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Equivariant homotopy: KR-theory

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Abstract

The goal of this project is the construction of a spectral sequence for computing KR-theory. The setting is equivariant homotopy theory and KR-theory is a variant of K-theory for spaces with an action of the group $\mathbb{Z}/2$. The spectral sequence starts at a certain equivariant version of cohomology and converges to KR-theory. We introduce equivariant homotopy theory and review all the concepts which are needed for the construction. As an application, we use the spectral sequence to compute KR-theory for a single point space and we give a proof of the fact that the homotopy fixed-points of the spectrum KR coincide with KO.

Resumé

Målet med dette speciale er konstruktionen af en spektralfølge, som udregner KR-teori. Vi arbejder i ækvivariant homotopiteori, og KR-teori er en variant af K-teori for rum med en gruppevirkning af $\mathbb{Z}/2$. Spektralfølgen begynder fra ækvivariant kohomologi og konvergerer til KR-teori. Vi introducerer ækvivariant homotopiteori, og vi gennemgår alle de koncepter, som vi har brug for i konstruktionen af spektralfølgen. Derefter anvender vi spekralfølgen til at udregne KR-teori af et enkelt punkt, og vi beviser at homotopi-fikspunkter af spektret KR er spektret KO.

Contents

Introduction

The main topic of this thesis is the construction of an Atiyah-Hirzebruch spectral sequence for computing KR-theory of spaces. The setting is equivariant homotopy theory, which deals with topological spaces endowed with a (continuous) action of a group. KR-theory is a variant of K-theory for equivariant spaces with an involution or, in other terms, an action of the group $\mathbb{Z}/2$. The purpose of this thesis is to describe in detail Dugger's [6] construction of the spectral sequence, which computes KR-theory of an equivariant space from its equivariant cohomology.

We fix a groups G; often this will simply be $\mathbb{Z}/2$. The first sections of the thesis contain a brief introduction to equivariant homotopy, including the definition of equivariant homotopy groups and Bredon homology and cohomology. Then we introduce a generalization of Bredon cohomology theories, the so-called $RO(G)$-graded cohomology theories, denoted $H^*_G(-)$, which are graded on the virtual representations of G, instead of the integers. To construct these cohomology theories, we define Mackey functors, which work as coefficients for these theories, in an algebraic way. A more extensive treatment of these topics can be found in [12].

The first step towards the construction of the spectral sequence is the computation of $H^*_G(G)$ and $H^*_G(pt)$ when $G = \mathbb{Z}/2$ (see 4.8). The result of this computation is stated by Dugger [6], and here we give a proof, using basic properties of $RO(G)$-graded cohomology. This computation will be useful to understand the homotopy of one of the main ingredients for the construction of the spectral sequence: equivariant Eilenberg-MacLane spaces.

Then, we will construct an equivariant version of Postnikov section functors. These functors are constructed similarly to the non-equivariant ones, but one has to take care of the fact that, in equivariant setting, the indexing of homotopy groups is no longer a total order, and one must decide what to mean by "killing higher homotopy groups". In practice we define Postnikov section functors in two ways: one of them has in general better properties, similar to the non-equivariant one, while the other one has the good property that when we compute it for certain G-spheres it has the homotopy type of an Eilenberg-MacLane space. Postnikov sections are used to build certain towers of homotopy fibrations, which are the main input for the construction of the spectral sequence.

In particular we work on the tower obtained from the space $\mathbb{Z} \times BU$, equipped with the involution coming from complex conjugation on BU. We show that the fibers of successive Postnikov sections are Eilenberg-MacLane spaces, and this requires a certain amount of work, and to do this we need to understand the Bott element in the KR-theory of a representation sphere. The tower of homotopy fibrations constructed this way gives rise to the wanted spectral sequence, which, under mild conditions, converges to KR-theory.

We also translate the construction of the spectral sequence to the stable setting: in this case one has to work with G-spectra rather than with G-spaces. This is contained in Section 8.2.

We have tried to make the topics presented here as clear as possible, including the proofs for almost all the results. It turns out in fact that there are several facts and lemmas in this subject, often used by many authors, whose proofs are very difficult to find in literature.

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Notation and conventions

To comply with the use of several texts in equivariant homotopy theory [12], we will use a notation for category names that diverges from the one commonly used in algebraic topology.

$\mathcal{T}$ The category of basepointed topological spaces and basepoint preserving continuous maps (otherwise denoted by $\mathcal{TOP}_*$).
$\mathcal{U}$ The category of (non basepointed) topological spaces and continuous maps (otherwise denoted by $\mathcal{TOP}$).
$\mathcal{G}$ The orbit category of a group G, with objects the orbits G/H for $H \leq G$ and G-maps between objects.

Throughout the thesis, G denotes a finite group; often we will look at the case $G = \mathbb{Z}/2$.

As usual in algebraic topology, we will assume all the spaces to be compactly generated weak Hausdorff. This is not a very restrictive hypothesis and it is satisfied, for instance, by CW-complexes.

Another standard assumption in equivariant homotopy theory is that the subgroups of G are closed. As mentioned, we will only work with finite groups, and points are closed by the weak Hausdorff assumption, hence this requirement is automatically satisfied by any subgroup.

The reader should be aware that two different conventions for the indexing of $\mathbb{Z}/2$-representations are present in literature, and this can lead to confusion. We will stick to the one for which $S^{p+q,q}$ denotes the representation sphere of topological dimension $p+q$, with q sign components. The reader is referred to the first Chapter for the definitions of the terms of the last sentence and to Remark 1.5.4 for further details on this issue.

1 Equivariant homotopy

1.1 G-spaces

The central objects in equivariant homotopy are topological spaces with the action of a group.

Definition 1.1.1. A *G-space*, or *equivariant space* is topological space with a continuous (left) action of a topological group G. The action is a continuous map:

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto g \cdot x, \end{aligned}$$

with the properties that $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ and $e \cdot x = x$, for every $g_1, g_2 \in G$ and $x \in X$.

We will soon begin to omit the dot when writing the action of a group element on a point of the space.

A good notion of map between G-spaces should take in account for the group action.

Definition 1.1.2. A continuous map $f\colon X \to Y$ is a *G-map* (or an *equivariant map*) if $f(g \cdot x) = g \cdot f(x)$ for every $g \in G$ and $x \in X$.

G-spaces and G-maps form the category $G\mathcal{U}$ of the non-basepointed G-spaces. The group G is itself a G-space, with the action given by the group operation. Moreover we can give a G-space structure to the product of two G-spaces X and Y, via the diagonal action of G on $X \times Y$.

If X and Y are G-spaces, the space $\mathcal{U}(X, Y)$ of continuous maps $X \to Y$ (not necessarily equivariant) is a G-space, where G acts by conjugation: for $f\colon X \to Y$, we have $g \cdot f$ defined by $(g \cdot f)(x) = g \cdot f(g^{-1} \cdot x)$.

Often it will be more convenient to work in the basepointed setting. Analogous definitions can be given for that case:

Definition 1.1.3. A *basepointed G-space* is a G-space with a basepoint which is fixed by the G-action. A G-map of basepointed G-space is a G-map which preserves the basepoint.

We can turn a G-space X into a basepointed one, by taking the topological sum of the space with a basepoint where G acts trivially: the space we obtain is denoted X_+. As usual, in the basepointed case, we will consider smash products as products. Given basepointed G-spaces (X, x_0) and (Y, y_0), their smash product is:

$$X \wedge Y = (X \times Y)/(X \times \{y_0\} \cup \{x_0\} \times Y),$$

and the action is induced by the diagonal action of G on the quotient. The category formed by basepointed G-spaces and basepoint preserving G-maps is denoted by $G\mathcal{T}$.

1.2 Homotopies of G-maps

To define homotopies of G-maps $X \to Y$, we consider the product of $X \times I$ (or $X \wedge I_+$ in the basepointed case) where the interval I is seen as a G-space with trivial action.

Two G-maps $f_0, f_1 \colon X \to Y$ are G-homotopic if there exists a G-map $X \times I \to Y$ whose compositions with the inclusions of $X \times \{0\}$ and $X \times \{1\}$ are f_0 and f_1. Similarly, two basepointed G-maps $f_0, f_1 \colon X \to Y$ are G-homotopic if there exists a G-map $X \wedge I_+ \to Y$ whose compositions with the inclusions of $X \wedge \{0\}_+$ and $X \wedge \{1\}_+$ are f_0 and f_1.

The set of the G-homotopy classes of G-maps is denoted by $[X,Y]_G$, or $[X,Y]_G^*$ in the basepointed case. We will omit the indication of the group G if it is clear from the context.

Note that G-spaces (and G-maps) are in particular H-spaces for every subgroup H of G. Hence we write $[X,Y]_H$ for the set of the homotopy classes of H-maps $X \to Y$. $[X,Y]_e$ denotes the homotopy classes of continuous (non-equivariant) maps.

1.3 Fixed points

Let $H \leq G$ be a subgroup. We define the *fixed point space* of X to be:

$$X^H = \{x \in X \mid h \cdot x = x \text{ for all } h \in H\}.$$

For a fixed $H \leq G$, we have the fixed point functor $G\mathcal{U} \to \mathcal{TOP}$: a G-map $f \colon X \to Y$ is sent to the continuous map $f^H \colon X^H \to Y^H$, defined by restriction of f to X^H. Fixed points are central in the development of the theory of equivariant stable homotopy. In particular, many results about the equivariant homotopy of a space can be reduced to ordinary homotopy theory of its fixed point space.

Lemma 1.3.1. *If K is a space regarded as a G-space with trivial action and X is a G-space, we have the isomorphisms (of sets):*

(a) $G\mathcal{U}(K,X) \cong \mathcal{U}(K,X^G)$
(b) $G\mathcal{U}(X,K) \cong \mathcal{U}(X/G,K)$.

Proof. (a) We can simply associate to a G-map $f \colon K \to X$ its restriction to $K \to X^G$. In fact, since G acts trivially on K, we have $g \cdot f(k) = f(g \cdot k) =$

$f(k)$, i.e. $f(k) \in X^G$ for all $k \in K$. This association has clearly an inverse, so it is a bijection.
(b) Given $f\colon X \to K$ G-map, by the triviality of the action on K, all the points of the same G-orbit in X are mapped to the same point in K. Hence we get a continuous map $\bar{f}\colon X/G \to K$. Conversely, a map $X/G \to K$ can be extended in a unique way to X to respect the fact that G acts trivially on K. One can easily check that this gives a bijection. $\square$

1.4 Representations of G

Recall that a *representation* of a group G on a real vector space V is a group homomorphism:

$$\rho\colon G \to \mathrm{GL}_{\mathbb{R}}(V),$$

that is a map such that

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2) \qquad \text{for any } g_1, g_2 \in G.$$

The *dimension* of the representation is the dimension of V: it is denoted by $\dim V$ or $|V|$. Even though a representation has extra structure in addition to its associated vector space, often, by abuse of notation, the representation itself is denoted by V.

In this project we will only consider finite dimensional real representations, in which the representation space V is a finite dimensional real vector space. We will be mostly concerned with *orthogonal* representations: the space V for such representations is an inner-product space and the map ρ has target $O(V)$, the group of the orthogonal automorphisms of V.

A representation is said to be *trivial* if the action induced on V is the trivial action, i.e. if $\rho(g)(v) = v$ for all $g \in G$ and $v \in V$.

Given two G-representations (V, ρ) and (V', ρ') we can form their sum $V \oplus V'$, which is a representation of G on the vector space $V \oplus V'$, with the action

$$\rho \oplus \rho'\colon G \to \mathrm{GL}_{\mathbb{R}}(V \oplus V')$$

defined by

$$(\rho \oplus \rho')(g)(v, v') = \rho(g)(v) \oplus \rho'(g)(v')$$

for all $v \in V$ and $v' \in V'$.

We can do similarly for the tensor product of two representations: the product of $V \otimes V'$ is a representation of G over $V \otimes V'$, with the action:

$$\rho \otimes \rho'\colon G \to \mathrm{GL}_{\mathbb{R}}(V \otimes V')$$

defined (on the basis elements of $V \otimes V'$) by

$$(\rho \otimes \rho')(g)(v \otimes v') = \rho(g)(v) \otimes \rho'(g)(v').$$

A subspace of a representation that is preserved by the action of G is called a *subrepresentation*. A representation V is *irreducible* if its only subrepresentations are 0 and V. A result of representation theory (Maschke's theorem) guarantees that, if the characteristic of the field K does not divide the order of the group G, the representation can be decomposed as a sum of irreducible subrepresentations. As said, we work with fields of characteristic 0, hence this always holds.

1.5 Spheres and discs

It will be useful to consider spheres with a non-trivial G-action: to do so, we associate to a representation V of G the *representation sphere* S^V, which is defined as the one-point compactification of V. It is a G-space, with the action associated to the representation and with trivial action on the basepoint ∞.

If we denote with n the trivial representation of G on $\mathbb{R}^n$, we recover as S^n the usual n-dimensional sphere (with trivial action).

Notation 1.5.1. When considering the sum of an orthogonal representation V with a trivial representation k, we use the symbol $+$ to denote the sum, rather than $\oplus$, not to make the notation too heavy. Similarly, if V contains k as a subrepresentation, we write $V - k$ to denote the complement of k in V.

Definition 1.5.2. For an orthogonal representation V, the *unit disc* in the representation is the space:

$$D(V) = \{v \in V \mid \|v\| \leq 1\}$$

and the *unit sphere* is:

$$S(V) = \{v \in V \mid \|v\| = 1\}.$$

The spaces $D(V)$ and $S(V)$ are clearly unbased G-spaces, restricting the action on V. We have a homeomorphism

$$D(V)/S(V) \cong S^V,$$

induced by the map:

$$\begin{aligned} D(V) &\to S^V \\ v &\mapsto \begin{cases} \frac{v}{1-\|v\|} & \text{if } \|v\| < 1, \\ \infty & \text{if } \|v\| = 1. \end{cases} \end{aligned}$$

This homeomorphism gives a cofibration sequence of basepointed G-spaces:

$$S(V)_+ \to D(V)_+ \to S^V, \tag{1.5.1}$$

which will be useful in several arguments.

Example 1.5.3. When $G = \mathbb{Z}/2 = \{e, g\}$, there are only two irreducible real representations.

- The 1-dimensional trivial representation, usually denoted $\mathbb{R}$:

$$\begin{aligned} \mathbb{Z}/2 &\to O(\mathbb{R}) \\ e &\mapsto \mathrm{id}_{\mathbb{R}} \\ g &\mapsto \mathrm{id}_{\mathbb{R}}. \end{aligned}$$

- The 1-dimensional sign representation, denoted $\mathbb{R}_-$ or σ:

$$\begin{aligned} \mathbb{Z}/2 &\to O(\mathbb{R}) \\ e &\mapsto \mathrm{id}_{\mathbb{R}} \\ g &\mapsto -\mathrm{id}_{\mathbb{R}}. \end{aligned}$$

Therefore, any $\mathbb{Z}/2$-representation V can be written as $\mathbb{R}^p \oplus (\mathbb{R}_-)^q$, for some $p, q \geq 0$. The representation sphere of V is denoted as:

$$S^V = S^{p+q,q},$$

where the first index is the topological dimension of the sphere and the second one is called the *weight*, that is the number of sign components.

Remark 1.5.4. When considering $\mathbb{Z}/2$-representations, other authors (for example Caruso [4]) use a different indexing and write (m, n) to mean $\mathbb{R}^m \oplus (\mathbb{R}_-)^n$ (and $S^{m,n}$ for the associated representation sphere), so that the first index is the dimension of the fixed-point subspace and the second is the weight. Instead, we stick to the convention described in the example above, following Dugger [6] and others.

1.6 Induced and coinduced spaces

Let H be a subgroup of G. Note that the group G is both a left and a right G-space, in the sense that the group operation can be seen both as a left and a right action. If we are given a H-space Y, we can form the *induced G-space*:

$$G \times_H Y = (G \times Y)/\sim,$$

where $(gh, y) \sim (g, hy)$ for every $g \in G$, $h \in H$ and $y \in Y$. The action of G on $G \times_H Y$ is defined by: $q \cdot [g, y] = [qg, y]$, for $q \in G$ and $[g, y] \in G \times_H Y$.

Similarly, we have the *coinduced G-space*:

$$\mathrm{map}_H(G, Y) = \{f \colon G \to Y \mid f(gh^{-1}) = hf(g) \text{ for } h \in H \text{ and } g \in G\},$$

with the G-action defined by:

$$(\gamma f)(g) = f(\gamma^{-1} g) \qquad \text{for } \gamma \in G.$$

The following lemma relates these two constructions with the previous definitions:

Lemma 1.6.1. *If X is a G-space and $H \leq G$, we have the following two G-homeomorphisms:*

$$G \times_H X \cong (G/H) \times X, \qquad \mathrm{map}_H(G, X) \cong \mathcal{U}(G/H, X).$$

Proof. For the first one, we define two maps:

$$\begin{aligned} G \times_H X &\to (G/H) \times X \\ [(g, x)] &\mapsto ([g], gx), \end{aligned}$$

$$\begin{aligned} (G/H) \times X &\to G \times_H X \\ ([g], x) &\mapsto [(g, g^{-1}x)]. \end{aligned}$$

One checks immediately that they are well-defined and G-maps. Moreover they are inverse to each other, so the thesis is proved.

As to the other homeomorphism, we can define the maps:

$$\begin{aligned} \mathrm{map}_H(G, X) &\to \mathcal{U}(G/H, X) \\ f &\mapsto ([g] \mapsto g(f(g))), \end{aligned}$$

$$\begin{aligned} \mathcal{U}(G/H, X) &\to \mathrm{map}_H(G, X) \\ f &\mapsto (g \mapsto g^{-1} f([g])). \end{aligned}$$

Also in this case, one can easily check that the two maps satisfy the needed properties. $\square$

The constructions of induced and coinduced spaces are functorial and give us a pair of adjunctions:

$$\begin{tikzcd} G\mathcal{U} \arrow[r, "U"] & H\mathcal{U} \arrow[l, bend right, "\mathrm{map}_H(G,-)"'] \arrow[l, bend left, "G\times_H -"] \end{tikzcd}$$

Lemma 1.6.2. *For $H \leq G$, the induced and the coinduced G-space functors are respectively left and right adjoint to the forgetful functor $G\mathcal{U} \to H\mathcal{U}$. We have then the isomorphisms:*

$$G\mathcal{U}(G \times_H Y, X) \cong H\mathcal{U}(Y, X), \qquad H\mathcal{U}(X, Y) \cong G\mathcal{U}(X, \mathrm{map}_H(G, Y)),$$

for every G-space X and H-space Y.

Proof. We can show the first adjunction by exhibiting the unit and counit maps: the unit is the H-map:

$$\begin{aligned} Y &\to G \times_H Y \\ y &\mapsto [e, y], \end{aligned}$$

while the counit is the G-map:

$$\begin{aligned} G \times_H X &\to X \\ [g, x] &\mapsto gx. \end{aligned}$$

We do similarly for the second adjunction. The unit is the G-map:

$$\begin{aligned} X &\to \mathrm{map}_H(G, X) \\ x &\mapsto (g \mapsto x), \end{aligned}$$

and the counit is the H-map:

$$\begin{aligned} \mathrm{map}_H(G, Y) &\to Y \\ f &\mapsto f(e). \end{aligned}$$ □

Remark 1.6.3. If we combine the previous two lemmas with the result of Lemma 1.3.1, we get the following isomorphism, which will be very useful soon and gives a hint about the usefulness of fixed points. Let $H \leq G$, X a G-space and K a space, regarded as a trivial G-space. Then:

$$G\mathcal{U}(G/H \times K, X) \cong G\mathcal{U}(G \times_H K, X) \cong H\mathcal{U}(K, X) \cong \mathcal{U}(K, X^H). \quad (1.6.1)$$

Note that this isomorphism is natural, and that, for every $H \leq G$, it expresses an adjunction between the functors:

$$G\mathcal{U} \to \mathcal{U} \qquad X \mapsto X^H$$

$$\mathcal{U} \to G\mathcal{U} \qquad X \mapsto G/H \times X.$$

All the arguments here mentioned work equally well if we work with basepointed spaces. In this case the adjunction is between the functors:

$$G\mathcal{T} \to \mathcal{T} \qquad X \mapsto X^H$$

$$\mathcal{T} \to G\mathcal{T} \qquad X \mapsto G/H_+ \wedge X,$$

and gives the isomorphism:

$$G\mathcal{T}(G/H_+ \wedge K, X) \cong G\mathcal{T}(G_+ \wedge_H K, X) \cong H\mathcal{T}(K, X) \cong \mathcal{T}(K, X^H). \quad (1.6.2)$$

1.7 G-CW complexes

The notion of CW complex can be translated to the equivariant setting. The main difference is given by 0-cells, or points: in equivariant context the orbits G/H have the role of points and so every cell has an orbit type.

The first stage in the construction of a G-CW complex X, the 0-skeleton X^0, is given by a disjoint union of orbits G/H, for different subgroups H of G. Then, each skeleton X^{n+1} is obtained from the previous skeleton X^n by attaching G-cells $G/H \times D^{n+1}$ along the boundaries via attaching G-maps $G/H \times S^n \to X^n$.

Let us be more precise about this definition.

Definition 1.7.1. A *relative G-CW complex* is a pair of G-spaces (X, A), together with a filtration $(X^n)_{n\in\mathbb{Z}}$ of X such that:

(a) $A \subseteq X^0$ and $\cdots = X^{-2} = X^{-1} = A$.
(b) $X = \cup_{n\in\mathbb{Z}} X^n$.
(c) For each $n \geq 0$, X^{n+1} is obtained by X^n by attaching equivariant n-cells, via the pushout square (in the category $G\mathcal{U}$):

$$\begin{tikzcd}
\coprod_{j\in J} G/H_j \times S^{n-1} \arrow[r, "\coprod \varphi_j"] \arrow[d] & X^{n-1} \arrow[d] \\
\coprod_{j\in J} G/H_j \times D^n \arrow[r, "g"] & X^n
\end{tikzcd}$$

where S^{n-1} and D^n have the trivial G-action and $H_j \leq G$ is a closed subgroup.

(d) $X = \operatorname{colim} X_n$, so that X has the colimit topology.

The definition immediately specializes to the non-relative case:

Definition 1.7.2. A G-space X, together with a filtration $(X^n)_{n\in\mathbb{Z}}$ is a *G-CW complex* if the pair $(X, \varnothing)$ is a relative G-CW complex.

Now we can make use of the isomorphism (1.6.1): the attaching G-maps $G/H \times S^{n-1} \to X^{n-1}$ are determined by the non-equivariant maps given by their restrictions to $S^{n-1} \to (X^{n-1})^H$.

