropy H, we know that $\mathrm{Free}_{\mathbb{E}_V^i}^{\mathbb{E}_V^{\mathrm{i,e}}}(-)$ does not change those H-fixed points, so that

$$\mathrm{Free}^{\mathbb{E}_V^{\mathrm{i,e}}} X \longrightarrow \mathrm{fgt}_{\mathbb{E}_V^{\mathrm{i,e}}}^{\mathbb{E}_V} \Omega^V \Sigma^V X$$

is a group completion on those H-fixed points, too. Now we can deduce from Proposition 3.12 that the natural map

$$\mathrm{GrpCompl}^{\mathbb{E}_V^{\mathrm{i,e}}} \mathrm{Free}^{\mathbb{E}_V^{\mathrm{i,e}}} X \longrightarrow \mathrm{fgt}_{\mathbb{E}_V^{\mathrm{i,e}}}^{\mathbb{E}_V} \Omega^V \Sigma^V X$$

is an equivalence on those H-fixed points, too. From Proposition 3.14 we finally deduce that the same holds for the map (3).

Now we will turn towards the H-fixed points where H is not V-isotropy. In this step we will combine equivariant nonabelian Poincaré duality with a splitting of $(\Omega^V \Sigma^V X)^G$ due to Hauschild [Hau77]. We will again restrict to the case $H = G$, the other cases following from this by using that the free functor is a parameterized functor and hence restricts to a free functor on subgroups.

By Proposition 3.14, we must verify that

$$X^G \to (\Omega^V \Sigma^V X)^G$$

exhibits the target as the free space with an action of

$$\left(\int_{V \setminus \{0\}} \Omega^V \Sigma^V X \right)^G \cong \Omega \operatorname{map}_*(S(V)_+, \Sigma^V X)^G,$$

where the latter equivalence is due to Theorem 2.15. Let us unwind this acting on $(\Omega^V \Sigma^V X)^G$. As nonabelian Poincaré duality is natural, this action is equivalently described as follows: The action of $V \setminus \{0\}$ on V in $\underline{\mathrm{Man}}_V^G$ gives rise to an action of $V \setminus \{0\}^+ \cong \Sigma S(V)_+$ on $V^+ = S^V$ in $(\mathcal{S}_*^G)^\vee$. In other words, S^V is a comodule over the coalgebra $\Sigma S(V)_+$ in based spaces. Finally, mapping out of those spaces yields the action of the algebra

$$\operatorname{map}_*(\Sigma S(V)_+, \Sigma^V X)^G \cong \Omega \operatorname{map}_*(\Sigma S(V)_+, \Sigma^V X)^G$$

on $(\Omega^V \Sigma^V X)^G$.

This means that we need to check that the composite

$$\begin{aligned} X^G \times \Omega \operatorname{map}_*(S(V)_+, \Sigma^V X)^G &\longrightarrow (\Omega^V \Sigma^V X)^G \times \Omega \operatorname{map}_*(S(V)_+, \Sigma^V X)^G \\ &\cong \operatorname{map}_*(S^V \vee \Sigma S(V)_+, \Sigma^V X)^G \\ &\longrightarrow (\Omega^V \Sigma^V X)^G \end{aligned}$$

is an equivalence, where the first map is induced by the unit, and the second by the collapse map $S^V \to S^V \vee \Sigma S(V)_+$. Modeling X by a G-CW complex, we can model the mapping spaces by taking mapping spaces in the category of based topological G-spaces. Using that $\Sigma S(V)_+ = S^V/S^0$, we can furthermore model

$$\mathrm{map}(\Sigma S(V)_+, \Sigma^V X)$$

by those maps $S^V \to \Sigma^V X$ which are sent to the base point in a neighborhood of the north and south pol. In this language, this splitting is due to Hauschild [Hau77], proven in this precise form in [RS00, Thm. 4]. They only state that there is a splitting $(\Omega^V\Sigma^V X)^G \cong X^G \times Z$ for some group Z. However, in the proof [RS00, pp. 11], they do check that $Z = \mathrm{map}_*(S(V)_+, \Sigma^V X)$ and that the homotopy equivalence $v\colon Z \times X^G \to (\Omega^V\Sigma^V X)^G$ is the one described above. This finishes the proof. □

4 The recognition principle

In this section, we will deduce the recognition principle from the approximation theorem, using the monadicity theorem.

Proposition 4.1. *The inclusion*

$$\mathrm{Alg}^{\mathrm{grp}}_{\mathbb{E}_V}(\underline{\mathcal{S}}) \longrightarrow \mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathcal{S}})$$

admits a right adjoint.

Proof. We first show that the inclusion

$$\mathrm{Alg}^{\mathrm{grp}}_{\mathbb{E}_V}(\underline{\mathrm{Set}}) \longrightarrow \mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathrm{Set}})$$

admits a right adjoint where $\underline{\mathrm{Set}} \subset \underline{\mathcal{S}}$ is the G-symmetric monoidal subcategory of G-spaces which is given by spaces which are discrete on all fixed points.

Given $A \in \mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathrm{Set}})$, let A^{core} denote the $\mathbb{E}_V$-algebra obtained from A by passing on H-fixed points A^H to the subset of elements for which the image in A^K for any subconjugate K of H which is V-isotropy is invertible in A^K. The $\mathbb{E}_V$-structure of A does restrict to one on A^{core}. Moreover, the inclusion $A^{\mathrm{core}} \to A$ gives a natural transformation which is the identity if A is group-like and the source always is group-like. It hence exhibits $(-)^{\mathrm{core}}$ as the right adjoint to the inclusion.

Now for $A \in \mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathcal{S}})$, let A^{core} denote the pullback

$$\begin{array}{ccc} A^{\mathrm{core}} & \longrightarrow & A \\ \downarrow & & \downarrow \\ \pi_0(A)^{\mathrm{core}} & \longrightarrow & \pi_0(A) \end{array}$$

where the vertical morphisms are induced by the unit $A \to \pi_0(A)$ of the left adjoint $\pi_0\colon \underline{\mathcal{S}} \to \underline{\mathrm{Set}}$ (which is G-symmetric monoidal).

Using that pullbacks can be computed pointwise by Proposition 2.4, one verifies that $A^{\mathrm{core}} \to A$ again is a natural transformation which is an equivalence on group-like objects and the source always is group-like, exhibiting $(-)^{\mathrm{core}}$ as the right adjoint to the inclusion. □

Proof of Theorem A. We actually start by showing that $\Omega^V\colon (\mathcal{S}_*^G)_{\geq V} \to \mathrm{Alg}^{\mathrm{grp}}_{\mathbb{E}_V}(\underline{\mathcal{S}})$ is an equivalence. We will use the monadicity theorem [Lur17, Cor. 4.7.3.16] applied to the following situation

$$\begin{array}{ccc} (\mathcal{S}_*^G)_{\geq V} & \xrightarrow{\quad\Omega^V\quad} & \mathrm{Alg}^{\mathrm{grp}}_{\mathbb{E}_V}(\underline{\mathcal{S}}) \\ & {}_{\Omega^V}\searrow \quad \swarrow_{\mathrm{fgt}^{\mathbb{E}_V}} & \\ & \mathcal{S}^G & \end{array} .$$

We need to verify that both functors to $\mathcal{S}^G$ are monadic and that the third functor preserves free objects.

The left hand map is monadic by Proposition 3.5. The forgetful functor $\mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathcal{S}}) \to \mathcal{S}^G$ is monadic by Proposition 2.4. Since the inclusion $\mathrm{Alg}^{\mathrm{grp}}_{\mathbb{E}_V}(\underline{\mathcal{S}}) \to \mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathcal{S}})$ does commute with all colimits by Proposition 4.1 and also is conservative right adjoint by Proposition 2.7, the right arrow in the above diagram is monadic, too.

Finally, the comparison map between free objects is an equivalence by the approximation theorem, using that the two functors to $\mathcal{S}^G$ factor through the forgetful functor $\mathcal{S}_*^G \to \mathcal{S}^G$ which admits a left adjoint, i.e. we apply Theorem 3.15 to the case where X arises from a non-based G-space by adding a disjoint base point. It follows that

$$\Omega^V\colon (\mathcal{S}_*^G)_{\geq V} \longrightarrow \mathrm{Alg}^{\mathrm{grp}}_{\mathbb{E}_V}(\underline{\mathcal{S}})$$

is an equivalence, as desired.

As limits are computed pointwise, the inclusion $(\mathcal{S}_*^G)_{\geq V} \subset \mathcal{S}_*^G$ admits a right adjoint, the V-connective cover. As S^V is V-connective, the functor $\Omega^V : \mathcal{S}_*^G \to \mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathcal{S}})$ factors through this V-connective cover functor.

It follows from composing left adjoints that $\Omega^V : \mathcal{S}_*^G \longrightarrow \mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathcal{S}})$ admits a left adjoint $B^V\colon \mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathcal{S}}) \to \mathcal{S}_*^G$ with the desired properties stated in Theorem A which is given by first group completing, then using the equivalence between group-like $\mathbb{E}_V$-algebras and V-connective based G-spaces established above and then finally regarding the resulting V-connective based G-spaces as just a based G-spaces. We will however describe this adjoint more explicitly in the upcoming Remark 4.2 ☐

Remark 4.2. Given an $\mathbb{E}_V$-algebra A, recall its bar complex $\mathrm{Bar}(A)\colon \Delta^{\mathrm{op}} \to \mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathcal{S}})$ given by resolving A by free algebras, that is

$$\mathrm{Bar}(A)_n = \left(\mathrm{Free}^{\mathbb{E}_V}\right)^n (\mathrm{fgt}^{\mathbb{E}_V} A)$$

as B^V is a left adjoint we have

$$B^V A \cong B^V \operatorname*{colim}_{\Delta^{\mathrm{op}}} \mathrm{Bar}(A) \cong \operatorname*{colim}_{\Delta^{\mathrm{op}}} B^V \,\mathrm{Bar}(A)$$

where

$$\left(B^V \,\mathrm{Bar}(A)\right)_n = \left(B^V \circ \left(\mathrm{Free}^{\mathbb{E}_V}\right)^n\right)\left(\mathrm{fgt}^{\mathbb{E}_V} A\right) \cong \Sigma^V \left(\mathrm{Free}^{\mathbb{E}_V}\right)^{n-1} (\mathrm{fgt}^{\mathbb{E}_V} A)\,,$$

that is we can compute the delooping B^V using a two-sided bar construction.

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On orthogonal factorization systems and double categories

Branko Juran

with an Appendix joint with Natalie Stewart

We prove that the ∞-category of orthogonal factorization systems embeds fully faithfully into the ∞-category of double ∞-categories. Moreover, we prove an (un)straightening equivalence for double ∞-categories, which restricts to an (un)straightening equivalence for op-Gray fibrations and curved orthofibrations of orthogonal factorization systems.

Contents

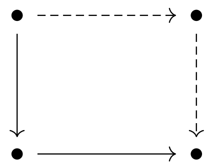
1. Introduction

Orthogonal factorization systems and double categories are very classical objects in category theory, their study goes back to work of MacLane [Mac50] and Ehresmann [Ehr63], respectively. Their ∞-categorical analogs, introduced by Joyal [Joy08] and by Haugseng [Hau13], play an equally important role in higher category theory. Both concepts deal with categorical structures equipped with two distinguished classes of morphisms. An orthogonal factorization system is a category together with the choice of two classes of morphisms, so that any morphism can uniquely be factored as a composite of one morphism from the first class, followed by one from the second. Similarly, we can think of a double category as a category with two different types of morphisms, *vertical* and *horizontal* morphisms. These two different types of morphisms cannot be composed with each other but part of the data are so-called *squares* which witness compatibilities between them. From this point of view, it seems like double categories are a generalization of orthogonal factorization systems.

The goal of this paper is to make this precise in the context of ∞-categories. We will construct a functor

$$\mathrm{Fact}\colon \mathrm{OFS} \hookrightarrow \mathrm{DCat}$$

from the ∞-category of orthogonal factorization systems into the ∞-category of double ∞-categories (Construction 3.3). It sends an orthogonal factorization system to the double ∞-category which has the objects of the underlying ∞-category as its objects, horizontal morphisms are morphisms belonging to the first class, vertical morphisms are morphisms from the second class and squares are commutative squares. The essential image of this functor consists of the double ∞-categories where there is a unique square filling every choice of a "bottom left corner":



This precisely encodes that the "wrong order" composition of morphisms from each class can uniquely be rewritten as composition in the "correct order", arguably the most important feature of an orthogonal factorization system. We will then prove the following:

Theorem A (Theorem 3.19)**.** *The functor*

$$\mathrm{Fact}\colon \mathrm{OFS} \hookrightarrow \mathrm{DCat}$$

from the ∞-category of orthogonal factorization systems into the ∞-category of double ∞-categories is fully faithful. The essential image consists of the double ∞-categories fulfilling the equivalent conditions from Proposition 3.1.

A similar result in the context of ordinary 1-categories has recently been obtained by Štěpán [Ště24].

We use Theorem A to deduce several other results about orthogonal factorization systems. Firstly, we verify that for $\mathcal{C}^\dagger$ an orthogonal factorization system, the functor Fact from Theorem A induces an equivalence between curved orthofibrations (or op-Gray fibrations, respectively) over $\mathcal{C}^\dagger$, defined in [HHLN23b], and (cocart,right)-fibrations (or (cart,right)-fibrations, respectively) over $\mathrm{Fact}(\mathcal{C}^\dagger)$, defined in [Nui24], (Proposition 4.5):

$$\mathrm{Ortho}\left(\mathcal{C}^\dagger\right) \cong \mathrm{CoR}\left(\mathrm{Fact}\left(\mathcal{C}^\dagger\right)\right) , \tag{1}$$

showing that we can regard the former as a special case of the latter. Then we prove the following (un)straightening equivalence for these fibrations:

Theorem B (Theorem 4.6)**.** *For $\mathbb{C}$ a double ∞-category, let* $\mathrm{CoR}(\mathbb{C})^{\simeq}$ *denote the space of (cocart,right)-fibrations. There is a natural equivalence*

$$\mathrm{CoR}(\mathbb{C})^{\simeq} \cong \mathrm{map}_{\mathrm{DCat}}\left((\mathbb{C})^{2\,\mathrm{op}}, \mathrm{Sq}^{\mathrm{oplax}}\left(\mathrm{Cat}_1^{(2)}\right)\right)$$

of functors from double ∞-categories to spaces. Here, $\mathrm{Sq}^{\mathrm{oplax}}\left(\mathrm{Cat}_1^{(2)}\right)$ *is the large double ∞-category having ∞-categories as its objects, functors of ∞-categories as horizontal and vertical morphisms and natural transformations as squares.*

This confirms an expectation outlined in [Nui24, Remark 2.14]. Combining (1) and Theorem B yields an (un)straightening equivalence for fibrations of orthogonal factorization systems.

