PhD Thesis

Separability in homotopy theory and topological Hochschild homology

University of Copenhagen

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August 2024

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PhD thesis submitted to:
PhD School of Science
Faculty of Science
University of Copenhagen

Date of submission:	30th August 2024
Date of defence:	4th October 2022
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This work was supported by the Danish National Research Foundation through the Copenhagen Centre for Geometry & Topology (CPH-GEOTOP-DNRF151). Part of the work was also supported by the Swedish Research Council (grant no. 2016-06596) during the research program "Higher algebraic structures in algebra, topology and geometry" held at Institut Mittag–Leffler, Sweden in the spring of 2022.

ISBN 978-87-7125-229-3

Abstract

This thesis consists of two essentially independent parts.

The first part is concerned with the notion of *separable algebras* in homotopy theory, specifically in the context of monoidal stable ∞-categories. Separable algebras are a common generalization of étale algebras in the commutative setting and Azumaya algebras in the noncommutative setting, and in Part I, I study their foundational properties: I prove rigidity results with respect to the homotopy category which generalize previously known rigidity results, and I try to bring the well-developed theory from classical algebra to homotopical algebra.

The second part is devoted to the study of *topological Hochschild homology* (THH) and related invariants. In the first chapter of Part II, I explain how to rephrase a theorem of Dundas and McCarthy relating THH and algebraic K-theory in terms of Kaledin and Nikolaus' *trace theories*, and how to use this formalism to extend the theorem to nonconnective ring spectra and their bimodules, as well as to more general invariants than algebraic K-theory. In the second chapter of the second part, I explain how to use this result to compute invariants of THH itself, such as its endomorphism ring spectrum, and variants thereof.

Resumé

Denne afhandling består af to uafhængige dele.

Den første del handler om konceptet *separable algebraer* i homotopiteori, specifikt i monoidale stabile ∞-kategorier. Separable algebraer er en fælles generalisering af étalealgebraer i den kommutative kontekst, og Azumayaalgebraer i den ikkekommutative kontekst, og i Del I studerer jeg deres grundlæggende egenskaber: jeg beviser rigiditetssætninger med hensyn til homotopikategorien som generaliserer tidligere kendte resultater, og jeg prøver at udvide den veludviklede teori fra klassisk algebra op til homotopisk algebra.

Den anden del er dedikeret til studiet af *topologisk Hochschild homologi* (THH) og relaterede invarianter. I første kapitel af Del II, forklarer jeg hvordan man kan interpretere en sætning af Dundas og McCarthy der relaterer THH og algebraisk K-teori ud fra perspektivet af Kaledin og Nikolaus' spor teorier, og hvordan man kan bruge deres formalisme til at udvide sætningen til ikkekonnektive ringspektre og bimoduler såvel som til mere generelle invarianter end algebraisk K-teori. I andet kapitel af Del II, forklarer jeg hvordan man kan bruge dette resultat til at beregne invarianter af THH selv, som dens endomorfismeringspektrum, og varianter deraf.

Remerciements

There is a long list of people to acknowledge for this thesis, or more properly, to thank, too long a list to include everyone. I will follow the tradition of first thanking math related folks, and later move on to my personal life. Lots of people should be thanked several times over, in particular because they belong to several categories. For simplicity of notation, I'll sometimes forget to thank them the appropriate amount.

Of course, first and foremost I have to thank my advisors Jesper Grodal and Markus Land. Both of you have trusted and supported me fully in all my endeavours, and have always been there to help when it was needed. You have also accepted to listen to me rant about whatever was going through my mind with patience, and always tried to make me clarify my thoughts. In different ways, both of you helped me with my fear of examples, and in all possible ways, you have shaped me to be the mathematician I am now, and I could not be more grateful to have had you as my advisors.

I also have to thank Jesper specifically for welcoming me in Copenhagen a first time in 2019, and renewing your welcome for my prethesis and finally, for my PhD. Copenhagen in general has been very welcoming. The mathematical community there has been wonderful, and I have everyone there to thank for their presence, their help and their jokes : Alexis, Dagur, Nanna, Bai, Andrea, Thomas B, Robert, Shachar, Dustin, Jonathan, Adriano, Marie-Camille, Pierre, Søren, Desirée, Oscar, Lars, Nuno, Branko, Dani, Marius K, Joachim, Isaac, Fadi, Marius N, Fabien, Azélie, Florian R, Vignesh, Nathalie. I particularly want to thank Kaif for welcoming me from the start, always bringing your laughter with you to work, and for letting me shamelessly steal your thesis template; and Jan among other things for being my most consistent co-swimmer.

Obviously I also need to thank Jan T for the incredible work he put into the center, allowing me to pretend to organize things while he was doing all of the heavy lifting.

Beyond Copenhagen, I must also thank the homotopy theory community at large for being such a warm environment, where I have been able to thrive and share my excitement. Before my PhD, I have always had doubts about the field I wanted to get into, and thanks to the community in and around homotopy theory, I no longer have any doubts. It's impossible to name everyone from whom I have benefitted in those three years, and from whom I will most likely benefit in the future, but outside of the Copenhagen crowd, outliers that I have to explicitly thank are Florian N, Lior, Bastiaan, Achim, Vova, Thomas N, Konrad, Anish, Piotr, Marco, Ishan, Jonas, Dominik, Christian, Shai, Shay, Fabian, Elden, Ryomei, and Mura.

I've also benefitted a lot from my interactions and collaborations with Christoph, Niko, Luca, Sil, Itamar, Shaul, Tom, Victor, Sebastian, Dhilan, Nick, Peter, Lucas, William, Rune, Cary and John.

Special thanks go to everyone who has invited or welcomed me in their groups or even house: Thomas and Lisa, Victor, Uli, Emanuele, Mike, Ben and Achim, Mona, Lucy, David, Niranjan, Akhil, Paul, and Elden.

Of course, many thanks to Ben, Sasha and Nathalie for agreeing to be on my jury.

Bien avant ma thèse, j'ai toujours eu le soutien et les encouragements de mon entourage mathématique, et je souhaite remercier toustes celleux qui m'ont aidé à arriver jusqu'à la thèse, et en particulier jusqu'à là où je suis maintenant: Luc Figuière, Daniel Franco, Alain Troesch, Marc Becker, Roger Mansuy, Ariane Mézard, Ilia Itenberg, Bruno Vallette, et bien sûr Geoffroy Horel. Il me faut aussi mentionner Christophe Chalons et Claude Quitté pour leur aide "extra-scolaire" au fil des années, et Muriel Livernet pour sa suggestion, il y a maintenant quelques années, que je visite Copenhague.

Bien sûr, durant ces années je n'ai pas fait *que* des maths, et je dois remercier mes ami-e-s (français-es, pour le moment) pour m'avoir supporté[1], et avoir toujours été là pour moi, alors même que j'ai disparu dans un pays lointain, et que ma communication n'a jamais été la meilleure. Pour avoir toujours été là, merci à Math, Louis, Lucie, Clément, Clémence, Enki, Stella, Emma, Thomas, Basile, Tess, Adrien, Rémi, Joffrey, Gabriel, Damien, Solène, Théo L, Baltho, Orphée, Théo D, mais j'en oublie, et je ne compte pas faire une liste exhaustive.

Jeg skal også takke mine danske venner - København, og den tid jeg har været her kunne aldrig have været det samme uden jer. Jeg skal give særlig tak til Anna for at have været den bedste bedste ven, og altid have været der når jeg havde brug for dit hjælp og din støtte.

Je conclus avec l'évidence : ma famille. Merci à mes parents d'avoir été et d'être ce que vous êtes; de m'avoir élevé comme vous l'avez fait; de m'avoir toujours soutenu et aimé sans conditions. Merci d'avoir toujours pris soin de moi. Merci à toute ma famille, et particulièrement mes frères et ma presque-soeur d'avoir créé un environnement heureux et précieux autour de moi.

[1] Ce n'est pas un anglicisme.

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Introduction

Contextual overview

Both parts of this thesis fit into the general framework of homotopical algebra[2], that is, the algebra of structures "up to coherent homotopy" - this means that equality is replaced with homotopy, and more precisely, axioms of algebraic structures are replaced by systems of homotopies that are given as part of the structure. This kind of algebra has two distinct but related origins: algebraic topology and category theory.

In category theory, this can be concretely seen in structures like (symmetric) monoidal categories, where properties like classical associativity and commutativity are replaced by associators and commutators which satisfy possibly subtle coherence conditions.

In algebraic topology, cohomology theories are powerful tools to study spaces and the Brown representability theorem guarantees that they are all represented by *spectra*. Often, additional structure on these cohomology theories is reflected in additional structure on these spectra. For example, multiplicative cohomology theories, that is, cohomology theories with a cup product, are often represented by ring spectra, or even commutative ring spectra. These are analogues of rings and commutative rings, where associativity and commutativity constraints are only "up to homotopy" - again, this means that specific homotopies witnessing associativity and commutativity are given as part of the structure. "Brave new algebra" is the term coined by Waldhausen to describe the kind of algebra where spectra and variants thereof (e.g. multiplicative structures on spectra) take the place of abelian groups, though Quillen's "homotopical algebra" is probably a more descriptive name.

Part I, about the theory of separable algebras, is part of the by-now-classical trend of taking structures and concepts from classical algebra, and making them homotopical. Separable algebras are a classical notion, introduced by Auslander and Goldman in [AG60], that encompasses both étale algebras in their commutative variant, and Azumaya algebras in their noncommutative variant. My work here draws inspiration from, and is related to many classical approaches to both of these notions.

In the commutative setting, the work of Lurie made étale algebras a natural part of homotopical algebra - in [Lur12, Section 7.5], Lurie proves that many of the classical properties of étale algebras extend to the setting of commutative ring spectra, and in fact, the latter often *reduce* to the classical properties through deformation theory. In Lurie's setting, the "derived direction" (as recorded through, e.g. higher homotopy groups) is seen as an infinitesimal thickening of classical algebra. This setting is extremely powerful and has been very successful in making (connective) spectral algebraic geometry a workable theory, more or less as understandable as classical algebraic geometry.

Another source of inspiration for my work, and indeed my original motivation for it, is Balmer's work on tensor-triangulated (henceforth tt-) geometry, wherein he studies a tt-

[2]The term "higher algebra" is often used, but "homotopical" seems more accurate and more descriptive.

category by means of what is known as its "Balmer spectrum". Just as in the classical situation, where the étale theory of a scheme contains information about the geometry of that scheme, the "étale" theory of a tt-category, as encoded through its commutative separable algebras, contains information about the geometry of that category.

Both of these approaches to what "étale" should mean, while powerful, suffer from problems, solved partially by the other. On the one hand, Lurie's approach to étale algebras works best in the ∞-category of spectra, or more generally in a stable symmetric monoidal ∞-category equipped with a t-structure, and is very largely based on the intuition that homotopy groups encode an infinitesimal thickening of π_0. This kind of intuition tends to fail in genuinely non-connective settings, such as chromatically localized categories of spectra. In particular, key features and important examples are not covered by this theory - such as my favourite Galois extension: $\mathrm{KO} \to \mathrm{KU}$.

On the other hand, Balmer's approach is inherently geometric, and does not put such a heavy weight on homotopy groups. Nonetheless, unlike Lurie's flexible ∞-categorical approach, Balmer works in the setting of tt-categories, where many coherence problems arise, and are best solved using ∞-categories.

In Part I of this thesis, I adress the latter issue by explaining in what precise sense the étale theory of a stably symmetric monoidal ∞-category is controlled by its homotopy category, and in what sense the work of Balmer naturally upgrades to the setting of stable ∞-categories, in a sense "bridging the gap" between those two approaches.

In the noncommutative setting, separable algebras are best thought of as generalizations of Azumaya algebras. There is an increasing body of work on "derived" or homotopical Azumaya algebras, started by Toën's work in [Toë12], and followed up on in the spectral case for example by Baker–Richter–Szymik [BRS12], Antieau–Gepner [AG14] and many others (see e.g. [GL16] and [HL17] for work in the nonconnective setting). The original connection between (classical) Azumaya algebras and (classical) separable algebras dates back to 1960 [AG60] and another goal of Part I is to clarify the extent to which this connection extends to this newer setting of homotopical algebra. My work in this direction is more incomplete, although I prove that in many relatively general cases the classical connections extend to homotopical algebra.

Part II is about topological Hochschild homology (henceforth THH). This is an invariant of stable ∞-categories, a version of classical Hochschild homology "linear over the sphere spectrum $\mathbb{S}$". In this part, I try to treat THH as a kind of cohomology theory, and explore a natural question from that perspective: what are the natural operations of this cohomology theory ? To adress this question, I first extend on a classical theorem of Dundas and McCarthy [DM94] relating THH to algebraic K-theory. I extend it by proving a kind of uniqueness theorem concerning THH, which immediately implies the Dundas–McCarthy theorem, as well as variants thereof where algebraic K-theory is allowed to be replaced by more general "cohomology theories", more specifically, by more general localizing invariants.

This extension uses the formalism of trace theories, as introduced by Kaledin [Kal20] and Nikolaus [HS19]. This formalism encodes the cyclic invariance of THH, namely the fundamental property that for $f : C \to D, g : D \to C$, we have canonical equivalences $\mathrm{THH}(C; gf) \simeq \mathrm{THH}(D; fg)$. After setting up the basics of this theory which has not been recorded in the ∞-categorical setting yet beyond the Oberwolfach report [HS19], I prove a classification result for certain trace theories, and use this to prove the desired extension of the Dundas–McCarthy theorem. Finally, I use this to explore the question of operations on THH, and fully answer some of its variants.

Technical overview

This thesis consists of two independent parts, each consisting of three chapters.

- Part I is a variation on my preprint [Ram23]: the sections in Chapter 1 corresponds to Sections 1-5 in [Ram23], in particular Sections 4 and 5 are reproductions of these sections with little to no modifications, while Sections 1-3 are mostly reproduced from *loc. cit.* with modifications that we discuss below.
- Part II is new material, though its second chapter has been presented in a talk [Ram24].

We now move on to a more detailed overview of the contents of this thesis. All of the work in this thesis takes place in the context of ∞-categories, as thoroughly developed in Lurie's [Lur09; Lur12].

Part I

Part I is a slight variation on my preprint [Ram23]. Let me mention the few changes from that preprint: the most significant one is that while in [Ram23] I work with a symmetric monoidal ∞-category from the start, many of the results work in greater generality. Namely, in this thesis, I work in the setting of an $\mathbb{E}_m$-monoidal category, for $1 \leq m \leq \infty$ (although the specific m may vary from result to result): all of the noncommutative results from [Ram23] work for any m, and most of the commutative results already work with $m = 3$. This change may be relevant for future applications[3].

Apart from that, the other changes are in the amount of content: the preprint [Ram23] has varied material with sometimes "separability" as the only common theme – I have thus decided to split up this paper in future revisions, and to only include in this thesis the foundations of the theory, leaving for future work the relationship between separable algebras and Hochschild homology, as well as their relationship to traces.

They will of course be stated appropriately in the relevant sections, but let me state here the main results of Part I.

The first key result is the following theorem in the associative case:

Theorem A. *Let $\mathbf{C}$ be an additively monoidal[4] ∞-category. Given an algebra A in $\mathrm{ho}(\mathbf{C})$, i.e. a homotopy algebra, which is separable in $\mathrm{ho}(\mathbf{C})$, the moduli space*

$$\mathrm{Alg}(\mathbf{C})^{\simeq} \times_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))^{\simeq}} \{A\}$$

of lifts of A to a homotopy coherent algebra in $\mathbf{C}$ is simply-connected (and in particular non-empty). Furthermore, any lift $\tilde{A}$ of A is separable as an algebra in $\mathbf{C}$.

In this situation, the canonical functor

$$\mathrm{ho}(\mathrm{LMod}_{\tilde{A}}(\mathbf{C})) \to \mathrm{LMod}_A(\mathrm{ho}(\mathbf{C}))$$

is an equivalence.

Finally, for any $R \in \mathrm{Alg}(\mathbf{C})$, the canonical map

$$\pi_0 \operatorname{map}_{\mathrm{Alg}(\mathbf{C})}(\tilde{A}, R) \to \hom_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))}(A, R)$$

is an isomorphism.

[3] Forthcoming work of Burklund–Clausen–Levy seems to involve finite m's.

[4] This means that $\mathbf{C}$ is additive and the tensor product commutes with finite direct sums in each variable.

Remark 0.0.1. The simple-connectedness of the moduli space of lifts cannot be improved to a contractibility statement, cf. Example 1.2.29. Similarly, the π_0-statement at the end cannot be improved to a space level statement, even if $\tilde{A}$ is commutative and the target R is separable, cf. Example 1.3.32. See however below for the case where the target is (homotopy) commutative. ◁

This theorem simply gathers Theorem 1.2.16 , Proposition 1.2.9, Theorem 1.2.6 and Theorem 1.2.14 in a single statement.

In the commutative case, both the obstructions to contractibility and to discreteness vanish, and I prove:

Theorem B. *Let* $\mathbf{C}$ *be an additively* $\mathbb{E}_m$*-monoidal* ∞*-category,* $3 \leq m \leq \infty$ *and let* $A \in \mathrm{Alg}(\mathrm{ho}(\mathbf{C}))$*, which is homotopy separable and homotopy commutative. In this case, the moduli space of lifts to an associative algebra in* $\mathbf{C}$ *is contractible. More generally, for any* $1 \leq d \leq m$*, the moduli space*

$$\mathrm{Alg}_{\mathbb{E}_d}(\mathbf{C})^{\simeq} \times_{\mathrm{Alg}_{\mathbb{E}_d}(\mathrm{ho}(\mathbf{C}))^{\simeq}} \{A\}$$

of lifts to an $\mathbb{E}_d$*-algebra in* $\mathbf{C}$ *is contractible - we let* $\tilde{A}$ *denote the unique lift.*

Furthermore, in this case, for any algebra R *which is homotopy commutative, the canonical map*

$$\mathrm{map}_{\mathrm{Alg}(\mathbf{C})}(\tilde{A}, R) \to \hom_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))}(A, R)$$

is an equivalence - the source is discrete.

If $R \in \mathrm{Alg}_{\mathbb{E}_d}(\mathbf{C})$*, the same holds with* $\mathrm{map}_{\mathrm{Alg}_{\mathbb{E}_d}(\mathbf{C})}(\tilde{A}, R)$ *in place of* $\mathrm{Alg}(\mathbf{C})$*.*

I have again stated it as a single theorem, but it is a combination of Proposition 1.3.11, Theorem 1.3.19, Corollary 1.3.33, Corollary 1.3.39.

As a sample application of these results, in Section 1.5.4, I use Ravenel and Wilson's computations from [RW80] to prove (a corrected version of) Sati and Westerland's main results from [SW15]. In some sense, my proof is simpler as it does not involve any obstruction theory.

These results are reminiscent of the Goerss–Hopkins–Miller theorem [GH05], and although Morava E-theory is not separable in $K(n)$-local spectra, I prove that it is close enough to being separable that some of my results still apply to it. In more detail, I introduce the notion of an (homotopy) *ind-separable* algebra in Section 1.4, and prove an analogue of Theorem B for ind-separable algebras (though subject to some caveats). I further prove, using as only input a computation of $\pi_*(L_{K(n)}(E \otimes E))$, that Morava E-theory is homotopy ind-separable, and I thus recover the Goerss–Hopkins–Miller theorem (I prove a more precise version, also for morphisms, cf. Corollary 1.4.53):

Theorem C. *Let* $E = E(k, \mathbf{G})$ *be a Morava* E*-theory, where* k *is a perfect field of characteristic* p *and* $\mathbf{G}$ *a formal group over* k*. For any* $d \geq 1$*, the moduli space* $\mathrm{Alg}_{\mathbb{E}_d}(\mathrm{Sp})^{\simeq} \times_{\mathrm{Alg}(\mathrm{ho}(\mathrm{Sp}))^{\simeq}} \{E\}$ *is contractible.*

Remark 0.0.2. When $d = 1$, this is the Hopkins-Miller theorem, and when $d = \infty$, this is its extension to the Goerss-Hopkins-Miller theorem. This result, for intermediary values of d, is well-known to experts, but does not seem to have been recorded in the literature. ◁

Remark 0.0.3. My proof of *this* theorem is also based on obstruction theory - I refer to Remark 1.4.41 for a discussion of the difference between my proof and previous proofs. ◁

In Chapter 2 of Part I, I study Auslander-Goldman theory in the context of homotopical algebra. In this direction, my results are only partial. A special case of what I prove is:

Theorem D. *Let R be a commutative ring spectrum satisfying the assumptions of Theorem 2.1.14. In this case, any dualizable central separable algebra over R is Azumaya.*

Conversely, in any additive presentably symmetric monoidal ∞-category $\mathbf{C}$, an Azumaya algebra A is separable if and only if its unit $\mathbf{1_C} \to A$ admits a retraction.

This is a combination of Theorem 2.1.14 and Proposition 2.1.10.

Remark 0.0.4. The assumptions of Theorem 2.1.14 cover all connective ring spectra, and all ring spectra "coming from chromatic homotopy theory", but they are nonetheless a bit restrictive. $\triangleleft$

I also study the question of whether centers of separable algebras are separable, although I only reach results in more restricted generality - the following is Theorem 2.2.1 in the body of the text:

Theorem E. *Let R be a connective commutative ring spectrum and let A be an almost perfect R-algebra. If A is separable, then so is its center.*

The same holds for separable algebras in $K(n)$-local E-modules, where E is Morava E-theory, and for separable algebras in $K(n)$-local spectra.

Part II

Part II of the thesis is more recent work that has not appeared in writing yet. It starts with a review of the notion of "trace theories", introduced by Kaledin [Kal20] and later Nikolaus in the context of ∞-categories [HS19]. My first main result is a classification of cocontinuous trace theories (see Section 4.1 for definitions):

Theorem F. *Evaluation at $(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})$ induces, for every cocomplete stable ∞-category $\mathcal{E}$, an equivalence*

$$\mathrm{TrThy}^L(\mathcal{E}) \to \mathcal{E}^{BS^1}$$

between $\mathcal{E}$-valued (fiberwise) cocontinuous trace theories and objects of $\mathcal{E}$ with an S^1-action.

The inverse to that equivalence is implemented by $E \mapsto \mathrm{THH} \otimes E$ with a certain trace theory structure.

I further prove, following Nikolaus, that localizing invariants give rise to trace theories by taking Goodwillie derivatives, or linearizations. I deduce from this a nonconnective extension of the Dundas–McCarthy theorem [DM94], as well as a generalization to general finitary localizing invariants (see Section 4.1 for definitions):

Theorem G. *Let E be a finitary localizing invariant with values in $\mathcal{E}$, a cocomplete stable ∞-category. The object $X_E := P_1 E^{\mathrm{cyc}}(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})$ admits a canonical S^1-action, and with this we have an equivalence of trace theories:*

$$P_1 E^{\mathrm{cyc}} \simeq \mathrm{THH} \otimes X_E$$

In fact, I further explain how this can be viewed as an equivalence of cyclotomic spectra when we plug in objects of the form (C, id_C), and more generally, of polygonic spectra.

Using the case where $E = K$ is (nonconnective) algebraic K-theory, which is simply a nonconnective extension of the classical Dundas–McCarthy theorem [DM94], I compute variants of "endomorphisms of THH". Indeed, the Dundas–McCarthy theorem can be interpreted as giving a "presentation" of THH in terms of functors of the form $K(\mathrm{Fun}^{\mathrm{ex}}(A,-))$, maps out of which are easy to describe.

As a plain functor or a symmetric monoidal functor I obtain:

Theorem H. *As a plain functor* $\mathrm{THH} : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Sp}$*, the* S^1*-action induces an equivalence*

$$\mathbb{S}[S^1] \simeq \mathrm{end}(\mathrm{THH})$$

As a symmetric monoidal functor, there is an equivalence

$$\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp})}(\mathbb{S}^{S^1}, \mathbb{S}) \simeq \mathrm{End}^{\otimes}(\mathrm{THH})$$

and the space $\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp})}(\mathbb{S}^{S^1}, \mathbb{S})$ *can be described.*

I also obtain k-linear versions which can be completely computed when $\mathrm{THH}(k)$ is well-understood, e.g. for $k = \mathbb{F}_p$:

Theorem I. *As a functor* $\mathrm{HH}_k : \mathrm{Cat}_k^{\mathrm{perf}} \to \mathrm{Mod}_k$*, there is a canonical equivalence*

$$\mathrm{end}_{k\otimes \mathrm{THH}(k)}(k,k)[S^1] \simeq \mathrm{end}(\mathrm{HH}_k)$$

Finally, I explore variants of these results when THH is regarded as a functor $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathrm{perf}}) \to \mathrm{Sp}$. After rationalization, or $T(n)$-localization for any height n and implicit prime, I give a complete description of the endomorphisms of THH, but I also show that the corresponding description fails integrally. A summary of my results in this direction is the following, where L denotes either $T(n)$ localization for some height $n \geq 1$ and some implicit prime p, or rationalization - here, γ_k denotes the (unique up to conjugacy) length k cycle in Σ_k, and $\mathcal{O}(k)^{\gamma_k}$ denotes the fixed points of $\mathcal{O}(k)$ for the $\mathbb{Z}$-action induced by γ_k, with its residual C_k-action (coming from the fact that C_k is the centralizer of γ_k in Σ_k):

Theorem J. *For any single-colored* ∞*-operad* $\mathcal{O}$*, there is a canonical map*

$$\bigoplus_{k\geq 1} \mathbb{S}[(\mathcal{O}(k)^{\gamma_k} \times S^1)_{hC_k}] \to \mathrm{end}_{\mathrm{Fun}(\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathrm{perf}}),\mathrm{Sp})}(\mathrm{THH})$$

such that :

(i) For any finite set $S \subset \mathbb{N}_{\geq 1}$*, the restriction to* $\bigoplus_{k\in S}$ *admits a splitting;*

(ii) The induced map on L*-localization, specifically:*

$$L(\bigoplus_{k\geq 1} \mathbb{S}[(\mathcal{O}(k)^{\gamma_k} \times S^1)_{hC_k}]) \to \mathrm{end}_{\mathrm{Fun}(\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathrm{perf}}),\mathrm{Sp})}(L\mathrm{THH})$$

is an equivalence.

One might wonder whether the second item is optimal - namely, whether the map is an equivalence integrally. I prove that it is not so: for example, we will see that for every prime p, $\mathrm{map}_{\mathrm{Fun}(\mathrm{Cat}^{\mathrm{perf}},\mathrm{Sp})}(\mathrm{THH}, \mathrm{THH}_{hC_p})$ is a direct summand of $\mathrm{end}_{\mathrm{Fun}(\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{perf}}),\mathrm{Sp})}(\mathrm{THH})$, and that it is equivalent to $\mathbb{S}[S^1/C_p] \oplus \Omega\mathbb{S}_p[S^1/C_p]$, where $\mathbb{S}_p$ is the p-complete sphere. I do not know a full description in the integral case, but believe that, with more work, my methods could in principle provide a full answer.

Relation to other work

As already mentioned, Part I is inspired by, and analogous to Lurie's study of étale algebras in [Lur12, Section 7.5]. As proved in Proposition 1.3.53, separable algebras are more general than étale algebras, which makes my results in some sense more general (they apply, for example, to all finite Galois extensions); but it is important to note that in general, we do not use (or indeed, have!) homotopy groups which gives the results a slightly different flavour, and they are not necessarily directly comparable to Lurie's.

Some of the foundational results of Part I are also closely related to similar results in the work of Dell'Ambrogio–Sanders (cf. [DS18, Theorem 1.6]), as well as that of Naumann and Pol [NP23], the latter of which was developed mostly independently and approximately simultaneously.

The notion of trace theory introduced in Part II goes back to Kaledin [Kal15], and was reintroduced in homotopy theory by Nikolaus [HS19], although the latter work has not yet appeared in print beyond *loc. cit.*. The observation that this formalism can be used to prove the nonconnective extension of the Dundas–McCarthy theorem was already known to Nikolaus, and will also appear in a slightly different form in work of Harpaz, Nikolaus and Saunier [HNSb].

However the classification of cocontinuous trace theories and its consequence for general localizing invariants seems to be new. The use of the Dundas–McCarthy theorem to compute endomorphisms of THH was independently discovered by Sasha Efimov and announced in a lecture series in Chicago.

The notion of trace theories is closely related to that of shadows, due to Ponto [Pon07], which was made ∞-categorical by Hess–Rasekh [HR21a]. While I do not do so in this thesis, I intend to prove in future work that the two formalisms, while definitionally different, are in some sense equivalent.

Conventions

We work with ∞-categories throughout, as developed by Lurie in [Lur09; Lur12]. Categorical notions (functors, subcategories etc.) are to be understood in this context unless explicitly stated.

$\mathcal{S}$ denotes the ∞-category of spaces[5], Sp the ∞-category of spectra, Cat that of (small) ∞-categories ($\widehat{\mathrm{Cat}}$ possibly denoting the ∞-category of large ∞-categories), Pr^{L} that of presentable ∞-categories and left adjoints between them, as well as its stable variant $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Fun^{L} denotes the ∞-category of cocontinuous functors (equivalently, left adjoints, as we only consider presentable ∞-categories). We use pt for the terminal ∞-category (or space), and $\mathbb{S}$ for the sphere spectrum. Abelian groups are often implicitly considered as Eilenberg-MacLane spectra, and we do not distinguish notationally between the two.

$\mathrm{Cat}^{\mathrm{ex}}$ denotes the ∞-category of small stable ∞-categories and exact functors, and $\mathrm{Cat}^{\mathrm{perf}}$ the full subcategory thereof spanned by idempotent-complete ∞-categories. We let $\mathrm{Fun}^{\mathrm{ex}}$ denote the ∞-category of exact functors.

Δ always denotes the usual simplex category, and we use $[n]$ and Δ^n interchangeably to denote the n-simplex. "Geometric realization" is used to mean "colimit over Δ^{op}".

Throughout, Map denotes mapping spaces, while map is reserved for mapping spectra in stable ∞-categories. In additive ∞-categories, we may consider Map as a commutative

[5] Or homotopy types, ∞-groupoids, anima,...

group[6] in $\mathcal{S}$, or equivalently a connective spectrum - thus, in a stable ∞-category, we have $\mathrm{Map} = \mathrm{map}_{\geq 0}$. We use hom for hom sets in 1-categories, as well as for enriched mapping objects, e.g. internal homs in closed monoidal ∞-categories.

When drawing adjunctions, we draw the left adjoint on top, and the right adjoint on the bottom. The right adjoint of a functor f, if it exists, is denoted f^R, and similarly, the left adjoint of f, if it exists, is denoted f^L.

We follow the convention from [Lur09] for the meaning of "cofinal", and use "initial" for the dual notion.

For an algebra A in a monoidal ∞-category $\mathbf{C}$, we use $\mathrm{LMod}_A(\mathbf{C})$ (resp. $\mathrm{RMod}_A(\mathbf{C})$) to denote the ∞-category of left (resp. right) A-modules in $\mathbf{C}$. We try to be careful and distinguish left and right modules, except when A is commutative, in which case we use $\mathrm{Mod}_A(\mathbf{C})$ to denote $\mathbb{E}_\infty$-A-modules, the ∞-category of which is equivalent to (say) left A-modules. If $\mathbf{C}$ is clear from context, we may drop it from notation and simply write LMod_A (resp. $\mathrm{RMod}_A, \mathrm{Mod}_A$).

[6] Also known as $\mathbb{E}_\infty$-group.

Part I

Separability in homotopy theory

Introduction to Part I

In classical algebra, separable algebras, introduced by Auslander and Goldman in [AG60], are a generalization to arbitrary commutative rings of the classical notion of separable field extensions. They are R-algebras A for which the multiplication map $A \otimes_R A^{\mathrm{op}} \to A$ admits an (A, A)-bimodule section. Commutative separable algebras are closely related to étale algebras, while separable algebras whose center is the base commutative ring are also known as Azumaya algebras, introduced in the context of the Brauer group.

A typical trend in homotopical algebra is the attempt to mirror constructions and notions from classical algebra to "derived" contexts, and see what parts of the theory carry over, and what changes. For example, in [Bal11], Balmer initiated the study of separable algebras in tensor-triangulated categories (henceforth, tt-categories). A surprising feature of these algebras in this context is that they admit a good notion of module categories, even at the unstructured level of triangulated categories. In many situations, these module categories recover the "expected" result. For example, if A is an étale R-algebra, then modules over A in the derived category of R recover the derived category of A, a surprising result which is known to be wrong when A is a general R-algebra.

A natural source of tt-categories (to some extent, the only source of "natural" tt-categories) is stably symmetric monoidal ∞-categories: given any such gadget $\mathbf{C}$, its homotopy category $\mathrm{ho}(\mathbf{C})$ has a natural structure of a tt-category (and in fact, all the "enhancements" that appear in [Bal11, Section 5] - which, from this perspective, are trying to encode as extra structure on $\mathrm{ho}(\mathbf{C})$ the homotopical data contained in $\mathbf{C}$ - arise in this way). From this point of view, it makes sense to wonder what parts of the tensor-triangulated story extend to the ∞-categorical case, and also, what parts of the tt-story are (at least morally) *explained* by the ∞-story.

The goal of Part I of this thesis is to answer some of these questions and to clarify the connection between separable algebras in $\mathbf{C}$, and separable algebras in $\mathrm{ho}(\mathbf{C})$. Chapter 1 is devoted to exactly this: there, I argue that separable algebras in $\mathbf{C}$ and their modules are mostly controlled by the homotopy category $\mathrm{ho}(\mathbf{C})$, and even more so in the commutative setting. In particular, I answer a folk question by proving that the distinction between separable algebras and homotopy separable homotopy algebras is mild in the associative case, and inexistent in the commutative case (Theorem 1.2.16 and Theorem 1.3.19).

I further introduce a variant of separability that works in the commutative case in more "infinitary" situations, which I call ind-separability, and prove similar results about this variant; among other things leading to a somewhat new proof of the Goerss–Hopkins–Miller theorem (Corollary 1.4.53 - see Remark 1.4.41 for a discussion of the sense of the word "new").

Beyond their nice behaviour with respect to homotopy categories (or more generally, tt-categories), separable algebras are interesting in their own right: as we mentioned before, separability can be seen as an analogue of étale-ness. For example, Balmer proves in [Bal16] that étale maps of schemes induce separable algebras, and Neeman proves in [Nee18] that over a noetherian scheme, this is not far from an exhaustive list of commutative separable algebras. See also the recent work of Naumann and Pol [NP23] for another comparison of separable commutative algebras and another notion of "(finite) étale" due to Mathew [Mat16].

Now, classically, separable algebras can be neatly organized in the following way: if A is a separable algebra over R, its center C is separable over R and commutative, hence "étale", and A is separable over C and central, and hence Azumaya. Thus, separable algebras can be studied by studying separately the commutative, slightly more geometric case, and the central case, closely related to Brauer groups. In Chapter 2, I try to replicate this story, originally due to Auslander and Goldman [AG60] in the case of ring spectra. Along the way, I correct a mistake in [BRS12], namely I prove that not all Azumaya algebras are separable by giving a number of examples, and formulate a criterion for when a given Azumaya algebra *is* separable.

Local conventions

On top of the global conventions outlined at the beginning of the thesis, we have the following conventions.

- Throughout this part, $\mathbf{C}$ will denote a monoidal ∞-category, satisfying various extra conditions. When we say that an ∞-category is "(semi)additively" (resp. "stably", "presentably") monoidal, we mean that it is a monoidal ∞-category whose underlying ∞-category is (semi)additive (resp. stable, presentable), and where the tensor product is compatible with this structure, that is, commutes with coproducts in each variable (resp. finite colimits, all colimits).
- If there is no specified ∞-operad, the word "algebra" (resp. the notation Alg) means "associative or equivalently $\mathbb{E}_1$-algebra" (resp. denotes the ∞-category of associative algebras).
- We use the Dunn additivity theorem without comment - for our purposes, we might as well consider that the ∞-operad $\mathbb{E}_m$ is defined as $\mathbb{E}_1^{\otimes m}$, where $\otimes$ denotes the Boardman–Vogt tensor product.
- Often, we will consider an $\mathbb{E}_m$-monoidal ∞-category $\mathbf{C}$, and will consider algebras for certain ∞-operads $\mathcal{O}$ over $\mathbb{E}_m$. In [Lur12], Lurie denotes these ∞-categories by $\mathrm{Alg}_{\mathcal{O}/\mathbb{E}_m}(\mathbf{C})$. For simplicity of notation, we will simply write $\mathrm{Alg}_{\mathcal{O}}(\mathbf{C})$, the map $\mathcal{O} \to \mathbb{E}_m$ always being clear from context.
- We use $\mathrm{ho}(C)$ to denote the homotopy category of an ∞-category. When X is an object of C (possibly with some extra structure), we write hX for the same object viewed as an object of $\mathrm{ho}(C)$ (with the appropriate extra structure, in $\mathrm{ho}(C)$). We append the word "homotopy" to a type of structure to mean "that type of structure, considered in $\mathrm{ho}(C)$". For example, a "homotopy algebra" is an algebra in $\mathrm{ho}(C)$.
- A common trick consists in embedding a small ∞-category (possibly with some extra structure) in its presheaf ∞-category (or a variant thereof) to reduce to proving statements about presentable ∞-categories, or simply ∞-categories with suitable colimits. This is usually compatible with multiplicative structures, essentially by [Lur12, Section 4.8.1] (see, e.g., [Lur12, Proposition 4.8.1.10]). We will usually simply say "up to adding enough colimits" to mean "without loss of generality, assume $\mathbf{C}$ has these colimits", i.e., to refer to this trick.
- We use ${}_A\mathrm{BiMod}_B(\mathbf{C})$ to denote (A,B)-bimodules in $\mathbf{C}$, and $\mathrm{BiMod}_A(\mathbf{C})$ to denote (A,A)-bimodules in $\mathbf{C}$.
- When there is a t-structure floating around, e.g. the standard one in Sp, we use $\tau_{\leq n}$ (resp. $\tau_{\geq n}$, $\pi_0^\heartsuit$ or π_0) to denote the truncation functor[7] (resp. the other truncation functor, the π_0 relative to the given t-structure). We append categories with a $^\heartsuit$ to indicate that we are considering suitable subcategories of objects in the heart, e.g. $\mathrm{CAlg}(\mathbf{C})^\heartsuit := \mathrm{CAlg}(\mathbf{C}) \times_{\mathbf{C}} \mathbf{C}^\heartsuit$.

[7] Everything is in homological conventions.

Local acknowledgements

More detailed acknowledgements are available in the preprint [Ram23], but it is my pleasure to acknowledge the following people, for helpful discussions related to the topic at hand: Andy Baker, Özgür Bayındır, Robert Burklund, Anish Chedalavada, Jesper Grodal, Markus Land, Itamar Mor, Niko Naumann, Luca Pol, Piotr Pstrągowski and Jan Steinebrunner.

Thanks to q.uiver for help with the commutative diagrams.

On top of the general DNRF funding, a portion of this work was completed while I was in residence at the Institut Mittag-Leffler in Djursholm, Sweden in 2022 as part of the program 'Higher algebraic structures in algebra, topology and geometry' supported by the Swedish Research Council under grant no. 2016-06596.

Chapter 1

Foundations of separability

Introduction

The goal of this chapter is to discuss the relationship between separable algebras in a monoidal ∞-category $\mathbf{C}$ and its homotopy category. As mentioned in the introduction to Part I, I do this to conceptually explain the good behaviour of separable algebras in tt-categories, as well as to provide lifts of most constructions and results concerning separable algebras in tt-categories arising as homotopy categories of (symmetric) monoidal stable ∞-categories.

After setting up the definitions and the basics of the theory, I start proving rigidity results for separable algebras. The first such rigidity theorem is Theorem A:

Theorem. *Let $\mathbf{C}$ be an additively monoidal ∞-category. Given an algebra A in $\mathrm{ho}(\mathbf{C})$, i.e. a homotopy algebra, which is separable in $\mathrm{ho}(\mathbf{C})$, the moduli space*

$$\mathrm{Alg}(\mathbf{C})^{\simeq} \times_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))^{\simeq}} \{A\}$$

of lifts of A to a homotopy coherent algebra in $\mathbf{C}$ is simply-connected (and in particular non-empty). Furthermore, any lift $\tilde{A}$ of A is separable as an algebra in $\mathbf{C}$.

In this situation, the canonical functor

$$\mathrm{ho}(\mathrm{LMod}_{\tilde{A}}(\mathbf{C})) \to \mathrm{LMod}_A(\mathrm{ho}(\mathbf{C}))$$

is an equivalence.

Finally, for any $R \in \mathrm{Alg}(\mathbf{C})$, the canonical map

$$\pi_0 \operatorname{map}_{\mathrm{Alg}(\mathbf{C})}(\tilde{A}, R) \to \hom_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))}(A, R)$$

is an isomorphism.

Remark 1.0.1. We already mentioned that the simple-connectedness of the moduli space of lifts cannot be improved to a contractibility statement, cf. Example 1.2.29 and similarly, the π_0-statement at the end cannot be improved to a space level statement, even if $\tilde{A}$ is commutative and the target R is separable, cf. Example 1.3.32. ◁

This theorem and the following remark show that in this generality the comparison between the tt-setting and the stable ∞-setting, while reasonably good, is not perfect.

In a later section of this chapter, I study the commutative case. There, the situation is much better: the obstructions to contractibility and discreteness respectively vanish, so that the comparison becomes essentially perfect. This can be encapsulated in Theorem B:

Theorem. *Let $\mathbf{C}$ be an additively $\mathbb{E}_m$-monoidal ∞-category, $3 \leq m \leq \infty$ and let $A \in \mathrm{Alg}(\mathrm{ho}(\mathbf{C}))$, which is homotopy separable and homotopy commutative. In this case, the moduli space of lifts to an associative algebra in $\mathbf{C}$ is contractible. More generally, for any $1 \leq d \leq m$, the moduli space*

$$\mathrm{Alg}_{\mathbb{E}_d}(\mathbf{C})^{\simeq} \times_{\mathrm{Alg}_{\mathbb{E}_d}(\mathrm{ho}(\mathbf{C}))^{\simeq}} \{A\}$$

of lifts to an $\mathbb{E}_d$-algebra in $\mathbf{C}$ is contractible - let $\tilde{A}$ denote the unique lift.

Furthermore, in this case, for any algebra R which is homotopy commutative, the canonical map

$$\mathrm{map}_{\mathrm{Alg}(\mathbf{C})}(\tilde{A}, R) \to \hom_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))}(A, R)$$

is an equivalence - the source is discrete.

If $R \in \mathrm{Alg}_{\mathbb{E}_d}(\mathbf{C})$, the same holds with $\mathrm{map}_{\mathrm{Alg}_{\mathbb{E}_d}(\mathbf{C})}(\tilde{A}, R)$ in place of $\mathrm{Alg}(\mathbf{C})$.

As these results are reminiscent of the Goerss–Hopkins–Miller theorem [GH05], we also spend some time after the commutative section to discuss the relation with Morava E-theory - we caution the reader here that even in the ∞-category of $K(n)$-local spectra, Morava E-theory is *not* separable. However it does satisfy a weaker property which I call “ind-separability” and briefly study.

Sectionwise outline

- In Section 1.1, I outline the basics of the theory of separable algebras: definitions, stability properties and the basic properties of their module categories;
- In Section 1.2, I start comparing things with the homotopy category. This is where I prove Theorem A. Most of the proofs are relatively elementary, except for the proof that the moduli space of lifts is non-empty, where I use some deformation theory of ∞-categories;
- In Section 1.3, I move on to the commutative side of the picture, and I prove Theorem B. In this section, we also study the analogy between separable algebras and étale algebras in the sense of Lurie;
- In Section 1.4, I study a variant of separability, which I call ind-separability, and use it to recover the Goerss–Hopkins–Miller theorem, namely Theorem C;
- Finally, in Section 1.5, I gather a number of examples of separable algebras, to indicate the wealth of examples despite how strong this condition is.

1.1 Generalities

The goal of this section is to set the stage: we define separable algebras, and gather some of their basic properties.

Notation 1.1.1. Throughout this section, $\mathbf{C}$ is a monoidal ∞-category, with unit $\mathbf{1}$ and tensor product denoted by $\otimes$. $\triangleleft$

Following [Bal11], we define:

Definition 1.1.2. An algebra $A \in \mathrm{Alg}(\mathbf{C})$ is said to be *separable* if the multiplication map, $A \otimes A^{\mathrm{op}} \to A$ admits a section s, as a map of A-bimodules[1].

[1] The notation $A \otimes A^{\mathrm{op}}$ is potentially confusing if $\mathbf{C}$ is only monoidal. Note that in this case, we cannot describe bimodules as modules over a certain ring, and really have to stick to bimodules. I hope that this notational convention does not bring confusion.

In this case, we call the composite $\mathbf{1} \to A \xrightarrow{s} A \otimes A^{\mathrm{op}}$, or sometimes the section s itself, a *separability idempotent*. Equivalently, this is an A-bimodule idempotent map $A \otimes A^{\mathrm{op}} \to A \otimes A^{\mathrm{op}}$. ◁

Variant 1.1.3. An algebra $A \in \mathrm{Alg}(\mathbf{C})$ is said to be *homotopy separable* if it is separable, as an algebra in $\mathrm{ho}(\mathbf{C})$.

If we start with an algebra $A \in \mathrm{Alg}(\mathrm{ho}(\mathbf{C}))$ directly, we will say that we have a *homotopy separable homotopy algebra*.

Variant 1.1.4. Suppose $\mathbf{C}$ admits geometric realizations which are compatible with the tensor product. In particular, $\mathbf{C}$ admits relative tensor products.

Suppose further that $\mathbf{C}$ is symmetric monoidal and let $R \in \mathrm{CAlg}(\mathbf{C})$ be a commutative algebra, and $A \in \mathrm{Alg}(\mathrm{Mod}_R(\mathbf{C}))$ be an R-algebra. We say A is separable over R if A is separable as an algebra in $\mathrm{Mod}_R(\mathbf{C})$. If $A \in \mathrm{CAlg}(\mathrm{Mod}_R(\mathbf{C}))$, we will say that it is a *separable extension* of R. We will use more careful terminology if $\mathbf{C}$ is only $\mathbb{E}_m$-monoidal, for some finite m.

Remark 1.1.5. Assume $\mathbf{C}$ is symmetric monoidal. Recall that an algebra is said to be *smooth* if A is right or left dualizable over $A \otimes A^{\mathrm{op}}$ [Lur12, Definition 4.6.4.13.]. If $\mathbf{C}$ is idempotent-complete, dualizable objects are closed under retracts, and so a separable algebra is smooth.

One can therefore think of separability as a strenghtening of smoothness. ◁

1.1.1 Basic properties

Separable algebras enjoy a number of closure properties:

Lemma 1.1.6. *Let $A, B \in \mathrm{Alg}(\mathbf{C})$ be algebras.*

(i) The unit of $\mathbf{C}$, $\mathbf{1}$, is separable; more generally if A is an idempotent algebra [Lur12, Definition 4.8.2.8.], then it is separable.

(ii) Suppose $\mathbf{C}$ admits geometric realizations, compatible with the tensor product, and suppose that the map $B \otimes_A B \to B$ is an equivalence. If A is separable, then so is B.

(iii) Suppose $\mathbf{C}$ is semiadditively monoidal. The product $A \times B$ is separable if and only if both A and B are.

(iv) If there is a retraction $A \to B \to A$ in $\mathrm{Alg}(\mathbf{C})$, and B is separable, then so is A.

(v) If $f : \mathbf{C} \to \mathbf{D}$ is a monoidal functor, and A is separable, then so is $f(A)$.

(vi) If $\mathbf{C}$ is $\mathbb{E}_2$-monoidal, $\mathrm{Alg}(\mathbf{C})$ acquires a tensor product. In this case, if A and B are separable, then so is $A \otimes B$.

(vii) If $\mathbf{C}$ is $\mathbb{E}_2$-monoidal, algebras have opposite algebras[2] If A is separable, then so is A^{op}

Proof. (i) is clear, as the multiplication map $A \otimes A^{\mathrm{op}} \to A$ of an idempotent algebra is an equivalence (on underlying objects, and hence as bimodules).

For (ii), observe that basechange along f induces a functor $\mathrm{BiMod}_A \to \mathrm{BiMod}_B$ that sends $A \otimes A^{\mathrm{op}}$ to $B \otimes B^{\mathrm{op}}$ by design, and the bimodule A to the bimodule

$$B \otimes_A A \otimes_A B \simeq B \otimes_A B \simeq B$$

where the last equivalence is by assumption. Further, the multiplication map is sent to the multiplication map, and the existence of a section in the source guarantees the existence of a section in the target.

[2] In fact there are two ways of doing so, depending on a chosen orientation. What we say is valid for both.

For one direction of (iii), we note that the multiplication map

$$(A \times B) \otimes (A \times B)^{\mathrm{op}} \to A \times B$$

factors as

$$(A \times B) \otimes (A \times B)^{\mathrm{op}} \to (A \otimes A^{\mathrm{op}}) \times (B \otimes B^{\mathrm{op}}) \to A \times B$$

If A, B are separable, then the second map has a bimodule section, and so the claim follows from the fact that, in the semiadditive case, with no assumption on A, B, the first map also has a bimodule section.

Conversely, notice that in the semiadditive case, the projections $A \times B \to A$ (resp. B) satisfy the assumption of (ii) so that, by (ii), if $A \times B$ is separable, then so are A and B.

For (iv), let $A \xrightarrow{i} B \xrightarrow{r} A$ denote a retraction diagram. We view B-bimodules as A-bimodules via restriction along i. Then, $A \xrightarrow{i} B \xrightarrow{s} B \otimes B^{\mathrm{op}} \xrightarrow{r \otimes r^{\mathrm{op}}} A \otimes A^{\mathrm{op}}$ is an A-bimodule map, where s is the separability idempotent of B. Furthermore, the composite

$$A \xrightarrow{i} B \xrightarrow{s} B \otimes B^{\mathrm{op}} \xrightarrow{r \otimes r^{\mathrm{op}}} A \otimes A^{\mathrm{op}} \to A$$

is equivalent to

$$A \xrightarrow{i} B \xrightarrow{s} B \otimes B^{\mathrm{op}} \to B \xrightarrow{r} A$$

because r is an algebra map, and thus, because s is a separability idempotent, to $A \to B \to A$ and thus, because we started with a retraction diagram, to id_A, and so we are done.

(v) is clear.

(vi) and (vii) follow from (v) applied to the monoidal functors $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ (using that an algebra $(A, B) \in \mathrm{Alg}(\mathbf{C} \times \mathbf{D}) \simeq \mathrm{Alg}(\mathbf{C}) \times \mathrm{Alg}(\mathbf{D})$ is separable if and only if both A, B are) and $\mathbf{C}^{\mathrm{rev}} \simeq \mathbf{C}$ respectively. □

Remark 1.1.7. It follows from (vi) and (vii) that if $\mathbf{C}$ is $\mathbb{E}_2$-monoidal and $A \in \mathrm{Alg}(\mathbf{C})$ is separable, then so is $A \otimes A^{\mathrm{op}}$. ◁

Remark 1.1.8. Item (v) really requires a monoidal functor, and not just a lax monoidal one. ◁

Remark 1.1.9. We will see in Proposition 2.2.5, that Item (v) has a form of converse in the commutative case, if we assume that f is more than conservative, rather part of a limit decomposition of $\mathbf{C}$. ◁

Remark 1.1.10. The condition in Item (ii) implies that $A \to B$ is an epimorphism in $\mathrm{Alg}(\mathbf{C})$. We do not know whether being an epimorphism is sufficient. Note that if $\mathbf{C}$ is furthermore stable, then this condition is *equivalent* to being an epimorphism. ◁

Lemma 1.1.11. *Suppose $\mathbf{C}$ is semi-additively monoidal. Then an algebra $A \in \mathrm{Alg}(\mathbf{C})$ is separable if and only if A is projective as an A-bimodule, i.e. if and only if there exists some finite n and a retraction of $(A \otimes A^{\mathrm{op}})^n$ onto A.*

Proof. Clearly separability implies the projectivity condition, with $n = 1$.

For the converse, fix a retraction diagram $A \xrightarrow{i} (A \otimes A^{\mathrm{op}})^n \xrightarrow{p} A$. Write p as $(p_i)_i$, where each $p_i : A \otimes A^{\mathrm{op}} \to A$ is p on the ith summand.

Observe that the unit map $\mathbf{1} \to A$ lifts through $A \otimes A^{\mathrm{op}} \xrightarrow{\mu} A$, so that each $p_i : A \otimes A^{\mathrm{op}} \to A$ lifts as well, now as a bimodule map. Fix a lift $\tilde{p}_i$ to get

$$\tilde{p} : (A \otimes A^{\mathrm{op}})^n \to A \otimes A^{\mathrm{op}}$$

Then $\tilde{p} \circ i$ is a section of μ. □

Remark 1.1.12. A consequence of this characterization is that in the setting of classical algebra, as projectivity is Morita invariant, separability also is.

This is *wrong* in the generality that we are in. The following counterexample was pointed out to me by Robert Burklund: one can show that if X is a finite spectrum which generates Sp^ω as a thick subcategory, then $\mathrm{End}(X)$ is separable if and only if the unit map $\mathbb{S} \to \mathrm{End}(X)$ splits (cf. Proposition 2.1.10), while $\mathrm{End}(X)$ is always Morita equivalent to the sphere spectrum $\mathbb{S}$, which is of course separable. Yet there are such spectra such that the unit map does *not* split, such as $X = \mathbb{S}/\eta$, the cone of $\eta \in \pi_1(\mathbb{S})$.

One can analyze this example and make it more general - in particular, one can make a similar example in some category of representations of some group over $\mathbb{Q}$, and so have such examples in characteristic 0.

One could instead formulate a notion of "projective Morita equivalence", and prove that separability is projective-Morita invariant. ◁

1.1.2 Modules over separable algebras

We now move on to discussing modules over separable algebras. The main observation in this realm is the following:

Proposition 1.1.13. *Let A be an algebra in $\mathbf{C}$, and consider the free-forgetful adjunction*

$$A \otimes - : \mathbf{C} \rightleftarrows \mathrm{LMod}_A(\mathbf{C}) : U$$

A is separable if and only if the co-unit $A \otimes U(-) \to \mathrm{id}_{\mathrm{LMod}_A(\mathbf{C})}$ admits a natural right $\mathbf{C}$-linear section.

Proof. There is a functor $\mathrm{Fun}_{\mathbf{C}}(\mathrm{LMod}_A(\mathbf{C}), \mathrm{LMod}_A(\mathbf{C})) \to \mathrm{BiMod}_A(\mathbf{C})$ given informally by evaluation at the object $A \in \mathrm{LMod}_A(\mathbf{C})$ [Lur12, Remark 4.6.2.9., Theorem 4.8.4.1.][3].

This functor sends $A \otimes U(-)$ to $A \otimes A^{\mathrm{op}}$ as an A-bimodule, and $\mathrm{id}_{\mathrm{Mod}_A(\mathbf{C})}$ to A itself, with its canonical A-bimodule structure. In particular, the existence of a natural section as indicated implies the existence of a bimodule section.