Example 1.7.3. Let us describe the G-CW structure on the sphere

$$S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$$

with different actions of the group $\mathbb{Z}/2$. In the different cases listed below, the map $\alpha\colon S^2 \to S^2$ is the involution associated with the group action. Note that this example is somewhat easy, because when $G = \mathbb{Z}/2$, cells have orbit type either $\mathbb{Z}/2$ or $(\mathbb{Z}/2)/(\mathbb{Z}/2) = e$, or in other words, they are either free or fixed.

- $\alpha(x, y, z) = (-x, y, z)$. In this case the fixed subspace is the 1-sphere $\{y^2 + z^2 = 1\}$ and we have a G-CW structure with a fixed 0-cell, a fixed 1-cell and a free 2-cell, attached via the identity map. The fixed cells have orbit type $(\mathbb{Z}/2)/(\mathbb{Z}/2)$ (the trivial group), while the free ones have orbit type $\mathbb{Z}/2$. The G-CW structure can be written as:

$$\big((\mathbb{Z}/2)/(\mathbb{Z}/2) \times D^0\big) \cup \big((\mathbb{Z}/2)/(\mathbb{Z}/2) \times D^1\big) \cup \big((\mathbb{Z}/2) \times D^2\big)\,.$$

- $\alpha(x, y, z) = (-x, -y, z)$. In this case only the two poles of the sphere, $(0, 0, 1)$ and $(0, 0, -1)$, are fixed and they are two 0-cells of trivial orbit type in the G-CW structure. Then we have a free 1-cell with the two ends attached to the 0-cells and a free 2-cell attached to the 1-cell (which can be thought as a 1-sphere with the sign with the flip action on one coordinate). The G-CW structure can be written as:

$$\big((\mathbb{Z}/2)/(\mathbb{Z}/2) \times D^0\big) \cup \big((\mathbb{Z}/2)/(\mathbb{Z}/2) \times D^0\big) \cup \\ \big((\mathbb{Z}/2) \times D^1\big) \cup \big((\mathbb{Z}/2) \times D^2\big)\,.$$

- $\alpha(x, y, z) = (-x, -y, -z)$. Now the action has no fixed points, so all the cells are free. We can see very easily that the G-CW decomposition is:

$$\big((\mathbb{Z}/2) \times D^0\big) \cup \big((\mathbb{Z}/2) \times D^1\big) \cup \big((\mathbb{Z}/2) \times D^2\big)\,,$$

with the obvious attaching maps.

Recall that, in non-equivariant homotopy theory, a map $f\colon X \to Y$ is said to be n-connected if $\pi_q(f)\colon \pi_q(X) \to \pi_q(Y)$ is a bijection for $q < n$ and a surjection for $q = n$.

Let $\nu\colon \{H \leq G \mid H \text{ closed subgroup}\} \to \{-1, 0, 1, \dots\}$ be a function, constant on conjugacy classes of subgroups of G.

Definition 1.7.4. A G-map $f\colon X \to Y$ is a *ν-equivalence* if $f^H\colon X^H \to Y^H$ is $\nu(H)$-connected for all closed subgroups $H \leq G$.

Definition 1.7.5. We say that a G-CW complex X has *dimension* ν if all the cells of orbit type G/H have dimension less or equal to $\nu(H)$.

We have the following proposition, about extension and lifting of homotopies for G-CW complexes.

Proposition 1.7.6 (HELP: Homotopy Extension and Lifting Property)**.** *Let X, Y, Z be G-CW complexes and A a finite subcomplex of X of dimension ν. Let $e\colon Y \to Z$ be a ν-equivalence. If we have maps:*

$$\begin{aligned} &g\colon A \to Y, \\ &h\colon A \times I \to Z, \\ &f\colon X \to Z, \end{aligned}$$

such that, in the following diagram, $eg = hi_1$ and $fi = hi_0$:

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl, "h"] \arrow[dd] & & A \arrow[ll, "i_1"'] \arrow[dl, "g"] \arrow[dd, "i"] \\
& Z & & Y \arrow[ll, "e"'] & \\
X \arrow[rr, "i_0"'] \arrow[ur, "f"] & & X \times I \arrow[ul, dashed, "\tilde{h}"] & & X. \arrow[ll, "i_1"] \arrow[ul, dashed, "\tilde{g}"]
\end{tikzcd} \tag{1.7.1}$$

Then there are maps $\tilde{f}$ and $\tilde{g}$ making the diagram commute.

This proposition is proved in a very similar way to the non-equivariant case. We sketch the proof here, referring the reader to [12] or [14] for more details. One can prove the result for the case

$$(X, A) = (G/H \times D^n, G/H \times S^{n-1}),$$

for $n \leq \nu(H)$. For this one can pass to fixed points and use the fact that f is a ν-equivalence, reducing this way to the non-equivariant statement. Then one can use this case to show the result for a general G-CW complex, by doing induction on cells, considering their attaching maps.

Remark 1.7.7. The previous proposition is stated in this form by May in [11] and in [12], respectively in the ordinary and in the equivariant versions. It has the quality of summing up concisely in a unique diagram the two properties of extension and lifting of homotopies, but it might not be immediate to see where these two properties appear in the diagram, especially for the lifting part.

• For the extension property, we have to look at the left square in the diagram (1.7.1): the homotopy that gets extended is $h\colon A \times I \to Z$ and the map $f\colon X \to Z$ is one end of the extension. We assume that the upper triangle is commutative and the extension property says that the two other triangles are also commutative. The next diagram shows just the homotopy extension property: we assume that the square is commutative and deduce that there exists $\tilde{h}$ making the two triangles commute.

$$\begin{tikzcd} A \arrow[r, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "h"] \\ X \arrow[r, "f"] \arrow[ur, dashed, "\tilde{h}"] & Z \end{tikzcd}$$

• By definition, we say that $e\colon Y \to Z$ has the homotopy lifting property with respect to a space V if, for any homotopy $F\colon V \times I \to Z$ and any $\tilde{F}_0\colon V \times \{0\} \to Y$ making the following solid square commute:

$$\begin{tikzcd} V \times \{0\} \arrow[r, "\tilde{F}_0"] \arrow[d, "i"'] & Y \arrow[d, "e"] \\ V \times I \arrow[r, "F"] \arrow[ur, dashed, "\tilde{F}"] & Z, \end{tikzcd}$$

there exists $\tilde{F}\colon V \times I \to Y$ making the two triangles commute. This property is a special case of the *lift extension property for a pair* (X, A): we assume that the outer square commutes and the property says that there exists a dashed map making the two triangles commute.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i"'] & Y \arrow[d, "e"] \\ X \arrow[r, "f"] \arrow[ur, dashed, "\tilde{g}"] & Z. \end{tikzcd}$$

The homotopy lifting property is the case $(X, A) = (V \times I, V \times \{0\})$. In the HELP Proposition, if we take h to be the constant homotopy (so that

$hi_t = fi$ for all $t \in I$), then the square on the right in (1.7.1) shows that e has the lift extension property for the pair (X, A) and so, in particular, the homotopy lifting property.

As non-equivariantly [11], the HELP proposition can be used to prove Whitehead theorem, in its equivariant version.

Theorem 1.7.8 (Equivariant Whitehead)**.** *Let $f\colon Y \to Z$ be a ν-equivalence and X a G-CW complex. Then the induced map:*

$$f_*\colon [X, Y]_G \to [X, Z]_G$$

is a bijection if X has dimension less than ν and a surjection if X has dimension ν.

Proof. We can apply Proposition 1.7.6 to the pair $(X, \varnothing)$ to show surjectivity. To prove injectivity, we can apply the same proposition to the pair $(X \times I, X \times \{0, 1\})$. □

1.8 Equivariant homotopy groups

We would like to give an equivariant definition of homotopy groups. Let X be a basepointed G-space. Taking $[S^n, X]^*_G$ as definition is not what we need: in fact, S^n has trivial G-action, hence any equivariant map $S^n \to X$ maps into the G-fixed points of X. Hence we need to consider spheres with a G-action: to do this we smash a sphere with a G-orbit. For $n \in \mathbb{N}$ and $H \leq G$, we define the n^{th} *equivariant homotopy group*:

$$\pi_n^H(X) = [S^n \wedge G/H_+, X]^*_G.$$

Since $S^n \wedge G/H_+$ is H-fixed, the every G-map $S^n \wedge G/H_+ \to X$ has image contained in X^H.

Remark 1.8.1. The adjunction of Remark 1.6.3 can be proved in the homotopy categories, giving an isomorphism:

$$[G/H_+ \wedge K, X]^*_G \cong [K, X]^*_H \cong [K, X^H]^*.$$

In our case, when taking $K = S^n$, one has an isomorphism of groups (for $n \geq 1$):

$$\pi_n^H(X) = [S^n \wedge G/H_+, X]^*_G \cong [S^n, X]^*_H \cong \pi_n(X^H), \qquad (1.8.1)$$

for any $H \leq G$. This result is very useful because allows us to reduce many questions about equivariant homotopy groups to questions about non-equivariant homotopy groups of fixed points.

It does not take long to slightly generalize our definition, replacing n by any orthogonal G-representation V:

$$\pi_V^H(X) = [S^V \wedge G/H_+, X]_G^* \cong [S^V, X]_H^*.$$

This way we get the V^{th} *equivariant homotopy group.* The isomorphism is a consequence of Lemma 1.6.2, in its basepointed version.

1.9 V-connectedness and weak equivalences

In ordinary homotopy theory, one has the notion of *n-connected* space, that is a connected space whose first n homotopy groups are zero. It is reasonable to try to generalize this to the equivariant setting, where we can map into spaces not only from spheres S^n, but also from representation spheres S^V.

Let V be an orthogonal G-representation and X a G-space. If we consider $[S^{V+k} \wedge G/H_+, X]_*$, we note that we can give a meaning to this writing also for negative values of $k \geq -\dim(V^H)$. In fact V can be written as $V(H) \oplus V^H$, where V^H is the subspace fixed by H and $V(H)$ its orthogonal complement. So, for any subgroup $H \leq G$, we can regard V as a H-representation that has a fixed subspace of dimension $\dim(V^H)$ and so $V^H + k$ is still a genuine representation as long as k satisfies the bound $k \geq -\dim(V^H)$.

In more detail, recall that we have the G-homeomorphism[1] $G_+ \wedge_H X \cong G/H_+ \wedge X$ and the following isomorphism, as a consequence of Lemmas 1.6.1 and 1.6.2:

$$[S^{V+k} \wedge G/H_+, X]_G^* \cong [S^{V+k} \wedge_H G_+, X]_G^* \cong [S^{V+k}, X]_H^*.$$

The last term can be also written as $[S^{V(H)\oplus V^H+k}, X]_H^*$, since V^H is fixed by H. We are now ready to give our definition:

Definition 1.9.1. Let V be an orthogonal G-representation and X a G-space. X is said to be *V-connected* if $[S^{V+k} \wedge G/H_+, X]^* = 0$ for all $H \leq G$ and all $0 \geq k \geq -\dim(V^H)$.

As in ordinary homotopy theory, we say that a map is a weak equivalence if it induces isomorphisms on the homotopy groups, in this case the equivariant ones.

Definition 1.9.2. A G-map $f\colon X \to Y$ is an *(equivariant) weak equivalence* if $f^H\colon X^H \to Y^H$ is a weak homotopy equivalence (in the sense of ordinary homotopy theory) for every $H \leq G$.

[1] $G_+ \wedge_H X$ is the basepointed version of the induced G-space and the homeomorphism is proved as done in Lemma 1.6.1 for the non-basepointed case

Equivalently, by (1.8.1), we can say that f is an equivariant weak equivalence if it induces isomorphisms:

$$\pi_n^H(X) = [S^n \wedge G/H_+, X]_G^* \xrightarrow{f_*} [S^n \wedge G/H_+, Y]_G^* = \pi_n^H(Y),$$

for all $n \geq 0$ and $H \leq G$. The next lemma, which is stated by Lewis [10], gives an equivalent condition for being an equivariant weak equivalence. We give a proof in the case $G = \mathbb{Z}/2$.

Proposition 1.9.3. *Let V be an orthogonal $\mathbb{Z}/2$-representation containing at least one copy of the trivial representation and let X, Y be $(V-1)$-connected spaces. Let $f\colon X \to Y$ be an equivariant map. Then f is an equivariant weak equivalence if it induces isomorphisms:*

$$[S^{V+k} \wedge G/H_+, X]_G^* \xrightarrow{f_*} [S^{V+k} \wedge G/H_+, Y]_G^*$$

for all $k \geq 0$ and all $H \leq G$.

To prove this result we need a preliminary lemma.

Lemma 1.9.4. *Let V be a G-representation such that $1 \subseteq V$. Let X be a $(V-1)$-connected G-space and F be a free G-CW-complex of dimension not greater than* $\dim(V-1)$. *Then:*

$$[F, X]_H^* = \{*\}$$

for any $H \leq G$.

Proof. G acts freely on F, hence all the cells of F are free, i.e. they are discs of the form $D^n \wedge G_+$. We will prove the claim by induction on the cells of F. It is clearly true when F is a single 0-cell. Let $F^{(i-1)}$ be a stage in the construction of F by attaching cells, and let us assume the claim for it. Let $F^{(i)}$ be the sub-complex obtained by attaching a free cell $D^{n+1} \wedge G_+$. We have a homotopy cofiber sequence:

$$F^{(i-1)} \to F^{(i)} \to S^n \wedge G_+.$$

If we map it into the the space X (we consider homotopy classes of H-maps), we get the following exact sequence of pointed sets:

$$[F^{(i-1)}, X]_H^* \leftarrow [F^{(i)}, X]_H^* \leftarrow [S^n \wedge G_+, X]_H^*.$$

Recall that $[S^n \wedge G_+, X]_H^* \cong [S^n \wedge G_+ \wedge G/H_+, X]_G^*$, and the wedge product $G_+ \wedge G/H_+$ can be written as a union of G-orbits. Thus $[S^n \wedge G_+, X]_H^* = 0$ because X is $(V-1)$-connected. By induction we have that $[F^{(i-1)}, X]_H^* = \{*\}$, therefore, by exactness, $[F^{(i)}, X]_H^* = \{*\}$. □

Proof of Proposition 1.9.3. Let $l > 0$ and $H \leq G$. If we take k such that $l = \dim((V-1)^H) + k \geq 0$, we look at the inclusion $S^l \to S^{V+k-1}$ and form the cofiber sequence:

$$S^l \to S^{V+k-1} \to Z,$$

where Z is a free G-space, since we are quotienting out all the trivial components of the representation $V+k-1$. Note that the inclusion $S^l \to S^{V+k-1}$ is actually the l-suspension of the map $S^0 \to S^{V+k-1-l}$, since $V+k-1$ contains the l-dimensional trivial representation. Now we consider homotopy classes of pointed H-maps mapping into X and Y from the sequence, which give the following diagram, with exact rows:

$$\begin{tikzcd}
\cdots & {[S^l, X]} \arrow[l] \arrow[d] & {[S^{V+k-1}, X]} \arrow[l] \arrow[d, "\cong"] & {[Z, X]} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & {[S^l, Y]} \arrow[l] & {[S^{V+k-1}, Y]} \arrow[l] & {[Z, Y]} \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

The central vertical arrow is an isomorphism: if $k \leq 0$, this is a consequence of the fact that X and Y are $(V-1)$-connected, so domain and target are trivial. If $k-1 \geq 0$, it is an isomorphism by assumption.

The vertical map to the right is also an isomorphism: to see this, we use the fact that Z is a free G-space, and so in particular a free H-space, and it has a G-CW structure with dimension not greater than $\dim(V-1)$. Se we can apply Lemma 1.9.4.

Since our map $S^l \to S^{V+k-1}$ is a suspension for $l \geq 1$, the diagram extends to the left with other two maps that are isomorphisms, hence we can apply the five-lemma, obtaining that $[S^l, X]_H \to [S^l, Y]_H$, i.e. $\pi_l^H(X) \to \pi_l^H(Y)$ is an isomorphism.

It remains to show the same for $l = 0$. In this case we cannot apply the five-lemma, because the diagram does not extend two steps to the left (and, moreover, we don't have abelian groups, but just sets). For $l = 0$, we have $k = -\dim((V-1)^H)$, and so the cofibration has the form:

$$S^0 \to S^{n\sigma} \to S^{n\sigma}/S^0,$$

where σ is the sign representation of $\mathbb{Z}/2$. The representation sphere can be easily seen as the unreduced suspension of the unit sphere in the representation:

$$S^{n\sigma} \cong \operatorname{susp}(S(n\sigma)),$$

and the sphere S^0 sits into this space as the two collapsed bases of the cylinder. This identifies

$$S(n\sigma)_+ \to S^0 \to S^{n\sigma}$$

as a cofiber sequence.

If we map it into the space X, we get the exact sequence of pointed sets:

$$[S(n\sigma)_+, X]^*_H \leftarrow [S^0, X]^*_H \leftarrow [S^{n\sigma}, X]^*_H.$$

The set to the left is a trivial, by the previous lemma. The set to the right is trivial too, because of the hypothesis of $(V-1)$-connectedness on X. Therefore $[S^0, X]^*_H = 0$. The same holds for the space Y, hence the proof is complete also for $l = 0$. $\square$

The following corollary does not add anything new to what we know, but is just a restatement of the lemma for $G = \mathbb{Z}/2$ and $V = \mathbb{R}^n \oplus (\mathbb{R}_-)^n = \mathbb{C}^n$, that is the case where the result will be mostly used.

Corollary 1.9.5. *Let V be the $\mathbb{Z}/2$-representation $\mathbb{R}^n \oplus (\mathbb{R}_-)^n$, with $n \geq 1$. Let X and Y be $\mathbb{Z}/2$-spaces verifying the conditions:*

- $[S^{i,0} \wedge \mathbb{Z}/2_+, X]^*_{\mathbb{Z}/2} = [S^{i,0} \wedge \mathbb{Z}/2_+, Y]^*_{\mathbb{Z}/2} = 0$, *for* $0 \leq i < 2n$,
- $[S^{i,0}, X]^*_{\mathbb{Z}/2} = [S^{i,0}, Y]^*_{\mathbb{Z}/2} = 0$, *for* $0 \leq i < n$.

If $f \colon X \to Y$ is a $\mathbb{Z}/2$-map which induces isomorphisms:

- $[S^{n+i,n} \wedge \mathbb{Z}/2_+, X]^*_{\mathbb{Z}/2} \xrightarrow[\cong]{f_*} [S^{n+i,n} \wedge \mathbb{Z}/2_+, Y]^*_{\mathbb{Z}/2}$, *for* $i \geq 0$,
- $[S^{n+i,n}, X]^*_{\mathbb{Z}/2} \xrightarrow[\cong]{f_*} [S^{n+i,n}, Y]^*_{\mathbb{Z}/2}$, *for* $i \geq 0$,

then f is a weak equivalence.

2 Bredon homology and cohomology

We are interested in a suitable notion of cohomology in equivariant context. There are different approaches to this and Bredon's proposal gives cohomology theories graded on the integers, by constructing suitable "coefficient systems". This notion has later been extended to what is called Mackey functors, which are used to define more general cohomology theories, graded on representations rather than on integers.

2.1 Coefficient systems

Recall that we denote with $\mathcal{G}$ the orbit category of a group G. If $H, K \leq G$, note that we have a map $G/H \to G/K$ in this category if and only if H is conjugate to a subgroup of K, i.e. there is some $g \in G$ such that $g^{-1}Hg \leq K$.

Let $h\mathcal{G}$ be the homotopy category of $\mathcal{G}$.

In this section we define the Bredon homology and cohomology. They are defined on G-CW complexes and then the definition can be extended on every G-space via CW approximation (see [12] for details on this). The functors we obtain satisfy an equivariant version of the axioms for (co)homology.

Definition 2.1.1. A *contravariant coefficient system* M is a functor $h\mathcal{G}^{\mathrm{op}} \to \mathcal{AB}$. A *covariant coefficient system* N is a functor $h\mathcal{G} \to \mathcal{AB}$.

Homology and cohomology of equivariant spaces are defined to take coefficients in coefficient systems. Let us give an example of this notion:

Example 2.1.2. Let X be a G-space. We define:

$$\begin{aligned} \underline{\pi}_n(X)\colon h\mathcal{G}^{\mathrm{op}} &\to \mathcal{AB} \\ G/H &\mapsto \pi_n(X^H). \end{aligned}$$

For a G-space X we have a contravariant fixed-point functor $h\mathcal{G} \to h\mathcal{TOP}$ defined by:

$$G/H \mapsto X^H,$$

which sends a map $f\colon G/H \to G/K$ such that $f(eH) = gK$ to $\tilde{f}\colon X^K \to X^H$ with $\tilde{f}(x) = gx$. We can define a coefficient system by composing it with H_n: let X be a G-CW complex. and n an integer. We define a contravariant coefficient system $\underline{C}_n(X) = \underline{H}_n(X^n, X^{n-1}; \mathbb{Z})$ by:

$$\begin{aligned} \underline{C}_n(X)\colon h\mathcal{G}^{\mathrm{op}} &\to \mathcal{AB} \\ G/H &\mapsto H_n\left((X^n)^H, (X^{n-1})^H\right). \end{aligned}$$

We want to define a map of coefficient systems (i.e. a natural transformation of functors) $\underline{H}_n(X^n, X^{n-1}; \mathbb{Z}) \to \underline{H}_n(X^{n-1}, X^{n-2}; \mathbb{Z})$. To do this, we observe that, for each $H \leq G$, we have the long exact sequence in homology for the triple of spaces $\left((X^H)^n, (X^H)^{n-1}, (X^H)^{n-2}\right)$, which contains the differential:

$$H_n\left((X^H)^n, (X^H)^{n-1}\right) \xrightarrow{d_H} H_{n-1}\left((X^H)^{n-1}, (X^H)^{n-2}\right).$$

The homomorphisms d_H define the components of the natural transformation d and so we get the chain complex:

$$\cdots \to \underline{C}_n(X) \xrightarrow{d} \underline{C}_{n-1}(X) \xrightarrow{d} \underline{C}_{n-2}(X) \to \ldots.$$

One can easily check that it is actually a chain complex, by verifying that $d^2 = 0$.

Given two coefficient systems M, M', we denote with $\mathrm{Hom}_{\mathcal{G}}(M, M')$ the set of the maps of coefficient systems $M \to M'$. Let

$$C^n_G(X; M) = \mathrm{Hom}_{\mathcal{G}}(\underline{C}_n(X), M).$$

$C^*_G(X; M)$ is a cochain complex, with the maps:

$$\delta = \mathrm{Hom}_{\mathcal{G}}(d, id) \colon C_n(X; M) \to C_{n+1}(X; M).$$

2.2 Definition of homology and cohomology

Definition 2.2.1. Let X be a G-CW complex and $M \colon h\mathcal{G}^{\mathrm{op}} \to \mathcal{AB}$ a contravariant coefficient system. The *Bredon cohomology* of X with coefficients M is the homology of the cochain complex $C^*_G(X; M)$. It is denoted $H^*_G(X; M)$.