Among orthogonal factorization there are those which are called *adequate* (in the sense of Barwick [Bar17]), those admitting certain pullbacks that make it possible to define the span category. We check that under the embedding from Theorem A, an orthogonal factorization system $\mathcal{C}^\dagger$ is adequate if and only if $\mathrm{Fact}(\mathcal{C}^\dagger)^{1\,\mathrm{op}}$, the opposite in the horizontal direction of the associated double ∞-category, is also contained in the essential image of Fact. In this case, there is an equivalence

$$\mathrm{Fact}\left(\mathcal{C}^\dagger\right)^{1\,\mathrm{op}} \cong \mathrm{Fact}\left(\mathrm{Span}\left(\mathcal{C}^\dagger\right)\right).$$

We hence recover the span category construction of adequate orthogonal factorization systems, which usually involves a certain amount of simplicial combinatorics. In fact we do so by computing the whole automorphism group of the ∞-category of adequate orthogonal factorization systems $\mathrm{OFS}^{\perp}$, refining [HHLN23b, Cor. 5.7]:

Theorem C (Theorem 5.5)**.** *There is an equivalence of groups*

$$\mathrm{Aut}\left(\mathrm{OFS}^{\perp}\right) \cong \mathbb{Z}/2\mathbb{Z}$$

with generator given by the span category functor.

Finally, in an appendix, which is joint with Natalie Stewart, we will use the results from this paper to construct a G-symmetric monoidal structure on the category of G-manifolds for G a compact Lie group.

Relation to 1- and 2-categorical constructions

To end this introduction, let us explain the relationship between our work and results in classical category theory.

Double categories for which a square is uniquely determined by one corner have been studied in the special cases of double groupoids. They were introduced by Mackenzie [Mac92], who was inspired by questions from Poisson geometry, under the name *vacant double groupoids*. He already showed that those double groupoids are equivalent to groupoids with a certain factorization system. Andruskiewitsch and Natale [AN05] showed that they are also equivalent to matched pairs of groupoids, a generalization of Takeuchi's matched pairs of groups [Tak81]. They also discuss how those matched pairs of groupoids give rise to weak Hopf algebras (also called quantum groupoids), generalizing Takeuchi's construction which associates a Hopf algebra (also called quantum group) to a matched pair of groups.

The category of corners, also used in this paper, first appeared in work by Weber [Web15]. Using this construction, Štěpán [Ště24] established an equivalence between factorization systems and certain double categories, which we already mentioned after Theorem A. We will now elaborate on the relation of his work and the results presented in this paper.

In [Ště24, Thm 3.7], he establishes an equivalence between strict factorization systems and double categories in which every square is uniquely (in the strict sense) determined by one corner. This can be seen as a strict 1-categorical version of Theorem A. However, even in 1-category theory, one often encounters factorization systems for which the factorization is only unique up to unique isomorphism, i.e. orthogonal factorization systems. Štěpán also proves that the 1-category of orthogonal factorization systems embeds fully faithfully into the 1-category of double 1-categories [Ště24, Thm. 3.30], but the description of the essential image becomes more complicated.

In contrast, when restricting the ∞-category OFS to the full subcategory of orthogonal factorization systems which are defined on a 1-category, one obtains the $(2,1)$-category of orthogonal factorization systems with natural equivalences as invertible 2-morphisms. Restricting Theorem A to this full subcategory then yields an embedding of this $(2,1)$-category into a certain $(2,1)$-subcategory of the ∞-category DCat.

Note however that restricting the ∞-category DCat to those double categories for which all vertical and horizontal categories are 1-categories, does not recover the classical notion of a double category, as we require the groupoid of vertical objects to agree with the groupoid of horizontal objects. For a classical double category, there is not even a natural comparison functor between those two groupoids. This difference in the definition of a double category enables us to give the straightforward description of the essential image in Definition 3.2. It would be interesting to explore those kinds of double categories in the 1-categorical setting and to study the relation of our result and Štěpán's theorem, as neither of them directly implies the other.

Organization of the paper

In Section 2, we collect some background material on double ∞-categories and orthogonal factorization systems. Section 3 is dedicated to the proof of Theorem A. In Section 4, we study fibrations, proving Theorem B. Finally, we discuss adequate orthogonal factorization systems in Section 5 and prove Theorem C.

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Conventions

- We use the theory of ∞-categories as developed by Lurie in [Lur09b] and [Lur17]. However, we make no essential use of the concrete model and the arguments should apply in most other models. Since most of the categories appearing in this paper are higher categories, we use the term *category* to refer to an ∞-category and say $(1,1)$-category if the mapping spaces are discrete.
- We fix three nested Grothendieck universes $U \in V \in W$. We call a category *very large* if it is defined in W and *large* if is defined in V. If we just say category, we assume that it is defined in U.
- We write $\mathcal{S}$ for the large category of spaces (or animae, homotopy types, ∞-groupoids, ...) and Cat_1 for the large category of categories. Moreover, we write $(-)^{\simeq}\colon \mathrm{Cat}_1 \to \mathcal{S}$ for the core functor.
- The simplex category is denoted by Δ. Its objects are finite posets $[n] = \{0, 1, \ldots, n\}$. We write $d_i\colon [n-1] \to [n]$ for the injective map omitting i and $\rho_i\colon [1] \to [n]$ for the map sending 0 to $i-1$ and 1 to i.

  We use the same notation to denote the category obtained from this poset via the inclusions $\mathrm{Poset} \hookrightarrow \mathrm{Cat}_1$.
- We use $\Lambda^2_2 = (0 \to 2 \leftarrow 1)$ for the cospan $(1,1)$-category (i.e. the 2-horn of the 2-simplex).
- For $\mathcal{C}$ a category, we write $\mathrm{Ar}(\mathcal{C}) = \mathrm{Fun}([1], \mathcal{C})$ for the *arrow category* of $\mathcal{C}$.
- We write $\mathrm{PSh}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ for the large presheaf category of a category $\mathcal{C}$.
- A full subcategory is called *reflective* if the inclusion functor admits a left adjoint. By [Lur09b, Prop. 5.5.4.15], a subcategory of a large presentable category is reflective if it can be characterized as the class of local objects with respect to a small set of morphisms.

2. Preliminaries

Definition 2.1. A *simplicial space* is a presheaf on Δ, i.e. a functor

$$X\colon \Delta^{\mathrm{op}} \longrightarrow \mathcal{S}\,.$$

We call it a *Segal space* if

$$\prod_{i=1}^{n} \rho_i \colon X_n \longrightarrow X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

is an equivalence.

We call a Segal space *complete* if

$$\begin{array}{ccc} X_0 & \xrightarrow{\Delta} & X_0 \times X_0 \\ \downarrow & & \downarrow \\ X_3 & \longrightarrow & X_1 \times X_1 \end{array}$$

is a pullback. Here, the vertical morphisms are induced by the unique degeneracy map to $[0]$ and the bottom map is the one induced by the inclusions $[1] \to [3]$ onto $\{0, 2\}$ and $\{1, 3\}$, respectively.

Remark 2.2. Let X be a Segal space and let $x, y \in X_0$. Let us write

$$\mathrm{map}_X(x, y) = \{x\} \times_{X_0, d_1} X_1 \times_{d_0, X_0} \{y\} \ .$$

Using the higher Segal conditions, one can define composition maps. It is proved in [Rez01] that a Segal space is complete if and only if the degeneracy map

$$X_0 \longrightarrow X_1$$

is fully faithful with essential image being the equivalences, i.e. the maps $f \colon x \to y$ such that composition with f induces equivalences

$$f_* \colon\ \mathrm{map}_X(z, x) \longrightarrow \mathrm{map}(z, y)$$

and

$$f^* \colon\ \mathrm{map}_X(y, z) \longrightarrow \mathrm{map}_X(x, z)$$

for every object z of X_0.

Theorem 2.3 (Joyal-Tierney [JT07])**.** *The restricted Yoneda embedding*

$$\begin{aligned} \mathrm{Cat}_1 &\longrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S}) \\ \mathcal{C} &\longmapsto \mathrm{map}([n], \mathcal{C}) \end{aligned}$$

is fully faithful with essential image the complete Segal spaces.

The idea of the above theorem goes back to Rezk [Rez01]. A simple proof of the above theorem can be found in [HS23].

Before turning towards the definition of a double category, we recall the following:

Proposition 2.4 ([Bri18, Prop. 1.7])**.** *Let $F\colon \mathcal{D} \to \mathcal{C}$ be a map of Segal spaces. The following are equivalent:*

- *The square*

$$\begin{array}{ccc} \mathcal{D}_n & \xrightarrow{F_n} & \mathcal{C}_n \\ \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d_0} \\ \mathcal{D}_0 & \xrightarrow{F_0} & \mathcal{C}_0 \end{array}$$

  is cartesian for all $n \geq 1$.

- *The above square is cartesian for $n = 1$.*

In this case, the map is called a *right fibration*. The dual notion (replacing d_0 by d_n) is called *left fibration*.

Definition 2.5. A *double category* is a bisimplicial space

$$\mathbb{C}\colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \longrightarrow \mathcal{S}$$
$$([m]\,, [n]) \longmapsto \mathbb{C}(m, n)$$

such that for all $n \geq 0$, $\mathbb{C}(n, -)$ and $\mathbb{C}(-, n)$ are complete Segal spaces.

We write $\mathrm{DCat} \subset \mathrm{PSh}(\Delta \times \Delta)$ for the large full subcategory of double categories.

Remark 2.6. This definition is not used consistently in the literature. We follow the terminology used in [Nui24]. Some authors only require $\mathbb{C}(-, n)$ to be a (not necessarily complete) Segal space.

In particular, our definition is not the direct higher categorical analog of the classical notion of a double category. In our definition, the space of objects in the vertical and horizontal category must agree.

Notation 2.7. We write

$$(-)^{1\,\mathrm{op}}, (-)^{2\,\mathrm{op}}\colon \mathrm{DCat} \longrightarrow \mathrm{DCat}$$

for the functors precomposing with the opposite functor $(-)^{\mathrm{op}}\colon \Delta \to \Delta$ in the first or second coordinate, respectively, and

$$(-)^{\mathrm{swap}}\colon \mathrm{DCat} \longrightarrow \mathrm{DCat}$$

for the functor which exchanges the two coordinates.

Construction 2.8. Given two categories $\mathcal{C}$ and $\mathcal{D}$, we obtain a double category $\mathcal{C} \boxtimes \mathcal{D}$ as follows:

$$(\mathcal{C} \boxtimes \mathcal{D})(k, l) = \mathrm{map}([k]\,, \mathcal{C}) \times \mathrm{map}([l]\,, \mathcal{D})$$

Definition 2.9. A double category if called a 2-*category* if $\mathbb{C}(0, -)$ is a constant simplicial space. We write $\mathrm{Cat}_2 \subset \mathrm{DCat}$ for the large full subcategory of double categories.

Remark 2.10. We may regard a strict $(2,2)$-category, i.e. a category enriched in small $(1,1)$-categories, as a 2-category as follows: Taking nerves on mapping spaces, we obtain a category enriched in simplicial spaces from which we obtain a 2-category using the equivalence of models for $(\infty,2)$-categories from [Lur09a, Thm. 0.0.4]. We will only make use of this embedding when we use the Gray tensor product of $(1,1)$-categories in the upcoming Construction 2.11 and the only thing we need to know about this embedding is that the Gray tensor product of $[m]$ and $[n]$ agrees with the one used in [HHLN23a] which is ensured by [HHLN23a, Prop. 5.1.9].

Construction 2.11. Restricting the Yoneda embedding along the bicosimplicial object

$$\begin{aligned} \Delta \times \Delta &\longrightarrow \mathrm{Cat}_2 \\ ([m]\,,[n]) &\longmapsto [n] \otimes [m] \end{aligned}$$

sending $([m]\,,[n])$ to the Gray tensor product $[n] \otimes [m]$ (a strict $(2,2)$-category), induces the *oplax square functor*:

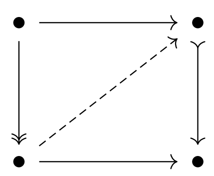
$$\begin{aligned} \mathrm{Sq}^{\mathrm{oplax}} \colon \mathrm{Cat}_2 &\longrightarrow \mathrm{DCat} \\ \mathbb{C} &\longmapsto (([m]\,,[n]) \mapsto \mathrm{Fun}_{\mathrm{Cat}_2}([n] \otimes [m]\,,\mathbb{C})) \end{aligned}$$

Here, $\mathrm{Sq}^{\mathrm{oplax}}(\mathbb{C})(n,-)$ is indeed complete Segal as the Gray tensor product preserves colimits in each variable separately.

Next we discuss orthogonal factorization systems, which we will simply call factorization system because there is no other notion of factorization systems studied in this paper.

Definition 2.12 (Joyal)**.** A category $\mathcal{C}$ together with two subcategories $\mathcal{C}_{\mathrm{eg}}$ and $\mathcal{C}_{\mathrm{in}}$ is called an *factorization system* if

- The subcategories $\mathcal{C}_{\mathrm{eg}}$ and $\mathcal{C}_{\mathrm{in}}$ contain every equivalence and
- for any commutative square



in which the left morphism is contained in $\mathcal{C}_{\mathrm{eg}}$ and the right morphism is $\mathcal{C}_{\mathrm{in}}$, there is a unique dashed filler.

The morphisms in $\mathcal{C}_{\mathrm{eg}}$ are called *egressive*, the ones in $\mathcal{C}_{\mathrm{in}}$ *ingressive*. The category of factorization systems OFS is defined as the subcategory of the category of functors $\mathrm{Fun}(\Lambda^2_2, \mathrm{Cat}_1)$ from the cospan $\Lambda^2_2 = (0 \to 2 \leftarrow 1)$ to Cat_1 where $0 \to 2$ and $2 \leftarrow 1$ are sent to the inclusion of the class of egressive (and ingressive, respectively) morphisms of a factorization system.

Remark 2.13. In [Lur09b, Def. 5.2.8.8], Lurie also requires the two classes of morphisms two be closed under retracts. This condition is redundant by [GKT18, Sec. 1.1].