Conversely, suppose that A is separable. Up to embedding $\mathbf{C}$ in a monoidal ∞-category admitting geometric realizations compatible with the tensor product, we may assume that $\mathbf{C}$ has these properties. In that case, the above restriction functor induces an equivalence $\mathrm{Fun}^{\Delta}_{\mathbf{C}}(\mathrm{LMod}_A(\mathbf{C}), \mathrm{LMod}_A(\mathbf{C})) \simeq \mathrm{BiMod}_A(\mathbf{C})$, and so we can reverse the argument from above.

Concretely, the section is described as follows :

$$M \xleftarrow{\simeq} A \otimes_A M \xrightarrow{s \otimes_A M} (A \otimes A^{\mathrm{op}}) \otimes_A M \simeq A \otimes M$$

□

Corollary 1.1.14. *If A is a separable algebra in $\mathbf{C}$, then any A-module M is a retract of the free A-module $A \otimes M$.*

It will be convenient to have a generalization of this observation in the following direction: if $\mathbf{M}$ is equipped with a coherent tensoring by $\mathbf{C}$, i.e. $\mathbf{M}$ is a left $\mathbf{C}$-module, the notion of left A-module *in* $\mathbf{M}$ makes sense. We have:

[3] Under relatively mild hypotheses on $\mathbf{C}$, this can be made into an equivalence by restricting the domain a little: if $\mathbf{C}$ admits geometric realizations compatible with the tensor product, then $\mathbf{C}$-linear endofunctors of $\mathrm{LMod}_A(\mathbf{C})$ that commute with geometric realizations are exactly given by bimodules [Lur12, Theorem 4.8.4.1.]. For this part of the proof, we do not need an equivalence.

Corollary 1.1.15. *Let* $\mathbf{M}$ *be a left* $\mathbf{C}$*-module,* A *an algebra in* $\mathbf{C}$*, and consider the free-forgetful adjunction* $A \otimes - : \mathbf{M} \rightleftarrows \mathrm{LMod}_A(\mathbf{M}) : U$*. If* A *is separable, then the co-unit* $A \otimes U(-) \to \mathrm{id}_{\mathrm{LMod}_A(\mathbf{M})}$ *admits a natural section. If* $\mathbf{M}$ *is a* $\mathbf{C}$*-bimodule, this section is right* $\mathbf{C}$*-linear*[4]*.*

In particular, any A*-module in* $\mathbf{M}$*,* M*, is a retract of the free* A*-module* $A \otimes M$*.*

Proof. As in the previous proof - by embedding $\mathbf{C}, \mathbf{M}$ in ∞-categories that have geometric realizations compatible with the tensor product (resp. the tensoring of $\mathbf{C}$), we may assume that they have these properties.

In this case, $\mathrm{LMod}_A(\mathbf{M}) \simeq \mathrm{LMod}_A(\mathbf{C}) \otimes_{\mathbf{C}} \mathbf{M}$, and the free-forgetful adjunction for $\mathbf{M}$ is identified with $- \otimes_{\mathbf{C}} \mathbf{M}$ applied to the free forgetful adjunction for $\mathbf{C}$. The result follows.

We note that the section has the same concrete description as in the case of $\mathbf{M} = \mathbf{C}$. □

Remark 1.1.16. We will have several results that hold for an arbitrary $\mathbf{C}$-module $\mathbf{M}$. While this always implies the result for the special case $\mathbf{M} = \mathbf{C}$, we will typically state this special case explicitly, to help with intuition. However, note that the general case (specifically with $\mathbf{C}$-modules such as $\mathrm{RMod}_A(\mathbf{C})$) will be relevant.

Furthermore, all the results can straightforwardly (and usefully!) be dualized to right $\mathbf{C}$-modules, and we will use the right-module version here and there with no further comment (except perhaps a reference to this remark). ◁

Thus, separability allows us to deduce things about A-modules based on underlying properties. For instance:

Corollary 1.1.17. *Let* $\mathbf{M}$ *be a left* $\mathbf{C}$*-module and let* A *be a separable algebra in* $\mathbf{C}$*. Consider a map* $M \to N$ *of left* A*-modules in* $\mathbf{M}$*. If it has a retraction in* $\mathbf{M}$ *(resp. a section), then it does so in* $\mathrm{LMod}_A(\mathbf{M})$ *as well.*

Proof. The map $M \to N$ is a retract of the map $A \otimes M \to A \otimes N$, and the property of having a section (resp. a retraction) is closed under retracts. □

Similarly, we have:

Corollary 1.1.18. *Let* $\mathbf{C}$ *be a pointed monoidal* ∞*-category in which* $\otimes$ *preserves the zero object, and* $A \in \mathrm{Alg}(\mathbf{C})$ *a separable algebra. Let* $f : M \to N$ *be a morphism in* $\mathrm{LMod}_A(\mathbf{C})$*, whose underlying map in* $\mathbf{C}$ *is nullhomotopic, i.e. factors through* 0*.*

In this case, f *is nullhomotopic in* $\mathrm{LMod}_A(\mathbf{C})$*.*

The same holds for morphisms in $\mathrm{LMod}_A(\mathbf{M})$ *whose underlying morphism in* $\mathbf{M}$ *is nullhomotopic, for any left* $\mathbf{C}$*-module* $\mathbf{M}$*.*

Proof. The proof is the same: retracts of nullhomotopic maps are nullhomotopic. □

Remark 1.1.19. Note that this fact is famously not true in general ∞-categories if we do not assume separability. For example, let $A = \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}/p)$, and view $\mathbb{Z}/p$ as an A-module. Then $p : \mathbb{Z}/p \to \mathbb{Z}/p$ is not zero as an A-module map, but its underlying map is 0. ◁

Corollary 1.1.14, as well its extension to Corollary 1.1.15 will be crucial in the next section, where we analyze the relation of separable algebras to homotopy categories, but we can already make good use of it to analyze relative tensor products and internal homs.

We recall that, for a right (resp. left) A-module M (resp. N) in $\mathbf{C}$, we can form a simplicial object $\mathrm{Bar}(M, A, N)_\bullet : \Delta^{\mathrm{op}} \to \mathbf{C}$, compatibly with monoidal functors $\mathbf{C} \to \mathbf{D}$,

[4]For example, if $\mathbf{C}$ is symmetric monodal one can consider $\mathbf{M}$ itself as an $\mathbb{E}_\infty$-$\mathbf{C}$-module and $\mathrm{LMod}_A(\mathbf{C})$ remains a $\mathbf{C}$-module, and this section is $\mathbf{C}$-linear.

and that its colimit, if it exists, is the relative tensor product $M \otimes_A N$, cf. [Lur12, Section 4.4.2.]. Given a monoidal functor $f : \mathbf{C} \to \mathbf{D}$, we have a canonical equivalence $f \circ \mathrm{Bar}(M, A, N)_\bullet \simeq \mathrm{Bar}(f(M), f(A), f(N))_\bullet$.

Definition 1.1.20. Let $f : \mathbf{C} \to \mathbf{D}$ be a monoidal functor, $A \in \mathrm{Alg}(\mathbf{C})$ an algebra in $\mathbf{C}$, M (resp. N) a right (resp. left) A-module.

We say that f *preserves* the relative tensor product $M \otimes_A N$ if it exists in $\mathbf{C}$, and f preserves the colimit $\mathrm{colim}_{\Delta^{\mathrm{op}}} \mathrm{Bar}(M, A, N)_\bullet$. ◁

We can then state:

Proposition 1.1.21. *Assume* $\mathbf{C}$ *admits geometric realizations which are compatible with the tensor product. Let* $A \in \mathrm{Alg}(\mathbf{C})$ *be a separable algebra, and* M, N *be a right* A*-module and a left* A*-module respectively.*

In this case, the relative tensor product $M \otimes_A N$ *is a (natural,* $\mathbf{C}$*-linearly on both sides) retract of* $M \otimes N$.

In particular, if we now remove the assumption that $\mathbf{C}$ *admits geometric realizations and replace it with* $\mathbf{C}$ *being idempotent complete, then* $\mathbf{C}$ *still admits relative tensor products of (right with left)* A*-modules; and they are preserved by any monoidal functor* $\mathbf{C} \to \mathbf{D}$.

Proof. The second part can be deduced from the first as follows: freely add geometric realizations to $\mathbf{C}$ to obtain a fully faithful monoidal functor $\mathbf{C} \to \mathbf{D}$ where $\mathbf{D}$ satisfies the hypotheses of the first part. The image of A in $\mathbf{D}$ is still separable, and so the tensor product $M \otimes_A N$, computed in $\mathbf{D}$, lives in $\mathbf{C}$, because $M \otimes N$ does and $\mathbf{C}$ is idempotent complete (here, we use the first part). Therefore, this colimit of the bar construction is a colimit in $\mathbf{C}$ as well.

For the first part, we note that N is a (natural, $\mathbf{C}$-linear) retract of $A \otimes N$, so that $M \otimes_A N$ is a (natural, $\mathbf{C}$-linearly on both sides) retract of $M \otimes_A (A \otimes N) \simeq M \otimes N$, as was claimed.

From the proof, it is clear that these relative tensor products are preserved by any monoidal functor, because retractions are; this proves the final part. □

Remark 1.1.22. One could instead phrase this, and the next proof, in terms of the canonical resolution of the left A-module N, namely a simplicial object which looks like $[n] \mapsto A^{\otimes n+1} \otimes N$. In those terms, the statement would be that the corresponding colimit diagram $N \simeq \mathrm{colim}_{\Delta^{\mathrm{op}}} A^{\otimes n+1} \otimes N$ is an absolute colimit diagram, as it is a retract of the corresponding diagram for $A \otimes N$, which is split augmented and hence an absolute colimit diagram. See the proof of [NP23, Lemma 4.7] for an argument in this direction. ◁

We now move on to hom objects. Given two left A-modules M, N, the hom-object from M to N, $\hom_A(M, N)$, is, if it exists the object of $\mathbf{C}$ equipped with a map of left A-modules $ev : M \otimes \hom_A(M, N) \to N$ which satisfies the following universal property: restriction along ev induces an equivalence

$$\mathrm{map}_{\mathbf{C}}(c, \hom_A(M, N)) \simeq \mathrm{map}_{\mathrm{LMod}_A(\mathbf{C})}(M \otimes c, N)$$

Remark 1.1.23. Since we have only assumed $\mathbf{C}$ to be monoidal, we must be careful about left and right internal homs. However, because in that situation, $\mathrm{LMod}_A(\mathbf{C})$ is only *right* tensored over $\mathbf{C}$, the handedness of the homs is forced upon us. To clarify this, we make the following definition for $\mathbf{C}$ itself. ◁

Definition 1.1.24. Let $\mathbf{C}$ be a monoidal ∞-category. We say $\mathbf{C}$ admits right internal homs if it admits internal homs as an ∞-category right tensored over $\mathbf{C}$, that is, if for any $x, y \in \mathbf{C}$, there exists an object $\hom(x, y)$ with a map $ev : x \otimes \hom(x, y) \to y$ satisfying the following universal property: for any $c \in \mathbf{C}$, composition with ev induces an equivalence $\mathrm{map}_{\mathbf{C}}(c, \hom(x, y)) \to \mathrm{map}_{\mathbf{C}}(x \otimes c, y)$. ◁

If $f : \mathbf{C} \to \mathbf{D}$ is a monoidal functor, and if $\hom_A(M, N)$ exists, then we obtain a map of left $f(A)$-modules $f(M) \otimes f(\hom_A(M, N)) \to f(N)$.

Definition 1.1.25. Let $f : \mathbf{C} \to \mathbf{D}$ be a monoidal functor, $A \in \mathrm{Alg}(\mathbf{C})$ an algebra in $\mathbf{C}$ and M, N left A-modules. We say that f preserves the internal hom $\hom_A(M, N)$ if it exists, and the induced map $f(M) \otimes f(\hom_A(M, N)) \to f(N)$ exhibits $f(\hom_A(M, N))$ as a hom object from $f(M)$ to $f(N)$. ◁

We begin with a well-known lemma:

Lemma 1.1.26. *Let* $\mathbf{C}$ *be monoidal, and assume it admits totalizations of cosimplicial objects as well as right internal homs in the sense of Definition 1.1.24. Finally, we assume that* $- \otimes c$ *preserves colimits for any* $c \in \mathbf{C}$.

In this case, for any algebra $A \in \mathrm{Alg}(\mathbf{C})$*, the right-*$\mathbf{C}$*-module* $\mathrm{LMod}_A(\mathbf{C})$ *admits hom-objects in* $\mathbf{C}$.

Remark 1.1.27. If $\mathbf{C}$ is $\mathbb{E}_2$-monoidal (in fact, a natural equivalence $x \otimes y \simeq y \otimes x$ suffices), then for any $c \in \mathbf{C}$, $- \otimes c \simeq c \otimes -$ has a right adjoint by assumption on right internal homs, and thus preserves all colimits. Thus in this setting the condition is superfluous. ◁

Proof. Now, given $Y \in \mathrm{LMod}_A(\mathbf{C})$, we note the following two things: first, if $\mathbf{C}$ admits I^{op}-shaped limits, then the property that $\hom_A(X, Y)$ exist is closed under I-shaped colimits in X, and second, for any $X \in \mathbf{C}$, $\hom_A(A \otimes X, Y)$ exists.

For the first one, we note that indeed, the condition that $\hom_A(X, Y)$ exists is by definition the condition that the presheaf

$$\mathbf{C}^{\mathrm{op}} \to \mathcal{S}, c \mapsto \mathrm{map}_A(X \otimes c, Y)$$

be representable. Representable presheaves are closed under I^{op}-shaped limits by assumption, and $- \otimes c$ preserves any colimits that exist in $\mathbf{C}$, so the claim follows at once.

For the second one, we note that

$$\mathrm{map}_A(A \otimes X \otimes c, Y) \simeq \mathrm{Map}(X \otimes c, Y) \simeq \mathrm{Map}(c, \hom(X, Y))$$

so for any X, $\hom_A(A \otimes X, Y)$ exists and is equivalent to $\hom(X, Y)$.

With these two things in hand, we can conclude: any A-module is the colimit of a Δ^{op}-shaped diagram, all of whose terms are of the form $A \otimes X$ for some X [Lur12, Proposition 4.7.3.14]. □

Proposition 1.1.28. *Assume* $\mathbf{C}$ *admits totalizations of cosimplicial objects, and right internal homs. Let* $A \in \mathrm{Alg}(\mathbf{C})$ *be a separable algebra, and* $M, N \in \mathrm{LMod}_A(\mathbf{C})$*. In this case,* $\hom_A(M, N) \in \mathbf{C}$ *exists, and is a retract of* $\hom(M, N)$*. Furthermore, any monoidal functor* $\mathbf{C} \to \mathbf{D}$ *which is also* closed*, or more generally, which preserves* $\hom(M, N)$*, preserves* $\hom_A(M, N)$.

If we remove the assumption that $\mathbf{C}$ *admits totalizations, while keeping the existence of* $\hom(M, N)$ *and we assume that* $\mathbf{C}$ *is idempotent complete, then we get the same conclusion about* $\hom_A(M, N)$.

Proof. We begin under the assumption that $\mathbf{C}$ admits totalizations and internal homs.

In this case, by Lemma 1.1.26, $\hom_A(M, N)$ exists and is a retract of

$$\hom_A(A \otimes M, N) \simeq \hom(M, N)$$

It is clear that this is preserved by any monoidal functor which preserves $\hom(M, N)$.

Now, we go back to a general idempotent-complete $\mathbf{C}$. There is a monoidal embedding $\mathbf{C} \to \mathbf{D}$ where $\mathbf{D}$ admits totalizations, and which preserves all homs that exist in $\mathbf{C}$: in fact, the Yoneda embedding into the Day convolution monoidal structure on presheaves has this property. In particular, $\hom_A(M,N)$ in $\mathcal{E}$ is a retract of $\hom(M,N)$ in $\mathbf{C}$, and thus is in $\mathbf{C}$ by idempotent-completeness. The conclusion about preservation follows similarly. □

An internal hom of specific interest is the center of A:

Corollary 1.1.29. *Let $\mathbf{C}$ be an idempotent complete $\mathbb{E}_2$-monoidal ∞-category, and $A \in \mathrm{Alg}(\mathbf{C})$ a separable algebra. Since $\mathbf{C}$ is $\mathbb{E}_2$-monoidal, $\mathrm{BiMod}_A(\mathbf{C})$ can be considered as a left $\mathbf{C}$-module*[5]*. In this case, $Z(A) = \hom_{\mathrm{BiMod}_A}(A,A)$ exists and is a retract of A.*

Furthermore, it is preserved by any $\mathbb{E}_2$-monoidal functor $\mathbf{C} \to \mathbf{D}$.

Notation 1.1.30. We introduce here the notation $Z(A) = \hom_{A\otimes A^{\mathrm{op}}}(A,A)$ - this is the $\mathbb{E}_1$-center of A. If $\mathbf{C}$ is $\mathbb{E}_m$-monoidal, $m \geq 2$, this is an $\mathbb{E}_2$-algebra in $\mathbf{C}$ [Lur12, Section 5.3.][6]. ◁

Remark 1.1.31. Note that the retraction $A \to Z(A)$ is given by precomposition by $s : A \to A \otimes A^{\mathrm{op}}$:

$$A \simeq \hom_{A\otimes A^{\mathrm{op}}}(A \otimes A^{\mathrm{op}}, A) \to \hom_{A\otimes A^{\mathrm{op}}}(A,A) = Z(A)$$

In particular, it has a canonical left $Z(A)$-linear structure. ◁

We conclude this section with the following classical fact:

Proposition 1.1.32. *Assume $\mathbf{C}$ is idempotent-complete and $\mathbb{E}_{m+1}$-monoidal, $m \geq 1$, and let $R \in \mathrm{Alg}_{\mathbb{E}_{m+1}}(\mathbf{C})$ be separable. In this case, relative tensor products over R exist and so $\mathrm{LMod}_R(\mathbf{C})$ is $\mathbb{E}_m$-monoidal.*

Let $A \in \mathrm{Alg}(\mathrm{LMod}_R(\mathbf{C}))$. If A is separable in $\mathrm{LMod}_R(\mathbf{C})$, then it is separable in $\mathbf{C}$.

Proof. Suppose A is separable over R. We then have section $A \to A \otimes_R A^{\mathrm{op}}$ in ${}_A\mathrm{BiMod}_A(\mathrm{LMod}_R(\mathbf{C}))$, and hence in ${}_A\mathrm{BiMod}_A(\mathbf{C})$ through the lax monoidal forgetful functor $\mathrm{LMod}_R(\mathbf{C}) \to \mathbf{C}$. But now, because R is separable, the latter is a retract of $A \otimes A^{\mathrm{op}}$ in A-bimodules. Composing the two retraction gives the claim. □

We prove the converse in the case of an additive ∞-category in Proposition 1.2.12.

1.2 Separable algebras and homotopy categories

In this section, I explain how separable algebras in $\mathbf{C}$ are controlled by the homotopy category $\mathrm{ho}(\mathbf{C})$. This suggests that a big chunk of the study of separable algebras can be performed in the homotopy category, and thus explains morally why separable algebras work so well in tt-categories. This also allows to *lift* many results about separable algebras in the tt-setting to the stable ∞-setting.

[5] As a left $\mathbf{C}$-module, it is equivalent to $\mathrm{RMod}_{A^{\mathrm{op}}\otimes A}(\mathbf{C})$.

[6] Much of [Lur12, Section 5.3] is only stated for *symmetric* monoidal ∞-categories. Since we will not need this structure on $Z(A)$ in the remainder of the thesis, the reader should feel free to ignore this subtlety. Let us simply point out a simple way of making $Z(A)$ an $\mathbb{E}_2$-algebra: if $\mathbf{C}$ is presentably $\mathbb{E}_2$-monoidal, $\mathrm{LMod}_{\mathbf{C}}(\mathrm{Pr}^{\mathrm{L}})$ is monoidal, and ${}_A\mathrm{BiMod}_A(\mathbf{C}) \simeq \mathrm{Fun}^L_{\mathbf{C}}(\mathrm{RMod}_A(\mathbf{C}), \mathrm{RMod}_A(\mathbf{C}))$ can thus be interpreted as an internal endomorphism object therein, so that it is canonically an $\mathbb{E}_1$-algebra in $\mathrm{LMod}_{\mathbf{C}}(\mathrm{Pr}^{\mathrm{L}})$, and thus the endomorphism object of its unit (A) is canonically an $\mathbb{E}_2$-algebra in $\mathbf{C}$.

1.2.1 Modules, separability and algebras

The key property that will drive our analysis is Corollary 1.1.14, which, recall, states that over a separable algebra, any module is a retract of a free module of the form $A \otimes M$. We will use it together with the following general fact:

Lemma 1.2.1. *Let $A \in \mathrm{Alg}(\mathbf{C})$ be any algebra, and let $X \in \mathrm{LMod}_A$ be a retract of a module of the form $A \otimes M$.*

For any $N \in \mathrm{LMod}_A$, the functor $\mathrm{ho}(\mathrm{LMod}_A(\mathbf{C})) \to \mathrm{LMod}_{hA}(\mathrm{ho}(\mathbf{C}))$ induces an isomorphim

$$\pi_0 \operatorname{map}_{\mathrm{LMod}_A(\mathbf{C})}(X, N) \xrightarrow{\cong} \hom_{\mathrm{LMod}_{hA}(\mathrm{ho}(\mathbf{C}))}(hX, hN)$$

More generally, if $\mathbf{M}$ is a left $\mathbf{C}$-module, and $X \in \mathrm{LMod}_A(\mathbf{M})$ is a retract of some module of the form $A \otimes M$, then for any $N \in \mathrm{LMod}_A(\mathbf{M})$, the functor $\mathrm{ho}(\mathrm{LMod}_A(\mathbf{M})) \to \mathrm{LMod}_{hA}(\mathrm{ho}(\mathbf{M}))$ induces an isomorphism

$$\pi_0 \operatorname{map}_{\mathrm{LMod}_A(\mathbf{M})}(X, N) \xrightarrow{\cong} \hom_{\mathrm{LMod}_{hA}(\mathrm{ho}(\mathbf{M}))}(hX, hN)$$

Proof. The collection of X's for which this map is an isomorphism is clearly closed under retract, so we may assume that X is free on some M. But then hX is free on the same hM, with the same unit map, from which the claim follows. □

Remark 1.2.2. Applying this (or making the same argument) with $\mathbf{M} = \mathrm{RMod}_B(\mathbf{C})$ shows that if X is a retract of a free (A, B)-bimodule, then for any (A, B)-bimodule N, the canonical map $\pi_0 \operatorname{map}_{{}_A\mathrm{BiMod}_B(\mathbf{C})}(X, N) \to \hom_{{}_{hA}\mathrm{BiMod}_{hB}(\mathrm{ho}(\mathbf{C}))}(hX, hN)$ is an isomorphism. ◁

Corollary 1.2.3. *Let $A \in \mathrm{Alg}(\mathbf{C})$ be a separable algebra. The functor $\mathrm{ho}(\mathrm{LMod}_A(\mathbf{C})) \to \mathrm{LMod}_{hA}(\mathrm{ho}(\mathbf{C}))$ is fully faithful.*

More generally, if $\mathbf{M}$ is a left $\mathbf{C}$-module, then

$$\mathrm{ho}(\mathrm{LMod}_A(\mathbf{M})) \to \mathrm{LMod}_{hA}(\mathrm{ho}(\mathbf{M}))$$

is fully faithful.

Proof. This follows from the previous lemma together with Corollary 1.1.14 (resp. Corollary 1.1.15). □

Convention 1.2.4. In the rest of this thesis, $\mathbf{C}$ will be assumed to be additive. We will however repeat it in the statements of results for self-containedness. ◁

It is not clear to the author whether this condition is necessary, and exactly where, but we use it in some key instances, so it is certainly necessary *for our proofs*, if not the results. Note that additivity is a place where both ∞-categories and their homotopy categories can live, so it is a suitable inbetween between ∞-categories and 1-categories. We use this assumption together with the following lemma:

Lemma 1.2.5. *Let $\mathbf{C}$ be an additive ∞-category and $e : X \to X$ an idempotent in $\mathrm{ho}(\mathbf{C})$. There exists a coherent idempotent $\mathrm{Idem} \to \mathbf{C}$ which lifts e (cf. [Lur09, Section 4.4.5.]).*

In particular, if $\mathbf{C}$ is idempotent-complete, then so is $\mathrm{ho}(\mathbf{C})$.

Proof. This is [Lur12, Lemma 1.2.4.6., Remark 1.2.4.9.] - note that as stated, the assumption is that $\mathbf{C}$ is stable, but the proof works just as well if $\mathbf{C}$ is additive. Alternatively, one can deduce the additive case from the stable case by embedding any additive ∞-category in a stable one. □

With all of this, we can show:

Theorem 1.2.6. *Suppose* $\mathbf{C}$ *is additively monoidal, and let* $A \in \mathrm{Alg}(\mathbf{C})$ *be a separable algebra. The forgetful functor* $\mathrm{ho}(\mathrm{LMod}_A(\mathbf{C})) \to \mathrm{LMod}_{hA}(\mathrm{ho}(\mathbf{C}))$ *is an equivalence.*

More generally, if $\mathbf{M}$ *is a left* $\mathbf{C}$*-module, then the forgetful functor* $\mathrm{ho}(\mathrm{LMod}_A(\mathbf{M})) \to \mathrm{LMod}_{hA}(\mathrm{ho}(\mathbf{M}))$ *is an equivalence.*

In the stable case, and for $\mathbf{C} = \mathbf{M}$, this is also a consequence of the main theorem of [DS18] (together with [Bal11]).

Proof. We have already shown it is fully faithful, so we are left with proving that it is essentially surjective. Note that we can assume without loss of generality that $\mathbf{M}$ is idempotent-complete: indeed, assume for a second the claim holds for idempotent complete ∞-categories, and let $\mathbf{M} \to \mathbf{M}'$ be the idempotent-completion of $\mathbf{M}$, which is in particular a fully faithful functor.

Let $M \in \mathrm{LMod}_{hA}(\mathrm{ho}(\mathbf{M})) \subset \mathrm{LMod}_{hA}(\mathrm{ho}(\mathbf{M}'))$. By the idempotent complete case, this can be lifted to an A-module in $\mathbf{M}'$. But the underlying object of M is in $\mathrm{ho}(\mathbf{M})$, and therefore the underlying object of this lift is in $\mathbf{M}$, which proves the claim.

So we now assume $\mathbf{M}$ is idempotent complete. It follows that $\mathrm{LMod}_A(\mathbf{M})$ is also idempotent-complete, cf. [Lur12, Corollary 4.2.3.3.] and [Lur09, Remark 4.4.5.13.]. It is also additive and therefore by Lemma 1.2.5, $\mathrm{ho}(\mathrm{LMod}_A(\mathbf{M}))$ is also idempotent complete. So to prove that a fully faithful functor $\mathrm{ho}(\mathrm{LMod}_A(\mathbf{M})) \to \mathbf{D}$ is essentially surjective, it suffices to show that any object in $\mathbf{D}$ is a retract of some object in the image; but here any object of $\mathrm{LMod}_{hA}(\mathrm{ho}(\mathbf{M}))$ is a retract of some $hA \otimes N$, by separability, and $hA \otimes N$ is the image of $A \otimes N$, so we are done. □

This is the first instance of how separable algebras behave nicely with respect to homotopy categories.

Remark 1.2.7. By Proposition 1.1.21 and Proposition 1.1.28 applied to the monoidal functor $\mathbf{C} \to \mathrm{ho}(\mathbf{C})$, the functors $\mathrm{LMod}_A(\mathbf{C}) \to \mathrm{LMod}_{hA}(\mathrm{ho}(\mathbf{C}))$ and $\mathrm{RMod}_A(\mathbf{C}) \to \mathrm{RMod}_{hA}(\mathrm{ho}(\mathbf{C}))$ are compatible with relative tensor products over A and internal homs over A that is, the tensor product $M \otimes_A N$ is the coequalizer in $\mathrm{ho}(\mathbf{C})$ over the two maps $M \otimes A \otimes N \rightrightarrows M \otimes N$, and the internal hom $\hom_A(M, N)$ is the equalizer in $\mathrm{ho}(\mathbf{C})$ of $\hom(M, N) \rightrightarrows \hom(A \otimes M, N)$.

This is therefore compatible with Balmer's construction in the triangulated setting [Bal14, Section 1].

Because the functor $\mathrm{ho}(\mathrm{LMod}_A(\mathbf{C})) \to \mathrm{LMod}_{hA}(\mathrm{ho}(\mathbf{C}))$ is also an equivalence, this gives an explanation, at least in the case of tensor triangulated categories which are homotopy categories of monoidal stable ∞-categories, of the fact that module categories over separable algebras are still triangulated, and why their tensor product behaves nicely.

We note that this is also discussed in the recent work of Naumann and Pol, see [NP23, Lemma 4.7, Remark 4.9]. ◁

Remark 1.2.8. The previous remark, as well as the equivalence

$$\mathrm{ho}(\mathrm{LMod}_A(\mathbf{M})) \simeq \mathrm{LMod}_{hA}(\mathrm{ho}(\mathbf{M}))$$

and the analogous one for RMod_B, for separable algebras A and B, shows that for separable algebras, there is a reasonable notion of Morita equivalence at the level of the homotopy category, that can be phrased in terms of bimodules in the "naive" way. See also Theorem 1.2.14 for an application of this idea to *morphisms* between separable algebras. ◁

We next show that the picture is even more rigid: separability is detected at the level of the homotopy category, more precisely:

Proposition 1.2.9. *Suppose* $\mathbf{C}$ *is additively monoidal and let* $A \in \mathrm{Alg}(\mathbf{C})$ *be a homotopy separable algebra. In this case,* A *is separable.*

For this, we use the following categorical facts:

Lemma 1.2.10 ([Lur22, Example 1.4.7.10. (Tag 00JC)])**.** *Let* $\Delta^1/\partial\Delta^1 \to B\mathbb{N}$ *be the canonical map. It is a categorical equivalence.*

Lemma 1.2.11. *[Lur09, p. 4.4.5.15.] The canonical map* $\mathbb{N}_{\geq} \to \mathrm{Idem}$ *is cofinal.*

Moreover, note that this canonical map is given by the following composite : $\mathbb{N}_{\geq} \to B\mathbb{N} \to \mathrm{Idem}$ so that we have the following commutative diagram for any ∞-category $\mathbf{D}$:

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1/\partial\Delta^1, \mathbf{D}) & \mathrm{Fun}(B\mathbb{N}, \mathbf{D}) \arrow[l, "\simeq"'] \arrow[r] & \mathrm{Fun}(\mathbb{N}_{\geq}, \mathbf{D}) \\
& \mathrm{Fun}(\mathrm{Idem}, \mathbf{D}) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

With this in hand, we can prove the claim:

Proof of Proposition 1.2.9. Without loss of generality, we assume $\mathbf{C}$ is idempotent complete.

Consider the idempotent $s : hA \otimes hA^{\mathrm{op}} \to hA \to hA \otimes hA^{\mathrm{op}}$ in the category of homotopy A-bimodules. Note that its source and target are free A-bimodules, and so that by Remark 1.2.2, the functor $\mathrm{ho}({}_A\mathrm{BiMod}_A(\mathbf{C})) \to_{hA} \mathrm{BiMod}_{hA}(\mathrm{ho}(\mathbf{C}))$ is fully faithful on the full subcategory spanned by $A \otimes A^{\mathrm{op}}$. By Lemma 1.2.5, this implies that this idempotent lifts to a coherent idempotent in ${}_A\mathrm{BiMod}_A(\mathbf{C})$. That is, we have a functor $\tilde{s} : \mathrm{Idem} \to_A \mathrm{BiMod}_A(\mathbf{C})$ that classifies s.

In the diagram

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1/\partial\Delta^1, {}_A\mathrm{BiMod}_A(\mathbf{C})) & \mathrm{Fun}(B\mathbb{N}, {}_A\mathrm{BiMod}_A(\mathbf{C})) \arrow[l, "\simeq"'] \arrow[r] & \mathrm{Fun}(\mathbb{N}_{\geq}, {}_A\mathrm{BiMod}_A(\mathbf{C})) \\
& \mathrm{Fun}(\mathrm{Idem}, {}_A\mathrm{BiMod}_A(\mathbf{C})) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

if we follow $\tilde{s}$ up and then left, we simply get $A \otimes A^{\mathrm{op}} \xrightarrow{s} A \otimes A^{\mathrm{op}}$. Now, in $\mathrm{Fun}(\Delta^1/\partial\Delta^1, \mathbf{C})$, we have an arrow that corresponds to the following commutative square in ${}_A\mathrm{BiMod}_A(\mathbf{C})$:

$$\begin{tikzcd}
A \otimes A^{\mathrm{op}} \arrow[r, "s"] \arrow[d, "\mu"'] & A \otimes A^{\mathrm{op}} \arrow[d, "\mu"] \\
A \arrow[r, "\mathrm{id}_A"'] & A
\end{tikzcd}$$

Note that there exists such a commutative square in ${}_A\mathrm{BiMod}_A(\mathbf{C})$, because there is one in ${}_{hA}\mathrm{BiMod}_{hA}(\mathrm{ho}(\mathbf{C}))$, and the source is a free A-bimodule (cf. Lemma 1.2.1 and Remark 1.2.2).

In particular, if we now go from $\mathrm{Fun}(\Delta^1/\partial\Delta^1, {}_A\mathrm{BiMod}_A(\mathbf{C}))$ to $\mathrm{Fun}(\mathbb{N}_{\geq}, {}_A\mathrm{BiMod}_A(\mathbf{C}))$, we get a map from $A \otimes A^{\mathrm{op}} \xrightarrow{s} A \otimes A^{\mathrm{op}} \xrightarrow{s} \dots$ to $A \xrightarrow{\mathrm{id}_A} A \xrightarrow{\mathrm{id}_A} \dots$. By commutativity of the diagram, the source is simply the restriction of $\tilde{s}$ along the map $\mathbb{N}_{\geq} \to \mathrm{Idem}$.

In particular, the source has a colimit given by the splitting of that idempotent, and we get a map from this colimit to A in ${}_A\mathrm{BiMod}_A(\mathbf{C})$. But splitting of idempotents are *absolute* colimits [Lur09, Corollary 4.4.5.12.], so they are preserved by the forgetful functor

${}_A\mathrm{BiMod}_A(\mathbf{C}) \to {}_{hA}\mathrm{BiMod}_{hA}(\mathrm{ho}(\mathbf{C}))$. If we redo this story in the latter category, A is the splitting of the idempotent in question, and so the canonical map from this colimit to A is an equivalence.

It is therefore an equivalence in $\mathrm{ho}(\mathbf{C})$, and therefore in $\mathbf{C}$, and therefore in ${}_A\mathrm{BiMod}_A(\mathbf{C})$. This proves that A is the splitting of some idempotent on $A \otimes A^{\mathrm{op}}$ in ${}_A\mathrm{BiMod}_A(\mathbf{C})$, along $A \otimes A^{\mathrm{op}} \xrightarrow{\mu} A$, which is exactly saying that A is separable, and so we are done. □

We can now prove the converse of Proposition 1.1.32, namely:

Proposition 1.2.12. *Assume $\mathbf{C}$ is additively $\mathbb{E}_m$-monoidal, $m \geq 3$ and idempotent-complete, and let $R \in \mathrm{Alg}_{\mathbb{E}_k}(\mathbf{C}), k \geq 2$ be separable. In this case, relative tensor products over R exist and so $\mathrm{LMod}_R(\mathbf{C})$ is monoidal.*

Let $A \in \mathrm{Alg}(\mathrm{LMod}_R(\mathbf{C}))$. In this case, A is separable in $\mathrm{LMod}_R(\mathbf{C})$ if and only if it is separable in $\mathbf{C}$.

Warning 1.2.13. Classically, if A is separable, then it is so over R, with no separability assumption on R. This is wrong in homotopical algebra and if one tries to run the classical proof, one will encounter the issue that $A \otimes A^{\mathrm{op}} \to A \otimes_R A^{\mathrm{op}}$ is not an epimorphism.

A counterexample is given by the $\mathbb{Q}$-algebras $R = \mathbb{Q}[x]$ and $A =$ any nonzero separable commutative $\mathbb{Q}$-algebra, all of this in the ∞-category of $\mathbb{Q}$-module spectra. ◁

Proof. We have proved in Proposition 1.1.32 that if A was separable over R, it was separable. Now, assume A is separable.

We observe that by Proposition 1.2.9, it suffices to show that A is separable in $\mathrm{ho}(\mathrm{Mod}_R(\mathbf{C}))$. But because R is separable, $\mathrm{ho}(\mathrm{Mod}_R(\mathbf{C})) \simeq \mathrm{Mod}_{hR}(\mathrm{ho}(\mathbf{C}))$, monoidally as the relative tensor products are preserved. In other words, we may work in $\mathrm{ho}(\mathbf{C})$ and thereby assume that $\mathbf{C}$ is a symmetric monoidal 1-category (an $\mathbb{E}_3$-monoidal 1-category is automatically symmetric monoidal).

But now $A \otimes A^{\mathrm{op}} \to A \otimes_R A^{\mathrm{op}}$ is a split epimorphism because R is separable, and in a 1-category, split morphisms are epimorphisms. It follows that an $A \otimes A^{\mathrm{op}}$-linear map between $A \otimes_R A^{\mathrm{op}}$-modules is automatically $A \otimes_R A^{\mathrm{op}}$-linear.

The following composite $A \xrightarrow{s} A \otimes A^{\mathrm{op}} \to A \otimes_R A^{\mathrm{op}}$, where s is a witness that A is separable, is therefore an $A \otimes_R A^{\mathrm{op}}$-linear section of the multiplication map, which proves the claim. □

We now exploit what we did so far to analyze morphisms between separable algebras. Our main result in the noncommutative world is:

Theorem 1.2.14. *Assume $\mathbf{C}$ is additively monoidal, and let $A, R \in \mathrm{Alg}(\mathbf{C})$. If A is separable, the canonical map*

$$\pi_0 \operatorname{map}_{\mathrm{Alg}(\mathbf{C})}(A, R) \to \hom_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))}(hA, hR)$$

is an isomorphism.

Warning 1.2.15. In general, $\operatorname{map}_{\mathrm{Alg}(\mathbf{C})}(A, R)$ is not discrete, even if R is also separable and $\mathbf{C}$ symmetric monoidal, see Proposition 1.2.27 and Example 1.2.29. ◁

The situation is better in the commutative world, as we will see in Proposition 1.3.11 and Theorem 1.3.19.

Proof. Up to embedding $\mathbf{C}$ monoidally and additively in a presentably additively monoidal ∞-category, we may assume $\mathbf{C}$ is presentably additively monoidal.

There is then a fully faithful functor

$$\mathrm{Alg}(\mathbf{C}) \to (\mathrm{RMod}_{\mathbf{C}})_{\mathbf{C}/}, A \mapsto (\mathrm{LMod}_A(\mathbf{C}), A)$$

by [Lur12, Theorem 4.8.5.11] (see also [Lur12, Remark 4.8.3.25]), so that $\mathrm{map}_{\mathrm{Alg}(\mathbf{C})}(A, R)$ can be described as the fiber of

$$\mathrm{map}_{\mathrm{RMod}_{\mathbf{C}}}(\mathrm{LMod}_A(\mathbf{C}), \mathrm{LMod}_R(\mathbf{C})) \to \mathrm{map}_{\mathrm{RMod}_{\mathbf{C}}}(\mathbf{C}, \mathrm{LMod}_R(\mathbf{C}))$$

at the map $\mathbf{C} \to \mathrm{LMod}_R(\mathbf{C})$ classifying the R-module R.

By [Lur12, Theorem 4.8.4.1][7], this map can be rewritten as the forgetful map

$${}_R\mathrm{BiMod}_A(\mathbf{C})^{\simeq} \to \mathrm{LMod}_R(\mathbf{C})^{\simeq}$$

For simplicity of notation, we simply write $\mathrm{LMod}_R(\mathbf{C}) = \mathrm{LMod}_R$, and then by [Lur12, Theorem 4.3.2.7], this can again be rewritten as

$$\mathrm{RMod}_A(\mathrm{LMod}_R)^{\simeq} \to \mathrm{LMod}_R^{\simeq}$$

In more concrete terms: an algebra map $A \to R$ is the same data as a right A-module structure on the left R-module R. It is easy to check that the same holds for 1-categories, with no presentability assumption[8].

Consider now the diagram:

$$\begin{tikzcd}
\mathrm{RMod}_A(\mathrm{LMod}_R) \arrow[r] \arrow[d] & \mathrm{RMod}_{hA}(\mathrm{ho}(\mathrm{LMod}_R)) \arrow[r] \arrow[d] & \mathrm{RMod}_{hA}(\mathrm{LMod}_{hR}) \arrow[d] \\
\mathrm{LMod}_R \arrow[r] & \mathrm{ho}(\mathrm{LMod}_R) \arrow[r] & \mathrm{LMod}_{hR}
\end{tikzcd}$$

When restricted to the full subcategory of LMod_R (resp. $\mathrm{ho}(\mathrm{LMod}_R)$, resp. $\mathrm{LMod}_{hR}(\mathrm{ho}(\mathbf{C}))$) spanned by $R, R \otimes A^{\otimes n}$, the horizontal maps are fully faithful on homotopy categories: for the left square, this follows from Corollary 1.2.3 applied to the right $\mathbf{C}$-module $\mathbf{M} = \mathrm{LMod}_R$ (cf. Remark 1.1.16), and for the right square, this follows from Lemma 1.2.1.

Passing to groupoid cores and restricting to these components, we see that the horizontal maps are therefore 1-equivalences, and so the induced maps on fibers are 0-equivalences - the fiber of the leftmost map is, by the previous argument, $\mathrm{map}_{\mathrm{Alg}(\mathbf{C})}(A, R)$, while the fiber of the rightmost map is $\hom_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))}(hA, hR)$, so that this proves the claim. $\square$

1.2.2 From homotopy algebras to algebras

The final goal of this section is to prove that not only is separability detected in the homotopy category, but that separability of a homotopy algeba is strong enough to guarantee that it can be lifted to an $\mathbb{E}_1$-algebra in $\mathbf{C}$. In other words, we now aim to prove:

Theorem 1.2.16. *Let $\mathbf{C}$ be an additive monoidal ∞-category, and $A \in \mathrm{Alg}(\mathrm{ho}(\mathbf{C}))$ a homotopy separable homotopy algebra.*

There exists an algebra $\widetilde{A} \in \mathrm{Alg}(\mathbf{C})$, necessarily separable, which lifts A. In fact, the moduli space of such lifts is simply-connected.

[7]See also [Lur12, Theorem 4.3.2.7]

[8]In fact, we can deduce that it also holds in general with no presentability assumption from the presentable case.

Warning 1.2.17. The moduli space of lifts is not contractible in general, cf. Example 1.2.29. We will later see however that it *is* contractible in the case of a homotopy commutative separable algebra, see Proposition 1.3.11. ◁

Let us first specify explicitly what we mean by "the moduli space of lifts".

Definition 1.2.18. Let $A \in \mathrm{Alg}(\mathrm{ho}(\mathbf{C}))$ be a homotopy algebra. We define the moduli space of lifts of A to $\mathrm{Alg}(\mathbf{C})$ to be the space $\mathrm{Alg}(\mathbf{C})^{\simeq} \times_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))^{\simeq}} \{A\}$, i.e. the fiber of $\mathrm{Alg}(\mathbf{C}) \to \mathrm{Alg}(\mathrm{ho}(\mathbf{C}))$ at A. ◁

Observation 1.2.19. By Theorem 1.2.14 together with the fact that

$$\mathrm{Alg}(\mathbf{C}) \to \mathrm{Alg}(\mathrm{ho}(\mathbf{C}))$$

is conservative, we find that $\mathrm{Alg}(\mathbf{C})^{\simeq} \to \mathrm{Alg}(\mathrm{ho}(\mathbf{C}))^{\simeq}$ is injective on π_0, and an isomorphism on π_1 at any point of $\mathrm{Alg}(\mathbf{C})^{\simeq}$. Furthermore, $\pi_2(\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))^{\simeq}) = 0$ at every point, so that to prove that the moduli space of lifts is simply connected, it really suffices to prove that it is non-empty. ◁

Observation 1.2.20. A lies in its connected component in $\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))$, which is equivalent to $B\mathrm{Aut}_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))}(A)$. If $\widetilde{A}$ is a lift of A, then the connected component of this lift in the moduli space is a connected component of the fiber of the map $B\mathrm{Aut}_{\mathrm{Alg}(\mathbf{C})}(\widetilde{A}) \to B\mathrm{Aut}_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C})}(A)$.

In particular, if we already know that the moduli space is connected, then the loop space of this moduli space at $\widetilde{A}$ is the fiber of $\mathrm{Aut}_{\mathrm{Alg}(\mathbf{C})}(\widetilde{A}) \to \mathrm{Aut}_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))}(A)$ at id_A. ◁

Our proof relies on deformation theory and Theorem 1.2.14, as well as the following lemma (which one could make more precise, but the following version is enough for our purposes):

Lemma 1.2.21. *Let $m \geq 1$ and let $\mathbf{C}$ be an additively $\mathbb{E}_m$-monoidal ∞-category. There exists an additively $\mathbb{E}_m$-monoidal ∞-category $\mathbf{D}$ and a commutative diagram of additively $\mathbb{E}_m$-monoidal ∞-categories:*

$$\begin{tikzcd} \mathrm{ho}_{\leq n+1}(\mathbf{C}) \arrow[r] \arrow[d] & \mathrm{ho}_{\leq n}(\mathbf{C}) \arrow[d] \\ \mathrm{ho}_{\leq n}(\mathbf{C}) \arrow[r] & \mathbf{D} \end{tikzcd}$$

such that the induced map $\mathrm{ho}_{\leq n+1}(\mathbf{C}) \to \mathrm{ho}_{\leq n}(\mathbf{C}) \times_{\mathbf{D}} \mathrm{ho}_{\leq n}(\mathbf{C})$ is fully faithful.

Here, $\mathrm{ho}_{\leq n}(\mathbf{C})$ denotes the homotopy n-category of $\mathbf{C}$.

Taking this lemma for granted, the proof of Theorem 1.2.16 is not hard.

Proof of Theorem 1.2.16. Fix a homotopy separable $A \in \mathrm{Alg}(\mathrm{ho}(\mathbf{C}))$. By Observation 1.2.19, it suffices to prove that A admits some lift $\tilde{A}$ to $\mathrm{Alg}(\mathbf{C})$.

As $\mathbf{C} \simeq \lim_n \mathrm{ho}_{\leq n}(\mathbf{C})$ and $\mathrm{Alg}(-)$ preserves limits (see Lemma 1.3.17), it suffices to prove that any given separable $A_n \in \mathrm{Alg}(\mathrm{ho}_{\leq n}(\mathbf{C}))$ admits a lift to $\mathrm{Alg}(\mathrm{ho}_{\leq n+1}(\mathbf{C}))$. For this, we apply Lemma 1.2.21: fix a square

$$\begin{tikzcd} \mathrm{ho}_{\leq n+1}(\mathbf{C}) \arrow[r] \arrow[d] & \mathrm{ho}_{\leq n}(\mathbf{C}) \arrow[d, "d_1"] \\ \mathrm{ho}_{\leq n}(\mathbf{C}) \arrow[r, "d_0"'] & \mathbf{D} \end{tikzcd}$$

as in the conclusion of that lemma.

Viewing A_n as a homotopy algebra in $\mathbf{C}$, we find that $d_0(A_n) \simeq d_1(A_n)$ as homotopy algebras in $\mathbf{D}$. But both are separable, as they are the image of the separable A_n under monoidal functors d_0, d_1.

In particular, by Theorem 1.2.14, this equivalence can be lifted to an equivalence of algebras, and thus we get a lift in the pullback. But the underlying object of this lift is (the underlying object of) A_n in $\mathrm{ho}_{\leq n+1}(\mathbf{C})$, so that it provides an algebra object A_{n+1} in $\mathrm{ho}_{\leq n+1}(\mathbf{C})$, by fully faithfulness, and this clearly lifts A_n. □

Finally, let us prove Lemma 1.2.21. We will need a slight modification in the $\mathbb{E}_m$-case which we sketch later. On first reading, the reader can pretend that $m = \infty$.

Proof of Lemma 1.2.21. One could in principle use the methods of [HNP18], but one would still have to make a number of additional verifications. Instead, let us use the synthetic objects of [HL17].

We first deal with the case where $\mathbf{C}$ is stable: let $\mathcal{E}$ be a stably $\mathbb{E}_m$-monoidal ∞-category. We let $\mathrm{Syn}_{\mathcal{E}}$ denote $\mathrm{Fun}^{\times}(\mathcal{E}^{\mathrm{op}}, \mathrm{Sp}_{\geq 0})$, this is an additive presentably $\mathbb{E}_m$-monoidal ∞-category, receiving a fully faithful, additive $\mathbb{E}_m$-monoidal functor

$$\mathcal{E} \to \mathrm{Syn}_{\mathcal{E}}, c \mapsto \mathrm{Map}(-,c) = \mathrm{map}(-,c)_{\geq 0}$$

By [HL17, Proposition 7.3.6.], we obtain a pullback square of additively $\mathbb{E}_m$- monoidal ∞-categories:

$$\begin{tikzcd}
\mathrm{Mod}_{\mathbf{1}^{\leq n}}(\mathrm{Syn}_{\mathcal{E}}) \arrow[r] \arrow[d] & \mathrm{Mod}_{\mathbf{1}^{\leq n-1}}(\mathrm{Syn}_{\mathcal{E}}) \arrow[d] \\
\mathrm{Mod}_{\mathbf{1}^{\leq n-1}}(\mathrm{Syn}_{\mathcal{E}}) \arrow[r] & \mathbf{D}
\end{tikzcd}$$

and the composite $\mathcal{E} \to \mathrm{Syn}_{\mathcal{E}} \to \mathrm{Mod}_{\mathbf{1}^{\leq k}}(\mathrm{Syn}_{\mathcal{E}})$ factors through $\mathrm{ho}_{\leq k+1}(\mathcal{E})$ in a fully faithful way - this is essentially saying that

$$\tau_{\leq k} \mathrm{Map}(-, \mathbf{1}_{\mathcal{E}}) \otimes \mathrm{Map}(-,e) \simeq \tau_{\leq k} \mathrm{Map}(-,e)$$

which follows from [HL17, Lemma 7.1.1. and Corollary 7.3.7.(c)] (alternatively, the proof of [HL17, Lemma 7.1.1.] works just as well for this statement).

To make sense of $\mathrm{Mod}_{\mathbf{1}^{\leq n}}(\mathrm{Syn}_{\mathcal{E}})$ as an $\mathbb{E}_m$-monoidal ∞-category when $m < \infty$, and to make sense of $\mathbf{D}$, we use Lemma 1.2.24. We note that the proof of [HL17, Proposition 7.3.6] works just as well in this context[9].

In other words, we have a commuting diagram:

$$\begin{tikzcd}
\mathrm{ho}_{\leq n+1}(\mathcal{E}) \arrow[r] \arrow[d] \arrow[dr] & \mathrm{ho}_{\leq n}(\mathcal{E}) \arrow[dr] & \\
\mathrm{ho}_{\leq n}(\mathcal{E}) \arrow[dr] & \mathrm{Mod}_{\mathbf{1}^{\leq n}}(\mathrm{Syn}_{\mathcal{E}}) \arrow[r] \arrow[d] & \mathrm{Mod}_{\mathbf{1}^{\leq n-1}}(\mathrm{Syn}_{\mathcal{E}}) \arrow[d] \\
& \mathrm{Mod}_{\mathbf{1}^{\leq n-1}}(\mathrm{Syn}_{\mathcal{E}}) \arrow[r] & \mathbf{D}
\end{tikzcd}$$

where the diagonal arrows are fully faithful. It follows that the map

$$\mathrm{ho}_{\leq n+1}(\mathcal{E}) \to \mathrm{ho}_{\leq n}(\mathcal{E}) \times_{\mathbf{D}} \mathrm{ho}_{\leq n}(\mathcal{E})$$

[9]In fact, as soon as the $\mathbb{E}_m$-monoidal square is set up, one can check that it is a pullback square on underlying ∞-categories, and thus this does not depend on the extra structure on $\mathbf{1}^{\leq n}$ from Lemma 1.2.24.

is fully faithful.

Now, let $\mathbf{C}$ be a general additive ∞-category. Using the Yoneda embedding[10]

$$\mathbf{C} \to \mathrm{Fun}^{\times}(\mathbf{C}^{\mathrm{op}}, \mathrm{Sp}_{\geq 0}) \to \mathrm{Fun}^{\times}(\mathbf{C}^{\mathrm{op}}, \mathrm{Sp})$$

and considering a small stable subcategory of $\mathrm{Fun}^{\times}(\mathbf{C}^{\mathrm{op}}, \mathrm{Sp})$ containing the image of the Yoneda embedding, we find a fully faithful additive $\mathbb{E}_m$-monoidal embedding $\mathbf{C} \to \mathcal{E}$ with $\mathcal{E}$ stably $\mathbb{E}_m$-monoidal.

The following diagram allows us to conclude:

$$\begin{tikzcd}
\mathrm{ho}_{\leq n+1}(\mathbf{C}) \arrow[r] \arrow[d] \arrow[dr, hook] & \mathrm{ho}_{\leq n}(\mathbf{C}) \arrow[dr, hook] & \\
\mathrm{ho}_{\leq n}(\mathbf{C}) \arrow[dr, hook] & \mathrm{ho}_{\leq n+1}(\mathcal{E}) \arrow[r] \arrow[d] & \mathrm{ho}_{\leq n}(\mathcal{E}) \arrow[d] \\
 & \mathrm{ho}_{\leq n}(\mathcal{E}) \arrow[r] & \mathbf{D}
\end{tikzcd}$$

□

Remark 1.2.22. In a previous version of the preprint [Ram23], we used obstruction theory to prove Theorem 1.2.16. Unravelling the proof in question, one would arrive at an essentially equivalent proof as the one proposed here. It simply seems to the author that this version is much simpler to parse, and to understand what is going on. ◁

In the $\mathbb{E}_m$-monoidal case, while $\mathbf{1}^{\leq n}$ is still an $\mathbb{E}_m$-algebra in $\mathrm{Syn}_{\mathcal{E}}$, this piece of structure alone does not allow us to define $\mathrm{Mod}_{\mathbf{1}^{\leq n}}(\mathrm{Syn}_{\mathcal{E}})$ as an $\mathbb{E}_m$-monoidal ∞-category itself, which is needed for the construction.

Remark 1.2.23. For $m \geq 2$, this structure on $\mathbf{1}^{\leq n}$ (and more generally on the relevant square zero extensions) is enough to make this ∞-category $\mathbb{E}_1$-monoidal, and this is enough for Theorem 1.2.16: indeed, a version of Lemma 1.2.21 where $\mathbf{D}$ and the relevant square is only monoidal is sufficient for Theorem 1.2.16.