The definition of homology will be in terms of a covariant coefficient system $N \colon h\mathcal{G} \to \mathcal{AB}$. We want to tensor our chain complex $\underline{C}_n(X)$ on the right with N. To do this, we use the following “coend” construction:

If M and N are respectively a contravariant and a covariant coefficient system, we form the abelian group:

$$M \otimes_{\mathcal{G}} N = \sum_{H \leq G} (M(G/H) \otimes N(G/H)) / \approx,$$

where $(mf^*, n) \approx (m, f_* n)$ for a G-map $f \colon G/H \to G/K$ and $m \in M(G/K)$, $n \in M(G/H)$.

Our cellular chain complex is then:

$$C^G_n(X; N) = \underline{C}_n(X) \otimes_{\mathcal{G}} N,$$

with boundary maps $\partial = d \otimes 1$.

Definition 2.2.2. Let X be a G-CW complex and $N \colon h\mathcal{G} \to \mathcal{AB}$ a covariant coefficient system. The homology of the chain complex $C^G_*(X; N)$ is called the *Bredon homology* of X with coefficients N and is denoted by $H^G_*(X; N)$.

3 Equivariant spectra

3.1 The non-equivariant definition

In non-equivariant context, we have the usual notion of prespectrum: a sequence of (basepointed) spaces $(E_n)_{n \in \mathbb{N}}$ with structure maps $\sigma_n \colon \Sigma E_n \to E_{n+1}$ for all $n \in \mathbb{N}$, going from the suspension of one space to the next space in the sequence. A spectrum is a prespectrum such that the adjoint structure maps $\tilde{\sigma}_n \colon X_{n+1} \to \Omega X_n$ are homeomorphisms.

To get an equivariant version of these definitions, we might just require the spaces to be G-spaces and the structure maps to be equivariant, but this does not exploit completely our equivariant structure: in fact, this way we would only consider spheres with trivial G-action, as $\Sigma^n X = S^n \wedge X$, and we have no reason to restrict our attention only to them, when we know S^V as a G-space for every representation V of G.

3.2 Definition

To get a notion of equivariant spectrum with spaces indexed on representations, we need first a definition:

Definition 3.2.1. A *G-universe* $\mathscr{U}$ is a real inner product space of countable infinite dimension with a G-action such that:

(a) $\mathscr{U}$ contains the trivial representation of G.
(b) $\mathscr{U}$ contains infinitely many copies of each of its finite dimensional subrepresentation.

If every subrepresentation of $\mathscr{U}$ is trivial, i.e. $\mathscr{U} = \mathbb{R}^\infty$ with trivial G-action, then $\mathscr{U}$ is said to be *trivial*. A G-universe is *complete* if it contains every irreducible representation of G, i.e. up to isomorphism it is a direct sum of $(V_i)^\infty$, where $\{V_i\}_i$ is the set of all the irreducible representations of G.

The finite dimensional subrepresentation of $\mathscr{U}$ work as indexing spaces for equivariant spectra:

Definition 3.2.2. A *G-prespectrum E* on a G-universe $\mathscr{U}$ is a collection of basepointed G-spaces EV for each finite dimensional indexing space (i.e. subrepresentation) $V \subseteq \mathscr{U}$ together with basepoint-preserving structure G-maps:

$$\sigma_{V,W} \colon \Sigma^{W-V} EV \to EW,$$

whenever $V \subseteq W \subseteq \mathscr{U}$, where $W - V$ denotes the orthogonal complement of V in W. The structure maps are also required to make the appropriate transitivity diagram commute when $V \subseteq W \subseteq X \subseteq \mathscr{U}$.

Definition 3.2.3. A *G-spectrum E* on a G-universe $\mathscr{U}$ is a G-prespectrum such that the adjoint structure maps:

$$\tilde{\sigma}_{V,W} \colon EV \to \Omega^{W-V} EW,$$

for $V \subseteq W \subseteq \mathscr{U}$ are homeomorphisms.

A G-spectrum indexed on a complete G-universe is called *genuine*. If the G-universe is trivial, the G-spectrum is called *naive*. Note that a naive G-spectrum is equivalent to a sequence of G-spaces E_n with G-maps $\Sigma E_n \to E_{n+1}$, i.e. a usual spectrum with a G-action on every space.

A *map of G-spectra indexed on $\mathscr{U}$* $E \to E'$ is a collection of basepoint-preserving G-maps $f_V \colon EV \to E'V$ for every subrepresentation $V \subseteq \mathscr{U}$, commuting with the structure maps:

$$\begin{tikzcd} \Sigma^{W-V} EV \arrow[r] \arrow[d, "\sigma_{V,W}"] & \Sigma^{W-V} E'V \arrow[d, "\sigma'_{V,W}"] \\ EW \arrow[r] & E'W \end{tikzcd}$$

for every $W \subseteq \mathscr{U}$.

G-spectra indexed on a G-universe $\mathscr{U}$ and maps between them form a category, denoted $G\mathcal{S}\mathscr{U}$.

Given a G-space X we can form the *suspension G-prespectrum* $\pi^\infty X$ of X indexed on $\mathscr{U}$, defined by

$$(\pi^\infty X)V = \Sigma^V X = S^V \wedge X$$

for every finite dimensional $V \subseteq \mathscr{U}$. The forgetful functor from G-spectra to G-prespectra has an adjoint, the "spectrification" functor. Applying it to $\pi^\infty X$, we get the suspension spectrum of X, denoted $\Sigma^\infty X$, as one does non-equivariantly.

Remark 3.2.4. This functor produces a G-spectrum from a G-space:

$$G\mathcal{U} \xrightarrow{\Sigma^\infty} G\mathcal{S}\mathscr{U}.$$

It has an adjoint:

$$\begin{aligned} G\mathcal{S}\mathscr{U} &\xrightarrow{\Omega^\infty} G\mathcal{U} \\ E &\mapsto E_0, \end{aligned}$$

which takes a spectrum to its 0^{th} space, i.e. the space corresponding to the representation 0.

A map of G-spectra $f \colon E \to E'$ is said to be a *weak equivalence* if every $f_V \colon EV \to E'V$ is an equivariant weak equivalence, i.e. it induces isomorphisms $\pi_*(EV^H) \to \pi_*(E'V^H)$ for every closed subgroup $H \leq G$.

3.3 Different interpretation when $G = \mathbb{Z}/2$

In the case $G = \mathbb{Z}/2$, any real representation V is contained in $\mathbb{C}^n$ (with the conjugation action) for n large enough. Hence we can simplify the definition of $\mathbb{Z}/2$-spectrum by giving only a map:

$$\mathbb{C}^n \mapsto E_{2n,n} = E_n$$

and structure maps

$$S^{2,1} \wedge E_n \to E_{n+1}.$$

This way we get a simplified definition of equivariant spectra in the case we are most interested in.

4 $RO(G)$-graded cohomology

4.1 Introduction

The non-equivariant notion of suspension and loop-space are defined in terms of the trivial spheres S^n. Since we want to look at those operations in our equivariant settings, we will define them in terms of the representation spheres S^V, for an orthogonal G-representation $G \to O(V)$:

$$\Sigma^V(X) = S^V \wedge X$$
$$\Omega^V(X) = \mathcal{T}(S^V, X),$$

for any basepointed G-spaces X. The group G acts on $\Omega^V(X)$ by conjugation (as seen in 1.1). Equivariant suspension and loop-space functors are adjoint, just as in the non-equivariant case. In particular, Σ^V is left adjoint to Ω^V ([12, IX.1]), and the unit of the adjunction is the map:

$$\eta_Y : Y \to \Omega^V \Sigma^V Y.$$

It is possible to prove a Freudenthal suspension theorem for G-CW complexes. We will only give the statement here, the proof can be found in [14].

Theorem 4.1.1. *The map $\eta : Y \to \Omega^V \Sigma^V Y$ is a ν-equivalence if:*

(a) $\nu(H) \leq 2c(Y^H) + 1$ for all $H \leq G$ with $|V^H| > 0$,
(b) $\nu(H) \leq c(Y^K)$ for all the pairs of subgroups $H, H \leq G$ with $V^K > V^H$.

The cited adjunction gives the following commutative diagram:

$$\begin{tikzcd}
{[X,Y]^*_G} \arrow[r, "\Sigma^V"] \arrow[dr, "\eta_*"'] & {[\Sigma^V X, \Sigma^V Y]^*_G} \arrow[d, "\cong"] \\
& {[X, \Omega^V \Sigma^V Y].}
\end{tikzcd}$$

Combining the previous result and Whitehead theorem, we get a sufficient condition for the map Σ^V to be a bijection.

4.2 Grading cohomology on representations

The goal of this section is to define a suitable notion of cohomology for G-equivariant spaces, graded on G-representations rather than on integers. In literature, this takes the name of $RO(G)$-graded cohomology. $RO(G)$ is the ring of the isomorphism classes of orthogonal G-representations. To construct it, we consider at first the semi-ring of isomorphism classes of real orthogonal G-representations, with the direct sum as sum and the tensor product as product. Then we make the Grothendieck construction to obtain a ring from it. The elements of $RO(G)$ can be described as formal differences $V - W$ of isomorphism classes of representations.

As pointed out by Adams in [1], what is usually called $RO(G)$-graded cohomology is not actually graded on the ring $RO(G)$. In fact, if we take two representations which are isomorphic, the cohomology is the same for both on spaces, but it is not in general the same on the maps between spaces.

4.3 Mackey functors

It is possible to give a convenient algebraic description of Mackey functors, as done in [12, XIX.3]. We say that a *Mackey functor* for a finite group G is a pair of functors M_* and M^* from G-sets to abelian groups, M_* covariant and M^* contravariant, which are identical on objects and convert disjoint unions to direct sums. Moreover, they are such that, for any pullback square of G-sets:

$$\begin{tikzcd}
P \arrow[r, "\delta"] \arrow[d, "\gamma"'] & X \arrow[d, "\alpha"] \\
Y \arrow[r, "\beta"'] & Z,
\end{tikzcd}$$

we have $M^*(\alpha) \circ M_*(\beta) = M_*(\delta) \circ M^*(\gamma)$.

In what follows we will be interested in the values assumed by Mackey functors on the orbit category $\mathcal{G}$, subcategory of the G-sets. If we take

$G = \mathbb{Z}/2$, we get a very simple orbit category $\mathcal{G}$. The diagram shows the maps different from the identity:

$$\begin{tikzcd} G/e \arrow[r, "i"] \arrow[loop below, "t"] & G/G \end{tikzcd}$$

The maps satisfy the relations: $it = i$ and $t^2 = id$. Therefore, giving a Mackey functor for $\mathbb{Z}/2$ amounts to giving abelian groups $M(\mathbb{Z}/2)$ and $M(e)$ satisfying the following conditions. To simplify the notation, we write f_* and f^* in place of $M_*(f)$ and $M^*(f)$.

(a) $(t^*)^2 = id$, $t^*i^* = i^*$;
(b) $(t_*)^2 = id$, $i_*t_* = i_*$;
(c) $t_*t^* = id$;
(d) $i^*i_* = id + t^*$.

The first two conditions come from contravariant and covariant functoriality. The third condition can be deduced by considering the pullback square:

$$\begin{tikzcd} \mathbb{Z}/2 \arrow[r, "t"] \arrow[d, "t"'] & \mathbb{Z}/2 \arrow[d, "\mathrm{id}"] \\ \mathbb{Z}/2 \arrow[r, "\mathrm{id}"'] & \mathbb{Z}/2. \end{tikzcd}$$

Note that this is a pullback, since t is an isomorphism in the category $\mathcal{G}$. The last condition comes from the axioms for Mackey functors in a less immediate way. To check it, we consider the pullback square:

$$\begin{tikzcd} \mathbb{Z}/2 \times \mathbb{Z}/2 \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathbb{Z}/2 \arrow[d, "i"] \\ \mathbb{Z}/2 \arrow[r, "i"'] & e. \end{tikzcd}$$

In the category of G-sets, we have that $\mathbb{Z}/2 \times \mathbb{Z}/2$ is isomorphic to the disjoint union $\mathbb{Z}/2 \coprod \mathbb{Z}/2$, via the map:

$$\begin{array}{ccccc} \mathbb{Z}/2 \times \mathbb{Z}/2 & \to & \mathbb{Z}/2 & \coprod & \mathbb{Z}/2 \\ (0,0) & \mapsto & 0 & & \\ (1,1) & \mapsto & 1 & & \\ (0,1) & \mapsto & & & 0 \\ (1,0) & \mapsto & & & 1 \end{array}$$

When giving this isomorphism, we are making a choice in the values of the last two elements: we would have also gotten an isomorphism flipping the two; this choice introduces the asymmetry in the formula we are trying to prove. Under this identification, we can apply the Mackey functor M to the previous pullback square, obtaining:

$$\begin{tikzcd} M(\mathbb{Z}/2) \oplus M(\mathbb{Z}/2) \arrow[r, "\mathrm{id}+t^*"] & M(\mathbb{Z}/2) \\ \mathbb{Z}/2 \arrow[u, "\mathrm{id}+\mathrm{id}"] \arrow[r, "i_*"'] & M(e). \arrow[u, "i^*"'] \end{tikzcd}$$

The last axiom for Mackey functors says that this square commutes, proving the relation (d).

We can represent a Mackey functor for $\mathbb{Z}/2$ with a diagram of the form:

$$\begin{tikzcd} M(\mathbb{Z}/2) \arrow[r, shift left, "i_*"] \arrow[loop below, "t*"] & M(e) \arrow[l, shift left, "i^*"] \end{tikzcd}$$

Example 4.3.1. One of the Mackey functors we will use most often is the *constant Mackey functor* $\underline{\mathbb{Z}}$:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, shift left, "2"] \arrow[loop below, "id"] & \mathbb{Z} \arrow[l, shift left, "id"] \end{tikzcd} \tag{4.3.1}$$

The functor $\underline{\mathbb{Z}}^{\mathrm{op}}$:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, shift left, "id"] \arrow[loop below, "id"] & \mathbb{Z} \arrow[l, shift left, "2"] \end{tikzcd}$$

is closely related to the previous one.

4.4 Transfer maps

Let us show how to understand the induced maps i_* and i^* of a Mackey functor M on $\mathbb{Z}/2$. As noted above, $i\colon \mathbb{Z}/2 \to *$ is the G-map in the orbit category.

In general, for a finite covering space $\pi\colon E \to B$, with B compact, we can do the following construction: we can find an embedding $E \hookrightarrow B \times \mathbb{R}^n$ for some n, such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d, "\pi"] & B \times \mathbb{R}^n \arrow[dl, "p_1"] \\ B. & \end{tikzcd} \tag{4.4.1}$$

This fact is not immediate to see: we will sketch here the argument to see it. Let $\pi\colon E \to B$ be a m-fold covering. Since it is a covering map, for every point $x \in B$ we can find an open neighbourhood of x, $U_x \subset B$, such that $\pi^{-1}(U_x) \cong U_x \times \{1, \dots, m\}$. The set $\{U_x\}_{x \in B}$ is an open covering of B, which is compact, so we can take a finite sub-covering $U_1, \dots, U_n$. Let $e\colon \{1, \dots, m\} \to \mathbb{R}$ be the inclusion map. For $1 \leq i \leq n$, let

$$f_i\colon \pi^{-1}(U_i) \cong U_i \times \{1, \dots, m\} \xrightarrow{I \times e} U_i \times \mathbb{R} \xrightarrow{p_2} \mathbb{R}.$$

We can take $\{\varphi_1, \dots, \varphi_n\}$ to be a partition of unity associated to $U_1, \dots, U_n$ and define the map:

$$\begin{aligned} F\colon E &\to \mathbb{R}^n \\ x &\mapsto (f_1(x)\varphi_1(\pi(x)), \dots, f_n(x)\varphi_n(\pi(x))), \end{aligned}$$

which is continuous and defined on the entire E, if we understand each f_i as the zero map outside of $\pi^{-1}(U_i)$. Finally we define the wanted embedding:

$$\begin{aligned} \tilde{F}\colon E &\to B \times \mathbb{R}^m \\ x &\mapsto (\pi(x), F(x)). \end{aligned}$$

We can see easily that $\tilde{F}$ is injective: if $e, e' \in E$ are such that $\tilde{F}(e) = \tilde{F}(e')$, then in particular $\pi(e) = \pi(e')$. We know that $U_1, \dots, U_n$ form a covering of B, hence there is some i such that $\pi(e) \in U_i$ and so $e, e' \in \pi^{-1}(U_i)$. By the definition of $\tilde{F}$, we have that $f_i(e) = f_i(e')$: the map $f_i\colon \pi^{-1}(U_i) \to \mathbb{R}$ identifies the sheet of the covering in which a point is. Therefore e and e' are in the same sheet and so $e = e'$.

Once we have the embedding $E \hookrightarrow B \times \mathbb{R}^n$, we can proceed with the construction of the transfer maps. Let U be a fiberwise tubular neighbourhood of E in $B \times \mathbb{R}^n$, $U \cong E \times \mathbb{R}^n$. We have the following map, collapsing $B \times \mathbb{R}^n \setminus U$, where $(-)^+$ denotes the 1-point compactification:

$$\begin{tikzcd} (B \times \mathbb{R}^n)^+ \arrow[r] \arrow[d, "\cong"] & U^+ \arrow[r, "\cong"] & (E \times \mathbb{R}^n)^+ \arrow[d, "\cong"] \\ \Sigma^n(B_+) \arrow[rr, "\mathrm{trf}_\pi"] & & \Sigma^n(E_+) \end{tikzcd}$$

Equivariantly, we replace the embedding $\tilde{F}$ of (4.4.1) by an equivariant embedding into some representation:

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d, "\pi"] & B \times V \arrow[dl, "p_1"] \\ B. & \end{tikzcd}$$

If we start with our $i\colon \mathbb{Z}/2 \to *$, we can embed equivariantly $\mathbb{Z}/2$ into $\mathbb{R}^-$ ($\mathbb{Z}/2$ acts by sign) as $\{-1, 1\}$ and take two disjoint open intervals around the two points as the neighbourhood U. The 1-point compactification of $\mathbb{R}^-$ is S^σ where σ is the sign-representation and the collapsing map above maps it to the wedge $S^\sigma \vee S^\sigma$ where the G-action also swaps the two factors. Using the fact $\Sigma^\sigma *_+ = \Sigma^1\mathbb{Z}/2$, we have

$$\mathrm{trf}_i\colon \Sigma^\sigma(*)_+ \to \Sigma^\sigma(\mathbb{Z}/2)_+.$$

The suspension of the transfer map i_* of the Mackey functor M is the map induced on degree zero cohomology by trf_i, while i^* in the Mackey functor is the map induced by i:

$$\begin{tikzcd}
M(\Sigma^\sigma(\mathbb{Z}/2)_+) \arrow[r, "\Sigma^\sigma i_*"] \arrow[d, "="] & M(\Sigma^\sigma(*)_+) \arrow[r, "\Sigma^\sigma i^*"] \arrow[d, "="] & M(\Sigma^\sigma(\mathbb{Z}/2)_+) \arrow[d, "="] \\
H_G^{0,0}(\Sigma^\sigma(\mathbb{Z}/2)_+) \arrow[r, "(\mathrm{trf}_i)^*"] & H_G^{0,0}(\Sigma^\sigma(*)_+) \arrow[r, "\Sigma^\sigma i^*"] & H_G^{0,0}(\Sigma^\sigma(\mathbb{Z}/2)_+).
\end{tikzcd}$$

4.5 Eilenberg-MacLane spaces

The equivariant version of Eilenberg-MacLane spaces is defined in terms of representations and Mackey functors. The definition we use here is found originally in [10].

Definition 4.5.1. Let V be a G-representation and M a Mackey functor. An *equivariant Eilenberg-MacLane space* of type M,V, denoted $K(M, V)$, is a basepointed G-space with the G-homotopy type of a G-CW complex such that $K(M, V)$ is $(V - 1)$-connected and

$$\underline{\pi}_{V+k}(K(M, V)) = \begin{cases} M & \text{if } k = 0, \\ 0 & \text{if } k > 0. \end{cases}$$

Recalling the definition of the Mackey functor $\underline{\pi}$, we can unwind the last condition in the definition: we require that

$$\pi^H_{V+k}(K(M, V)) = [S^{V+k} \wedge G/H_+, K(M, V)] = 0 \qquad \text{for } k > 0 \text{ and } H \leq G,$$

$$\pi^H_V(K(M, V)) = [S^V \wedge G/H_+, K(M, V)] = M(G/H) \qquad \text{for all } H \leq G,$$

It can be proved that, for any G-representation $V \supseteq 1$ and any Mackey functor M, there exists an associated equivariant Eilenberg-MacLane space $K(M, V)$ ([10, Theorem 1.5]).

4.6 Definition of $RO(G)$-graded cohomology

Given a Mackey functor M, we have a $RO(G)$-graded cohomology theory associated to it:

$$V \mapsto H^V(-; M),$$

for every virtual representation V of G. The theory is characterized by the two properties:

- The cohomology groups of the orbit spaces G/H:

$$H^n(G/H; M) = \begin{cases} M(G/H) & \text{if } n = 0 \\ 0 & \text{if } n \neq 0. \end{cases}$$

with the usual convention that n denotes the n-dimensional trivial real representation of G.
- The restriction map:

$$H^0(G/K; M) \xrightarrow{i^*} H^0(G/H; M),$$

induced by $i\colon G/H \to G/K$ is the transfer map i^* of the Mackey functor.

4.7 Representing cohomology with equivariant spectra

Having the notion of G-spectra, which we have introduced in 3.2.1, we can also view $RO(G)$-graded cohomology on the represented level: in fact, G-spectra represent such cohomology theories. Let E be a G-spectrum: the cohomology theory associated to it is defined by:

$$E^{\nu}_G(X) = [S^{-\nu} \wedge X, E]_G,$$

where $\nu = V - W$ is a virtual representation of G and X is a G-spectrum. Note that, since we are in stable setting, we can rewrite this as $E^{\nu}_G(X) = [S^W \wedge X, S^V \wedge E]_G$.

If we start with a Mackey functor M, we can associate to it an equivariant spectrum HM, so that the cohomology theory represented by this spectrum is the same to the one associated to M. In particular, for $G = \mathbb{Z}/2$, if we take $M = \underline{\mathbb{Z}}$, the constant Mackey functor that we have defined in Example 4.3.1, we get the Eilenberg-MacLane spectrum $H\underline{\mathbb{Z}}$, representing the cohomology theory we use in this project to construct the spectral sequence.