Notation 2.14. Given two categories $\mathcal{C}$ and $\mathcal{D}$, there is a factorization system $\mathcal{C}\bar{\times}\mathcal{D}$ on $\mathcal{C} \times \mathcal{D}$ with $(\mathcal{C}\bar{\times}\mathcal{D})_{\mathrm{eg}} = \mathcal{C} \times \mathcal{D}^{\simeq}$ and $(\mathcal{C}\bar{\times}\mathcal{D})_{\mathrm{in}} = \mathcal{C}^{\simeq} \times \mathcal{D}$.

Proposition 2.15 ([BS24, Prop. A.0.4]). *Let $\mathcal{C}_{\mathrm{eg}}, \mathcal{C}_{\mathrm{in}} \subset \mathcal{C}$ be subcategories containing every equivalence. The following are equivalent:*

- *The triple forms a factorization system.*
- *The restricted composition map*

$$\begin{aligned} \mathrm{Ar}\,(\mathcal{C}_{\mathrm{eg}})^{\simeq} \times_{\mathcal{C}^{\simeq}} \mathrm{Ar}\,(\mathcal{C}_{\mathrm{in}})^{\simeq} &\longrightarrow \mathrm{Ar}\,(\mathcal{C})^{\simeq} \\ (f, g) &\longmapsto g \circ f \end{aligned}$$

  is an equivalence.

3. Double categories and factorization systems

In this section, we will first define a subcategory of *factorization double categories* $\mathrm{DCat}^{\mathrm{OF}} \subset \mathrm{DCat}$ (Definition 3.2). We will construct the functor

$$\mathrm{Fact}\colon\ \mathrm{OFS} \longrightarrow \mathrm{DCat}^{\mathrm{OF}}$$

in Construction 3.3 and Lemma 3.4. Then, we construct its inverse functor

$$\mathrm{Cnr}\colon\ \mathrm{DCat}^{\mathrm{OF}} \longrightarrow \mathrm{OFS}$$

in Construction 3.14. Afterwards, we will explicitly define unit and counit transformations and check that they are equivalences, proving Theorem 3.19.

Proposition 3.1. *For $\mathbb{C}$ a double category, the following are equivalent*

1. *The square*

$$\begin{array}{ccc} \mathbb{C}(1,1) & \xrightarrow{\mathbb{C}(\mathrm{id},d_0)} & \mathbb{C}(1,0) \\ {\scriptstyle \mathbb{C}(d_1,\mathrm{id})}\downarrow & & \downarrow{\scriptstyle \mathbb{C}(d_1,\mathrm{id})} \\ \mathbb{C}(0,1) & \xrightarrow{\mathbb{C}(\mathrm{id},d_0)} & \mathbb{C}(0,0) \end{array}$$

   is cartesian.

2. *The functor*

$$\mathbb{C}(-, d_0)\colon \mathbb{C}(-,1) \longrightarrow \mathbb{C}(-,0)$$

   is a left fibration.

3. *The functor*

$$\mathbb{C}(d_1, -) \colon \mathbb{C}(1, -) \longrightarrow \mathbb{C}(0, -)$$

is a right fibration.

4. *The functor*

$$\mathbb{C}(-, d_0) \colon \mathbb{C}(-, n) \longrightarrow \mathbb{C}(-, 0)$$

is a left fibration for all $n \geq 0$*.*

5. *The functor*

$$\mathbb{C}(d_n, -) \colon \mathbb{C}(n, -) \longrightarrow \mathbb{C}(0, -)$$

is a right fibration for all $n \geq 0$*.*

Proof. The first condition is the weakest. It implies the second and third condition by Proposition 2.4. The second condition implies the fourth by evaluating at n and using Proposition 2.4. Similarly, the third condition implies the fifth. □

Definition 3.2. If a double category satisfies one of the equivalent conditions from Proposition 3.1, we call it a *factorization double category.* We write $\mathrm{DCat}^{\mathrm{OF}} \subset \mathrm{DCat}$ for the full subcategory spanned by the factorization double categories.

Construction 3.3. Restricting the Yoneda embedding along the bicosimplicial object

$$\begin{aligned} \Delta \times \Delta &\longrightarrow \mathrm{OFS} \\ ([m], [n]) &\longrightarrow [m] \bar{\times} [n] \end{aligned}$$

yields a functor

$$\mathrm{Fact} \colon \mathrm{OFS} \longrightarrow \mathrm{PSh}(\Delta \times \Delta)\,.$$

Lemma 3.4. *The functor* Fact *takes values in factorization double categories.*

Proof. Let $\mathcal{C}^\dagger = (\mathcal{C}, \mathcal{C}_{\mathrm{eg}}, \mathcal{C}_{\mathrm{in}})$ be a factorization system. The simplicial space $\mathrm{Fact}(n, -)$ is a complete Segal space by Theorem 2.3: It arises as the Rezk nerve of the (non-full) subcategory of $\mathrm{Fun}([n], \mathcal{C})$ spanned by the functors sending each morphism of $[n]$ to a morphism in $\mathcal{C}_{\mathrm{eg}}$ and by the natural transformations which (pointwise) take values in $\mathcal{C}_{\mathrm{in}}$. Analogously, $\mathrm{Fact}(-, n)$ is a complete Segal space.

The category $[1] \times [1]$ is the pushout of two copies of $[2]$ along $[1]$. Mapping out of this pushout, passing to subspaces and also using that $[2]$ is a pushout of $[1]$ and $[1]$ along $[0]$ yields the following pullback:

$$\begin{array}{ccc} \mathrm{map}_{\mathrm{OFS}}([1] \bar{\times} [1], \mathcal{C}^\dagger) & \xrightarrow{(\mathrm{id} \times d_1), (d_0 \times \mathrm{id})} & \mathrm{Ar}(\mathcal{C}_{\mathrm{eg}})^{\simeq} \times_{\mathcal{C}^{\simeq}} \mathrm{Ar}(\mathcal{C}_{\mathrm{in}})^{\simeq} \\ {\scriptstyle (d_1 \times \mathrm{id}), (\mathrm{id} \times d_0)} \big\downarrow & & \big\downarrow {\scriptstyle \circ} \\ \mathrm{Ar}(\mathcal{C}_{\mathrm{in}})^{\simeq} \times_{\mathcal{C}^{\simeq}} \mathrm{Ar}(\mathcal{C}_{\mathrm{eg}})^{\simeq} & \xrightarrow{\quad \circ \quad} & \mathrm{Ar}(\mathcal{C})^{\simeq} \end{array}$$

The right hand map is an equivalence by Proposition 2.15, implying that the left map is an equivalence too. Unwinding definitions, this shows that $\mathrm{Fact}(\mathcal{C}^\dagger)$ is a factorization double category. □

Example 3.5. Using that the projection $[k] \times [m] \to [k]$ is a localization at the class of morphisms which are constant in $[k]$, we find that for two categories $\mathcal{C}$ and $\mathcal{D}$, there is a natural equivalence

$$\mathrm{Fact}(\mathcal{C} \bar{\times} \mathcal{D}) \cong \mathcal{C} \boxtimes \mathcal{D} \,.$$

Remark 3.6. Actually, the above construction makes sense more generally: One can assign a double category to any category equipped with two subcategories containing every equivalence. The resulting double category is a factorization double category if and only if the subcategories form a factorization system.

Remark 3.7. It follows from [CH21, Prop. 5.2] together with [RS22, Thm. 1.1] that OFS is presentable. The functor

$$\mathrm{Fact}\colon \mathrm{OFS} \longrightarrow \mathrm{DCat}$$

therefore admits a left adjoint. However, this left adjoint uses colimits in the category of factorization systems, which are hard to compute. We will therefore not show directly that the counit of the adjuntion is an equivalence. Instead, we will explicitly construct an inverse functor defined on the subcategory of factorization double categories. It follows a posteriori that this inverse functor is the restriction of the left adjoint obtained by the adjoint functor theorem.

We will now turn towards the definition of the inverse functor $\mathrm{Cnr}\colon \mathrm{DCat}^{\mathrm{OF}} \to \mathrm{OFS}$.

Construction 3.8. For $\mathcal{C}$ a category, the arrow category $\mathrm{Ar}\,(\mathcal{C})$ admits the structure of a factorization system with egressive morphisms the ones which are an equivalence in the source and ingressive morphisms the ones which are an equivalence in the target, see [HHLN23b, Exa. 4.7]. We write

$$\mathbb{A}\mathrm{r}\,(\mathcal{C}) = \mathrm{Fact}(\mathrm{Ar}\,(\mathcal{C}))$$

for the *double arrow category.*

The combined source and target functor $(t, s)\colon \mathrm{Ar}\,(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ enhances to a functor of factorization systems to $\mathcal{C} \bar{\times} \mathcal{C}$. This induces a functor

$$(t, s)\colon \mathbb{A}\mathrm{r}\,(\mathcal{C}) \longrightarrow \mathrm{Fact}(\mathcal{C} \bar{\times} \mathcal{C}) \cong \mathcal{C} \boxtimes \mathcal{C} \,.$$

Postcomposing with the map $\mathcal{C} \to [0]$ gives maps

$$s\colon \mathbb{A}\mathrm{r}\,(\mathcal{C}) \longrightarrow [0] \boxtimes [\mathcal{C}]$$

and

$$t\colon \mathbb{A}\mathrm{r}\,(\mathcal{C}) \longrightarrow [\mathcal{C}] \boxtimes [0] \,.$$

Remark 3.9. It follows from the comparison of thin and fat joins, [Lur09b, Prop. 4.2.1.2] that there is a natural map $[m] \times [n] \times [1] \longrightarrow [n] \star [m]$ which exhibits the target as the localization of the source at the morphisms which are either constantly 0 in the [1]-coordinate and constant in the $[n]$-coordinate or constantly 1 and constant in the $[m]$-coordinate. Unwinding definitions, we find that

$$\mathbb{A}\mathrm{r}\,(\mathcal{C})\,(m,n) \cong \mathrm{map}([n] \star [m]\,, \mathcal{C})\,,$$

which is the definition used in [Nui24].

Construction 3.10. Restricting the Yoneda embedding along the cosimplicial object

$$\begin{aligned} \Delta &\longrightarrow \mathrm{DCat} \\ [n] &\longmapsto \mathbb{A}\mathrm{r}\,([n]) \end{aligned}$$

yields a functor:

$$\begin{aligned} \mathrm{cnr}\colon \mathrm{DCat} &\longrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S}) \\ \mathbb{C} &\longmapsto (n \mapsto \mathrm{map}_{\mathrm{DCat}}(\mathbb{A}\mathrm{r}\,([n])\,, \mathbb{C})) \end{aligned}$$

We now want to check that, when restricted to factorization double categories, this functor takes values in complete Segal spaces and that it enhances to a functor to factorization systems.

Construction 3.11. For fixed $n \geq 2$, let P be the poset of subsets of $[n] = \{0, \ldots, n\}$ which are either a singleton $\{i\}$ for $1 \leq i \leq n-1$ or a two-element subset containing two consecutive elements. Using that all these subsets are linearly ordered posets and all the inclusions preserve the order, we obtain a functor $p\colon P \to \Delta$ and the inclusion of the subsets into $[n]$ induces a natural transformation $p \Rightarrow \mathrm{const}\,[n]$. Let $\mathcal{C}$ be a category with finite colimits and let $X\colon \Delta \to \mathcal{C}$ be a cosimplicial object in $\mathcal{C}$. We write $I_n\,(X) = \mathrm{colim}_P(X \circ p)$ for the colimit of the composite functor. The natural transformation induces a morphism $I_n\,(X) \to X(n)$, which we call the spine inclusion.

If $X\colon \Delta \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S})$ is the Yoneda embedding, we recover the inclusion of simplicial sets usually referred to as the "spine inclusion". We will also simply denote this spine by I_n. In this case, we can also describe $I_n \to [n]$ more explicitly as the inclusion of the simplicial subset spanned by those 1-simplices $[1] \to [n]$ which increase by at most 1. The local objects with respect to those morphisms are precisely the Segal spaces.

Lemma 3.12. *The saturation (under pushouts and retracts) of the following two sets of functors of double categories agree:*

- *the one morphism*

$$(d_1, \mathrm{id}), (\mathrm{id}, d_0)\colon\ ([0] \boxtimes [1]) \cup_{[0]\boxtimes[0]} ([1] \boxtimes [0]) \longrightarrow [1] \boxtimes [1]\ . \tag{2}$$

- *the spine inclusions*

$$I_n\,(\mathbb{A}\mathrm{r}\,(-)) \longrightarrow \mathbb{A}\mathrm{r}\,(n) \tag{3}$$

  for $n \geq 2$.

Proof. It is easy to see that the map (2) is a retract of the map (3) for $n = 2$. We leave the details to the reader, since we will not make use of this direction in the rest of the paper.

For the other direction, we will proceed as follows:

- We will define a certain bicosimplicial subspace $P \subset \mathbb{A}\mathrm{r}([n])$ such that the inclusion is sent to an equivalence under the localization functor $\mathrm{PSh}(\Delta \times \Delta) \to \mathrm{DCat}$.
- We will show that P can be obtained from the source of (3) via iterated pushouts of (2) in $\mathrm{PSh}(\Delta \times \Delta)$.

The claim follows because the localization $\mathrm{PSh}(\Delta \times \Delta) \to \mathrm{DCat}$ preserves colimits.

Let $P \subset \mathbb{A}\mathrm{r}([n])$ be the bicosimplicial subspace spanned by the elements

$$\mathbb{A}\mathrm{r}([n])(k, m) = \mathrm{map}([m] \star [k], [n])$$

for which every morphism in the image of the natural inclusions $[m] \to [m] \star [k]$ and $[k] \to [m] \star [k]$ is sent to a morphism in $[n]$ which increases by at most 1. Let $Q \subset \mathbb{A}\mathrm{r}([n])$ be the bicosimplicial subspace spanned by all the elements where this holds just for the morphisms in the image of the natural inclusions $[m] \to [m] \star [k]$.

Let $\mathrm{map}^{\leq 1}([m], [n]) \subset \mathrm{map}([m], [n])$ denote the subset of maps where each morphism in $[m]$ is sent to a morphism in $[n]$ which increases by at most 1.