Thus the reader who does not care about maximal generality can safely skip the next construction and pretend that $m \geq 2$ throughout. ◁

The following lemma guarantees that $\mathrm{Mod}_{\mathbf{1}^{\leq n}}(\mathrm{Syn}_{\mathcal{E}})$ makes sense as an $\mathbb{E}_m$-monoidal ∞-category, and that the pullback square constructed in [HL17] generalizes to this context. Indeed, given an $\mathbb{E}_m$-algebra A in an $\mathbb{E}_m$-monoidal ∞-category $\mathbf{C}$ with compatible geometric realizations, to promote $\mathrm{LMod}_A(\mathbf{C})$ to an $\mathbb{E}_m$-monoidal ∞-category, it suffices to produce a lift $\tilde{A} \in \mathrm{Alg}_{\mathbb{E}_{m+1}}(Z_{\mathbb{E}_m}(\mathbf{C}))$ of A along the canonical functor $Z_{\mathbb{E}_m}(\mathbf{C}) \to \mathbf{C}$, where $Z_{\mathbb{E}_m}(\mathbf{C})$ is the $\mathbb{E}_m$-center of $\mathbf{C} \in \mathrm{Alg}_{\mathbb{E}_m}(\mathrm{Cat}(\Delta))$ in the sense of [Lur12, Definition 5.3.1.12, Remark 5.3.1.13].

Lemma 1.2.24. *Let $\mathcal{E}$ be a small stable $\mathbb{E}_m$-monoidal ∞-category, $\mathrm{Syn}_{\mathcal{E}}^{\mathrm{st}} = \mathrm{Sp}(\mathrm{Syn}_{\mathcal{E}})$ be the stabilization of $\mathrm{Syn}_{\mathcal{E}}$, where $\mathrm{Syn}_{\mathcal{E}}$ is as in the proof of Lemma 1.2.21, and finally let $Z_{\mathbb{E}_m}(\mathrm{Syn}_{\mathcal{E}}^{\mathrm{st}})$ denote the $\mathbb{E}_m$-center of $\mathrm{Syn}_{\mathcal{E}}^{\mathrm{st}}$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, considered as an $\mathbb{E}_{m+1}$-monoidal ∞-category.*

[10] We implicitly use here that any additive ∞-category has a fully faithful Yoneda embedding into its presheaves of connective spectra. This follows from the fact that $\mathrm{Sp}_{\geq 0} \simeq \mathbf{Grp}_{\mathbb{E}_\infty}$ by the recognition principle.

The Postnikov square-zero extensions

$$\begin{tikzcd} \mathbf{1}^{\leq n+1} \arrow[r] \arrow[d] & \mathbf{1}^{\leq n} \arrow[d] \\ \mathbf{1}^{\leq n} \arrow[r] & \mathbf{1}^{\leq n} \oplus \Sigma^{n+2}\pi_{n+1}^{\heartsuit}(\mathbf{1}) \end{tikzcd}$$

from [Lur12, Corollary 7.4.1.28] lift to $\mathbb{E}_{m+1}$-square zero extensions in $Z_{\mathbb{E}_m}(\mathrm{Syn}_{\mathcal{E}}^{\mathrm{st}})$.

Remark 1.2.25. We state this lemma in the case of $\mathrm{Syn}_{\mathcal{E}}$, as we are not sure what is the correct maximal generality. What happens here is that $\mathbf{1}^{\leq n}$ happens to be in the center of $\mathrm{Syn}_{\mathcal{E}}$, essentially because of the formula

$$\mathbf{1}^{\leq n} \otimes \mathrm{Map}(-, e) \simeq \tau_{\leq n} \mathrm{Map}(-, e)$$

There does not seem to be an analogous formula in a general $\mathbb{E}_m$-monoidal prestable ∞-category, and so it is not clear that $\mathbf{1}^{\leq n}$ should always be in the center in that generality. In fact, one can produce counterexamples to $\pi_0(\mathbf{1})$ being in the center from examples of bimodules being left flat but not right flat over rings. $\triangleleft$

Remark 1.2.26. With this lemma, one can make sense of $\mathrm{Mod}_{\mathbf{1}^{\leq n}}(\mathrm{Syn}_{\mathcal{E}})$ as an $\mathbb{E}_m$-monoidal ∞-category, as well as of the pullback square used in the proof of Lemma 1.2.21. As mentioned in the proof, it is not hard to check that the proofs from [HL17] extend to this setting, essentially because the result only depends on the underlying square of ∞-categories, and basic formal properties of the (relative) tensor product. $\triangleleft$

Proof. Let $Z_{\mathbb{E}_m}(\mathcal{E})$ denote the $\mathbb{E}_m$-center of $\mathcal{E}$ in small stable ∞-categories. By the universal property of centers and the lax symmetric monoidality of the Syn-construction, this induces an $\mathbb{E}_{m+1}$-monoidal functor $\mathrm{Syn}^{\mathrm{st}}_{Z_{\mathbb{E}_m}(\mathcal{E})} \to Z_{\mathbb{E}_m}(\mathrm{Syn}^{\mathrm{st}}_{\mathcal{E}})$ such that the composite

$$\mathrm{Syn}^{\mathrm{st}}_{Z_{\mathbb{E}_m}(\mathcal{E})} \to Z_{\mathbb{E}_m}(\mathrm{Syn}^{\mathrm{st}}_{\mathcal{E}}) \to \mathrm{Syn}^{\mathrm{st}}_{\mathcal{E}}$$

is the left Kan extended from the canonical functor $Z_{\mathbb{E}_m}(\mathcal{E}) \to \mathcal{E}$. It thus suffices to find appropriate lifts in $\mathrm{Syn}_{Z_{\mathbb{E}_m}}(\mathcal{E})$.

Note that an easy calculation shows that in $\mathrm{Syn}_{\mathcal{F}}$, $\tau_{\geq n}\mathbf{1}_{\mathrm{Syn}_{\mathcal{F}}}$ can be described as $\Sigma^n \operatorname{map}(-, \Omega^n \mathbf{1}_{\mathcal{F}})_{\geq 0}$[11] and thus the functor $\mathrm{Syn}_{Z_{\mathbb{E}_m}(\mathcal{E})} \to \mathrm{Syn}_{\mathcal{E}}$ sends $\tau_{\geq n+1}\mathbf{1}$ to $\tau_{\geq n+1}\mathbf{1}$ for every n, since it preserves colimits and representables - it also preserves the corresponding map to $\mathbf{1}$.

Since it also preserves cofibers, it follows that it sends $\mathbf{1}^{\leq n} := \tau_{\leq n}\mathbf{1}$ to $\mathbf{1}^{\leq n}$. Finally, the relevant pushout/pullback square in $\mathrm{Syn}^{\mathrm{st}}_{Z_{\mathbb{E}_m}(\mathcal{E})}$ consists only of connective objects so it is also a pushout in $\mathrm{Syn}_{Z_{\mathbb{E}_m}(\mathcal{E})}$ and is thus sent to a pushout in $\mathrm{Syn}_{\mathcal{E}}$, and thus also to the appropriate pushout/pullback there.

[Lur12, Corollary 7.4.1.28] allows us to conclude that the Postnikov tower of the unit in $\mathrm{Syn}_{Z_{\mathbb{E}_m}(\mathcal{E})}$ upgrades to a tower of $\mathbb{E}_{m+1}$-square zero extensions, as claimed. $\square$

We now analyze the moduli space of lifts a bit further to show that it is not typically contractible. Theorem 1.2.16 shows that the moduli space is simply-connected, and we now explain how to describe its 2-fold loopspace.

[11] This is true more generally for any object $x \in \mathcal{F}$ in place of $\mathbf{1}_{\mathcal{F}}$.

Proposition 1.2.27. *Let $A \in \mathrm{Alg}(\mathbf{C})$, and hA the corresponding algebra in $\mathrm{ho}(\mathbf{C})$. Let $\mathcal{M}$ be the moduli space of lifts of hA to $\mathrm{Alg}(\mathbf{C})$, and L_A the $\mathbb{E}_1$-cotangent complex of A.*

The double-loop space of $\mathcal{M}$ at A is equivalent to the following spaces:

(i) $\Omega(\mathrm{Map}_{\mathrm{Alg}(\mathbf{C})}(A, A), \mathrm{id}_A)$*;*

(ii) the fiber over id_A *of* $\mathrm{Map}_{\mathrm{BiMod}_A}(A, A) \to \mathrm{Map}_A(A, A)$*, or equivalently if* $\mathbf{C}$ *is additive, its fiber over* 0*;*

(iii) $\mathrm{Map}_{\mathrm{BiMod}_A}(\Sigma L_A, A)$ *if* $\mathbf{C}$ *is stable.*

Proof. Without loss of generality, we assume $\mathbf{C}$ is presentably monoidal.

Recall that $\mathcal{M} = \mathrm{Alg}(\mathbf{C}) \times_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))} \{hA\}$ by definition. An equivalent description is the fiber sequence $\mathcal{M} \to \mathrm{Alg}(\mathbf{C})^{\simeq} \to \mathrm{Alg}(\mathrm{ho}(\mathbf{C}))^{\simeq}$ at the point $hA \in \mathrm{Alg}(\mathrm{ho}(\mathbf{C}))^{\simeq}$.

Looping once at A, we find the fiber sequence

$$\Omega\mathcal{M} \to \Omega(\mathrm{Alg}(\mathbf{C})^{\simeq}, A) \to \Omega(\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))^{\simeq}, hA)$$

and thus

$$\Omega\mathcal{M} \to \mathrm{Aut}_{\mathrm{Alg}(\mathbf{C})}(A) \to \mathrm{Aut}_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))}(hA)$$

As an endomorphism of A which is the identity in $\mathrm{ho}(\mathbf{C})$ must be an equivalence, this also yields a fiber sequence

$$\Omega\mathcal{M} \to \mathrm{Map}_{\mathrm{Alg}(\mathbf{C})}(A, A) \to \hom_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))}(hA, hA)$$

at id_{hA}. As the latter is discrete, we deduce point (i).

Next, we use the fully faithful embedding[12]

$$\mathrm{Alg}(\mathbf{C}) \to (\mathrm{RMod}_{\mathbf{C}}(\mathrm{Pr}^{\mathrm{L}}))_{\mathbf{C}/}, A \mapsto (\mathrm{LMod}_A, A)$$

to describe $\mathrm{Map}_{\mathrm{Alg}(\mathbf{C})}(A, A)$ as the fiber of

$$\mathrm{Map}_{\mathrm{RMod}_{\mathbf{C}}}(\mathrm{LMod}_A, \mathrm{LMod}_A) \to \mathrm{Map}_{\mathrm{RMod}_{\mathbf{C}}}(\mathbf{C}, \mathrm{LMod}_A)$$

over the canonical functor $\mathbf{C} \to \mathrm{LMod}_A$ classified by $A \in \mathrm{LMod}_A$. Using [Lur12, Theorem 4.8.4.1], we rewrite this map as the functor $\mathrm{BiMod}_A^{\simeq} \to \mathrm{LMod}_A^{\simeq}$ that forgets the right A-module structure. Taking loops at the identity of $A \in \mathrm{Alg}(\mathbf{C})$ yields the fiber sequence

$$\Omega(\mathrm{Map}_{\mathrm{Alg}(\mathbf{C})}(A, A), \mathrm{id}_A) \to \Omega(\mathrm{BiMod}_A^{\simeq}, A) \to \Omega(\mathrm{LMod}_A^{\simeq}, A)$$

We rewrite the latter two terms as $\mathrm{Aut}_{\mathrm{BiMod}_A}(A) \to \mathrm{Aut}_A(A)$ and use again the fact that any A-bimodule endomorphism of A which is an underlying equivalence is an equivalence to rewrite the fiber of this map as the fiber of

$$\mathrm{Map}_{\mathrm{BiMod}_A}(A, A) \to \mathrm{Map}_A(A, A)$$

over the identity. This proves the first half of (ii), and using the additivity of $\mathbf{C}$ and the fact that this is a map of grouplike $\mathbb{E}_\infty$-monoids which has both id_A and 0 in its image, we deduce that the fiber is the same over 0, which is the second half of (ii).

Finally, (iii) follows from the second half of (ii): we rewrite $\mathrm{Map}_A(A, A)$ as $\mathrm{Map}_{\mathrm{BiMod}_A}(A \otimes A^{\mathrm{op}}, A)$ and then the restriction map

$$\mathrm{Map}_{\mathrm{BiMod}_A}(A, A) \to \mathrm{Map}_A(A, A) \simeq \mathrm{Map}_{\mathrm{BiMod}_A}(A \otimes A^{\mathrm{op}}, A)$$

[12] Cf. again [Lur12, Theorem 4.8.5.11].

is identified with precomposition by the multiplication map $\mu : A \otimes A^{\mathrm{op}} \to A$ (indeed, the multiplication is just the co-unit of the adjunction that forgets the right A-module structure), so that by (ii), our double-loop space is identified with $\mathrm{map}_{\mathrm{BiMod}_A}(\mathrm{cofib}(\mu), A)$. To conclude, we use that $\mathrm{cofib}(\mu) \simeq \Sigma L_A$ [Lur12, Theorem 7.3.5.1.][13]. □

Remark 1.2.28. If $\mathbf{C}$ is $\mathbb{E}_2$-monoidal so that ${}_A\mathrm{BiMod}_A(\mathbf{C})$ can be considered a left $\mathbf{C}$-module, then $\mathrm{Map}_{{}_A\mathrm{BiMod}_A}(X, A)$ can also be described as $\mathrm{Map}(\mathbf{1}, \hom_{{}_A\mathrm{BiMod}_A}(X, A))$. So unless $\mathrm{Map}(\mathbf{1}, Z(A)) \to \mathrm{Map}(\mathbf{1}, A)$ is an inclusion of components, $\Omega^2(\mathcal{M}, A)$ is not contractible, and therefore $\mathcal{M}$ isn't either. Here, $Z(A)$ is the $\mathbb{E}_1$-center of A, from Notation 1.1.30. ◁

Example 1.2.29. Consider the commutative differential graded $\mathbb{Q}$-algebra $R = \mathbb{Q}[t]$, where t is in degree 2 as a commutative ring spectrum and let $\mathbf{C} = \mathrm{Mod}_R$. For any $n \geq 1$, the matrix ring $M_n(R) = \mathrm{End}_R(R^n)$ is separable, and its center is R itself. Therefore $\mathrm{Map}_{A \otimes A^{\mathrm{op}}}(A, A) \to \mathrm{Map}_A(A, A)$ is the map $\Omega^\infty(\mathbb{Q}[t] \to M_n(\mathbb{Q}[t]))$ and this is clearly not an inclusion of components as long as $n \geq 2$ (it is not surjective in any $\pi_{2k}, k \geq 1$). By Remark 1.2.28, the moduli space of lifts of $M_n(\mathbb{Q}[t])$ is not contractible. ◁

Remark 1.2.30. In [KT17], the authors also study a certain moduli space of algebra structures. Note, however, that this is a different moduli space in that it is the moduli space of algebra structures on an *object* of $\mathbf{C}$, while ours is the moduli space of algebra structures extending a homotopy algebra. The answers we get for the double loop space are thus different, even if there is some similarity in that both involve some version of Hochschild cohomology. ◁

We apply these results in the case of ring spectra. For an $\mathbb{E}_2$-ring spectrum R[14], let $\mathrm{ProjSep}(R)$ denote the full subgroupoid of $\mathrm{Alg}(\mathrm{Mod}_R(\mathrm{Sp}))$ spanned by separable algebras whose underlying R-module is finitely generated projective. This is clearly functorial along basechange.

The following is our version of [BRS12, Theorem 6.1] (cf. also [GL21, Proposition 3.12, Theorem 3.15]):

Proposition 1.2.31. *Let R be an $\mathbb{E}_2$-ring spectrum. In the span*

$$\mathrm{ProjSep}(R) \leftarrow \mathrm{ProjSep}(R_{\geq 0}) \to \mathrm{ProjSep}(\pi_0(R))$$

the left leg is an equivalence, and the right leg is essentially surjective, with simply-connected fibers.

In particular, any separable algebra over $\pi_0(R)$ can be (weakly uniquely) realized as π_0 of a separable algebra over R.

Proof. For any $\mathbb{E}_2$-ring spectrum R, $\mathrm{ProjSep}(R)$ can equivalently be described as the space of separable algebras in $\mathrm{Proj}(R)$, the additive monoidal ∞-category of projective R-modules. In particular, this only depends on this additive ∞-category, and it is a classical fact that $\mathrm{Proj}(R_{\geq 0}) \to \mathrm{Proj}(R)$ is an equivalence. This proves the statement about the left leg.

For the right leg, we observe that $\pi_0 : \mathrm{Proj}(R_{\geq 0}) \to \mathrm{Proj}(\pi_0(R))$ witnesses the latter as the homotopy category of the former, and thus, passing to algebras, $\mathrm{Alg}(\mathrm{Proj}(R_{\geq 0})) \to \mathrm{Alg}(\mathrm{Proj}(\pi_0(R))$ is equivalently

$$\mathrm{Alg}(\mathrm{Proj}(R_{\geq 0})) \to \mathrm{Alg}(\mathrm{ho}(\mathrm{Proj}(R_{\geq 0})))$$

The statement thus follows from Theorem 1.2.16. □

[13] Strictly speaking, [Lur12, Theorem 7.3.5.1.] is stated for symmetric monoidal ∞-categories, but in the $\mathbb{E}_m$-case it works over $\mathbb{E}_m$-monoidal ∞-categories. More to the point, in the rest of this thesis we will never actually need to know that this L_A has anything to do with the actual cotangent complex, only its description as $\mathrm{fib}(\mu)$ is relevant to us, so the skeptical reader can safely ignore this description.

[14] So that $\mathrm{Mod}_R(\mathrm{Sp})$ is a monoidal ∞-category.

Remark 1.2.32. Note that if R is a discrete commutative ring, any central separable algebra over R is necessarily finitely generated projective [AG60, Theorem 2.1]. In particular, this allows us to lift all central separable algebras. ◁

Remark 1.2.33. In [BRS12, Theorem 6.1], the lift along the right leg is said to be "unique". Remark 1.2.28 and Example 1.2.29 show that this unicity is to be taken with a grain of salt. ◁

The results of this section suggest the slogan that "Separable algebras and their modules are controlled by the homotopy category". From the perspective of homotopy theory, this justifies to some extent the study of separable algebras and their modules in tensor triangulated categories, but also suggests that many results in the unstructured setting can be lifted for free to a more structured or coherent setting. Because of the non-unicity pointed out in Remark 1.2.28, we see that not *everything* can be lifted for free.

In the next section we will see that the situation in the commutative world is much better. The moduli spaces become contractible (even the $\mathbb{E}_1$ ones!), and the mapping spaces become discrete.

1.3 Commutative separable algebras

In this section, we study commutative separable algebras. Unsurprisingly, this situation is much better behaved than in the associative case. In the commutative case, we will see that the obstructions to contractibility from Section 1.2 vanish - in fact, even in the homotopy commutative case. The key difference with the general case is that now, the multiplication map $\mu : A \otimes A \to A$ is an (homotopy) algebra map.

With a view towards future applications, it seems relevant to allow a bit more flexibility - namely, just as $\mathbf{C}$ was allowed to be only monoidal in the previous sections, in this one, we will allow $\mathbf{C}$ to be only $\mathbb{E}_m$-monoidal for some $m \geq 3$, rather than symmetric monoidal - note that this includes $m = \infty$[15]. For some of the results, $\mathbb{E}_2$ is sufficient, but $\mathbb{E}_3$ is much simpler to handle, because in that case $\mathrm{ho}(\mathbf{C})$ is actually symmetric monoidal, rather than only braided monoidal. There are also some results where we are not sure whether they work for $m = 2$ - thus, in total we will only deal with $m \geq 3$, except where it is easy to put $m = 2$.

We begin this section with a study of certain moduli spaces, and of mapping spaces from commutative separable algebras, and we then apply Lurie's deformation theory from [Lur12, Section 7.4] to compare étale algebras and separable commutative algebras.

Definition 1.3.1. Let $\mathbf{C}$ be $\mathbb{E}_m$-monoidal for some $m \geq 1$. For $1 \leq d \leq m$, a separable $\mathbb{E}_d$-algebra in $\mathbf{C}$ is an $\mathbb{E}_d$-algebra whose underlying algebra is separable.

When $d = m = \infty$, we simply say commutative, so a separable commutative algebra or perhaps commutative separable algebra. ◁

We begin with a general proposition:

Proposition 1.3.2. *Suppose $\mathbf{C}$ is additively $\mathbb{E}_m$-monoidal for some $m \geq 2$, and idempotent-complete.*

Let $A \in \mathrm{Alg}_{\mathbb{E}_m}(\mathbf{C})$ be an $\mathbb{E}_m$-algebra in $\mathbf{C}$ and $B \in \mathrm{Alg}(\mathbf{C})$ an algebra, and finally $f : A \to B$ a morphism of algebras. If f admits an A-module splitting, then B is the localization of A at an idempotent e.

In particular $\mathrm{LMod}_A(\mathbf{C})$ splits right $\mathbf{C}$-linearly as a product

$$\mathrm{LMod}_A \simeq \mathrm{LMod}_B(\mathbf{C}) \times \mathrm{LMod}_B(\mathbf{C})^{\perp}$$

[15] Thus, in every assumption of the form "$\mathbb{E}_m$, for some m", m should be interpreted as varying in $\{0, ..., \infty\}$ together with specified extra restrictions.

This proposition relies on the following result, which is simply a variant of the discussion of idempotent algebras in [Lur12, Section 4.8.2.] in the setting of *non-symmetric* monoidal ∞-categories. Parts of it work exactly the same as in the monoidal setting, but some of it does not, so we simply record it here.

First, recall the definition:

Definition 1.3.3. Let $\mathbf{M}$ be an $\mathbb{E}_k$-monoidal ∞-category with unit $\mathbf{1}$ and $k \geq 1$. An $\mathbb{E}_0$-object e therein, that is, an object in $\mathbf{M}_{\mathbf{1}/}$, is called idempotent, if both $e \otimes (\mathbf{1} \to e)$ and $(\mathbf{1} \to e) \otimes e$ are equivalences.

For $0 \leq d \leq k$, an $\mathbb{E}_d$-algebra A in $\mathbf{M}$ is called idempotent if its underlying $\mathbb{E}_0$-algebra is. ◁

Remark 1.3.4. In the case of an $\mathbb{E}_k$-monoidal ∞-category, $k \geq 2$, these two maps are homotopic and so it suffices to require one of them to be an equivalence. ◁

The main result is:

Proposition 1.3.5. *Let $\mathbf{M}$ be an $\mathbb{E}_k$-monoidal ∞-category with $1 \leq k \leq \infty$, and let $0 \leq d \leq k$. The forgetful functor from $\mathbb{E}_d$-algebras to $\mathbb{E}_0$-algebras restricts to an equivalence between the respective full subcategories of idempotent algebras:*

$$\mathrm{Alg}_{\mathbb{E}_d}(\mathbf{M})^{\mathrm{Idem}} \xrightarrow{\simeq} \mathrm{Alg}_{\mathbb{E}_0}(\mathbf{M})^{\mathrm{Idem}}$$

We begin with an easy lemma:

Lemma 1.3.6. *Let $\mathbf{M}$ be an $\mathbb{E}_k$-monoidal ∞-category, and $e \in \mathbf{M}_{\mathbf{1}/}$ an idempotent $\mathbb{E}_0$-object. The full subcategory $\mathbf{M}_e$ of $\mathbf{M}$ spanned by those m's for which both $m \otimes (\mathbf{1} \to e)$ and $(\mathbf{1} \to e) \otimes m$ are equivalences determines a full sub-$\mathbb{E}_k$-operad of $\mathbf{M}$, which is itself an $\mathbb{E}_k$-monoidal ∞-category.*

Warning 1.3.7. For $k \geq 2$, the inclusion $\mathbf{M}_e \to \mathbf{M}$ actually admits an $\mathbb{E}_k$-monoidal left adjoint, given by tensoring with e. For $k = 1$, this left adjoint is only oplax monoidal, as the canonical map $e \otimes x \otimes y \otimes e \to e \otimes x \otimes e \otimes y \otimes e$ need not be an equivalence. This is the key difference between $k = 1$ and higher k's. ◁

Proof sketch. Firstly, $\mathbf{M}_e$ is stable under the formation non-empty tensor products, so we only need to prove that it admits a unit, which we claim is e.

In other words, we need to show that for $m \in \mathbf{M}_e$, the restriction map $\mathrm{Map}(e, m) \to \mathrm{Map}(\mathbf{1}, m)$ is an equivalence. The inverse is given by

$$\mathrm{Map}(\mathbf{1}, m) \to \mathrm{Map}(e, e \otimes m) \to \mathrm{Map}(e, m)$$

- it is a diagram chase to check that this is indeed an inverse. □

Proof of Proposition 1.3.5. Forgetting down to its underlying $\mathbb{E}_d$-monoidal ∞-category, we may assume without loss of generality that $d = k$.

We first prove essential surjectivity: fix an idempotent $\mathbb{E}_0$-algebra e. By Lemma 1.3.6, there is a lax $\mathbb{E}_k$-monoidal inclusion $\mathbf{M}_e \to \mathbf{M}$, which therefore sends $\mathbb{E}_k$-algebras to $\mathbb{E}_k$-algebras, and e is the unit in the source, so it has an $\mathbb{E}_k$-algebra structure in the target as well, which proves essential surjectivity.

For fully faithfulness, fix two idempotent $\mathbb{E}_k$-algebras e, e'. We aim to prove that in both ∞-categories, the mapping space from e to e' is empty or contractible, in the same case. Clearly if the mapping space in $\mathbb{E}_0$-algebras is empty, the same holds in $\mathbb{E}_k$-algebras, so we may assume it's non-empty, and in this case, we need to prove that both are contractible.

So suppose there exists a factorization $\mathbf{1} \to e \to e'$. We claim that in this case, e' is in $\mathbf{M}_e$. By fully faithfulness of the inclusion $\mathbf{M}_e^{\otimes} \to \mathbf{M}^{\otimes}$, this will then prove the claim, as e is the unit in $\mathbf{M}_e$.

But now, note that the map $e' \to e \otimes e'$ can be composed with $e \otimes e' \to e' \otimes e' \simeq e'$, so that e' is a retract of $e \otimes e'$, which is an object m for which the map $e \to e \otimes m$ is an equivalence. It follows that e' is also such an object. Similarly for the map $e' \to e' \otimes e$. □

With this in mind, we can prove the following preliminary version of the desired result:

Lemma 1.3.8. *Let* $\mathbf{D}$ *be a semiadditively* $\mathbb{E}_k$*-monoidal* ∞*-category with unit* $\mathbf{1}$*, where* $1 \leq k \leq \infty$*. Suppose* $\mathbf{1}$ *splits as* $a \oplus b$*.*

There are essentially unique $\mathbb{E}_k$*-algebra structures on* a, b *in* $\mathbf{D}$ *for which the unit maps* $\mathbf{1} \to a, b$ *are the projections coming from this decomposition. In particular,* $\mathbf{1} \to a \times b$ *is an equivalence of algebras.*

More precisely, a, b *are idempotent algebras in* $\mathbf{D}$ *in the sense of Definition 1.3.3, and therefore have unique algebra structures extending their unit map by Proposition 1.3.5.*

Proof. We show that the projections witness a, b as idempotent $\mathbb{E}_0$-algebras.

For this, observe that

$$a \otimes a \oplus a \otimes b \oplus b \otimes a \oplus b \otimes b \simeq (a \oplus b) \otimes (a \oplus b) \simeq \mathbf{1} \otimes \mathbf{1} \simeq \mathbf{1} \simeq a \oplus b$$

Second, observe that the morphism

$$a \otimes b \to \mathbf{1} \otimes \mathbf{1} \simeq \mathbf{1}$$

factors as $a \otimes b \to a \otimes \mathbf{1} \simeq a \to \mathbf{1}$, but also as $a \otimes b \to \mathbf{1} \otimes b \simeq b \to \mathbf{1}$. In particular, $a \otimes b \to \mathbf{1}$ factors through 0, but it has a retraction, so $a \otimes b$ must be 0.

Similarly, $b \otimes a \simeq 0$. It is then just a matter of diagram chasing to see that a, b are idempotents. (Note that this diagram chases can be made in $\mathrm{ho}(\mathbf{D})$, as $\mathbf{D} \to \mathrm{ho}(\mathbf{D})$ is conservative, monoidal, and biproduct preserving). □

Proof of Proposition 1.3.2. We start by assuming $\mathbf{C}$ has geometric realizations that commute with the tensor product in each variable. Thus, because $m \geq 2$, we can make $\mathrm{LMod}_A(\mathbf{C})$ into a monoidal ∞-category with the relative tensor product[16].

We can now apply Lemma 1.3.8 to the ∞-category $\mathrm{LMod}_A(\mathbf{C})$: A is the unit, and it splits as $B \oplus C$ for some C, as $\mathbf{C}$ is additive and idempotent-complete, where the projection $A \to B$ is chosen to be f. Lemma 1.3.8 in the case $k = 1$ tells us exactly that C admits a unique algebra structure in $\mathrm{LMod}_A(\mathbf{C})$ extending its unit $A \to C$, and then $A \simeq B \times C$ as algebras, which is exactly saying that B is the localization of A at an idempotent.

As $\mathbf{C}$ is additive compatibly with the tensor product, it follows that $\mathrm{LMod}_A(\mathbf{C}) \to \mathrm{LMod}_B(\mathbf{C}) \times \mathrm{LMod}_C(\mathbf{C})$ is an equivalence, and under this identification we clearly have $\{0\} \times \mathrm{Mod}_C(\mathbf{C}) = \mathrm{Mod}_B(\mathbf{C})^{\perp}$, where for a subcategory $\mathcal{E}$, $\mathcal{E}^{\perp} := \{f \in \mathrm{Mod}_A(\mathbf{C}) \mid \forall e \in \mathcal{E}, \mathrm{Map}(e, f) \simeq \mathrm{pt} \simeq \mathrm{Map}(f, e)\}$.

To deduce the statement for general $\mathbf{C}$, we note that if $\mathbf{C} \to \mathbf{D}$ is an additive, $\mathbb{E}_m$-monoidal embedding where $\mathbf{D}$ has geometric realizations compatible with the tensor product, then

[16] This is well known if $\mathbf{C}$ is symmetric monoidal, but the following construction works in this generality: $(\mathrm{Pr}^{\mathrm{L}})^{\mathrm{Alg}} \to \mathrm{Pr}^{\mathrm{L}}, (\mathbf{C}, A) \mapsto \mathrm{LMod}_A(\mathbf{C})$ is a symmetric monoidal functor by [Lur12, Theorem 4.8.5.16], and so one can plug in algebras in $(\mathrm{Pr}^{\mathrm{L}})^{\mathrm{Alg}}$ to obtain algebras on the output - by Dunn additivity, algebras in $(\mathrm{Pr}^{\mathrm{L}})^{\mathrm{Alg}}$ correspond exactly to $\mathbb{E}_2$-monoidal ∞-categories equipped with an $\mathbb{E}_2$-algebra.

because $\mathbf{C}$ was assumed idempotent complete, the decomposition $M \simeq B \otimes_A M \oplus C \otimes_A M$ for any $M \in \mathrm{LMod}_A(\mathbf{C}) \subset \mathrm{LMod}_A(\mathbf{D})$ shows that $B \otimes_A M$ (resp. $C \otimes_A M$) is in fact in $\mathrm{LMod}_B(\mathbf{C}) \subset \mathrm{LMod}_B(\mathbf{D})$ (resp. $\mathrm{LMod}_C(\mathbf{C}) \subset \mathrm{LMod}_{\mathbf{C}}(\mathbf{D})$), which concludes the proof. □

The first consequence we wanted to reach is the following:

Corollary 1.3.9. *Suppose $\mathbf{C}$ is additively $\mathbb{E}_m$-monoidal, $m \geq 3$, and let $A \in \mathrm{Alg}_{\mathbb{E}_m}(\mathbf{C})$ be a separable $\mathbb{E}_m$-algebra, with L_A its $\mathbb{E}_1$-cotangent complex. In this case, the mapping space $\mathrm{map}_{\mathrm{BiMod}_A}(L_A, A)$ is trivial. In fact, one can make BiMod_A into a right $\mathbf{C}$-module and the corresponding hom-object in $\mathbf{C}$ is 0.*

Proof. We assume without loss of generality that $\mathbf{C}$ admits geometric realizations compatible with the tensor product, in particular it admits basechange along algebra maps.

Since $m \geq 2$, for any $B \in \mathrm{Alg}(\mathbf{C})$, $\mathrm{BiMod}_B(\mathbf{C})$ can be equipped with a right $\mathbf{C}$-module structure, for which we have a right $\mathbf{C}$-linear equivalence

$$\mathrm{BiMod}_B \simeq \mathrm{LMod}_{B \otimes B^{\mathrm{rev}}}$$

Thus the statement about the hom-object makes sense. To prove that it is 0, we assume $m \geq 3$. In that case, $\mathrm{ho}(\mathbf{C})$ is *symmetric* monoidal, and furthermore by Proposition 1.1.28, the hom-object in $\mathbf{C}$ is the same as in $\mathrm{ho}(\mathbf{C})$, so that we may assume without loss of generality that $\mathbf{C}$ is symmetric monoidal, and A is commutative.

In that case, we may consider $\mu : A \otimes A \to A$ as a commutative algebra map, and under the identification $A^{\mathrm{op}} \simeq A$, this corresponds to the A-bimodule multiplication map $A \otimes A^{\mathrm{op}} \to A$. Let L'_A denote the $A \otimes A$-module fiber of μ. Under the same identification, this corresponds to L_A.

In particular, as an $A \otimes A$-module, A can be described as $\mu^* A$, so that $\hom_{A \otimes A}(L'_A, \mu^* A) \simeq \hom_A(\mu_! L_A, A)$.

By Proposition 1.3.2, $\mu : A \otimes A \to A$ is a localization at an idempotent, so that $\mu_!$ of the fiber is trivial, and the claim follows. □

Remark 1.3.10. This corollary is the crucial difference between the commutative and the associative case. In Remark 1.2.28, we saw that it was precisely the nontriviality of $\mathrm{map}_{\mathrm{BiMod}_A}(L_A, A)$ that makes the moduli space of $\mathbb{E}_1$-algebra structures non-trivial. ◁

As a corollary, we find that in the presence of homotopy commutativity, the obstruction theory for $\mathbb{E}_1$-structures simplifies greatly. We have:

Proposition 1.3.11. *Suppose $\mathbf{C}$ is additively $\mathbb{E}_m$-monoidal, $m \geq 3$. Let $A \in \mathrm{CAlg}(\mathrm{ho}(\mathbf{C}))$ be a homotopy commutative, homotopy separable homotopy algebra. The moduli space of lifts of A to an $\mathbb{E}_1$-algebra in $\mathbf{C}$ is contractible.*

Proof. Using again the embedding $\mathrm{Sp}_{\geq 0} \to \mathrm{Sp}$, we may assume without loss of generality that $\mathbf{C}$ is stable and admits internal hom's.

Theorem 1.2.16 proves that this moduli space is simply-connected, so it suffices to prove that its double loopspace at any given point is contractible. So we fix an $\mathbb{E}_1$-algebra $\tilde{A}$ extending $A \in \mathrm{Alg}(\mathrm{ho}(\mathbf{C}))$.

By point 3. in Proposition 1.2.27, it suffices to prove that $\mathrm{map}_{\mathrm{BiMod}_{\tilde{A}}}(\Sigma L_{\tilde{A}}, \tilde{A})$ is contractible, or better, it suffices to prove that the hom object $\hom_{\mathrm{BiMod}_{\tilde{A}}}(\Sigma L_{\tilde{A}}, \tilde{A})$ is zero. The algebra $\tilde{A} \otimes \tilde{A}^{\mathrm{op}}$ is separable, so by Proposition 1.1.28, this hom object can be computed in $\mathrm{ho}(\mathbf{C})$.

It now follows from Corollary 1.3.9 that it is 0 - note that as $\tilde{A} \otimes \tilde{A}^{\mathrm{op}} \to \tilde{A}$ is split, its fiber can be computed in $\mathbf{C}$ or in $\mathrm{ho}(\mathbf{C})$ equivalently. □

We now explore other consequences of this orthogonality, namely the uniqueness of the separability idempotent of a commutative separable algebra, and we deduce from it nice descent properties of separable algebras. This uniqueness is well-known classically, and Naumann and Pol have also isolated it, as well as the resulting descent properties, in their recent work, cf. [NP23, Lemma 6.2, Proposition 6.3].

Corollary 1.3.12. *Assume $\mathbf{C}$ is additively $\mathbb{E}_m$-monoidal, $m \geq 3$. Let $A \in \mathrm{Alg}_{\mathbb{E}_m}(\mathbf{C})$ be a separable $\mathbb{E}_m$-algebra. The space of separability idempotents for A is contractible. More precisely, we define this space is the fiber of*

$$\mathrm{map}_{\mathrm{BiMod}_A}(A, A \otimes A^{\mathrm{op}}) \to \mathrm{map}_{\mathrm{BiMod}_A}(A, A)$$

over id_A.

Proof. This is a map of grouplike $\mathbb{E}_\infty$-monoids, and id_A is in the image by separability, so the fiber is the same as the fiber over 0, so it is equivalent to $\mathrm{map}_{\mathrm{BiMod}_A}(A, L_A)$.

This is contractible by the same reasoning as in Corollary 1.3.9. □

A corollary of this uniqueness of separability idempotents is the fact that for commutative algebras, separability can be checked locally. We first make the following definition:

Definition 1.3.13. We let $\mathrm{CSep}_m(\mathbf{C}) \subset \mathrm{Alg}_{\mathbb{E}_m}(\mathbf{C})^{\simeq}$ denote the subspace spanned by separable algebras. ◁

The statement of locality can then be phrased as follows:

Corollary 1.3.14. *Let $m \geq 3$. The functor $\mathbf{C} \mapsto \mathrm{CSep}_m(\mathbf{C})$, defined on additively $\mathbb{E}_m$-monoidal ∞-categories, is limit-preserving.*

Remark 1.3.15. The corresponding statement for separable $\mathbb{E}_1$-algebras is wrong. We will give a counterexample involving Azumaya algebras and based on [GL21] in Example 2.1.11. ◁

Proof. The proof is similar to the corresponding claim for dualizable objects, cf. [Lur12, Proposition 4.6.11.].

Consider the space of "$\mathbb{E}_m$-algebras equipped with a separability idempotent", namely the space of tuples $(A, s : A \to A \otimes A^{\mathrm{op}}, h)$ where s is a map of bimodules $A \to A \otimes A^{\mathrm{op}}$, and h a homotopy witnessing that $\mu \circ s \simeq \mathrm{id}_A$ in A-bimodules.

The functor that assigns this space to $\mathbf{C}$ is clearly limit preserving in $\mathbf{C}$ (it can be written as a limit of spaces that are limit-preserving functors of $\mathbf{C}$), and the projection down to $\mathrm{Alg}_{\mathbb{E}_m}(\mathbf{C})^{\simeq}$, which is natural in $\mathbf{C}$, establishes, by Corollary 1.3.12, an equivalence with $\mathrm{CSep}_m(\mathbf{C})$. The claim thus follows. □

Remark 1.3.16. If one thinks of "descent"-type statements as statements about recovering a ($\mathbb{E}_m$-monoidal) ∞-category as a limit of other ($\mathbb{E}_m$-monoidal) ∞-categories, this result can be interpreted as saying that commutative separable algebras satisfy descent.

For instance, if $\mathbf{1} \to A$ is a universal descent morphism in the sense of [Lur18b, Definition D.3.1.1] (e.g. an étale cover in $\mathrm{CAlg}(\mathrm{Sp})$), one sees that an algebra $R \in \mathrm{CAlg}(\mathbf{C})$ is separable if and only if $A \otimes R \in \mathrm{CAlg}(\mathrm{Mod}_A)$ is separable: one can check separability (of *commutative* algebras) after passing to a (universal descent) cover. ◁

In the above proof, we have used implicitly the following lemma, which we record explicitly:

Lemma 1.3.17. *The functor* $\mathrm{Alg}_{\mathbb{E}_m} : \mathrm{Alg}_{\mathbb{E}_m}(\mathrm{Cat}) \to \mathrm{Cat}$ *preserves limits. This is more generally true for the functor* $\mathrm{Alg}_{\mathcal{O}}$*, for any* ∞*-operad* $\mathcal{O}$ *over* $\mathbb{E}_m$*.*

Furthermore, given a limit diagram $\mathbf{C}_\bullet : I^{\triangleleft} \to \mathrm{Alg}_{\mathbb{E}_m}(\mathrm{Cat})$*, and an algebra object* $A \in \mathrm{Alg}_{\mathbb{E}_1}(\mathbf{C}_\infty)$[17]*, the canonical map* $\mathrm{LMod}_A(\mathbf{C}_\infty) \to \lim_I \mathrm{LMod}_{A_i}(\mathbf{C}_i)$ *is an equivalence, where* A_i *is the image of* A *under the induced functor* $\mathrm{Alg}(\mathbf{C}_\infty) \to \mathrm{Alg}(\mathbf{C}_i)$

Proof. The first part follows from the existence of envelopes, see [Lur12, Proposition 2.2.4.9]. In more detail, for any ∞-operad $\mathcal{O}$ over $\mathbb{E}_m$, there is an $\mathbb{E}_m$-monoidal ∞-category $\mathrm{Env}_{\mathbb{E}_m}(\mathcal{O})$ with an $\mathcal{O}$-algebra $U_{\mathcal{O}} \in \mathrm{Alg}_{\mathcal{O}}(\mathrm{Env}_{\mathbb{E}_m}(\mathcal{O}))$ such that evaluation at $U_{\mathcal{O}}$ induces an equivalence

$$\mathrm{Fun}^{\otimes}_{\mathbb{E}_m}(\mathrm{Env}_{\mathbb{E}_m}(\mathcal{O}), \mathbf{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathbf{C}),$$

natural in $\mathbf{C}$. Since the source of this equivalence clearly preserves limits in $\mathbf{C}$, the claim follows.

The second part is a corollary of the first: let $\mathcal{LM}$ denote the ∞-operad that classifies left modules [Lur12, Section 4.2.1], and Ass the associative operard, with its canonical inclusion $\mathrm{Ass} \to \mathcal{LM}$ which induces the canonical forgetful functor $\mathrm{Alg}_{\mathcal{LM}} \to \mathrm{Alg}$, and similarly over $\mathbb{E}_m$.

We can then write $\mathrm{LMod}_A(\mathbf{C}) = \mathrm{Alg}_{\mathcal{LM}}(\mathbf{C}) \times_{\mathrm{Alg}(\mathbf{C})} \{A\}$. By the first part of the statement, it follows that $(\mathbf{C}, A) \mapsto \mathrm{LMod}_A(\mathbf{C})$ is a pullback of limit-preserving functors of the pair $(\mathbf{C}, A)$, and is thus itself a limit-preserving functor. $\square$

Corollary 1.3.18. *Let* $m \geq 3$*. The functor* $\mathrm{Alg}^{sep}_{\mathbb{E}_m}(-)$*, that assigns to an additively* $\mathbb{E}_m$*-monoidal* ∞*-category* $\mathbf{C}$ *the full subcategory of* $\mathrm{Alg}_{\mathbb{E}_m}(\mathbf{C})$ *spanned by separable algebras, is limit-preserving.*

Proof. Generally, if $f, g : S \to \mathrm{Cat}_\infty$ are functors, g preserves limits and $i : f \to g$ is a pointwise fully faithful natural transformation, then f preserves limits if and only if $f^{\simeq} : E \to \mathcal{S}$ does. "Only if" is clear as $(-)^{\simeq} : \mathrm{Cat}_\infty \to \mathcal{S}$ preserves limits.

To prove "if", we note that limits of fully faithful functors are fully faithful. It follows that for any diagram $X : I \to E$, in the following commutative square

$$\begin{tikzcd}
f(\lim_I X) \arrow[r] \arrow[d, "i"] & \lim_I f(X) \arrow[d, "i"] \\
g(\lim_I X) \arrow[r] & \lim_I g(X)
\end{tikzcd}$$

the vertical arrows and the bottom horizontal arrow are all fully faithful. Therefore, so is the top horizontal arrow. Thus, to prove that it is an equivalence, we simply need to check that it is essentially surjective, but this follows from $f^{\simeq}$ preserving limits.

We apply this to $f = \mathrm{Alg}^{sep}_{\mathbb{E}_m}, g = \mathrm{Alg}_{\mathbb{E}_m}(-)$: we observed above that g and $f^{\simeq} = \mathrm{CSep}_m(-)$ preserved limits. $\square$

We now move on to the main theorem of this section, which concerns highly coherent commutative structures on separable algebras.

Theorem 1.3.19. *Let* $\mathbf{C}$ *be an additively* $\mathbb{E}_m$*-monoidal* ∞*-category where* $m \geq 3$*, and* $A \in \mathrm{Alg}(\mathbf{C})$ *a separable algebra.*

If A *is homotopy commutative, then it has an essentially unique* $\mathbb{E}_m$*-structure extending its given* $\mathbb{E}_1$*-structure.*

[17] We use "∞" to denote the cone point in $I^{\triangleleft}$.

Remark 1.3.20. More generally, but as a consequence of the way the theorem is stated, for any $1 \leq n \leq m$, A has an essentially unique $\mathbb{E}_n$-structure extending its given $\mathbb{E}_1$-structure. $\triangleleft$

Remark 1.3.21. We note that this is an obvious commutative analogue of Theorem 1.2.16, but that, as with Proposition 1.3.11, the situation is better in the commutative world. $\triangleleft$

We also note the important corollary that all the previous work in the section, about commutative separable algebras therefore also applies in the case of homotopy commutative separable algebras. Combined with Proposition 1.3.11, this yields:

Corollary 1.3.22. *Let* $\mathbf{C}$ *be an additively* $\mathbb{E}_m$*-monoidal* ∞*-category with* $m \geq 3$*, and* $A \in \mathrm{Alg}(\mathrm{ho}(\mathbf{C}))$ *a homotopy separable homotopy commutative homotopy algebra.*

It has an essentially unique $\mathbb{E}_m$*-structure extending its given homotopy algebra structure.*

Remark 1.3.23. This theorem is consistent with the experience that all commutative separable algebras in tensor triangulated categories coming from stably symmetric monoidal ∞-categories admit highly coherent structures. $\triangleleft$

Remark 1.3.24. This corollary should be reminiscent of the Goerss–Hopkins–Miller theorem [GH05]. However, Morava E-theory is *not* separable. In Section 1.4.3, we introduce the notion of an *ind-separable* algebra to make up for this defect, and observe that Morava E-theories are examples of such things. We deduce extensions of the Goerss–Hopkins–Miller theorem to other ∞-operads than $\mathbb{E}_1$ and $\mathbb{E}_\infty$ (cf. Theorem 1.3.28 below and Corollary 1.4.53) - these are well-known to experts but do not seem to be recorded in the literature. $\triangleleft$

In fact, we deduce Theorem 1.3.19 from a more general statement. To state it, we introduce a certain class of ∞-operads over $\mathbb{E}_m$ which contains the $\mathbb{E}_n, 1 \leq n \leq m$.

Notation 1.3.25. Let $\mathcal{O}$ be an ∞-operad, with a single color x, i.e. $\mathcal{O}^{\otimes}_{\langle 1 \rangle}$ has a single object up to equivalence. By definition of an ∞-operad, it follows that $\mathcal{O}^{\otimes}_{\langle n \rangle}$ has a unique object up to equivalence too, denoted $x \oplus ... \oplus x$ (see [Lur12, Remark 2.1.1.15] for the notation). In this case, we let $\mathcal{O}(n)$ denote the space of n-ary operations. In more detail, letting $\mu_n : \langle n \rangle \to \langle 1 \rangle$ denote the unique active morphism in Fin_*, we put:

$$\mathcal{O}(n) := \mathrm{map}_{\mathcal{O}^{\otimes}}(x \oplus ... \oplus x, x) \times_{\mathrm{hom}_{\mathrm{Fin}_*}(\langle n \rangle, \langle 1 \rangle)} \{\mu_n\}$$

. $\triangleleft$

Definition 1.3.26. Let $\mathcal{O}$ be an ∞-operad. We say it is *weakly reduced* if:

- It has a single color, i.e. its underlying ∞-category has a unique object x up to equivalence.
- Both $\mathcal{O}(0)$ and $\mathcal{O}(1)$ are connected.

$\triangleleft$

Example 1.3.27. The $\mathbb{E}_n$-operads, $1 \leq n \leq \infty$ are weakly reduced ∞-operads. $\triangleleft$

Our more general statement can thus be stated as:

Theorem 1.3.28. *Let* $\mathbf{C}$ *be an additively* $\mathbb{E}_m$*-monoidal* ∞*-category,* $m \geq 3$*, and* $A \in \mathrm{Alg}(\mathbf{C})$ *a separable algebra which is homotopy commutative.*

For any weakly reduced ∞*-operad* $\mathcal{O}$ *equipped with a morphism of* ∞*-operads* $\mathcal{O} \to \mathbb{E}_{m-1}$*, the space of* $\mathcal{O} \otimes \mathbb{E}_1$*-structures on* A *extending the given* $\mathbb{E}_1$*-structures is contractible. Here,* $\otimes$ *denotes the Boardman-Vogt tensor product of* ∞*-operads following [Lur12, section 2.2.5], and* $\mathcal{O} \otimes \mathbb{E}_1$ *is viewed as an* ∞*-operad over* $\mathbb{E}_m$ *using the Dunn additivity equivalence* $\mathbb{E}_{m-1} \otimes \mathbb{E}_1 \simeq \mathbb{E}_m$*.*

Remark 1.3.29. We note that the Boardmann–Vogt tensor product has the relevant universal property also in the relative setting, essentially by definition - see [Lur12, Construction 3.2.4.1, Proposition 3.2.4.3]. Thus for $\mathcal{O} \to \mathbb{E}_{m-1}$, and $\mathbf{C}$ an $\mathbb{E}_m$-monoidal ∞-category, there is a canonical $\mathbb{E}_{m-1}$-monoidal structure on $\mathrm{Alg}_{\mathbb{E}_1}(\mathbf{C})$ and a canonical equivalence

$$\mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathbf{C}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathbb{E}_1}(\mathbf{C}))$$

As usual, this is well known in the case of $m = \infty$, but the constructions from [Lur12] are made to work in this generality too. ◁

Let us briefly describe the strategy of proof of Theorem 1.3.28, so that we can also explain the hypotheses on $\mathcal{O}$. We will expand on Theorem 1.2.14, by proving that under the homotopy commutativity assumption, $\mathrm{map}_{\mathrm{Alg}(\mathbf{C})}(A, R)$ has lots of discrete components, in fact, enough to guarantee that each $\mathrm{map}_{\mathrm{Alg}(\mathbf{C})}(A^{\otimes n}, A)$ is discrete and therefore[18] equivalent to $\hom_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))}(hA^{\otimes n}, hA)$.

From this, it follows at once that $\mathcal{O}$-algebra structures on A in $\mathrm{Alg}_{\mathbb{E}_1}(\mathbf{C})$, i.e. $\mathcal{O} \otimes \mathbb{E}_1$-algebra structures on A extending the given algebra structure, are equivalent to $\mathcal{O}$-algebra structures on hA in $\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))$. As hA is commutative and $\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))$ is a 1-category, the assumptions on $\mathcal{O}$ will then guarantee that there is a unique such structure.

We thus begin with:

Proposition 1.3.30. *Assume $\mathbf{C}$ is additively $\mathbb{E}_m$-monoidal, with $m \geq 2$. Let $A \in \mathrm{Alg}(\mathbf{C})$ be separable and homotopy commutative, and let $R \in \mathrm{Alg}(\mathbf{C})$ be arbitrary. Let $f : A \to R$ be a map in $\mathrm{Alg}(\mathbf{C})$, and suppose that it is homotopy-central, i.e. the following two maps[19] are equivalent in $\mathbf{C}$:*

$$A \otimes R \xrightarrow{f\otimes \mathrm{id}} R \otimes R \to R$$

and

$$A \otimes R \simeq R \otimes A \xrightarrow{\mathrm{id}\otimes f} R \otimes R \to R$$

In this situation, $\Omega(\mathrm{map}_{\mathrm{Alg}(\mathbf{C})}(A, R), f)$ is contractible, i.e. the component of f in $\mathrm{map}_{\mathrm{Alg}(\mathbf{C})}(A, R)$ is contractible.

Proof. Recall from [Lur12, Theorems 4.8.4.1 and 4.8.5.11] that $\mathrm{map}_{\mathrm{Alg}(\mathbf{C})}(A, R)$ is equivalent to the fiber over R of the forgetful map ${}_R\mathrm{BiMod}_A^{\simeq} \to \mathrm{LMod}_R^{\simeq}$.

It follows that $\Omega(\mathrm{map}_{\mathrm{Alg}(\mathbf{C})}(A, R), f)$ is equivalent to the fiber of $\mathrm{Aut}_{{}_R\mathrm{BiMod}_A}(R, R) \to \mathrm{Aut}_R(R, R)$ over id_R, where R has the (R, A)-bimodule structure induced by f. As the forgetful functor ${}_R\mathrm{BiMod}_A \to \mathrm{LMod}_R$ is conservative, this is equivalently the fiber of the corresponding mapping spaces, again at id_R. Because id_R is in the image and this map is a map of grouplike $\mathbb{E}_\infty$-spaces, the fiber over id_R is equivalent to the fiber over 0.

In other words, it suffices to prove that for every n,

$$\pi_n(\mathrm{map}_{{}_R\mathrm{BiMod}_A}(R, R), 0) \to \pi_n(\mathrm{map}_R(R, R), 0)$$

is an isomorphism, or equivalently, that

$$\pi_0(\mathrm{map}_{{}_R\mathrm{BiMod}_A}(R, \Omega^n R)) \to \pi_0(\mathrm{map}_R(R, \Omega^n R))$$

[18] By Theorem 1.2.14.

[19] In the case of $m = 2$, we have to pick an orientation for the equivalence $A \otimes R \simeq R \otimes A$. Either choice makes the statement correct.

is an isomorphism.

By adjunction, this forgetful map is equivalent to the map given by precomposition with $R \otimes A \to R$: $\pi_0(\mathrm{map}_{{}_R\mathrm{BiMod}_A}(R, \Omega^n R)) \to \pi_0(\mathrm{map}_{{}_R\mathrm{BiMod}_A}(R \otimes A, \Omega^n R))$, and because A is separable, $R \otimes A \to R$ is split, so that this map is always injective. It thus suffices to prove that it is surjective.

Note that this is a map between hom sets in $\mathrm{ho}({}_R\mathrm{BiMod}_A(\mathbf{C}))$, and by Corollary 1.2.3, it is thus equivalent to a map between hom sets in $\mathrm{RMod}_{hA}(\mathrm{ho}(\mathrm{LMod}_R))$. Furthermore, the source in both cases is free as an R-module, so by Lemma 1.2.1 it is equivalent to a map between hom sets in ${}_{hR}\mathrm{BiMod}_{hA}(\mathrm{ho}(\mathbf{C}))$. In other words, we are trying to prove that every map $R \to \Omega^n R$ of left hR-modules is right hA-linear. We note that $R, \Omega^n R$ are R-bimodules, and the right A-module structure is induced from the right R-module structure by restricting along $f : A \to R$. In other words, for both R and $\Omega^n R$, the right A-module structure is given by $M \otimes A \to M \otimes R \to M$.