Remark 4.7.1. In the case of $H\mathbb{Z}$, we can write down the cohomology in represented way on the level of G-spaces and G-maps, by means of the equivariant Eilenberg-MacLane spaces. If V is a $\mathbb{Z}/2$-representation and X is a $\mathbb{Z}/2$-space, we have the formula:

$$H^V_{\mathbb{Z}/2}(X, \underline{\mathbb{Z}}) = H\underline{\mathbb{Z}}\,^V_{\mathbb{Z}/2}(X) = [X, K(\underline{\mathbb{Z}}, V)]_{\mathbb{Z}/2}.$$

4.8 A computation of $RO(G)$-graded cohomology

We will now compute the $RO(G)$-graded cohomology of a point, over the constant Mackey functor $\underline{\mathbb{Z}}$, defined i (4.3.1), in the case $G = \mathbb{Z}/2 = \{-1, 1\}$. This is not as trivial as it may appear thinking at the non-equivariant case. A representation V of $\mathbb{Z}/2$ can be written as $(\mathbb{R})^p \oplus (\mathbb{R}_-)^q$, i.e. as a direct sum of a number of copies of the trivial representation $\mathbb{R}$ and a number of copies of the sign representation $\mathbb{R}_-$, as we have observed earlier. We will write $H^V_G(-; M)$ as $H^{p+q,q}_G(-; M)$.

The following result is stated by Dugger in [6, 2.7]: the author refers to [4, Appendix] for the proof, where the computation is carried out for the cohomology with coefficients in $\underline{\mathbb{Z}/p}$. Here we include a complete proof.

Theorem 4.8.1. *Let $G = \mathbb{Z}/2$. The cohomology groups of a point over the constant Mackey functor $\underline{\mathbb{Z}}$ are:*

$$H^{a,b}_G(pt, \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z}/2 & \text{if } a - b \text{ is even and } b \geq a > 0, \\ \mathbb{Z} & \text{if } a = 0 \text{ and } b \text{ is even}, \\ \mathbb{Z}/2 & \text{if } a - b \text{ is odd and } b + 1 < a \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The graded multiplication in $H^{*,*}_G(pt, \underline{\mathbb{Z}})$ is commutative.

Lemma 4.8.2. *Let $G = \mathbb{Z}/2$. $H^{*,*}_G(G; \underline{\mathbb{Z}})$ is a graded commutative polynomial algebra on the generator $t \in H^{0,1}_G(G; \underline{\mathbb{Z}})$.*

Proof. Note that, for every G-space X, we have a G-homeomorphisms $G_+ \wedge X \cong G_+ \wedge X^e$, where, X^e is X with trivial G-action. From this, we get the chain of isomorphisms:

$$H^{a,b}_G(G) \xrightarrow[\cong]{\text{suspension}} \tilde{H}^{a+n,b+n}_G(\Sigma^{n,n}(G_+)) \xrightarrow[\cong]{}$$
$$\tilde{H}^{a+n,b+n}_G(\Sigma^{n,0}(G_+)) \xleftarrow[\cong]{\text{suspension}} H^{a,b+n}_G(G). \quad (4.8.1)$$

This is an isomorphism of graded $H^{*,*}_G(G)$-modules, so it is given by multiplication by an invertible element $t_n \in H^{0,n}_G$. In other words, the cohomology groups associated to representations with the same dimension are all isomorphic.

By the cohomology axioms, we have $H^{0,0}_G(G) = \mathbb{Z}$ and $H^{m,0}_G(G) = 0$ if $m \neq 0$, so we can use the above isomorphism to get:

$$H^{a,b}_G(G) = \begin{cases} \mathbb{Z} & \text{if } a = 0, \\ 0 & \text{if } a \neq 0. \end{cases}$$

$\square$

Proof of Theorem 4.8.1. Let $G = \mathbb{Z}/2$ and let σ denote the $\mathbb{Z}/2$ sign representation, as usual. From the cofiber sequence

$$\mathbb{Z}/2 = S(\sigma) \xrightarrow{i} pt \xrightarrow{r} S^{\sigma},$$

we get the following exact triangle. The maps are labeled with their degree.

$$\begin{tikzcd}
\tilde{H}^{*,*}_G(S^{\sigma}) \arrow[rr, "r^*", "(0{,}0)"'] & & H^{*,*}_G(pt) \arrow[dl, "(0{,}0)"', "i^*"] \\
& H^{*,*}_G(G) \arrow[ul, "(1{,}0)"] &
\end{tikzcd}$$

We can change degrees via the suspension isomorphism, with the purpose of obtaining $H^{*,*}_G(pt)$ also in the top-left entry of the diagram: to do this we replace the map r^*, in every degree, by the compositions:

$$\begin{tikzcd}
H^{a+1,b+1}_G(S^{\sigma}) \arrow[r, "r^*"] \arrow[d, "\cong"'] & H^{a+1,b+1}_G(pt) \\
H^{a,b}_G(pt) \arrow[ur] &
\end{tikzcd}$$

and similarly for the connecting homomorphism of the exact triangle. This way we obtain the following exact triangle, where two maps have non-zero degree. Again we label each map with its degree in the chart:

$$\begin{tikzcd}
\tilde{H}^{*,*}_G(pt) \arrow[rr, "\cdot\tau", "(1{,}1)"'] & & H^{*,*}_G(pt) \arrow[dl, "(0{,}0)"', "\varphi"] \\
& H^{*,*}_G(G) \arrow[ul, "(0{,}-1)", "\psi"'] &
\end{tikzcd}$$

By the module structure, one can see that the map on the top row of the diagram is given by multiplication by an element $\tau \in H^{1,1}_G(\mathbb{Z}/2)$.

Now we can combine the information we have about $H^{*,*}_G(G)$ with this exact sequence to compute $H^{p,q}_G(pt)$ for certain degrees. The following diagram shows the generators for the cohomology of G. Recall that we write the representation $\mathbb{R}^p \oplus (\mathbb{R}_-)^q$ as $(p+q, q)$; the diagram has the $p+q$ coordinate on the horizontal axis and the q coordinate on the vertical one.

$q \uparrow$					
3			t^{-3}		
2			t^{-2}		
1			t^{-1}		
0			1		
-1			t		
-2			t^2		
-3			t^3		
	-2	-1	0	1	2

$\xrightarrow[p+q]{}$

This implies that multiplication by τ is an isomorphism of the groups:

$$H_G^{a,b}(pt) \to H_G^{a+1,b+1}(pt),$$

except when $H_G^{a,b+1}(\mathbb{Z}/2)$ or $H_G^{a+1,b+1}(\mathbb{Z}/2)$ are non-zero. By the dimension axiom, $H_G^{a,0}(pt) = 0$ for $a \neq 0$ and $H_G^{0,0}(pt) = \mathbb{Z}$. So we have already some information about $H_G^{*,*}(pt)$ (the axes are as indexed as in the previous diagram): the 0's in the diagram indicate cells which we know correspond to the trivial group. The empty cells are the ones for which we do not know yet.

$q \uparrow$											
5	0	0	0	0	0						
4	0	0	0	0	0						0
3	0	0	0	0	0					0	0
2	0	0	0	0	0				0	0	0
1	0	0	0	0	0			0	0	0	0
0	0	0	0	0	0	$\mathbb{Z}$	0	0	0	0	0
-1	0	0	0	0			0	0	0	0	0
-2	0	0	0				0	0	0	0	0
-3	0	0					0	0	0	0	0
-4	0						0	0	0	0	0
-5							0	0	0	0	0
	-5	-4	-3	-2	-1	0	1	2	3	4	5

$\xrightarrow[p+q]{}$

The next step is to identify the map ψ in the exact sequence:

$$0 \to H_G^{0,1}(pt) \xrightarrow{\varphi} H_G^{0,1}(G) \xrightarrow{\psi} H_G^{0,0}(pt) \xrightarrow{\cdot\tau} H_G^{1,1}(pt) \to 0. \qquad (4.8.2)$$

Substituting:

$$0 \to H_G^{0,1}(pt) \xrightarrow{\varphi} \mathbb{Z}\langle t^{-1}\rangle \xrightarrow{\psi} \mathbb{Z}\langle 1\rangle \xrightarrow{\cdot\tau} H_G^{1,1}(pt) \to 0.$$

Modulo the suspension isomorphism, the map ψ is the map induced on cohomology by $r\colon S^\sigma \to \Sigma\mathbb{Z}/2$, under the identification $S(\sigma) = \mathbb{Z}/2$:

$$\begin{tikzcd}
\tilde{H}^{1,1}_G(\Sigma\mathbb{Z}/2_+) \arrow[r, "r^*"] \arrow[d, "\cong"'] & \tilde{H}^{1,1}_G(S^\sigma) \arrow[d, "\cong"] & \\
\tilde{H}^{0,1}_G(\mathbb{Z}/2_+) \arrow[r] \arrow[d, "\cong"'] & \tilde{H}^{0,0}_G(S^0) \arrow[d, "\cong"] & \\
H^{0,1}_G(\mathbb{Z}/2) \arrow[r, "\psi"] \arrow[d, "\cong"', "(4.8.1)"] & H^{0,0}_G(pt) \arrow[r, "\cong"] & \mathbb{Z} \\
H^{0,0}_G(\mathbb{Z}/2) \arrow[ur, "i_*"'] \arrow[d, "\cong"'] & & \\
\mathbb{Z} & &
\end{tikzcd}$$

Our interpretation of the transfer maps of Mackey functor shows that the map ψ in the degree shown by the diagram can be identified with the transfer map i_* of the Mackey functor, i.e. multiplication by 2. Hence, by the exact sequence (4.8.2), we get:

$$H^{0,1}_G(pt) = 0 \qquad H^{1,1}_G(pt) \cong \mathbb{Z}/2\langle\tau\rangle.$$

By the exact sequence (4.8.2), in different degrees, we get that the map

$$H^{0,1}_G(pt) \to H^{1,2}_G(pt)$$

is a surjection, therefore the continuation of the diagonal $b = a + 1$ has zero groups everywhere. Again by the exact sequence, $H^{b,b}_G(pt) \to H^{b+1,b+1}_G(pt)$ is an isomorphism for $b \geq 1$ and it is multiplication by τ. Hence we have $H^{b,b}_G(pt) \cong \mathbb{Z}/2\langle\tau^b\rangle$, for $b > 0$.

The next step is to compute $H^{0,2}_G(pt)$: let us write the involved part of the exact sequence:

$$0 = H^{-1,1}_G(pt) \xrightarrow{\cdot\tau} H^{0,2}_G(pt) \xrightarrow{\varphi} H^{0,2}_G(\mathbb{Z}/2) \cong \mathbb{Z}\langle t^{-2}\rangle \xrightarrow{\psi} H^{0,1}_G(pt) = 0.$$

Hence $H^{0,2}_G(pt) \cong \mathbb{Z}$: let x be a generator of it such that $i^*(x) = \varphi(x) = t^{-2}$.

Let us consider the map

$$\psi\colon \mathbb{Z}\langle t^{-3}\rangle = H^{0,3}_G(\mathbb{Z}/2) \to H^{0,2}_G(pt) = \mathbb{Z}\langle x\rangle.$$

We have the following relation, as a consequence of the module structure of $H^*_G(pt)$:

$$i_*(i^*(x)y) = x i_*(y). \tag{4.8.3}$$

So, in our case we can compute:

$$i_*(t^{-3}) = i_*(i^*(x)t^{-1}) = xi_*(t^{-1}) = 2x$$

and, using the exact sequence once again, we get:

$$H_G^{0,3}(pt) = 0 \qquad H_G^{1,3}(pt) \cong \mathbb{Z}/2\langle x\tau\rangle.$$

Then the argument repeats and we get that, for $b > 0$ and $a \geq 0$, the diagonals $b = a + k$ are all zero for k odd and they have a copy of $\mathbb{Z}$ in the position $(0, k)$ and an infinite tower of $\mathbb{Z}/2$ above that for k even.

The argument to compute $H_G^{a,b}(pt)$ for the remaining quadrant ($b < 0$ and $a \leq 0$) is similar: the first observation is that $i^* = \varphi\colon H^{0,0}(pt) \to H_G^{0,0}(\mathbb{Z}/2)$ is an isomorphism, because it coincides with the restriction map i^* in the Mackey functor $\underline{\mathbb{Z}}$. Hence:

$$0 \to H_G^{-1,-1}(pt) \xrightarrow{\cdot\tau} H_G^{0,0}(pt) \underset{\cong}{\xrightarrow{\varphi}} H_G^{0,0}(\mathbb{Z}/2) \xrightarrow{\psi} H_G^{0,-1}(pt) \to H_G^{1,0}(pt).$$

Recall that $H_G^{1,0}(pt)$ by the initial observation; the sequence implies that:

$$H_G^{-1,-1}(pt) = 0 \qquad H_G^{0,-1}(pt) = 0.$$

Moreover, by the exact sequence (4.8.2), $H^{0,-1}(\mathbb{Z}/2) \overset{\psi}{\twoheadrightarrow} H^{0,-2}(pt)$ is an isomorphism, and so $\psi(t) = y$, where $H^{0,-2}(pt) \cong \mathbb{Z}\langle y\rangle$. One can check with a geometric argument that the diagram:

$$\begin{tikzcd}
H_G^{0,-1}(\mathbb{Z}/2) \arrow[r, "\psi", "\cong"'] \arrow[d, "\cong"', "\cdot t"] & H_G^{0,-2}(pt) \\
H_G^{0,-2}(\mathbb{Z}/2) \arrow[ur, "i_*"'] &
\end{tikzcd}$$

is commutative and so $i_*(t^2) = y$. Hence:

$$i^*(y) = \varphi(y) = i^*i_*(t^2) = (1 + t_*)(t^2) = 2t^2.$$

In fact, the map t_* in the Mackey functor is the identity. Thus, by the same argument with the exact sequence used before, we have:

$$H_G^{-1,-3}(pt) = 0 \qquad H_G^{0,-3}(pt) \cong \mathbb{Z}/2.$$

To compute $H_G^{-1,-4}(pt)$ we proceed as follows: by the exact sequence we have

$$0 \xrightarrow{\psi} H_G^{-1,-4}(pt) \hookrightarrow H_G^{0,-3}(pt) \xrightarrow{\varphi} H_G^{0,-3}(\mathbb{Z}/2).$$

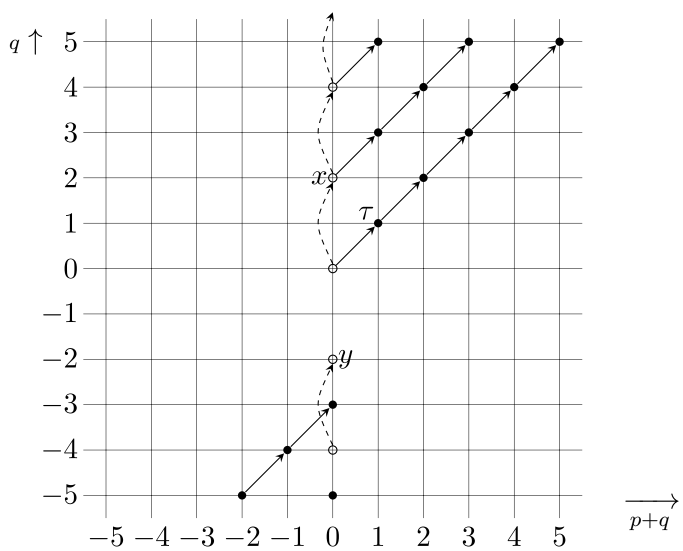
Hence $H_G^{-1,-4}(pt)$ is either 0 or $\mathbb{Z}/2$, as it is isomorphic to a subgroup of $\mathbb{Z}/2$. If it was the former, then we would have that $\mathbb{Z}/2$ injects in $\mathbb{Z}$. This is not possible, hence we have $H_G^{-1,-4}(pt) \cong \mathbb{Z}/2$.

Now the argument repeats, as it happened in the other quadrant. We get that, in the quadrant we are considering, the diagonal $b = a$ and $b = a - 1$ have only trivial cohomollogy groups. If $k > 1$ and odd, the diagonal $b = a - k$ has a single copy of $\mathbb{Z}$ at the coordinates $(0, -k)$. If $k > 1$ is even, we have a tower of groups $\mathbb{Z}/2$ beginning at $(0, -k)$ and infinite to the bottom left. All isomorphisms between consecutive elements along the diagonals are given by multiplication by τ.

Let us still check something about the multiplicative structure: we can write $H^{0,-4}(pt) \cong \mathbb{Z}\langle u\rangle$, for some generator u, such that $\psi\colon H_G^{0,-3}(\mathbb{Z}/2) \to H_G^{0,-4}(pt)$ maps t^3 to u. Then:

$$y = i_*(t) = i_*(t^3 t^{-2}) = i_*(t^3 i^*(x)) = ux,$$

which means that multiplication by x gives an isomorphism $H_G^{0,-4}(pt) \to H_G^{0,-2}(pt)$. In general, a generator of $H_G^{0,-2k}(pt)$ is obtained from a generator of $H_G^{0,-2k-2}(pt)$ by multiplying by $x \in H_G^{0,2}(pt)$. The rest of the multiplicative structure can be obtained easily with similar computations. The following diagram shows the structure of $H_G^{*,*}(pt)$: for typographical reasons, we use a circle to indicate the group $\mathbb{Z}$, and a dot to indicate $\mathbb{Z}/2$. The solid arrows are multiplication by τ, while the curved dashed arrows are multiplication by x. This time all the empty cells indicate trivial groups.



$\square$

This theorem allows us to get easily some information about the (non-equivariant) homotopy of the equivariant Eilenberg-MacLane spaces.

Corollary 4.8.3. *(a)* $K(\underline{\mathbb{Z}}, (2n, n))$ *is (non-equivariantly) homotopy equivalent to* $K(\mathbb{Z}, 2n)$.
(b) $K(\underline{\mathbb{Z}}, (2n, n))^{\mathbb{Z}/2}$ *is (non-equivariantly) homotopy equivalent to a product of Eilenberg-MacLane spaces and has the homotopy of:*
$K(\mathbb{Z}, 2n) \times K(\mathbb{Z}/2, 2n-2) \times K(\mathbb{Z}/2, 2n-4) \times \cdots \times K(\mathbb{Z}/2, n)$ *(for n even)*
$K(\mathbb{Z}/2, 2n-1) \times K(\mathbb{Z}/2, 2n-3) \times \cdots \times K(\mathbb{Z}/2, n)$ *(for n odd).*

Proof. Let $G = \mathbb{Z}/2$ and $X = K(\underline{\mathbb{Z}}, (2n, n))$. Then X and $X^{\mathbb{Z}/2}$ are products of Eilenberg-MacLane spaces as a consequence of a theorem of dos Santos [5]. We can compute the homotopy groups, using the fact that Eilenberg-MacLane spaces represent $RO(G)$-graded cohomology:

$$\begin{aligned}\pi_i(X) = [S^i, X]^* \cong [S^i \wedge G_+, K(\underline{\mathbb{Z}}, (2n, n))]^*_G \\ \cong \tilde{H}^{2n,n}(S^i \wedge G_+) \cong H_G^{2n-i,n}(G).\end{aligned}$$

And the thesis follows reading the groups in the previous theorem.

For the second part:

$$\pi_i(X^{\mathbb{Z}/2}) = [S^i, X^{\mathbb{Z}/2}]^* \cong [S^i, K(\underline{\mathbb{Z}}, (2n, n))]_G \cong \tilde{H}^{2n,n}(S^i) \cong H_G^{2n-i,n}(pt).$$

□

5 KR-theory

5.1 Definition

KR-theory is a variant of K-theory for spaces with an involution or, in other words, an action of $\mathbb{Z}/2$. It was introduced first by Atiyah in [2]. It shares many features with K-theory, including periodicity.

KR-theory is represented by an Ω-spectrum, whose spaces are all equal to $\mathbb{Z} \times BU$. This space has a $\mathbb{Z}/2$-action coming from the complex conjugation on the infinite unitary group U.

On the represented level, we have:

$$KR(X) = [X, \mathbb{Z} \times BU]^*_{\mathbb{Z}/2},$$

and the other groups are defined as one does for K-theory. For $p + q \leq 0$, $q \geq 0$ we set:

$$KR^{p+q,q}(X) = [\Sigma^{-p-q,q} X, \mathbb{Z} \times BU]^*_{\mathbb{Z}/2},$$

and for the remaining values of the indices, we can extend the definition by periodicity. In fact, we have a Bott periodicity also for KR-theory: this is shown, for instance, in [2]. The periodicity gives an equivariant weak equivalence:

$$\mathbb{Z} \times BU \xrightarrow{\simeq} \Omega^{2,1}(\mathbb{Z} \times BU).$$

KR-theory is graded on the representations of $\mathbb{Z}/2$: each of them can be written uniquely as $\mathbb{R}^p \oplus (\mathbb{R}_-)^q$, which we denote with the pair $(p+q, q)$, and the periodicity is expressed by the isomorphism:

$$KR^{p+2,q+1}(X) \cong KR^{p,q}(X).$$

5.2 The $\mathbb{Z}/2$-spectrum KR

Of course we can view the spectrum representing KR-theory as a $\mathbb{Z}/2$-spectrum, by using the interpretation of equivariant spectra we have for the case $G = \mathbb{Z}/2$. The $\mathbb{Z}/2$-spectrum KR is defined by the assignment:

$$\mathbb{C}^n \mapsto \mathbb{Z} \times BU$$

with the structure maps

$$S^{2,1} \wedge \mathbb{Z} \times BU \to \mathbb{Z} \times BU,$$

that are adjoint to the Bott map.

6 Postnikov systems

6.1 Relations with the non-equivariant case

In the non-equivariant setting, one defines the *Postnikov tower* of a connected CW complex X: the n^{th} space of the tower, P_nX, is a space such that:

$$\pi_i(P_nX) = \begin{cases} \pi_i(X) & \text{if } i \leq n, \\ 0 & \text{if } i > n, \end{cases}$$

equipped with a natural map $X \to P_nX$, which realizes the isomorphism on homotopy group of order less or equal than n.

To construct P_nX, one at first kills π_{n+1}, by building a CW complex $Y = X \cup_{\varphi_\alpha} D^{n+2}_\alpha$ in which we attach $n+2$ cells to Y via the maps φ_α, which generate the groups $\pi_{n+1}(X)$. The complex Y has the same homotopy groups of X up to degree n, by cellular approximation, and $\pi_{n+1}(Y) = 0$. Then one builds a new complex by attaching $n+3$ cells to Y via the generators of π_{n+2} and so on, until we obtain a space with the properties of P_nX.

This construction works because, for non-equivariant spaces, there are no non-trivial homotopy classes of maps $S^k \to S^n$ if $k < n$: this can be proved by cellular approximation. This is not true for G-spaces, where one can have a non G-homotopically trivial G-map $S^V \to S^W$ for $V \subseteq W$ (V subrepresentation of W).