Every map $\alpha\colon [m] \to [n]$ gives an element in

$$\mathrm{Fact}(\mathrm{Ar}([n]))(n - \alpha(m), m) = \mathrm{map}([m] \star [n - \alpha(m)], [n])$$

whose restriction to $[m]$ is α and whose restriction to $[n - \alpha(m)]$ is the inclusion of the last $n - \alpha(m) + 1$ elements. This induces equivalences

$$\mathbb{A}\mathrm{r}([n])(-, m) \cong \coprod_{\alpha \in \mathrm{map}([m],[n])} [n - \alpha(m)]$$

and similarly

$$\mathbb{A}\mathrm{r}([n])(k, -) \cong \coprod_{\alpha \in \mathrm{map}([k],[n])} [\alpha(0)] ,$$

inducing equivalences on simplicial subspaces

$$Q(-, m) \cong \coprod_{\alpha \in \mathrm{map}^{\leq 1}([m],[n])} [n - \alpha(m)] ,$$

$$Q(k, -) \cong \coprod_{\alpha \in \mathrm{map}([k],[n])} I_{\alpha(0)}$$

and

$$P(-, m) \cong \coprod_{\alpha \in \mathrm{map}^{\leq 1}([m],[n])} I_{n-\alpha(m)}$$

From this description it follows that the inclusion $P \to Q$ exhibits the target as the localization of the source with respect to the reflective inclusion

$$\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Cat}_1) \longrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{PSh}(\Delta)) \cong \mathrm{PSh}(\Delta \times \Delta)$$

and so is $Q \to \mathbb{A}\mathrm{r}([n])$ for the other coordinate.

But both these inclusions are intermediate inclusions of $\mathrm{DCat} \subset \mathrm{PSh}(\Delta\times\Delta)$, it follows that the inclusion $P \to \mathbb{A}\mathrm{r}([n])$ is sent to an equivalence by the localization functor to double categories.

For $0 \leq j \leq i \leq n-1$, let $P^{i,j}$ denote the bicosimplicial subspace of P spanned by all objects $\alpha\colon [1] \to [n]$ of

$$P(0,0) = \mathbb{A}\mathrm{r}([n])(0,0) = \mathrm{map}([0] \star [0], [n])$$

such that $\alpha(1) - \alpha(0) \leq 1$, or $\alpha(1) \leq i$, or $\alpha(1) = i+1$ and $\alpha(0) \geq j$. We get a nested sequence

$$P^{1,1} \subset P^{1,0} = P^{2,2} \subset P^{2,1} \subset P^{2,0} = P^{3,3} \subset \ldots P^{n-1,0} = P\,,$$

see Figure 1 for an example. Note that the inclusion $P^{1,1} \subset \mathbb{A}\mathrm{r}([n])$ is equivalent to the spine inclusion $I_n(\mathbb{A}\mathrm{r}(-)) \longrightarrow \mathbb{A}\mathrm{r}(n)$. This can be seen using that colimits in $\mathrm{PSh}(\Delta \times \Delta)$ are computed pointwise and the colimit used to define $I_n(\mathbb{A}\mathrm{r}(-))$ gives a bicosimplicial space which already is a double category.

$$\begin{array}{ccccccc}
00 & \longrightarrow & 01 & \longrightarrow & 02 & \longrightarrow & 03 \\
 & & \downarrow & P^{1,0} & \downarrow & P^{2,0} & \downarrow \\
 & & 11 & \longrightarrow & 12 & \longrightarrow & 13 \\
 & & & & \downarrow & P^{2,1} & \downarrow \\
 & & & & 22 & \longrightarrow & 23 \\
 & & & & & & \downarrow \\
 & & & & & & 33
\end{array}$$

Figure 1: $\mathbb{A}\mathrm{r}([3])$, the squares are labeled by the smallest bicosimplicial subspace $P^{i,j}$ in which they are contained

Again using that pushouts in $\mathrm{PSh}(\Delta \times \Delta)$ are computed pointwise, one verifies that the following square is a pushout for all $1 \leq j+1 \leq i \leq n-1$:

$$\begin{array}{ccc}
([0] \boxtimes [1]) \cup_{[0]\boxtimes[0]} ([1] \boxtimes [0]) & \longrightarrow & P^{i,j+1} \\
\big\downarrow & & \big\downarrow \\
[1] \boxtimes [1] & \longrightarrow & P^{i,j}
\end{array}$$

where the lower map is the map classifying the element in

$$P^{i,j}(1,1) \subset \mathbb{A}\mathrm{r}([n])(1,1) = \mathrm{map}([1] \star [1], [n])$$

restricting to the inclusion of $\{j, j+1\}$ on the first copy of $[1]$ and the inclusion of $\{i, i+1\}$ in the second. In fact, in this case the diagram above is pointwise a diagram of sets. Moreover, an explicit combinatorial argument shows that pointwise all maps in the above square are mapped to inclusions of sets and the upper left corner is precisely the intersection of the two inclusion into the lower right corner. We leave the details to the reader.

This finishes the proof of the claim that P can be obtained from $P^{1,1}$ via iterated pushouts of (2). □

Proposition 3.13. *The simplicial space* $\mathrm{cnr}(\mathbb{C})$ *is a complete Segal space if and only if* $\mathbb{C}$ *is a factorization double category.*

Proof. A double category $\mathbb{C}$ is a factorization category if it is local with respect to the morphism (2). Unwinding definitions, $\mathrm{cnr}(\mathbb{C})$ is a Segal space if and only if $\mathbb{C}$ is local with respect to the morphisms (3). Therefore, the Segal claim follows from Lemma 3.12. It is left to show that in this case the Segal space is automatically complete. We will use the criterion from Remark 2.2.

First note that degeneracy map

$$\mathrm{cnr}(\mathbb{C})_0 = \mathbb{C}(0,0) \cong \mathbb{C}(0,0) \times_{\mathbb{C}(0,0)} \mathbb{C}(0,0) \longrightarrow \mathbb{C}(1,0) \times_{\mathbb{C}(0,0)} \mathbb{C}(0,1) = \mathrm{cnr}(\mathbb{C})_1$$

is an inclusion of path components because it is a pullback of such.

We need to show that every equivalence is contained in the essential image, which consists of the objects in $\mathbb{C}(1,0) \times_{\mathbb{C}(0,0)} \mathbb{C}(0,1) = \mathrm{cnr}(\mathbb{C})_1$ which are equivalences in both components.

Let $c, d \in \mathbb{C}(0,0) = \mathrm{cnr}(\mathbb{C})_0$ be two objects and

$$(f,g) \in \left(\mathbb{C}(-,0)_{c/}\right)^{\simeq} \times_{\mathbb{C}(0,0)} \left(\mathbb{C}(0,-)_{/d}\right)^{\simeq} = \mathrm{map}_{\mathrm{cnr}(\mathbb{C})}(c,d)\,.$$

be an equivalence.

Then postcomposition with (f,g) induces an equivalence

$$\mathrm{map}_{\mathrm{cnr}(\mathbb{C})}(d,c) \longrightarrow \mathrm{map}_{\mathrm{cnr}(\mathbb{C})}(d,d)$$

Picking an inverse of the identity and unwinding composition in $\mathrm{cnr}(\mathbb{C})$, we see that g has a left inverse. Similarly, one can show that it has a left inverse and so does f. □

Construction 3.14. Proposition 3.13 together with Theorem 2.3 implies that we obtain a functor

$$\mathrm{cnr}\colon \mathrm{DCat}^{\mathrm{OF}} \longrightarrow \mathrm{Cat}_1\,.$$

We now refine this to a functor

$$\mathrm{Cnr}\colon \mathrm{DCat}^{\mathrm{OF}} \longrightarrow \mathrm{OFS}\,,$$

which assigns to a factorization double category its *category of corners*.

The target functor

$$\mathbb{A}\mathrm{r}\left([m]\right) \longrightarrow [m] \boxtimes [0]$$

gives rise to a natural functor

$$\mathbb{C}(-,0) \longrightarrow \mathrm{cnr}(\mathbb{C})$$

which induces an equivalence on cores and an inclusion of path components

$$\mathbb{C}(1,0) \cong \mathbb{C}(1,0) \times_{\mathbb{C}(0,0)} \mathbb{C}(0,0) \longrightarrow \mathbb{C}(1,0) \times_{\mathbb{C}(0,0)} \mathbb{C}(0,1) = \mathrm{Ar}\left(\mathrm{cnr}(\mathbb{C})\right)^{\simeq}$$

on cores of arrow categories.

Similarly, we obtain a natural functor $\mathbb{C}(0,-) \to \mathrm{cnr}(\mathbb{C})$. The triple

$$\mathrm{Cnr}(\mathbb{C}) = (\mathrm{cnr}(\mathbb{C}), \mathbb{C}(-,0), \mathbb{C}(0,-))$$

forms a factorization system by Proposition 2.15.

Construction 3.15. There are natural equivalences

$$\mathrm{map}([k]\,, [m] \times [n]) \cong \mathrm{map}([k]\,, [m]) \times \mathrm{map}([k]\,, [n]) \cong \mathrm{map}([k] \boxtimes [k]\,, [m] \boxtimes [n])$$

Precomposition with $(t,s)\colon \mathbb{A}\mathrm{r}\left([k]\right) \to [k] \boxtimes [k]$ from Construction 3.8 hence induces a natural functor

$$[m] \times [n] \longrightarrow \mathrm{cnr}([m] \boxtimes [n])$$

and this functor enhances to a functor of factorization systems

$$[m] \mathbin{\bar{\times}} [n] \longrightarrow \mathrm{Cnr}([m] \boxtimes [n])\,.$$

We hence obtain an element in

$$\mathrm{Fact}(\mathrm{Cnr}([m] \boxtimes [n]))(m,n) = \mathrm{map}_{\mathrm{DCat}}([m] \boxtimes [n]\,, \mathrm{Fact}(\mathrm{Cnr}([m] \boxtimes [n])))\,.$$

Applying $\mathrm{Fact} \circ \mathrm{Cnr}$ and precomposition with this morphism gives a natural map

$$\mathrm{map}_{\mathrm{DCat}}([m] \boxtimes [n]\,, \mathbb{C}) \longrightarrow \mathrm{map}_{\mathrm{DCat}}([m] \boxtimes [n]\,, \mathrm{Fact}(\mathrm{Cnr}(\mathbb{C}))) \tag{4}$$

and hence a natural transformation

$$\mathrm{id}_{\mathrm{DCat}^{\mathrm{OF}}} \Longrightarrow \mathrm{Fact} \circ \mathrm{Cnr}\ .$$

Proposition 3.16. *The above natural transformation is a natural equivalence.*

Proof. We need to prove that (4) is an equivalence for all m, n and $\mathbb{C}$. By the Segal condition, it is enough to check this for $m, n \leq 1$. Because both sides are factorization double categories, we can also exclude the case $m = n = 1$. In the remaining three cases, one verifies that the map is an equivalence by unwinding the definitions, e.g. in the case $m = 1, n = 0$ the map identifies with the natural map from $\mathbb{C}(1,0)$ into the core of the $\mathrm{Ar}\left(\mathrm{Cnr}(\mathbb{C})_{\mathrm{eg}}\right)$, which was defined to be $\mathbb{C}(1,0)$. □

Construction 3.17. Let $\mathcal{C}^\dagger$ be a factorization system on a category $\mathcal{C}$. There is a natural map

$$\mathrm{map}_{\mathrm{OFS}}\left(\mathrm{Ar}\left([n]\right), \mathcal{C}^\dagger\right) \longrightarrow \mathrm{map}_{\mathrm{Cat}_1}\left([n], \mathcal{C}\right)$$

given by passing to the functor of underlying categories and then embedding $[n]$ into $\mathrm{Ar}\left([n]\right)$ by sending each element to the identity arrow. It follows from [HHLN23b, Prop. 4.8] that this map is an equivalence. The result is stated for orthogonal adequate triples but the proof makes no essential use of the adequate triple property.

Applying the functor Fact yields a map

$$\mathrm{map}_{\mathrm{OFS}}\left(\mathrm{Ar}\left([n]\right), \mathcal{C}^\dagger\right) \longrightarrow \mathrm{map}_{\mathrm{DCat}}\left(\mathbb{A}\mathrm{r}\left([n]\right), \mathrm{Fact}\left(\mathcal{C}^\dagger\right)\right) .$$

Combining these two map yields a natural functor

$$\mathcal{C} \longrightarrow \mathrm{cnr}\left(\mathrm{Fact}\left(\mathcal{C}^\dagger\right)\right) .$$

Proposition 3.18. *The above natural transformation refines to a natural transformation of automorphisms of the category of factorization systems*

$$\mathrm{id}_{\mathrm{OFS}} \Longrightarrow \mathrm{Cnr} \circ \mathrm{Fact}$$

and this natural transformation is a natural equivalence.

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{map}_{\mathrm{OFS}}\left(([1], [1], [1]^{\simeq}), \mathcal{C}^\dagger\right) & \xrightarrow{\mathrm{Fact}} & \mathrm{map}_{\mathrm{DCat}}\left([1] \boxtimes [0], \mathrm{Fact}\left(\mathcal{C}^\dagger\right)\right) \\ \big\downarrow t^* & & \big\downarrow t^* \\ \mathrm{map}_{\mathrm{OFS}}\left(\mathrm{Ar}\left([1]\right), \mathcal{C}^\dagger\right) & \xrightarrow{\mathrm{Fact}} & \mathrm{map}_{\mathrm{DCat}}\left(\mathbb{A}\mathrm{r}\left([1]\right), \mathrm{Fact}\left(\mathcal{C}^\dagger\right)\right) \end{array} .$$

The bottom arrow is the one used Construction 3.17 to define the functor on cores of arrow categories $\mathrm{Ar}\left(\mathcal{C}^\dagger\right)^{\simeq} \to \mathrm{Ar}\left(\mathrm{Cnr}(\mathrm{Fact}(\mathcal{C}^\dagger))\right)^{\simeq}$, and the vertical arrows induce the inclusion of the class of egressive morphisms in $\mathcal{C}^\dagger$ and $\mathrm{Cnr}(\mathrm{Fact}(\mathcal{C}^\dagger))$, respectively. The top arrow is an equivalence, it can be identified with the identity functor on the subspace of $\mathrm{map}_{\mathrm{Cat}_1}\left([1], \mathcal{C}\right)$ spanned by the functors sending the morphism to an egressive morphism.

This shows that the functor in question preserves the class of egressive morphisms. Analogously, one proves that it preserves ingressive morphisms. The argument actually shows that it induces an equivalence on the space of the egressive (or ingressive, respectively) morphisms. Moreover, it follows from the definitions that it induces an equivalence on cores. It follows from Proposition 2.15 that the functor is an equivalence of factorization systems. □

We can now deduce the main theorem:

Theorem 3.19. *The functor* Fact *induces an equivalence of categories*

$$\mathrm{OFS} \cong \mathrm{DCat}^{\mathrm{OF}}$$

Proof. This follows from Proposition 3.16 and Proposition 3.18. □

This is an ∞-categorical analog of [Ště24, Theorem 3.7].