It thus suffices to prove that the right A-action on $\Omega^n R$ agrees with the following map[20]: $\Omega^n R \otimes A \simeq A \otimes \Omega^n R \to R \otimes \Omega^n R \to \Omega^n R$. Indeed, this is the case for R by assumption, and it will thus follow immediately that any left hR-linear map $R \to \Omega^n R$ is also right hA-linear.

Now, by assumption, we already know that this is the case for the A-action on R, so it suffices to show that the action map of A on $\Omega^n R$ is given (up to homotopy) by

$$\Omega^n R \otimes A \simeq \Omega^n (R \otimes A) \to \Omega^n R$$

where the second map is $\Omega^n \rho$, ρ being the right action of A on R. But this is clear, as the left A-action on $\Omega^n R$ is obtained via restiction of scalars from the left R-action, which *is* given this way. □

Remark 1.3.31. Note that the end of this proof really identifies maps in $\mathrm{ho}(\mathbf{C})$, and there is no coherence claim. This is what the results from Section 1.2 buy us. ◁

In the proof, we really use the homotopy centrality of $f : A \to R$. The result is not true in general if we drop this hypothesis, as the following example shows (in fact, in this example, R is also separable, and $\mathbf{C}$ is symmetric monoidal, i.e. $m = \infty$):

Example 1.3.32. We start by a computation in a homotopy category, namely, consider $\mathbf{D}$ the symmetric monoidal 1-category of $\mathbb{Z}/2d$-graded $\mathbb{Q}$-vector spaces, where d is some odd integer different from 1. Let H be some group with a nontrivial automorphism α of order d, and consider the corresponding semi-direct product $G := H \rtimes \mathbb{Z}/d$, with projection map $p : G \to \mathbb{Z}/d$ and section $i : \mathbb{Z}/d \to G$.

We let $A = \mathbb{Q}[u]$ where $|u| = 2$, and $u^d = 1$, and $R = \bigoplus_{g \in G} \mathbb{Q}[2p(g)]$, where the algebra structure is the natural one, namely, given by

$$\mathbb{Q}[2p(g)] \otimes \mathbb{Q}[2p(g')] \to \mathbb{Q}[2p(gg')]$$

(note that A is given by the same construction, replacing G by $\mathbb{Z}/d$). Both A and R are separable in $\mathbf{D}$. Let $f : A \to R$ be given by the section i, i.e. u maps to the generator of the $i(\sigma)$ summand in R - this is easily checked to be an algebra map.

Given $h \in H$ such that $\alpha(h) \neq h$, let $g_0 = (h, 1)$, and consider the corresponding element r_0, corresponding to $1 \in \mathbb{Q}[2p(g_0)] = \mathbb{Q}[2]$ which corresponds to a left R-linear map $R \to R[-2]$. We claim that this left R-linear map is not right A-linear. Indeed, it is given by $r \mapsto rr_0$, and right A-linearity would be the claim that $rar_0 = rr_0a$ which, when $r = 1$, is

[20] In the case $m = 2$, the first swap map is the one in the *opposite* direction as the one in the statement of the proposition.

the claim that $ar_0 = r_0 a$, which can be checked to be wrong, essentially because i does not land in the center of G. More precisely, when $a = u$, $r_0 a, ar_0$ live in different summands of $\bigoplus_{g\in G} \mathbb{Q}[2p(g)]$, one of them in the summand corresponding to (h,σ), and the other in the summand corresponding to $(\alpha(h),\sigma)$.

We claim that this is now enough to give a counterexample to the previous proposition when f is not homotopy central. Indeed, consider $S = \mathbb{Q}[t^{\pm 1}]$ as a commutative algebra in $\mathrm{Mod}_{\mathbb{Q}}$, where t has degree $2d$. The homotopy category of Mod_S is symmetric monoidally equivalent to $\mathbf{D}$, and the suspension on Mod_S corresponds to shifting in $\mathbf{D}$. Further, because A, R are separable in $\mathbf{D}$, they can be lifted to algebras in Mod_S in a weakly unique way by Theorem 1.2.16, similarly for the map $f : \tilde{A} \to \tilde{R}$[21], and because A is commutative in $\mathbf{D}$, Theorem 1.3.19 implies that $\tilde{A}$ is commutative in a unique way. We are now left with proving that $\Omega(\mathrm{map}_{\mathrm{Alg}_S}(\tilde{A},\tilde{R}), f)$ is not discrete. By the analysis in the previous proof, it suffices to prove that $\hom_{R\otimes A}(R, \Omega^2 R) \to \hom_R(R,R)$ is not surjective. We have just done this! $\triangleleft$

As an immediate corollary, we find:

Corollary 1.3.33. *Assume* $\mathbf{C}$ *is additively* $\mathbb{E}_m$*-monoidal,* $m \geq 2$*, and let* $A \in \mathrm{Alg}(\mathbf{C})$ *be separable and homotopy commutative, and* $R \in \mathrm{Alg}(\mathbf{C})$ *be homotopy commutative.*

In this case, $\mathrm{map}_{\mathrm{Alg}(\mathbf{C})}(A,R)$ *is discrete and equivalent (via the canonical map) to* $\hom_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))}(hA, hR)$.

Proof. This follows from Proposition 1.3.30 as any map $f : A \to R$ is homotopy central, by homotopy commutativity of R. $\square$

Corollary 1.3.34. *Assume* $\mathbf{C}$ *is additively* $\mathbb{E}_m$*-monoidal,* $m \geq 3$*, and let* $A \in \mathrm{Alg}(\mathbf{C})$ *be separable and homotopy commutative.*

In this case, for any $n \geq 0$*,* $\mathrm{map}_{\mathrm{Alg}(\mathbf{C})}(A^{\otimes n}, A)$ *is discrete and equivalent to* $\hom_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C})}(hA^{\otimes n}, hA)$.

Proof. This is an immediate corollary of Corollary 1.3.33, using that because $m \geq 2$, $A^{\otimes n}$ is also separable by Lemma 1.1.6, and because $m \geq 3$, $A^{\otimes n}$ is also homotopy commutative. $\square$

We can now prove Theorem 1.3.28:

Proof of Theorem 1.3.28. By Corollary 1.3.34, the ($\mathbb{E}_{m-1}$ monoidal) forgetful functor $\mathrm{Alg}(\mathbf{C}) \to \mathrm{Alg}(\mathrm{ho}(\mathbf{C}))$ restricts to a ($\mathbb{E}_{m-1}$-monoidal) equivalence between the full subcategories spanned by $A^{\otimes n}, n \geq 0$ and $hA^{\otimes n}, n \geq 0$ respectively.

It therefore induces an equivalence between the space of $\mathcal{O}$-algebra structures on A, and the space of $\mathcal{O}$-algebra structures on hA, for any single-colored operad $\mathcal{O}$ over $\mathbb{E}_{m-1}$.

Now, $m \geq 3$, so $\mathrm{ho}(\mathbf{C})$ is symmetric monoidal. Furthermore, hA is commutative, so that the $\mathcal{O}$-algebra structures on hA in $\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))$ are the same thing as $\mathcal{O}$-algebra structures in $\mathrm{CAlg}(\mathrm{ho}(\mathbf{C}))$, which is cocartesian $\mathbb{E}_{m-1}$-monoidal (and hence symmetric monoidal). It follows that these are simply $\mathcal{O}$-algebra structures on hA in $\mathrm{CAlg}(\mathrm{ho}(\mathbf{C}))$.

Therefore, by [Lur12, Proposition 2.4.3.9], the assumption that $\mathcal{O}$ is weakly reduced guarantees that the space of such structures is contractible[22]. $\square$

[21] In this case, the algebras in question are just Thom spectra so one can actually give a relatively easy construction both of the algebras and the map, as well as a proof that they are separable. This is expanded upon in Example 1.5.17.

[22] In this case, $\mathrm{CAlg}(\mathrm{ho}(\mathbf{C}))$ is a 1-category, so we do not really need anything as sophisticated as [Lur12, Proposition 2.4.3.9]

Proof of Theorem 1.3.19. The operad $\mathbb{E}_\infty$ is clearly weakly reduced, and by [Lur12, Corollary 5.1.1.5, Theorem 5.1.2.2], $\mathbb{E}_\infty \otimes \mathbb{E}_1 \simeq \mathbb{E}_\infty$.

Similarly, by Example 1.3.27, the operads $\mathbb{E}_n$ are weakly reduced for $1 \leq n \leq \infty$, and again by [Lur12, Theorem 5.1.2.2], $\mathbb{E}_{n+1} \simeq \mathbb{E}_n \otimes \mathbb{E}_1$. □

In fact, for $m \geq 3$ we could have guessed ahead of time that A could be made at least $\mathbb{E}_2$ in a canonical way, via the following elementary observation:

Observation 1.3.35. Assume $\mathbf{C}$ is additively $\mathbb{E}_m$-monoidal. Let $A \in \mathrm{Alg}(\mathbf{C})$ be a separable algebra, which is homotopy commutative. The forgetful map

$$Z(A) = \hom_{A \otimes A^{\mathrm{op}}}(A, A) \to A$$

(cf. Notation 1.1.30) is an equivalence of algebras, and in particular A admits an $\mathbb{E}_2$-algebra structure, as $Z(A)$ always does for $m \geq 3$. ◁

Indeed, by Proposition 1.1.28, this internal hom is preserved by passage to the homotopy category, so it suffices to prove the claim there. But now, hA is literally a commutative algebra in a symmetric monoidal 1-category, so the claim is obvious.

Remark 1.3.36. We can rephrase Theorem 1.3.28 as saying that for any weakly reduced operad $\mathcal{O}$, the forgetful map $\mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathbf{C})^{\simeq} \to \mathrm{Alg}(\mathbf{C})^{\simeq}$ has trivial fibers over separable, homotopy commutative algebras, and in particular is an equivalence when restricted to the appropriate components. ◁

A special case of the previous remark is that $\mathrm{CAlg}(\mathbf{C})^{\simeq} \to \mathrm{Alg}(\mathbf{C})^{\simeq}$ is an equivalence when restricted to the components of algebras that are homotopy commutative and separable in the target. We note a corollary of this:

Corollary 1.3.37. *Let $h\mathrm{CSep}(\mathbf{C})$ denote the full subspace of $\mathrm{Alg}(\mathbf{C})^{\simeq}$ spanned by the separable, homotopy commutative algebras. The functor $\mathbf{C} \mapsto h\mathrm{CSep}(\mathbf{C})$, defined on additively $\mathbb{E}_m$-monoidal ∞-categories, $m \geq 3$, is limit-preserving.*

Proof. This follows from Corollary 1.3.14 and Remark 1.3.36 in the case $\mathcal{O} = \mathrm{Comm}$: the natural map $\mathrm{CSep}(\mathbf{C}) \to h\mathrm{CSep}(\mathbf{C})$ is an equivalence. □

Corollary 1.3.38. *The functor $\mathbf{C} \mapsto \mathrm{CSep}_m(\mathbf{C})$, defined on additively $\mathbb{E}_m$-monoidal ∞-categories, $m \geq 3$, preserves filtered colimits.*

Proof. By Corollary 1.3.22, Proposition 1.2.9 and Corollary 1.3.33, it is equivalent to $\mathbf{C} \mapsto \mathrm{CSep}(\mathrm{ho}(\mathbf{C}))$.

Now $\mathbf{C} \mapsto \mathrm{ho}(\mathbf{C})$ preserves filtered colimits and the fact that $\mathrm{CSep}(-)$ also does on 1-categories is elementary: the definition of a commutative separable algebra in a 1-category involves finitely many objects, morphisms and equations. □

We also note the following corollary of Corollary 1.3.33:

Corollary 1.3.39. *Assume $\mathbf{C}$ is additively $\mathbb{E}_m$-monoidal, $m \geq 3$.*

Let $A, R \in \mathrm{Alg}_{\mathbb{E}_m}(\mathbf{C})$. If A is separable, then the forgetful maps

$$\mathrm{map}_{\mathrm{Alg}_{\mathbb{E}_m}(\mathbf{C})}(A, R) \to \mathrm{map}_{\mathrm{Alg}(\mathbf{C})}(A, R) \to \hom_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))}(hA, hR)$$

are equivalences.

More generally, if $\mathcal{O}$ is any single-colored ∞-operad equipped with a map $\mathcal{O} \to \mathbb{E}_{m-1}$ and $R \in \mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathbf{C})$ is an algebra whose underlying $\mathbb{E}_1$-algebra is homotopy commutative, then, viewing A as an $\mathcal{O} \otimes \mathbb{E}_1$-algebra using the map of ∞-operads $\mathcal{O} \otimes \mathbb{E}_1 \to \mathbb{E}_m$, we find that the canonical map

$$\mathrm{map}_{\mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathbf{C})}(A, R) \to \hom_{\mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathrm{ho}(\mathbf{C}))}(hA, hR)$$

is an equivalence.

Remark 1.3.40. The condition on R in the second part of the statement is automatic if $\mathcal{O}$ is weakly reduced and has at least one operation in arity 2, e.g. for $\mathcal{O} = \mathbb{E}_n, 1 \leq n \leq \infty$. ◁

To see that this really is a corollary, we first record the following classical lemma:

Lemma 1.3.41. *Let $f : \mathbf{C} \to \mathbf{D}$ be an $\mathbb{E}_m$-monoidal functor, and $A, B \subset \mathbf{C}$ two full subcategories. Assume A is closed under tensor products in $\mathbf{C}$, and furthermore assume that for every $a \in A, b \in B$, the canonical map*

$$\mathrm{map}_{\mathbf{C}}(a, b) \to \mathrm{map}_{\mathbf{D}}(f(a), f(b))$$

is an equivalence.

In this case, for any ∞-operad $\mathcal{O}$ over $\mathbb{E}_m$ and any $R \in \mathrm{Alg}_{\mathcal{O}}(A)$, viewed as an $\mathcal{O}$-algebra in $\mathbf{C}$, and any $S \in \mathrm{Alg}_{\mathcal{O}}(\mathbf{C})$ whose underlying objects are in B, the canonical map $\mathrm{map}_{\mathrm{Alg}_{\mathcal{O}}(\mathbf{C})}(R, S) \to \mathrm{map}_{\mathrm{Alg}_{\mathcal{O}}(\mathbf{D})}(f(R), f(S))$ is an equivalence.

This is in turn a consequence of:

Lemma 1.3.42. *Let $f : C \to D$ be a functor between two ∞-categories, and let $A, B \subset C$ be full subcategories such that for each $a \in A, b \in B$, the canonical map $\mathrm{map}_C(a, b) \to \mathrm{map}_D(f(a), f(b))$ is an equivalence.*

Let $X, Y : I \to C$ be two functors such that for each $i \in I, X_i \in A, Y_i \in B$. In this case, the canonical map

$$\mathrm{map}_{\mathrm{Fun}(I,C)}(X, Y) \to \mathrm{map}_{\mathrm{Fun}(I,D)}(f \circ X, f \circ Y)$$

is an equivalence.

Proof. This follows directly from the description of mapping spaces in $\mathrm{Fun}(I, C)$ as ends, cf. [GHN15, Proposition 5.1], but for the sake of completeness, we give here a more elementary proof.

Let $C_{A,B} \subset C^{\Delta^1}$ be the full subcategory spanned by arrows $a \to b$ where $a \in A, b \in B$. We note that our assumption guarantees that the following is a pullback square:

$$\begin{tikzcd} C_{A,B} \arrow[r] \arrow[d] & D^{\Delta^1} \arrow[d] \\ A \times B \arrow[r] & D \times D \end{tikzcd}$$

In particular, it remains so after taking $\mathrm{Fun}(I, -)$. We now note that

$$\mathrm{Fun}(I, C_{A,B}) \simeq \mathrm{Fun}(I, C)_{\mathrm{Fun}(I,A),\mathrm{Fun}(I,B)}$$

and $\mathrm{Fun}(I, D^{\Delta^1}) \simeq \mathrm{Fun}(I, D)^{\Delta^1}$ compatibly.

Now for X, Y as in the statement, $\mathrm{Map}(X,Y)$ is the fiber of

$$\mathrm{Fun}(I,C)^{\Delta^1} \to \mathrm{Fun}(I,C) \times \mathrm{Fun}(I,C)$$

over (X,Y), so that by fullness of $A, B \subset C$, it is also the fiber of $\mathrm{Fun}(I,C)_{\mathrm{Fun}(I,A),\mathrm{Fun}(I,B)} \to \mathrm{Fun}(I,A) \times \mathrm{Fun}(I,B)$ over (X,Y) and thus, because the above square is a pullback, the fiber of $\mathrm{Fun}(I,D)^{\Delta^1} \to \mathrm{Fun}(I,D) \times \mathrm{Fun}(I,D)$ over $(f \circ X, f \circ Y)$, i.e. $\mathrm{Map}(f \circ X, f \circ Y)$, as claimed. □

Proof of Lemma 1.3.41. For simplicity of notation, we deal with the case of $m = \infty$, but the general case is strictly analogous. The ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathbf{C})$ (resp. $\mathrm{Alg}_{\mathcal{O}}(\mathbf{D})$) is a full subcategory of $\mathrm{Fun}(\mathrm{Fin}_*, \mathbf{C}^{\otimes}) \times_{\mathrm{Fun}(\mathrm{Fin}_*,\mathrm{Fin}_*)} \{\mathrm{id}\}$ (resp. $\mathrm{Fun}(\mathrm{Fin}_*, \mathbf{D}^{\otimes}) \times_{\mathrm{Fun}(\mathrm{Fin}_*,\mathrm{Fin}_*)} \{\mathrm{id}\}$).

We can thus apply Lemma 1.3.42 here, by taking $A^{\otimes}$ to be the full suboperad of $\mathbf{C}^{\otimes}$ spanned by objects of A, and taking $B^{\otimes}$ to be the full suboperad of $\mathbf{C}^{\otimes}$ spanned by objects of B.

We simply need to check the assumptions on $f^{\otimes} : \mathbf{C}^{\otimes} \to \mathbf{D}^{\otimes}$, i.e. we need to prove that for any tuples $A_1, ..., A_n \in A, B_1, ..., B_m \in B$ with corresponding objects $\underline{A} \in \mathbf{C}^{\otimes}_{\langle n \rangle}, \underline{B} \in \mathbf{C}^{\otimes}_{\langle m \rangle}$, the canonical map

$$\mathrm{map}_{\mathbf{C}^{\otimes}}(\underline{A}, \underline{B}) \to \mathrm{map}_{\mathbf{D}^{\otimes}}(f^{\otimes}\underline{A}, f^{\otimes}\underline{B})$$

is an equivalence.

This map is a map of spaces over $\hom_{\mathrm{Fin}_*}(\langle n \rangle, \langle m \rangle)$ and so we can take fibers over a given morphism $\alpha : \langle n \rangle \to \langle m \rangle$, and because $f^{\otimes}$ is symmetric monoidal, it is compatible with the equivalences $\mathrm{map}_{\mathbf{C}^{\otimes}}(\underline{A}, \underline{B}) \simeq \prod_{i \in \langle m \rangle^o}(\bigotimes_{j \in \alpha^{-1}(i)} A_j, B_i)$; and so the claim follows from the assumption on A, B, and the fact that A is closed under tensor products. □

Proof of Corollary 1.3.39. This follows again from Corollary 1.3.33, using the (definitional) equivalence $\mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathbf{C}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathrm{Alg}_{\mathbb{E}_1}(\mathbf{C}))$, and Lemma 1.3.41 - we apply the latter to the $\mathbb{E}_{m-1}$-monoidal functor $\mathrm{Alg}(\mathbf{C}) \to \mathrm{Alg}(\mathrm{ho}(\mathbf{C}))$, with the full subcategories A, B spanned on the one hand by the commutative separable algebras, and on the other hand by the homotopy commutative algebras. □

We also record the following special case explicitly:

Corollary 1.3.43. *Assume $\mathbf{C}$ is additively symmetric monoidal. Let $A \in \mathrm{CAlg}(\mathbf{C})$ be a commutative separable algebra, and $R \in \mathrm{CAlg}(\mathbf{C})$ an arbitrary commutative algebra. In this case, $\mathrm{map}_{\mathrm{CAlg}(\mathbf{C})}(A, R)$ is 0-truncated, i.e. discrete.*

It is of course a special case of the above, but to make this consequence more concrete, we give an alternative, more elementary proof that could be useful in different contexts.

Alternative proof of Corollary 1.3.43. It suffices to argue that the diagonal map $\mathrm{map}_{\mathrm{CAlg}(\mathbf{C})}(A, R) \to \mathrm{map}_{\mathrm{CAlg}(\mathbf{C})}(A, R) \times \mathrm{map}_{\mathrm{CAlg}(\mathbf{C})}(A, R)$ is an inclusion of components, i.e. a monomorphism.

Since $\mathrm{CAlg}(\mathbf{C})$ admits coproducts given by tensor products, this amounts to the claim that the multiplication map $A \otimes A \to A$ is an epimorphism in $\mathrm{CAlg}(\mathbf{C})$. But this follows immediately from it being a localization at an idempotent, cf. Proposition 1.3.2. □

1.3.1 Deformation theory and étale algebras

In the specific case where $\mathbf{C} = \mathrm{Mod}_R$, for some connective ring spectrum, we can try, as in the étale case, to relate connective commutative separable algebras to their π_0, rather than to their corresponding homotopy algebra $\mathrm{ho}(\mathbf{C})$. In that regard, the usual techniques of deformation theory work just as well as in the étale case, cf. [Lur12, Section 7.5]. We explain how this works in our situation. In fact, a big chunk of the deformation theory works for general 0-cotruncated commutative algebras. For a little while, we will therefore be in the setting of spectra and no longer a general $\mathbf{C}$ (although many of these results could be phrased more genreally in the presence of a t-structure).

Proposition 1.3.44. *Let S be a connective $\mathbb{E}_{m+1}$-ring spectrum, $m \geq 1$, R a connective $\mathbb{E}_m$ S-algebra[23], and A a 0-cotruncated connective $\mathbb{E}_m$-S-algebra.*

In this case, the canonical maps

$$\mathrm{map}_{\mathrm{Alg}_{\mathbb{E}_m,S}}(A,R) \to \mathrm{map}_{\mathrm{Alg}_{\mathbb{E}_m,\pi_0(S)}}(A \otimes_S \pi_0(S), \pi_0(R)) \to \mathrm{map}_{\mathrm{CAlg}^{\heartsuit}_{\pi_0(S)}}(\pi_0(A), \pi_0(R))$$

are equivalences.

Proof. Note that, for both maps, by adjunction it suffices to prove that the map $\mathrm{map}_{\mathrm{Alg}_{\mathbb{E}_m,S}}(A,R) \to \mathrm{map}_{\mathrm{Alg}_{\mathbb{E}_m,S}}(A,\pi_0(R))$ is an equivalence.

As $R \simeq \lim_n R_{\leq n}$, it suffices to prove that each $R_{\leq n+1} \to R_{\leq n}$ induces an equivalence $\mathrm{map}_{\mathrm{Alg}_{\mathbb{E}_m,S}}(A, R_{\leq n+1}) \to \mathrm{map}_{\mathrm{Alg}_{\mathbb{E}_m,S}}(A, R_{\leq n})$.

For this, we note that $R_{\leq n+1} \to R_{\leq n}$ is a square zero extension [Lur12, Corollary 7.4.1.28], so it suffices to prove this claim for arbitrary square zero extensions of $\mathbb{E}_m$-S-algebras by connective modules.

So let $\widetilde{R} \to R$ denote such a square zero extension, classified by a pullback square

$$\begin{tikzcd} \widetilde{R} \arrow[r, "p"] \arrow[d, "p"'] & R \arrow[d, "d_0"] \\ R \arrow[r, "d_\eta"'] & R \oplus \Sigma M \end{tikzcd}$$

where M is connective. As this is a pullback square, it remains so after applying $\mathrm{map}_{\mathrm{Alg}_{\mathbb{E}_m,S}}(A,-)$, and so, to prove that the left vertical map becomes an equivalence, it suffices to prove that this is so for the right vertical map. But the right vertical map has a left inverse, namely, the projection $R \oplus \Sigma M \to R$, so it suffices to prove that this left inverse gets sent to an equivalence.

However, this projection map $R \oplus \Sigma M \to R$ is a trivial square zero extension, so it suffices to prove the claim for these ones, i.e., extensions where the corresponding pullback square has $d_\eta \simeq d_0$. This is where we use 0-cotruncatedness: applying $\mathrm{map}_{\mathrm{Alg}_{\mathbb{E}_m,S}}(A,-)$ yields a pullback square of *sets* of the form:

$$\begin{tikzcd} \mathrm{map}_{\mathrm{Alg}_{\mathbb{E}_m,S}}(A,\widetilde{R}) \arrow[r, "p"] \arrow[d, "p"'] & \mathrm{map}_{\mathrm{Alg}_{\mathbb{E}_m,S}}(A,R) \arrow[d, "d_0"] \\ \mathrm{map}_{\mathrm{Alg}_{\mathbb{E}_m,S}}(A,R) \arrow[r, "d_0"'] & \mathrm{map}_{\mathrm{Alg}_{\mathbb{E}_m,S}}(A,R \oplus \Sigma M) \end{tikzcd}$$

[23]By which we mean an $\mathbb{E}_m$-algebra in the $\mathbb{E}_m$-monoidal ∞-category Mod_S.

Where the two d_0's really are the same map, and further are (split) injections. The pullback of sets along injections is given by the intersection of the images. But if the maps are equal, then their images are equal too, so that the intersection is the whole thing. This implies that the left vertical map is an equivalence, as was to be proved. $\square$

Proposition 1.3.45. *Let* $\mathbf{C} \mapsto \mathcal{Q}(\mathbf{C})$ *be a limit-preserving subfunctor of the functor* $\mathbf{C} \mapsto \mathrm{Alg}_{\mathbb{E}_m}(\mathbf{C})$ *defined on the* ∞*-category of* $\mathbb{E}_m$*-monoidal* ∞*-categories,* $m \geq 1$.

Let $\mathcal{Q}^{\mathrm{cn}}$ *denote the restriction of* $\mathcal{Q}$ *to the* ∞*-category* $\mathrm{Alg}^{\mathrm{cn}}_{\mathbb{E}_{m+1}}$ *of connective* $\mathbb{E}_{m+1}$*-ring spectra along* $R \mapsto \mathrm{Mod}^{\mathrm{cn}}_R$*. Suppose that the image of*

$$\mathcal{Q}^{\mathrm{cn}}(R) \to \mathrm{Alg}_{\mathbb{E}_m}(\mathrm{Mod}^{\mathrm{cn}}_R)$$

consists of 0*-cotruncated* $\mathbb{E}_m$*-algebras.*

In this case, for any connective $\mathbb{E}_{m+1}$*-ring spectrum* R*, the canonical map* $R \to \pi_0(R)$ *induces an equivalence* $\mathcal{Q}^{\mathrm{cn}}(R) \to \mathcal{Q}^{\mathrm{cn}}(\pi_0(R))$.

Proof. The argument is essentially the same as before. We use the fact that $\mathrm{Mod}^{\mathrm{cn}}_{\bullet}$ preserves the inverse limits involved in Postnikov towers, namely the limit diagrams of the form $R \simeq \lim_n R_{\leq n}$ [Lur18b, p. 19.2.1.5], and pullback squares defining square zero extensions by connective modules [Lur18b, Theorem 16.2.0.2.].

As $\mathcal{Q}$ preserves all limits, $\mathcal{Q}^{\mathrm{cn}}$ preserves these specific limits, so that, to prove that $\mathcal{Q}^{\mathrm{cn}}(R) \to \mathcal{Q}^{\mathrm{cn}}(\pi_0(R))$ is an equivalence, it suffices to prove that $\mathcal{Q}^{\mathrm{cn}}(\widetilde{R}) \to \mathcal{Q}^{\mathrm{cn}}(R)$ is an equivalence for all square zero extensions by connective modules and hence, as before, it suffices to prove it for trivial square zero extensions by connective modules.

For these ones, we note that the same argument as before using pullbacks of sets along equal injections implies, by looking at mapping spaces, that $\mathcal{Q}^{\mathrm{cn}}(R \oplus \Sigma M) \to \mathcal{Q}^{\mathrm{cn}}(R)$ is fully faithful. In more detail, we can fit this map in a pullback square:

$$\begin{tikzcd}
\mathcal{Q}^{\mathrm{cn}}(R \oplus \Sigma M) \arrow[r] \arrow[d] & \mathcal{Q}^{\mathrm{cn}}(R) \arrow[d] \\
\mathcal{Q}^{\mathrm{cn}}(R) \arrow[r] & \mathcal{Q}^{\mathrm{cn}}(R \oplus \Sigma^2 M)
\end{tikzcd}$$

where the two maps $\mathcal{Q}^{\mathrm{cn}}(R) \to \mathcal{Q}^{\mathrm{cn}}(R \oplus \Sigma^2 M)$ are equivalent, and so, looking at mapping spaces, we find pullback squares of *sets* of the form:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}$$

where the two maps $Y \to Z$ are equal and (split) injective. It follows that the map $X \to Y$ is an isomorphism as before, and hence, that $\mathcal{Q}^{\mathrm{cn}}(R \oplus \Sigma M) \to \mathcal{Q}^{\mathrm{cn}}(R)$ is fully faithful.

Because it also has a right inverse (namely $\mathcal{Q}^{\mathrm{cn}}(R) \to \mathcal{Q}^{\mathrm{cn}}(R \oplus \Sigma M)$), it follows that it is also essentially surjective, hence an equivalence. $\square$

Corollary 1.3.46. *Let* R *be a connective* $\mathbb{E}_{m+1}$*-ring spectrum,* $m \geq 3$*. Basechange along* $R \to \pi_0(R)$ *induces an equivalence* $\mathrm{CSep}^{\mathrm{cn}}_m(R) \to \mathrm{CSep}^{\mathrm{cn}}_m(\pi_0(R))$.

Proof. Combine Corollary 1.3.43 and Proposition 1.3.45. $\square$

In the case of étale extensions, however, one can go further: flatness (which is part of the definition of étale) forces étale extensions of $\pi_0(R)$ to also be discrete. We do not know if this is so for arbitrary commutative separable extensions, however, in the noetherian case, Neeman proved the following:

Theorem 1.3.47 ([Nee18, Lemma 2.1., Remark 2.2.]). *Let R be a discrete commutative noetherian ring. Any commutative separable algebra in Mod_R is coconnective*[24]*. In particular, any connective commutative separable algebra is discrete.*

A discrete separable commutative algebra is also flat.

Remark 1.3.48. In that last sentence, we are considering separable algebras in Mod_R, which are discrete; and not separable algebras in $\mathrm{Mod}_R^{\heartsuit}$. The latter can be non-flat: for example, any quotient $R \to R/I$ is separable in $\mathrm{Mod}_R^{\heartsuit}$, as $R/I \otimes_R R/I \cong R/I$. $\triangleleft$

Remark 1.3.49. Neeman proves more than Theorem 1.3.47, he completely classifies commutative separable algebras over noetherian schemes. $\triangleleft$

We will deduce the following:

Corollary 1.3.50. *Let R be a connective $\mathbb{E}_{m+1}$-ring spectrum, $m \geq 3$, with noetherian π_0. The functor $\pi_0 : \mathrm{Mod}_R^{\mathrm{cn}} \to \mathrm{Mod}_R^{\heartsuit} \simeq \mathrm{Mod}_{\pi_0(R)}^{\heartsuit} \subset \mathrm{Mod}_{\pi_0(R)}$ preserves separable $\mathbb{E}_m$-algebras.*

Furthermore, any connective separable $\mathbb{E}_m$-algebra over R is flat.

The functor $\pi_0(-)$, or equivalently $\pi_0(R) \otimes_R -$ induces an equivalence

$$\mathrm{Alg}_{\mathbb{E}_m}^{sep}(R)^{\mathrm{cn}} \to \mathrm{Alg}_{\mathbb{E}_m}^{sep}(\pi_0(R))^{\heartsuit,\flat}$$

We have used the following notation:

Notation 1.3.51. We use the superscript $^\flat$ to indicate flatness - we can use this for Mod_R, where R is a ring spectrum [Lur12, Definition 7.2.2.10.], and by extension for ∞-categories that admit natural forgetful functors to it, such as $\mathrm{Alg}_{\mathbb{E}_m}(R)$. $\triangleleft$

To prove this, we use the following standard lemma:

Lemma 1.3.52. *Let R be a connective ring spectrum and M a right R-module. Suppose $M \otimes_R \pi_0(R)$ is a flat (in particular discrete) $\pi_0(R)$-module. In this case, M is a flat R-module.*

Proof. For any discrete left R-module N, $M \otimes_R N \simeq M \otimes_R \pi_0(R) \otimes_{\pi_0(R)} N$ is discrete by assumption. This is enough by [Lur12, Theorem 7.2.2.15.(5)]. $\square$

Proof of Corollary 1.3.50. Let A be a connective, homotopy commutative separable algebra over R. Basechange along $R \to \pi_0(R)$ is $\mathbb{E}_m$-monoidal, so that $A \otimes_R \pi_0(R)$ is separable and hence, by Neeman's theorem (Theorem 1.3.47), coconnective. As it is also connective, it is therefore discrete.

It follows that it is isomorphic to its π_0, which is also $\pi_0(A)$, and hence, $\pi_0(A)$ is separable, *as an algebra in* $\mathrm{Mod}_{\pi_0(R)}$[25].

Furthermore, $A \otimes_R \pi_0(R)$ is flat, again by Theorem 1.3.47, and hence A is flat, by the previous lemma.

Now, as commutative separable algebras are 0-truncated, by Corollary 1.3.43, this implies that the two functors (which we just explained are equivalent) $\pi_0(-)$ and $\pi_0(R) \otimes_R -$ are

[24] In [Nee18], Neeman says "connective", but he is working with cohomological conventions.

[25] It is obviously separable in $\mathrm{Mod}_{\pi_0(R)}^{\heartsuit}$, because $\pi_0 : \mathrm{Mod}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\pi_0(R)}^{\heartsuit}$ is strong $\mathbb{E}_m$-monoidal.

fully faithful as functors $\mathrm{CAlg}^{sep}(R)^{\mathrm{cn}} \to \mathrm{CAlg}^{sep}(\pi_0(R))^{\heartsuit,\flat}$. We are left with proving that they are essentially surjective.

But the inclusion $\mathrm{Mod}_{\pi_0(R)}^{\heartsuit,\flat} \to \mathrm{Mod}_{\pi_0(R)}$ is strong symmetric[26] monoidal, and hence preserves separable algebras. Thus, any object in $\mathrm{CAlg}^{sep}(\pi_0(R))^{\heartsuit,\flat}$ lifts to $\mathrm{CAlg}^{sep}(\pi_0(R))^{\mathrm{cn}}$, and by Corollary 1.3.46, anything there can be lifted to $\mathrm{CAlg}^{sep}(R)^{\mathrm{cn}}$. □

In other words, under this noetherian-ness assumption, connective commutative separable algebras in $\mathrm{Mod}_{\pi_0(R)}$ are exactly the flat ordinary commutative separable algebras. Note that given a flat ordinary commutative separable algebra A_0, the corresponding commutative separable algebra over R is flat, and hence has homotopy groups $\pi_*(A) \cong A_0 \otimes_{\pi_0(R)} \pi_*(R)$.

This allows us to compare separability with étale-ness in the sense of Lurie in the noetherian case. Namely, we have:

Proposition 1.3.53. *Let R be a commutative ring spectrum and A a commutative R-algebra. If A is étale in the sense of [Lur12, Definition 7.5.0.4.], then A is separable.*

Conversely, if R is connective and $\pi_0(R)$ is noetherian, then if A is separable, connective and $\pi_0(A)$ is finitely presented over $\pi_0(R)$ then A is étale in the same sense.

Remark 1.3.54. It is not clear to the author what the optimal statement is. Clearly, one cannot drop all connectivity assumptions: for instance, Galois extensions are separable (see Proposition 1.5.3), but many of them, such as $\mathrm{KO} \to \mathrm{KU}$ are not étale.

It is reasonable to expect that one can drop the noetherian assumption, and possibly the connectivity assumption on A. ◁

Proof. Assume A is étale. By definition, $\pi_0(A)$ is étale over $\pi_0(R)$. In particular, $\pi_0(A)$ is flat over $\pi_0(R)$, and separable in the classical sense. So let $e \in \pi_0(A) \otimes_{\pi_0(R)} \pi_0(A) \cong \pi_0(A \otimes_R A)$ be a separability idempotent. This is in turn an idempotent in $A \otimes_R A$ which gets sent to $1 \in \pi_0(A)$ under the multiplication map.

In particular, it induces a map $(A \otimes_R A)[e^{-1}] \to A$. Because homotopy groups commute with localizations, and because A is flat, this map can be identified, on homotopy groups, with

$$(\pi_0(A) \otimes_{\pi_0(R)} \pi_0(A) \otimes_{\pi_0(R)} \pi_*(R))[e^{-1}] \to \pi_0(A) \otimes_{\pi_0(R)} \otimes \pi_*(R)$$

Because $(\pi_0(A) \otimes_{\pi_0(R)} \pi_0(A))[e^{-1}] \to \pi_0(A)$ is an isomorphism, this map is also an isomorphism, which proves that $(A \otimes_R A)[e^{-1}] \simeq A$. As e is idempotent, it follows that $A \otimes_R A \to A$ has an $A \otimes_R A$-linear splitting, thus proving that it is separable.

For the converse, we already know $\pi_0(A)$ is separable, flat and finitely presented, which means that it is étale over $\pi_0(R)$ in the classical sense. Furthermore, we also know that A is flat over R, which altogether means that A is étale in the sense of [Lur12, Definition 7.5.0.4.]. □

Question 1.3.55. Do these results (Theorem 1.3.47, Corollary 1.3.50 and Proposition 1.3.53) continue to hold without a noetherian-ness assumption ? ◁

In their recent work [NP23], Naumann and Pol partially answer this question by removing the noetherian assumption and adding the assumption that the algebra A is perfect as an R-module. One easily sees that their proof only uses the assumption that A is *almost perfect* [Lur12, Definition 7.2.4.10]. In fact, as it turns out, it follows that in this case, almost perfect implies perfect. We record it here for the convenience of the reader, but the proof is the same as that of [NP23, Proposition 10.5]:

[26] Note that $\pi_0(R)$ is commutative.

Proposition 1.3.56. *The answer to Question 1.3.55 is yes, when restricted to almost perfect separable algebras. More precisely, fix a connective commutative ring spectrum R. Let A be a commutative separable R-algebra, whose underlying R-module is almost perfect.*

In this case, A is flat, and hence connective; $\pi_0(A)$ is separable as a $\pi_0(R)$-algebra in $\mathrm{Mod}_{\pi_0(R)}$ *(and not only in* $\mathrm{Mod}_{\pi_0(R)}^{\heartsuit}$*). In particular, A is étale in the sense of Lurie over R.*

In particular, A is perfect, *and even finitely generated projective as an R-module. In other words, almost perfect commutative separable algebras are always perfect/finitely generated projective, and they correspond exactly to finite étale extensions. The functor $\pi_0(-)$, or equivalently $\pi_0(R) \otimes_R -$, induces an equivalence between these and finite étale $\pi_0(R)$-algebras.*

Proof. By Lemma 1.3.52, to prove that A is flat, it suffices to prove that $A \otimes_R \pi_0(R)$ is flat, and in particular discrete, over $\pi_0(R)$.

We reduce to the case of a field using [Sta23, Tag 068V]. In more detail, we note that the word "pseudo-coherent" used in [Sta23] is equivalent to "almost perfect" in the case of discrete rings, so that this lemma does apply to our situation. Then, we note that basechange preserves almost perfect modules, so that $A \otimes_R \pi_0(R)$ is almost perfect over $\pi_0(R)$, i.e. pseudo-coherent. Finally, we note that being flat is equivalent to being of Tor-amplitude in $[0, 0]$, so that by [Sta23, Tag 068V] it suffices to prove that $A \otimes_R \pi_0(R) \otimes_{\pi_0(R)} k$ is concentrated in degree 0 for any field k and morphism $\pi_0(R) \to k$.

But now $A \otimes_R k$ is the basechange of A along the commutative ring map

$$R \to \pi_0(R) \to k$$

so it is a separable commutative algebra over the field k, and by [Nee18, Proposition 1.6], these are all discrete, as was to be shown.

Now, we have proved that A is flat (and in particular connective), so that $\pi_0(R) \otimes_R A \simeq \pi_0(A)$ - the former is obviously separable, and therefore, so is the latter.

To prove that A is étale, as in the proof of Proposition 1.3.53, because we already know that it is flat, it suffices to prove that $\pi_0(A)$ is étale over $\pi_0(R)$. We know that it is separable and flat, so it suffices to prove that it is finitely presented, but it is finitely presented *as a module* over $\pi_0(R)$, which immediately implies that it is finitely presented as an algebra as well, and hence étale.

Furthermore, $\pi_0(A)$ is a finitely presented flat $\pi_0(R)$-module, hence it is finitely generated projective, and thus, by flatness of A, A is a finitely generated projective R-module, and in particular perfect.

By Corollary 1.3.46, the functor

$$\mathrm{CAlg}^{sep}(R)^{\mathrm{aperf}} \subset \mathrm{CAlg}^{sep}(R)^{\mathrm{cn}} \to \mathrm{CAlg}^{sep}(\pi_0(R))^{\mathrm{cn}}$$

is fully faithful, and it lands in the full subcategory $\mathrm{CAlg}^{sep}(\pi_0(R))^{\heartsuit,\flat}$. To conclude the proof, it thus suffices to prove that its essential image is exactly the finite étale extensions. But if A_0 is a finite étale extension of $\pi_0(R)$, it is in particular finitely generated projective, and so its unique lift to a flat $\pi_0(R)$-module is also finitely generated projective, and hence almost perfect. □

Question 1.3.55 remains however open in full generality.

1.4 A variant: ind-separability

The goal of this section is to study a variant of the notion of separability, which I call "ind-separability", and which is better suited in some "infinitary" situations. We will see that they share many of the properties of separable algebras, in particular concerning highly structured multiplicative structures.

We will see that, in the ∞-category of $K(n)$-local spectra, Morava E-theory is ind-separable - the proof of this will require as its only input the computation of the ring of cooperations of Morava E-theory, by Hopkins–Ravenel, Baker, and revisited by Hovey in [Hov04]. As a corollary, we will obtain a relatively simple proof of the Goerss–Hopkins–Miller theorem, or in some sense a reorganization of the classical proof; as well as an extension to the folklore claim that E-theory admits a unique $\mathbb{E}_d$-structure for any $1 \leq d \leq \infty$[27].

Because of the infinitary nature of the notion of ind-separability, and because separable algebras only have strong enough rigidity properties in the commutative case, the variant we introduce here only really works in the (homotopy) commutative case. For similar reasons, this variant is best suited in the compactly generated case, and we will mostly stick to this assumption.

Furthermore, unlike the previous sections where the proofs were carried out very elementarily, in this case we will actually use obstruction theory, specifically the obstruction theory from [PV22], which is developed in the setting of *symmetric* monoidal ∞-categories. While it is very likely that most of what we need can be made to work in the $\mathbb{E}_m$-monoidal case for m large enough, I did not have the time to go over the work from [PV22] to "make it $\mathbb{E}_m$-monoidal". Thus, for this section, and for now, I will stick to the symmetric monoidal case, though perhaps future versions will include the $\mathbb{E}_m$-monoidal case as well.

1.4.1 Ind-separability

Recall that, if A is a commutative separable algebra, the multiplication map $A \otimes A \to A$ witnesses the target as the localization of the source at an idempotent. The key observation of this section is that many of our results only really need it to be a localization at some set of elements. We thus define:

Definition 1.4.1. Let $\mathbf{C}$ be a presentably, stably symmetric monoidal ∞-category, and $A \in \mathrm{CAlg}(\mathbf{C})$ a commutative algebra in $\mathbf{C}$. We say that A is *ind-separable* if there is a set $S \subset \pi_0 \operatorname{map}(\mathbf{1}, A \otimes A)$ such that the multiplication map witnesses A as the localization $(A \otimes A)[S^{-1}]$ in the ∞-category of $A \otimes A$-modules. ◁

Remark 1.4.2. Note that a priori, an ind-separable algebras has no particular reason to be a filtered colimit of separable algebras, i.e. an ind-(separable algebra). This will, however, be our main source of examples, cf. the subsequent sections. ◁

This notion is relatively well-suited if we want to study the moduli space of commutative structures extending the underlying $\mathbb{E}_1$-algebra structure of A, at least when $\mathbf{C}$ is compactly generated.

However, if we also want to get off the ground and go from a homotopy algebra structure to an actual algebra structure, we need to phrase this in "up-to-homotopy" terms. Because in a stably symmetric monoidal ∞-category with filtered colimits, localizing a commutative algebra at a set of elements is a relatively well understood procedure, namely it is given by a

[27] The $d = 1$ case is known as the Hopkins–Miller theorem, and the $d = \infty$ case as the Goerss–Hopkins–Miller theorem.

telescope (see e.g. [BNT18, Appendix C]), we can in fact give the following definition in the compactly generated case:

Definition 1.4.3. Let $\mathbf{C}$ be a compactly generated presentably, stably symmetric monoidal ∞-category, and $A \in \mathrm{CAlg}(\mathrm{ho}(\mathbf{C}))$ a homotopy commutative homotopy algebra in $\mathbf{C}$. We say that A is *homotopy ind-separable* if there exists a set $S \subset \pi_0 \operatorname{map}(\mathbf{1}, A \otimes A)$ such that the multiplication map induces, for each compact object $c \in \mathbf{C}^\omega$, an isomorphism

$$\pi_*(\operatorname{map}(c, A \otimes A))[S^{-1}] \to \pi_* \operatorname{map}(c, A)$$

◁

In this definition, the localization is taken outside of π_*, and by it we simply mean the usual telescope construction in abelian groups.

Remark 1.4.4. Because filtered colimits are exact in Ab, the condition that the above map be an isomorphism *at* c is closed under co/fiber sequences in $\mathbf{C}^\omega$, and thus it suffices to check it on generators, e.g. on R if $\mathbf{C} = \mathrm{Mod}_R(\mathrm{Sp})$ for some commutative ring spectrum R. ◁

Again because localizations of commutative algebras in stable ∞-categories are computed as telescopes [BNT18, Appendix C], the following is an immediate consequence of the definition:

Lemma 1.4.5. *Let* $\mathbf{C}$ *be a compactly generated presentably, stably symmetric monoidal* ∞*-category, and* $A \in \mathrm{CAlg}(\mathbf{C})$ *a commutative algebra in* $\mathbf{C}$*. If* A *is ind-separable, then its underlying homotopy algebra is homotopy ind-separable.*

In some cases of interest though, the compacts of $\mathbf{C}$ are complicated to calculate, and so it can be useful to formulate a criterion at the level of $\mathbf{C}$. It can be hard to phrase in this generality, because the diagram that defines the telescope has no reason to lift to $\mathbf{C}$ in general. However, if S is particularly nice, the diagram *can* be lifted to $\mathbf{C}$ even if A is only a homotopy algebra.

To explain this in more detail, we begin with a construction.

Construction 1.4.6. Let $\mathbf{C}$ be a symmetric monoidal additive ∞-category admitting sequential colimits. Let $B \in \mathrm{CAlg}(\mathrm{ho}(\mathbf{C}))$ be a homotopy commutative homotopy algebra, and let $s : \mathbf{1} \to B$ be an "element" of B. This induces a (homotopy-)B-module map $B \to B$ given by multiplication by s, and thus, an $\mathbb{N}$-shaped diagram $B \to B \to B \to \dots$ *in* $\mathbf{C}$.

We call its colimit the telescope of B at s, $\mathrm{Tel}_s(B)$.

Suppose instead given an $\mathbb{N}$-indexed family $s = (s_i)$ of elements of B. We can then form its telescope as the colimit of the diagram $B \xrightarrow{s_1} B \xrightarrow{s_1 s_2} B \xrightarrow{s_1 s_2 s_3} B \dots$, and we still denote it by $\mathrm{Tel}_s(B)$. ◁

Remark 1.4.7. In this construction, the diagram $B \to B \to B \to \dots$ can really be constructed in $\mathbf{C}$ and not only in $\mathrm{ho}(\mathbf{C})$, because $\mathbb{N}$ is free as an ∞-category (cf., e.g., [Lur09, Proof of Proposition 4.4.2.6]). ◁

Definition 1.4.8. Let $\mathbf{C}$ be a symmetric monoidal additive ∞-category admitting sequential colimits. Let $B \in \mathrm{CAlg}(\mathrm{ho}(\mathbf{C}))$ be a homotopy commutative homotopy algebra and let M be a (homotopy-)B-module with a (homotopy-)B-module map $f : B \to M$. Let s be an $\mathbb{N}$-indexed family of elements of B.

We say that f witnesses M as a telescope of B at s if there exist homotopies

$$f \circ (s_1 ... s_n) \simeq f$$

for all n that induce an equivalence $\mathrm{Tel}_s(B) \simeq M$. ◁

Remark 1.4.9. Note that in the latter definition, because $\mathbb{N}$ is free as an ∞-category, the collection of homotopies $f \circ (s_1 ... s_n) \simeq f$ is sufficient to induce a map from the colimit to M. $\triangleleft$

Remark 1.4.10. If $\mathbf{C}$ is stable and compactly generated, the condition that the induced map $\mathrm{Tel}_s(B) \to M$ be an equivalence does not depend on the chosen homotopies $f \circ (s_1 ... s_n) \simeq f$. Indeed, it can be checked after applying $\pi_0 \operatorname{map}(c, -)$ for all compacts c, and the induced map there does not depend on the homotopies. $\triangleleft$

Definition 1.4.11. Let $\mathbf{C}$ be a stably symmetric monoidal ∞-category with sequential colimits compatible with the tensor product, and let $A \in \mathrm{CAlg}(\mathrm{ho}(\mathbf{C}))$ be a homotopy commutative homotopy algebra. We say that A is homotopy ω-separable if the multiplication map $A \otimes A^{\mathrm{op}} \to A$ witnesses A as a telescope of $A \otimes A^{\mathrm{op}}$ at some sequence $s = (s_i)$ of elements $s_i : \mathbf{1} \to A \otimes A^{\mathrm{op}}$. $\triangleleft$

The following is again an easy consequence of the definition:

Lemma 1.4.12. *Let $\mathbf{C}$ be a compactly generated presentably, stably symmetric monoidal ∞-category, and $A \in \mathrm{CAlg}(\mathrm{ho}(\mathbf{C}))$ a homotopy commutative homotopy algebra in $\mathbf{C}$. If A is homotopy ω-separable, then it is homotopy ind-separable.*

The way things will go is that we will prove things about (homotopy) ind-separable (homotopy) algebras, and our main example of a homotopy ind-separable homotopy algebra (Morava E-theory) will be proved to be so by proving that is a homotopy ω-separable algebra.

1.4.2 Obstruction theory

Having defined (homotopy) ind-separability, our first goal is to argue that most of our results about commutative separable algebras extend to the case of (homotopy) ind-separable algebras, at least when $\mathbf{C}$ is compactly generated, and under suitable assumption on phantom maps. To prove these results, we will use obstruction theory - we could in principle follow a similar approach as in Section 1.2, but because filtered colimits do not interact so well with the formation of homotopy (n-)categories, we would have to stick closer to the proof of Lemma 1.2.21 rather than just using it as a lemma. As a result, the proof would be more convoluted and the obstruction theory from [PV22] neatly packages the constructions anyway.

Warning 1.4.13. An earlier version of this document was missing a key assumption which will appear here, about phantom maps. We will discuss this assumption when it is relevant. $\triangleleft$

Unlike Section 1.2, we now start with a compactly generated stably symmetric monoidal $\mathbf{C}$.

Assumption 1.4.14. $\mathbf{C}$ is a compactly generated stably symmetric monoidal ∞-category, in which tensor products commute with colimits in each variable. Further, we assume that the compact objects of $\mathbf{C}$ are closed under non-empty tensor products[28] $\triangleleft$

We will rely heavily on [PV22], and so our first goal is to get ourselves in the setting of the obstruction theory from that paper.

The following construction is very similar to the construction in [HL17, Section 4.4], so we only briefly go over the details.

[28]We do *not* assume that the unit is compact.

Construction 1.4.15. Let $\mathbf{C}$ be as in Assumption 1.4.14.

Let $\mathrm{Syn}_{\mathbf{C}} \subset \mathrm{Fun}^{\times}((\mathbf{C}^{\omega})^{\mathrm{op}}, \mathrm{Sp}_{\geq 0})$. Similarly to [HL17, Section 4.4], the assumption that $\mathbf{C}^{\omega}$ is closed under non-empty tensor products makes $\mathrm{Syn}_{\mathbf{C}}$ into a non-unital symmetric monoidal ∞-category for which the Yoneda embedding $\mathbf{C}^{\omega} \to \mathrm{Syn}_{\mathbf{C}}$ is canonically non-unitally symmetric monoidal.

Again, similarly to [HL17, Section 4.4], we obtain an essentially unique non-unitally symmetric monoidal colimit-preserving preserving functor $f : \mathrm{Syn}_{\mathbf{C}} \to \mathbf{C}$ whose restriction to $\mathbf{C}^{\omega}$ is (non-unitally symmetric monoidally) equivalent to the inclusion. The right adjoint of f, given by the restricted Yoneda embedding $M : c \mapsto \mathrm{map}(-, c)_{\geq 0}$ thus acquires a canonical (non-unital) lax symmetric monoidal structure.

The structure maps $\mathrm{map}(-, c)_{\geq 0} \otimes \mathrm{map}(-, d)_{\geq 0} \to \mathrm{map}(-, c \otimes d)_{\geq 0}$ are equivalences whenever $c, d \in \mathbf{C}^{\omega}$, so because $\mathbf{C}$ is compactly generated and M preserves filtered colimits, we find that these structure maps are also equivalences for all $c, d \in \mathbf{C}$.

In particular, $\mathrm{map}(-, \mathbf{1})_{\geq 0} \otimes M(c) \simeq M(c)$ for all $c \in \mathbf{C}^{\omega}$ and thus, because of the universal property of $\mathrm{Syn}_{\mathbf{C}}$, $M(\mathbf{1}) \otimes - \simeq \mathrm{id}$ as functors $\mathrm{Syn}_{\mathbf{C}} \to \mathrm{Syn}_{\mathbf{C}}$ (see [HL17, Lemma 4.4.9]).

It follows that $\mathrm{Syn}_{\mathbf{C}}$ is in fact a *unital* symmetric monoidal ∞-category, and that $M : \mathbf{C} \to \mathrm{Syn}_{\mathbf{C}}$ is fully faithful and symmetric monoidal. Furthermore, $\mathrm{Syn}_{\mathbf{C}}$ is clearly Grothendieck prestable, complete and separated. We abuse notation and write $\mathbf{1}$ also for the unit of $\mathrm{Syn}_{\mathbf{C}}$, i.e. for $M(\mathbf{1})$.