Example 6.1.1. Consider the map of $\mathbb{Z}/2$-spaces:

$$S^{1,1} \to S^{2,2},$$

which embeds the 1-dimensional sign sphere $S^{1,1}$ as an equator of the 2-dimensional sphere $S^{2,2}$ with the sign action on two coordinates (the representations sphere for $\mathbb{R}_- \oplus \mathbb{R}_-$). This map is not homotopically trivial. To see this, note that $S^{1,1}$ has only two fixed-points, the north and the south pole, and the same is true for $S^{2,2}$. The inclusion map sends the north pole to the north pole and the south pole to the south pole: to make it nullhomotopic we should move one of the poles to the other, but this is cannot be done equivariantly, since fixed-points are always mapped to fixed-points.

The situation would be different if we had an extra G-fixed coordinate in the larger sphere, because we could build a homotopy moving the included sphere along this coordinate and push it to a single point.

When defining equivariant Postnikov section one has to take care of the property highlighted by this example. We are going to define two different versions of them, with different behaviours with respect to this property.

6.2 Definition of Postnikov section

A G-space A is *small with respect to closed inclusion* if, for any sequence of closed inclusions:

$$Z_0 \hookrightarrow Z_1 \hookrightarrow Z_2 \hookrightarrow \dots,$$

the canonical map $\operatorname{colim}_i G\mathcal{U}(A, Z_i) \to G\mathcal{U}(A, \operatorname{colim}_i Z_i)$ is an isomorphism. One can check that any Hausdorff space has this property. Recall that a basepointed G-space (X, x_0) is *well-pointed* if $\{x_0\} \to X$ is a cofibration.

Let $\mathcal{A}$ be a set of well-pointed compact Hausdorff G-spaces. We say that a G-space Z is $\mathcal{A}$-null if the maps

$$[*, Z] \to [\Sigma^n A, Z],$$

induced by $\Sigma^n A \to *$, are isomorphisms for every $n \geq 0$ and $A \in \mathcal{A}$. For Z connected, this is the same of saying $[\Sigma^n A, Z] = 0$.

The Postnikov section that we are going to define are particular nullification functors, as defined by Farjoun [7]:

Definition 6.2.1. Given a G-space X and $\mathcal{A}$ as above, $P_{\mathcal{A}}\colon G\mathcal{U} \to G\mathcal{U}$ is a *nullification functor* if, for every G-space X, we have:

(a) there exists a natural map $i\colon X \to P_{\mathcal{A}}(X)$,
(b) $P_{\mathcal{A}}(X)$ is $\mathcal{A}$-null,
(c) If Z is $\mathcal{A}$-null, for every $f\colon X \to Z$ there is an extension:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"] & Z \\ P_{\mathcal{A}}(X) \arrow[ur, dashed] & \end{tikzcd}$$

Our equivariant Postnikov section functors will be defined as nullification functors $P_{\mathcal{A}}$ for certain families $\mathcal{A}$. Let us see how we can construct $P_{\mathcal{A}}$: for a given G-space Y, we define another G-space $F_{\mathcal{A}}Y$ as the pushout:

$$\begin{tikzcd} \coprod_\sigma \Sigma^n A \arrow[r, "\coprod \sigma"] \arrow[d] & Y \arrow[d, dashed] \\ \coprod_\sigma C(\Sigma^n A) \arrow[r, dashed] & F_{\mathcal{A}}Y \end{tikzcd}$$

where σ is a map $\Sigma^n A \to Y$, for all $A \in \mathcal{A}$ and $n \geq 0$.

Denoting the composition of $F_{\mathcal{A}}$ for n-times by $F^n_{\mathcal{A}}$, we have the inclusions:

$$X \to F_{\mathcal{A}}(X) \to F^2_{\mathcal{A}}(X) \to \dots .$$

We take as $P_{\mathcal{A}}(X)$ the colimit of $\operatorname{colim}_n F^n_{\mathcal{A}}(X)$. One can check that this $P_{\mathcal{A}}$ satisfies the properties of nullification functors: we refer the reader to [7] for the proof of this. The following proposition gives certain properties of the functors $P_{\mathcal{A}}$: we omit the proof here, it can be found for example in [9].

Proposition 6.2.2. *(a) Let $f\colon X \to Z$ be a G-map and $g\colon X \to Y$ a cofibration. If the space Z is $\mathcal{A}$-null and*

$$g_*\colon P_{\mathcal{A}}(X) \to P_{\mathcal{A}}(Y)$$

is a weak equivalence, then there exists a map $\bar{f}$, making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & Z \\ Y. \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

Moreover $\bar{f}$ is unique up to homotopy equivalence.
(b) Let $X\colon \mathcal{C} \to \mathcal{T}$ be a functor. We denote by X_α the value of the functor at $\alpha \in \mathcal{C}$. The natural map

$$P_{\mathcal{A}}(\operatorname{hocolim}_\alpha X_\alpha) \to P_{\mathcal{A}}(\operatorname{hocolim}_\alpha P_{\mathcal{A}}(X_\alpha))$$

is a weak equivalence.
(c) If $X \to Y \to Z$ is a homotopy cofiber sequence and $P_{\mathcal{A}}(X)$ is contractible, then

$$P_{\mathcal{A}}(Y) \to P_{\mathcal{A}}(Z)$$

is a weak equivalence.

Remark 6.2.3. We can obtain the non-equivariant Postnikov sections by this construction: for G the trivial group and

$$\mathcal{A}_n = \{S^i \mid i \geq n+1\},$$

then $P_{\mathcal{A}_n}(X)$ is the n^{th} Postnikov section of X.

We will now define two different kinds of Postnikov sections for equivariant spaces. Let

$$\tilde{\mathcal{A}}_V = \{S^W \wedge G/H_+ \mid W \supseteq V+1, H \leq G\}. \tag{6.2.1}$$

We define $\mathbb{P}_V(X) = P_{\tilde{\mathcal{A}}_V}(X)$.

The second choice is:

$$\mathcal{A}_V = \{S^W \wedge G/H_+ \mid W \supsetneq V, H \leq G\}, \tag{6.2.2}$$

and we define $P_V(X) = P_{\mathcal{A}_V}(X)$.

The former functor, $\mathbb{P}$, has better properties and resembles the ordinary Postnikov section. In particular, $\mathbb{P}_V(X)$ has the lower homotopy groups isomorphic to the ones of X, in the sense which will be made precise by the following proposition, while this is not true for P_V.

6.3 Properties

The following proposition lists the main properties we are interested in. Dugger [6] states those properties in the case where G is any finite group, but here we prove them for $G = \mathbb{Z}/2$.

Proposition 6.3.1. *Let $G = \mathbb{Z}/2$. If X is a basepointed G-space and V a G-representation. Then:*

(a) The map $X \to \mathbb{P}_V(X)$ induces an isomorphism

$$\pi_k^H(X) = [S^k \wedge G/H_+, X]_G^* \to [S^k \wedge G/H_+, \mathbb{P}_V(X)]_G^* = \pi_k^H(\mathbb{P}_V(X)),$$

for $0 \leq k \leq \dim V^H$ and an epimorphism for $k = \dim V^H + 1$.
(b) If W is a representation such that $\dim W^H \leq \dim V^H$ for all $H \leq G$, then the map $[S^W, X]_* \to [S^W, \mathbb{P}_V(X)]_*$ is an isomorphism.
(c) The homotopy fiber of the map

$$\mathbb{P}_{V+1}X \to \mathbb{P}_V X$$

is an equivariant Eilenberg-MacLane space $K(\underline{\pi}_{V+1}X, V+1)$.
(d) The homotopy limit of the sequence $\cdots \to \mathbb{P}_{V+2}X \to \mathbb{P}_{V+1}X \to \mathbb{P}_V X$ is weakly equivalent to X.

Proof. (a) Let $H \leq G$ be a subgroup. Let us consider how we have constructed $\mathbb{P}_V(X)$: starting from the space X, we attached on it cones along all the maps

$$S^W \wedge G/J_+ \to X,$$

for any $W \in \tilde{\mathcal{A}}_V$ and $J \leq G$. Let Z be any step of this construction: when attaching the cone on one of the maps $S^W \wedge G/J_+ \to Z$, we get a new space Z_1 and we have an inclusion $Z \to Z_1$. Note that the fixed point $(Z_1)^H$ is obtained by Z^H by attaching a cone on the map $(S^W \wedge G/J_+)^H \to Z^H$. The space $(S^W \wedge G/J_+)^H$ is a wedge of spheres $S^{|W^H|}$, and so it is $|W^H|$-connected (non-equivariantly). Since we know that $W \supsetneq V+1$, the map is $(|V^H|+1)$-connected. This is true for every step of the construction of $\mathbb{P}_V(X)$, hence we have that $X \to \mathbb{P}_V$ is $(|V^H|+1)$-connected on the H-fixed points, and this is what we wanted to prove, by the usual isomorphism $\pi_k(X^H) \cong \pi_k^H(X)$.
(b) W can be written as $\mathbb{R}^p \oplus (\mathbb{R}_-)^q$ and so we can write S^W as

$$S^1 \wedge \cdots \wedge S^1 \wedge S^\sigma \wedge \cdots \wedge S^\sigma,$$

with p copies of S^1 and q of S^σ, where σ is the sign representation. It is true in general that the representation sphere S^W has a G-CW structure, but the above exhibits one in this case. Suppose that $S^k \wedge G/H_+$ is a cell in this structure: then it is a product of k 1-cells. Let l be the number of 1-cells fixed by the action and $(k-l)$ the number of free 1-cells. If $k-l>0$, then the entire cell $S^k \wedge G/H_+$ has to be free, because it has a positive number of free factors. Thus H is the trivial group, and so $k \leq |W^H| = |W|$. If $k-l=0$, we have that $S^k \wedge G/H_+$ is a cell fixed by G, hence it cannot be bigger than the G-fixed subspace of W and we have $k \leq |W^G|$. In both

cases we have proved that, for any cell of S^W of the form $S^k \wedge G/H_+$, we have $k \leq |W^H|$.
A k-cell is attached via push-out diagrams of the following form, where S' and S'' are subcomplexes of S^W:

$$\begin{tikzcd}
S^{k-1} \wedge G/H_+ \arrow[r] \arrow[d, hook] & S' \arrow[d] \\
D^k \wedge G/H_+ \arrow[r] & S'',
\end{tikzcd}$$

with $k \leq \dim(W^H) \leq \dim(V^H)$. Recall that the disc $D^k \wedge G/H_+$ is contractible. We proceed by induction: we assume that

$$[S', X]^*_G \to [S', \mathbb{P}_V(X)]^*_G$$

is an isomorphism and want to prove that

$$[S'', X]^*_G \to [S'', \mathbb{P}_V(X)]^*_G$$

also is. Note that we do not have a group structure on domain and target: we are dealing with pointed sets.
To prove surjectivity, consider $f \in [S'', \mathbb{P}_V(X)]^*_G$.

$$\begin{tikzcd}
S^{k-1} \wedge G/H_+ \arrow[r, "\alpha"] \arrow[d, hook, "\beta"'] \arrow[ddrr, dashed, "u"] & S' \arrow[d] \arrow[ddr, dashed, "\hat{f}"] \arrow[ddrr] & & \\
D^k \wedge G/H_+ \arrow[r] & S'' \arrow[drr, "f"] & & \\
& & X \arrow[r, "i"'] & \mathbb{P}_V(X)
\end{tikzcd}$$

The inductive hypothesis gives us a unique (up to homotopy) map $\hat{f}$ factoring f through X. The square is commutative, and we can compose f with its edges getting a map $S^{k-1} \wedge G/H_+ \to \mathbb{P}_V(X)$. This composition is null-homotopic, and we can factor it through X using part (a): this way we get again a null-homotopic map, labeled with u in the diagram. Since the disc $D^k \wedge G/H_+$ is contractible, we can get a map

$$\hat{u} \colon D^k \wedge G/H_+ \to X$$

such that $\hat{u}\beta = u$, as shown in the following diagram:

$$\begin{tikzcd}
S^{k-1}\wedge G/H_+ \arrow[r, "\alpha"] \arrow[d, hook, "\beta"'] \arrow[ddrr, dashed, "u"] & S' \arrow[d] \arrow[ddr, dashed, "\hat{f}"] \arrow[ddrr, "l"] & & \\
D^k\wedge G/H_+ \arrow[r] \arrow[drr, dashed, "\hat{u}"'] & S'' \arrow[drr, "f"] & & \\
& & X \arrow[r, "i"'] & \mathbb{P}_V(X)
\end{tikzcd}$$

By construction, $\hat{f}\alpha = \hat{u}\beta$, and so, by the universal property of the push-out, there exists a unique map $\tilde{f}\colon S'' \to X$ such that $\hat{f}$ and $\hat{u}$ factor through this map. Now we have built two maps: $i\tilde{f}, f\colon S'' \to \mathbb{P}_V(X)$; the maps $i\hat{u}$ and l factor through both of them. By the universal property of the push-out, there exists a unique map $S'' \to \mathbb{P}_V(X)$ with this property: therefore $i\tilde{f}$ and f must coincide, and this proves surjectivity.

As to injectivity, one can do a similar argument: let $f, g\colon S'' \to X$ be two maps which become homotopic when post-composing with $i\colon X \to \mathbb{P}_V(X)$. We can make a map:

$$(f,g)\colon \{0,1\}\times S'' \to X,$$

by mapping one copy of S'' with f and the other with g. The claim is that there is a homotopy $S''\times I \to X$ between f and g: this can be expressed by saying that the above map can be extended to $I\times S''$, which is shown as a dashed arrow in the following diagram:

$$\begin{tikzcd}
\{0,1\}\times S' \arrow[r] \arrow[d] & \{0,1\}\times S'' \arrow[r, "{(f,g)}"] \arrow[d] & X \arrow[d, "i"] \\
I\times S' \arrow[r] \arrow[urr] & I\times S'' \arrow[r] \arrow[ur, dashed] & \mathbb{P}_V(X).
\end{tikzcd}$$

We already know that they are homotopic when restricted to S', by the inductive hypothesis, and this gives the solid diagonal arrow in the diagram. To get the wanted homotopy, we can consider the push-out square:

$$\begin{tikzcd}
I\times(S^{k-1}\wedge G/H_+) \arrow[r, "\alpha"] \arrow[d, hook, "\beta"'] & I\times S' \arrow[d] \arrow[ddr, dashed] \arrow[ddrr] & & \\
I\times(D^k\wedge G/H_+) \arrow[r] \arrow[drr, dashed] & I\times S'' \arrow[drr] & & \\
& & X \arrow[r, "i"'] & \mathbb{P}_V(X).
\end{tikzcd}$$

The same argument used for surjectivity gives us a map $I \times S'' \to X$ with the wanted properties: we leave to the reader to check this. This map is a homotopy between f and g, and this shows that $[S'', X]_G^* \to [S'', \mathbb{P}_V(X)]_G^*$ is injective.
(c) Let $F_V X$ be the homotopy fiber of the map $\mathbb{P}_{V+1}X \to \mathbb{P}_V X$. As usual we write $V = \mathbb{R}^q \oplus (\mathbb{R}_-)^q$. We shall check the properties listed in Definition 4.5.1 for the space $F_V X$.

- $F_V X$ is $(V-1)$-connected. We have to check that

$$\pi^H_{V-1+k}(F_V X) = [S^{V-1+k} \wedge G/H_+, F_V X]_G^* = 0,$$

for $0 \geq k \geq |(V-1)^H|$ and $H \leq G = \mathbb{Z}/2$. If we map the space $S^{V-1+k} \wedge G/H_+$ into the homotopy fiber sequence, we get a long exact sequence:

$$\begin{aligned} \cdots \to [S^{V+k} \wedge G/H_+, \mathbb{P}_V X]_G^* \to [S^{V-1+k} \wedge G/H_+, F_V X]_G^* \\ \to [S^{V-1+k} \wedge G/H_+, \mathbb{P}_{V+1} X]_G^* \to \dots \end{aligned}$$

If $H = \mathbb{Z}/2$, by (b) we have that:

$$[S^{V-1+k}, \mathbb{P}_{V+1}X]_G^* \cong [S^{V-1+k}, X]_G^* \cong [S^{V-1+k}, \mathbb{P}_V X]_G^*$$

and

$$[S^{V+k}, \mathbb{P}_{V+1}X]_G^* \cong [S^{V+k}, X]_G^* \cong [S^{V+k}, \mathbb{P}_V X]_G^*$$

for $k \leq 0$. Hence $\pi^{\mathbb{Z}/2}_{V-1+k}(F_V X) = 0$, because it is enclosed between two isomorphisms in the exact sequence. In the other case, $H = 0$, we shall prove that

$$[S^{V-1+k} \wedge \mathbb{Z}/2_+, \mathbb{P}_{V+1}X]_G^* \cong [S^{V-1+k} \wedge \mathbb{Z}/2_+, X]_G^* \cong [S^{V-1+k} \wedge \mathbb{Z}/2_+, \mathbb{P}_V X]_G^*$$

and

$$[S^{V+k} \wedge \mathbb{Z}/2_+, \mathbb{P}_{V+1}X]_G^* \cong [S^{V+k} \wedge \mathbb{Z}/2_+, X]_G^* \cong [S^{V+k} \wedge \mathbb{Z}/2_+, \mathbb{P}_V X]_G^*.$$

To do that, one can use the $\mathbb{Z}/2$-homeomorphism (4.8.1)

$$S^{V+k} \wedge \mathbb{Z}/2_+ \cong S^{|V|+k} \wedge \mathbb{Z}/2_+,$$

which untwists the action, and then apply (a). This shows that $F_V X$ is $(V-1)$-connected.

- To prove that $F_V X$ is the wanted Eilenberg-MacLane space, we need to show that:

$$\pi^H_{V+1}(F_V X) \cong \pi^H_{V+1}(X).$$

We can use again the long exact sequence:

$$\cdots \to [S^{V+2} \wedge G/H_+, \mathbb{P}_V X]^*_G \to [S^{V+1} \wedge G/H_+, F_V X]^*_G$$
$$\to [S^{V+1} \wedge G/H_+, \mathbb{P}_{V+1} X]^*_G \to [S^{V+1} \wedge G/H_+, \mathbb{P}_V X]^*_G \to \dots$$

The first and the last groups are trivial, by definition of $\mathbb{P}$, since $V+2, V+1 \in \tilde{\mathcal{A}}_V$. Then

$$[S^{V+1} \wedge G/H_+, F_V X]^*_G \cong [S^{V+1} \wedge G/H_+, \mathbb{P}_{V+1} X]^*_G,$$

and, by the same argument used to show that $F_V X$ is $(V-1)$-connected, we get that

$$[S^{V+1} \wedge G/H_+, \mathbb{P}_{V+1} X]^*_G \cong [S^{V+1} \wedge G/H_+, X]^*_G,$$

for any $H \leq G = \mathbb{Z}/2$. Hence $\pi^H_{V+1}(F_V X) \cong \pi^H_{V+1}(X) = \underline{\pi}_{V+1}(X)(H)$.

• To complete the proof, we have to show that the higher homotopy groups of $F_V X$ are trivial:

$$[S^{V+k} \wedge G/H_+, F_V X]^*_G = 0 \qquad \text{for } k > 1.$$

This can be checked immediately using again the long exact sequence, using the fact that:

$$\pi_{V+1+k}(\mathbb{P}_{V+1} X) = 0 = \pi_{V+k}(\mathbb{P}_V X) \qquad \text{for } k > 1.$$

(d) X maps to each $\mathbb{P}_{V+i} X$, hence there is a map $X \to \operatorname{holim}(\mathbb{P}_{V+i} X)$. Note that $\dim(V+n)^H = \dim V^H + n$. By (a), for every k and H, if n is big enough, we have $\dim(V+n)^H \geq k$ and so $\pi^H_k(X) \cong \pi^H_k(\mathbb{P}_{V+n} X)$. It follows that $\pi^H_k(X) \cong \pi^H_k(\operatorname{holim}(\mathbb{P}_{V+i} X))$. □

Corollary 6.3.2. *Non-equivariantly, $\mathbb{P}_V X$ is weakly equivalent to the ordinary Postnikov section $P_{\dim V} X$.*

Proof. Taking as H the trivial group in Proposition 6.3.1 (a), we get that $[S^k \wedge G_+, X]^*_G \cong [S^k \wedge G_+, \mathbb{P}_V X]^*_G$ for $0 \leq k \leq \dim(V)$. By (1.6.1) (for basepointed spaces), we have that $[S^k \wedge G_+, X]_*$ coincides with the set of the non-equivariant homotopy classes of maps $S^k \to X$, and similarly for $\mathbb{P}_V X$. So the lower homotopy groups of X and $\mathbb{P}_V X$ coincide. For $k > \dim V$, $\pi_k(\mathbb{P}_V X)$ is in bijection with the set of the equivariant homotopy classes of maps $[S^k \wedge G_+, \mathbb{P}_V X]^*_G = 0$, because k (as a representation) lies in $\tilde{\mathcal{A}}_V$. □

The construction of the spectral sequence will make use both of P and of $\mathbb{P}$. In particular, we will prove a result connecting the two for the space we are interested in to construct the spectral sequence. As said, the latter functor has better properties, but we are also interested in the former because of an useful property of it. The next section is devoted to the proof of this property.

7 The V-th Postnikov section of S^V

The result we are aiming at is the following:

Theorem 7.0.3. *Let $G = \mathbb{Z}/2$ and let $V \supseteq 1$. Then the space $P_V(S^V)$ has the equivariant weak homotopy type of $K(\underline{\mathbb{Z}}, V)$.*

The proof of this result is quite involved and we are going to split the process in several steps.

7.1 The infinite symmetric product

The main idea behind the proof of our theorem is a clever use of the infinite symmetric product Sp^∞. Let us introduce briefly this object: let X be a space with a basepoint and $k > 0$. We denote by Σ_k the symmetric group of the permutations of k objects. The *k-fold symmetric product* of X is the space:

$$\mathrm{Sp}^k(X) = X^k/\sim,$$

where $(x_1, \ldots, x_k) \sim (x'_1, \ldots, x'_k)$ if and only if there is a permutation $\sigma \in \Sigma_k$ which sends the coordinates of the first point to the second. In other words, we consider the action of Σ_k on X^k given by permuting coordinates and we quotient it out. Note that $\mathrm{Sp}^1(X)$ is naturally isomorphic to X. There is a natural map $\mathrm{Sp}^k(X) \to \mathrm{Sp}^{k+1}(X)$, given by

$$(x_1, \ldots, x_k) \mapsto (e, x_1, \ldots, x_k),$$

where e is the basepoint of X. This gives us a sequence:

$$X \to \mathrm{Sp}^2(X) \to \mathrm{Sp}^3(X) \to \mathrm{Sp}^4(X) \to \ldots.$$

The colimit of the sequence is the *infinite symmetric product* of X and is denoted by $\mathrm{Sp}^\infty(X)$.