4. Fibrations

In this section, we will first recall the definition of *op-Gray fibrations* (and *curved orthofibrations*) of factorization systems as well as *(cart,right)-fibrations* (and *(cocart,right)-fibrations*) of double categories. We will then verify in Proposition 4.5 that the two notions agree under the equivalence between factorization systems and factorization double categories from the previous section. In Theorem 4.6, we will prove an (un)straightening equivalence for those fibrations.

Definition 4.1 ([HHLN23b, Def. 5.14])**.** A functor $F\colon \mathcal{D}^\dagger \to \mathcal{C}^\dagger$ of factorization systems is called an *ingressive cartesian fibration* if ingressive morphisms admit F-cartesian lifts and those lifts precisely make up the subcategory of ingressive morphisms in $\mathcal{D}^\dagger$.

A functor $F\colon \mathcal{D}^\dagger \to \mathcal{C}^\dagger$ of factorization systems is called

- a *curved orthofibration* if it is an ingressive cartesian fibration and $F_{\mathrm{eg}}\colon \mathcal{D}_{\mathrm{eg}} \to \mathcal{C}_{\mathrm{eg}}$ is a cocartesian fibration and
- an *op-Gray fibration* if it is an ingressive cartesian fibration and $F_{\mathrm{eg}}\colon \mathcal{D}_{\mathrm{eg}} \to \mathcal{C}_{\mathrm{eg}}$ is a cartesian fibration.

We write $\mathrm{Ortho}(\mathcal{C}^\dagger)$ for the non-full subcategory of the large category $\mathrm{OFS}_{/\mathcal{C}^\dagger}$ spanned by the curved orthofibrations over $\mathcal{C}^\dagger$ and functors over $\mathcal{C}^\dagger$ preserving cocartesian egressive lifts. Similarly, we define the large category $\mathrm{opGray}(\mathcal{C}^\dagger)$ of curved op-Gray fibrations.

As the category of orthogonal factorization systems OFS has pullbacks which commute with the forgetful functor to Cat_1 by [CH21, Prop. 5.2] and both types of fibrations are preserved by pullbacks, [HHLN23b, Obs. 5.2(3)], we obtain functors

$$\mathrm{opGray},\ \mathrm{Ortho}\colon\ \mathrm{OFS}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_1\ .$$

Lemma 4.2. *A functor $F\colon \mathcal{D}^\dagger \to \mathcal{C}^\dagger$ is an ingressive cartesian fibration if and only if $F_{\mathrm{in}}\colon \mathcal{D}_{\mathrm{in}} \to \mathcal{C}_{\mathrm{in}}$ is a right fibration.*

Proof. It is observed in [HHLN23b, Obs. 5.2(2)] that an ingressive morphism in $\mathcal{D}$ is F-cartesian if and only if it is F_{in}-cartesian (while not making any use of the additional assumption that the factorization systems are adequate). It follows that F is an ingressive cartesian fibration if and only if $F_{\mathrm{in}}\colon \mathcal{D}_{\mathrm{in}} \to \mathcal{C}_{\mathrm{in}}$ is a cartesian fibration for which every morphisms in the source is F_{in}-cartesian. But that is equivalent to F_{in} being a right fibration. □

Definition 4.3. A functor $F\colon \mathbb{D} \to \mathbb{C}$ of double categories is called a *(cocart,right)-fibration* if $F(-,0)$ is a cocartesian fibration and $F(n,-)$ is a right fibration for all $n \geq 0$.

Let $\mathrm{CoR}\,(\mathbb{C})$ denote the non-full subcategory of the large category $\mathrm{DCat}_{/\mathbb{C}}$ spanned by the (cocart,right)-fibrations and functors preserving cocartesian edges. We analogously define the category $\mathrm{CaR}\,(\mathbb{C})$ of *(cart,right)-fibrations* over $\mathbb{C}$.

Cocartesian fibrations, functors preserving cocartesian edges and right fibrations are preserved by pullbacks. Therefore, pullbacks also preserve (cocart,right)-fibrations between double categories and functors between them, we therefore obtain functors

$$\mathrm{CoR},\, \mathrm{CaR}\colon\, \mathrm{DCat}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_1\ .$$

Lemma 4.4. *Let $F\colon \mathbb{D} \to \mathbb{C}$ be a functor of double categories and assume that $\mathbb{C}$ is a factorization double category and that $F(0,-)$ is a right fibration. Then $F(n,-)$ is a right fibration for all $n \geq 0$ if and only if $\mathbb{D}$ is a factorization double category.*

Proof. We use characterization 5 from Proposition 3.1 for factorization double categories. Consider the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{D}(n,-) & \xrightarrow{F(n,-)} & \mathbb{C}(n,-) \\
{\scriptstyle \mathbb{D}(d_n,-)}\big\downarrow & & \big\downarrow{\scriptstyle \mathbb{C}(d_n,-)} \\
\mathbb{D}(0,-) & \xrightarrow[F(0,-)]{} & \mathbb{C}(0,-)
\end{array}$$

It follows from our assumption that the bottom functor and the functor on the right are right fibrations. Using left cancellation of right fibrations, we deduce that the upper functor is a right fibration for all $n \geq 0$ if and only if the functor on the left is a right fibration for all $n \geq 0$. $\square$

Proposition 4.5. *Let $\mathcal{C}^\dagger$ be a factorization system. The functor* $\mathrm{Fact}\colon\, \mathrm{OFS} \to \mathrm{DCat}$ *induces equivalences*

$$\mathrm{Ortho}(\mathcal{C}^\dagger) \cong \mathrm{CoR}\left(\mathrm{Fact}\left(\mathcal{C}^\dagger\right)\right)$$

and

$$\mathrm{opGray}(\mathcal{C}^\dagger) \cong \mathrm{CaR}\left(\mathrm{Fact}\left(\mathcal{C}^\dagger\right)\right)\ .$$

Proof. We discuss the first equivalence, the other one can be proven analogously.

It follows from Theorem 3.19, that Fact induces a fully faithful inclusion

$$\mathrm{OFS}_{/\mathcal{C}^\dagger} \hookrightarrow \mathrm{DCat}_{/\mathrm{Fact}\,\mathcal{C}^\dagger}\ .$$

We need to check that this equivalence restricts to an equivalence on the subcategories $\mathrm{Ortho}(\mathcal{C}^\dagger)$ and $\mathrm{CoR}\,\big(\mathrm{Fact}(\mathcal{C}^\dagger)\big)$.

Let $F\colon \mathscr{D}^\dagger \to \mathscr{C}^\dagger$ be a functor of factorization systems. Unwinding definitions, we see that F_{eg} is naturally equivalent to $\mathrm{Fact}(F)(-,0)$ and F_{in} is naturally equivalent to $\mathrm{Fact}(F)(0,-)$. It hence follows from Lemma 4.2 that F is an ingressive cartesian fibration if and only if $\mathrm{Fact}(F)(0,-)$ is a right fibration. But Lemma 4.4 implies that this automatically forces $\mathrm{Fact}(F)(n,-)$ to be a right fibration for all $n \geq 0$. We conclude that $\mathrm{Fact}(F)$ is a (cocart,right)-fibration if and only F is a curved orthofibration and that a functor between two curved othofibrations preserves cocartesian lifts of egressive morphisms if and only if the associated functor of (cocart,right)-fibrations preserves cocartesian lifts. All in all, we have shown that Fact induces a fully faithful functor $\mathrm{Ortho}(\mathscr{C}^\dagger) \hookrightarrow \mathrm{CoR}\left(\mathrm{Fact}(\mathscr{C}^\dagger)\right)$. But this functor is also essentially surjective because the source of any (cart,right)-fibration over a factorization double category is a factorization double category by Lemma 4.4. □

We will now state the (un)straightening equivalence for (cocart,right)-fibrations. We will denote the 2-category of categories, functors and natural transformations by $\mathrm{Cat}_1^{(2)}$, for example defined in [HHLN23a, Def. 5.1.6].

Theorem 4.6. *There is a natural equivalence*

$$\mathrm{CoR}\,(-)^{\simeq} \cong \mathrm{map}_{\mathrm{DCat}}\left((-)^{2\,\mathrm{op}}, \mathrm{Sq}^{\mathrm{oplax}}\left(\mathrm{Cat}_1^{(2)}\right)\right)$$

of functors from double categories to spaces.

Remark 4.7. Since $\mathrm{Cat}_1^{(2)}$ is a large category, the mapping space actually has to be taken in the very large category of large categories. We omit this distinction from the notation.

The proof strategy follows the one from [AF20, Thm. 1.26]. We are grateful to Jaco Ruit for pointing us to this reference.

Proposition 4.8. *The functor*

$$\mathrm{CoR}\,(-)^{\simeq}\colon\ \mathrm{DCat}^{\mathrm{op}} \longrightarrow \mathcal{S}$$

is right Kan extended from its restriction along $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{DCat}^{\mathrm{op}}$.

Proof. Let $\mathbb{C}$ be a double category. Using the pointwise formula for right Kan extension, we must show that the upper map in the following commutative diagram is an equivalence

$$\begin{array}{ccc}
\mathrm{CoR}\,(\mathbb{C})^{\simeq} & \longrightarrow & \lim\left(\left(\Delta \times \Delta_{/\mathbb{C}}\right)^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{DCat} \to \mathcal{S}\right) \\
\downarrow & & \downarrow \\
\left(\mathrm{PSh}(\Delta \times \Delta)_{/\mathbb{C}}\right)^{\simeq} & \longrightarrow & \lim\left(\left(\Delta \times \Delta_{/\mathbb{C}}\right)^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{PSh}(\Delta \times \Delta) \to \mathcal{S}\right)
\end{array}$$

where the last functor in the limit in the upper right corner is $\mathrm{CoR}\,(-)^{\simeq}$ and on the lower right corner it is $\mathrm{DCat}^{\simeq}_{/\_}$. The left arrow is an inclusion of path components by definition and the right arrow is an inclusion of path components because it is a limit of such. The bottom arrow is an equivalence because $\mathrm{PSh}(\Delta \times \Delta)$ is a topos and the slice functor $\mathrm{PSh}(\Delta \times \Delta)^{\mathrm{op}} \longrightarrow \mathrm{Cat}_1$ hence does preserve limits by [Lur09b, Thm. 6.1.3.9, Prop. 6.1.3.10]. The claim hence follows from Lemma 4.9. ☐

Lemma 4.9. *Let $\mathbb{C}$ be a double category and $F\colon \mathbb{D} \to \mathbb{C}$ be a morphism of bisimplicial spaces such that the pullback along any map $[m] \boxtimes [n] \to \mathbb{C}$ is a (cocart,right)-fibration of double categories. Then F is a (cocart,right)-fibration of double categories.*

Proof. Let us first check that $\mathbb{D}$ is a double category. The large category of double categories is the category of local objects in $\mathrm{PSh}(\Delta \times \Delta)$ with respect to certain morphisms $\mathbb{E} \to \mathbb{E}'$ (corepresenting the Segal and completeness condition), i.e. we need to check that for those morphisms, the induced map

$$\mathrm{map}_{\mathrm{PSh}(\Delta\times\Delta)}(\mathbb{E}', \mathbb{D}) \longrightarrow \mathrm{map}_{\mathrm{PSh}(\Delta\times\Delta)}(\mathbb{E}, \mathbb{D})$$

is an equivalence. This can equivalently be checked on fibers over the same morphism for $\mathbb{C}$ (where we know that it is an equivalence), i.e. that

$$* \times_{\mathrm{map}(\mathbb{E}',\mathbb{C})} \mathrm{map}(\mathbb{E}', \mathbb{D}) \longrightarrow * \times_{\mathrm{map}(\mathbb{E},\mathbb{C})} \mathrm{map}(\mathbb{E}, \mathbb{D}) \tag{5}$$

is an equivalence for every point in $\mathrm{map}(\mathbb{E}', \mathbb{C})$. Now we use that

$$* \times_{\mathrm{map}(\mathbb{E}',\mathbb{C})} \mathrm{map}(\mathbb{E}', \mathbb{D}) \cong * \times_{\mathrm{map}(\mathbb{E}',\mathbb{E}')} \mathrm{map}(\mathbb{E}', \mathbb{D} \times_{\mathbb{C}} \mathbb{E}')$$

and

$$* \times_{\mathrm{map}(\mathbb{E},\mathbb{C})} \mathrm{map}(\mathbb{E}, \mathbb{D}) \cong * \times_{\mathrm{map}(\mathbb{E},\mathbb{E}')} \mathrm{map}(\mathbb{E}, \mathbb{D} \times_{\mathbb{C}} \mathbb{E}')$$

where the fiber on the right side is taken over the identity and the morphism $\mathbb{E} \to \mathbb{E}'$, respectively. This shows that (5) is an equivalence for $\mathbb{D} \to \mathbb{C}$ if it holds when replacing the functor with the pullback along an arbitrary map $\mathbb{E}' \to \mathbb{C}$.

In our concrete situation, we have that $\mathbb{E}' = [m] \boxtimes [n]$ for any morphism appearing in the definition of a double category. But we did assume that the total space is a double category when being pulled back along a map from $[m] \boxtimes [n]$.

Now we check that F is a (cocart,right)-fibration. Let $[2] \to \mathbb{C}(-, 0)$ be an arbitrary functor. This functor is also represented by a functor $[2] \boxtimes [0] \to \mathbb{C}$. The pullback of F along this map is a (cocart,right)-fibration. Evaluating at $(-, 0)$, we see that the pullback of $F(-, 0)$ along the auxiliary map $[2] \to \mathbb{C}(-, 0)$ is a cocartesian fibration. It follows from [AF20, Prop. 2.23(1)(c)] that $F(-, 0)$ is a cocartesian fibration. Similarly, one shows that $F(n, -)$ is a right fibration by pulling back along maps out of $[n]\boxtimes[2]$. ☐

Proof of Theorem 4.6. Both functors are right Kan extended from their restriction along $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{DCat}^{\mathrm{op}}$ by Proposition 4.8 and because representable functors of reflective subcategories of presheaves are right Kan extended.

Therefore, it is enough to prove that the functors are equivalent when restricted along $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{DCat}^{\mathrm{op}}$. There will be a unique equivalence extending the equivalence on this restriction.