The grading given by $F[1] := F(\Omega -)$, where $\Omega : \mathbf{C} \to \mathbf{C}$ is the loop functor (equivalently the suspension functor on $\mathbf{C}^{\mathrm{op}}$) makes it into a graded Grothendieck prestable ∞-category in the sense of [PV22], and the assembly map induces a shift structure on the unit

$$\tau : \Sigma \mathbf{1}[-1] \to \mathbf{1}$$

again in the sense of [PV22]. This is the only shift algebra we will consider in this section, so "periodic module" in the sense of [PV22] should always be understood with respect to this shift algebra. ◁

Warning 1.4.16. When specialized to the case of $\mathbf{C} = \mathrm{Mod}_E(\mathrm{Sp}_{K(n)})$ where E is Morava E-theory at height n, our definition of $\mathrm{Syn}_{\mathbf{C}}$ is related to the Syn_E appearing in [HL17], but they are not the same: Syn_E is also defined as an ∞-category of product-preserving presheaves, but on something smaller than $\mathbf{C}^{\omega}$. ◁

We recall the following definition from [PV22]:

Definition 1.4.17 ([PV22, Definition 2.17])**.** An object $M \in \mathrm{Syn}_{\mathbf{C}}$ is a *periodic* module over $\mathbf{1}$ if the canonical map induces an isomorphism $\pi_0(M) \otimes_{\pi_0(\mathbf{1})} \pi_*(\mathbf{1}) \to \pi_*(M)$; equivalently if $\tau : \Sigma M[-1] \to M$ is a 1-connective cover[29]. ◁

Lemma 1.4.18. *For any $c \in \mathbf{C}$, $M(c)$ is a periodic module over $\mathbf{1}$.*

Proof. We use the second characterization for this: $\tau : \Sigma M(c)[-1] \to M(c)$ identifies with the map $\Sigma(\mathrm{map}(\Sigma -, c))_{\geq 0} \to \mathrm{map}(-, c)_{\geq 0}$.

Furthermore, $\mathrm{map}(\Sigma -, c) \simeq \Omega\, \mathrm{map}(-, c)$. Now, for any spectrum X, the canonical map $\Sigma(\Omega X)_{\geq 0} \to X_{\geq 0}$ is a 1-connective cover, so we are done. □

Lemma 1.4.19. *Let $M, N \in \mathrm{Syn}_{\mathbf{C}}$ be periodic modules over $\mathbf{1}$. If $f : M \to N$ is a morphism which induces an isomorphism on π_0, then f is an equivalence.*

[29] The equivalence between these two conditions is proved as [PV22, Proposition 2.16]

Proof. It follows from the first definition of periodic modules that f induces an isomorphism on all homotopy groups. Given the definition of $\mathrm{Syn}_{\mathbf{C}}$ and of the homotopy groups, it is clear that this implies that it is an equivalence. □

Lemma 1.4.20. *The functor $M : \mathbf{C} \to \mathrm{Syn}_{\mathbf{C}}$ identifies $\mathbf{C}$ with the full subcategory of $\mathrm{Syn}_{\mathbf{C}}$ spanned by periodic modules over $\mathbf{1}$.*

Proof. By [PV22, Proposition 2.22], we have an equivalence $\mathrm{Syn}_{\mathbf{C}}^{per} \simeq \mathrm{Sp}(\mathrm{Syn}_{\mathbf{C}})^{\tau^{-1}}$, where the superscript τ^{-1} means "τ-local", i.e. the $M \in \mathrm{Sp}(\mathrm{Syn}_{\mathbf{C}}) \simeq \mathrm{Fun}^{\times}((\mathbf{C}^{\omega})^{\mathrm{op}}, \mathrm{Sp})$ such that the canonical map $\Sigma M(\Sigma -) \to M$ is an equivalence.

By definition, this canonical map is an equivalence if and only if M sends suspensions in $(\mathbf{C}^{\omega})^{\mathrm{op}}$ to loops, i.e. if and only if M is an exact functor [Lur12, Corollary 1.4.2.14.]. But $M : \mathbf{C} \to \mathrm{Fun}((\mathbf{C}^{\omega})^{\mathrm{op}}, \mathrm{Sp})$ identifies $\mathbf{C}$ with the full subcategory of $\mathrm{Fun}((\mathbf{C}^{\omega})^{\mathrm{op}}, \mathrm{Sp})$ spanned by exact functors, i.e. $\mathrm{Ind}(\mathbf{C}^{\omega})$. □

Corollary 1.4.21. *Let $\mathbf{C}$ be as in Assumption 1.4.14, and let $A \in \mathrm{CAlg}(\mathrm{ho}(\mathbf{C}))$ be a homotopy-ind-separable homotopy commutative homotopy algebra.*

In this case, $\pi_0 M(A) \in \mathrm{Mod}_{\pi_0(\mathbf{1})}(\mathrm{Syn}_{\mathbf{C}})$ is an ind-separable commutative algebra.

If $A \in \mathrm{Alg}(\mathbf{C})$ is homotopy commutative and ind-separable, the same holds.

Proof. We first observe that the canonical map

$$\pi_0 M(A) \otimes_{\pi_0(\mathbf{1})} \pi_0 M(A) \to \pi_0 M(A \otimes A)$$

is an equivalence. Granted this observation, the claim simply follows from the definition of (homotopy) ind-separable and the fact that $\pi_0 : \mathrm{Syn}_{\mathbf{C}} \to \mathrm{Syn}_{\mathbf{C}}$ preserves filtered colimits (and again, the fact that localizations of commutative algebras are given by telescopes).

To prove the observation, we combine Lemma 1.4.18 and [PV22, Proposition 2.16] to get that $\pi_0 M(-) \simeq M(-) \otimes_{\mathbf{1}} \pi_0(\mathbf{1})$, from which the claim follows as $M(-)$ is symmetric monoidal, and so is basechange along $\mathbf{1} \to \pi_0(\mathbf{1})$ in $\mathrm{Syn}_{\mathbf{C}}$. □

Lemma 1.4.22. *Let $\mathbf{D}$ be a stably symmetric monoidal ∞-category admitting filtered colimits that are compatible with the tensor product, and let $A \in \mathrm{CAlg}(\mathbf{D})$ be an ind-separable commutative algebra.*

Let L_A be the fiber of the multiplication map $A \otimes A \to A$, viewed as an A-bimodule. For any A-module M, viewed as an A-bimodule via restriction along the multiplication map, we have that the mapping spectrum $\mathrm{map}_{A \otimes A}(L_A, M)$ vanishes.

Proof. Basechange along the multiplication map is left adjoint to restriction, so it suffices to prove that the basechange of L_A is zero, and for this it suffices to prove that the co-unit $A \otimes_{A \otimes A} A \to A$ is an equivalence. This follows immediately from ind-separability. □

Theorem 1.4.23. *Let $\mathbf{C}$ be as in Assumption 1.4.14, and let $A \in \mathrm{Alg}(\mathbf{C})$ be homotopy commutative and homotopy ind-separable. For any homotopy commutative algebra $R \in \mathrm{Alg}(\mathbf{C})$, the mapping space $\mathrm{map}_{\mathrm{Alg}(\mathbf{C})}(A, R)$ is discrete and equivalent to $\hom_{\mathrm{Alg}(\mathrm{Syn}_{\mathbf{C}}^{\heartsuit})}(\pi_0 M(A), \pi_0 M(R))$.*

Proof. First, note that $M(A), M(R)$ are periodic algebras over $\mathbf{1}$, so that we can use [PV22, Proposition 5.7] to study the mapping space

$$\mathrm{map}_{\mathrm{Alg}(\mathbf{C})}(A, R) \simeq \mathrm{map}_{\mathrm{Alg}(\mathrm{Syn}_{\mathbf{C}})}(M(A), M(R))$$

By [PV22, Proposition 5.7], the fiber of

$$\mathrm{map}_{\mathrm{Alg}(\mathrm{Mod}_{\mathbf{1}^{\leq n+1}}(\mathrm{Syn}_{\mathbf{C}}))}(\mathbf{1}^{\leq n+1}\otimes M(A), \mathbf{1}^{\leq n+1}\otimes M(R)) \to \mathrm{map}_{\mathrm{Alg}(\mathrm{Mod}_{\mathbf{1}^{\leq n}}(\mathrm{Syn}_{\mathbf{C}}))}(\mathbf{1}^{\leq n}\otimes M(A), \mathbf{1}^{\leq n}\otimes M(R))$$

at any point in the target is a space of paths in a certain space. We claim that this space is contractible. Indeed, this space is

$$\mathrm{map}_{\mathrm{BiMod}_{\pi_0 M(A)}(\mathrm{Syn}_{\mathbf{C}})}(L^{\mathbb{E}_1}_{\pi_0 M(A)/\pi_0(\mathbf{1})}, \Sigma^{n+2}\pi_0 M(R)[-(n+1)])$$

Now because $M(R)$ is homotopy commutative, the $M(R)$-bimodule $\pi_0 M(R)[-(n+1)]$ is pulled back along the multiplication map

$$\pi_0 M(R)\otimes_{\pi_0(\mathbf{1})}\pi_0 M(R) \to \pi_0 M(R)$$

and so the same holds when we see this bimodule as a $\pi_0 M(A)$-bimodule. Thus by Lemma 1.4.22 applied to $\mathbf{D} = \mathrm{Mod}_{\pi_0(\mathbf{1})}(\mathrm{Syn}_{\mathbf{C}})$ (and Corollary 1.4.21), this space is contractible, and hence so is the path space between any two points therein. Here we use that by [Lur12, Theorem 7.3.5.1], $L^{\mathbb{E}_1}_{\pi_0 M(A)/\pi_0(\mathbf{1})}$ is what we have called $L_{\pi_0 M(A)}$, computed in $\mathrm{Mod}_{\pi_0(\mathbf{1})}(\mathrm{Syn}_{\mathbf{C}})$.

This proves that the map

$$\mathrm{map}_{\mathrm{Alg}(\mathrm{Mod}_{\mathbf{1}^{\leq n+1}}(\mathrm{Syn}_{\mathbf{C}}))}(\mathbf{1}^{\leq n+1}\otimes M(A), \mathbf{1}^{\leq n+1}\otimes M(R)) \to \mathrm{map}_{\mathrm{Alg}(\mathrm{Mod}_{\mathbf{1}^{\leq n}}(\mathrm{Syn}_{\mathbf{C}}))}(\mathbf{1}^{\leq n}\otimes M(A), \mathbf{1}^{\leq n}\otimes M(R))$$

is an equivalence, and thus by [PV22, Remark 5.2], the composite map

$$\begin{aligned}\mathrm{map}_{\mathrm{Alg}(\mathrm{Syn}_{\mathbf{C}})}(M(A), M(R)) \to \mathrm{map}_{\mathrm{Alg}(\mathrm{Mod}_{\pi_0(\mathbf{1})})}&(\pi_0(\mathbf{1})\otimes M(A), \pi_0(\mathbf{1})\otimes M(R))\\ &\simeq \mathrm{map}_{\mathrm{Alg}(\mathrm{Syn}^{\heartsuit}_{\mathbf{C}})}(\pi_0 M(A), \pi_0 M(R))\end{aligned}$$

is an equivalence. □

As in the proof of Corollary 1.3.39, and because $\mathrm{Syn}^{\heartsuit}_{\mathbf{C}}$ is a 1-category, we obtain:

Corollary 1.4.24. *Let* $\mathbf{C}$ *be as in Assumption 1.4.14 and let* $A \in \mathrm{Alg}(\mathbf{C})$ *be an ind-separable homotopy commutative algebra. Let* $\mathcal{O}$ *be an arbitrary one-colored* ∞*-operad. In this case, the canonical forgetful map*

$$\mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathbf{C})^{\simeq}\times_{\mathrm{Alg}(\mathbf{C})^{\simeq}}\{A\} \to \mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathrm{Syn}^{\heartsuit}_{\mathbf{C}})^{\simeq}\times_{\mathrm{Alg}(\mathrm{Syn}^{\heartsuit}_{\mathbf{C}})^{\simeq}}\{\pi_0 M(A)\}$$

is an equivalence. In particular, if $\mathcal{O}$ *is weakly reduced,* $\mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathbf{C})^{\simeq}\times_{\mathrm{Alg}(\mathbf{C})^{\simeq}}\{A\}$ *is contractible. This is the case e.g. if* $\mathcal{O} = \mathbb{E}_d, d \geq 1$.

More generally, again arguing as in the proof of Corollary 1.3.39:

Corollary 1.4.25. *Let* $\mathbf{C}$ *be as in Assumption 1.4.14 and let* $A \in \mathrm{Alg}(\mathbf{C})$ *be an ind-separable homotopy commutative algebra. If* $\mathcal{O}$ *is any* ∞*-operad and* $R \in \mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathbf{C})$ *is an algebra whose underlying* $\mathbb{E}_1$*-algebra is homotopy commutative, then, viewing* A *as an* $\mathcal{O}\otimes\mathbb{E}_1$*-algebra using the unique map of* ∞*-operads* $\mathcal{O}\otimes\mathbb{E}_1\to\mathbb{E}_\infty$*, we find that the canonical map*

$$\mathrm{map}_{\mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathbf{C})}(A, R) \to \hom_{\mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathrm{Syn}^{\heartsuit}_{\mathbf{C}})}(\pi_0 M(A), \pi_0 M(R))$$

is an equivalence.

As showcased in the proof of Corollary 1.4.24, our obstruction theory based on $\mathrm{Syn}_{\mathbf{C}}$ really only knows about the π_0 of $M(A)$, which can be understood as the cohomology theory represented by A *on* $\mathbf{C}^\omega$, rather than the whole of $\mathbf{C}$. The following assumption on $\mathbf{C}$ will thus be needed if we want to lift information about $\mathrm{Syn}_{\mathbf{C}}^{\heartsuit}$ to information about $\mathrm{ho}(\mathbf{C})$:

Assumption 1.4.26. Let $X, Y \in \mathbf{C}$ and let $f : \pi_0 \operatorname{map}(-, X) \cong \pi_0 \operatorname{map}(-, Y)$ be an isomorphism between the cohomology theories they represent on $\mathbf{C}^\omega$. There exists a map $\tilde{f} : X \to Y$ which lifts f (and is therefore an equivalence). $\triangleleft$

Remark 1.4.27. By [Hoy23, Theorem 7], this assumption is often satisfied in practice. Specifically, for $\mathbf{C}$ stable, it suffices for $\mathrm{ho}(\mathbf{C}^\omega)$ to be "countable", in the sense that it has countably many isomorphism classes, and mapping sets between any two objects are countable.

Because of long exact sequences induced by fiber sequences, to check this it suffices to check that there is a countable number of generators, and that the mapping sets between (shifts) of these generators are all countable. $\triangleleft$

Theorem 1.4.28. *Let* $\mathbf{C}$ *be as in Assumption 1.4.14.*

Let $A \in \mathrm{CAlg}(\mathrm{ho}(\mathbf{C}))$ *a homotopy ind-separable homotopy commutative homotopy algebra. In this case, the moduli space* $\mathrm{Alg}(\mathbf{C})^{\simeq} \times_{\mathrm{Alg}(\mathrm{Syn}_{\mathbf{C}}^{\heartsuit})^{\simeq}} \{\pi_0 M(A)\}$ *is contractible.*

If $\mathbf{C}$ *satisfies Assumption 1.4.26, then any lift* $\tilde{A}$ *of* $\pi_0 M(A)$ *is equivalent, as an object of* $\mathbf{C}$*, to* A*.*

Proof. As in Observation 1.2.19, Corollary 1.4.24 shows that the real content here is the non-emptiness of this moduli space.

We first replace $\mathrm{Alg}(\mathbf{C})$ by $\mathrm{Alg}(\mathrm{Syn}_{\mathbf{C}})^{per}$ (algebras in $\mathrm{Syn}_{\mathbf{C}}$ whose underlying object is periodic), and we use [PV22, Theorem 5.4] to prove the existence of a lift - by Lemma 1.4.20, the map $\mathrm{Alg}(\mathbf{C}) \to \mathrm{Alg}(\mathrm{Syn}_{\mathbf{C}})^{per}$ is an equivalence.

In more detail, [PV22, Theorem 5.4] tells us that the obstructions to the existence of a lift live in $\mathrm{Ext}^{n+2}(L^{\mathbb{E}_1}_{\pi_0 M(A)/\pi_0(\mathbf{1})}, \pi_0 M(A)[-n])$, so it suffices to prove that these groups vanish. By [Lur12, Theorem 7.3.5.1], $L^{\mathbb{E}_1}_{\pi_0 M(A)/\pi_0(\mathbf{1})}$ is what we have called $L_{\pi_0 M(A)}$, computed in $\mathrm{Mod}_{\pi_0(\mathbf{1})}(\mathrm{Syn}_{\mathbf{C}})$. As the $\pi_0 M(A)$-bimodule structure on $\pi_0 M(A)[-n]$ is obtained by shifting the bimodule structure on $\pi_0 M(A)$, and in particular by restriction along the multiplication map, Lemma 1.4.22 implies that these Ext-groups vanish (using that $\pi_0 M(A)$ is ind-separable by Corollary 1.4.21). We thus find a periodic algebra $\tilde{A}$ in $\mathrm{Syn}_{\mathbf{C}}$ whose π_0 is $\pi_0 M(A)$ as was claimed.

Now for the second part, by Lemma 1.4.20, we may in fact write $M(\tilde{A})$ for some algebra $\tilde{A} \in \mathbf{C}$, with $\pi_0 M(\tilde{A}) \cong \pi_0 M(A)$ as algebras. By Assumption 1.4.26, it follows that $\tilde{A} \simeq A$, and so $M(\tilde{A}) \simeq M(A)$, all of this lifting the isomorphism $\pi_0 M(\tilde{A}) \cong \pi_0 M(A)$, and so we do get an algebra structure on $M(A)$ lifting the one on $\pi_0 M(A)$. By fully faithfulness of $\mathbf{C} \to \mathrm{Syn}_{\mathbf{C}}$, this is what we wanted. $\square$

Remark 1.4.29. Without Assumption 1.4.26, this proof constructs a homotopy ind-separable algebra $\tilde{A}$ such that $\pi_0 M(\tilde{A}) \cong \pi_0 M(A)$ as algebras in $\mathrm{Syn}_{\mathbf{C}}^{\heartsuit}$, i.e. as multiplicative cohomology theories on $\mathbf{C}^\omega$, but there is no way to guarantee that $A \simeq \tilde{A}$.

Even with Assumption 1.4.26, there is no way to guarantee that the multiplication we obtain on A is the one we started with, they only agree up to phantom maps. $\triangleleft$

One might be tempted to conclude that the same sort of result holds for $\mathrm{Alg}(\mathbf{C})^{\simeq} \times_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))^{\simeq}} \{A\}$, because "$\mathrm{ho}(\mathbf{C}) \to \mathrm{Syn}_{\mathbf{C}}^{\heartsuit}$ is fully faithful"[30]. However,

[30] This is what an earlier version of this document claimed without justification.

$\mathrm{ho}(\mathbf{C}) \to \mathrm{Syn}_{\mathbf{C}}^{\heartsuit}$ is generally not fully faithful, for similar reasons that Assumption 1.4.26 was needed: the hom-set between $\pi_0 M(X)$ and $\pi_0 M(Y)$ is the set of natural transformation $\pi_0 \operatorname{map}(-,X) \to \pi_0 \operatorname{map}(-,Y)$ as functors on $\mathbf{C}^\omega$, not on $\mathbf{C}$. For example, any phantom map from X to Y is sent to 0 in $\mathrm{Syn}_{\mathbf{C}}^{\heartsuit}$. Recall:

Definition 1.4.30. A phantom map $X \to Y$ in $\mathbf{C}$ is a map such that for any compact $c \in \mathbf{C}^\omega$, the composite $c \to X \to Y$ is nullhomotopic. ◁

Remark 1.4.31. Of couse if the nullhomotopy is natural in $c \in \mathbf{C}^\omega$, then $X \to Y$ is nullhomotopic itself, but $\mathrm{Syn}_{\mathbf{C}}^{\heartsuit} = \mathrm{Fun}^\times(\mathrm{ho}(\mathbf{C}^\omega)^{\mathrm{op}}, \mathrm{Ab})$ cannot see this. ◁

Similarly, in the above theorem, if $\mathbf{C}$ satisfies Assumption 1.4.26, we obtain a lift on $M(A)$, or equivalently A, of the algebra structure on $\pi_0 M(A)$, but this tells us that the homotopy algebra structure that we obtained on A need only agree with the original one *up to phantom* maps. And in fact, this is no surprise: our assumption that A be ind-separable only depends on the multiplication μ up to phantom maps, because it is tested after mapping in from compact objects. The above theorem shows that if two homotopy algebra structures on A agree up to phantom maps, then at most one of them can be lifted to an actual algebra structure.

In particular, to properly get statements about $\mathrm{ho}(\mathbf{C})$ rather than $\mathrm{Syn}_{\mathbf{C}}^{\heartsuit}$, one needs to make assumptions about phantom maps.

Observation 1.4.32. If there are no phantom maps $A \to R$, then

$$\hom_{\mathrm{ho}(\mathbf{C})}(hA, hR) \to \hom_{\mathrm{Syn}_{\mathbf{C}}^{\heartsuit}}(\pi_0 M(A), \pi_0 M(R))$$

is injective, and thus so is $\hom_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))}(hA, hR) \to \hom_{\mathrm{Alg}(\mathrm{Syn}_{\mathbf{C}}^{\heartsuit})}(\pi_0 M(A), \pi_0 M(R))$.

For any lift $\tilde{A}$ of A, the map

$$\operatorname{map}_{\mathrm{Alg}(\mathbf{C})}(\tilde{A}, R) \to \hom_{\mathrm{Alg}(\mathrm{Syn}_{\mathbf{C}}^{\heartsuit})}(\pi_0 M(A), \pi_0 M(R))$$

factors through $\hom_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))}(A, hR)$, and is an equivalence by Theorem 1.4.28. If there are no phantom maps $A \to R$, the above implies that both maps

$$\operatorname{map}_{\mathrm{Alg}(\mathbf{C})}(\tilde{A}, R) \to \hom_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))}(A, hR)$$

and

$$\hom_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))}(A, hR) \to \operatorname{map}_{\mathrm{Alg}(\mathrm{Syn}_{\mathbf{C}}^{\heartsuit})}(\pi_0 M(A), \pi_0 M(R))$$

are equivalences. ◁

The above observation buys us the following version of Theorem 1.4.28:

Theorem 1.4.33. *Let* $\mathbf{C}$ *be as in Assumption 1.4.14 and Assumption 1.4.26. Let* $A \in \mathrm{CAlg}(\mathrm{ho}(\mathbf{C}))$ *a homotopy ind-separable homotopy commutative homotopy algebra, and assume that* A *receives no phantom map from any tensor power of* A*. In this case, the moduli space*

$$\mathrm{Alg}(\mathbf{C})^{\simeq} \times_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))^{\simeq}} \{A\}$$

is contractible, i.e. A *admits a unique lift to an algebra in* $\mathbf{C}$*.*

Let $A \in \mathrm{Alg}(\mathbf{C})$ *be a homotopy commutative, ind-separable algebra, and let* $R \in \mathrm{Alg}(\mathbf{C})$ *be a homotopy commutative algebra. If there are no phantom maps from* A *to* R*, then the mapping space* $\operatorname{map}_{\mathrm{Alg}(\mathbf{C})}(A, R)$ *is equivalent to* $\hom_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))}(hA, hR)$[31].

[31] One could weaken the assumption to "Any two homotopy algebra maps that differ by a phantom map are homotopic", but it does not seem like this is a checkable criterion.

We could also run an obstruction-theory argument to get to highly structured commutative structures on homotopy ind-separable homotopy commutative homotopy algebras, but as in **??**, we can also deduce it by more elementary means.

Corollary 1.4.34. *Let* $\mathbf{C}$ *be as in Assumption 1.4.14 and Assumption 1.4.26, and let* $A \in \mathrm{Alg}(\mathbf{C})$ *be an ind-separable homotopy commutative algebra. Let* $\mathcal{O}$ *be an arbitrary one-colored* ∞*-operad. If* A *receives no phantom maps from tensor powers of* A*, the canonical forgetful map*

$$\mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathbf{C})^{\simeq} \times_{\mathrm{Alg}(\mathbf{C})^{\simeq}} \{A\} \to \mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathrm{ho}(\mathbf{C}))^{\simeq} \times_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))^{\simeq}} \{hA\}$$

is an equivalence.

More generally, again as a corollary of Lemma 1.3.41, we obtain:

Corollary 1.4.35. *Let* $\mathbf{C}$ *be as in Assumption 1.4.14 and Assumption 1.4.26, and let* $A \in \mathrm{CAlg}(\mathbf{C})$ *be ind-separable. If* $\mathcal{O}$ *is any* ∞*-operad and* $R \in \mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathbf{C})$ *is an algebra whose underlying* $\mathbb{E}_1$*-algebra is homotopy commutative, and which receives no phantom maps from* A*, then, viewing* A *as an* $\mathcal{O}\otimes\mathbb{E}_1$*-algebra using the unique map of* ∞*-operads* $\mathcal{O}\otimes\mathbb{E}_1 \to \mathbb{E}_\infty$*, the canonical map*

$$\mathrm{map}_{\mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathbf{C})}(A,R) \to \mathrm{hom}_{\mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathrm{ho}(\mathbf{C}))}(hA,hR)$$

is an equivalence.

Remark 1.4.36. As in the separable case, a consequence of this corollary is the discreteness of $\mathrm{map}_{\mathrm{CAlg}(\mathbf{C})}(A,R)$, and, just as in that case, we could give a more elementary proof of this specific fact, cf. Corollary 1.3.43 and its alternative proof. ◁

Corollary 1.4.37. *Let* $\mathbf{C}$ *be as in Assumption 1.4.14 and Assumption 1.4.26, and let* $A \in \mathrm{CAlg}(\mathrm{ho}(\mathbf{C}))$ *be a homotopy commutative, homotopy ind-separable homotopy algebra in* $\mathbf{C}$ *which receives no phantom maps from any tensor power of* A*. For any* $1 \leq d \leq \infty$*, the moduli space* $\mathrm{Alg}_{\mathbb{E}_d}(\mathbf{C})^{\simeq} \times_{\mathrm{Alg}(\mathrm{ho}(\mathbf{C}))^{\simeq}} \{A\}$ *is contractible.*

The upshot of this discussion is that, at least in the compactly-generated case, with some assumption on phantom maps, and using slightly less elementary methods, we are able to recover most of the results from the commutative separable case in the commutative (homotopy) ind-separable case.

1.4.3 Examples

We now discuss examples of ind-separable algebras.

Ind-(separable algebras)

The first natural source of examples is filtered colimits of (commutative) separable algebras. Of course, separable algebras are ind-separable (one can pick the set S to consist of the single separability idempotent).

Lemma 1.4.38. *Let* $\mathbf{C}$ *be as in Assumption 1.4.14, and let* $A_\bullet : I \to \mathrm{CAlg}(\mathbf{C})$ *be a filtered diagram of commutative separable algebras. In this case,* $\mathrm{colim}_I A_i$ *is ind-separable.*

If I *is countable, one can choose* S *in the definition of ind-separable to be countable.*

Proof. For every $i \in I$, let $s_i : \mathbf{1} \to A_i \otimes A_i \to A \otimes A$ be the image in $A \otimes A$ of the separability idempotent of A_i, and let S be the set of the s_i's. It is easy to verify that this does the job. □

Example 1.4.39. Let X be a profinite set. The algebra $C(X;\mathbb{Z})$ of continuous functions on X is ind-separable, as the filtered colimit of $i \mapsto C(X_i;\mathbb{Z})$ for any presentation of X as $\lim_i X_i$, where each X_i is finite. However, if X is not finite, it is not separable. ◁

Example 1.4.40. More generally, Rognes' pro-Galois extension [Rog08, Definition 8.1.1] are ind-separable, by the above lemma together with Proposition 1.5.3. In particular, taking the Goerss–Hopkins–Miller theorem for granted, Devinatz and Hopkins prove in [DH04a] that Morava E-theory is a pro-Galois extension of the $K(n)$-local sphere $\mathbb{S}_{K(n)}$ in the ∞-category of $K(n)$-local spectra. As we wish to give a non-circular proof of the Goerss–Hopkins–Miller theorem, we will give a different proof that Morava E-theory is ind-separable below. ◁

Morava E-theory

In this section, we study Morava E-theory. Example 1.4.40 together with its description as a profinite Galois extension [Rog08] show that it is ind-separable in the ∞-category of $K(n)$-local spectra. However, the proof that it is a pro-Galois extension relies on its highly commutative multiplicative structure, cf. [DH04a], i.e. on the Goerss–Hopkins–Miller theorem.

We offer here a proof of the latter based on our earlier work on ind-separable algebras. The key (and in fact, essentially only) input that we need about Morava E-theory is the computation of $\pi_*(L_{K(n)}(E \otimes E))$, as done by Hopkins–Ravenel, Baker, and revisited by Hovey in [Hov04] (we refer to *loc. cit.* for a brief history of this computation).

Remark 1.4.41. Our results on ind-separable algebras rely on the obstruction theory from [PV22], an obstruction theory which was designed and used to give a proof of the Goerss–Hopkins–Miller theorem, so one might wonder to what extent our proof is actually different. It is not completely clear to the author - it however seems that it is at the very least a re-organization of that proof. Indeed, we first prove a single result about E-theory, namely its ind-separability, and then let the obstruction theory machine take its course, with no further input needed, unlike in [PV22, Section 7], where calculations about Morava E-theory show up alongside the obstruction theory (among other things, Ext-group computations in E_*E-comodules).

Furthermore, as is clear from our proofs, we only really need the obstruction theory to get an $\mathbb{E}_1$-structure and describe $\mathbb{E}_1$-maps to other algebras - our proof clarifies the formal aspect of going from there to higher $\mathbb{E}_d$'s (including $d = \infty$). In particular, we obtain a proof of the folklore fact that Morava E-theory admits a unique $\mathbb{E}_d$-structure also for $1 < d < \infty$ that does not require computing the corresponding $\mathbb{E}_d$-cotangent complexes - while this computation is not complicated (they all vanish, for $d > 1$), it does not allow for generalizations to more general operads of the form $\mathcal{O} \otimes \mathbb{E}_1$.

Finally, while we use the same obstruction theory as in [PV22, Section 7], we apply it to a much simpler ∞-category: our $\mathrm{Syn}_{\mathbf{C}}$ has no completion/localization coming into its definition.

In other words, it is not clear to what extent our proof is really new, but it is a re-packaging of the classical proof which has several advantages. ◁

Fix a (from now on, implicit) prime p and a height n. For a perfect field k of characteristic p, and a formal group $\mathbf{G}$ of height n over k, we have a spectrum $E(k, \mathbf{G})$, called Morava E-theory (or Lubin-Tate theory), usually denoted E or E_n. It can for instance be constructed using the Landweber exact functor theorem, and has a homotopy associative, homotopy commutative ring structure. It is also $K(n)$-local, so we can consider it as an object in $\mathrm{CAlg}(\mathrm{ho}(\mathrm{Sp}_{K(n)}))$. We refer to [Rez98, Part 1] for an introduction to these homotopy ring spectra.

As we mentioned, the only input we need is a computation of $\pi_*(L_{K(n)}(E \otimes E))$. In the statement, we write $\hat{\otimes}$ for the $K(n)$-local tensor product, and $C(X, R)$ for the graded ring of continuous functions from a topological space X to a graded topological ring R. For k algebraic over $\mathbb{F}_p$ (and perfect), Hovey proves:

Theorem 1.4.42 ([Hov04, Theorem 4.11]). *There is an isomorphism*

$$\pi_*(E\hat{\otimes}E) \cong C(\Gamma, E_*)$$

for which the multiplication map $\pi_*(E\hat{\otimes}E) \to E_*$ is identified with evaluation at the neutral element $e \in \Gamma$, $C(\Gamma, E_*) \to E_*$. Here, Γ is the (profinite) Morava stabilizer group, equivalently, the group of automorphisms of E in $\mathrm{Alg}(\mathrm{ho}(\mathrm{Sp}_{K(n)}))$.

Let X be a profinite space with a point $x \in X$. Write $X = \lim_i X_i$ where the X_i's are finite sets, with projection maps $p_i : X \to X_i$, and let $\delta_i : X \to X_i \to \{0,1\}$ denote the indicator function of $(p_i)^{-1}(p_i(x))$.

Lemma 1.4.43. *Composing the δ_i's with the inclusion $\{0,1\} \to \mathbb{Z}$, form the subset S of $C(X,\mathbb{Z})$ consisting of the δ_i's.*

Then evaluation at x, as a ring map $e : C(X,\mathbb{Z}) \to \mathbb{Z}$, witnesses the target as the localization of the source at S.

Proof. As $\mathbb{Z}$ is discrete, $C(X,\mathbb{Z})$ is the colimit of the $C(X_i,\mathbb{Z})$ along restriction maps. Now, the localization of $C(X_i,\mathbb{Z})$ at the indicator function of $p_i(x)$ is clearly $\mathbb{Z}$, and the result follows easily. □

We also recall the following lemma from [Hov04]:

Lemma 1.4.44 ([Hov04, Proposition 2.5]). *Suppose G is a profinite group and R is a graded commutative ring that is complete in the $\mathfrak{a}$-adic topology for some homogeneous ideal $\mathfrak{a}$. Then there is a natural isomorphism $R\hat{\otimes}C(G,\mathbb{Z}) \to C(G,R)$, where $\hat{\otimes}$ is the $\mathfrak{a}$-adically completed tensor product.*

Corollary 1.4.45. *The homotopy algebra $E \in \mathrm{CAlg}(\mathrm{ho}(\mathrm{Sp}_{K(n)}))$ is homotopy ind-separable.*

Proof. We prove that it is in fact homotopy ω-separable.

Let $\Gamma \cong \lim_k \Gamma/U_k$ be a description of the Morava stabilizer group as a countable inverse limit of its finite quotients (we implicitly use here that Γ is first countable, cf. [Hov04, Theoem 1.4], and let δ_k denote the indicator function of U_k (this corresponds to δ_i in Lemma 1.4.43 with $x =$ the neutral element of Γ).

Let $S \subset \pi_0(E\hat{\otimes}E) \cong C(\Gamma, E_0)$ correspond to the set of the δ_k's. We claim that the multiplication map $E\hat{\otimes}E \to E$ witnesses the latter as a telescope of $E\hat{\otimes}E$ at S in $\mathrm{Sp}_{K(n)}$. Indeed, this telescope is the $K(n)$-localization of the same telescope *in* Sp, and we can compute that the homotopy groups of the latter are simply $\pi_*(E\hat{\otimes}E)[S^{-1}] \cong C(\Gamma, E_*)[S^{-1}]$. In particular, they are concentrated in even degrees and the sequence $(p, u_1, ..., u_{n-1})$ is a regular sequence on them. To express this precisely, we can e.g. observe that $E\hat{\otimes}E$ can be viewed as an MU-module, and so we can make sense of $(p, u_1, ..., u_{n-1})$ on it, and they agree with the ones coming from E_*. The same can be said for $E\hat{\otimes}E[S^{-1}]$.

Now, for an MU-module M on which u_n acts invertibly, the $K(n)$-localization is given by $\lim_k M \otimes_{\mathrm{MU}} \mathrm{MU}/(p^k, ...u_{n-1}^k)$, and so, if M is concentrated in even degrees and the sequence $(p, u_1, ..., u_{n-1})$ is regular on M, then the homotopy groups of $L_{K(n)}M$ are simply the $\mathfrak{m} = (p, u_1, ..., u_{n-1})$-adic completion of the homotopy groups of M.

In particular, $\pi_*(L_{K(n)}((E\hat{\otimes}E)[S^{-1}])$ is the $\mathfrak{m}$-adic completion of $C(\Gamma, E_*)[S^{-1}]$, i.e., by Lemma 1.4.44 the $\mathfrak{m}$-adic completion of $E_* \otimes C(\Gamma,\mathbb{Z})[S^{-1}]$, and so, by Lemma 1.4.43, just E_*. This is only a verification on homotopy groups, but it is not hard to see that it implies the desired statement. □

Remark 1.4.46. Note that $\mathrm{Sp}_{K(n)}$ *is* compactly generated, and since its compacts are also dualizable, they are closed under non-empty tensor products. However, the unit is not compact. ◁

Lemma 1.4.47. *The ∞-category of $K(n)$-local spectra satisfies Assumption 1.4.14 and Assumption 1.4.26.*

Proof. Assumption 1.4.14 is clear, in fact $\mathrm{Sp}_{K(n)}$ is compactly generated (as a stable ∞-category) by $L_{K(n)}X$ for any finite spectrum X of type n.

Therefore, by Remark 1.4.27, it suffices to prove that for a type n spectrum X, $\pi_* \operatorname{map}(L_{K(n)}X, L_{K(n)}X)$ is countable. Because X is a finite spectrum, this reduces to proving that $\pi_*(L_{K(n)}X)$ is countable. For this, we refer to the discussion about finite type in the introduction of [Dev07]. We sketch the argument below for the convenience of the reader. In what follows, we let E_n denote Morava theory at height n over $\mathbb{F}_{p^n}$.

The argument is essentially that there is a strongly convergent spectral sequence of signature

$$E_2^{s,t} = H^s(\mathbb{G}_n, (E_n)_t X) \implies \pi_{t-s}(L_{K(n)}X)$$

by [DH04b, Proposition 6.7], using that X has type n.

Using again that X has type n, we observe that $E_n \otimes X$ is in the thick subcategoy generated by $K(n)$. Now the spectral sequence with E_2-page $H^s(\mathbb{G}_n, K(n)_t)$ consists of countable groups: $\mathbb{G}_n$ is a profinite group with a countable basis, and each $K(n)_t$ is a discrete countable group. Thus, the same holds for $H^s(\mathbb{G}_n, (E_n)_t X)$.

Finally, this spectral sequence has a vanishing line, i.e. for a fixed $r \geq 2$, $E_r^{s,t} = 0$ for $s >> 0$ by the smashing theorem, so the countability of the E_2 terms implies the countability of the groups it converges to (there are no infinite extensions because of the vanishing line). □

Remark 1.4.48. Alternatively, the proof of the analogous result for $\mathrm{Sp}_{T(n)}$ is simpler because for a type n finite spectrum X, $L_{T(n)}X$ is a telescope of a v_n-self map on X. One can then simply observe that $L_{T(n)}(E \otimes E) \simeq L_{K(n)}(E \otimes E)$ because E is an MU-module (note that in $\mathrm{Sp}_{T(n)}$, $L_{K(n)}$ is smashing, so this equivalence between $L_{T(n)}$ and $L_{K(n)}$ for MU-modules follows from the same one for MU, which in turn follows from [Rav93, Theorem 2.7.(iii)]). ◁

It already follows from Theorem 1.4.28 and Corollary 1.4.24 that there is a unique commutative algebra in $\mathrm{Sp}_{K(n)}$, $\tilde{E}$, which represents $E^*(-)$ on compact $K(n)$-local spectra, and is equivalent to E as a ($K(n)$-local) spectrum; and furthermore its endomorphism operad is entirely determined by the corresponding one for $E^*(-)$, which one can compute - for exemple its endomorphism space is discrete and isomorphic to the Morava stabilizer group.

For completeness, to reassure the reader about phantom maps and to relate our work to algebra structures in the homotopy category, we spend some time discussing phantom maps to Morava E-theory.

Lemma 1.4.49. *Let E, E' be Landweber exact spectra.*

- *$E \otimes E'$ is Landweber exact;*
- *There are no nonzero phantom maps $E \to E'$.*

Proof. The first part is a consequence of [Rez98, §15], and the second is [Lur10, Lecture 17, Corollary 7]. □

Warning 1.4.50. This lemma is about phantom maps in Sp. There are more phantom maps in $\mathrm{Sp}_{K(n)}$, because the compact objects are of the form $L_{K(n)}X$ for X a finite type $\geq n$ spectrum, so there are fewer compact objects. ◁

Lemma 1.4.51. *Fix a perfect $\mathbb{F}_p$-algebra k, and a formal group $\mathbf{G}$ of height n over k, and let $E = E(k, \mathbf{G})$ be the corresponding Morava E-theory, and let $X \to E$ be a $K(n)$-locally phantom map. It is also phantom in* Sp*.*

Proof. Let V be a finite spectrum, and $f : V \to X$ a map. We wish to show that $V \to X \to E$ is null, or equivalently that $L_{K(n)}V \to L_{K(n)}X \to E$ is null (as E is $K(n)$-local). Now note that $L_{K(n)}V$, being ω_1-compact, is $K(n)$-locally a sequential colimit of finite type n spectra, say $L_{K(n)}V \simeq L_{K(n)}\mathrm{colim}_{\mathbb{N}} V_k$. Now each composite

$$V_k \to L_{K(n)}X \to L_{K(n)}E$$

is null because of our assumption, so the only obstruction to $L_{K(n)}V \to E$ being null is in $\lim^1_{\mathbb{N}} \pi_1 \mathrm{map}(V_k, E)$. It therefore suffices to argue that this $\lim^1$ is 0, by e.g. showing that it satisfies the Mittag-Leffler condition.

But each V_k is a type n complex, so $\mathrm{map}(V_k, E)$ is in the thick subcategory generated by $K(n)$, and thus its π_m, for any fixed m, is an Artinian $\pi_0(E)$-module[32]. This automatically implies the Mittag-Leffler condition. ☐

Corollary 1.4.52. *There are no nonzero phantom maps in $K(n)$-local spectra from any tensor powers of Morava E-theories to any Morava E-theory.*

Proof. Morava E-theories are Landweber exact, so there are no nonzero phantom maps in Sp of the form $\bigotimes_{i=1}^k E(k_i, \mathbf{G}_i) \to E$ by Lemma 1.4.49. By Lemma 1.4.51, this implies that there are no nonzero phantoms in $\mathrm{Sp}_{K(n)}$. ☐

We thus obtain the Goerss–Hopkins–Miller theorem, and its variants for other operads, namely:

Corollary 1.4.53. *Fix a perfect algebraic extension k of $\mathbb{F}_p$, and a formal group $\mathbf{G}$ of height n over k, and let $E = E(k, \mathbf{G})$ be the corresponding Morava E-theory, considered as a (homotopy commutative) homotopy algebra. We have:*

(i) For any weakly reduced ∞-operad $\mathcal{O}$ (e.g. $\mathbb{E}_d, 1 \leq d \leq \infty$), the moduli space

$$\mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathrm{Sp})^{\simeq} \times_{\mathrm{Alg}(\mathrm{ho}(\mathrm{Sp}))^{\simeq}} \{E\}$$

is contractible.

(ii) For any ∞-operad $\mathcal{O}$ and any $R \in \mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathrm{Sp}_{K(n)})$ whose underying algebra is homotopy commutative and which receives no phantom map from E, viewing E as an $\mathcal{O} \otimes \mathbb{E}_1$-algebra using the unique map of ∞-operads $\mathcal{O} \otimes \mathbb{E}_1 \to \mathbb{E}_\infty$, the canonical map

$$\mathrm{map}_{\mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(Sp_{K(n)})}(E, R) \to \hom_{\mathrm{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathrm{ho}(\mathrm{Sp}_{K(n)}))}(hE, hR)$$

is an equivalence. This is the case e.g. if R is a Lubin-Tate theory.

In particular, if we consider the underlying spectrum of E, its space of $\mathbb{E}_d$-structures, for any $1 \leq d \leq \infty$, is equivalent to $B\mathrm{Aut}(\Gamma)$.

[32] If k is a finite field, it is in fact a *finite* abelian group; but for k a large perfect field, even $\pi_0(K(n))$ is not finite - however the homotpy groups of $K(n)$ are finite dimensional vector spaces over $\pi_0(K(n)) \cong k$.

Proof. We first note that $\mathrm{Sp}_{K(n)}$ satisfies Assumption 1.4.14 and Assumption 1.4.26 by Lemma 1.4.47.

Now, by Theorem 1.4.33, Corollary 1.4.34 and Corollary 1.4.35, the only thing left to comment on is why we could write Sp in place of $\mathrm{Sp}_{K(n)}$ in item (i). The point is that E is $K(n)$-local, and $\mathrm{Sp}_{K(n)}$ is a symmetric monoidal Bousfield localization of Sp, so that the space of $\mathcal{O}$-algebra structures on E in Sp is equivalent to the one in $\mathrm{Sp}_{K(n)}$. There, E is ind-separable and so the results from the previous subsection apply. □

The right hand side of these equivalences, i.e. homotopy algebra maps from hE to hR can also be computed, at least under favourable circumstances, e.g. if R is also a Morava E-theory, or more generally if it is even 2-periodic, cf. e.g. [Rez98].

Remark 1.4.54. At some point, the Goerss–Hopkins–Miller theorem was the only known way to construct a commutative ring structure on Morava E-theory. Lurie proposed an alternative construction in [Lur18a] where he directly gives a construction of E-theory with its commutative ring structure. ◁

In the case of $E(k, \mathbf{G})$ for an algebraic extension of $\mathbb{F}_p$, k, we wanted to give a self-contained argument for the ind-separability, for our proof to be at least a somewhat new proof of the Goerss–Hopkins–Miller theorem. We now move on to the case of a general perfect commutative $\mathbb{F}_p$-algebra - for this, we use an analogue of the computation of $\pi_*(L_{K(n)}(E \otimes E))$ for general perfect $\mathbb{F}_p$-algebras which follows from Lurie's work [Lur18a]. We prove:

Theorem 1.4.55. *Let R be a perfect (discrete) commutative $\mathbb{F}_p$-algebras, $\mathbf{G}$ a formal group of height exactly n over R. Assume that R is ind-separable over $\mathbb{F}_p$, i.e. that the multiplication $R \otimes_{\mathbb{F}_p} R \to R$ is a localization at a set S of elements.*

In this situation, $E(R, \mathbf{G})$ is ind-separable in $\mathrm{Sp}_{K(n)}$.

Remark 1.4.56. The ∞-category $\mathrm{Sp}_{K(n)}$ is a smashing localization of $\mathrm{Sp}_{T(n)}$, so that this result implies the same one in $\mathrm{Sp}_{T(n)}$. ◁

As mentioned above, the key ingredient is again a computation of $E(R, \mathbf{G}) \otimes E(R, \mathbf{G})$ (to be taken in $\mathrm{Sp}_{K(n)}$) - this computation is a combination of the universal property of Morava E-theory [Lur18a, Theorem 5.1.5.] together with an explicit analysis of the coproduct in Lurie's $\mathcal{FG}$ (cf. [Lur18a, Remark 5.1.6.]), by way of an analysis of the stack of isomorphisms between formal groups, cf. [Goe08, Theorem 5.23] (Goerss credits Lazard for this result).

Proof. Consider $R \otimes_{\mathbb{F}_p} R$ with the two formal groups induced from $\mathbf{G}$ along the two inclusions of R, say $\mathbf{G}_1$ and $\mathbf{G}_2$.

By [Goe08, Theorem 5.23], we can find a sequence R_k of finite étale extensions of $R \otimes_{\mathbb{F}_p} R$ whose colimit R_∞ classifies isomorphisms of formal groups between $\mathbf{G}_1$ and $\mathbf{G}_2$ (in particular, it acquires one specific formal group, $\mathbf{G}_\infty$, which comes with isomorphisms to the basechanges of $\mathbf{G}_i, i = 1, 2$)[33].

Furthermore, it is not hard to deduce from [Lur18a, Theorem 5.1.5.] (see also [Lur18a, Remark 5.1.6.]) that $E(R, \mathbf{G}) \otimes E(R, \mathbf{G}) \simeq E(R_\infty, \mathbf{G}_\infty)$ as commutative algebras (the tensor product is taken in $K(n)$-local spectra), and the multiplication map to $E(R, \mathbf{G})$ corresponds to the map $R_\infty \to R$ classifying the identity isomorphism of $\mathbf{G}$.

Let S be a set of elements of $R \otimes_{\mathbb{F}_p} R$ such that the multiplication map

$$R \otimes_{\mathbb{F}_p} R \to R$$

[33] Note that in [Goe08], Goerss $\mathrm{Iso}(\mathbf{G}_1, \mathbf{G}_2)_k \to \mathrm{Spec}(R \otimes_{\mathbb{F}_p} R)$ is finite étale, and hence it is also affine - the corresponding ring is our R_k.

witnesses the target as the localization of the domain at S (this exists by assumption). We claim that each induced map $R_k[S^{-1}] \to R$ is a localization, in fact, a split localization. Indeed, R_k is a finite étale extension of $R \otimes_{\mathbb{F}_p} R$, so that $R_k[S^{-1}]$ is a finite étale extension of R. But via $R_k \to R_\infty \to R$, this is an étale extension which admits a section, and thus a splitting, and this splitting gives an R-algebra isomorphism $R_k[S^{-1}] \cong R \times T$, for some R-algebra T. Let e_k be the corresponding idempotent in $R_k[S^{-1}]$. It follows that $R_\infty[(S \cup \{e_k, k \in \mathbb{N}\})^{-1}] \cong R$ via the canonical map $R_\infty \to R$.

Thus, to conclude, it suffices to prove the following: if $(R, \mathbf{G}) \to (A, \mathbf{H})$ be a morphism of formal groups over perfect $\mathbb{F}_p$-algebras which witnesses the target as a localization of the source at a set S of elements, then there is a set of elements $\tilde{S} \subset \pi_0 E(R, \mathbf{G})$ such that $E(R, \mathbf{G}) \to E(A, \mathbf{H})$ witnesses the target as the localization of the source at this set of elements (in $\mathrm{Sp}_{K(n)}$).

But this follows from [Lur18a, Theorems 5.1.5. and 5.4.1.]: indeed, consider any subset $\tilde{S} \subset \pi_0 E(R, \mathbf{G})$ whose image under the surjective morphism $\pi_0 E(R, \mathbf{G}) \to R$ is S, and let $T = E(R, \mathbf{G})[\tilde{S}^{-1}]$ (computed in $\mathrm{Sp}_{K(n)}$). By [Lur18a, Remark 4.1.10.], T is complex periodic. It is then clear that T and $E(A, \mathbf{H})$ have the same universal property in the ∞-category of complex periodic $K(n)$-local commutative ring spectra.

□

Remark 1.4.57. When R is strictly henselian, one can prove a converse to this theorem, that is, if $E(R, \mathbf{G})$ is ind-separable in $\mathrm{Sp}_{K(n)}$, then R is ind-separable over $\mathbb{F}_p$. It is also reasonable to expect a converse in general, but I have not found a proof.

In particular, as there are strictly henselian rings that are not ind-separable over $\mathbb{F}_p$, one sees that the result is not true in full generality. ◁

Remark 1.4.58. We make a final note that the results in this subsection say nothing about the homotopy algebra structures on the spectra $E(R, \mathbf{G})$. In particular, while for a given perfect field k, and a given height n, the various spectra $E(k, \mathbf{G}), \mathrm{ht}(\mathbf{G}) = n$ are homotopy equivalent [LP22], they are not equivalent as ring spectra if the formal groups are not isomorphic. But this is already the case at the level of homotopy algebras (the formal group only depends on the homotopy algebra structure). ◁

1.5 Examples

We take a bit of time away from theory to look at some examples of separable algebras. All the examples we mention here are fairly standard. We begin with Galois extensions, and then move on to certain "cochain algebras" which appear among other places in equivariant stable homotopy theory, and can be organized through ∞-categories of spans. We later go to the setting of group rings under certain assumptions on the "cardinality" of the group - these appear among other places in ambidexterity theory, and can also be organized through ∞-categories of spans. We later mention examples related to algebraic geometry, namely we recall that étale maps of schemes induce separable algebras, and that (certain) Azumaya algebras are separable. Finally, we conclude with a non-example, by pointing out that separability is really a "linear" story, namely that there are no interesting examples in cartesian cases.

Warning 1.5.1. In the cases of ∞-categories of spans, it is convenient to use $(\infty, 2)$-categorical technology to organize the proofs that the relevant algebras are separable, by going through the $(\infty, 2)$-category of correspondences. However, some of this technology has not been developed yet, and is only really known in the case of 2-categories. The reader can thus view these examples as either sketches ("a complete proof is left to the reader"), conjectures, or as proving less than what we claim, in the following sense : our proofs will still be valid at the homotopy category level, because there we only need the 2-categorical version of the aforementioned technology. We note that because of the results of the previous sections, for most purposes, this is not a real restriction: as long as one maps those span ∞-categories to an *additive* symmetric monoidal ∞-category, homotopy separability guarantees full-fledged separability.

We will indicate with a (*) the statements that are subject to this warning. $\triangleleft$

1.5.1 Galois theory

In this section, we review one of the main examples of separability, namely Galois extensions. Originally introduced in field theory, they were later studied in the more general context of commutative rings [AG60], and later, by work of Rognes [Rog08], in the setting of commutative ring spectra. His definition extends verbatim to more general stable homotopy theories. We recall the definition for the convenience of the reader:

Definition 1.5.2. Let $\mathbf{C}$ be a cocompletely, stably symmetric monoidal ∞-category, and let $A \in \mathrm{CAlg}(\mathbf{C})$. For an $\mathbb{E}_1$-group G, an object $B \in \mathrm{CAlg}(\mathrm{Mod}_A)^{BG}$ is called a G-Galois extension of A if:

- The induced map $A \to B^{hG}$ is an equivalence (of commutative algebras);
- the natural map $B \otimes_A B \to F(G_+, B)$, adjoint to the action map

$$A[G] \otimes_A B \otimes_A B \to B \otimes_A B \to B$$

  is an equivalence (informally, this map is given by $x \otimes y \mapsto (g \mapsto g(x)y)$).

$\triangleleft$

When G is a discrete group, $F(G_+, B) \simeq \prod_G B$ and the multiplication map $B \otimes_A B \to B$ becomes identified with evaluation at $e \in G, \prod_G B \to B$. In particular, this clearly has a section as $\prod_G B$-modules, and we obtain:

Proposition 1.5.3 ([Rog08, Lemma 9.1.2.]). *Let G be a discrete finite group, and $A \to B$ a G-Galois extension in* $\mathrm{CAlg}(\mathbf{C})$. *In this case, B is a separable A-algebra.*

Remark 1.5.4. In the case of a Galois extension, a proof of Theorem 1.3.19 was already sketched by Mathew in [Mat16, Theorem 6.25]. $\triangleleft$

Example 1.5.5. Any Galois extension of fields $K \to L$ is $\mathrm{Gal}(L/K)$-Galois. More generally, Galois extensions of commutative rings are Galois, this follows from [Rog08, Proposition 2.3.4.(c)]. $\triangleleft$

Example 1.5.6. Profinite Galois extension in the sense of [Rog08, Definition 8.1.1] are in general only ind-separable, cf. Section 1.4. $\triangleleft$

1.5.2 Spans and equivariant stable homotopy theory

In this subsection and the next, we will deal with span ∞-categories. For an account, see [Bar17; BGS19] [34]. For the proofs, it will also be convenient to use the $(\infty, 2)$-categories of correspondences that extend them [Ste20],[Mac22].

[34] Where the ∞-category of spans is called the "effective Burnside category"

One reason to be interested in span ∞-categories is their relation to equivariant stable homotopy theory: the ∞-category of genuine G-spectra, Sp_G, can be described as the ∞-category of spectral Mackey functors, i.e. direct sum preserving functors $\mathrm{Span}(\mathrm{Fin}_G)^{\mathrm{op}} \to \mathrm{Sp}$.