Remark 7.1.1. The the symmetric group Σ_k of the permutations of k elements has a standard representation R, with $\mathbb{R}^k$ as representation space, and Σ_k-action given by permuting the coordinates of the points of $\mathbb{R}^k$. Clearly R has a subrepresentation, i.e. a fixed subspace, given by the vectors:

$$(x, \ldots, x) \in \mathbb{R}^k, \qquad \text{for } x \in \mathbb{R},$$

which can be written as $\langle(1, \ldots, 1)\rangle$. If we take the quotient of R by this subspace, we get the reduced standard representation of Σ_k, denoted $\tilde{R}$.

As for any representation, we may look at R and $\tilde{R}$ as representations of $\mathbb{Z}/2$, with a trivial action of this group.

Remark 7.1.2. We are going to use the fact that $\mathrm{Sp}^\infty(S^V)$ is equivalent to $K(\underline{\mathbb{Z}}, V)$, for any representation $V \supseteq 1$. This fact is known in the non-equivariant context, as a consequence of the Dold-Thom theorem. It is also true when we consider G-spaces, for any finite group G, and one can prove it showing that both spaces are equivalent to $AG(S^V)$, the free abelian group on the points of S^V. A reference for the equivalence of $K(\underline{\mathbb{Z}}, V)$ and $AG(S^V)$ is [5], while the other part is worked out in [6, Appendix A]. We state the result we will use in the following proposition.

Proposition 7.1.3. *Let V be a $\mathbb{Z}/2$-representation such that $V \supseteq 1$. Then $\mathrm{Sp}^\infty(S^V)$ is weakly equivalent to $K(V, \underline{\mathbb{Z}})$.*

7.2 The Postnikov section of Sp^∞

Lemma 7.2.1. *Let $V \supseteq 1$ be an orthogonal representation. Then the natural map $i\colon \mathrm{Sp}^\infty(S^V) \to P_V(\mathrm{Sp}^\infty(S^V))$ is an equivariant weak equivalence.*

Proof. Let $r, s \geq 0$ and let W denote the $\mathbb{Z}/2$-representation $\mathbb{R}^r \oplus (\mathbb{R}_-)^s$. By Lemma 4.8.2 and Theorem 4.8.1, we have the following isomorphisms:

$$[S^{V \oplus W}, K(\underline{\mathbb{Z}}, V)]^*_G \cong H^V(S^{V+W}; \underline{\mathbb{Z}}) \cong H^{-r-s,-s}(pt; \underline{\mathbb{Z}}),$$

$$[S^{V \oplus W} \wedge \mathbb{Z}/2_+, K(\underline{\mathbb{Z}}, V)]^*_G \cong H^V(S^{V+W} \wedge \mathbb{Z}/2_+; \underline{\mathbb{Z}}) \cong H^{-r-s,-s}(\mathbb{Z}/2; \underline{\mathbb{Z}}),$$

where the groups to the right in both lines are zero, unless $r = s = 0$. The representations $V + W$, with r and s not both zero, are precisely the representations strictly containing V, which are used to define $\mathcal{A}_V$ (compare with (6.2.2)). Hence, the isomorphisms above say that $K(\underline{\mathbb{Z}}, V) \simeq \mathrm{Sp}^\infty(S^V)$ is $\mathcal{A}_V$-null. Hence, by the universal property of P_V, we get a map

$$t\colon P_V(\mathrm{Sp}^\infty(S^V)) \to \mathrm{Sp}^\infty(S^V),$$

such that $ti = \mathrm{id}$. Hence $\pi_n^H(\mathrm{Sp}^\infty(S^V)) \to \pi_n^H(P_V(\mathrm{Sp}^\infty(S^V)))$ is injective and this implies that i is an equivariant weak equivalence. $\square$

Remark 7.2.2. During the proof of the next proposition, we are going to use the fact that there is an homeomorphism between the spaces:

$$\underbrace{V \oplus \cdots \oplus V}_{k \text{ times}} \cong V \otimes R,$$

both as $\mathbb{Z}/2$- and as Σ_k-spaces. Let us say a few words about the actions we have on the two spaces. $\mathbb{Z}/2$ acts on $V \oplus \cdots \oplus V$ via the diagonal action and Σ_k acts by permuting coordinates. This means that, for $\sigma \in \Sigma_k$ and $(w_1, \ldots, w_k) \in V \oplus \cdots \oplus V$, we have:

$$(w_1, \ldots, w_k) \cdot \sigma = (w_{\sigma^{-1}(1)}, \ldots, w_{\sigma^{-1}(k)}).$$

In other words, the entry in coordinate 1 goes to coordinate $\sigma(1)$ and so on. Note that Σ_k acts on the right via this action.

If V, as a vector space, has basis $(v_1, \ldots, v_n)$ and $(e_1, \ldots, e_k)$ is the standard basis of $R \cong \mathbb{R}^k$, the actions of the two groups on a generating vector $v_i \otimes e_j$ of $V \otimes R$ is defined by:

$$\begin{aligned} g \cdot (v_i \otimes e_j) &= (g \cdot v_i) \otimes e_j, && \text{for } g \in \mathbb{Z}/2, \\ \sigma \cdot (v_i \otimes e_j) &= v_i \otimes e_{\sigma(j)}, && \text{for } \sigma \in \Sigma_k. \end{aligned}$$

We can explicitly write the map defining the homeomorphism:

$$\begin{aligned} &V \oplus \cdots \oplus V \to V \otimes R \\ &(w_1, \ldots, w_k) \to w_1 \otimes e_1 + \cdots + w_k \otimes e_k. \end{aligned}$$

One can easily check that this map is an isomorphism of vector spaces and that is preserves the actions of the two groups.

Proposition 7.2.3. *The map* $\mathrm{Sp}^{k-1}(S^V) \to \mathrm{Sp}^k(S^V)$ *is part of a homotopy cofibration of the form:*

$$S^V \wedge ([V \otimes \tilde{R}] - 0)/\Sigma_k \to \mathrm{Sp}^{k-1}(S^V) \to \mathrm{Sp}^k(S^V).$$

Proof. Let $B = B(V)$ and $S = S(V)$ denote respectively the ball and the sphere inside the representation V. Then we have the usual relative homeomorphism:

$$(B, S) \xrightarrow{\cong} (S^V, *),$$

given by collapsing the subspace S. By relative homeomorphism we mean that it maps $B \setminus S$ onto $S^V \setminus *$ homeomorphically. Now we can apply Sp^k to the pairs and get:

$$(\mathrm{Sp}^k(B), \mathrm{Sp}^k(S)) \xrightarrow{\cong} (\mathrm{Sp}^k(S^V), \mathrm{Sp}^k(*)).$$

We can replace the two subspaces of the pairs with bigger subspaces, choosing them so that the map is still a relative homeomorphism:

$$(\mathrm{Sp}^k(B), Z/\Sigma_k) \xrightarrow{\cong} (\mathrm{Sp}^k(S^V), \mathrm{Sp}^{k-1}(S^V)). \tag{7.2.1}$$

Let us try to explain what we mean with this writing:

(a) Z is the space:

$$(S \times B \times \cdots \times B) \cup (B \times S \times B \times \cdots \times B) \cup \cdots \cup (B \times \cdots \times B \times S) \subseteq B^k.$$

and it can be described by saying that $(b_1, \ldots, b_k) \in B^k$ is a point of Z/Σ_k if at least one of the b_i's is in S. Clearly $Z/\Sigma_K \supseteq \mathrm{Sp}^k(S)$.
(b) Doing the same process for the pair $(\mathrm{Sp}^k(S^V), \mathrm{Sp}^k(*))$, we consider the subspace of $\mathrm{Sp}^k(S^V)$ containing the k-tuples of points for which at least one coordinate is in $\mathrm{Sp}^k(*) = \{*\}$. This is the same of choosing $k-1$ points of S^V, modulo the action, so we get the subspace $\mathrm{Sp}^{k-1}(S^V)$.

Z/Σ_k is mapped to $\mathrm{Sp}^{k-1}(S^V)$, and the map (7.2.1) is still a relative homeomorphism. One can easily verify that, in general, a relative homeomorphism gives a push-out square. In our case we get the push-out:

$$\begin{tikzcd}
\mathbb{Z}/\Sigma_k \arrow[r] \arrow[d] & \mathrm{Sp}^{k-1}(S^V) \arrow[d] \\
\mathrm{Sp}^k(B) \arrow[r] & \mathrm{Sp}^k(S^V).
\end{tikzcd}$$

Note that the space $\mathrm{Sp}^k(B)$ is contractible, since the ball B is, hence we have a homotopy cofiber sequence:

$$\mathbb{Z}/\Sigma_k \to \mathrm{Sp}^{k-1}(S^V) \hookrightarrow \mathrm{Sp}^k(S^V),$$

and so, to show our claim, it is enough to show that

$$\mathbb{Z}/\Sigma_k \simeq S^V \wedge ([V \otimes \tilde{R}] - 0)/\Sigma_k.$$

We can include Z into $(V \oplus \cdots \oplus V) \setminus 0$ and this last space can be identified with $(V \otimes R) \setminus 0$, as shown in Remark 7.2.2. Moreover, we have the following homeomorphism, which is $\mathbb{Z}/2$- and Σ_k-equivariant:

$$\begin{aligned} Z &\to S(V \oplus \cdots \oplus V) \\ z &\mapsto \frac{v}{\|v\|}. \end{aligned}$$

Hence we have that $Z/\Sigma_k \cong S(V \otimes R)/\Sigma_k$. As observed earlier, R has a trivial subrepresentation and we can write it as $R = 1 + \tilde{R}$. Therefore,

$$V \otimes R = V \otimes (1 + \tilde{R}) = V \oplus (V \otimes \tilde{R}).$$

Now we can apply Lemma A.1.3 to get the following $\mathbb{Z}/2$-homeomorphism:

$$S(V \oplus (V \otimes \tilde{R}))/\Sigma_k \cong S(V) * (S(V \otimes \tilde{R})/\Sigma_k),$$

where the symbol $*$ denotes the join of the two spaces, whose definition is recalled in Appendix A.1.

We want to apply Lemma A.1.2 to conclude. Note that we are assuming that $1 \subseteq V$, then the two spaces $S(V)$ and $S(V \otimes \tilde{R})/\Sigma_k$ both have a subspace fixed by $\mathbb{Z}/2$ and so they can be made basepointed (this is needed since the group has to act trivially on basepoints). Our lemma about the join and the suspension deals with basepointed spaces and one can easily check that it works equally well when we take G-spaces. Hence we have:

$$\begin{aligned} Z/\Sigma_k &\cong S(V \otimes R)/\Sigma_k \cong S(V \oplus (V \otimes \tilde{R}))/\Sigma_k \\ &\simeq S(V) * (S(V \otimes \tilde{R})/\Sigma_k) \simeq \Sigma(S(V) \wedge (S(V \otimes \tilde{R})/\Sigma_k) \\ &\simeq \Sigma^1 \wedge S(V) \wedge (S(V \otimes \tilde{R})/\Sigma_k) \simeq S^V \wedge ((V \otimes \tilde{R}) \setminus 0)/\Sigma_k. \end{aligned}$$

In the last passage we have used the fact that $\Sigma^1 \wedge S(V) \simeq S^V$: the reader can easily check that this is true in general. □

The next step in order to prove Theorem 7.0.3 is to show that applying the functor P_V to the space $S^V \wedge ((V \otimes \tilde{R}) \setminus 0)/\Sigma_k$ we get a $\mathbb{Z}/2$-contractible space.

Proposition 7.2.4. *Let V be the $\mathbb{Z}/2$-representation $\mathbb{R}^p \oplus (\mathbb{R}_-)^q$, with $p \geq 1$ and let $X = ((V \otimes \tilde{R}) \setminus 0)/\Sigma_k$. Then:*

(a) If $k \geq 3$, then $X^{\mathbb{Z}/2}$ is path-connected.
(b) If $k = 2$, then the fixed point $X^{\mathbb{Z}/2}$ has one of the following homotopy types:

$$X^{\mathbb{Z}/2} \simeq \begin{cases} \mathbb{R}P^{p-1} & \text{if } q = 0, \\ \mathbb{R}P^{p-1} \coprod * & \text{if } q = 1, \\ \mathbb{R}P^{p-1} \coprod \mathbb{R}P^{q-1} & \text{if } q \geq 2. \end{cases}$$

When $q \geq 1$, there exists a map $S^{1,1} \to X$, which induces an isomorphism on π_0 of the fixed sets.

Proof. As usual, we can decompose the representation V as a sum of irreducible representations in a unique way:

$$V = U_0 \oplus U_1 \oplus \cdots \oplus U_{n-1},$$

with $n = p + q$. Of course, since $G = \mathbb{Z}/2$, every factor U_i will be either a trivial or a sign representation. In particular, let us order the factors so that U_0 is a trivial representation. Each point of X can be written as

$$[(u_0, \ldots, u_{n-1})],$$

where $u_i \in U_i \otimes R$ for every i, and not all the u_i's are zero.

(a) Let $k \geq 3$. If $[(u_0, \ldots, u_{n-1})] \in X^{\mathbb{Z}/2}$ is a fixed point with $u_1 \neq 0$, then we can make the following path:

$$t \in [0,1] \mapsto [(tu_0, \ldots, tu_{i-1}, u_i, tu_{i+1}, \ldots, tu_{n-1})],$$

connecting it to $[(0, \ldots, u_i, \ldots, 0)]$. Here we exploit the underlying structure of real vector spaces of V and $\tilde{R}$. Note that all the points in the path are fixed by $\mathbb{Z}/2$, since each u_j is fixed. This means that we can connect every point of X with a point of either

$$(\mathbb{R} \otimes \tilde{R}) \cong (\mathbb{R} \otimes \tilde{R}) \oplus 0 \cdots \oplus 0 \subseteq V \times \tilde{R} \quad \text{or}$$

$$(\mathbb{R}_- \otimes \tilde{R}) \cong (\mathbb{R} \otimes \tilde{R}) \oplus 0 \cdots \oplus 0 \subseteq V \times \tilde{R}.$$

If we check the two following conditions, the proof of (a) is complete.

(i) The subspace $[((\mathbb{R} \otimes \tilde{R}) \setminus 0)/\Sigma_k]^{\mathbb{Z}/2}$ is path connected. Since the $\mathbb{Z}/2$-action is trivial, we have that $\mathbb{R} \otimes \tilde{R} \cong \tilde{R}$ and $\tilde{R}$ is path connected, being a real vector space. If we remove the origin to get $\tilde{R} \setminus 0$ we do not disconnect the space, since $\dim(\tilde{R}) = \dim(R) - 1 = k - 1 \geq 2$.

(ii) The subspace $[((\mathbb{R} \otimes \tilde{R} \oplus \mathbb{R}_- \otimes \tilde{R}) \setminus 0)/\Sigma_k]^{\mathbb{Z}/2}$ is path connected. Note that, since $V \supseteq 1$, this subspace is always contained in X, if V is not trivial. In case V is trivial, every point is connected by a path to a point of the subspace analysed in (i).

Let $V' = \mathbb{R} \oplus \mathbb{R}_-$ and $X' = ((V' \otimes \tilde{R}) \setminus 0)/\Sigma_k$. We are trying to show that $(X')^{\mathbb{Z}/2}$ is path connected. We can identify V' with $\mathbb{C}$, with the conjugation action and, as noted earlier, $(V' \otimes R) \setminus 0 \cong V' \oplus \cdots \oplus V' \cong \mathbb{C}^k$. The space $\tilde{R}$ is a quotient of R in which we quotient out the the subspace of the vectors having the same component on each basis vector. In this view, $(V' \otimes R) \setminus 0$ is identified with

$$\{(z_1, \ldots, z_k) \in \mathbb{C}^k \mid \sum z_i = 0\},$$

since we can subtract from any vector $(z_1, \ldots, z_k)$ the vector $(\frac{\sum z_i}{k}, \ldots, \frac{\sum z_i}{k})$. The class of $(z_1, \ldots, z_k)$ in X' is denoted $[z_1, \ldots, z_k]$. The space $(X')^{\mathbb{Z}/2}$ has the subspace of the real vectors, which are fixed under conjugation:

$$X'' = \{[a_1, \ldots, a_k] \mid a_i \in \mathbb{R}\} \cong \{[z_1, \ldots, z_k] \mid z_i \in \mathbb{C}^k, \sum z_i = 0, \mathrm{Im}(z_i) = 0\}$$

This subspace is path connected: in fact it corresponds to the subspace analysed in (i). Hence it will be sufficient to show that any point of $(X')^{\mathbb{Z}/2}$ can be connected with a path of a point in X''. Let us consider a point $[z_1, \ldots, z_k] \in (X')^{\mathbb{Z}/2}$. The fact that is is fixed by $\mathbb{Z}/2$ means that

$$[z_1, \ldots, z_k] = [\overline{z}_1, \ldots, \overline{z}_k].$$

This is equivalent to saying that there exists a permutation $\sigma \in \Sigma_k$ for which:

$$(z_{\sigma(1)}, z_{\sigma(k)}) = (\overline{z}_1, \dots, \overline{z}_k).$$

The permutation σ decomposes as a product of disjoint cycles. If we have a cycle of length l, it means that (up to a change of indices):

$$z_1 = \overline{z}_2, \quad z_2 = \overline{z}_3, \quad z_l = \overline{z}_1.$$

These relations imply that either $\{z_1, \dots, z_l\} \subseteq \{w, \overline{w}\}$ for some $w \in \mathbb{C}$ (in case l is even), or that all the z_i's are real. In any case, this argument shows that, modulo permutation, $(z_1, \dots, z_k)$ can be written as

$$(w_1, \overline{w}_1, \dots, w_u, \overline{w}_u, r_1, \dots, r_v),$$

for certain $w_i \in \mathbb{C}$ and $r_i \in \mathbb{R}$. If all the w_i's are real, then the point is already contained in X'', so we are done. Else, we can assume $\mathrm{Im}(w_1) \neq 0$. We want to build a path in $(X')^{\mathbb{Z}/2}$ connecting our point with

$$[w_1 - \mathrm{Re}(w_1), \overline{w}_1 - \mathrm{Re}(w_1), 0, \dots, 0].$$

Let us define a path $\gamma\colon I \to (X')^{\mathbb{Z}/2}$ by:

$$t \mapsto [w_1 + \alpha(t), \overline{w}_1 + \alpha(t), w_2, \overline{w}_2, \dots, w_u, \overline{w}_u, r_1, \dots, r_v].$$

We have not defined α yet: we are going to use it to impose the condition of having the sum of the components equal to zero. The sum of the components of a point in our path at t is:

$$2\,\mathrm{Re}(w_1) + 2t \sum_{i=2}^{u} \mathrm{Re}(w_i) + t \sum_{j=0}^{v} r_j + 2\alpha(t) = 0.$$

We can use this relation to define $\alpha(t)$: clearly $\alpha(t) \in \mathbb{R}$, and this implies that the points of the path are still fixed by $\mathbb{Z}/2$. We have:

$$\begin{aligned} \gamma(0) &= [w_1 - \mathrm{Re}(w_1), \overline{w}_1 - \mathrm{Re}(w_1), 0, \dots, 0], \\ \gamma(1) &= [w_1, \overline{w}_1, \dots, w_u, \overline{w}_u, r_1, \dots, r_v]. \end{aligned}$$

So this is actually a the path we wanted. Let us note that the components of $\gamma(0)$ are pure imaginary numbers, so we can write it as

$$[bi, -bi, 0, \dots, 0]$$

for some $b \in \mathbb{R} \setminus 0$. The last step to complete the proof of (a) is to show how to connect this point with a point of X''. We define another path:

$$t \in [0,1] \mapsto [t + (1-t)bi, t - (1-t)bi, -2t, 0, \dots, 0].$$

It connects our point $[bi, -bi, 0, \dots, 0]$ with $[1, 1, -2, 0, \dots, 0]$ and is contained in $(X')^{\mathbb{Z}/2}$. Note that here we are using 3 coordinates to build the path, hence the assumption $k \geq 3$. Composing the paths we have connected the initial point with $[1, 1, -2, 0, \dots, 0] \in X''$, showing that $(X')^{\mathbb{Z}/2}$ is path connected.

(b) Now we take $k = 2$. In this case $R = \mathbb{R}^2$, with the action of $\Sigma_2 = \{e, \sigma\}$ in which σ exchanges coordinates. To get $\tilde{R}$ we quotient out the subspace $\langle(1, 1)\rangle$ and so we have that $\tilde{R} \cong \mathbb{R}$ and σ acts by changing sign. We are considering a representation $\mathbb{R}^p \oplus (\mathbb{R}_-)^q$. Then we have:

$$X = [(V \otimes \tilde{R}) \setminus 0]/\Sigma_2 = [(\mathbb{R}^p \oplus (\mathbb{R}_-)^q) \setminus 0]/\Sigma_2.$$

As said, Σ_2 acts changing sign on each coordinate, in other words we have the antipodal action on $(\mathbb{R}^p \oplus (\mathbb{R}_-)^q) \setminus 0$ and so, by quotienting out the action, we obtain a space homotopy equivalent to $\mathbb{P}(\mathbb{R}^p \oplus (\mathbb{R}_-)^q) = \mathbb{R}P^{p+q-1}$. A homotopy equivalence to $\mathbb{R}P^{p+q-1}$ can be obtained by collapsing each the (contractible) open half-lines based at the origin to one point. A point in the projective space can be written, in homogeneous coordinates, as:

$$[x_1, \dots, x_p, x_{p+1}, x_{p+q}],$$

and the $\mathbb{Z}/2$-action maps this point to

$$[x_1, \dots, x_p, -x_{p+1}, -x_{p+q}].$$

Thus, a point is fixed if

$$(x_1, \dots, x_p, x_{p+1}, x_{p+q}) = \lambda(x_1, \dots, x_p, -x_{p+1}, -x_{p+q}),$$

and this can happen either if $x_1 = \cdots = x_p = 0$ and $\lambda = -1$, or if $x_{p+1} = \cdots = x_{p+q} = 0$ and $\lambda = 1$. Hence we have:

$$\begin{aligned} X^{\mathbb{Z}/2} &= \{[x_1, \dots, x_p, 0, \dots, 0] \mid x_i \in \mathbb{R} \text{ not all zero}\} \\ &\quad \coprod \{[0, \dots, 0, x_{p+1}, \dots, x_{p+q}] \mid x_i \in \mathbb{R} \text{ not all zero}\} \\ &= \mathbb{R}P^{p-1} \coprod \mathbb{R}P^{q-1}. \end{aligned}$$

The last thing to show is the existence of a map $S^{1,1} \to X$ which induces a bijection on π_0 of the fixed-points, under the assumption $q \geq 1$. Let us define:

$$\begin{aligned} ((\mathbb{R} \oplus \mathbb{R}_-) \setminus 0)/\sim &\to X \\ [a, b] &\mapsto [a, 0, \dots, b, 0, \dots, 0], \end{aligned}$$

where $\sim$ is the antipodal action and b is mapped in the $(p+1)^{\text{th}}$ coordinate. As noted earlier, the domain of the map is equivalent to $\mathbb{R}P^1$. This space is, non-equivariantly, homeomorphic to the sphere S^1; the $\mathbb{Z}/2$-action is trivial on the first coordinate and changes the sign on the second one, hence $((\mathbb{R} \oplus \mathbb{R}_-) \setminus 0) \simeq S^{1,1}$. Clearly the map is well defined, because in both the domain and the target we quotient out the antipodal action. With the same reasoning used before, one can see that the fixed point space $[((\mathbb{R} \oplus \mathbb{R}_-) \setminus 0)/\sim]^{\mathbb{Z}/2}$ is made by two path components. We can write it as:

$$\{[a,0] \mid a \in \mathbb{R} \setminus 0\} \coprod \{[0,b] \mid b \in \mathbb{R} \setminus 0\}.$$

The two path components are sent to the two fixed-points of $S^{1,1}$ via the homotopy equivalence. It is immediate to check that, on the fixed points, the map defined above induces a bijection on the set of the path components. $\square$

Lemma 7.2.5. *Let $\mathcal{A}$ be the family containing the objects:*

$$\begin{aligned} &S^{n,0} \quad n \geq 1, \\ &S^{n,0} \wedge \mathbb{Z}/2_+ \quad n \geq 1, \\ &S^{1,1}. \end{aligned}$$

Let X be the space defined in the previous proposition. Then $P_{\mathcal{A}}(X)$ is contractible.