By definition, we have natural equivalences

$$\mathrm{map}_{\mathrm{DCat}}\left(([m]\boxtimes[n])^{2\,\mathrm{op}},\mathrm{Sq}^{\mathrm{oplax}}\left(\mathrm{Cat}_1^{(2)}\right)\right)\cong\mathrm{map}_{\mathrm{Cat}_2}\left([n]^{\mathrm{op}}\otimes[m]\,,\mathrm{Cat}_1^{(2)}\right).$$

Combining [HHLN23a, Prop. 5.2.10], [HHLN23b, Cor. 6.5] and [HHLN23a, Thm. 2.5.1], there are also natural equivalences

$$\mathrm{map}\left([n]^{\mathrm{op}}\otimes[m]\,,\mathrm{Cat}_1^{(2)}\right)\cong\mathrm{Ortho}([m]\,\bar{\times}\,[n])\,.$$

The claim hence follows from Proposition 4.5 and Example 3.5. □

A functor $F\colon\mathbb{D}\to\mathbb{C}$ of double categories is called a (left,cart)-fibration if $F^{12\,\mathrm{op},\mathrm{swap}}$ is a (cocart,right)-fibration. We denote the large category of (left,cart)-fibrations over $\mathbb{C}$ by $\mathrm{LCa}\,(\mathbb{C})$.

The following result, combined with Theorem 4.6, establishes an equivalence of spaces of (cocart,right)-fibrations and (left,cart)-fibrations. In [Nui24, Thm. 3.1], it is shown that this enhances to an equivalence of categories.

Corollary 4.10. *There is a natural equivalence*

$$\mathrm{LCa}\,(-)^{\simeq}\cong\mathrm{map}_{\mathrm{DCat}}\left((-)^{2\,\mathrm{op}},\mathrm{Sq}^{\mathrm{oplax}}\left(\mathrm{Cat}_1^{(2)}\right)\right)$$

of functors from factorization systems to spaces.

Proof. By Theorem 4.6, we have natural equivalences

$$\begin{aligned}\mathrm{LCa}\,(\mathbb{C})^{\simeq}&\cong\mathrm{CoR}\left(\mathbb{C}^{12\,\mathrm{op},\mathrm{swap}}\right)^{\simeq}\\&\cong\mathrm{map}_{\mathrm{DCat}}\left(\left(\mathbb{C}^{12\,\mathrm{op},\mathrm{swap}}\right)^{2\,\mathrm{op}},\mathrm{Sq}^{\mathrm{oplax}}\left(\mathrm{Cat}_1^{(2)}\right)\right)\\&\cong\mathrm{map}_{\mathrm{DCat}}\left(\mathbb{C}^{2\,\mathrm{op}},\left(\mathrm{Sq}^{\mathrm{oplax}}\left(\mathrm{Cat}_1^{(2)}\right)\right)^{\mathrm{swap}}\right)\end{aligned}$$

because $\left(\left(\mathbb{C}^{12\,\mathrm{op},\mathrm{swap}}\right)^{2\,\mathrm{op}}\right)^{\mathrm{swap}}=\mathbb{C}^{2\,\mathrm{op}}$.

Taking opposite categories induces an equivalence

$$\mathrm{Cat}_1^{(2)}\cong\left(\mathrm{Cat}_1^{(2)}\right)^{\mathrm{co}}$$

where co denotes the operation of taking opposites of 2-morphisms (see e.g. [HHLN23a, Rem. 3.1.10]).

Unwinding definitions and using that $[n]\otimes[m]\cong([m]\otimes[n])^{\mathrm{co}}$ (see e.g. [AGH24, Obs. 2.2.10]), we obtain an induced equivalence

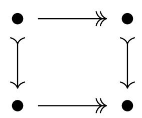
$$\mathrm{Sq}^{\mathrm{oplax}}\left(\mathrm{Cat}_1^{(2)}\right)\cong\mathrm{Sq}^{\mathrm{oplax}}\left(\mathrm{Cat}_1^{(2)}\right)^{\mathrm{swap}}.$$

Combining the two above facts, the claim follows. □

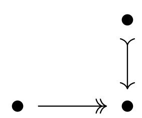
5. Adequate triples and span categories

In this section, we will recall the definition of an *adequate factorization system* and classify when a factorization double category comes from an adequate factorization system (Proposition 5.4). We will use this to prove that the category of adequate factorization systems has a unique non-trivial automorphism, the span category (Theorem 5.5, Corollary 5.6).

Definition 5.1 ([HHLN23b, Def. 4.2])**.** Let $\mathcal{C}^\dagger$ be a factorization system. We call a square



ambigressive if the horizontal morphisms are egressive and the vertical morphisms are ingressive. Similarly, we call a cospan



ambigressive if one leg is egressive and the other one is ingressive.

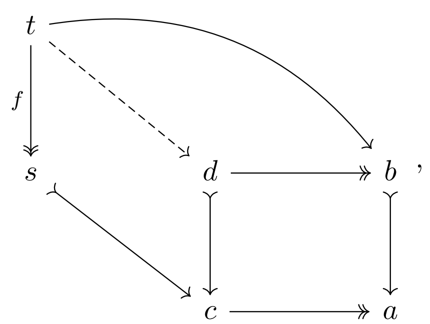
The factorization system $\mathcal{C}^\dagger$ is called *adequate* if every ambigressive square is a pullback and if every ambigressive cospan admits a pullback.

We write $\mathrm{OFS}^\perp \subset \mathrm{OFS}$ for the full subcategory of adequate factorization systems.

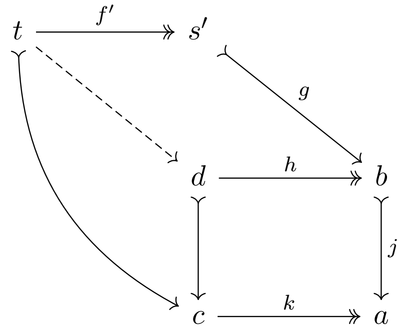
Lemma 5.2. *A factorization system is adequate if and only if every ambigressive cospan can uniquely be extended to an ambigressive square.*

Proof. The "only if"-part follows from the uniqueness of pullbacks.

Now assume that every ambigressive cospan can uniquely be extended to an ambigressive square. We must show that this square is a pullback. Equivalently, we must check that the space of fillers of the following diagram is contractible



(here we already factored an arbitray morphism from t to c into an egressive morphism f followed by an ingressive morphism). We can apply the lifting criterion for factorization systems to the left square, to see that the dashed arrow would admit a unique factorization through f. We might therefore reduce to the situation where f is an equivalence, i.e.



,

(where we now factored the morphism from t to b). The span $(d \xrightarrow{h} b \xleftarrow{g} s')$ can be extended uniquely to an ambigressive square. This in particular gives a morphism from t to d. The composition of the lower square with the upper square gives an extension of the ambigressive cospan $(k \xrightarrow{k} a \xleftarrow{j \circ g} s')$. But the outer square already provided such an extension, which we assumed to be unique. This provides the necessary homotopies, making the diagram commute. □

Definition 5.3. A factorization double category $\mathbb{C}$ is called *adequate* if $\mathbb{C}^{1\,\mathrm{op}}$ is a factorization double category. We denote the full subcategory of adequate factorization double categories by $\mathrm{DCat}^{\perp} \subset \mathrm{DCat}^{\perp}$.

Proposition 5.4. *A factorization system $\mathcal{C}^{\dagger}$ is adequate if and only if* $\mathrm{Fact}(\mathcal{C}^{\dagger})$ *is adequate, i.e. the equivalence from Theorem 3.19 restricts to an equivalence*

$$\mathrm{OFS}^{\perp} \cong \mathrm{DCat}^{\perp} \ .$$

Proof. Unwinding definitions, we find that $\mathrm{Fact}(\mathcal{C}^{\dagger})$ is adequate if and only if $\mathcal{C}^{\dagger}$ fulfills the condition from Lemma 5.2, which is equivalent to being adequate. □

We will now verify that $(-)^{1\,\mathrm{op}}$ recovers the span construction from [HHLN23b, Def. 2.12, Prop. 4.9] under the equivalence from Proposition 5.4.

We will actually compute the entire automorphism group of $\mathrm{DCat}^{\perp}$, hence also generalizing [HHLN23b, Thm. 5.21], saying that the span category construction refines to a $\mathbb{Z}/2\mathbb{Z}$-action on $\mathrm{OFS}^{\perp} \cong \mathrm{DCat}^{\perp}$.

Theorem 5.5. *There is an equivalence of groups*

$$\mathrm{Aut}\left(\mathrm{DCat}^{\perp}\right) \cong \mathbb{Z}/2\mathbb{Z}$$

with generator $(-)^{1\,\mathrm{op}}$ on the left side.

Proof. Note that the image of the Yoneda embedding $\Delta \times \Delta \to \mathrm{PSh}(\Delta \times \Delta)$ is contained in $\mathrm{OFS}^\perp$. We first prove that any equivalence restricts to an equivalence of this subcategory.

Any such equivalence F must preserve the terminal object $[0] \boxtimes [0]$. From fully-faithfulness it follows that F sends $[0] \boxtimes [1]$ and $[1] \boxtimes [0]$ to double categories with two objects.

There must be a vertical or horizontal morphism between these two objects because we could otherwise write the double category as a disjoint union of two non-initial double categories, which is not the case for $[0] \boxtimes [1]$ and $[1] \boxtimes [0]$.

We hence obtain a morphism

$$\mathbb{C} \longrightarrow F([1] \boxtimes [0]) \tag{6}$$

which is an equivalence on objects where $\mathbb{C}$ is either $[1] \boxtimes [0]$ or $[0] \boxtimes [1]$. Applying the same argument to F^{-1} we obtain a morphism

$$\mathbb{D} \longrightarrow F^{-1}(\mathbb{C})$$

which is an equivalence on objects where $\mathbb{D}$ is either $[1] \boxtimes [0]$ or $[0] \boxtimes [1]$. Applying F to this morphism yields a morphism

$$F(\mathbb{D}) \longrightarrow \mathbb{C} \tag{7}$$

which is still an equivalence on objects because F preserves $[0] \boxtimes [0]$ and is fully faithful. Composing (7) with (6), yields an element in $\mathrm{map}(F(\mathbb{D}), F([1] \boxtimes [0])) \cong \mathrm{map}(\mathbb{D}, [1] \boxtimes [0])$ which is an equivalence on objects. From this we learn that $\mathbb{D} \cong [1] \boxtimes [0]$ and that this composition actually is an equivalence of double categories.

Composing (7) with (6) the other way around now yields an endomorphism of $\mathbb{C}$ which is an equivalence on objects, and therefore also an equivalence. We conclude that (6) is an equivalence.

A similar argument shows that $F([0] \boxtimes [1])$ is either $[0] \boxtimes [1]$ or $[1] \boxtimes [0]$ (and different from $F([1] \boxtimes [0])$).

A computation shows that any pushout of $[0] \boxtimes [1]$ and $[1] \boxtimes [0]$ along $[0] \boxtimes [0]$ must either be $[1] \boxtimes [1]$ (in case where this is literally the condition of being an adequate factorization system, i.e. the pushout in (2) or a similar one obtained by pre-composing with $(-)^{1\,\mathrm{op}}$) or it only contains 3 objects (because in this case, it turns out that the pushout can be computed in $\mathrm{PSh}(\Delta \times \Delta)$). Because F preserves pushouts as well as the space of objects, we conclude that F preserves $[1] \boxtimes [1]$ and that F must either restrict to the identity or to $(-)^{1\,\mathrm{op}}$ on $\Delta_{\leq 1} \times \Delta_{\leq 1} \subset \Delta \times \Delta \subset \mathrm{OFS}^\perp$.

By the Segal condition, every object in $\Delta \times \Delta$ is a colimit of objects in $\Delta_{\leq 1} \times \Delta_{\leq 1}$, the value $F([m] \boxtimes [n])$ hence is determined by the restriction of F to $\Delta_{\leq 1} \times \Delta_{\leq 1}$. A calculation shows that if F restricts to $(-)^{1\,\mathrm{op}}$ on $\Delta_{\leq 1} \times \Delta_{\leq 1}$, it will still send $[m] \boxtimes [n]$ to $[m] \boxtimes [n]$ (and also if F restricts to the identity, obviously).

We moreover claim that any automorphism of $\Delta \times \Delta$ is already determined by its restriction to $\Delta_{\leq 1} \times \Delta_{\leq 1}$. Indeed, the $k+1$ inclusions $[0] \to [k]$ and the $m+2$ maps

$[m] \to [1]$ induce an injective map

$$\mathrm{map}_\Delta([k]\,, [m]) \longhookrightarrow (\mathrm{map}_\Delta([0]\,, [1]))^{\times(k+1)(m+2)}\ .$$

The category $\mathrm{DCat}^\perp$ is a reflective subcategory of $\mathrm{PSh}(\Delta\times\Delta)$, every colimit–preserving functor is hence left Kan extended along the inclusion $\Delta \times \Delta \to \mathrm{DCat}^\perp$. Therefore, we have inclusions

$$\mathrm{Aut}(\mathrm{DCat}^\perp) \subset \mathrm{Fun}^L(\mathrm{DCat}^\perp, \mathrm{DCat}^\perp) \subset \mathrm{Fun}(\Delta \times \Delta, \mathrm{DCat}^\perp)\,.$$

It follows from the above discussion that we actually have that

$$\mathrm{Aut}(\mathrm{OFS}^\perp) \subset \mathrm{Aut}(\Delta \times \Delta, \Delta \times \Delta)$$

is the subspace spanned by the two components $(-)^{1\,\mathrm{op}}$ and the identity.

Finally, note that the identity functor on $\Delta \times \Delta$ has no non-trivial automorphisms because none of the objects in $\Delta \times \Delta$ has. □

Corollary 5.6. *Let $\mathcal{C}^\dagger$ be an adequate factorization system. Then there is a natural equivalence*

$$\mathrm{Fact}(\mathcal{C}^\dagger)^{1\,\mathrm{op}} \cong \mathrm{Fact}(\mathrm{Span}(\mathcal{C}^\dagger))$$

where $\mathrm{Span}\colon \mathrm{OFS}^\perp \to \mathrm{OFS}^\perp$ *denotes Barwick's span category functor, [Bar17, Def. 5.7].*

Proof. By [HHLN23b, Thm. 4.12], Span is an automorphism of $\mathrm{OFS}^\perp$ which is not naturally equivalent to the identity and so is $\mathrm{Cnr} \circ (-)^{1\,\mathrm{op}} \circ \mathrm{Fact}$ by Proposition 5.4, so they agree by Theorem 5.5 (using the equivalence from Proposition 5.4 again). □

Combining the above result with Proposition 4.5, we recover the equivalence between op-Gray fibration over an orthogonal adequate triple and curved orthofibrations between its span category from [HHLN23b, Thm. 5.21]:

Theorem 5.7 (Haugseng, Hebestreit, Nuiten, Linskens)**.** *Let $\mathcal{C}^\dagger$ be an adequate factorization system. The functor* $\mathrm{Span}\colon \mathrm{OFS}^\perp \to \mathrm{OFS}^\perp$ *induces a natural equivalence*

$$\mathrm{opGray}(\mathcal{C}^\dagger) \cong \mathrm{Ortho}(\mathrm{Span}(\mathcal{C}^\dagger))$$

Proof. This follows from combining Corollary 5.6 and Proposition 4.5. □

A. Normed structures on the category of equivariant manifolds

In this appendix, which is jointly written with Natalie Stewart, we use the results from this paper to construct a *G-symmetric monoidal category of G-manifolds* for G a compact Lie group, generalizing the construction for finite groups by Horev [Hor19].