In [BDS14], Balmer, Dell'Ambrogio and Sanders describe, for a subgroup $H \leq G$, the ∞-category Sp_H as the ∞-category of modules over some algebra $A_H^G \in \mathrm{CAlg}(\mathrm{Sp}_G)$ which they prove is separable. In particular, all their work at the level of homotopy categories works at the level of stable ∞-categories by Section 1.2.

Note that A_H^G is the image under the symmetric monoidal Yoneda embedding $\mathrm{Span}(\mathrm{Fin}_G) \to \mathrm{Sp}_G$ of an algebra *in* $\mathrm{Span}(\mathrm{Fin}_G)$. The object of this subsection is to prove that this algebra is already separable there. Note that $\mathrm{Span}(\mathrm{Fin}_G)$ is not additive, so we cannot apply [BDS14, Theorm 1.1] directly and work in $\mathrm{ho}(\mathrm{Span}(\mathrm{Fin}_G))$, where the result is simpler to prove.

More generally, we prove

Theorem 1.5.7 (*). *Let C be a small ∞-category with finite limits, and $X \in C$. We view X as a commutative algebra in C^{op}, and thus, $X^\vee$ as a commutative algebra in $\mathrm{Span}(C)$ under the canonical symmetric monoidal functor $C^{\mathrm{op}} \to \mathrm{Span}(C)$.*

If the evaluation map from the cotensoring $X^{S^1} \to X$ is an equivalence, then $X^\vee$ is a separable commutative algebra in $\mathrm{Span}(C)$.

Remark 1.5.8. This applies in particular if C is a 1-category such as Fin_G. ◁

In the course of this proof, we use the following:

Conjecture 1.5.9. *Let $\mathfrak{B}$ be a symmetric monoidal $(\infty, 2)$-category, $A \in \mathrm{Alg}(\iota_1\mathfrak{B})$ an algebra in (the underlying ∞-category of) $\mathfrak{B}$, M, N A-modules in $\mathfrak{B}$, and $f : M \to N$ an A-module map. If f admits a right adjoint f^R, and the square:*

$$\begin{array}{ccc} A \otimes M & \xrightarrow{A\otimes f} & A \otimes N \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

is horizontally right-adjointable, then f^R is canonically A-linear; and more precisely f admits a right adjoint in $\mathrm{Mod}_A(\mathfrak{B})$. ◁

We note that in the case where $\mathfrak{B}$ is the $(\infty, 2)$-category of ∞-categories, this conjecture is essentially proved in [Lur12, Remark 7.3.2.9].

Remark 1.5.10. This conjecture should also have a more general form, similarly to the calculus of mates in [HHL+20]. Namely, in the above, if we only assume that f admits a right adjoint f^R, then this right adjoint should be canonically *lax* A-linear, that is, come with suitably compatible and coherent maps

$$\text{“}a \otimes f^R(m) \to f^R(a \otimes m)\text{''}$$

and it should then be a property (namely, adjointability) that these maps are equivalences. Conversely, the left adjoint of a lax A-linear morphism should always be oplax A-linear, and this should be a perfect correspondence between oplax A-linear left adjoints, and lax A-linear right adjoints. In the case $\mathfrak{B} = \mathrm{Cat}$, this can be deduced from [HHL+20], but below we need it for $\mathfrak{B}$ being an $(\infty, 2)$-category of correspondences. ◁

As explained in the introduction to this section, this conjecture is well-known (and classical) in the case of 2-categories, so the arguments that we give apply unconditionally to the homotopy category $\mathrm{ho}(\mathrm{Span}(C))$, and thus to any additive ∞-category $\mathbf{C}$ with a symmetric monoidal map $\mathrm{Span}(C) \to \mathbf{C}$.

Proof. We use the $(\infty, 2)$-category of correspondences, $\mathrm{Corr}(C)$, see [Ste20], [Mac22]. In particular, its underlying ∞-category is $\mathrm{Span}(C)$.

We note that the multiplication map of X is given by the span

$$X \times X \xleftarrow{\Delta} X \xrightarrow{=} X$$

We note that, as a morphism in $\mathrm{Corr}(C)$, it admits a left adjoint, cf. [Ste20]. Because the multiplication map is $X \times X$-linear, it is a *property* that this left adjoint is actually $X \times X$-linear, namely that the square from be left adjointable (by the dual of Conjecture 1.5.9).

Let us assume for now that we have checked this - the composite is then the composite of spans

$$X \xleftarrow{=} X \xrightarrow{\Delta} X \times X \xleftarrow{\Delta} X \xrightarrow{=} X$$

which is easily seen to be given by the span $X \xleftarrow{ev} X^{S^1} \xrightarrow{ev} X$. Our assumption guarantees that this is an equivalence, hence an equivalence of $X \times X$-modules, and so up to composing by its inverse, we find that X is separable as an algebra.

Let us now check the property : we need to check that the oplax-X-linear structure maps are strict, we do it for the left-X-linear one, and the right-X linear one follows by symmetry. The left X-linearity of the multiplication map is given by the following commutative diagram in $\mathrm{Corr}(C)$:

$$\begin{tikzcd} X \times X \times X \arrow[r, "X \times \mu"] \arrow[d, "\mu \times X"'] & X \times X \arrow[d, "\mu"] \\ X \times X \arrow[r, "\mu"] & X \end{tikzcd}$$

In this diagram, all maps are in C^{op}, so this is just the image under $C^{\mathrm{op}} \to \mathrm{Corr}(C)$ of the canonical coassociativity diagram for X, and this canonical coassociativity diagram is a pullback square:

$$\begin{tikzcd} X \times X \times X & X \times X \arrow[l, "X \times \Delta"'] \\ X \times X \arrow[u, "\Delta \times X"] & X \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd}$$

In particular, it is adjointable in $\mathrm{Corr}(C)$ e.g. by [Ste20],[Mac22], so we are done. □

1.5.3 Spans, ambidexterity and Thom spectra

In this subsection, we study a situation similar to the one of the previous subsection, except that we start with a monoid G in C, and view it as a monoid in $\mathrm{Span}(C)$.

The result that we prove is:

Theorem 1.5.11 (*). *Let $f : \mathrm{Span}(C) \to \mathbf{C}$ be a symmetric monoidal functor, and suppose it sends the span $\mathrm{pt} \leftarrow G \to \mathrm{pt}$ to an equivalence. Then $f(G)$ is a separable algebra in $\mathbf{C}$.*

Example 1.5.12. Consider the case where $C = \mathrm{Fin}$, the category of finite sets. In this case, $\mathrm{Span}(\mathrm{Fin})$ is the initial semiadditively symmetric monoidal ∞-category. In particular, for any semiadditively symmetric monoidal $\mathbf{C}$, there is an essentially unique symmetric monoidal, semiadditive functor $\mathrm{Span}(\mathrm{Fin}) \to \mathbf{C}$. It sends a finite set X to $\bigoplus_X \mathbf{1}$.

In this case, a G as in the theorem is simply a finite group. The theorem is saying that if its order $|G|$ is invertible in $\mathbf{C}$, then $\mathbf{1}[G]$ is separable. This is typical from classical algebra: the group algebra $\mathbb{Q}[G]$ is always separable, and more generally, for a field k, $k[G]$ is separable over k if and only if $|G| \in k^\times$. $\triangleleft$

A generalization of the previous example, and our motivating example for this section, comes from the theory of higher semi-additivity, cf. [Har20; CSY18]. This is a context where one can sum not only over finite sets, but also over finite groupoids, or more generally, m-finite spaces, i.e. spaces X with finitely many components, and with, at every point, $\pi_k(X) = 0$ for $k > m$ (or possibly only the m-finite spaces, all of whose homotopy groups are p-groups, for some fixed prime p). We refer to the above references for a more detailed account of this theory. We let $\mathcal{S}_m^{(p)}$ denote the ∞-category of m-finite spaces all of whose homotopy groups are p-groups.

In that case, when $\mathbf{C}$ is (p-typically) m-semiadditive [CSY21a, Definition 3.1.1], there is a unique symmetric monoidal functor $\mathrm{Span}(\mathcal{S}_m^{(p)}) \to \mathbf{C}$ which preserves p-typical m-finite colimits, which we denote by $\mathbf{1}[-]$. The span $\mathrm{pt} \leftarrow G \to \mathrm{pt}$ is sent to the *cardinality* $|G|_{\mathbf{C}}$ of G, as a morphism $\mathbf{1} \to \mathbf{1}$. The property that this be an equivalence is related to the so-called *semi-additive height* of $\mathbf{C}$. For example, "height 0" corresponds to the rational case, where all these cardinalities are invertible. Higher heights are also related to chromatic height - we refer to [CSY21a] for more details.

Proof. The proof again makes use of the higher categorical structure of $\mathrm{Corr}(C)$. Just as before, we observe that $\mu : G \times G \to G$ has a right adjoint, and by Conjecture 1.5.9, it is simply a property for it to be $G \times G^{\mathrm{op}}$-linear, which we can check in the exact same way as in the proof of Theorem 1.5.7. The key point is that the associativity diagram for G in C (which is also the "left G-linearity" diagram) is a pullback diagram in C:

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mu \times G"] \arrow[d, "G \times \mu"'] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"'] & G \end{tikzcd}$$

and hence, it is adjointable in $\mathrm{Corr}(C)$. This is exactly what we need for the adjoint of μ to be $G \times G^{\mathrm{op}}$-linear.

Now, this gives us a $G \times G^{\mathrm{op}}$-linear morphism $G \to G \times G^{\mathrm{op}}$ in $\mathrm{Span}(C)$. The composition $G \to G \times G^{\mathrm{op}} \to G$ is given by the span $G \xleftarrow{\mu} G \times G \xrightarrow{\mu} G$, and as a morphism in $\mathrm{Span}(C)$, this is equivalent to $G \xleftarrow{pr_1} G \times G \xrightarrow{pr_1} G$ because of the shear map $G \times G \to G \times G$. We can rewrite the latter span as $(\mathrm{pt} \leftarrow G \to \mathrm{pt}) \times G$. The claim now follows in the same way: up to inverting the span $\mathrm{pt} \leftarrow G \to \mathrm{pt}$, we have a separability idempotent. $\square$

Example 1.5.13. In [CSY21b, Definition 4.7], the authors introduce, for any stable ∞-semiadditive presentably symmetric monoidal ∞-category $\mathbf{C}$ a height n p^rth-cyclotomic extension $\mathbf{1}[\omega_{p^r}^{(n)}]$, which is a higher height analogue of the usual cyclotomic extensions.

This cyclotomic extension is defined as the splitting of an idempotent on $\mathbf{1}[B^n C_{p^r}]$ and the definition of "height n" guarantees that $|B^n C_{p^r}|$ is invertible in $\mathbf{C}$, in other words, that the previous theorem applies. So $\mathbf{1}[B^n C_{p^r}]$ is separable, and hence so is $\mathbf{1}[\omega_{p^r}^{(n)}]$. This shows that, even if it is not always Galois (cf. [Yua22, Proposition 3.9]), it is separable, which is a notion not too far from "étale" in the commutative setting. ◁

For a group G, the group algebra $\mathbf{1}[G]$ can be seen as the colimit of the constant diagram with value $\mathbf{1}$, indexed by G. We saw in Example 1.5.12 that when $|G|$ is invertible, this algebra is separable - we now describe a slight extension of this result, namely to Thom objects. First, we recall the following construction:

Construction 1.5.14. Let X be a space, and $f : X \to \mathrm{Pic}(\mathbf{C})$ a map, where $\mathrm{Pic}(\mathbf{C}) \subset \mathbf{C}^{\simeq}$ is the maximal subgroupoid spanned by the invertible objects in $\mathbf{C}$. One may take the colimit of the composite $X \to \mathrm{Pic}(\mathbf{C}) \to \mathbf{C}$, if it exists.

If $\mathbf{C}$ is, say, cocomplete, this corresponds to the unique colimit-preserving functor $\mathcal{S}_{/\mathrm{Pic}(\mathbf{C})} \to \mathbf{C}$ which restricts to the canonical inclusion along the Yoneda embedding $\mathrm{Pic}(\mathbf{C}) \to \mathcal{S}_{/\mathrm{Pic}(\mathbf{C})} \to \mathbf{C}$.As a consequence, this functor $\mathcal{S}_{/\mathrm{Pic}(\mathbf{C})} \to \mathbf{C}$ is symmetric monoidal, so it sends groups G equipped with a group map $G \to \mathrm{Pic}(\mathbf{C})$ to an algebra object in $\mathbf{C}$. ◁

Proposition 1.5.15 (*). *Assume* $\mathbf{C}$ *is* m*-semiadditive for some* $0 \leq m \leq \infty$. *The above construction extends uniquely to an* m*-semiadditive, symmetric monoidal functor* $\mathrm{Span}((\mathcal{S}_m)_{/\mathrm{Pic}(\mathbf{C})}) \to \mathbf{C}$.

In particular, if $f : G \to \mathrm{Pic}(\mathbf{C})$ *is a group map from an* m*-finite group* G*, where* $|G|_{\mathbf{C}}$ *is invertible in* $\mathbf{C}$*, then its Thom object* $\mathrm{colim}_G f$ *is a separable algebra in* $\mathbf{C}$.

Proof. The "in particular" part follows from Theorem 1.5.11, together with the observation that the span $(\mathrm{pt}, \mathbf{1}) \leftarrow (G, \mathbf{1}) \to (\mathrm{pt}, \mathbf{1})$ is indeed sent to $|G|_{\mathbf{C}}$ in $\mathbf{C}$.

Now, for the first part, namely the existence of the map, we use [Har20, Theorem 5.28] in the special case where $\mathcal{C} = \mathrm{Pic}(\mathbf{C})$. We note that the canonical symmetric monoidal structure on $\mathrm{Span}((\mathcal{S}_m)_{/X})$, when X is a symmetric monoidal ∞-groupoid, is the one induced by the universal property of [Har20, Theoem 5.28] because the natural map $X \to \mathrm{Span}((\mathcal{S}_m)_{/X})$ is symmetric monoidal for this symmetric monoidal structure.

The cited theorem thus implies that a symmetric monoidal map $X \to \mathbf{C}$ (here the inclusion $\mathrm{Pic}(\mathbf{C}) \subset \mathbf{C}$) extends essentially uniquely to an m-semiadditive symmetric monoidal functor $\mathrm{Span}((\mathcal{S}_m)_{/X}) \to \mathbf{C}$. □

Example 1.5.16. In the case $m = 0$, an m-finite group is simply an ordinary finite group, and if furthermore every point in G is sent to the unit $\mathbf{1} \in \mathrm{Pic}(\mathbf{C})$, then the Thom object is simply a twisted group ring $\mathbf{1}_\alpha[G]$. ◁

Example 1.5.17. The algebra from Example 1.3.32 is an example of this construction. Indeed, let $\mathbf{D} = \mathrm{Mod}_{\mathbb{Q}[t^{\pm 1}]}$ with t in degree $2d$ for some odd $d \neq 1$. We let $G = H \rtimes \mathbb{Z}/d$ as in Example 1.3.32, and $G \to \mathrm{Pic}(\mathbf{D})$ is the map $G \to \mathbb{Z}/d \to \mathrm{Pic}(\mathbf{D})$, where the latter map picks out $\Sigma^2\mathbb{Q}[t^{\pm 1}]$. Let us briefly explain why $\mathbb{Z}/d \to \mathrm{Pic}(\mathbf{D})$ can be made into a map of commutative groups. This picard element is clearly classified by a map $\mathbb{S} \to \mathrm{Pic}(\mathbf{D})$, and because it is d-torsion, by a map $\mathbb{S}/d \to \mathrm{Pic}(\mathbf{D})$. The homotopy groups of $\mathrm{Pic}(\mathbf{D})$ are rational above π_2, and the homotopy groups of $\mathbb{S}/d$ are finite, so this map canonically factors through $\tau_{\leq 1}(\mathbb{S}/d)$, which is $\mathbb{Z}/d$ because d is odd.

Now colimits over $G, \mathbb{Z}/d$ are just coproducts, so it is easy to check that the algebra structure in the homotopy category of $\mathbf{D}$ is the one we described in Example 1.3.32. Because $|G|$ and $|\mathbb{Z}/d|$ are invertible in $\mathbf{D}$, we find that these algebras are indeed separable (note that this does not depend on Conjecture 1.5.9 because $\mathbf{D}$ is additive). ◁

Along the way, we record the following result we have sketched in the previous example (cf. also [Law20, Example 2.30] and the surrounding discussion):

Lemma 1.5.18. *Let $\mathbf{D}$ be a symmetric monoidal ∞-category, and $L \in \mathrm{Pic}(\mathbf{D})$ be an invertible element with $L^{\otimes d} \simeq \mathbf{1}_{\mathbf{D}}$. Assume that d is odd, and invertible in $\pi_* \operatorname{map}(\mathbf{1}_{\mathbf{D}}, \mathbf{1}_{\mathbf{D}}), * \geq 1$. The space of maps of commutative groups $\mathbb{Z}/d \to \mathrm{Pic}(\mathbf{D})$ classifying L is equivalent to the space of equivalences $L^{\otimes d} \simeq \mathbf{1}$.*

1.5.4 Twists of Morava K- and E-theories

In this subsection, we recover the main results of [SW15] (namely [SW15, Theorems 1.1 and 1.2]), and correct along the way [SW15, Theorem 1.2], as well as get rid of any need for obstruction theory. For the convenience of the reader, we recall these theorems:

Theorem 1.5.19 ([SW15, Theorem 1.1]). *Let $n \geq 1$ and $K(n)$ be ($2(p^n - 1)$-periodic) Morava K-theory at height n and an implicit prime p. The canonical map is an equivalence:*

$$\mathrm{Map}_{\mathcal{S}_*}(K(\mathbb{Z}, n+2), BGL_1(K(n))) \to \hom_{\mathrm{Alg}(K(n)_*)}(K(n)_* K(\mathbb{Z}, n+1), K(n)_*)$$

We warn the reader that, as stated, the following theorem contains a small mistake, which we correct later:

Theorem 1.5.20 ([SW15, Theorem 1.2]). *Let $n \geq 1$ and E_n be Morava E-theory at height n and an implicit prime p. The canonical maps are equivalences:*

$$\begin{aligned}\mathrm{Map}_{\mathrm{CAlg}(\mathcal{S})}(K(\mathbb{Z}, n+2), BGL_1(E_n)) &\to \mathrm{Map}_{\mathcal{S}_*}(K(\mathbb{Z}, n+2), BGL_1(E_n)) \\ &\to \hom_{\mathrm{Alg}((E_n)_*)}((E_n)_* K(\mathbb{Z}, n+1), (E_n)_*)\end{aligned}$$

Their proof relies on a computation of the Morava K-theories of Eilenberg-MacLane spaces due to Ravenel and Wilson [RW80], together with Goerss–Hopkins obstruction theory. We aim to explain how one can get rid of the latter, and recover this result based only on Ravenel and Wilson's calculations.

Warning 1.5.21. Our proof proceeds by proving that $E_n[K(\mathbb{Z}/p^k, n)]$ is separable in $K(n)$-local spectra – in particular, it involves $K(n)$-localization, and thus we note that [SW15, Theorem 1.2] is wrong as stated. Namely, the correct equivalence is

$$\mathrm{Map}_{\mathrm{CAlg}(E_n)}(E_n[K(\mathbb{Z}, n+1)], E_n) \simeq \hom_{\mathrm{CAlg}(\pi_*(E_n))}((E_n)^\vee_* K(\mathbb{Z}, n+1), \pi_*(E_n))$$

where $(E_n)^\vee_*(X) := \pi_*(L_{K(n)}(E_n \otimes X))$ is completed Morava E-theory. One can prove that without completion, this equivalence does not hold. For example if n is even (so that $n+1$ is odd, and $H_*(K(\mathbb{Z}, n+1); \mathbb{Q}) = \mathbb{Q}[\epsilon], |\epsilon| = 1$), it is not so hard[35] to prove that

$$\hom_{\mathrm{CAlg}(\pi_*(E_n))}((E_n)_* K(\mathbb{Z}, n+1), (E_n)_*) \cong \hom_{\mathrm{CAlg}(\pi_*(E_n))}((E_n)_*, (E_n)_*) \cong \mathrm{pt}$$

On the other hand, with completed Morava E-theory, one does find a set isomorphic to $\mathbb{Z}_p$, as claimed in [SW15]. $\triangleleft$

Thus, the corrected version of Theorem 1.5.20 is:

Theorem 1.5.22 ([SW15, Theorem 1.2]). *Let $n \geq 1$ and E_n be Morava K-theory at height n and an implicit prime p. The canonical maps are equivalences:*

$$\begin{aligned}\mathrm{Map}_{\mathrm{CAlg}(\mathcal{S})}(K(\mathbb{Z}, n+2), BGL_1(E_n)) &\to \mathrm{Map}_{\mathcal{S}_*}(K(\mathbb{Z}, n+2), BGL_1(E_n)) \\ &\to \hom_{\mathrm{Alg}((E_n)_*)}((E_n)^\vee_* K(\mathbb{Z}, n+1), (E_n)_*)\end{aligned}$$

[35] The proof uses the rational computation together with the fact that $(E_n)_*$ is torsion free and concentrated in even degrees.

We prove Theorem 1.5.19, indicating along the way the necessary changes for Theorem 1.5.22.

Proof. Let

$$M_K := \mathrm{Map}_*(K(\mathbb{Z},n+2), BGL_1(K(n))), M_E^{d+1} := \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_d}(\mathcal{S})}(K(\mathbb{Z},n+2), BGL_1(E_n))$$

(including for $d=0$ and ∞). We begin by noting that

$$M_K \simeq \mathrm{Map}_{\mathrm{Alg}(\mathcal{S})}(K(\mathbb{Z},n+1), GL_1(K(n)) \simeq \mathrm{Map}_{\mathrm{Alg}(\mathrm{Sp}_{K(n)})}(\mathbb{S}_{K(n)}[K(\mathbb{Z},n+1)], K(n))$$

In the case of E-theory, one can similarly move to

$$M_E^d \simeq \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_d}(\mathrm{Mod}_{E_n}(\mathrm{Sp}_{K(n)}))}(E_n[K(\mathbb{Z},n+1)], E_n)$$

Now, $K(\mathbb{Z},n+1)$ is p-adically equivalent to $\mathrm{colim}_k K(\mathbb{Z}/p^k,n)$ so that we can rewrite our space as inverse limits of similar spaces with $K(\mathbb{Z}/p^k,n)$ in place of $K(\mathbb{Z},n+1)$. By [CSY21a, Lemma 5.3.3.], we may apply Theorem 1.5.11 to $G = K(\mathbb{Z}/p^k,n)$ and $\mathbf{C} = \mathrm{Sp}_{K(n)}$ (resp. $\mathrm{Mod}_{E_n}(\mathrm{Sp}_{K(n)})$) and obtain that $\mathbb{S}_{K(n)}[K(\mathbb{Z}/p^k,n)]$ (resp. $E_n[K(\mathbb{Z}/p^k,n)]$, computed in $\mathrm{Sp}_{K(n)}$) is separable as a $K(n)$-local (E_n-)algebra[36].

In the case of E-theory, we can directly conclude that

$$\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_d}(E_n)}(E_n[K(\mathbb{Z}/p^k,n)], E_n) \simeq \hom_{\mathrm{CAlg}(\mathrm{ho}(\mathrm{Mod}_{E_n}))}(E_n[K(\mathbb{Z}/p^k,n)], E_n)$$

by Corollary 1.3.39. By [HL13, Proposition 3.4.3., Proposition 2.4.10.], $\pi_*(E_n[K(\mathbb{Z}/p^k,n)])$ is a free $(E_n)_*$-module, so that

$$\hom_{\mathrm{CAlg}(\mathrm{ho}(\mathrm{Mod}_{E_n}))}(E_n[K(\mathbb{Z}/p^k,n)], E_n) \cong \hom_{\mathrm{CAlg}((E_n)_*)}(\pi_*(E_n[K(\mathbb{Z}/p^k,n)]), (E_n)_*)$$

Thus

$$M_E^d \simeq \hom_{\mathrm{Alg}((E_n)_*)}(\pi_*(\mathrm{colim}_k L_{K(n)}(E_n[K(\mathbb{Z}/p^k,n)])), (E_n)_*)$$

Finally, $(E_n)_*$ is $\mathfrak{m}$-adically complete (where $\mathfrak{m} = (p, v_1, ..., v_{n-1})$ is the maximal ideal in the local ring $\pi_0(E_n)$), and $\mathrm{colim}_k L_{K(n)}(E_n[K(\mathbb{Z}/p^k,n))$ is concentrated in even degrees, so that the $\mathfrak{m}$-adic completion of its homotopy groups is the same as the homotopy groups of its $K(n)$-localization, and so we get:

$$M_E^d \simeq \hom_{\mathrm{Alg}((E_n)_*)}(\pi_*(L_{K(n)}(E_n[K(\mathbb{Z},n+1)])), (E_n)_*)$$

which was to be proved.

In the case of K-theory, the approach is the same but more care must be taken at the prime 2, as Morava K-theory is not homotopy commutative there[37].

In the case of odd primes, Morava K-theory is homotopy commutative, and so by combining Theorem 1.2.14 and Corollary 1.3.33 we obtain

$$M_K \simeq \hom_{\mathrm{Alg}(\mathrm{ho}(\mathrm{Sp}))}(\mathbb{S}_{K(n)}[K(\mathbb{Z},n+1)], K(n))$$

[36] For the connection to Theorem 1.5.11, see the discussion following Example 1.5.12.

[37] Since we are working over the sphere, there is only one $\mathbb{E}_1$-structure on Morava K-theory for each choice of a height n formal group over $\mathbb{F}_p$, and it is homotopy commutative at odd primes. In the case of odd primes, this is the only place where we use that we are considering $2(p^n-1)$-periodic Morava K-theory as opposed to the 2-periodic version.

In ho(Sp), $K(n)$ is now a commutative algebra, so this is equivalent to

$$\hom_{\mathrm{Alg}(\mathrm{Mod}_{K(n)}(\mathrm{ho}(\mathrm{Sp})))}(K(n)[K(\mathbb{Z},n+1)],K(n))$$

and $\mathrm{Mod}_{K(n)}(\mathrm{ho}(\mathrm{Sp}))$ is monoidally equivalent to $\mathrm{Mod}_{K(n)_*}(\mathbf{GrVect}_{\mathbb{F}_p})$[38] so that the latter is equivalent to $\hom_{\mathrm{Alg}(K(n)_*)}(K(n)_*K(\mathbb{Z},n+1),K(n)_*)$, as claimed.

The case of the prime 2 is a bit more subtle, but it can be approached using the work of Würgler [Wür86]. More specifically, [Wür86, Proposition 2.4, Remark 2.6.(b)] shows that there is a map $Q : K(n) \to \Sigma^{2^n-1}K(n)$ such that the multiplication map $\mu : K(n) \otimes K(n) \to K(n)$ differs from its twist

$$K(n) \otimes K(n) \overset{\tau}{\simeq} K(n) \otimes K(n) \to K(n)$$

by $v_n \cdot \mu \circ (Q \otimes Q)$.

Note that Q is of odd degree. Ravenel and Wilson's computation [RW80] shows, in particular, that $K(n)^*(K(\mathbb{Z},n+1))$ is concentrated in even degrees, so that for any map $f : \mathbb{S}[K(\mathbb{Z},n+1)] \to K(n)$, the composition

$$\mathbb{S}[K(\mathbb{Z},n+1)] \to K(n) \to \Sigma^{2^n-1}K(n)$$

is 0 ($2^n - 1$ is odd as $n \geq 1$). In particular, for any such map, the composites

$$\mathbb{S}[K(\mathbb{Z},n+1)] \otimes K(n) \to K(n) \otimes K(n) \to K(n)$$

and

$$\mathbb{S}[K(\mathbb{Z},n+1)] \otimes K(n) \simeq K(n) \otimes \mathbb{S}[K(\mathbb{Z},n+1)] \to K(n) \otimes K(n) \to K(n)$$

agree. We may thus apply Proposition 1.3.30 even though $K(n)$ is not homotopy commutative, and conclude in the same way as before, using Theorem 1.2.14 to compute π_0. □

1.5.5 Scheme theory

In [Bal16], Balmer proves the following (compare Proposition 1.3.53):

Theorem 1.5.23 ([Bal16, Theorem 3.5])**.** *Let $f : V \to X$ be a separated étale morphism of quasicompact, quasiseparated schemes. In this case, $f_*\mathcal{O}_V$ is a separable algebra in* $\mathrm{QCoh}(X)$*.*

As already mentioned, Neeman proved in [Nee18] that, at least in the noetherian case, this is not far from exhausting all examples:

Theorem 1.5.24 ([Nee18, Theorem 7.10])**.** *Let X be a noetherian scheme and $A \in$ $\mathrm{QCoh}(X)$ a commutative separable algebra. There exists an étale morphism $g : U \to X$ and a specialization-closed subset $V \subset U$ such that $A \simeq g_*L_V\mathcal{O}_U$ of commutative algebras*[39]*.*

Here, L_V is the Bousfield-localization of $\mathrm{QCoh}(U)$ *associated to the specialization-closed subset V.*

In other words, up to idempotent algebras and under a noetherianity assumption, all commutative separable algebras come from étale maps.

[38]Every (homotopy) $K(n)$-module is free up to shifts.

[39]Neeman only proves that this is an equivalence of algebras in the homotopy category, but Theorem 1.3.19 tells us that this suffices.

1.5.6 Azumaya algebras

In Section 2.1, we will see that there is a strong connection between Azumaya algebras and separable algebras. We will prove that many Azumaya algebras are separable, specifically (cf. Proposition 2.1.10):

Proposition 1.5.25. *Assume* $\mathbf{C}$ *is presentably symmetric monoidal. Let* $A \in \mathrm{Alg}(\mathbf{C})$ *be an algebra. If* A *is Azumaya* and *the unit* $\eta : \mathbf{1} \to A$ *admits a retraction, then* A *is separable.*

We will recall the definition of Azumaya algebras in higher algebra in Section 2.1. Doing so, we will along the way correct a mistake in [BRS12, Proposition 1.4], which states this result without the assumption that the unit splits - we will provide counterexamples to this statement, cf. Example 2.1.8 and Example 2.1.9.

In classical algebra, the assumption on the unit is automatic, and we have:

Theorem 1.5.26 ([AG60, Theorem 2.1])**.** *Let* R *be an ordinary commutative ring. An Azumaya algebra in* $\mathrm{Mod}_R^{\heartsuit}$ *is separable.*

1.5.7 Cartesian symmetric monoidal ∞-categories

We conclude this Examples section with a situation where there are no interesting examples. The unit of a symmetric monoidal ∞-category is of course always separable, and we show:

Proposition 1.5.27. *Let* $\mathbf{C}$ *be cartesian symmetric monoidal. The only separable algebra in* $\mathbf{C}$ *is the unit, i.e. the terminal object.*

Proof. As $\mathbf{C} \to \mathrm{ho}(\mathbf{C})$ preserves products, and as the unit object has an essentially unique algebra structure, we may assume $\mathbf{C}$ is a 1-category. Using the (classical) Yoneda embedding, we may even assume $\mathbf{C} = \mathrm{Set}$.[40]

Let M be a monoid with multiplication map $\mu : M \times M \to M$ and neutral element $\eta : \mathrm{pt} \to M$, which we assume to be separable, with section $s : M \to M \times M$. We write s as (s_1, s_2). Left M-linearity of s guarantees that s_2 is constant. Indeed, for any $x \in M$, we have

$$(s_1(x), s_2(x)) = s(x) = s(x \cdot 1) = x \cdot s(1) = x \cdot (s_1(1), s_2(1)) = (x \cdot s_1(1), s_2(1))$$

and similarly right M-linearity guarantees that s_1 is constant. This proves that s is constant and hence $\mu \circ s$ is, i.e. id_M is constant, from which it follows that $M = \mathrm{pt}$. □

[40] These reductions are purely æsthetic, the proof goes through more or less unchanged in the general case.

Chapter 2

Auslander–Goldman theory in homotopical algebra

Introduction

In [AG60], Auslander and Goldman lay the foundations of a systematic study of separable algebras (in classical algebra). One of the key results that they prove is the following: for a (discrete) commutative ring R, an R-algebra A is separable if and only if its center C is separable over R, and A is Azumaya over its center. This allows one to reduce the study of general separable algebras to two special cases : the commutative case, which is closely related to étale algebras, and the central case, which is closely related to the theory of Azumaya algebras and the Brauer group.

Our goal in this section is to raise two questions such that a positive answer to both, or at least reasonable conditions under which they have positive answers, would allow one to give a similar treatment of separable algebras in homotopical algebra.

We separate this key result in two parts: first, the center C of A is separable over R (and commutative), and second, A is Azumaya over its center.

In homotopical algebra, the center $Z(A)$ of A is in general an $\mathbb{E}_2$-algebra, but if it is separable, it is therefore canonically $\mathbb{E}_\infty$, i.e. commutative, by Theorem 1.3.19. This raises the following question (cf. [AG60, Theorem 2.3]):

Question 2.0.1. Let $A \in \mathrm{Alg}(\mathbf{C})$ be a separable algebra. Is its center $Z(A)$ separable too ? $\triangleleft$

The second key result is that A is Azumaya over its center $Z(A)$. We start by offering a few recollections about Azumaya algebras, along the way correcting an error in [BRS12] about the relation between separable algebras and Azumaya algebras. Once this is done, we can phrase the second main question of this section:

Question 2.0.2. Let $A \in \mathrm{Alg}(\mathbf{C})$ be a dualizable separable algebra which is *central*, i.e. the unit map $\mathbf{1} \to Z(A)$ is an equivalence. In particular, A is full, as it retracts onto $Z(A) \simeq \mathbf{1}$.

Is A necessarily Azumaya ? $\triangleleft$

I start this section by answering Question 2.1.13 in certain cases; the main result in this direction is Theorem D - the assumptions on the ring R are slightly technical so we defer them to Theorem 2.1.14:

Theorem. *Let R be a commutative ring spectrum satisfying the assumptions of Theorem 2.1.14. In this case, any dualizable central separable algebra over R is Azumaya.*

Conversely, in any additive presentably symmetric monoidal ∞-category $\mathbf{C}$*, an Azumaya algebra* A *is separable if and only if its unit* $\mathbf{1}_{\mathbf{C}} \to A$ *admits a retraction.*

We then attack Question 2.0.1, again answering it in certain cases. The main result there is Theorem E:

Theorem. *Let* R *be a connective commutative ring spectrum and let* A *be an almost perfect* R*-algebra. If* A *is separable, then so is its center.*

The same holds for separable algebras in $K(n)$*-local* E*-modules, where* E *is Morava* E*-theory, and for separable algebras in* $K(n)$*-local spectra.*

As is clear from these statements, I fall short of answering both questions in the generality of $\mathrm{Mod}_R(\mathrm{Sp})$, where R is some commutative ring spectrum - this is essentially because we lack "residue fields", as will be clear from our discussion.

Here, our main examples and main sources of positive answers to our questions are in symmetric monoidal ∞-categories, I did not try to extend the discussion to $\mathbb{E}_m$-monoidal ones for m large enough, though large swaths of it could work in that generality.

Sectionwise outline

- In Section 2.1, I do some recollections concerning Azumaya algebras and the interaction between the Azumaya condition and separability. This is where I prove Theorem D;
- In Section 2.2, I study Question 2.0.1 and this is where I prove Theorem E.

2.1 Azumaya algebras

We start by recalling a possible definition of Azumaya algebras:

Definition 2.1.1. Let $\mathbf{C}$ be presentably symmetric monoidal. An algebra $A \in \mathrm{Alg}(\mathbf{C})$ is *Azumaya* if $\mathrm{LMod}_A(\mathbf{C})$ is invertible in $\mathrm{Mod}_{\mathbf{C}}(\mathrm{Pr}^{\mathrm{L}})$. ◁

We also recall several equivalent characterizations. For this, we need the following proposition/definition:

Proposition 2.1.2 ([HL17, Proposition 2.1.3., Corollary 2.1.4.])**.** *Let* $\mathbf{C}$ *be presentably symmetric monoidal. Let* M *be a dualizable object of* $\mathbf{C}$*. The following are equivalent:*

(i) M *generates* $\mathbf{C}$ *under* $\mathbf{C}$*-colimits, that is, the smallest tensor ideal of* $\mathbf{C}$ *closed under colimits and containing* M *is the whole of* $\mathbf{C}$*;*

(ii) The ($\mathbf{C}$*-linear) functor* $\hom(M,-) : \mathbf{C} \to \mathrm{RMod}_{\mathrm{End}(M)}(\mathbf{C})$ *is an equivalence;*

(iii) $\mathrm{End}(M)$ *is (*$\mathbf{C}$*-linearly) Morita equivalent to the unit* $\mathbf{1}$*;*

(iv) $M \otimes -$ *is conservative.*

If M *satisfies one (and hence all) of these properties, it is called* full*. Furthermore, any (and hence all) of these properties are stable under passing to the dual* $M^\vee$*.*

Proof. We prove (i) $\implies$ (iv) $\implies$ (ii) $\implies$ (iii) $\implies$ (i).

Assume (i), and let $f : X \to Y$ be a map such that $M \otimes f$ is an equivalence. The collection of Z's such that $Z \otimes f$ is an equivalence is certainly a tensor ideal of $\mathbf{C}$, closed under colimits, so that by (i), it contains $\mathbf{1}$. In particular, f is an equivalence, thus proving (iv).

Let us now assume (iv). The functor $G = \hom(M,-) : \mathbf{C} \to \mathrm{RMod}_{\mathrm{End}(M)}$ preserves limits and colimits hence has a left adjoint $F = M \otimes_{\mathrm{End}(M)} -$. The unit map at $\mathrm{End}(M)$,

$\mathrm{End}(M) \to \hom(M, M \otimes_{\mathrm{End}(M)} \mathrm{End}(M))$ is easily seen to be an equivalence, and both the source and the target of the unit $\mathrm{id} \to GF$ are $\mathbf{C}$-linear and colimit-preserving, hence the unit is an equivalence at all $\mathrm{End}(M)$-modules.

To prove that the counit is an equivalence, by the triangle identities, it thus suffices to show that the right adjoint $\hom(M, -)$ is conservative, and because the forgetful functor $\mathrm{RMod}_{\mathrm{End}(M)} \to \mathbf{C}$ is conservative, it suffices to show that $\hom(M, -) : \mathbf{C} \to \mathbf{C}$ is conservative. By dualizability, this is equivalent to $M^\vee \otimes -$. Now if $M^\vee \otimes f$ is an equivalence, so is $M \otimes M^\vee \otimes M \otimes f$; and thus, so is $M \otimes f$, as M is a retract of $M \otimes M^\vee \otimes M$. By conservativity of M, it follows that f is an equivalence, and hence $\hom(M, -)$ is conservative. This proves (ii).

(ii) clearly implies (iii), by definition of Morita equivalence.

So let us now assume (iii). The existence of a Morita equivalence yields a right $\mathrm{End}(M)$-module X and a left $\mathrm{End}(M)$-module Y such that $X \otimes_{\mathrm{End}(M)} Y \simeq \mathbf{1}$. The smallest $\mathbf{C}$-linear subcategory of $\mathrm{RMod}_{\mathrm{End}(M)}$ closed under colimits and containing $\mathrm{End}(M)$ contains X, so that the smallest $\mathbf{C}$-linear subcategory of $\mathbf{C}$ closed under colimits and containing $Y \simeq \mathrm{End}(M) \otimes_{\mathrm{End}(M)} Y$ also contains $\mathbf{1}$. But now Y is a retract (in $\mathbf{C}$) of $\mathrm{End}(M) \otimes Y \simeq M \otimes M^\vee \otimes Y$ so that (i) follows. □

We also briefly need:

Definition 2.1.3. Let $\mathbf{C}$ be presentably symmetric monoidal, and let $\mathbf{M}$ be a $\mathbf{C}$-module in Pr^{L}. An object $x \in \mathbf{M}$ is called $\mathbf{C}$-*atomic* if the canonical map

$$c \otimes \hom(x, y) \to \hom(x, c \otimes y)$$

is an equivalence for all $c \in \mathbf{C}, y \in \mathbf{M}$, and $\hom(x, -) : \mathbf{M} \to \mathbf{C}$ preserves all colimits. Here, hom denotes the $\mathbf{C}$-valued hom object of $\mathbf{M}$. ◁

Remark 2.1.4. This definition appears in [BS21, Definition 2.2] in the case where $\mathbf{C}$ is a *mode*, so that it actually suffices to assume that $\hom(x, -)$ preserves colimits, cf. [BS21, Remark 2.4]. ◁

The following is immediate from the definitions:

Lemma 2.1.5. *Let $\mathbf{C}$ be presentably symmetric monoidal.*

- *$\mathbf{C}$-atomic objects in $\mathbf{C}$ are exactly dualizable objects.*
- *If $f : \mathbf{M}_0 \to \mathbf{M}_1$ is an equivalence of $\mathbf{C}$-modules in Pr^{L}, it carries $\mathbf{C}$-atomic objects to $\mathbf{C}$-atomic objects.*

We can now prove:

Proposition 2.1.6 ([HL17, Corollary 2.2.3.]). *Let $\mathbf{C}$ be presentably symmetric monoidal, and let $A \in \mathrm{Alg}(\mathbf{C})$ be an algebra. The following are equivalent:*

(i) A is Azumaya;

(ii) A is dualizable, full, and the canonical map $A \otimes A^{\mathrm{op}} \to \mathrm{End}(A)$ is an equivalence;

(iii) A is dualizable, full, and there is an equivalence of algebras $A \otimes A^{\mathrm{op}} \simeq \mathrm{End}(A)$;

(iv) There is some full dualizable object M and an equivalence of algebras $A \otimes A^{\mathrm{op}} \simeq \mathrm{End}(M)$;

(v) $A \otimes A^{\mathrm{op}}$ is ($\mathbf{C}$-linearly) Morita equivalent to the unit $\mathbf{1}$;

(vi) There exists an algebra B, a full dualizable object M, and an equivalence $A \otimes B \simeq \mathrm{End}(M)$

(vii) There exists an algebra B and a ($\mathbf{C}$-linear) Morita equivalence between $A \otimes B$ and $\mathbf{1}$

Proof. We prove (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv) $\implies$ (v) $\implies$ (vii) $\implies$ (i), and we prove (vi) $\iff$ (vii).

Note that (ii) $\implies$ (iii) $\implies$ (iv) are just each specializations of the previous one, so these implications are obvious, same for (v) $\implies$ (vii).

For (iv) $\implies$ (v) (resp. (vi) $\implies$ (vii)), we simply observe that for a full dualizable object M, $\mathrm{End}(M)$ is Morita equivalent to $\mathbf{1}$ by the previous proposition/definition.

(vii) $\implies$ (i) follows from the observation that $\mathrm{LMod}_A \otimes_{\mathbf{C}} \mathrm{LMod}_B \simeq \mathrm{LMod}_{A\otimes B}$, and hence (vii) implies that $\mathrm{LMod}_A \otimes_{\mathbf{C}} \mathrm{LMod}_B \simeq \mathbf{C}$, which is the definition of Azumaya.

We are left with (i) $\implies$ (ii) and (vii) $\implies$ (vi). The proof of (vii) $\implies$ (vi) poceeds by observing that any algebra Morita equivalent to $\mathbf{1}$ is of the form $\mathrm{End}(M)$ for some full dualizable M. Indeed, suppose A is such an algebra, and fix a Morita equivalence $F : \mathrm{LMod}_A \simeq \mathbf{C}$. Note that $A \in \mathrm{LMod}_A$ is $\mathbf{C}$-atomic, i.e. $\hom_A(A, -) : \mathrm{LMod}_A \to \mathbf{C}$ is $\mathbf{C}$-linear and colimit-preserving - indeed, $\hom_A(A, -)$ is $\mathbf{C}$-linearly equivalent to the forgetful functor. As F is a $\mathbf{C}$-linear equivalence, $F(A) \in \mathbf{C}$ is also atomic by the previous lemma, and it is therefore dualizable, also by the previous lemma. Furthermore, we have

$$A \simeq \mathrm{End}_A(A)^{\mathrm{op}} \simeq \mathrm{End}(F(A))^{\mathrm{op}} \simeq \mathrm{End}(F(A)^\vee)$$

It follows that $F(A)^\vee$ is a dualizable object with $\mathrm{End}(F(A)^\vee)$ Morita equivalent to the unit, so by the previous proposition/definition, it is full, which proves the claim.

Finally, we need to prove that (i) implies (ii). The observation here is that LMod_A is always dualizable in $\mathrm{Mod}_{\mathbf{C}}$, so that invertibility is the property that the evaluation and coevaluation maps,

$$\mathrm{LMod}_{A^{\mathrm{op}}} \otimes_{\mathbf{C}} \mathrm{LMod}_A \to \mathbf{C}$$

and

$$\mathbf{C} \to \mathrm{LMod}_A \otimes_{\mathbf{C}} \mathrm{LMod}_{A^{\mathrm{op}}}$$

respectively, be equivalences.

For the second one, it implies in particular that A is proper, i.e. that A is dualizable as an object of $\mathbf{C}$.

Next, note that the map $\mathrm{LMod}_{A\otimes A^{\mathrm{op}}} \simeq \mathrm{LMod}_A \otimes_{\mathbf{C}} \mathrm{LMod}_{A^{\mathrm{op}}} \to \mathbf{C}$ is given by tensoring over $A \otimes A^{\mathrm{op}}$ with the $A \otimes A^{\mathrm{op}}$-module A. It therefore sends $A \otimes A^{\mathrm{op}}$ to A, and as it is an equivalence, it induces an equivalence of algebras

$$A \otimes A^{\mathrm{op}} \simeq \mathrm{End}_{A\otimes A^{\mathrm{op}}}(A \otimes A^{\mathrm{op}}) \xrightarrow{\simeq} \mathrm{End}(A)$$

It is easy to check that this is the canonical map.

To prove (ii), we are left with checking that A is full. But it follows from what we just said that $\mathrm{LMod}_{\mathrm{End}(A)}$ was equivalent to $\mathbf{C}$, and A is dualizable, so by the previous proposition/definition, A is full, and so we are done. □

A further key property of Azumaya algebras is their *centrality*.

Lemma 2.1.7. *Let $A \in \mathrm{Alg}(\mathbf{C})$ be an Azumaya algebra. In this case, the center of A is equivalent to the unit $\mathbf{1}$.*

Proof. The center of A is equivalent to the endomorphism object of the $\mathbf{C}$-module $\mathrm{LMod}_A(\mathbf{C})$. Since the latter is invertible, the functor

$$\mathbf{C} \to \mathrm{Fun}^L_{\mathbf{C}}(\mathrm{LMod}_A(\mathbf{C}), \mathrm{LMod}_A(\mathbf{C}))$$

is a $\mathbf{C}$-linear equivalence, and it sends $\mathbf{1}$ to $\mathrm{id}_{\mathrm{LMod}_A(\mathbf{C})}$. It follows that

$$Z(A) \simeq \mathrm{End}(\mathrm{id}_{\mathrm{LMod}_A(\mathbf{C})}) \simeq \mathrm{End}(\mathbf{1}) \simeq \mathbf{1}$$

as claimed. $\square$

In [BRS12, Proposition 1.4], it is claimed that an Azumaya algebra is necessarily separable, in analogy with [AG60, Theorem 2.1.]. Unfortunately, there is an error in their argument: in their notation, the module $\tilde{F}(A) = A \wedge_R A$ is *not* the canonical bimodule $A \wedge_R A^{\mathrm{op}}$, but rather the bimodule obtained by tensoring the canonical bimodule A with the object A. There are, in fact, counterexamples to this statement. We give two: a local one, and a global one.

Example 2.1.8. There are some associative ring structures on Morava K-theory $K(n)$ in the ∞-category $\mathrm{Sp}_{K(n)}$ of $K(n)$-local spectra, which are Azumaya algebras, cf. [HL17]. However, none of these are separable: a bimodule splitting as in Definition 1.1.2 would yield a retraction of

$$E_n \simeq \mathrm{Map}_{K(n)\otimes K(n)^{\mathrm{op}}}(K(n), K(n)) \to K(n)$$

and there is clearly no such thing (the first equivalence follows from Lemma 2.1.7). $\triangleleft$

Example 2.1.9. The same example as in Remark 1.1.12 also provides a global example here, that is, without needing to localize. Namely, if X is a type 0 spectrum, such as the cofiber of η, $\mathrm{End}(X)$ is Morita equivalent to $\mathbb{S}$ and hence Azumaya, but we already argued that it is not separable. $\triangleleft$

We can now state the corrected version of [BRS12, Proposition 1.4]

Proposition 2.1.10. *Let $A \in \mathrm{Alg}(\mathbf{C})$ be an algebra. If A is Azumaya, then A is separable if and only if the unit $\eta : \mathbf{1} \to A$ admits a retraction.*

Proof. First note that the multiplication map $A \otimes A^{\mathrm{op}} \to A$ factors as

$$A \otimes A^{\mathrm{op}} \to \mathrm{End}(A) \to A$$

where the second map is evaluation at the unit $\eta : \mathbf{1} \to A$, as a map of bimodules.

If A is Azumaya, it follows that this multiplication admits a bimodule section if and only if $ev_\eta : \mathrm{End}(A) \to A$ has an A-bimodule section. Now $\mathrm{End}(A) \simeq A \otimes A^\vee$ as A-bimodules, where the latter has the structure of A-bimodule coming from A, so that this map is really $A \otimes (A^\vee \xrightarrow{ev_\eta} \mathbf{1})$.

Finally, $A^\vee \to \mathbf{1}$ is dual to $\eta : \mathbf{1} \to A$. So, if the unit has a retraction, then $ev_\eta : A^\vee \to \mathbf{1}$ has a section, and therefore so does $A \otimes (A^\vee \to \mathbf{1})$, as a map of bimodules, and by the previous discussion, so does $A \otimes A^{\mathrm{op}} \to A$, so that A is separable.

Conversely, if A is separable, then the canonical map $Z(A) \to A$ admits a section (cf. Corollary 1.1.29). As A is Azumaya, we can combine this with Lemma 2.1.7 to obtain that $\mathbf{1} \simeq Z(A) \to A$ admits a retraction. $\square$

We use this proposition to prove that, unlike in the commutative case, separability cannot be checked locally:

Example 2.1.11. In [GL21, Proposition 7.17], Gepner and Lawson construct a twisted form of $M_2(\mathrm{KU})$, that is, a KO-algebra Q, necessarily Azumaya, for which $Q \otimes_{\mathrm{KO}} \mathrm{KU} \simeq M_2(\mathrm{KU})$ as algebras. This algebra is not $M_2(\mathrm{KO})$, in fact $\pi_* Q \cong \mathrm{KU}_*\langle C_2\rangle$, a twisted group ring.

It follows that Q is not separable: it is Azumaya so by the above proposition, if it were separable, its unit would split. But it has no π_1, so such a splitting is impossible as $\pi_1(\mathrm{KO}) \neq 0$.

Therefore we have a non-separable algebra, Q, whose basechange along a Galois-extension is separable - it follows that $\mathrm{Alg}^{sep}(\mathrm{Mod}_{\mathrm{KU}}^{hC_2}) \to \mathrm{Alg}^{sep}(\mathrm{Mod}_{\mathrm{KU}})^{hC_2}$ is not an equivalence: the former is equivalent to $\mathrm{Alg}^{sep}(\mathrm{Mod}_{\mathrm{KO}})$, and the latter to the full subgroupoid of $\mathrm{Alg}(\mathrm{Mod}_{\mathrm{KO}})$ consisting of those algebras whose basechange to KU is separable. $\triangleleft$

Remark 2.1.12. Let us mention, without too much detail, the following interpretation of separable Azumaya algebras. Let $\mathbf{C}$ be an additively symmetric monoidal ∞-category, and A an Azumaya algebra therein. In this case, A is separable (equivalently, its unit splits) if and only if it remains Azumaya in $\mathrm{Syn}_{\mathbf{C}} = \mathrm{Fun}^{\times}(\mathbf{C}^{\mathrm{op}}, \mathrm{Sp})$ under the symmetric monoidal Yoneda embedding $\mathbf{C} \to \mathrm{Syn}_{\mathbf{C}}$, if and only if it is "absolutely Azumaya", i.e. it remains Azumaya after applying any additive symmetric monoidal functor (the part of the definition of "Azumaya" which is not clearly preserved by any functor is the "fullness" property, but is here guaranteed by the retraction onto the unit).

Sven van Nigtevecht has independently observed[1] that the obstruction theory from [PV22][2] can be used in the case where A is an Azumaya algebra which remains Azumaya in $\mathrm{Syn}_{\mathbf{C}}$, for then the mapping spectrum in $\mathrm{Mod}_{A\otimes A^{\mathrm{op}}}(\mathrm{Syn}_{\mathbf{C}})$ from A to itself is simply $\mathrm{map}_{\mathrm{Syn}_{\mathbf{C}}}(\mathbf{1}_{\mathrm{Syn}_{\mathbf{C}}}, \mathbf{1}_{\mathrm{Syn}_{\mathbf{C}}}) \simeq \mathrm{map}(\mathbf{1}_{\mathbf{C}}, \mathbf{1}_{\mathbf{C}})_{\geq 0}$. From our perspective, this is explained by the fact that under this assumption, A is actually separable. $\triangleleft$

In the setting of classical rings, a stronger result holds: if A is dualizable, separable, and *central*, i.e. its center $Z(A)$ is the unit $\mathbf{1}$, then A is Azumaya. We do not know whether the converse holds in our generality, and we therefore raise it as a question:

Question 2.1.13. Let $A \in \mathrm{Alg}(\mathbf{C})$ be a dualizable separable algebra which is *central*, i.e. the unit map $\mathbf{1} \to Z(A)$ is an equivalence. In particular, A is full, as it retracts onto $Z(A) \simeq \mathbf{1}$.
Is A necessarily Azumaya ? $\triangleleft$

I provide a positive answer in the following cases:

Theorem 2.1.14. *If* $\mathbf{C}$ *is one of the following:*

- $\mathrm{QCoh}(X)$ *for some (connective) spectral Deligne-Mumford stack* X *[Lur18b, Definition 1.4.4.2];*
- $\mathrm{Mod}_R(\mathrm{Sp})$*, where* R *is some commutative ring spectrum for which* $R \otimes \mathbb{F}_p = 0$ *for all primes* p*;*
- $\mathrm{Mod}_R(\mathrm{Sp})$*, where* R *is a commutative ring spectrum which is even,* 2*-periodic and whose* π_0 *is regular noetherian, and in which* 2 *is invertible.*

then Question 2.1.13 has a positive answer for $\mathbf{C}$*: if* $A \in \mathrm{Alg}(\mathbf{C})$ *is a dualizable separable algebra which is* central*, i.e. the unit map* $\mathbf{1} \to Z(A)$ *is an equivalence, then* A *is Azumaya.*

Remark 2.1.15. The second situation of Theorem 2.1.14 is somewhat orthogonal to the first one: such a commutative ring R, unless it is rational, must be non-connective, and of "chromatic" flavour. For instance, Morava E-theories fall into this category.