Proof. By the Whitehead theorem, it is enough to show:

$$[S^{n,0}, P_{\mathcal{A}}(X)]^*_G = 0, \qquad [S^{n,0} \wedge \mathbb{Z}/2_+, P_{\mathcal{A}}(X)]^*_G = 0,$$

for $n \geq 0$. This is clear for $n \geq 1$, because of the choice of the elements in $\mathcal{A}$, by the definition of the functor $P_{\mathcal{A}}$. So it remains to check the claim for $n = 0$: the statement is equivalent to say that both $P_{\mathcal{A}}(X)$ and $P_{\mathcal{A}}(X)^{\mathbb{Z}/2}$ are path-connected.

We have that the space X is connected, therefore $P_{\mathcal{A}}(X)$ is too: in fact $P_{\mathcal{A}}$ attaches cones on the maps from elements of $\mathcal{A}$, and this operation does not disconnect the space. As to the fixed-points, we know by the previous proposition that it is path-connected when $k \geq 3$ or when $k = 2$ and $q = 0$: if we attach to the space cones on maps we do not disconnect the fixed-points. In the remaining cases, i.e. for $K = 2$ and $q \geq 1$, the space X is not connected, but has two path-components and we know that there is a map $S^{1,1} \to X$ inducing an isomorphism on π_0 of the fixed point set. This map hits the two path components of the fixed point: when we construct $P_{\mathcal{A}}(X^{\mathbb{Z}/2})$ we attach the cone on this map, so we get a connected space again. $\square$

Proposition 7.2.6. *Let X be again the space of Proposition 7.2.4. Then $P_V(S^V \wedge X)$ is contractible.*

Proof. Let us consider any map $S^{1,0} \to X$. We can make it into a homotopy fibration:

$$S^{1,0} \to X \to X'$$

and then smash it with S^V, obtaining the homotopy fiber sequence:

$$S^{V+1} \to X \wedge S^V \to X' \wedge S^V.$$

By Proposition 6.2.2(c), it follows that we have an equivalence:

$$P_V(S^V \wedge X) \simeq P_V(S^V \wedge X').$$

In other words, if we attach the cone on any map $S^{1,0} \to X$ we don't change the homotopy type of $P(S^V \wedge X)$. We can repeat the argument with any map $S^{n,0} \to X$ ($n \geq 1$), $S^{n,0} \wedge \mathbb{Z}/2_+ \to X$ ($n \geq 1$) and $S^{1,1} \to X$. The space $P_{\mathcal{A}}(X)$ considered in the previous lemma is, by definition, built from X by attaching cones on all such maps. Then we obtain a homotopy equivalence:

$$P_V(S^V \wedge X) \simeq P_V(S^V \wedge P_{\mathcal{A}}(X)).$$

The previous proposition showed that $P_{\mathcal{A}}(X) \simeq *$, hence the smash product $S^V \wedge P_{\mathcal{A}}(X)$ is also contractible. Applying P_V we get again a contractible space, so the claim is proved. □

Proof of Theorem 7.0.3. Combining the previous propositions, we have that the map

$$P_V(\mathrm{Sp}^{k-1}(S^V)) \to P_V(\mathrm{Sp}^k(S^V))$$

is a weak equivalence for all k. These maps form a sequence:

$$P_V(S^V) \to P_V(\mathrm{Sp}^2(S^V)) \to P_V(\mathrm{Sp}^3(S^V)) \to \dots$$

of equivalences, giving that $P_V(S^V) \to \operatorname{hocolim} P_V(\mathrm{Sp}^i(S^V))$ is also a weak equivalence.

By Proposition 6.2.2(b), we have that

$$P_V(\operatorname{hocolim}_i(\mathrm{Sp}^i(S^V))) \simeq P_V(\operatorname{hocolim}_i P_V(\mathrm{Sp}^i(S^V))).$$

It is possible to see that the second term of the equivalence is weakly equivalent to $\operatorname{hocolim}_i P_V(\mathrm{Sp}^i(S^V))$: in fact, when we apply P_V to the hocolim, we are applying it to the homotopy colimit of a sequence of spaces which are already P_V of something, hence we are not killing any new homotopy group. Thus we have the composition of maps:

$$\begin{aligned} P_V(S^V) &\xrightarrow{\simeq} \operatorname{hocolim} P_V(\mathrm{Sp}^i(S^V)) \simeq P_V(\operatorname{hocolim}_i(\mathrm{Sp}^i(S^V))) \\ &\to P_V(\operatorname{colim}_i(\mathrm{Sp}^i(S^V))) = P_V(\mathrm{Sp}^\infty(S^V)). \end{aligned}$$

We have that the natural map $\operatorname{hocolim}_i(\mathrm{Sp}^i(S^V)) \to \operatorname{colim}_i(\mathrm{Sp}^i(S^V))$ is a weak equivalence: this is true because the inclusions

$$\mathrm{Sp}^{i-1}(S^V) \to \mathrm{Sp}^i(S^V)$$

are cofibrations, since the spaces are well pointed. Then the last map in the composition is also a weak equivalence, and finally we get the equivalence

$$P_V(S^V) \simeq P_V(\mathrm{Sp}^\infty(S^V)). \qquad \square$$

8 The construction of the spectral sequence

8.1 Unstable case

We are constructing an Atiyah-Hirzebruch spectral sequence to compute KR-theory. From now on, we will just be concerned with spaces with an action of $G = \mathbb{Z}/2$. We will often consider the representation $n\mathbb{C}$, i.e. the complex space $\mathbb{C}^n$ seen as a $2n$-dimensional real vector space, with the action given by complex conjugation. In the notation of the previous sections, this would correspond to $(2n, n)$. To simplify the notation, following [6], we will write P_{2n} and $\mathbb{P}_{2n}$ for the Postnikov functors associated to this representation. This choice introduces a possible ambiguity of notation, because $2n$ also denotes the $2n$-dimensional trivial representation. We are not going any more to consider trivial representations in what follows: this should avoid the confusion.

The aim of this section is to construct a tower of homotopy fibrations:

$$\begin{tikzcd} \cdots \arrow[r] & P_4(\mathbb{Z}\times BU) \arrow[r] & P_2(\mathbb{Z}\times BU) \arrow[r] & P_0(\mathbb{Z}\times BU) \arrow[r] & * \\ & K(\underline{\mathbb{Z}},(4,2)) \arrow[u] & K(\underline{\mathbb{Z}},(2,1)) \arrow[u] & \mathbb{Z} \arrow[u] & \end{tikzcd} \tag{8.1.1}$$

The reason for which we are concentrating on the space $\mathbb{Z} \times BU$ is that we are interested in KR-theory, which is represented by such spaces. Let us explain what are the maps in the diagram above. Let $\beta\colon S^{2,1} \to \mathbb{Z} \times BU$ be the Bott map in $\widetilde{KR}^{0,0}(S^{2,1})$ and let $\beta^n\colon S^{2n,n} \to \mathbb{Z} \times BU$ be its n^{th} power. The homotopy fiber sequence is:

$$P_{2n}(S^{2n,n}) \xrightarrow{\beta^n} P_{2n}(\mathbb{Z} \times BU) \to P_{2n-2}(\mathbb{Z} \times BU),$$

where $P_{2n}(\mathbb{Z} \times BU) \to P_{2n-2}(\mathbb{Z} \times BU)$ is the structure map of the Postnikov section.

The next goal is to show that the homotopy fiber

$$F_n \to P_{2n}(\mathbb{Z} \times BU) \to P_{2n-2}(\mathbb{Z} \times BU)$$

is weakly equivalent to $P_{2n}(S^{2n,n})$, It turns out that, in order to show this result, it is more convenient to show an intermediate result for $\mathbb{P}$ and then try to connect $\mathbb{P}$ and P.

Proposition 8.1.1. *The homotopy fiber of* $\mathbb{P}_{2n}(\mathbb{Z} \times BU) \to \mathbb{P}_{2n-2}(\mathbb{Z} \times BU)$ *is homotopy equivalent to* $K(\underline{\mathbb{Z}}, (2n, n))$*.*

Proof. Let $\mathbb{F}_n$ be the homotopy fiber. By Proposition 6.3.1(a),

$$\pi_k^G(\mathbb{P}_{2n}(\mathbb{Z} \times BU)) \cong \pi_k^G(X) \cong \pi_k^G(\mathbb{P}_{2n-2}(\mathbb{Z} \times BU))$$

for $k \leq n-1$ and $\pi_n(\mathbb{P}_{2n-2}(\mathbb{Z} \times BU)) = 0$. So, by the long exact sequence for equivariant homotopy groups, we get that $\pi_k^G(\mathbb{F}_n) = 0$ for $k \leq n-1$. Moreover, using the fact that the equivariant homotopy group $\pi_k^e(-)$ is isomorphic to the non-equivariant one $\pi_k(-)$ and since, by Corollary 6.3.2, $\mathbb{P}_{2n}$ is equivalent to the non-equivariant Postnikov section P_{2n}, we get that $\pi_k^e(\mathbb{F}_n) = 0$ for $k \leq 2n-1$. These two facts combined mean that $\mathbb{F}_n$ is $(n\mathbb{C}-1)$-connected.

By construction, looking at the definition of $\tilde{\mathcal{A}}_V$, we have that

$$[S^{2n+k,n+l} \wedge G/H_+, \mathbb{P}_{2n}(\mathbb{Z} \times BU)]_G^* = 0$$

for all $k > 0$, $l \geq 0$ and $H \leq G$. Then, by the long exact sequence, we get that $[S^{2n+k,n} \wedge G/H_+, \mathbb{F}_n]_* = 0$ for all $k > 0$ and $H \leq G$. To conclude, we still need to show that the Mackey functor $\underline{\pi}_{2n,n}(\mathbb{F}_n)$ coincides with $\underline{\mathbb{Z}}$, and this is proved true using again the long exact sequence of homotopy groups. □

Proposition 8.1.2. *(a) Let X be a $\mathbb{Z}/2$-space for which the forgetful map:*

$$[S^{2n,n}, X]_{\mathbb{Z}/2}^* \to [S^{2n,n}, X]_e^*$$

is injective. Then $\mathbb{P}_{2n}(X)$ is weakly equivalent to $P_{2n}(X)$.

(b) The space $X = \mathbb{Z} \times BU$ satisfies the hypothesis of (a).

Proof. (a) Recall that $[S^{2n,n}, X]_e$ denotes the non-equivariant homotopy classes of maps. We have in general a map $\mathbb{P}_{2n}X \to P_{2n}X$. By definition, $\mathcal{A}_V$ and $\tilde{\mathcal{A}}_V$ differ by the elements of the form $S^{2n+k,n+k} \wedge G/H_+$ for $k > 0$, i.e. the representation spheres that strictly containing $V = (2n, n)$, but with the same number of trivial components. Then, to prove the weak equivalence between $\mathbb{P}_{2n}(X)$ and $P_{2n}(X)$, it is enough to show the two following conditions:

$$[S^{2n+k,n+k} \wedge \mathbb{Z}/2_+, \mathbb{P}_{2n}X]^*_{\mathbb{Z}/2} = 0 \qquad \text{for all } k > 0,$$
$$[S^{2n+k,n+k}, \mathbb{P}_{2n}X]^*_{\mathbb{Z}/2} = 0 \qquad \text{for all } k > 0.$$

The first one is easily verified: we have the isomorphism

$$[S^{2n+k,n+k} \wedge \mathbb{Z}/2_+, \mathbb{P}_{2n}X]^*_{\mathbb{Z}/2} \cong [S^{2n+k,n+k}, \mathbb{P}_{2n}X]^*_e,$$

and the second group is trivial: in fact it is the same as the homotopy classes of non-equivariant maps $S^{2n+k} \to \mathbb{P}_{2n}X$. By Corollary 6.3.2 it is the trivial group.

Now we need to check the second condition: this case is more involved than the previous one. We begin with the case $k = 1$: we can take the cofibration sequence

$$\mathbb{Z}/2_+ \to S^{0,0} \to S^{1,1} \to \mathbb{Z}/2_+ \wedge S^{1,0} \to \dots, \tag{8.1.2}$$

and smash it with $S^{2n,n}$, so that we get

$$\mathbb{Z}/2_+ \wedge S^{2n,n} \to S^{2n,n} \to S^{2n+1,n+1} \to \mathbb{Z}/2_+ \wedge S^{2n+1,2n}.$$

If we map the sequence into $\mathbb{P}_{2n}X$ we get the top exact row of four terms in the diagram, in which it is understood that we consider basepoint-preserving G-homotopy classes of equivariant maps:

$$\begin{tikzcd}
 & {[S^{2n+1,n+1}, \mathbb{P}_{2n}X]} \arrow[d] & {[\mathbb{Z}/2_+ \wedge S^{2n+1,n}, \mathbb{P}_{2n}X] = 0} \arrow[l] \\
{[\mathbb{Z}/2_+ \wedge S^{2n,n}, \mathbb{P}_{2n}X]} & {[S^{2n,n}, \mathbb{P}_{2n}X]} \arrow[l] & \\
{[\mathbb{Z}/2_+ \wedge S^{2n,n}, X]} \arrow[u, "\cong"] & {[S^{2n,n}, X]} \arrow[l] \arrow[u, "\cong"] &
\end{tikzcd}$$

The two vertical maps in the lower square are isomorphisms, because of Proposition 6.3.1. The bottom map in the square corresponds to the forgetful map $[S^{2n,n}, X]^*_{\mathbb{Z}/2} \to [S^{2n,n}, X]^*_e$, as $[G_+ \wedge S^V, X]^*_G \cong [S^V, X]^*_e$, hence the top map in the square is injective. We already know that

$$[G_+ \wedge S^{2n+1,n}, \mathbb{P}_{2n}X]^*_G$$

is trivial, because $S^{2n+1,n} \in \tilde{\mathcal{A}}_{(2n,n)}$, then we deduce by exactness that $[S^{2n+1,n+1}, \mathbb{P}_{2n}X]^*_{\mathbb{Z}/2} = 0$.
To get the thesis for $k > 1$, we proceed by induction. Assume that

$$[S^{2n+k,n+k}, \mathbb{P}_{2n}X]^*_{\mathbb{Z}/2} = 0.$$

We can smash the cofibration sequence (8.1.2) with $S^{2n+k,n+k}$, getting:

$$S^{2n+k,n+k} \to S^{2n+k+1,n+k+1} \to \mathbb{Z}/2_+ \wedge S^{2n+k+1,n+k}.$$

If we map into $\mathbb{P}_{2n}X$, we obtain the exact sequence:

$$[\mathbb{Z}/2_+ \wedge S^{2n+k+1,n+k}, \mathbb{P}_{2n}X] \to [S^{2n+k+1,n+k+1}, \mathbb{P}_{2n}X] \to [S^{2n+k,n+k}, \mathbb{P}_{2n}X].$$

We know that the first term is zero, because of the properties of $\mathbb{P}$, and that the third one also is, by inductive hypothesis. Hence our thesis follows.
(b) We have to check that the forgetful map

$$[S^{2n,n}, \mathbb{Z} \times BU]^*_{\mathbb{Z}/2} \to [S^{2n,n}, \mathbb{Z} \times BU]^*_e$$

is injective. The domain of the map is precisely $\widetilde{KR}^{0,0}(S^{2n,n})$, because equivariantly $\mathbb{Z} \times BU$ represents $\widetilde{KR}^{0,0}$ and the target is $\widetilde{K}^{0,0}(S^{2n,n})$, since non-equivariantly $\mathbb{Z} \times BU$ represents $\widetilde{K}^{0,0}$. Hence we are checking that the forgetful map

$$\mathbb{Z} \cong \widetilde{KR}^{0,0}(S^{2n,n}) \to \widetilde{K}^{0,0}(S^{2n,n}) \cong \mathbb{Z}$$

is injective. And this is true, because this map is even an isomorphism, since the Bott element β is a generator for both groups. □

The two previous propositions imply that we have the tower of homotopy fiber sequences (8.1.1). By Proposition 6.3.1, we have that the homotopy limit of the tower is $\mathbb{Z} \times BU$. The homotopy spectral sequence associated to the tower (8.1.1) is the one we are interested in. For a $\mathbb{Z}/2$-space X, it has the form:

$$H_G^{p,-\frac{q}{2}}(X; \underline{\mathbb{Z}}) \Rightarrow [S^{-p-q,0} \wedge X_+, \mathbb{Z} \times BU]^*_G.$$

Our tower of homotopy fibrations is bounded below: in this case the spectral sequence is conditionally convergent (see Boardman [3]). Moreover, if the condition "$RE_\infty = 0$", where RE_∞ is the derived E_∞ term, holds, then the spectral sequence converges strongly, by [3, 7.4].

Remark 8.1.3. There is a multiplicative structure on the pages of the homotopy spectral sequence, which can be useful for computations. The proof and the details of this fact are out of the scope of this project. The reader is referred to [6, 5.6] for more details about the multiplication in our spectral sequence.

8.2 The stable case

The spectral sequence that we have built in the previous section is *fringed*, meaning that the entries close to $p = 0$ and $q = 0$ are not abelian groups and so things work well only away from the axes. This can be fixed easily, producing a stable version of the spectral sequence, with G-spectra in place of G-spaces.

We are going to create a G-spectrum kr representing *connective* KR-theory, in analogy to the ordinary spectrum ku represents connective complex K-theory. More in detail, we are aiming at constructing a homotopy cofiber sequence:

$$\Sigma^{2,1} kr \xrightarrow{\beta} kr \to H\underline{\mathbb{Z}},$$

that will allow us to get the stable version of the spectral sequence.

Let W_n be the homotopy fiber of the map:

$$\mathbb{Z} \times BU \to P_{2n-2}(\mathbb{Z} \times BU).$$

These spaces come equipped with natural maps $W_{n+1} \to W_n$ connecting them, given by the corresponding maps $P_{2n}(\mathbb{Z} \times BU) \to P_{2n-2}(\mathbb{Z} \times BU)$. Recall that $\Omega^{2,1}$ denotes the loop-space functor with respect to the representation $\mathbb{R} \oplus \mathbb{R}_-$.

The next proposition will be the key to build the wanted G-spectrum.

Proposition 8.2.1. *Let W_n be the homotopy cofiber:*

$$W_n \to \mathbb{Z} \times BU \to P_{2n-2}(\mathbb{Z} \times BU). \tag{8.2.1}$$

There exists a weak equivalence $\varphi\colon W_n \to \Omega^{2,1} W_{n+1}$, which commutes with the Bott map β in the following diagram:

$$\begin{tikzcd}
W_n \arrow[r, "\varphi"] \arrow[d] & \Omega^{2,1} W_{n+1} \arrow[d] \\
\mathbb{Z} \times BU \arrow[r, "\beta"] & \Omega^{2,1}(\mathbb{Z} \times BU).
\end{tikzcd}$$

Moreover, φ is unique up to homotopy.

Proof. Let us consider the natural inclusion map $\alpha\colon \mathbb{Z}\times BU \to P_n(\mathbb{Z}\times BU)$ and apply to it the functor $\Omega^{2,1}$.

Note that, if $X \in \mathcal{A}_{(2n-2,n-1)}$, then $S^{2,1} \wedge X \in \mathcal{A}_{(2n,n)}$. By definition $P_{2n}(\mathbb{Z}\times BU)$ is $\mathcal{A}_{2n,n}$-null, and so, by the adjunction of loop-space and suspension, we deduce that $\Omega^{2,1}P_n(\mathbb{Z}\times BU)$ is $\mathcal{A}_{(2n-2,n-1)}$-null. By one property of nullification functors (see Proposition 6.2.2), this implies that there is a lift, unique up to homotopy:

$$\begin{tikzcd}
\Omega^{2,1}(\mathbb{Z}\times BU) \arrow[r, "\Omega^{2,1}\alpha"] \arrow[d, hook] & \Omega^{2,1}P_{2n}(\mathbb{Z}\times BU) \\
P_{2n-2}(\Omega^{2,1}(\mathbb{Z}\times BU)) \arrow[ur, dashed, "\psi"'] &
\end{tikzcd}$$

This lift fits into the following diagram:

$$\begin{tikzcd}
W_n \arrow[r] \arrow[dd, dashed] & \mathbb{Z}\times BU \arrow[r] \arrow[d, "\beta"] & P_{2n-2}(\mathbb{Z}\times BU) \arrow[d, "P_{2n-2}(\beta)"] \\
& \Omega^{2,1}(\mathbb{Z}\times BU) \arrow[r] \arrow[d, "\mathrm{id}"] & P_{2n-2}(\Omega^{2,1}(\mathbb{Z}\times BU)) \arrow[d, "\psi"] \\
\Omega^{2,1}W_{n+1} \arrow[r] & \Omega^{2,1}(\mathbb{Z}\times BU) \arrow[r] & \Omega^{2,1}P_{2n}(\mathbb{Z}\times BU).
\end{tikzcd}$$

By general properties of fibrations, it follows that there is a map $W_n \to \Omega^{2,1}W_{n+1}$ between the homotopy fibers, making the diagram commute. Now the goal is to show that this map is a weak homotopy equivalence, and we will do this using Corollary 1.9.5. We have check the following.