Let us recall some basic notions from equivariant homotopy theory:

Definition A.1. We write $\mathcal{T}op^G$ for the topologically enriched category of topological spaces with a continuous G-action and G-equivariant maps. We define

$$\mathbb{O}_G^{\sqcup} \subset \mathcal{T}op^G$$

as the full topologically enriched subcategory spanned by finite disjoint unions of transitive G-spaces G/H for some closed subgroup H.

We write $O_G^{\sqcup} = \mathrm{N}^{\Delta}(\mathbb{O}^{\sqcup})$ for the homotopy coherent nerve of this category. We moreover write $O_G \subset O_G^{\sqcup}$ for the subcategory only consisting of single orbits G/H. This category is called the orbit category of G. We denote by $O_G^{f.i.,\sqcup} \subset O_G^{\sqcup}$ the wide subcategory of maps which are finite coverings.

The importance of the orbit category comes from Elmendorf's theorem [Elm83], stating that the homotopy theory of topological spaces with a continuous G-action with respect to maps inducing weak equivalences on all fixed points, is modeled by the presheaf category of *G-spaces*

$$\mathrm{Fun}(O_G^{\mathrm{op}}, \mathcal{S}) .$$

It follows from pullback stability of finite coverings that $(O_G^{\sqcup}, O_G^{f.i.,\sqcup}, O_G^{\sqcup})$ is an *adequate triple* in the sense of Barwick [Bar17], so that we can form its span category

$$\mathrm{Span}_G^{f.i.} = \mathrm{Span}\left(O_G^{\sqcup}, O_G^{f.i.,\sqcup}, O_G^{\sqcup}\right) .$$

We warn the reader that this is just a span of an adequate triple, not of an orthogonal factorization system. However, this span category admits the structure of an orthogonal factorization system with backwards maps as ingressives and forwards maps as egressives.

Definition A.2. A G-symmetric monoidal G-category is a product–preserving functor

$$\mathrm{Span}_G^{f.i.} \longrightarrow \mathrm{Cat}$$

Here, the products in the category $\mathrm{Span}_G^{f.i.}$ are given by disjoint unions. Given a G-symmetric monoidal G-category $\mathcal{C}$, we denote $\mathcal{C}(G/H)$ by $\mathcal{C}^H$ and call it the H-fixed points of $\mathcal{C}$. For any subgroup H, precomposition with the product–preserving functor $\mathrm{Span}(\mathrm{Fin}) \to \mathrm{Span}_G^{f.i.}$ sending A to $G/H \times A$ equips the H-fixed points category $\mathcal{C}^H$ with a symmetric monoidal structure. Any inclusion $K < H < G$ gives rise to a map $G/K \to G/H$. The backwards functoriality yields a functor

$$\mathcal{C}^H \longrightarrow \mathcal{C}^K ,$$

usually referred to as the *restriction*. If $K < H$ is a finite index inclusion, we obtain a functor in the other direction

$$\mathcal{C}^K \to \mathcal{C}^H ,$$

called the norm.

Definition A.3. In the following, a manifold is always assumed to be a smooth manifold without boundary. A G-manifold is a smooth manifold without boundary together with a smooth action by a compact Lie group G.

The goal of this appendix is to define the G-symmetric monoidal G-category of G-manifolds, i.e. to construct a product–preserving functor

$$\underline{\mathrm{Mfld}}\colon \mathrm{Span}_G^{f.i.} \longrightarrow \mathrm{Cat}$$

which sends G/H to the category of H-manifolds and H-equivariant embeddings, where restriction along $K < H$ is given by restricting the action and norming along a finite index inclusion $K < H$ sends a K-manifold M to $H \times_K M$.

This construction certainly requires geometric input, so it is convenient to define the ∞-category of manifolds as a homotopy coherent nerve of a topologically enriched category.

The main technical difficulty arises from dealing with the two different directions in the span category. It is hard to write down the unstraightening of this functor directly, as the source $\mathrm{Span}_G^{f.i.}$ does not admit a natural model as a topologically enriched category, since pullbacks are only unique up to *contractible* choice. This precisely is why it is useful to work with double categories, where the two directions are clearly separated.

Construction A.4. Let $\mathbb{S}\mathrm{pan}^G$ denote the double category where $\mathbb{S}\mathrm{pan}^G(m,n) \subset \mathrm{map}([m] \times [n], O_G^{\sqcup})$ is given by the subspace of elements where

- the map $A_{i,j-1} \to A_{i,j}$ has finite fibers for all $0 \leq i \leq m$ and all $1 \leq j \leq n$ and
- the induced map $A_{i-1,j-1} \to A_{i-1,j} \times_{A_{i,j}} A_{i,j-1}$ is an equivalence for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Here, the bisimplicial structure is given by forgetting/composing maps and inserting identities.

We would like to topologically model this using the following construction.

Construction A.5. For every n consider the topologically enriched category $\mathcal{S}pan_n^G$ in which

- objects are composable sequences $A_0 \to \cdots \to A_n$ in $\mathcal{O}_G^{\sqcup}$ which are all finite coverings.
- morphisms between $(A_0 \to \cdots \to A_n)$ and $(B_0 \to \cdots \to B_n)$ are given by $A_i \to B_i$ for all $0 \leq i \leq n$ such that for all $1 \leq i \leq n$, the following diagram commutes

$$\begin{array}{ccc} A_{i-1} & \longrightarrow & B_{i-1} \\ \downarrow & & \downarrow \\ A_i & \longrightarrow & B_i \end{array}$$

and is a pullback square. We equip the mapping spaces with the subspace topology of $\prod \mathrm{map}(A_i, B_i)$.

This is functorial in maps of finite linearly ordered posets via composing/forgetting maps and inserting identities, i.e. it assembles into a functor

$$\Delta^{\mathrm{op}} \longrightarrow \mathrm{Cat}^{\mathrm{Top}}$$

into the category of topologically enriched categories.

Proposition A.6. *The homotopy coherent nerve of $\mathcal{S}pan_n^G$ is equivalent to the functor $\Delta^{\mathrm{op}} \to \mathrm{Cat}$ sending $[n] \mapsto \mathbb{S}\mathrm{pan}(\bullet, n)$.*

Proof. The category $\mathcal{S}pan_n^G$ is a topologically enriched subcategory of the topologically enriched functor category

$$\mathcal{S}pan_n^G \subset \mathcal{F}un\big([n], \mathcal{O}_G^{\sqcup}\big)$$

The evaluation

$$\mathcal{F}un([n], \mathcal{O}_G^{\sqcup}) \times [n] \to \mathcal{O}_G^{\sqcup}$$

induces a comparison functor

$$\begin{array}{ccc} \mathrm{N}^{\Delta}\mathcal{S}pan_n^G & \overset{\gamma'}{\dashrightarrow} & \mathbb{S}\mathrm{pan}(\bullet, n) \\ \big\downarrow & & \big\downarrow \\ \mathrm{N}^{\Delta}\mathcal{F}un([n], \mathcal{O}_G^{\sqcup}) & \xrightarrow{\gamma} & \mathrm{Fun}\left([n], O_G^{\sqcup}\right). \end{array}$$

which restricts (essentially uniquely) to a functor γ'; moreover, the above diagram is natural in n, i.e. it is a simplicial diagram of commutative squares.

We claim that γ and γ' are equivalences, beginning with γ. Indeed, essential surjectivity of γ is clear and fully faithfulness follows from a computation on mapping spaces, for which homotopy pullbacks and pullbacks agree as any any map in $O_G^{\sqcup}$ is a G-Serre fibration. For γ', it follows by unwinding definitions that the image of the composite $\mathrm{N}^{\Delta}\mathcal{S}pan_n^G \to \mathrm{Fun}\left([n], O_G^{\sqcup}\right)$ is $\mathbb{S}\mathrm{pan}(\bullet, n)$, giving the equivalence. □

Construction A.7. Given a non-negative integer n, let $\mathcal{M}f\ell d_n^G$ be the topologically enriched category with

- objects the composable sequences $(M \to A_0 \to \cdots \to A_n)$ of equivariant maps where $A_0 \to \cdots \to A_n$ is a composable sequence of finite coverings in $\mathcal{O}_G^{\sqcup}$ and M is a smooth manifold equipped with a smooth action by G, and
- morphisms $(M \to A_0 \to \cdots \to A_n) \longrightarrow (N \to B_0 \to \cdots \to B_n)$ given by the tuples of maps $A_i \to B_i$ in $O_G^{\sqcup}$ for all $0 \leq i \leq n$ and smooth G-equivariant maps $M \to N$, making the apparent diagrams commute and such that
  - the map $A_{i-1} \to A_i \times_{B_i} B_{i-1}$ is an equivalence for all $1 \leq i \leq n$, and
  - the map $M \to N \times_{A_0} B_0$ is an embedding (or equivalently $M \to N \times_{A_i} B_i$ is an embedding for all $0 \leq i \leq n$ by the first condition).

The mapping spaces are topologized using the subset topology of the product $\prod_i \mathrm{map}(A_i, B_i) \times \mathrm{map}(M, N)$ where we equip the mapping space $\mathrm{map}(M, N)$ with the strong C^∞-topology.

This construction is functorial in maps of finite linearly ordered posets by forgetting/composing maps and adding identities, so that we obtain a functor

$$\mathcal{Mfld}^G_\bullet \colon \Delta^{\mathrm{op}} \longrightarrow \mathrm{Cat}^{\mathrm{Top}}$$

Moreover, forgetting the manifold M yields a natural transformation

$$p\colon \mathcal{Mfld}^G_\bullet \longrightarrow \mathcal{Span}^G_\bullet$$

of functors $\Delta^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{Top}}$. Taking homotopy coherent nerves, followed by the Rezk nerve, we obtain a functor

$$\begin{aligned} \mathbb{M}\mathrm{fld}^G \colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} &\longrightarrow \mathcal{S} \\ (m, n) &\longmapsto \mathrm{map}\left([m], \mathrm{N}^\Delta(\mathcal{Mfld}^G_n)\right) \end{aligned}$$

along with a natural transformation $p\colon \mathbb{M}\mathrm{fld}^G \to \mathbb{S}\mathrm{pan}^G$.

We know by construction that $\mathbb{M}\mathrm{fld}^G(\bullet, n)$ is a complete Segal space. Our next goal is to establish that it also is a complete Segal space in the other direction and that the functor p is a cartesian fibration, starting with the following.

Lemma A.8. *The topologically enriched functor*

$$p\colon \mathcal{Mfld}^G_n \longrightarrow \mathcal{Span}^G_n$$

induces Serre fibrations on topological mapping spaces.

Moreover, given objects $(M \to A_0 \to \cdots \to A_n)$ and $(N \to B_0 \to \cdots \to B_n)$ in $\mathcal{Mfld}^G_n$ and a map f from $(A_0 \to \cdots \to A_n)$ to $(B_0 \to \cdots \to B_n)$, the topological space of lifts of f to map in $\mathcal{Mfld}^G_n$ is homeomorphic to the topological space of embeddings from M into $N \times_{B_0} A_0$ over A_0.

Proof. We will actually check that the induced map on mapping spaces is a fiber bundle; the remaining claim follows by unwinding definitions. Let us start with the case $n = 0$.

Let $M \to A$ and $N \to B$ be two objects. Using that a manifold with a map to a disjoint union can naturally be written as a disjoint union, we can reduce to the case where A and B are transitive.

Moreover, restricting to path components in the base, we can moreover restrict to a distinguished neighborhood of the projection map $A = G/K \to G/H = B$ in the mapping space $\mathrm{map}(G/K, G/H)$; by [Sch18, Prop. B.17] such a neighborhood is given by the quotient of centralizers $C_G(K)/G_H(K)$ mapping to $(G/H)^K = \mathrm{map}(G/K, G/H)$ via the inclusion $C_G(K) \subset G$. In particular, there is an action of $C_G(K)$ on that space. We will extend this action to an action on $\mathrm{map}_{\mathcal{Mfld}^G_0}(M \to G/K, N \to G/H)$ so that the map to $\mathrm{map}(G/K, G/H)$ becomes $C_G(K)$-equivariant. This will imply the claim, as any $C_G(K)$-equivariant map to a transitive $C_G(K)$-space automatically is a fiber bundle.

For this, we use that any G-manifold M over G/K is of the form $G \times_K M'$ for a K-manifold M' (obtained as the fiber over eK). Given an element $g \in C_G(K)$, we associate to it the map $G \times_K M \to G \times_K M$ sending $[\overline{g}, m]$ to $[\overline{g}g, m]$ (which is well-defined as g is contained in the centralizer of K). This map lives over the map $G/K \to G/K$ induced by multiplication with g, so that we did construct a map of topological groups

$$C_G(K) \longrightarrow \mathrm{Aut}_{\mathcal{Mfld}^G_n}(M \to G/K)\,.$$

The target acts on $\mathrm{map}_{\mathcal{Mfld}^G_0}(M \to G/K, N \to G/H)$ and this yields our desired action.

For the general case observe that by definition we have a pullback of topological spaces

$$\begin{array}{ccc}
\mathrm{map}_{\mathcal{Mfld}^G_n}(M \to \cdots \to A_n, N \to \cdots \to B_n) & \longrightarrow & \mathrm{map}_{\mathcal{Mfld}^G_0}(M \to A_0, N \to B_0) \\
\downarrow & & \downarrow \\
\mathrm{map}_{\mathcal{Span}^G_n}(A_0 \to \cdots \to A_n, B_0 \to \cdots \to B_n) & \longrightarrow & \mathrm{map}_{\mathcal{Span}^G_0}(A_0, B_0)
\end{array}$$

The claim follows as fiber bundles are stable under pullbacks. □

Let $a_0 \colon [0] \to [n]$ denote the inclusion of $\{0\}$.