The third situation allows for certain non-connective commutative $\mathbb{F}_p$-algebras at odd primes, such as $\mathbb{F}_p^{tS^1}$, but not, e.g., $\mathbb{F}_p^{tC_p}$. $\triangleleft$

Proof. Combine Corollary 2.1.25, Corollary 2.1.37 and Proposition 2.1.28. $\square$

[1] Private communication.
[2] Which used to be used in Section 1.2.2

The strategy of proof in all cases of Theorem 2.1.14, which is also the one we will use in Section 2.2 to adress Question 2.0.1, is to try to *descend* the question to simpler and simpler $\mathbf{C}$'s, until we reach a classical algebraic $\mathbf{C}$, where the usual proofs just go through.

The "descent" statement in this case is the following:

Lemma 2.1.16. *Let $f : \mathbf{C} \to \mathbf{D}$ be a* conservative *symmetric monoidal functor. For a dualizable, full algebra $A \in \mathrm{Alg}(\mathbf{C})$, if $f(A)$ is Azumaya, then so is A.*

More generally, if $f_i : \mathbf{C} \to \mathbf{D}_i$ is a jointly conservative family of symmetric monoidal functors, if each $f_i(A)$ is Azumaya, then so is A.

Proof. We deal with the case of a single functor, the other case being similar (or simply a consequence, by taking $f = (f_i)_{i\in I} : \mathbf{C} \to \prod_I \mathbf{D}_i$).

A is already assumed to be dualizable and full, so by point 2. in Proposition 2.1.6, it suffices to show that the canonical map $A \otimes A^{\mathrm{op}} \to \mathrm{End}(A)$ is an equivalence.

The functor f is symmetric monoidal, and A is rigid, so that applying f to this map yields the canonical map $f(A) \otimes f(A)^{\mathrm{op}} \to \mathrm{End}(f(A))$. By conservativity of f, if this is an equivalence, then so was the canonical map. □

The key example we try to reduce to is the category of modules over a graded field.

Notation 2.1.17. We consider the category of graded abelian groups as symmetric monoidal using the Koszul convention: the symmetry isomorphism $A \otimes B \cong B \otimes A$ is $a \otimes b \mapsto (-1)^{|a||b|} b \otimes a$ for homogeneous elements a, b of respective degrees $|a|, |b|$. ◁

Definition 2.1.18. A graded field is a commutative algebra k in graded abelian groups such that every homogeneous element $x \in k_*$ is invertible.

A graded division algebra is similar, except we do not require commutativity. ◁

Remark 2.1.19. Graded fields are easy to classify: they are either fields concentrated in degree 0, or of the form $k[t^{\pm 1}]$ for some t of positive degree - necessarily even if the characteristic of k is not 2.

On the other hand, graded division algebras are more complicated to classify: even if the degree 0 part is a field (i.e. commutative), the non-commutativity of the multiplication in higher degrees allows for a wealth of examples. ◁

Proposition 2.1.20. *Let k be a graded field, and $\mathbf{D}$ the category of graded k-vector spaces. Any central separable algebra in $\mathbf{D}$ is Azumaya.*

One way to go about this proof is to prove the following lemma, which is classical in the ungraded case and most likely well-known in the graded case too. There is, however, an easier proof under our assumption, so I will simply mention the lemma here and let the reader fill in the details of this proof if they are interested.

Lemma 2.1.21. *Let k be a graded field, and D, D' central graded division algebras over k. The algebra $D \otimes_k D'$ is graded simple, i.e. it has no nontrivial homogeneous ideal.*

We are in a simpler situation, as we assume separability:

Lemma 2.1.22. *Let $\mathbf{D}$ be a symmetric monoidal abelian category which is semi-simple, and let $A \in \mathrm{Alg}(\mathbf{C})$ be a separable algebra. Any (bilateral) ideal I in A splits: there is an isomophism of algebras $A \cong I \times A/I$.*

In particular, if A is central, i.e. $\mathbf{1} \cong Z(A)$, and $\mathrm{End}_{\mathbf{D}}(\mathbf{1})$ has no nontrivial idempotents, then any (bilateral) ideal is 0 or A.

Proof. As $\mathbf{D}$ is semi-simple, the inclusion $I \to A$, which is a morphism of $A \otimes A^{\mathrm{op}}$-modules, admits a section in $\mathbf{D}$. Because $A \otimes A^{\mathrm{op}}$ is separable, Corollary 1.1.17 implies that it admits a section of A-bimodules. The result follows by Lemma 1.3.8.

The "in particular" follows from the fact that $Z(A \times B) \cong Z(A) \times Z(B)$. □

Lemma 2.1.23. *Let $\mathbf{C}$ be idempotent-complete, and let $A, B \in \mathrm{Alg}(\mathbf{C})$ be separable. The canonical map $Z(A) \otimes Z(B) \to Z(A \otimes B)$ is an equivalence.*

Proof. Note that this a map of the form

$$\hom_R(M, N) \otimes \hom_S(P, Q) \to \hom_{R \otimes S}(M \otimes P, N \otimes Q)$$

and the latter is natual in M, N, P, Q. Furthermore, for $M = N = R, P = Q = S$, it is clearly an equivalence. Hence it is so for any tuple (M, N, P, Q) which is a retract of (R, R, S, S).

By separability of A, B, (A, A, B, B) is a retract of $(A \otimes A^{\mathrm{op}}, A \otimes A^{\mathrm{op}}, B \otimes B^{\mathrm{op}}, B \otimes B^{\mathrm{op}})$ and so we are done. □

Remark 2.1.24. In fact an easy modification of this proof shows that it suffices that A is separable if we also assume that B is smooth, or that A is proper, see [Lur12, Section 4.6.4] for definitions. ◁

Proof of Proposition 2.1.20. As k is a graded field, the category $\mathbf{D}$ of graded k-vector spaces is semi-simple, and $\mathrm{End}_{\mathbf{D}}(\mathbf{1}) = \mathrm{End}(k) = k$ has no nontrivial idempotents.

In particular, by Lemma 2.1.22 if A is central and separable, then it is simple: it has no nontrivial ideals.

We apply this to $A \otimes A^{\mathrm{op}}$ instead: it is still separable (Lemma 1.1.6) and central (Lemma 2.1.23), and therefore by the above it is simple.

It follows that the canonical map $A \otimes A^{\mathrm{op}} \to \mathrm{End}(A)$, which is an algebra map, has no kernel, i.e. it is injective. Comparing the dimensions of both sides implies that it is an isomorphism[3]. By Proposition 2.1.6, point 2., we are done.

□

The case of ordinary fields is enough to bootstrap to all connective Deligne-Mumford stacks:

Corollary 2.1.25. *Let X be a (connective) Deligne-Mumford stack. Any dualizable central separable algebra in $\mathrm{QCoh}(X)$ is Azumaya.*

Proof. By [Lur18b, Proposition 6.2.4.1], $\mathrm{QCoh}(X)$ is a limit, in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$, of ∞-categories of the form $\mathrm{Mod}_R(\mathrm{Sp})$, where R is a connective commutative ring spectrum.

Since for any diagram $f : I \to \mathrm{Cat}$ and any essentially surjective map from a set $I_0 \to I$, the forgetful functor $\lim_I f \to \prod_{I_0} f$ is conservative, we can apply Lemma 2.1.16 to reduce to the case of $\mathrm{Mod}_R(\mathrm{Sp})$, where R is a connective commutative ring spectrum.

Since we assumed the algebra was dualizable, we can in fact reduce to $\mathbf{Perf}(R)$. Now, for a connective ring spectrum R, the restriction of $\pi_0(R) \otimes_R -$ to bounded below R-modules is symmetric monoidal and conservative, so we can reduce to the case where R is discrete, again by Lemma 2.1.16.

For a discrete commutative ring R, the basechange functors along all ring maps $R \to k$, where k is a field are jointly conservative on perfect R-modules, so we can reduce to the case of a field.

[3]We are in a graded setting, but over a graded field, so dimensions still make sense.

Now note that for any $\mathbf{C}$, $\mathbf{C} \to \mathrm{ho}(\mathbf{C})$ is conservative and symmetric monoidal. For a field k, $\mathrm{ho}(\mathrm{Mod}_k)$ is symmetric monoidally equivalent to the 1-category of graded k-vector spaces, and so Proposition 2.1.20 allows us to conclude. □

To deal with the second case of Theorem 2.1.14, we first specialize to $R =$ Morava E-theory - the nilpotence theorem [HS98] and the chromatic Nullstellensatz [BSY22] will be our tools to reduce to this key case.

The situation is simpler at odd primes than at the even prime, so we first deal with the odd primes, even though the proof we will give for the prime 2 also works for odd primes.

Proposition 2.1.26. *Let $R = E = E(k, \mathbf{G})$ be a Morava E-theory*[4] *over some field k of odd characteristic, and at some height $n > 0$, and let $A \in \mathrm{Alg}(\mathrm{Mod}_E)$ be a rigid separable E-algebra. If A is furthermore central, then A is Azumaya.*

Remark 2.1.27. In contrast to Example 2.1.8, these Azumaya algebras are not "atomic" in the sense of [HL17], precisely because they are separable and therefore retract onto E. ◁

Proof. Because we are working at an odd prime, there exist ring structures on Morava K-theory $K(n)$ that are homotopy commutative [Str99, Section 3]. In this case, $K(n)_* : \mathrm{Mod}_E \to \mathrm{Mod}_{K(n)_*}(\mathbf{GrVect}_k)$ is a symmetric monoidal functor, and it is conservative when restricted to $K(n)$-local E-modules, in particular when restricted to perfect, or equivalently dualizable, E-modules.

As $K(n)_*$ is a graded field, Proposition 2.1.20 applies again, and we are done, again by Lemma 2.1.16. □

In fact, thanks to work of Mathew [Mat15], the same argument works more generally:

Proposition 2.1.28. *Let R be a commutative ring spectrum which is even, 2-periodic, with regular noetherian π_0, and such that $2 \in \pi_0(R)^\times$. Let $A \in \mathrm{Alg}(\mathrm{Mod}_R)$ be a dualizable separable R-algebra. If A is furthermore central, then A is Azumaya.*

Proof. The same proof as above works, where we replace $K(n)$ by the $K(\mathfrak{p})$'s, cf. [Mat15, Definition 2.5]. Indeed, each $K(\mathfrak{p})_*$ is a graded field by *loc. cit.*, they are jointly conservative on perfect R-modules [Mat15, Proposition 2.8] (in fact on all modules), and finally by [Str99, Section 3], if $2 \in \pi_0(R)^\times$, they can be chosen to be homotopy commutative. □

We now deal with the even prime. The point is that in this situation, Morava K-theory cannot be chosen to be homotopy commutative, so that $K(n)_*$ is only monoidal, but not symmetric monoidal, which means that it is possibly not compatible with the map

$$A \otimes A \to \mathrm{End}(A) \simeq A^\vee \otimes A$$

and in particular we cannot check Azumaya-ness through this functor.

There is a way out, using the notion of Milnor modules [HL17, Section 6]. The main takeaway of this notion for us is the following:

Theorem 2.1.29. *Let $E = E(k, \mathbf{G})$ be a Morava E-theory at height n at the prime p, possibly even. There is a symmetric monoidal* 1-category Mil_E *of* Milnor-modules *together with a (strong) symmetric monoidal homology theory $h_* : \mathrm{Mod}_E \to \mathrm{Mil}_E$.*

For any choice of a Morava K-theory $K \in \mathrm{Alg}(\mathrm{ho}(\mathrm{Mod}_E))$, the monoidal homology theory $K_* : \mathrm{Mod}_E \to \mathrm{coMod}_{K_*^E K}(\mathrm{Mod}_{K_*}((\mathbf{GrVect}_k))$ factors through a monoidal[5] *equivalence $\mathrm{Mil}_E \simeq \mathrm{coMod}_{K_*^E K}(\mathrm{Mod}_{K_*}((\mathbf{GrVect}_k))$.*

[4]See [BSY22, Section 2.4] for a modern introduction

[5]At the prime 2, there is no choice of Morava K-theory that makes this symmetric monoidal.

Note that the notion of separable algebra, and of rigidity can be phrased completely in monoidal terms (for duality, one needs to worry about left vs right duality, but these notions still make sense). The only part of "rigid central separable algebra" that requires symmetry is the centrality part.

In particular, if $A \in \mathrm{Alg}(\mathrm{Mod}_E)$ is rigid and separable, $K_*(A)$ is a rigid separable algebra in K_*-modules in $\mathbf{GrVect}_k$. This will turn out to be enough for us.

We begin with a lemma:

Lemma 2.1.30. *Let* $\mathbf{C}$ *be a symmetric monoidal* 1*-category,* H *a commutative Hopf algebra in* $\mathbf{C}$*, i.e. a group object in* $\mathrm{CAlg}(\mathbf{C})^{\mathrm{op}}$*.*

Let I *be a non-unital algebra in* $\mathrm{coMod}_H(\mathbf{C})$ *such that the underlying non-unital algebra* I *in* $\mathbf{C}$ *admits a unit [Lur12, Definition 5.4.3.1]. In this case,* I *admits a unit in* $\mathrm{coMod}_H(\mathbf{C})$*.*

Remark 2.1.31. We state and prove this lemma for 1-categories because in the proof, we use a description of comodules as "algebraic representations" of an "algebraic group" (see below). This description is elementary for 1-categories, while for ∞-categories, it is highly expected to hold completely analogously, but we did not want to get into the intricacies of its proof.

We later only use it for 1-categories, so this is not an issue, but it would be interesting to prove the corresponding description for symmetric monoidal ∞-categories (the lemma, for instance, would follow immediately in the same generality). ◁

As mentioned in the remark, to prove this lemma, it is convenient to use the usual description of $\mathrm{coMod}_H(\mathbf{C})$ as "algebraic representations of $\mathrm{Spec}_{\mathbf{C}}(H)$". Let us make this a bit more precise. The corepresented functor $M = \mathrm{Spec}_{\mathbf{C}}(H) : \mathrm{CAlg}(\mathbf{C}) \to \mathcal{S}$ given by $\mathrm{Map}(H, -)$ is canonically a monoid whenever H is a comonoid in $\mathrm{CAlg}(\mathbf{C})$. The category of transformations $BM(R) \to \mathrm{Mod}_R(\mathbf{C})$, natural in $R \in \mathrm{CAlg}(\mathbf{C})$ can be viewed as a category of "algebraic representations of M" - here, $R \mapsto \mathrm{Mod}_R(\mathbf{C})$ is functorial along base-change[6].

It is an instructive exercise to prove that this category is symmetric monoidally equivalent to the category of H-comodules, compatibly with the forgetful functor to $\mathbf{C}$. We use this fact without further comment.

Notation 2.1.32. Let $M : \mathrm{CAlg}(\mathbf{C}) \to \mathrm{Mon}$ be a functor from commutative algebras in $\mathbf{C}$ to (discrete) monoids. We let $\mathbf{Rep}_M(\mathbf{C})$ denote the symmetric monoidal category of algebraic representations of M, as described above. ◁

Remark 2.1.33. We note that here, H needs to be a Hopf algebra - the lemma is not true for general commutative bialgebras. For example, let H be the bialgebra in abelian groups whose underlying algebra is $\mathbb{Z} \times \mathbb{Z}$. The functor on $\mathrm{CAlg}(\mathrm{Ab})$ it corepresents is simply

$$\mathrm{Idem} : R \mapsto \mathrm{Idem}(R)$$

the functor mapping a ring to its set of idempotents, and we can make it a commutative monoid under multiplication, thus making H into a bialgebra. In this case, one can make $\mathbb{Z}$ into an algebraic representation of $\mathrm{Spec}_{\mathbf{C}}(H)$, i.e. an H-comodule, via the canonical action of $\mathrm{Idem}(R)$ on R by multiplication. It is easy to check that this makes it into a non-unital algebra whose underlying algebra is unital, but it is not unital. ◁

Proof. By [Lur12, Theorem 5.4.3.5], the unit of a non-unital algebra, if it exists, is unique. More precisely, the forgetful functor $\mathrm{Alg}(\mathbf{D})^{\simeq} \to \mathrm{Alg}^{nu}(\mathbf{D})^{\simeq}$ is fully faithful[7].

[6] Because every such natural transformation has a value c at $R = \mathbf{1}$, the value at every other R is of the form $R \otimes c$, and so all the required basechanges exist, along arbitrary maps $R \to S$, therefore, to make this definition, we do not actually need $\mathbf{C}$ to have arbitrary relative tensor products.

[7] Note that this is not true if one removes the symbol $^{\simeq}$, it is only faithful.

In particular, for any group G, if $A \in \mathrm{Alg}^{nu}(\mathrm{Fun}(BG, \mathbf{D}))$ is a non-unital algebra such that the underlying $A \in \mathrm{Alg}^{nu}(\mathbf{D})$ admits a unit, then A admits a unit too. More precisely, the canonical map

$$\mathrm{Alg}(\mathrm{Fun}(BG, \mathbf{D})) \to \mathrm{Alg}^{nu}(\mathrm{Fun}(BG, \mathbf{D})) \times_{\mathrm{Alg}^{nu}(\mathbf{D})} \mathrm{Alg}(\mathbf{D})$$

is an equivalence.

It follows that the same holds for the category of representations of any functor $G : \mathrm{CAlg}(\mathbf{C}) \to \mathrm{Grp}$, i.e. the canonical map

$$\mathrm{Alg}(\mathbf{Rep}_G(\mathbf{C})) \to \mathrm{Alg}^{nu}(\mathbf{Rep}_G(\mathbf{C})) \times_{\mathrm{Alg}^{nu}(\mathbf{C})} \mathrm{Alg}(\mathbf{C})$$

is an equivalence for any such G.

The result now follows from the symmetric monoidal equivalence

$$\mathrm{coMod}_H(\mathbf{C}) \simeq \mathbf{Rep}_{\mathrm{Spec}_{\mathbf{C}}(H)}(\mathbf{C})$$

compatible with the forgetful functor as discussed before the proof. □

Proposition 2.1.34. *Let $R = E = E(k, \mathbf{G})$ be a Morava E-theory over some field k of positive, possibly even characteristic, and at some height $n > 0$, and let $A \in \mathrm{Alg}(\mathrm{Mod}_E)$ be a dualizable separable E-algebra. If A is furthermore central, then A is Azumaya.*

Proof. Fix an atomic E-algebra K [HL17, Definition 1.0.2], i.e. a Morava K-theory.

By Theorem 2.1.29, K_* factors through $h_* : \mathrm{Mod}_E \to \mathrm{Mil}_E$, and K_* is conservative on perfect E-modules, hence by Lemma 2.1.16, it suffices to prove the result in Mil_E.

We prove the following intermediary result: let A be a central separable algebra in Mil_E, then A is simple, i.e. any (bilateral) ideal $I \hookrightarrow A$ is 0 or A. Notice that the functor $\mathrm{Mil}_E \to \mathrm{coMod}_{K_*^E K}(\mathrm{Mod}_{K_*}((\mathbf{GrVect}_k)) \to \mathrm{Mod}_{K_*}(\mathbf{GrVect}_k)$ is (strong) monoidal, and conservative, so it sends ideals to ideals, and dualizable separable algebras to dualizable separable algebras.

By Lemma 2.1.22, there is a central idempotent e in A such that $I = eA$. Furthermore, we started with an ideal in $\mathrm{coMod}_{K_*^E K}(\mathrm{Mod}_{K_*})$, and Lemma 2.1.30 will in fact imply that e is a morphism $K_* \to A$ in comodules, and not only in Mod_{K_*} (note that $K_*^E K$ is a commutative Hopf algebra by [BP21, Lemma 2.6]- this is so even at the prime 2).

The algebra we apply Lemma 2.1.30 to is I, viewed as a non-unital algebra in comodules. The existence of the central idempotent e in A such that $I = eA$ guarantees that I is unital in Mod_{K_*}, and thus, the lemma guarantees that it is unital in comodules.

This means that its unit is a morphism of $K_*^E K$-comodules $K_* \to I$, i.e., that the idempotent e is a map of $K_*^E K$-comodules $K_* \to A$.

This further implies that A splits as an algebra in $\mathrm{coMod}_{K_*^E K}$ as $I \times A/I$. This being a statement only about the monoidal structure of $\mathrm{coMod}_{K_*^E K}$, it holds also in the monoidal category of Milnor modules, i.e. Mil_E. But there, A is central by assumption, and so $I = 0$ or A, as was to be proved. We have thus proved that A was simple.

We now apply this to $A \otimes A^{\mathrm{op}}$, which is dualizable, central and separable as well, and hence simple. It follows that the canonical map $A \otimes A^{\mathrm{op}} \to \mathrm{End}(A)$ is injective. Now, the two sides have the same (finite) dimension as K_*-modules, so it follows that this map is an isomorphism, which is what was to be proved. □

To prove the general case of a commutative ring spectrum for which $R \otimes \mathbb{F}_p = 0$ for all p, we use the nilpotence theorem [HS98]. Let us recall an important consequence of it:

Proposition 2.1.35. *Let R be a commutative ring spectrum and P a dualizable R-module. Suppose that for all implicit primes p and and all $0 \leq n \leq \infty$, $L_{K(n)}P = 0$. In this case, $P = 0$.*

Here, $K(0) = \mathbb{Q}, K(\infty) = \mathbb{F}_p$. In particular, if $R \otimes \mathbb{F}_p = 0$ for all primes p, then it suffices to check that $L_{K(n)}P = 0$ for all $0 \leq n < \infty$.

Proof. By definition, $L_{K(n)}P = 0$ if and only if $K(n) \otimes P = 0$. As P is dualizable and $K(n)$ admits a ring structure, $K(n) \otimes P = 0$ if and only if $K(n) \otimes \mathrm{End}(P) = 0$: one direction is always true, as $K(n) \otimes P$ is a module over $K(n) \otimes \mathrm{End}(P)$. For the other direction, note that $\mathrm{End}(P) \simeq P \otimes_R P^\vee \simeq \mathrm{colim}_{\Delta^{\mathrm{op}}} P \otimes R^{\otimes n} \otimes P^\vee$.

Similarly, $P = 0$ if and only if $\mathrm{End}(P) = 0$.

Now, $\mathrm{End}(P)$ is an $\mathbb{E}_1$-ring, so the result follows from [HS98, Theorem 3].

The "in particular" part follows from the fact that if $R \otimes \mathbb{F}_p = 0$, then $P \otimes \mathbb{F}_p = 0$ too. □

We also recall an important consequence of the Chromatic Nulstellensatz [BSY22].

Proposition 2.1.36. *Fix an implicit prime p. Let R be a $K(n)$-local commutative ring spectrum, and P a nonzero dualizable R-module. There exists a field L as well as a map of commutative ring spectra $R \to E(L)$ such that $E(L) \otimes_R P \neq 0$.*

To prove this from the results of [BSY22], we need a bit of work. Before doing so, let us deduce the desired result from this.

Corollary 2.1.37. *Let R be a commutative ring spectrum such that $R \otimes \mathbb{F}_p = 0$ for all p. Question 2.1.13 has a positive answer in $\mathrm{Mod}_R(\mathrm{Sp})$, that is, every dualizable central separable algebra is Azumaya.*

Proof. Let A be a dualizable central separable algebra over R, and let P denote the cofiber of $A \otimes A^{\mathrm{op}} \to \mathrm{End}(A)$. We aim to prove that $P = 0$. To reach a contradiction, we assume $P \neq 0$.

As A is dualizable, P is dualizable too. By Proposition 2.1.35, there exists a prime p and an n such that $L_{K(n)}P \neq 0$. By Proposition 2.1.36, there exists a field and a map of commutative ring spectra $L_{K(n)}R \to E(L)$ such that $E(L) \otimes_{L_{K(n)}R} L_{K(n)}P \neq 0$. Note that P is dualizable over R, so that $L_{K(n)}R \otimes_R P \simeq L_{K(n)}P$.

Therefore, $E(L) \otimes_R P \neq 0$. As $E(L) \otimes_R -$ is symmetric monoidal, we find that $E(L) \otimes_R A$ is not Azumaya. This contradicts Proposition 2.1.34. □

We now explain how to deduce Proposition 2.1.36 from [BSY22].

First, a definition [CSY18, Definition 4.4.1]:

Definition 2.1.38. A monoidal functor $f : \mathbf{D} \to \mathcal{E}$ between stably monoidal ∞-categories is said to be *nil-conservative* if for all $R \in \mathrm{Alg}(\mathbf{D})$, $f(R) = 0$ implies that $R = 0$. ◁

Lemma 2.1.39 ([BSY22, Lemma 4.32]). *Let $\mathbf{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ be compactly generated, with the property that every compact in $\mathbf{C}$ is dualizable, and let $A \to B$ a morphism in $\mathrm{CAlg}(\mathbf{C})$. If it detects nilpotence, then $B \otimes_A - : \mathrm{Mod}_A(\mathbf{C}) \to \mathrm{Mod}_B(\mathbf{C})$ is nil-conservative.*

Lemma 2.1.40 ([CSY18, Proposition 4.4.4]). *A nil-conservative monoidal exact functor between stably monoidal ∞-categories is conservative when restricted to dualizable objects.*

Corollary 2.1.41. *Let $\mathbf{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ be compactly generated, with the property that every compact in $\mathbf{C}$ is dualizable, and $A \to B$ a morphism in $\mathrm{CAlg}(\mathbf{C})$. If it detects nilpotence, then $B \otimes_A - : \mathrm{Mod}_A(\mathbf{C}) \to \mathrm{Mod}_B(\mathbf{C})$ is conservatie when restricted to dualizable objects.*

One of the main results of [BSY22] is:

Theorem 2.1.42 ([BSY22, Theorem 5.1]). *Let R be a nonzero $T(n)$-local ring. There exists a perfect $\mathbb{F}_p$-algebra A of Krull dimension 0 and a nilpotence detecing map $R \to E(A)$ in $\mathrm{Sp}_{T(n)}$.*

If R is $K(n)$-local, then it is also $T(n)$-local. If P is furthermore dualizable over R, then for any map of commutative algebras $R \to S$ to a $K(n)$-local ring S, $E(A) \otimes_R P$ is already $K(n)$-local.

Corollary 2.1.43. *In order to prove Proposition 2.1.36, it suffices to prove the special case where $R = E(A)$ for A a perfect $\mathbb{F}_p$-algebra of Krull dimension 0.*

Proof. Suppose Proposition 2.1.36 holds whenever $R = E(A)$, A a perfect $\mathbb{F}_p$-algebra of Krull dimension 0, and let R be an arbitrary $K(n)$-local commutative ring spectrum, and P a nonzero dualizable R-module.

By Theorem 2.1.42 ([BSY22, Theorem 5.1]), we can find a nilpotence detecting map $R \to E(A)$ in $\mathrm{Sp}_{T(n)}$ for some perfect $\mathbb{F}_p$-algeba of Krull dimension 0, A. By Corollary 2.1.41, the $T(n)$-local tensor product with $E(A)$ over R is conservative on dualizable objects, hence $E(A) \otimes_R P$ is nonzero, since its $T(n)$-localization is nonzero (note that P is dualizable over R, so it is already $T(n)$-local, and hence it is nonzero as a $T(n)$-local R-module), and dualizable over $E(A)$, thus Proposition 2.1.36 follows for R. □

The proof of this special case is in fact implicit in the proof of [BSY22, Theorem 4.47] - we reproduce the proof nonetheless, for the convenience of the reader, as it is not explicitly spelled out:

Proof of Proposition 2.1.36. By the previous corollary, we may assume $R = E(A)$ for some perfect $\mathbb{F}_p$-algebra A of Krull dimension 0.

Let P be a dualizable $E(A)$-module. For any field k and any map $A \to k$, $E(k) \otimes_{E(A)} P$ is $K(n)$-local, and thus equivalent to its $K(n)$-localization.

Assume $E(k) \otimes_{E(A)} P = 0$ for all such $A \to k$, we wish to prove that $P = 0$. The ∞-category $L_{K(n)}\mathrm{Mod}_{E(A)}$ is compactly generated so it suffices to show that $[c, P]_{E(A)} = 0$ for any compact c. As c is compact in a p-complete ∞-category, p acts nilpotently on it. It follows that if $[c, P]_{E(A)} \otimes_{W(A)} A \cong [c, P]_{E(A)} \otimes_{\mathbb{Z}} \mathbb{F}_p$ is zero, then so is $[c, P]_{E(A)}$. Here, $W(A)$ is the ring of Witt vectors of A.

Now, by [BSY22, Lemma 4.45], if $[c, P]_{E(A)} \otimes_{W(A)} A$ is nonzero, there is a perfect field k and a map $A \to k$ such that $[c, P]_{E(A)} \otimes_{W(A)} A \otimes_A k \neq 0$. By [BSY22, Lemma 4.46], this tensor product is $[c \otimes_{E(A)} E(k), P \otimes_{E(A)} E(k)]_{E(k)}$, and this is 0 by assumption.

Here, we have used [BSY22, Lemma 4.46] with $\mathcal{C} = L_{K(n)}\mathrm{Mod}_{E(\mathbb{F}_p)}$ so that, by [BSY22, Lemma 2.37], $\mathbb{W}_{\mathcal{C}}(B) \simeq E(B)$ for any perfect $\mathbb{F}_p$-algebra B (in particular $B = \mathbb{F}_p, A$), and so that this really is an application of [BSY22, Lemma 4.46]. □

This concludes the proof of Theorem 2.1.14. As is clear from the proof, if one wants to answer Question 2.1.13 positively for $\mathrm{Mod}_R(\mathrm{Sp})$ for an arbitrary commutative ring spectrum R, one may without loss of generality assume R is an $\mathbb{F}_p$-algebra for some prime p. In this case, residue fields are harder to come by, and are the subject of ongoing work.

A first issue is that, away from characteristic 2, one cannot hope for graded fields, cf. [Mat17, Example 3.9]. One could still try to find enough "residue fields" and analyze their homotopy categories in enough detail to answer the question there. A good test-case would be to start with $R = k^{tC_p}$, where k is a field of characteristic p and the Tate construction is taken with respect to the trivial action. In this case, $\mathrm{Mod}_{k^{tC_p}} \simeq \mathrm{StMod}_{kC_p}$, the stable module ∞-category, and it seems possible to study the separable algebras therein an try to prove that they are Azumaya - for instance, the commutative case was studied in [BC18] (of course, the commutative case is orthogonal to our discussion, but Balmer and Carlson's result shows that such an analysis is not completely impossible).

We note that Question 2.1.13 (both its inputs and its answers, positive or negative) can be phrased in the homotopy category $\mathrm{ho}(\mathbf{C})$, and so one can also try to approach it using the homological residue fields of Balmer [Bal20]. Thus the question becomes completely about (graded) abelian symmetric monoidal 1-categories, over $\mathbb{F}_p$. It is not clear to the author whether one can say anything in this generality.

2.2 Centers of separable algebras

In this subsection, we study Question 2.0.1. Just as in the previous subsection, our approach is via descent. As the center of an algebra is $\mathbb{E}_2$, and in particular homotopy commutative, Corollary 1.3.37 tells us that separability can be tested locally.

In the previous subsection however, we used a much weaker notion of "local", namely, we tested Azumaya-ness against *conservative functors*, of which there is a larger supply than "descendable" functors. I was not able to phrase separability in terms of certain maps being equivalences, and so I am not able to use this technique.

For this reason, our positive answer is in a more restricted generality. The goal of this section is to prove:

Theorem 2.2.1. *Let $A \in \mathrm{Alg}(\mathbf{C})$ be a separable algebra. Question 2.0.1 has a positive answer, i.e. the center $Z(A)$ is separable, in the following cases:*

(i) If $\mathbf{C} = \mathrm{Mod}_R(\mathrm{Sp})$ for some connective commutative ring spectrum R, and A is almost perfect [Lur12, Definition 7.2.4.10]. More generally, this holds if $\mathbf{C} = \mathrm{QCoh}(X)$ for some (connective) Deligne-Mumford stack X and if A is locally almost perfect.

(ii) If $\mathbf{C} = \mathrm{Mod}_E(\mathrm{Sp}_{K(n)})$ is the ∞-category of $K(n)$-local E-modules, where E is Morava E-theory at height n, for some height n and some odd implicit prime p. In particular, the same is true if $\mathbf{C} = \mathrm{Sp}_{K(n)}$.

Remark 2.2.2. If we have some *a priori* control over $Z(A)$, one can get sometimes phrase separability in terms of certain maps being equivalences, and then get a more general positive answer. This is the case if, for instance, we assume that A is sufficiently finite over its center. For instance, in the ordinary category of (discrete) R-modules for some (discrete) commutative ring R, Auslander and Goldman prove in [AG60, Theorem 2.1] that a separable algebra is always dualizable over its center. I do not know in what generality this can be expected, and as I was not able to formulate general criteria for this to happen, I did not include results along these lines here. ◁

Remark 2.2.3. We note that if A is separable and almost perfect over a connective commutative ring spectrum R, $Z(A)$ is also almost perfect, and thus, by Proposition 1.3.56, it is separable if and only if it is étale. Under these finiteness assumptions, étaleness can be checked "conservative locally", and this is how we will be able to actually prove Item (i). In an earlier draft, the assumptions on R were more restrictive, and I am grateful to Niko Naumann and Luca Pol for sharing a draft of their work which allowed us to prove Proposition 1.3.56, and subsequently, this version of the above theorem. ◁

Before moving on to the proof of Theorem 2.2.1, we note that under a positive answer to Question 2.0.1, we can somewhat recreate the picture from [AG60]:

Lemma 2.2.4. *Suppose $A \in \mathrm{Alg}(\mathbf{C})$ is separable, and that $Z(A)$ is also separable.*

In this case, the center of A as a $Z(A)$-algebra, C, is equivalent to $Z(A)$, i.e. A is a central $Z(A)$-algebra.

Proof. Note that Theorem 1.3.19 implies, together with the homotopy commutativity of $Z(A)$, that $Z(A)$ has an essentially unique commutative algebra structure extending its ($\mathbb{E}_2$-)algebra structure.

Furthermore, by Theorem 1.2.6 and Proposition 1.1.21, we have that

$$\mathrm{ho}(\mathrm{Mod}_{Z(A)}(\mathbf{C})) \simeq \mathrm{Mod}_{hZ(A)}(\mathrm{ho}(\mathbf{C}))$$

as symmetric monoidal categories, compatibly with the lax symmetric monoidal functor to $\mathrm{ho}(\mathbf{C})$.

As A is separable over $Z(A)$ by Proposition 1.2.12, its center in $\mathrm{Mod}_{Z(A)}(\mathbf{C})$ can be computed in $\mathrm{ho}(\mathrm{Mod}_{Z(A)}(\mathbf{C}))$ by Corollary 1.1.29, and thus we may assume that $\mathbf{C}$ is a 1-category.

In particular, $A \otimes A^{\mathrm{op}} \to A \otimes_{Z(A)} A^{\mathrm{op}}$ is then an epimorphism, as it is split, so that $\hom_{A\otimes_{Z(A)}A^{\mathrm{op}}}(A, A) \to \hom_{A\otimes A^{\mathrm{op}}}(A, A)$ is a a monomorphism, compatible with the forgetful map to A.

But the first one receives a map from $Z(A)$, as $Z(A)$ is commutative, also compatible with the forgetful map to A, and so, because all these maps to A are monomorphisms (as they admit retractions and we are in a 1-category), this implies the claim. □

We now move on to Theorem 2.2.1. The descent method here is based on:

Proposition 2.2.5. *Let A be a homotopy commutative algebra in $\mathbf{C}$. Assume that there is fully faithful symmetric monoidal functor $\mathbf{C} \to \lim_I \mathbf{D}_i$, where $i \mapsto \mathbf{D}_i$ is a diagram of additively symmetric monoidal ∞-categories, and where I has a weakly initial set of objects I_0.*

In this case, if the projection $p_{i_0}(A)$ is separable in $\mathbf{D}_{i_0}$ for all $i_0 \in I_0$, then A is separable in $\mathbf{C}$.

Here, a set of objects I_0 in I is weakly initial if any object in I receives a map from some object in I_0.

Proof. As I_0 is a weakly initial set of objects, the assumption on $p_{i_0}(A)$ implies that $p_i(A)$ is separable in every object i, and so by Corollary 1.3.37, the image of A in $\lim_I \mathbf{D}_i$ is separable. By fully faithfulness, it follows that A is also separable in $\mathbf{C}$. □

This explains the second half of Item (i) in Theorem 2.2.1: for any (connective) Deligne Mumford stack X, $\mathrm{QCoh}(X)$ can be expressed as a limit of ∞-categories of the form $\mathrm{Mod}_R(\mathrm{Sp})$, so if one can prove the result for those ones, it follows automatically for $\mathrm{QCoh}(X)$. So we will prove Item (i) from Theorem 2.2.1 only in the affine case. We begin with:

Lemma 2.2.6. *Let A be a separable algebra in $\mathrm{Mod}_k(\mathrm{Sp})$, where k is a field. In this case, the center of A is separable.*

Proof. The homotopy groups functor induces a symmetric monoidal equivalence $\mathrm{ho}(\mathrm{Mod}_k(\mathrm{Sp})) \simeq \mathbf{GrVect}_k$ with the category of graded k-vector spaces, so by Corollary 1.1.29 and Proposition 1.2.9, it suffices to prove the result in $\mathbf{GrVect}_k$, so let A be a separable algebra therein.

We note that by Remark 1.1.31, $Z(A)$ is a $Z(A)$-linear retract of A. It follows that for any ideal I in $Z(A)$, we have $IA \cap Z(A) = I$. But now, by Lemma 2.1.22, because $\mathbf{GrVect}_k$ is semisimple, IA must be principal, generated by a (graded) central idempotent e. In particular, $e \in IA \cap Z(A)$, and so $e \in I$. Thus, $Z(A)$ is semi-simple.

It follows that any module over $Z(A)$ is projective, and in particular A is projective over $Z(A)$. Thus, as $Z(A)$-bimodules, we have that $Z(A)$ is a retract of A, which is a retract of $A \otimes_k A^{\mathrm{op}}$, which is projective over $Z(A) \otimes_k Z(A)$. Hence $Z(A)$ is projective over $Z(A) \otimes_k Z(A)$, which implies that it is separable. □

Recall that by [Nee18, Proposition 1.6], this means in particular that $Z(A)$ is discrete and an étale algebra overr the field k, in the usual sense.

We then reduce the general case to the discrete case:

Proposition 2.2.7. *To prove Item (i) from Theorem 2.2.1, it suffices to prove it in the case where R is discrete.*

Proof. Note that the canonical functor $\mathrm{Mod}_R \to \lim_n \mathrm{Mod}_{R_{\leq n}}$ is fully faithful when restricted to bounded below objects by [Lur18b, p. 19.2.1.5], so it suffices to prove the result for each $R_{\leq n}$, by Proposition 2.2.5.

In particular, as $R_{\leq n+1} \to R_{\leq n}$ is a square zero extension by a connective spectrum, it suffices to prove that the result is stable under such, namely, that the result for $R_{\leq n}$ implies that for $R_{\leq n+1}$. This follows from [Lur18b, Theorem 16.2.0.2] and Proposition 2.2.5: the ∞-category of bounded below $R_{\leq n+1}$-modules can be expressed as a pullback where the two corners are the ∞-category of bounded below $R_{\leq n}$-modules. □

Remark 2.2.8. This proof in fact shows that to provide a positive answer to Question 2.0.1 for a connective R, and in the bounded below case, it suffices to provide one for $\pi_0(R)$. ◁

Proof of Item (i) from Theorem 2.2.1. As explained above, we may assume R is a discrete commutative ring. We fix an almost perfect separable algebra A over R. We first aim to prove that its center $Z(A)$ is a flat R-module.

By Corollary 1.1.29, $Z(A)$ is a retract of A and thus is also almost perfect. By [Sta23, Tag 068V], to prove that it is flat, we may therefore basechange to any field and check that the result is in degree 0.

Using again Corollary 1.1.29, we find that for any map $R \to k$ to a field, $Z(A) \otimes_R k \simeq Z(A \otimes_R k)$. Now $A \otimes_R k$ is a separable algebra over a field, so by the case of fields, i.e. Lemma 2.2.6 , $Z(A \otimes_R k)$ is separable. By [Nee18, Proposition 1.6], it follows that it is discrete. Thus, $Z(A)$ is indeed flat over R, as claimed.

In particular, it is also discrete. Because it is almost perfect, it follows that it is also finitely presented, as a module over R. It also follows that it is finitely presented as an algebra over R, and thus [Sta23, Tag 02GM] implies that, to prove that it is étale over R, we may check after basechange along any map $R \to k$, k a field. But there Lemma 2.2.6 kicks in again: $Z(A) \otimes_R k \simeq Z(A \otimes_R k)$ is separable and hence étale over k.

It follows that $Z(A)$ is étale over R. By Proposition 1.3.53, $Z(A)$ is separable. □

We now move on to Item (ii) from Theorem 2.2.1. The proof of this will rely, as in Section 2.1, on Milnor modules. However, because the center of an algebra is a notion that really relies on the symmetric monoidal structure of the ambient category, this time we were not able to use a trick as in the proof of Proposition 2.1.34 to use the (not-necessarily-symmetric) monoidal equivalence $\mathrm{Mil}_E \simeq \mathrm{coMod}_{K^E_* K}(\mathrm{Mod}_{K_*}((\mathbf{GrVect}_k))$, so we are only able to give a proof at odd primes, where this equivalence *can* be made symmetric monoidal.

We will need a bit more about Milnor modules, so we recommend the reader have a deeper look at [HL17, Section 6]. What we called Mil_E in Theorem 2.1.29 is denoted $\mathrm{Syn}_E^{\heartsuit}$ in [HL17], but there is also a larger ∞-category Syn_E[8] and a fully faithful (Proposition 4.2.5 in *loc. cit.*),

[8] cf. Warning 1.4.16

symmetric monoidal (Variant 4.4.11 in *loc. cit.*) embedding $\mathrm{Sy}[-] : \mathrm{Mod}_E(\mathrm{Sp}_{K(n)}) \to \mathrm{Syn}_E$. We let $\mathbf{1}$ denote the unit of Syn_E, and $\mathbf{1}^{\leq n}$ its truncations. The following is implicit in [HL17]:

Lemma 2.2.9. *Let $X \in \mathrm{Syn}_E$. The canonical map $X \to \lim_n \mathbf{1}^{\leq n} \otimes X$ is an equivalence. In particular, the canonical symmetric monoidal functor*

$$\mathrm{Syn}_E \to \lim_n \mathrm{Mod}_{\mathbf{1}^{\leq n}}(\mathrm{Syn}_E)$$

is fully faithful.

Proof. The second part of the statement follows from the first, as the canonical map $X \to \lim_n \mathbf{1}^{\leq n} \otimes X$ is the unit of the adjunction $\mathrm{Syn}_E \rightleftarrows \lim_n \mathrm{Mod}_{\mathbf{1}^{\leq n}}(\mathrm{Syn}_E)$.

For the first part, we simply note that the canonical map $X \to \mathbf{1}^{\leq n} \otimes X$ induces an equivalence upon n-truncation, and therefore so do the morphisms $\mathbf{1}^{\leq m} \otimes X \to \mathbf{1}^{\leq n} \otimes X$. Because limits and truncations in $\mathrm{Syn}_E = \mathrm{Fun}^{\times}(\mathrm{Mod}_E^{\mathrm{mol}}, \mathcal{S})$ are pointwise, the claim follows. □

The following will allow us to reduce to $\mathrm{Syn}_E^{\heartsuit}$:

Lemma 2.2.10 ([HL17, Proposition 7.3.6])**.** *For every $n \geq 0$, basechange along $\mathbf{1}^{\leq n+1} \to \mathbf{1}^{\leq n}$ fits in a pullback square of additively symmetric monoidal ∞-categories of the form:*

$$\begin{tikzcd}
\mathrm{Mod}_{\mathbf{1}^{\leq n+1}}(\mathrm{Syn}_E) \arrow[r] \arrow[d] & \mathrm{Mod}_{\mathbf{1}^{\leq n}}(\mathrm{Syn}_E) \arrow[d] \\
\mathrm{Mod}_{\mathbf{1}^{\leq n}}(\mathrm{Syn}_E) \arrow[r] & \mathcal{C}
\end{tikzcd}$$

Proof of Item (ii) from Theorem 2.2.1. We begin by proving the case of $\mathbf{C} = \mathrm{Mod}_E(\mathrm{Sp}_{K(n)})$.

Let $A \in \mathrm{Mod}_E(\mathrm{Sp}_{K(n)})$ be a separable algebra. By [HL17, Proposition 4.2.5, Variant 4.4.11] and Corollary 1.1.29, to prove that its center is separable, it suffices to prove that its image $\mathrm{Sy}[A] \in \mathrm{Syn}_E$ has the same property, and by Lemma 2.2.9 and Proposition 2.2.5, it suffices to prove the same result for each $\mathbf{1}^{\leq n} \otimes \mathrm{Sy}[A] \in Mod_{\mathbf{1}^{\leq n}}(\mathrm{Syn}_E)$.

By induction, Proposition 2.2.5 and by Lemma 2.2.10, it suffices to prove it for $\mathbf{1}^{\leq 0} \otimes \mathrm{Sy}[A] \in \mathrm{Mod}_{\mathbf{1}^{\leq 0}}(\mathrm{Syn}_E)$. By [HL17, Lemma 7.1.1], the latter is equivalent to $\pi_0\mathrm{Sy}[A]$[9][10], and the same is true for $\mathrm{Sy}[A\hat{\otimes}A]$. In other words, $\pi_0\mathrm{Sy}[A]$ is a separable algebra in $\mathrm{Syn}_E^{\heartsuit}$, so we are reduced to the case of separable algebras in $\mathrm{Syn}_E^{\heartsuit}$.

It is in this last analysis, i.e. that of separable algebras in $\mathrm{Syn}_E^{\heartsuit}$, that we really use that we were working with $\mathrm{Mod}_E(\mathrm{Sp}_{K(n)})$ and the precise Syn_E from [HL17]. Namely, [HL17, Proposition 6.9.1] states that, at an odd prime, there is a symmetric monoidal equivalence between $\mathrm{Syn}_E^{\heartsuit}$ and the category of graded modules over a (finite dimensional) cocommutative Hopf algebra over $K_* \cong k[t^{\pm 1}], |t| = 2$. The latter is equivalently described as a category of algebraic representations of an algebraic group, and so, by Corollary 1.3.14, one can check that an algebra is separable on underlying objects (this is similar to the proof of Lemma 2.1.30).

As the center is preserved by this forgetful functor, we are reduced to the case of the category of graded modules over a graded field, where the proof is essentially the same as that of Lemma 2.2.6.

This concludes the proof for $\mathbf{C} = \mathrm{Mod}_E(\mathrm{Sp}_{K(n)})$. The case of $\mathbf{C} = \mathrm{Sp}_{K(n)}$ follows from this, together with Proposition 2.2.5 and Galois descent for the $K(n)$-local Galois extension $\mathbb{S}_{K(n)} \to E$ (in more detail, see [Mat16, Proposition 10.10]). □

[9] Denoted $\mathrm{Sy}^{\heartsuit}[A]$ in *loc. cit.*.

[10] This result can be seen as a version of the statement "$\mathrm{Sy}[A]$ is flat", see also [PV22, Proposition 2.16]. Thus in a sense the beginning of this proof is very similar to the proof of Item (i).

As for Question 2.1.13 we note that all the parts involved in Question 2.0.1 (its inputs and its answers, positive or negative) only depend on the homotopy category of **C**. In particular, if one tries to answer the question in full generality, one can try to consider abelian categories, e.g. via Balmer's homological residue fields – this idea is very clearly apparent in the proof of Item (ii).

Ultimately, the questions in Chapter 2, in full generality, are questions about symmetric monoidal abelian categories.

Chapter 3

Questions and perspectives

3.1 The finite étale site

My work in this Part I tightens the analogy between commutative separable algebras and étale algebras. To get an even closer connection and not worry about finiteness questions, it is reasonable to restrict to the "finite étale" part of the theory, corresponding to dualizable commutative separable algebras.

A natural research direction is thus to extend much of the classical "étale geometry" from classical algebra, or even spectral algebraic geometry to this more general setting, particularly towards nonconnective geometry. Can we compute some natural étale sites ? Is étale cohomology in this generality interesting/computable ?

Burklund and Burklund–Clausen–Levy have work in progress in this direction in the $T(n)$- and $K(n)$-local settings, but the rational and $\mathbb{F}_p$-linear settings remain wildly open. For example, some of my proofs use the chromatic nullstellensatz from [NS18] and are therefore unable to reach $\mathbb{F}_p$-linear settings. An almost precise question would be:

Question 3.1.1. Can one describe nullstellensatzian commutative $\mathbb{F}_p$-algebras in the sense of [NS18] ? What about separably closed $\mathbb{F}_p$-algebras ?

What about nullstellensatzian/separably closed symmetric monoidal stable ∞-categories? ◁

Another question, related to the classification of separable algebras in general, which can be asked for finite separable algebras but also in general concerns the notion of *tt-degree*, for which I refer the reader to [Bal14]. I have not discussed this in this thesis, but it seems like a natural question to ask now that the foundations of the theory of separable algebras have been laid, and I hope to study it in future work - with Chedalavada, we have made some progress on this question:

Question 3.1.2. What natural assumptions on $\mathbf{C}$ guarantee that its separable algebras all have finite tt-degree ? ◁

Remark 3.1.3. In [Góm23], Gómez shows that these assumptions must be more stringent than "rigidly compactly generated". ◁

3.2 Ind-separability

The notion of "ind-separability" that I set up in Section 1.4 is partly *ad hoc*, though it is sufficient for the results I presented here. I believe it would be interesting to study it in more detail, and potentially refine the definition.

A question I essentially completely adressed in the separable case but not at all in the ind-separable case is:

Question 3.2.1. How does ind-separability interact with module categories ? ◁

There are also other reasonable notions of infinitary separability that one can study, at least once one already has a highly structured algebra (though see the discussion at the beginning of Section 1.4), it could be interesting to compare them to ind-separability:

Question 3.2.2. How does ind-separability precisely relate to formal étale-ness in the sense of a vanishing $\mathbb{E}_\infty$-cotangent complex ? How about to formal THH-étaleness in the sense of [Rog08] ? ◁

3.3 Auslander-Goldman theory

I have already raised these questions in the main body of the text, but I wish to recall them as interesting questions in the structure theory of general separable algebras:

Question 3.3.1. Is the center of a separable algebra necessarily separable ? ◁

Question 3.3.2. Is a central separable, dualizable algebra necessarily Azumaya ? ◁

Finally, a question which we have not seriously considered here but studied in some examples, is the amount by which the condition of Proposition 2.1.10 is restrictive, say up to Morita equivalence. A precise question could be:

Question 3.3.3. Can we describe the Morita closure of separable Azumaya algebras ? How about the "descent-closure", that is, the Azumaya algebras that are locally separable, or locally Morita equivalent to a separable algebra ? ◁

The word "locally" here may be interpreted liberally. A possibly interesting interpretation could be to ask it "separable-locally", and in particular study the theory of separable splitting rings for Azumaya algebras (note that not all Azumaya algebras are separable-locally trivial in the nonconnective world). More generally, and somewhat tangentially, the study of Brauer groups and Azumaya algebras away from the connective setting seems like a promising avenue for research.

Finally, I'll allow myself to give a "splitting" of a conjecture of Hopkins and Lurie [HL17, Conjecture 9.4.1] about Azumaya algebras into two possibly simpler conjectures - besides a vague hope and work at height 1, there is no evidence that these ought to be much simpler than their conjecture:

Question 3.3.4. Let A be an Azumaya algebra in Mod_E, where E is a Morava E-theory[1].

Is A necessarily Morita equivalent to a separable Azumaya algebra ?

If A is separable, is A Morita equivalent to an E-algebra which is (graded) free as an E-module ? ◁

A positive answer to both questions would imply a positive answer to [HL17, conjecture 9.4.1], but I believe that these questions may be more easily approachable.

[1]We are considering *non-$K(n)$*-local modules here, so Morava K-theories are *not* Azumaya in this context

Part II

The Dundas–McCarthy theorem and applications

Introduction to Part II

Topological Hochschild homology (henceforth, THH) is a central object in modern homotopical algebra, along with its many variants such as TR, TP, TC, TC^-. It can be used to study generalizations of traces of matrices to noncommutative settings, and it can also be seen as a noncommutative analog of the complex of de Rham forms, through the Hochschild–Kostant–Rosenberg theorem [KHK+09]. Here, we shall take a third point of view on THH, namely its relationship to algebraic K-theory: THH is some kind of *linearization* (or first derivative) of K-theory. This relationship is at the heart of so-called "trace methods" which can be used to reduce (some) K-theory calculations to linear algebra.

The goal of Part II of this thesis is two-fold. First, I give a modern account and slight generalization of this relationship - this is the content of Chapter 4, where I explore the Dundas–McCarthy theorem from the perspective of Blumberg, Gepner and Tabuada's localizing invariants and of Kaledin and Nikolaus's trace theories.

Second, I use this relationship to compute invariants *of* THH, more specifically, compute the endomorphisms of THH (and variants thereof). This is the content of Chapter 5.

I give more precise descriptions of their content, and more complete introductions at the start of the respective chapters.

Local conventions

On top of the global conventions outlined at the beginning of the thesis, we have the following conventions.

- We recall in Appendix B our conventions regarding localizing invariants and splitting[2] invariants.
- We use implicitly the equivalence $\mathrm{Ind} : \mathrm{Cat}^{\mathrm{perf}} \simeq \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega} : (-)^{\omega}$ between small idempotent complete stable ∞-categories and compactly generated stable ∞-categories and compact preserving morphisms between them.
- The symbol K denotes nonconnective algebraic K-theory, while K^{cn} denotes its connective variant. Note that, unlike in [BGT13], we do not require the latter to be invariant under idempotent-completion, and in particular, K^{cn} is only the connective cover of K for idempotent-complete stable ∞-categories (otherwise it agrees with it in degrees ≥ 1).
- Given a coCartesian fibration $p : E \to S$ and an edge $f : t \to s$ in S, we let $f_! : E_t \to E_s$ denote the associated coCartesian pushforward.
- Given a space X, LX denotes $\mathrm{map}(S^1, X)$, its free loop space.
- Given an $(\infty, 2)$-category B, we let $\iota_1 B$ denote the underlying ∞-category, that is the ∞-category obtained by forgetting the non-invertible 2-morphisms in B.

Local acknowledgements

I've had helpful conversations about the contents of this work with Sasha Efimov, Achim Krause, Ishan Levy, Zhouhang Mao, Thomas Nikolaus, Victor Saunier and Lior Yanovski. As usual, I also want to thank my advisors, Jesper Grodal and Markus Land for their support.

Some of this work was written while I was visiting various universities in North America, and I wish to thank them and the people who invited me or housed me there for their

[2] More often called "additive".

hospitality: in chronological order, those were Mike Hopkins at Harvard University, Ben Antieau and Achim Krause at the Institute for Advanced Study, David Gepner at Johns Hopkins Univerisity, Niranjan Ramachandran at the University of Maryland, Akhil Mathew at the University of Chicago, Paul Balmer and Morgan Opie at UCLA, and Elden Elmanto at the University of Toronto.

Finally, thanks to q.uiver for help with the commutative diagrams.

Chapter 4

Around the Dundas–McCarthy theorem

Introduction

The Dundas–McCarthy theorem [DM94] is at the heart of trace methods: it suggests that the infinitesimal behavior of K-theory is controlled by THH, and that therefore we may be able to understand K-theory in a "neighborhood" of some already understood ring or ring spectrum by completely "linear algebraic" methods, using THH or variants thereof.