(a) $[S^{k,0}, W_n]^*_G = 0$ for $0 \leq k < n$. To see this we can use the long exact sequence that we get by mapping $S^{k,0}$ into (8.2.1). By Propositions 6.3.1 and 8.1.2, we know that

$$[S^{k,0}, \mathbb{Z}\times BU]^*_G \to [S^{k,0}, P_{2n-2}(\mathbb{Z}\times BU)]^*_G$$

is an isomorphism for $k \leq |V^{\mathbb{Z}/2}| = n-1$ and an epimorphism for $k = n$. This implies the claim.

(b) $[S^{k,0} \wedge \mathbb{Z}/2_+, W_n]^*_G = 0$ for $0 \leq k < 2n$. As in the previous case, we use the analogous long exact sequence, but now we have an isomorphism for $k \leq |V^e| = 2n-2$ and an epimorphism for $k = 2n-1$, hence the claim is proved only for $k \leq 2n-1$. However, we know that

$$[S^{2n,0} \wedge \mathbb{Z}/2_+, P_{2n-2}(\mathbb{Z}\times BU)]^*_G = 0 = [S^{2n-1,0} \wedge \mathbb{Z}/2_+, P_{2n-2}(\mathbb{Z}\times BU)]^*_G,$$

by the properties of P_{2n-2}, hence the long exact sequence gives us an isomorphism:

$$[S^{2n-1} \wedge \mathbb{Z}/2_+, W_n]^*_G \cong [S^{2n-1} \wedge \mathbb{Z}/2_+, \mathbb{Z} \times BU]^*_G \cong \\ \cong [S^{2n-1}, \mathbb{Z} \times BU]^* = \pi_{2n-1}(\mathbb{Z} \times BU) = 0;$$

in fact $\pi_i(\mathbb{Z} \times BU) = 0$ for i odd, as a consequence of Bott periodicity.
(c) $[S^{k,0}, \Omega^{2,1} W_{n+1}]^*_G = 0$ for $0 \leq k < n$. Via the adjunction, we have isomorphisms:

$$[S^{k,0}, \Omega^{2,1} X]^*_G \cong [S^{k+2,1}, X]^*_G,$$

and we can use again the long exact sequence, as done in (a).
(d) $[S^{k,0} \wedge \mathbb{Z}/2_+, \Omega^{2,1} W_{n+1}]^*_G = 0$ for $0 \leq k < 2n$. By the isomorphism (4.8.1), we can write:

$$[S^k \wedge \mathbb{Z}/2_+, \Omega^{2,1} W_{n+1}]^*_G \cong [S^{k+2,1} \wedge \mathbb{Z}/2_+, W_{n+1}]^*_G \cong [S^{k+2} \wedge \mathbb{Z}/2_+, W_{n+1}]^*_G,$$

and, by (b), the last group is zero for $k+2 < 2n+2$, so our claim is proved.

We have that $[S^{2n+k,n} \wedge (\mathbb{Z}/2)/H_+, P_{2n-2}(\mathbb{Z} \times BU)]^*_G = 0$ for $k \geq 0$ and $H \leq \mathbb{Z}/2$, and so, by the long exact sequence associated to the homotopy fiber sequence

$$W_n \to \mathbb{Z} \times BU \to P_{2n-2}(\mathbb{Z} \times BU),$$

we have the two isomorphisms:

$$[S^{2n+k,n}, W_n]^*_G \cong [S^{2n+k,n}, \mathbb{Z} \times BU]^*_G$$
$$[S^{2n+k,n} \wedge \mathbb{Z}/2_+, W_n]^*_G \cong [S^{2n+k,n} \wedge \mathbb{Z}/2_+, \mathbb{Z} \times BU]^*_G.$$

Now we can use the square obtained before:

$$\begin{tikzcd} W_n \arrow[r] \arrow[d] & \mathbb{Z} \times BU \arrow[d, "\beta", "\simeq"'] \\ \Omega^{2,1} W_{n+1} \arrow[r] & \Omega^{2,1}(\mathbb{Z} \times BU) \end{tikzcd}$$

Now, if we map $S^{2n+k,2n}$ or $S^{2n+k,n} \wedge \mathbb{Z}/2_+$ (for $k \geq 0$) into the square and use the isomorphisms we have just obtained, we get isomorphisms:

$$[S^{2n+k,n}, W_n]^*_G \cong [S^{2n+k,n}, \Omega^{2,1} W_{n+1}]^*_G$$
$$[S^{2n+k,n} \wedge \mathbb{Z}/2_+, W_n]^*_G \cong [S^{2n+k,n} \wedge \mathbb{Z}/2_+, \Omega^{2,1} W_{n+1}]^*_G,$$

for $k \geq 0$. The hypotheses of Corollary 1.9.5 are now verified and so we can conclude that $W_n \to \Omega^{2,1}(W_{n+1})$ is a weak equivalence. $\square$

The previous lemma gives us the structure map for our G-spectrum of connective KR-theory:

Definition 8.2.2. The *connective KR-spectrum kr* is a G-spectrum with spaces W_n and the structure maps $W_n \to \Omega^{2,1}W_{n+1}$ obtained in Lemma 8.2.1.

As usual, we have a Ω-spectrum associated to kr, whose n^{th} space is W_n. The functor Ω^∞ associates to a G-spectrum the 0^{th} space of the associated Ω-spectrum, and so we have:

$$\Omega^\infty(\Sigma^{2,1}kr) = W_1, \qquad \Omega^\infty(\Sigma^{4,2}kr) = W_2.$$

The maps $W_{n+1} \to W_n$, which we have constructed for every $n \geq 1$, glue together to form a map $\beta : \Sigma^{2,1}kr \to kr$; in fact, the n^{th} space of the Ω-spectrum associated to $\Sigma^{2,1}kr$ is W_{n+1}.

To produce the stable spectral sequence, we want a homotopy cofiber sequence of the form:

$$\Sigma^{2,1}kr \xrightarrow{\beta} kr \to H\underline{\mathbb{Z}}. \tag{8.2.2}$$

The map of G-spectra $\beta : \Sigma^{2,1}kr \to kr$ can be turned in a homotopy cofibration with the stable version of the usual construction for spaces. Let C be the homotopy cofiber G-spectrum; we want to show that C is equivalent to $H\underline{\mathbb{Z}}$. We can apply suspension:

$$\Sigma^{4,2}kr \xrightarrow{\Sigma^{2,1}\beta} \Sigma^{2,1}kr \to \Sigma^{2,1}C$$

Since our objects are spectra, we can continue the sequence to the left by desuspending:

$$\Sigma^{-1}\Sigma^{2,1}C \to \Sigma^{4,2}kr \xrightarrow{\Sigma^{2,1}\beta} \Sigma^{2,1}kr \to \Sigma^{2,1}.$$

And we can switch to spaces by applying Ω^∞:

$$\Omega^\infty(\Sigma^{1,1}C) \to W_2 \to W_1.$$

Let us consider the following diagram:

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & W_2 \arrow[r] \arrow[d] & W_1 \arrow[d] \\
* \arrow[r] \arrow[d] & \mathbb{Z}\times BU \arrow[r, "="] \arrow[d] & \mathbb{Z}\times BU \arrow[d] \\
K(\underline{\mathbb{Z}},(2,1)) \arrow[r] & P_2(\mathbb{Z}\times BU) \arrow[r] & P_0(\mathbb{Z}\times BU),
\end{tikzcd}$$

in which rows and columns are all homotopy cofiber sequences. The space Z can be identified easily, since it is the fiber of $* \to K(\underline{\mathbb{Z}}, (2,1))$: it is the loop-space $\Omega^1 K(\underline{\mathbb{Z}}, (2,1))$. Hence we have:

$$\Omega^\infty(\Sigma^{1,1} C) \simeq \Omega^1 K(\underline{\mathbb{Z}}, (2,1)).$$

One can check easily the weak equivalence $\Omega^1 K(\underline{\mathbb{Z}}, (2,1)) \simeq K(\underline{\mathbb{Z}}, (1,1))$.

This implies that $C \simeq H\underline{\mathbb{Z}}$, which was our claim. We can paste together all the homotopy cofiber sequences (8.2.2) getting the following diagram:

$$\begin{tikzcd} \cdots \arrow[r] & \Sigma^{2,1} kr \arrow[r, "\beta"] \arrow[d] & kr \arrow[r, "\Sigma^{-2,-1}\beta"] \arrow[d] & \Sigma^{-2,-1} kr \arrow[r] \arrow[d] & \cdots \\ & \Sigma^{2,1} H\underline{\mathbb{Z}} & H\underline{\mathbb{Z}} & \Sigma^{-2,-1} H\underline{\mathbb{Z}}. & \end{tikzcd} \tag{8.2.3}$$

This tower of cofibration is what we need to produce the stable version of the spectral sequence: in fact, we build the *Bockstein spectral sequence* associated to this diagram. For details about this spectral sequence, we refer the reader to [13].

The upper row in our diagram has homotopy colimit KR, essentially by definition of KR. Moreover we can also see that the homotopy inverse limit is contractible. To see this, note that, for every i, $\pi_i(W_j) = 0$ if j is big enough: this implies that $\varprojlim_j(\pi_i(W_j)) = 0$. The same holds for $\varprojlim^1$, hence, by the Milnor exact sequence:

$$0 \to \varprojlim{}^1 \pi_{i-1}(W_j) \to \pi_i \operatorname{holim}(W_j) \to \varprojlim_j(\pi_i(W_j)) \to 0,$$

we get that $\pi_i \operatorname{holim}(W_j) = 0$.

This shows that we have a Bockstein spectral sequence, corresponding to the exact couple:

$$\begin{tikzcd} {[X, kr]^*_G} \arrow[rr, "(2{,}1)"] & & {[X, kr]^*_G} \arrow[dl] \\ & {[X, H\underline{\mathbb{Z}}]^*_G} \arrow[ul] & \end{tikzcd}$$

The spectral sequence goes from $RO(G)$-graded cohomology to KR-theory:

$$H^{p,-\frac{q}{2}}(X; \underline{\mathbb{Z}}) \Rightarrow KR^{p+q,0}(X),$$

for any $\mathbb{Z}/2$-space X. We can make considerations on the convergence similarly to the unstable case: the tower (8.2.3) has a contractible homotopy limit, so the spectral sequence is conditionally convergent Also in this case, if the condition $RE_\infty = 0$ holds, then the spectral sequence converges strongly, by [3, 8.10].

9 Computations

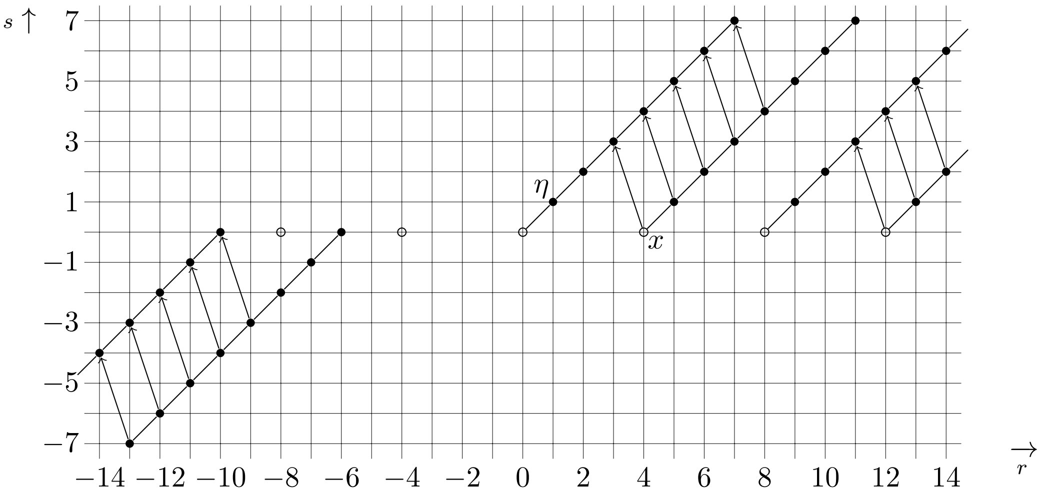
9.1 The spectral sequence for $X = *$

If we take X to be a single point, the spectral sequence has the form:

$$E_2^{p,q} = H^{p,-\frac{q}{2}}(pt;\underline{\mathbb{Z}}) \Rightarrow KR^{p+q}(pt) = KO^{p+q}(pt).$$

In fact, for a space with trivial $\mathbb{Z}/2$-action, KR-theory coincides with KO.

The following chart shows the E_2 page of the spectral sequence. The indexing is chosen so that the entry of coordinates (r,s) is $H_G^{s,\frac{r+s}{2}}(pt)$. The result of Theorem 4.8.1 gives us the groups on the page E_2.



Let $\beta^2 = x$ denote the generator of $E_2^{4,0} \cong \mathbb{Z}$ and η the generator of $E_2^{1,1} \cong \mathbb{Z}/2$.

To determine the differentials, we will use the fact that we know the groups $KO^*(*)$:

$i \pmod 8$	0	1	2	3	4	5	6	7
$KO^{-i}(*)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0

In particular, in the page E_2, we would like to know if one of the possibly non-trivial differential is actually not trivial:

$$d_2\colon E_2^{4,0} \to E_2^{3,3}.$$

The target of this differential is $E_2^{3,3} \cong \mathbb{Z}/2$, and we know that this cell in the spectral sequence has to be killed, because it converges to $KO^{-3}(*) = 0$.

The only differential in the spectral sequence that hits this cell is the one we are looking at, hence we deduce that d_2 is surjective:

$$d_2(x) = \eta^3 \in E_2^{3,3},$$

and $\ker(d_2) \cong \mathbb{Z}\langle 2x \rangle$. We can now compute the differential originating at the elements in the diagonal which begins at $E_2^{4,0}$, using the multiplicative structure and the derivation rule:

$$d_2(x\eta^j) = d_2(x)\eta^j + xd_2(\eta^j) = \eta^{j+3}.$$

The value of the differential at $E_2^{8,0}$ can be deduced similarly:

$$d_2(x^2) = 2xd_2(x) = 2x\eta^3 = 0 \in E_2^{7,3} \cong \mathbb{Z}/2.$$

The generators for the groups in the diagonal beginning at $E_2^{4,0}$ have the form $x^2\eta^j$ ($j \geq 0$). We can apply the derivation rule and compute:

$$d_2(x^2\eta^j) = x^2 d_2(\eta^j) + d_2(x^2)\eta^j = 0.$$

The things change for the next diagonal:

$$d_2(x^3) = 3x^2 d_2(x^2) = 3x^2\eta^3 = x^2\eta^3.$$

This means that the differential $E_2^{12,0} \to E_2^{11,3}$ is surjective and, by the same computation done before, so are the differential originating at the groups in the diagonal beginning at $E_2^{12,0}$.

This way we know all the differentials on the page E_2. The next pages have no non-trivial differentials, so the spectral sequence collapses at the page E_3.

9.2 Comparing fixed-points

Note 9.2.1. The treatment of G-spectra in this project has been limited to the most basic definitions. This last section presents an interesting (in our opinion) computation which is an aside from the main topic of the project, and uses some notions of stable equivariant homotopy theory which we have not introduced before. We refer the reader to [12] and [8] for a complete treatment of G-spectra.

The previous calculation allows us to compute rather easily the homotopy fixed-points of KR. To do this, we show that, for the spectrum KR, the homotopy fixed-points coincide with the actual fixed-points, which we know are the spectrum KO. Here we sketch the main points of the argument.

Let us start in more generality, taking X to be any G-spectrum. By definition, the homotopy fixed-points of X, denoted X^{hG}, are the actual fixed-points of $F(EG_+, X)$, the G-space of the basepoint-preserving G-maps $EG_+ \to X$:

$$X^{hG} = F(EG_+, X)^G,$$

where EG is the universal free G-space.

The space EG_+ fits into the cofibration sequence:

$$EG_+ \to S^0 \to \widetilde{EG},$$

where the first map comes from the map $EG \to *$ by adding a basepoint, and $\widetilde{EG}$ is the mapping cone of this map. If we map the sequence into the space X and take fixed-points, we obtain the following exact sequence:

$$X^{hG} \leftarrow X^G \leftarrow F(\widetilde{EG}, X)^G.$$

In fact, basepointed maps $S^0 \to X$ choose one point in X and so, the fixed-points of the space of these maps are the fixed-points of X.

To show the weak equivalence $X^G \to X^{hG}$ we need to show that the space $F(\widetilde{EG}, X)^G$ is contractible. We are going to show this in our case, where $G = \mathbb{Z}/2$ and $X = KR$. By Whitehead theorem, it is enough to show that the (non-equivariant) homotopy groups of $F(\widetilde{EG}, X)^G$ are trivial or, equivalently, that the equivariant homotopy groups of $F(\widetilde{EG}, X)$ are trivial. We shall prove that:

$$\pi_n^G(F(\widetilde{EG}, X)) = 0 \qquad \text{for any } n \geq 0.$$

For $G = \mathbb{Z}/2$, the space EG can be modelled as S^∞, with the antipodal action of the group. The space $\widetilde{EG}$ is the mapping cone of the map $EG_+ \to S^0$ and one can easily see that it is homotopy equivalent to the suspension of EG. Hence we have

$$\widetilde{EG} = \mathrm{hocolim}(S^{0,0} \hookrightarrow S^{1,1} \hookrightarrow S^{2,2} \hookrightarrow \dots),$$

and so: $F(\widetilde{EG}, X) = \mathrm{holim}_n(F(S^{n,n}, X))$. Then, for any $p, q \geq 0$, we have a Milnor short exact sequence (we omit 0 from each end):

$$\varprojlim_n{}^1(\pi_{p+q+n-1,q+n-1}(X)) \rightarrowtail \pi_{p+q,q}(F(\widetilde{EG}, X)) \twoheadrightarrow \varprojlim_n(\pi_{p+q+n,q+n}(X)).$$

In fact we have obtained $F(\widetilde{EG}, X)$ as the homotopy limit of:

$$F(S^{n,n}, X) \leftarrow F(S^{n+1,n+1}, X) \leftarrow F(S^{n+2,n+2}, X) \leftarrow \dots.$$

In the exact sequence, we have the limit of the sequence:

$$[S^{n,n}, X]^*_G \leftarrow [S^{n+1,n+1}, X]^*_G \leftarrow [S^{n+2,n+2}, X]^*_G \leftarrow \dots,$$

and the maps are given by composition with a suspension of the map $e : S^{0,0} \to S^{1,1}$, which includes the two points as the north and south poles of $S^{1,1}$.

Let us take $X = KR$. If we look at the diagram showing the homotopy groups $\pi_{p+q,q}(KR)$, we see that, for any value of p and q, if we go along the diagonal $(p+i, q+i)$, we find trivial groups. Moreover, by the periodicity of the homotopy groups, any diagonal contains an infinite number of trivial groups and this implies that the $\varprojlim$ and $\varprojlim^1$ terms of the exact sequence are zero. Hence $F(\widetilde{EG}, X)$ is equivariantly contractible and our claim follows.

A Topological matters

A.1 The topological join

In this section we recall the definition and some properties of the topological join of two spaces.

Definition A.1.1. Let X and Y be topological spaces. Their *join*, denoted $X * Y$, is the quotient space of the cartesian product $X \times Y \times I$ by the equivalence relation $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$.

This can be interpreted as the space of the line segments connecting a point of X and a point of Y. $X * Y$ contains a copy of X, sitting as the subspace $X \times \{y\} \times \{0\}$ (for any $y \in Y$, in view of the identity relation) and a copy of Y as $\{x\} \times Y \times \{1\}$.

One can show that, if the two spaces have a CW structure, then also their join is a CW complex.

It is possible to describe nicely the points of $X * Y$ as formal linear combinations

$$tx + (1-t)y \qquad x \in X,\ y \in Y,\ 0 \leq t \leq 1,$$

where $0x + y$ is identified with y and $x + 0y$ is identified with x.

Even if we start with two basepointed spaces (X, x_0) and Y are basepointed, for their join $X * Y$ there is not a canonical choice of a basepoint: in fact, in view of the above interpretation, $X * Y$ contains an entire segment connecting the basepoints. Then we can make a basepointed version of the join, the *reduced join*, by collapsing this segment $\{x_0\} * \{y_0\}$.

The following lemma shows one property of the join which we use during the proof of Proposition 7.2.3.

Lemma A.1.2. *Let (X, x_0) and (Y, y_0) be basepointed CW-complexes. Then*

$$X * Y \simeq \Sigma(X \wedge Y).$$

Proof. We will consider the reduced join, which is homotopy equivalent to the non-reduced one, since it is obtained by collapsing the contractible subcomplex $\{x_0\} * \{y_0\}$. The first thing to observe is that we have a homeomorphism:

$$(X * Y)/(X * \{y_0\} \cup \{x_0\} * Y) \cong \Sigma(X \wedge Y),$$

as one can check by looking at the identifications brought by the different equivalence relations that are in play.

Now we note that the subcomplexes $X * \{y_0\}$ and $\{x_0\} * Y$ of $X * Y$ are the cones on the copies of X and of Y contained in the join and, moreover, their intersection is the basepoint of the reduced join, $\{x_0\} * \{y_0\}$. Therefore their union is a contractible subcomplex: quotienting that out, we get the wanted homotopy equivalence. □

Lemma A.1.3. *Let V and W be orthogonal $\mathbb{Z}/2$-representations and let G be a finite group. If G acts orthogonally via $\mathbb{Z}/2$-automorphisms on W, and it acts trivially on V, then there exists a $\mathbb{Z}/2$-homeomorphism:*

$$S(V) * (S(W)/G) \cong S(V \oplus W)/G.$$

Proof. By saying that G acts on W via $\mathbb{Z}/2$-automorphisms, we mean that we have an action

$$G \to O(W) \qquad g \mapsto \alpha_g$$

for which α_g is a $\mathbb{Z}/2$-equivariant map for all g.

As observed earlier, a point in the join corresponds to a formal linear combination $tx + (1-t)y$ for $x \in S(V)$, $y \in S(W)/G$ and $0 \leq t \leq 1$, identifying $0x$ and $0y$ with 0. Let $v \in S(V)$, $w \in S(W)$ and $0 \leq t \leq 1$. We denote with square brackets the G-orbit of a point. We define the following map:

$$\begin{aligned} S(V) * (S(W)/G) &\to S(V \oplus W)/G \\ tv + (1-t)[w] &\mapsto [(\sqrt{t}v, \sqrt{1-t}w)] \end{aligned}$$

The map is well defined with respect to the G-action on $S(W)$, since the action is trivial on V and the target is actually $S(V \oplus W)/G$, since v and w are both unitary vectors. The reader can easily check that this map defines a $\mathbb{Z}/2$-homeomorphism. □

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