Lemma A.9. *The square*

$$\begin{array}{ccc}
\mathbb{M}\mathrm{fld}^G(0,n) & \longrightarrow & \mathbb{S}\mathrm{pan}(0,n) \\
\downarrow{\scriptstyle a_0^*} & & \downarrow{\scriptstyle a_0^*} \\
\mathbb{M}\mathrm{fld}^G(0,0) & \longrightarrow & \mathbb{S}\mathrm{pan}(0,0)
\end{array}$$

is cartesian for all $n \geq 0$.

Proof. We prove that the induced functor

$$\mathbb{M}\mathrm{fld}^G(0,n) \longrightarrow \mathrm{PB} := \mathbb{S}\mathrm{pan}(0,n) \times_{\mathbb{S}\mathrm{pan}(0,0)} \mathbb{M}\mathrm{fld}^G(0,0)$$

is an equivalence, starting with essential surjectivity (i.e. surjectivity on π_0). An object in PB is given by an object $M \to A$ in $\mathcal{Mfld}^G_0$ together with a composable sequence $A_0 \to A_1 \to \cdots \to A_n$ in $\mathcal{Span}^G_n$ and an equivalence $A \simeq A_0$ in $\mathcal{Span}^G_0$. This sequence is equivalent to $A \to A_1 \to \ldots A_n$ in $\mathbb{S}\mathrm{pan}^G(0,n)$ via $A \simeq A_0$, so that we might assume $A = A_0$. This element precisely is the image of $M \to A \to \cdots \to A_n$.

We now turn to fully faithfulness (i.e. equivalence on each path component). By Lemma A.8, we can compute this pullback on mapping spaces as the homotopy type of a (strict) pullback of the induced diagram on mapping topological spaces from the original functor of topologically enriched categories. The fibers in this square of topological mapping spaces are homeomorphic by Lemma A.8, so the fibers on the diagram of mapping spaces are equivalent, proving fully faithfulness. □

Lemma A.10. *The square*

$$\begin{array}{ccc} \mathbb{M}\mathrm{fld}^G(1,n) & \longrightarrow & \mathbb{S}\mathrm{pan}(1,n) \\ \downarrow{\scriptstyle a_0^*} & & \downarrow{\scriptstyle a_0^*} \\ \mathbb{M}\mathrm{fld}^G(1,0) & \longrightarrow & \mathbb{S}\mathrm{pan}(1,0) \end{array}$$

is cartesian for all $n \geq 0$.

Proof. Essential surjectivity follows as in the previous lemma.

Let us turn to fully faithfulness. We have to verify that the diagram induces pullbacks on mapping spaces. This can be checked by verifying that they induce equivalences on fibers along the horizontal maps. Let $F\colon X \to Y$ be an object of $\mathbb{M}\mathrm{fld}^G(1,n)$, i.e. an arrow in $\mathcal{M}f\ell d_n^G$. We want to compute the fiber

$$\mathrm{Fib} \longrightarrow \mathrm{Aut}_{\mathrm{Ar}(\mathbb{M}\mathrm{fld}^G(\bullet,n))}(F) \longrightarrow \mathrm{Aut}_{\mathrm{Ar}(\mathbb{S}\mathrm{pan}^G(\bullet,n))}(p(F))\,.$$

Expanding the pullback description of mapping spaces in the arrow category, we can write this fiber as as the pullback

$$\begin{array}{ccc} \mathrm{Fib} & \xrightarrow{\qquad\qquad} & \mathrm{Aut}_{\mathbb{M}\mathrm{fld}^G(\bullet,n)}(X) \times_{\mathrm{Aut}_{\mathbb{S}\mathrm{pan}^G(\bullet,n)}(p(X))} * \\ \downarrow & & \downarrow{\scriptstyle \circ F} \\ \mathrm{Aut}_{\mathbb{M}\mathrm{fld}^G(\bullet,n)}(Y) \times_{\mathrm{Aut}_{\mathbb{S}\mathrm{pan}^G(\bullet,n)}(p(Y))} * & \xrightarrow{F\circ} & \mathrm{map}_{\mathbb{M}\mathrm{fld}^G(\bullet,n)}(X,Y) \times_{\mathrm{map}_{\mathbb{S}\mathrm{pan}^G(\bullet,n)}(X,Y)} * \end{array}$$

Now we apply Lemma A.8 again to model each of the terms in this pullback diagram as a strict pullback in topological mapping spaces of the associated topological functor $\mathcal{M}f\ell d_n^G \to \mathcal{S}pan_n^G$. It follows from the description of the fiber in the same lemma that $a_0\colon [0] \to [n]$ induces homeomorphisms on those strict fibers, yielding an equivalence. ☐

Lemma A.11. *The maps $\mathcal{M}f\ell d^G(n,\bullet) \to \mathbb{S}\mathrm{pan}^G(n,\bullet)$ are right fibratons of complete Segal spaces for all $n \geq 0$.*

Proof. We first check that it is a right fibration of Segal spaces. As we already know that $\mathbb{S}\mathrm{pan}^G(n,\bullet)$ is Segal, the claim for $n = 0, 1$ follows immediately from the two previous lemmas and [Bri18, Prop. 1.7]. For $n \geq 2$, the claim now follows from pasting pullback diagrams, as we already know that $\mathcal{M}f\ell d^G(\bullet, m)$ is Segal.

The completness is automatic by [Bri18, Prop. 1.19] as we know that $\mathbb{S}\mathrm{pan}^G(n,\bullet)$ is is complete. ☐

Lemma A.12. *Given an arrow $\varphi\colon A \to B$ in the base and a lift $F\colon M \to B$ of the target to the $\mathbb{M}\mathrm{fld}^G(0,0)$, the topological fiber product yields a $p(\bullet,0)$-cartesian arrow:*

$$\begin{array}{ccc} M \times_A B & \xrightarrow{\varphi'} & M \\ \downarrow{\scriptstyle F'} & & \downarrow{\scriptstyle F} \\ A & \xrightarrow{\varphi} & B \end{array}$$

In particular, the functor $p(\bullet,0)\colon \mathbb{M}\mathrm{fld}^G(\bullet,0) \to \mathbb{S}\mathrm{pan}^G(\bullet,0)$ is a cartesian fibration.

Proof. Let $N \to C \in \mathbb{M}\text{fld}^G(0,0)$ be another object. We have to verify that the square

$$\begin{array}{ccc} \mathrm{map}_{\mathbb{M}\text{fld}^G(\bullet,0)}(N \to C, M \times_B A \to A) & \longrightarrow & \mathrm{map}_{\mathbb{M}\text{fld}^G(\bullet,0)}(N \to C, M \to B) \\ \downarrow & & \downarrow \\ \mathrm{map}_{\mathbb{S}\text{pan}^G(\bullet,0)}(C, A) & \longrightarrow & \mathrm{map}_{\mathbb{S}\text{pan}^G(\bullet,0)}(C, B) \end{array}$$

is cartesian. We can verify this by checking that the induced map on homotopy fibers in the vertical direction is an equivalence over all points. Using Lemma A.8, we see that we can model these homotopy fibers as strict fibers of the map on mapping spaces of topologically enriched category $\mathcal{M}fld_0^G \to \mathcal{S}pan_0^G$. The same lemma shows that these models are homeomorphic; indeed, both of them can be identified with embeddings of N into $M \times_B C \simeq M \times_A B \times_B C$ which commute with the map to A. □

Construction A.13. Lemma A.11 and Lemma A.12, together establish that

$$p\colon \mathbb{M}\text{fld}^G \longrightarrow \mathbb{S}\text{pan}^G$$

is a (cart,right)-fibration.

By the pullback condition in its definition, $\left(\mathbb{S}\text{pan}^G\right)^{1\,\text{op}}$ is a factorization double category. Moreover,

$$p^{1\,\text{op}}\colon \left(\mathbb{M}\text{fld}^G\right)^{1\,\text{op}} \longrightarrow \left(\mathbb{S}\text{pan}^G\right)^{1\,\text{op}}$$

is a (cocart,right)-fibration.

It follows from Proposition 4.5 that the source of $p^{1\,\text{op}}$ automatically is a factorization double category and that

$$\mathrm{Cnr}\left(p^{1\,\text{op}}\right)\colon \mathrm{Cnr}\left(\left(\mathbb{M}\text{fld}^G\right)^{1\,\text{op}}\right) \longrightarrow \mathrm{Cnr}\left(\left(\mathbb{S}\text{pan}^G\right)^{1\,\text{op}}\right)$$

is an op-Gray fibration.

Proposition A.14. *The functor*

$$\mathrm{Cnr}\left(p^{1\,\text{op}}\right)\colon \mathrm{Cnr}\left(\left(\mathbb{M}\text{fld}^G\right)^{1\,\text{op}}\right) \longrightarrow \mathrm{Cnr}\left(\left(\mathbb{S}\text{pan}^G\right)^{1\,\text{op}}\right)$$

is a cartesian fibration.

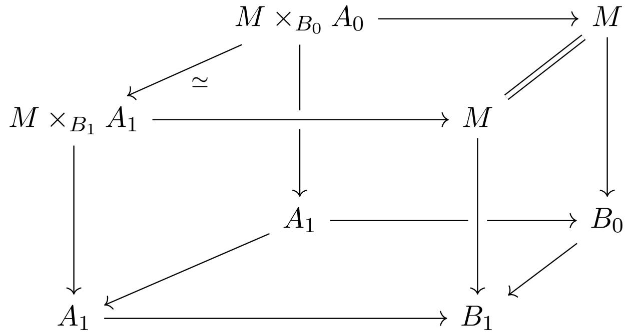
Proof. We use the same categorical argument as in [HHLN23b, Prop. 5.17]. By definition, op-Gray fibrations have cartesian lifts over ingressive morphisms and locally cartesian lifts over egressive morphisms. As the composition of a locally cartesian morphisms followed by a cartesian morphism is again cartesian, it follows that we have locally cartesian lifts over any morphism.

It remains to check that those locally cocartesian morphisms compose. We might reduce to the case where we compose an ingressive morphism with an egressive in the "wrong way", i.e. to diagrams coming from pullbacks

$$\begin{array}{ccc} A_0 & \longrightarrow & B_0 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & B_1 \end{array}$$

where $A_0 \to A_1$ and $B_0 \to B_1$ have finite fibers.

Given an object $M \to B_0$ in the fiber over B_0, consider the following diagram:



The front and back squares are cartesian and the left and right square are locally cartesian, showing that locally cartesian arrows compose, as claimed. □

Finally, we note that $\mathrm{Cnr}\left(\left(\mathbb{S}\mathrm{pan}^G\right)^{1\,\mathrm{op}}\right)^{\mathrm{op}} \simeq \mathrm{Span}_G^{f.i.}$. This follows from unwinding the respective definitions, as both sides are defined as a complete Segal space using maps out of the twisted arrow category.

In particular, we can straighten the functor $\mathrm{Cnr}\left(p^{1\,\mathrm{op}}\right)$ to obtain a functor

$$\mathrm{Span}_G^{f.i.} \longrightarrow \mathrm{Cat}\ .$$

Let us unwind this functor. The restriction to $(O_G^{\sqcup})^{\mathrm{op}} \subset \mathrm{Span}_G^{f.i.}$ is simply given by the straightening of the cartesian fibration $\mathcal{M}\mathit{fld}_0^G \to \mathbb{S}\mathrm{pan}_0 \simeq O_G^{\sqcup}$. The fiber over $A \in O_G^{\sqcup}$ is given by G-manifolds over A with embeddings between them. Given a map $f\colon A \to B$ and two manifolds $M \to B$ and $N \to B$ in $O_G^{\sqcup}$, we can use the description of the cartesian morphisms from Lemma A.12, to identify the induced map on mapping spaces as

$$\begin{aligned}
&\mathrm{Emb}_{/B}(M \to B, N \to B)\\
\simeq\ &\mathrm{map}_{\mathcal{M}\mathit{fld}_0^G}(M \to B, N \to B) \times_{\mathrm{map}_{O_G^{\sqcup}}(B,B)} \{\mathrm{id}_B\}\\
\longrightarrow\ &\mathrm{map}_{\mathcal{M}\mathit{fld}_0^G}(M \times_B A \to A, N \to B) \times_{\mathrm{map}_{O_G^{\sqcup}}(A,B)} \{f\}\\
\simeq\ &\mathrm{map}_{\mathcal{M}\mathit{fld}_0^G}(M \times_B A \to A, N \times_B A \to A) \times_{\mathrm{map}_{O_G^{\sqcup}}(A,A)} \{\mathrm{id}_A\}\\
\simeq\ &\mathrm{Emb}_{/A}(M \times_B A \to A, N \times_B A \to A)
\end{aligned} \tag{8}$$

with the pullback functoriality of embeddings.

For the forwards functoriality, we note that the lifts of the right fibration from Lemma A.11 of a manifold $M \to A$ along a finite covering $A \to B$ is given by $M \to A$. That is, norms are just given by post-composition.

Proposition A.15. *The straightening*

$$\mathrm{Span}_G^{f.i.} \longrightarrow \mathrm{Cat}$$

of $\mathrm{Cnr}\left(p^{1\,\mathrm{op}}\right)$ *is a G-symmetric monoidal G-category.*

Proof. We must check that the restriction to $(O_G^{\sqcup})^{\mathrm{op}}$ is product–preserving, i.e. the coassembly map

$$\mathrm{Cnr}\left(p^{1\,\mathrm{op}}\right)(A \sqcup B) \longrightarrow \mathrm{Cnr}\left(p^{1\,\mathrm{op}}\right)(A) \times \mathrm{Cnr}\left(p^{1\,\mathrm{op}}\right)(B)$$

is an equivalence. The essential surjectivity comes from the fact that a manifold with a map to a disjoint union is naturally a disjoint union of manifolds mapping to each factor individually. The fully faithfulness follows from the description of mapping spaces of (8). ☐

Definition A.16. We denote the straightening of $\mathrm{Cnr}\left(p^{1\,\mathrm{op}}\right)$ by

$$\underline{\mathrm{Mfld}} \colon \mathrm{Span}_G^{f.i.} \longrightarrow \mathrm{Cat}$$

and call it the G-symmetric monoidal G-category of G-manifolds.

We can moreover identify the category of manifolds equipped with a smooth map to G/H with the category of H-manifolds, by taking the fiber over eH.

Under this identification, restriction is indeed given by restricting the action. The norm along a finite index inclusion $K < H$ identifies with the functor sending a K-manifold M to the H-manifold $H \times_K M$.

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