This theorem can be stated and proved for connective ring spectra using connectivity estimates on the K-theory and THH spectra of these, and while trace methods have mostly been successful in the connective setting, it remains a natural question to wonder whether this result holds in the nonconnective setting; and in particular, to what extent this theorem can be proved within the conceptual framework provided by Blumberg, Gepner and Tabuada's perspective on K-theory and related invariants [BGT13]. Among other things, this nonconnective version of the result will be needed in Chapter 5 to compute endomorphisms of THH as a functor of stable ∞-categories by essentially reducing to K-theory, using non-connective ring spectra in an essential way.

A proof of the Dundas–McCarthy theorem closer to those lines is sketched in [Ras18], but the details of this proof are not particularly easy to fill. The approach I will follow here was pioneered by Nikolaus, using so-called *trace theories*, as sketched in [HS19].

My proof, though, is slightly different: while I use the language of trace theories and the universal property of K-theory, I prove a classification theorem for *cocontinuous* trace theories which essentially directly implies the result; but further provides a generalization of the Dundas–McCarthy theorem for arbitrary finitary localizing invariants.

The classification result is Theorem F from the introduction and can be stated as follows - here, $\mathrm{TrThy}(\mathcal{E})$ denotes the ∞-category of $\mathcal{E}$-valued *trace theories*, the superscript L indicates that we are considering the ones that are cocontinuous "in the bimodule variable", and TrThy_Δ is a slight variant of TrThy; precise definitions will appear later:

Theorem. *Let $\mathcal{E}$ be a stable cocomplete ∞-category. Evaluation at $(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})$ induces equivalences:*

$$\mathrm{TrThy}_\Delta^L(\mathcal{E}) \xrightarrow{\simeq} \mathcal{E}$$

and

$$\mathrm{TrThy}^L(\mathcal{E}) \xrightarrow{\simeq} \mathcal{E}^{BS^1}$$

I use this to prove the following version of the Dundas–McCarthy theorem, which is Theorem G from the introduction:

Theorem. *Let $E : \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{E}$ be a finitary localizing invariant with values in a cocomplete stable ∞-category $\mathcal{E}$. There exists an object of $\mathcal{E}$ with S^1-action X_E such that*

$$P_1 E^{\mathrm{cyc}} \simeq X_E \otimes \mathrm{THH}$$

where $P_1 E^{\mathrm{cyc}}$ is the linearization of E.

For E being algebraic K-theory, X_E is $\mathbb{S}$ with trivial S^1-action.

In this theorem, $X_E \otimes \mathrm{THH}$ is obtained using the canonical action of Sp on $\mathcal{E}$, and using the S^1-action on X_E to make the result into a trace theory. A more precise construction will also appear later.

I also use the classification result to sketch a comparison bettwen Nikolaus' construction of THH in [HS19] and Hoyois, Scherotzke and Sibilla's construction in [HSS17]; and finally I use methods from Land and Tamme's work [LT19] to recover the connectivity estimates from [DM94] in the connective case, showing that even working purely with universal properties may be used to recover these estimates (which can be useful for other purposes).

Outline

In Section 4.1, I introduce trace theories following [HS19], and explain how to go back and forth between trace theories and localizing invariants. In Section 4.2, I specialize the picture to cocontinuous trace theories and finitary localizing invariants, and I prove the classification theorem for cocontinuous trace theories. Finally, I use it to deduce the Dundas–McCarthy theorem and its generalization mentioned above. This section has three addenda: in the first one, I explain how to recover cyclotomic (and more generally polygonic) structures on THH and first derivatives of localizing invariants from the perspective of trace theories; in the second one, I explain how to define the trace functor from [HSS17] as a trace theory, and how to use my classification theorem to compare it to THH; and finally in the third one I explain how to recover connectivity estimates for the trace map $K \to \mathrm{THH}$ simply from the abstract Dundas–McCarthy theorem.

4.1 Trace theories and localizing invariants

In this section we introduce trace theories following Kaledin [Kal15] and Nikolaus [HS19] and relate them to localizing invariants in the sense of [BGT13][1].

We begin in the general setting of an $(\infty, 2)$-category, and are thereby approximately in the middle of these two presentations (the former being set in an ordinary 2-category, while the latter is in the specific $(\infty, 2)$-category of compactly generated stable ∞-categories).

First, a historical and contextual remark:

Remark 4.1.1. Kaledin seems to have defined trace theories around the same time that Ponto–Shulman defined shadows [PS13], and somewhat independently. The two definitions are remarkably close to one another, but subtly different.

I intend to adress this difference, as well as a comparison between the two in future work [Ram], where I prove that while they have different definitions, the resulting theories agree (I view this result as a homotopy-coherent enhancement of the Morita invariance of shadows, cf. [CP19, Proposition 4.8]). ◁

[1] With the by now standard omission of the filtered-colimit-preservation condition.

Trace theories are meant to encode the following fundamental property of Hochschild homology (which also holds for THH): if R, S are two rings, M, N are (R, S)- and (S, R)-bimodules respectively, then there is a canonical equivalence

$$\mathrm{HH}(R; M \otimes_S N) \cong \mathrm{HH}(S; N \otimes_R M)$$

It turns out to be convenient, for higher coherences (and expected structures like C_n-actions on $\mathrm{HH}(R, M^{\otimes_R n})$), to rather encode this in an "unbiased way" as follows: there will be an object on which we can evaluate our "trace theory" $\mathrm{HH}(-)$, loosely denoted by $(R, S; M, N)$ equipped with maps

$$(S, N \otimes_R M) \leftarrow (R, S; M, N) \rightarrow (R, M \otimes_S N)$$

and we will demand that these maps be sent to equivalences by $\mathrm{HH}(-)$. This is convenient as this third object is "unbiased" towards either composition order $N \otimes_R M$ or $M \otimes_S N$.

The natural context for these kinds of objects is that of 2-categories, or rather, in our case, $(\infty, 2)$-categories: rings will correspond to objects, bimodules to 1-morphisms, and morphisms of bimodules to 2-morphisms. So we fix for the remainder of the discussion[2] an $(\infty, 2)$-category B. Rather than just singling out the cyclic invariance for *two* bimodules, the generic object that we can plug in to the trace theory will look like

$$(b_0 \xrightarrow{f_0} b_1 \xrightarrow{f_1} \ldots\ldots \xrightarrow{f_{n-1}} b_n \xrightarrow{f_n} b_0)$$

where, again, the b_i's can be thought of as rings, and the f_i's as bimodules.

Maps between those, for a "fixed n" will be (co)lax morphisms between these diagrams, accounting for both (restricted) functoriality in the b_i's and in the f_i's[3]. The morphisms between varying n are supposed to encode composition, and will be encoded in the combinatorics of Connes' cyclic category Λ, which we review in Appendix A[4]. These objects, together with these kinds of morphisms, will assemble into an ∞-category Λ_B. A trace theory on B will, in turn, be a functor out of Λ_B, sending specific morphisms to equivalences.

Before giving the precise definitions involved in the concept of trace theories, we describe here our main example of an $(\infty, 2)$-category:

Notation 4.1.2. We let $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)}$ denote the *full* sub-$(\infty, 2)$-category of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ spanned by compactly generated stable ∞-categories. ◁

Remark 4.1.3. It is crucial here that we take the full subcategory, as opposed to what we would denote by $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$ where the morphisms are also restricted to be the compact-preserving cocontinuous functors. ◁

A reasonably large collection of trace theories on $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)}$ (the *exact* trace theories, that is, those that induce exact functors on $\mathrm{Fun}^L(C, C)$ for every C) is easily seen[5] to induce localizing invariants upon restriction to objects of the form (C, id_C). This is for example how one obtains THH as a localizing invariant from THH as a trace theory.

The remarkable phenomenon that we describe in this section after having given the basic definitions is that there is a way to go back: the first Goodwillie derivative of localizing invariants can be canonically given the structure of a trace theory. This structure will be particularly helpful to compare THH and P_1K^{cyc}, which is the announced goal of Chapter 4.

[2]Which is meant to be an informal description of the concept of "trace theory", a precise definition of which appearing shortly after.

[3]This is the difference between shadows and trace theories: in shadows, the lax maps have to fix the b_i's, so that the functoriality is *a priori* only in the f_i's.

[4]Our conventions are slightly different from some appearing in the literature, so we recommend the reader have a brief look at this appendix.

[5]See Proposition 4.1.54.

4.1.1 Trace theories

We can now start with precise definitions. We first recall the following definitions and notations:

Definition 4.1.4. Let B be an $(\infty,2)$-category. The ∞-category B^{ladj} is the wide subcategory of the underlying ∞-category $\iota_1 B$ of B spanned by left adjoint morphisms in B. ◁

Example 4.1.5. For $B = \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)}$, we find that $(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)})^{\mathrm{ladj}} = \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$, the ∞-category of compactly generated ∞-categories and compact-preserving cocontinuous functors between them.

If instead we had started with $B = \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$, then in B^{ladj}, the only morphisms allowed would be those whose right adjoint also preserves compacts! ◁

Definition 4.1.6. Let C be an ∞-category and B an $(\infty,2)$-category. The ∞-category $\mathrm{Fun}_{\mathrm{colax}}(C,B)$ is defined in [Hau20, Definition 3.9]. Its objects are functors $C \to \iota_1 B$, and its morphisms are colax natural transformations. ◁

Informally, a colax natural transformation from f to g is a collection of 1-morphisms $f(c) \to g(c)$ in B together with (not necessarily invertible) 2-morphisms

$$\begin{tikzcd} f(c_0) \arrow[r] \arrow[d] & g(c_0) \arrow[d] \\ f(c_1) \arrow[r] \arrow[ur, phantom, "\nearrow"] & g(c_1) \end{tikzcd}$$

for every morphism $c_0 \to c_1$, with (higher) coherences. We will call these "colax naturality squares".

Before giving the main definition, we first introduce a convenient notation:

Notation 4.1.7. For $C \in \mathrm{Cat}$, we let

$$\mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}(C,B) := \mathrm{Fun}_{\mathrm{colax}}(C,B) \times_{\mathrm{Fun}(C^{\simeq},B)} \mathrm{Fun}(C^{\simeq}, B^{\mathrm{ladj}})$$

◁

These are the functors $C \to B$ and colax transformations between them that are, objectwise, left adjoints - so in the example square from above, the maps $f(c_i) \to g(c_i)$ would be left adjoints in B, while the maps $f(c_0) \to f(c_1)$ and $g(c_0) \to g(c_1)$ have no extra conditions.

Warning 4.1.8. This notation has the potential for confusion: $\mathrm{Fun}_{\mathrm{colax}}(C,B)$ can be naturally upgraded to an $(\infty,2)$-category, and as such it has an internal notion of left adjoints. What we are describing is *not* the ∞-category of left adjoints in $\mathrm{Fun}_{\mathrm{colax}}(C,B)$. However, the latter will not play a role in this thesis, so there should be no confusion. ◁

In the following definition, we use the description of Λ from Corollary A.0.19 to identify Λ with a certain (non-full) subcategory of Cat:

Definition 4.1.9. Let B be an $(\infty,2)$-category. The functor $\Lambda^{\mathrm{op}} \to \mathrm{Cat}$, given by

$$C \mapsto \mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}(C,B)$$

has a cocartesian unstraightening, which we denote by $\Lambda_B \to \Lambda^{\mathrm{op}}$.

We can restrict it along $\Delta^{\mathrm{op}} \to \Lambda^{\mathrm{op}}$ and obtain $\Delta_B \to \Delta^{\mathrm{op}}$ (and similarly $(\Lambda_\infty)_B \to (\Lambda_\infty)^{\mathrm{op}}$, though we will not make much use of it).

More generally, whenever we have a functor $\Gamma \to \mathrm{Cat}$ for some ∞-category Γ, we may define an ∞-category Γ_B in the same way. We will not use this a lot either, but it will be convenient to have this flexibility. ◁

Here, the notation is slightly abusive since Γ_B depends on the specific functor $\Gamma \to \mathrm{Cat}$: for example, Δ_B is defined using *not* the standard embedding $\Delta \subset \mathrm{Cat}$ but the functor $(-)_\Lambda : \Delta \to \Lambda$ followed by the standard inclusion $\Lambda \to \mathrm{Cat}$. We hope no confusion arises from this.

In $\mathrm{Fun}_{\mathrm{colax}}(C, B) \times_{\mathrm{Fun}(C^{\simeq},B)} \mathrm{Fun}(C^{\simeq}, B^{\mathrm{ladj}})$, the objects are simply functors $C \to \iota_1 B$, and morphisms between them are colax natural transformations which are *object-wise* left adjoints.

Example 4.1.10. If $C = \mathrm{pt}$, this ∞-category is B^{ladj}.

If $C = \Delta^1$, this ∞-category has objects arbitrary arrows in B, and morphisms colax naturality squares where the horizontal maps are left adjoints in B. ◁

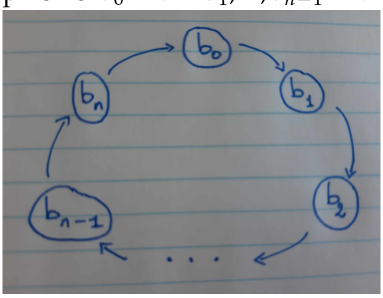
Example 4.1.11. In [HHL+20, Theorem E], the authors prove that for $B = \mathrm{Cat}$, $\mathrm{Fun}_{\mathrm{colax}}(C, \mathrm{Cat})$ is equivalent to the *full* subcategory of $\mathrm{Cat}_{/C^{\mathrm{op}}}$ spanned by cartesian fibrations.

Recall that under un/straightening, $\mathrm{Fun}(C, \mathrm{Cat})$ is equivalent to the subcategory of $\mathrm{Cat}_{/C^{\mathrm{op}}}$ spanned by cartesian fibrations and cartesian-morphism-preserving functors between those. This latter condition corresponds exactly to a colax natural transformation being strict, that is, to the colax naturality squares being given by invertible 2-cells.

In [HS19], Nikolaus takes this perspective rather than that of colax natural transformations to describe trace theories, but by *loc. cit.*, we get equivalent notions. ◁

Remark 4.1.12. One could also use a *cartesian* unstraightening of the functor $\Lambda^{\mathrm{op}} \to \mathrm{Cat}$. The total category of this cartesian fibration would be inequivalent but lead to an equivalent notion of trace theory; and in fact this alternative description can be useful for some purposes. However, we do not need it for the present work, so we do not give it a name or discuss it further. ◁

The typical element of Λ_B lying over $[n]_\Lambda$ is a tuple $(b_0, ..., b_n)$ equipped with morphisms $b_0 \to b_1, ..., b_{n-1} \to b_n, b_n \to b_0$. It is convenient to draw it as follows:



In particular, we have a functor $B^{\mathrm{ladj}} \to \Lambda_B$ given by $b \mapsto (b, \mathrm{id}_b)$ lying over $[0]_\Lambda$.

Cocartesian edges in Λ_B lying over maps $C \to D \in \Lambda$ correspond to precomposition. The reader is encouraged to write those out explicitly for the generating edges from Construction A.0.20, Construction A.0.22 and Construction A.0.21.

Definition 4.1.13. A *trace theory* on B with values in some ∞-category $\mathcal{E}$ is a functor $\Lambda_B \to \mathcal{E}$ which inverts edges that are coCartesian with respect ot the fibration[6] $\Lambda_B \to \Lambda$. Equivalently, it is a transformation

$$\mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}(C, B) = \mathrm{Fun}_{\mathrm{colax}}(C, B) \times_{\mathrm{Fun}(C^{\simeq},B)} \mathrm{Fun}(C^{\simeq}, B^{\mathrm{ladj}}) \to \mathcal{E}$$

[6]By default, if the fibration is not specified, this is always the one that is meant.

natural in C. A pre-trace theory is simply a functor $\Lambda_B \to \mathcal{E}$. We let $\mathrm{TrThy}(B;\mathcal{E})$, resp. $\mathrm{TrThy}^{\mathrm{pre}}(B;\mathcal{E})$ denote the ∞-categories of trace theories and pre-trace theories with values in $\mathcal{E}$.

A Δ-trace theory with values in $\mathcal{E}$ is similarly a functor $\Delta_B \to \mathcal{E}$ which inverts cocartesian edges. A pre-Δ-trace theory is similarly a functor $\Delta_B \to \mathcal{E}$. We similarly denote the corresponding ∞-categories by $\mathrm{TrThy}_\Delta(B;\mathcal{E}), \mathrm{TrThy}_\Delta^{\mathrm{pre}}(B;\mathcal{E})$. ◁

Remark 4.1.14. It would be convenient for some arguments, and also generally interesting to define a symmetric monoidal version of this notion when B is equipped with a symmetric monoidal structure, such as the Lurie tensor product on $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)}$, but we will not do so here. We will pay the price in the proof of Theorem 4.2.1, which will be (only slightly) less slick than it could be. ◁

In particular, the universal trace theory has values in $\Lambda_B[(\mathrm{cocart})^{-1}]$, which is equivalently[7] a colimit over Λ^{op} of the functor

$$C \mapsto \mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}(C, B)$$

Notation 4.1.15. When $B = \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)}$,we let Λ^{st} and Δ^{st} denote Λ_B and Δ_B respectively, and more generally Γ^{st} for Γ_B. ◁

Convention 4.1.16. When we use the word "trace theory" with no further context, we mean a trace theory over $B = \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)}$. ◁

Remark 4.1.17. Note that in the definition of Λ_B or Δ_B, we allow morphisms that are left adjoints *in* B between different labelled cyclic graphs. In $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)}$, this means functors that have left adjoints themselves in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, that is, functors whose right adjoint preserves colimits, or equivalently, compact preserving functors. ◁

Notation 4.1.18. Because of Example 4.1.11, we also think of an object of Λ^{st} lying over $[n]_\Lambda$ as a particular kind of presentable cartesian fibration over $([n]_\Lambda)^{\mathrm{op}}$ (specifically, one where the fibers are compactly generated and stable). ◁

Before moving on to examples, let us describe the kind of structure that comes from a trace theory. Specifically, we mention two features: first, the desired "cyclic invariance" which was the motivation to set all this structure up; second, C_n and S^1-actions on specific values of trace theories, which were part of the motivation for the higher coherences of the definition of trace theories. These constructions will be evidence that the notion of trace theory encodes what we wanted it to.

Example 4.1.19 (Cyclic invariance)**.** Let $b_0, b_1 \in B$ and $f_0 : b_0 \to b_1, f_1 : b_1 \to b_0$. We get an object $(\vec{b}, \vec{f}) := (b_0 \xrightarrow{f_0} b_1 \xrightarrow{f_1} b_0) \in \Lambda_B$ lying over $[1]_\Lambda$. We then have two morphisms $[1]_\Lambda \to [0]_\Lambda \in \Lambda^{\mathrm{op}}$, corresponding to the two morphisms $[0]_\Lambda \to [1]_\Lambda$ in Λ, which send the single object to either object of $[1]_\Lambda$, and the generating morphism to the corresponding composite of generating morphisms.

These two morphisms have coCartesian lifts, namely precomposition by the two maps in Λ, so these coCartesian lifts are maps

$$(\vec{b}, \vec{f}) \to (b_0, f_1 \circ f_0), (\vec{b}, \vec{f}) \to (b_1, f_0 \circ f_1)$$

[7] Modulo set theory.

respectively.

By definition, a trace theory T sends these to equivalences so that we find a canonical equivalence:

$$T(b_0, f_1 \circ f_0) \xleftarrow{\simeq} T(\vec{b}, \vec{f}) \xrightarrow{\simeq} T(b_1, f_0 \circ f_1)$$

◁

One can call this the "trace property" of T.

Since we will upgrade the following example in Section 4.2.1 and will not use it before then, we do not go into too much detail:

Example 4.1.20 (C_n-actions)**.** Our goal is to construct the following: given $b \in B$ and an endomorphism $f : b \to b$, for any trace theory T, a C_n-action on $T(b, f^{\circ n})$.

For this, we first use the previous example to rewrite $T(b, f^{\circ n})$ as $T(\vec{b}, \vec{f})$ where $(\vec{b}, \vec{f}) = (b \xrightarrow{f} ... \xrightarrow{f} b)$ with n times f, lying over $C = [n-1]_\Lambda$.

This a C_n-fixed point in $\mathrm{Fun}(C, B)$ where C_n-acts on $C = [n-1]_\Lambda$ by rotation, so it can be seen as a section

$$BC_n \to (\mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}(C, B))_{hC_n}$$

of the classifying fibration.

But the total category of this classifying fibration maps to Λ_B, as it is exactly the pullback $\Lambda_B \times_{\Lambda^{\mathrm{op}}} BC_n$. In other words, we have a functor $BC_n \to \Lambda_B$ classifying the rotation action on $(\vec{b}, \vec{f})$. This is the desired C_n-action. ◁

Example 4.1.21 (S^1-actions)**.** The S^1-action comes from "bundling up" together all the C_n-actions on $T(b, \mathrm{id}_b) = T(b, \mathrm{id}_b^{\circ n})$. In more detail, let us observe that we have a *coCartesian* section $\Lambda^{\mathrm{op}} \to \Lambda_B$ of the form

$$[n]_\Lambda \mapsto (\vec{b}, \vec{\mathrm{id}}_b) = (b \xrightarrow{\mathrm{id}_b} b \to \dots \xrightarrow{\mathrm{id}_b} b)$$

This can be constructed by noting that each $C \in \Lambda$ has a morphism in Cat to the terminal category pt, which can be used to get a coCartesian section as desired. Thus the composite $\Lambda^{\mathrm{op}} \to \Lambda_B[\mathrm{cocart}^{-1}]$ is constant and therefore its restriction to Δ^{op} has an absolute colimit, given by (b, id_b).

By Construction A.0.13, this implies that for any functor $T : \Lambda_B[\mathrm{cocart}^{-1}] \to \mathcal{E}$, i.e. any $\mathcal{E}$-valued trace theory, $T(b, \mathrm{id}_b)$ has a canonical S^1-action. This can clearly be upgraded to yield a functor $\mathrm{TrThy}_B(\mathcal{E}) \to \mathrm{Fun}(B^{\mathrm{ladj}}, \mathcal{E}^{BS^1})$ for any $(\infty, 2)$-category B. ◁

As it will be relevant in the near future, we also point out the following piece of structure:

Example 4.1.22 (Local functoriality)**.** Fix $b \in B$, and let T be a trace theory on B with values in $\mathcal{E}$. The fiber at $[0]_\Lambda$ of Λ_B is $\mathrm{Fun}_{\mathrm{colax}}(B\mathbb{N}, B) \times_B B^{\mathrm{ladj}}$, which is itself fibered over B^{ladj} with fiber $\hom_B(b, b)$ over b. Thus we obtain a functor

$$T(b, -) : \hom(b, b) \to \mathcal{E}$$

which is the "local" functoriality of T.

When $B = \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)}$ and $b = \mathrm{LMod}_A(\mathrm{Sp})$ for some ring spectrum A, $\mathrm{Fun}^L(\mathrm{LMod}_A, \mathrm{LMod}_A)$ is equivalent to the category of A-bimodules, and so we also say "bimodule functoriality" of T to talk about this local functoriality. When referring to properties that are local in this sense, we may say "in the bimodule variable". ◁

We will not need the fact that the forgetful functor $(\Lambda_B)_{[0]_\Lambda} \to B^{\mathrm{ladj}}$ is in fact a coCartesian fibration, but the proof of this is a simpler version of Proposition 4.2.21 to come, and so for the convenience of the reader, to get a better feel for the latter, we include a proof. The idea here (and later, there) is that restricting to adjoints in the target (the base) of the functor will typically provide the desired coCartesian lifts one needs.

Proof that $\mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}(B\mathbb{N}, B) \to B^{\mathrm{ladj}}$ *is a coCartesian fibration.* In fact we prove more generally that $\mathrm{Fun}_{\mathrm{colax}}(\Delta^1, B) \times_{\mathrm{Fun}(\{0\},B)} \mathrm{Fun}(\{0\}, B^{\mathrm{ladj}}) \to B^{\mathrm{ladj}} \times \iota_1 B$ is a fibration. The result is obtained by pulling back along the diagonal $B^{\mathrm{ladj}} \to B^{\mathrm{ladj}} \times \iota_1 B$.

We give two proofs: one "by hand", and one "abstract" proof.

First, by hand: consider a map $f : b_0 \to b_1$ in B, and a map $p_0 : b_0 \to c_0 \in B^{\mathrm{ladj}}$, a map $p_1 : b_1 \to c_1$ in B. We can then form $c_0 \xrightarrow{p_1 f p_0^R} c_1$, and we have an obvious diagram:

$$\begin{tikzcd} b_0 \arrow[r, "p_0"] \arrow[d, "f"'] \arrow[dr, phantom, "\nearrow\!\!\!\nearrow"] & c_0 \arrow[d, "p_1 f p_0^R"] \\ b_1 \arrow[r, "p_1"'] & c_1 \end{tikzcd}$$

given by the unit map $\mathrm{id} p_0^R p_0$. To check that this is a coCartesian edge, one observes that given maps $q_0 : c_0 \to d_0, q_1 : c_1 \to d_1$, and $g : d_0 \to d_1$, a filler 2-cell in:

$$\begin{tikzcd} c_0 \arrow[r, "q_0"] \arrow[d, "p_1 f p_0^R"'] \arrow[dr, phantom, "\nearrow\!\!\!\nearrow"] & d_0 \arrow[d, "g"] \\ c_1 \arrow[r, "q_1"'] & d_1 \end{tikzcd}$$

is definitionally the data of a map $q_1 p_1 f p_0^R \to g q_0$ which is, by adjunction[8], the same as a map $q_1 p_1 f \to g q_0 p_0$, i.e. the same as a filler in:

$$\begin{tikzcd} b_0 \arrow[r, "q_0 p_0"] \arrow[d, "f"'] \arrow[dr, phantom, "\nearrow\!\!\!\nearrow"] & d_0 \arrow[d, "g"] \\ b_1 \arrow[r, "q_1 p_1"'] & d_1 \end{tikzcd}$$

and this equivalence is "along the unit".

Now, an abstract proof. Consider the forgetful functor

$$\mathrm{Fun}_{\mathrm{colax}}(\Delta^1, B) \times_{\mathrm{Fun}(\{0\},B)} \mathrm{Fun}(\{0\}, B^{\mathrm{ladj}}) \to B^{\mathrm{ladj}} \times B \to B^{\mathrm{ladj}}$$

Essentially by definition, it is pulled back from $\mathrm{Fun}_{\mathrm{colax}}(\Delta^1, B) \xrightarrow{\mathrm{src}} \iota_1 B$ along the inclusion $B^{\mathrm{ladj}} \to \iota_1 B$. Now this latter functor is clearly a cartesian fibration: it classifies the functoriality of the lax slice, $(\iota_1 B)^{\mathrm{op}} \to \mathrm{Cat}, b \mapsto (B_{b/\!/})$. Furthermore, since B^{ladj} consists of left adjoints, each $b_0 \to c_0 \in B^{\mathrm{ladj}}$ induces a right adjoint $B_{c_0/\!/} \to B_{b_0/\!/}$. A cartesian fibration where all pullbacks are right adjoints is also a coCartesian fibration.

For much simpler reasons, $B^{\mathrm{ladj}} \times B \to B^{\mathrm{ladj}}$ is also a coCartesian fibration. Furthermore, on fibers over $b \in B^{\mathrm{ladj}}$, our functor induces the functor $B_{b/\!/} \to B$ which is also a coCartesian fibration.

It then becomes easy to check the hypotheses of [HMS22, Lemma A.1.8] to deduce that the global functor is indeed a coCartesian fibration. □

[8]Recall the $f \mapsto (- \circ f)$ reverses the order of adjunctions.

The following is a general fact about cocartesian fibrations, combined with the description of Λ as $(\Lambda_\infty)_{hS^1}$ (Definition A.0.9) and the cofinality of the functor $\Delta^{\mathrm{op}} \to \Lambda_\infty^{\mathrm{op}}$ (Proposition A.0.6):

Lemma 4.1.23. *Let B be a $(\infty, 2)$-category, and let E be a ∞-category. There is an S^1-action on $\mathrm{TrThy}_\Delta(B; E)$ and an equivalence*

$$\mathrm{TrThy}_\Delta(B; E)^{hS^1} \simeq \mathrm{TrThy}(B; E)$$

Let us now describe the key examples of pre-trace theories. We start by introducing some notation.

Notation 4.1.24. We denote by $(\vec{C}, \vec{T})$ a typical element of Λ^{st} or Δ^{st}, where, over $[n]_\Lambda$, $\vec{C} = (C_0, ..., C_n)$ is a list of (compactly generated, stable) ∞-categories, and $T_i : C_i \to C_{i+1}$ a list of (cocontinuous) functors (where $C_{n+1} := C_0$). $\triangleleft$

Example 4.1.25. Over $[0]_\Lambda$, this data amounts to some $C \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)}$ and an endomorphism T thereof. A map $(C, T) \to (D, S)$ lying over $[0]_\Lambda$ corresponds to the data of a compact-preserving cocontinuous functor $f : C \to D$ and a (not-necessarily invertible) 2-cell α:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "T"'] & D \arrow[d, "S"] \\ C \arrow[r, "f"'] \arrow[ur, phantom, "\nearrow\!\!\!\nearrow"] & D \end{tikzcd}$$

More generally, since $[0]_\Lambda = B\mathbb{N}$, the fiber over $[0]_\Lambda$ is exactly $\mathrm{End}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)})$ in the notation of [HSS17]. $\triangleleft$

Remark 4.1.26. This fiber at $[0]_\Lambda$ comes up quite naturally in the context of Goodwillie calculus, as it can be shown to be equivalent to the tangent ∞-category $T\mathrm{Cat}^{\mathrm{perf}}$ [Lur12, Definition 7.3.1.9]. We will not need or prove this here. $\triangleleft$

In the following notation, we follow the perspective of Example 4.1.11:

Notation 4.1.27. Given $(\vec{C}, \vec{T})$ lying over $[n]_\Lambda$, we let $\mathrm{End}(\vec{C}, \vec{T})$ denote the full subcategory of

$$\mathrm{Fun}_{/([n]_\Lambda)^{\mathrm{op}}}(\mathrm{Sp} \times ([n]_\Lambda)^{\mathrm{op}}, (\vec{C}, \vec{T}))$$

spanned by those functors such that over each $i \in [n]_\Lambda$, $\mathrm{Sp} \to C_i$ is in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)}$; and $\mathrm{End}_\omega(\vec{C}, \vec{T})$ the full subcategory thereof where each $\mathrm{Sp} \to C_i$ is in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$, that is, it preserves compacts. $\triangleleft$

Since Sp is free on a point as a presentable stable ∞-category, and $[n]_\Lambda$, as an ∞-category, is free on a cyclic graph, $\mathrm{End}(\vec{C}, \vec{T})$ can be informally described as follows: it's the ∞-category whose objects are tuples $x_i \in C_i, f_i : x_{i+1} \to T_i(x_i)$; while $\mathrm{End}_\omega(\vec{C}, \vec{T})$ is the full subcategory thereof where each x_i is compact, i.e. $x_i \in C_i^\omega$.

Example 4.1.28. Suppose $T : \mathrm{Ind}(C) \to \mathrm{Ind}(C)$ is an endomorphism, where C is a small stable idempotent-complete ∞-category. $\mathrm{End}_\omega(\mathrm{Ind}(C), T)$ is the ∞-category with the following informal description: its objects are a pair $(x, \alpha : x \to Tx)$ with $x \in C$.

When $C = \mathbf{Perf}(A), T = \Sigma M \otimes_A -$ for some ring spectrum A and some A-bimodule M, if both A, M are connective, then there is an equivalence

$$\mathrm{End}_\omega(\mathrm{Mod}_A(\mathrm{Sp}), \Sigma M \otimes_A -) \simeq \mathbf{Perf}(A \oplus M)$$

where $A \oplus M$ is the trivial square zero extension, see [Bar22] for a detailed proof, and [Ras18] for a more elementary sketch. $\triangleleft$

The following is more or less clear from the definition and the fact that colax limits are computed as global sections, which in turn follows from the description of colax functors as cartesian fibrations from Example 4.1.11:

Lemma 4.1.29. *Fix $[n]_\Lambda \in \Lambda$. The functor*

$$\Lambda^{st}_{[n]_\Lambda} \to \mathrm{Cat}^{\mathrm{perf}}, (\vec{C}, \vec{T}) \mapsto \mathrm{End}_\omega(\vec{C}, \vec{T})$$

from the fiber at $[n]_\Lambda$ is right adjoint to the functor $D \mapsto (\mathrm{Ind}(D) \xrightarrow{\mathrm{id}} \dots \xrightarrow{\mathrm{id}} \mathrm{Ind}(D))$ obtained by restriction along the unique functor $[n]_\Lambda \to \mathrm{pt}$ (where Λ is viewed as a subcategory of Cat).

We now globalize the previous lemma:

Corollary 4.1.30. *The functor $\mathrm{triv} : \Lambda^{op} \times \mathrm{Cat}^{\mathrm{perf}} \to \Lambda^{st}, (G, D) \mapsto p_G^* D$, where $p_G : G \to \mathrm{pt}$ is the unique functor to a point, admits a relative right adjoint in the sense of (the dual of) [Lur12, Definition 7.3.2.2], given on each fiber by End_ω.*

Proof. This follows directly from (the dual of) [Lur12, Proposition 7.3.2.6]: $\Lambda^{op} \times \mathrm{Cat}^{\mathrm{perf}} \to \Lambda^{op}$ and $\Lambda^{st} \to \Lambda^{op}$ are both coCartesian and hence locally coCartesian fibrations, and triv clearly preserves coCartesian edges. □

We record the the following construction, which is obtained from the previous corollary by composing the right adjoint with the projection $\Lambda^{op} \times \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Cat}^{\mathrm{perf}}$ - here, the various items come from unwinding the proof of [Lur12, Proposition 7.3.2.6]:

Construction 4.1.31. We have constructed a functor $\mathrm{End}_\omega : \Lambda^{st} \to \mathrm{Cat}^{\mathrm{perf}}$ such that for any $n \in \mathbb{N}_{\geq 0}$, the restriction of End_ω to the fiber over $[n]_\Lambda$ is the functor from Notation 4.1.27. Furthermore, it acts as follows on the "generating morphisms" of Λ:

(i) For each edge e of the form $(i \to i+1) \mapsto (i \to i+2)$ in Λ (cf. Construction A.0.20), and each object $(\vec{C}, \vec{T}) \in \Lambda^{st}$ lying over $[n+1]_\Lambda$ corresponding to a sequence $(C_0 \xrightarrow{T_0} \dots \xrightarrow{T_n} C_{n+1} \xrightarrow{T_{n+1}} C_0)$, the coCartesian lift of e is sent by End_ω to the functor informally described as

$$(\vec{x}, x_1 \xrightarrow{\alpha_1} T_0 x_0, \dots, x_0 \to T_{n+1} x_{n+1}) \mapsto (\hat{\vec{x}}, x_1 \to T_0 x_0, ..., x_{i+2} \xrightarrow{\alpha_{i+2}} T_{i+1} x_{i+1} \xrightarrow{T_{i+1}\alpha_{i+1}} T_{i+1} T_i x_i, \dots)$$

where $\hat{\vec{x}}$ is $\vec{x}$ minus the $(i+1)$st term.

(ii) For each edge e of the form $(i \to i+1) \mapsto i$ (cf. Construction A.0.22), and and each object $(\vec{C}, \vec{T}) \in \Lambda^{st}$ lying over $[n]_\Lambda$ corresponding to a sequence $(C_0 \xrightarrow{T_0} \dots \xrightarrow{T_{n-1}} C_n \xrightarrow{T_n} C_0)$, the coCartesian lift of e is sent by End_ω to the functor informally described as

$$(\vec{x}, x_1 \xrightarrow{\alpha_1} T_0 x_0, \dots, x_0 \to T_n x_n) \mapsto (\tilde{\vec{x}}, x_1 \to T_0 x_0, ..., x_i \to T_{i-1} x_{i-1}, x_i \xrightarrow{\mathrm{id}} x_i, x_{i+1} \to T_i x_i, \dots)$$

(iii) For the generating automorphism of $[n]_\Lambda$, i.e. the cyclic permutation σ sending $i \mapsto i+1$, and $(\vec{C}, \vec{T})$ lying over $[n]_\Lambda$, the corresponding coCartesian edge in Λ^{st} is sent by End_ω to the obvious equivalence

$$(\vec{x}, x_1 \to T_0 x_0, \dots, x_0 \to T_n x_n) \mapsto (\sigma \cdot \vec{x}, x_2 \to T_1 x_1, \dots, x_0 \to T_n x_n, x_1 \to T_0 x_0)$$

◁

Remark 4.1.32. By Proposition A.0.24 and the fact that any morphism can be decomposed as a coCartesian morphism followed by a morphism in a fiber, these examples determine the full behaviour of End_ω as a functor-up-to-homotopy, i.e. they determine its effect on any given morphism in Λ^{st}. ◁

Over Δ^{op}, we have the following simplification:

Lemma 4.1.33. *The functor End_ω, restricted to Δ^{st}, is right adjoint to the functor $\mathrm{Cat}^{\mathrm{perf}} \to \Delta^{\mathrm{st}}, D \mapsto (D, \mathrm{id}_D)$.*

Proof. The relative adjunction $\Lambda^{\mathrm{op}} \times \mathrm{Cat}^{\mathrm{perf}} \rightleftharpoons \Lambda^{\mathrm{st}}$ pulls back to a relative adjunction

$$\Delta^{\mathrm{op}} \times \mathrm{Cat}^{\mathrm{perf}} \rightleftharpoons \Delta^{\mathrm{st}}$$

by [Lur12, Proposition 7.3.2.5]. Now, as $[0]$ is initial in Δ^{op} (terminal in Δ), the inclusion of $\{[0]\}$ induces an adjunction $\mathrm{pt} \rightleftharpoons \Delta^{\mathrm{op}}$ which we can compose with the above one to get the desired result. □

We note that in particular, on Δ^{st}, the functor End_ω (or rather its core) is representable by $(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})$ lying over $[0]$.

Remark 4.1.34. We stress that this is not true in Λ^{st}: the crucial difference is that in Δ^{op}, there is only one map $[0] \to [n]$ for every n, so that a map $(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}}) \to (\vec{C}, \vec{T})$ in Δ^{st} has to lie over this map and in that case the coCartesian pushforward of $(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})$ to $[n]$ is given by $(\mathrm{Sp} \xrightarrow{\mathrm{id}_{\mathrm{Sp}}} \dots \xrightarrow{\mathrm{id}_{\mathrm{Sp}}} \mathrm{Sp})$.

Over Λ^{op}, the coCartesian pushforwards look the same, but there are more maps $[0]_\Lambda \to [n]_\Lambda$ (namely, $n+1$ of them) so that the mapping space from $(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})$ to $(\vec{C}, \vec{T})$ will be a disjoint union of $n+1$ copies of the one described above, as is clear from Corollary 4.1.30, since mapping spaces in $\Lambda^{\mathrm{op}} \times \mathrm{Cat}^{\mathrm{perf}}$ are products of the respective mapping spaces. ◁

While the adjunction statement is not true for Λ^{st}, it *is* important to know that the functor End_ω exists over Λ^{op} too, as we will see later.

We finally note that the adjunction $\Lambda^{\mathrm{op}} \times \mathrm{Cat}^{\mathrm{perf}} \rightleftharpoons \Lambda^{\mathrm{st}}$ exists for more general ∞-categories Γ equipped with a functor $\Gamma \to \mathrm{Cat}$. If Γ is a subcategory that has (for example) the same objects as Λ, then the same proof works to show that the right adjoint is also described in terms of End_ω, but for more complicated Γ, the right adjoint may be more complex (some kind of lax limit). We nonetheless record the special case as follows (it implies the Λ version simply by pulling back):

Corollary 4.1.35. *Let $\Gamma \subset \mathrm{Cat}$ be the essential image of the canonical functor $\Lambda \to \mathrm{Cat}$. There is also a functor $\mathrm{End}_\omega : \Gamma^{\mathrm{st}} \to \Gamma^{\mathrm{op}} \times \mathrm{Cat}^{\mathrm{perf}}$, right adjoint to the fiberwise diagonal functor.*

Another example of a pre-trace theory which will also be relevant for us comes from the following:

Lemma 4.1.36. *Fix $[n]_\Lambda \in \Lambda$. The functor $\Lambda^{\mathrm{st}}_{[n]_\Lambda} \to (\mathrm{Cat}^{\mathrm{perf}})^{n+1}, (\vec{C}, \vec{T}) \mapsto \vec{C^\omega}$ is fully faithful when restricted to the $(\vec{C}, \vec{T})$ where $\vec{T} = \vec{0}$, and thus induces an equivalence of this full subcategory with $(\mathrm{Cat}^{\mathrm{perf}})^{n+1}$.*

More generally, it induces an adjunction $(\mathrm{Cat}^{\mathrm{perf}})^{n+1} \rightleftharpoons \Lambda^{\mathrm{st}}_{[n]_\Lambda}$, and thus an adjunction $\mathrm{Cat}^{\mathrm{perf}} \rightleftharpoons \Lambda^{\mathrm{st}}_{[n]_\Lambda}$ where the right adjoint is $(\vec{C}, \vec{T}) \mapsto \prod_i C_i$.

Proof. The fully faithfulness claim is evident, which allows us to even define in the first place the functor $(\mathrm{Cat}^{\mathrm{perf}})^{n+1} \to \Lambda^{\mathrm{st}}_{[n]_\Lambda}$ simply as an inverse.

It is then not hard to verify the adjunction property as 0 is initial in every $\mathrm{Fun}^L(C_i, D_{i+1})$. $\square$

Just as for End_ω (with the same proof), we can globalize $\prod$ to be the right adjoint in some adjunction $\Lambda^{\mathrm{op}} \times \mathrm{Cat}^{\mathrm{perf}} \rightleftharpoons \Lambda^{\mathrm{st}}$, and just as before we can project onto the $\mathrm{Cat}^{\mathrm{perf}}$ coordinate. Using the natural inclusion $([n]_\Lambda)^{\simeq} \to [n]_\Lambda$, we can also produce a natural transformation $\mathrm{End}_\omega \to \prod$.

We record it as the following construction:

Construction 4.1.37. We have constructed a functor $\prod : \Lambda^{\mathrm{st}} \to \mathrm{Cat}^{\mathrm{st}}$ as well as a natural transformation $\mathrm{End}_\omega \to \prod$ such that on each fiber, the functor $\prod$ restricts to the one described in Lemma 4.1.36, and such that its behaviour on the generating edges from Construction A.0.20, Construction A.0.22 and Construction A.0.21 is described analogously to the one from Construction 4.1.31:

- On coCartesian edges lying over edges of the form $(i \to i+1) \mapsto (i \to i+2)$, the induced map between products is a projection away from the index $i+1$;
- On coCartesian edges lying over edges of the form $(i \to i+1) \mapsto i$, the induced map between products is given by a diagonal $C_i \to C_i \times C_i$;
- On coCartesian edges lying over edges of the form $\sigma \in \mathrm{Aut}([n]_\Lambda) = C_{n+1}$, the induced map between products is the canonical symmetry isomorphism between $\prod_i C_i$ and $\prod_i C_{\sigma(i)}$.

$\triangleleft$

With the exact same proof as for End, Π admits a left adjoint on Δ^{st} (but not on Λ^{st}, for the same reason):

Lemma 4.1.38. *The functor $\prod : \Delta^{\mathrm{st}} \to \mathrm{Cat}^{\mathrm{perf}}$, is right adjoint to the functor $D \mapsto (D, 0)$.*

4.1.2 Trace theories from localizing invariants

Our main example of trace theory comes from the following construction and variants thereof. The construction takes as input a functor $E : \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{E}$ (typically a localizing invariant), and outputs a (pre-)trace theory E^{cyc}, which we dub "cyclic E(-theory)".

Construction 4.1.39. Let $E : \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{E}$ be a functor with values in a stable ∞-category $\mathcal{E}$.

We can define a pre-trace theory $E^{\mathrm{cyc}} : \Lambda^{\mathrm{st}} \to \mathcal{E}$ by

$$(\vec{C}, \vec{T}) \mapsto \mathrm{fib}(E(\mathrm{End}_\omega(\vec{C}, \vec{T})) \to E(\prod(\vec{C}, \vec{T})))$$

$\triangleleft$

Proposition 4.1.40. *Let $E : \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{E}$ be a functor with values in a stable ∞-category $\mathcal{E}$. If E is a localizing invariant in the sense of [BGT13][9], then E^{cyc} is a trace theory.*

In the proof, we will use the following general lemma:

[9]Though we do not require our localizing invariants to be finitary, i.e. to preserve filtered colimits. See the end Appendix B for (very) brief reminders on localizing invariants.

Lemma 4.1.41. *Let $p : A \to B$ be a localization in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ with colimit-preserving right adjoint p^R, and let $A_0 \subset A^\omega$ be some stable subcategory of the compacts of A*[10]*. Suppose further that* $\ker(p) = \mathrm{Ind}(\ker(p)^\omega)$ *and that* $\ker(p)^\omega \subset A_0$.

In that case, the canonical functor $A_0/\ker(p)^\omega \to B$ is fully faithful: $A_0 \to B$ is a Verdier localization onto its image.

Proof. Let $i : \ker(p) \to A$ denote the kernel inclusion, with right adjoint i^R. Because of the cofiber sequence $ii^R \to \mathrm{id}_A \to p^R p$, we find that i^R is colimit-preserving and hence i preserves compacts.

For any $x \in \mathrm{Ind}(A_0)$, the cofiber sequence $ii^R(x) \to x \to p^R p(x)$ tells us that $p^R p(x)$ also lives in $\mathrm{Ind}(A_0) \subset A$. In other words, $p_{|\mathrm{Ind}(A_0)}$ lands in $B_0 := (p^R)^{-1}\mathrm{Ind}(A_0)$ and thus the adjunction $p \dashv p^R$ restricts to an adjunction $\mathrm{Ind}(A_0) \rightleftharpoons B_0$. Since the counit is compatible, it follows that the right adjoint is also fully faithful, and hence $p_{|\mathrm{Ind}(A_0)} : \mathrm{Ind}(A_0) \to B_0$ is a localization, and its kernel is $\ker(p) = \mathrm{Ind}(\ker(p)^\omega)$.

It follows that $B_0 \simeq \mathrm{Ind}(A_0/\ker(p)^\omega)$, and since $B_0 \subset B$, this proves the claim. □

Proof of Proposition 4.1.40. By Corollary A.0.25 it suffices to prove that the coCartesian morphisms lying over the edges of the form $(i \to i+1) \mapsto (i \to i+2)$[11] in Λ are sent to equivalences. For notational simplicity and for the clarity of the argument, we deal only with the special case of the map $[0]_\Lambda \to [1]_\Lambda$ sending 0 to 0, but the other cases are completely analogous (if more notationally tedious).

In this case, an object lying over $[1]_\Lambda$ is a pair

$$(C, D, F : C \to D, G : D \to C)$$

and by Construction 4.1.31, the corresponding coCartesian edge[12] $(C, D, F, G) \to (C, GF)$ is sent by End_ω to

$$\mathrm{End}_\omega(C, D, F, G) \to \mathrm{End}_\omega(C, GF), (x, y, x \to Gy, y \to Fx) \mapsto (x \to Gy \to GFx)$$

and by $\prod$ to the projection $C^\omega \times D^\omega \to C^\omega$.

To prove that $E^{\mathrm{cyc}}(C, D, F, G) \to E^{\mathrm{cyc}}(C, GF)$ is an equivalence, we thus need to prove that the vertical fibers in the following square are equivalent, i.e. that the following square is a pullback square:

$$\begin{tikzcd} E(\mathrm{End}_\omega(C, D, F, G)) \arrow[r] \arrow[d] & E(\mathrm{End}_\omega(C, GF)) \arrow[d] \\ E(C^\omega \times D^\omega) \arrow[r] & E(C^\omega) \end{tikzcd}$$

Since $\mathcal{E}$ is stable, it thus suffices to show that the horizontal fibers are equivalent. Since E is a localizing invariant, the fiber of the bottom map is D^ω, so it suffices to show:

(i) The functor $\mathrm{End}_\omega(C, D, F, G) \to \mathrm{End}_\omega(C, GF)$ is a Verdier localization,

(ii) The restriction of the projection $\mathrm{End}_\omega(C, D, F, G) \to C^\omega \times D^\omega$ to the kernel of this localization induces an equivalence with D^ω.

The second fact is easier: if $(x, x \to Gy \to GFx)$ is 0, then x is 0. So the kernel of $\mathrm{End}_\omega(C, D, F, G) \to \mathrm{End}_\omega(C, GF)$ is the ∞-category of tuples $(0, y, 0 \to Gy, y \to 0)$ and is clearly equivalent to D^ω under the projection to $C^\omega \times D^\omega$.

[10] We do not assume A is compactly generated.

[11] Cf. Construction A.0.20.

[12] It goes in the opposite direction because Λ^{st} lies over Λ^{op}.

For the first fact, we embed $\mathrm{End}_\omega(C,D,F,G)$ and $\mathrm{End}_\omega(C,GF)$ in $\mathrm{End}(C,D,F,G)$ and $\mathrm{End}(C,GF)$ respectively. For these ones, the corresponding map $\mathrm{End}(C,D,F,G) \to \mathrm{End}(C,GF)$ still clearly has D as a kernel (which *is* compactly generated by the kernel of the End_ω version).

Furthermore, the corresponding map admits a right adjoint, given by

$$(x, x \to GFx) \mapsto (x, Fx, x \to GFx, Fx = Fx)$$

for which the counit is clearly an equivalence[13]. Thus

$$\mathrm{End}(C,D,F,G) \to \mathrm{End}(C,GF)$$

is a Bousfield localization with compactly generated kernel and colimit preserving right adjoint.

This puts us in the setting of Lemma 4.1.41 with

$$A_0 = \mathrm{End}_\omega(C,D,F,G) \subset \mathrm{End}(C,D,F,G) = A$$

It follows that $\mathrm{End}_\omega(C,D,F,G) \to \mathrm{End}_\omega(C,GF)$ is a Verdier localization onto its image, so we are left with examining its essential image. Let $x \in C^\omega$ and $\alpha : x \to GFx$. Write $Fx \simeq \mathrm{colim}_I y_i, y_i \in D^\omega$ where I is filtered. Since x is compact, the map α factors through $G(y_i)$ for some i, so that (x,α) is the image of $(x, y_i, x \to G(y_i), y_i \to Fx)$, as was needed. □

Remark 4.1.42. Note that in this proof, the functor

$$\mathrm{End}_\omega(C,D,F,G) \to \mathrm{End}_\omega(C,GF)$$

is surjective, and not only up to retracts, and its kernel is exactly D^ω. Thus the result above also holds for invariants that are localizing on (non-split) Verdier sequences (as opposed to Karoubi sequences, which feature in the usual definition of "localizing invariant"). An example of such is connective K-theory. Thus a lot of what we will do/say will apply not only to localizing invariants, but also to connective K-theory. This is particularly true for Theorem 4.2.1. ◁

Remark 4.1.43. Despite how simple the kernel was to analyze[14], we needed something like Lemma 4.1.41 since it is not true in general that $\mathrm{Ind}(\mathrm{End}_\omega(C,T)) = \mathrm{End}(C,T)$, even for C compactly generated. For example, if $T = \mathrm{id}_C$, then for any object $x \in C$, the free endomorphism object on x has $\bigoplus_{\mathbb{N}} x$ as its underlying object of C, which is never compact unless $x = 0$. ◁

Definition 4.1.44. A (pre-,Δ-)trace theory F is reduced if for any $(\vec{C},\vec{T}) \in \Lambda^{\mathrm{st}}$ (resp.Δ^{st}) such that there exists i with $T_i = 0$, we have $F(\vec{C},\vec{T}) = 0$. ◁

Remark 4.1.45. We will not use the general fact, nor the explicit formula, but for formal reasons, there is a "reduction" functor which universally turns a (pre-,Δ-)trace theory F into a reduced one. ◁

Corollary 4.1.46. *For a localizing invariant E, E^{cyc} is reduced: if one of the T_i's is 0, then $E^{\mathrm{cyc}}(\vec{C},\vec{T}) = 0$.*

[13] At the End_ω level, we cannot use this since Fx need not be compact.

[14] So that there are no "telescope conjecture" types of questions appearing here

Proof. We use the trace property to obtain that

$$E^{\mathrm{cyc}}(\vec{C},\vec{T}) = E^{\mathrm{cyc}}(C_0, T_1 \circ \ldots \circ T_{n+1}) = E^{\mathrm{cyc}}(C_0, 0)$$

and now the claim follows immediately from the fact that $\mathrm{End}_\omega(C,0) = C^\omega$ and the definition of E^{cyc}. □

In fact, E^{cyc} is universal in that respect:

Corollary 4.1.47. *The map $E \circ \mathrm{End}_\omega \to E^{\mathrm{cyc}}$ is initial among maps from $E \circ \mathrm{End}_\omega$ to a reduced Δ-trace theory*[15].

Proof. The fiber of this map is $E \circ \prod$, so it suffices to prove that the reduction of $E \circ \prod$ is null, i.e. that for any reduced trace theory F, $\mathrm{map}(E \circ \Pi, F) = 0$.

Since $\prod$ is right adjoint to $C \mapsto (C,0)$ by Lemma 4.1.38, it suffices to prove that $\mathrm{map}(E, F \circ ((\vec{-}), \vec{0})) = 0$. As F is reduced, the target is already 0, which implies the desired claim. □

Remark 4.1.48. For "formulaic" reasons, this is also true for trace theories rather than Δ-trace theories, though we will not use that fact. Since proving it would require setting up the relevant "formulas" for reduction, we restrain from doing so. ◁

We explained how to go from localizing invariants to trace theories. There is a way to go back : for example, note that THH is more naturally seen as a trace theory [HS19] and plugging in identities allows us to see it as a localizing invariant.

Definition 4.1.49. Let $(C,T) \xrightarrow{i} (D,S) \xrightarrow{p} (E,Q)$ be a sequence of morphisms in Λ^{st} (resp. Δ^{st}) lying over the identity of $[0]_\Lambda$ (resp. $[0]$). It is called a localization sequence if the underlying sequence $C \to D \to E$ is a localization sequence and the induced null-sequence[16] $iTi^R \to S \to p^RQp$ is a co/fiber sequence in $\mathrm{Fun}^L(D,D)$. ◁

Example 4.1.50. Two key examples of localization sequences in Λ^{st} are those of the form

$$(C,T) \to (D, iTi^R) \to (D/C, 0)$$

and

$$(\ker(p), 0) \to (D, p^RQp) \to (E,Q)$$

where i is fully faithful, resp. p is a localization; but they are not the only ones. ◁

Definition 4.1.51. Let $T : \Lambda^{\mathrm{st}} \to \mathcal{E}$ (resp. $\Delta^{\mathrm{st}} \to \mathcal{E}$) be a trace theory (resp. Δ-trace theory) with values in a stable ∞-category $\mathcal{E}$. It is called localizing if it sends localization sequences in Λ^{st} (resp. Δ^{st}) to co/fiber sequences in $\mathcal{E}$. ◁

Warning 4.1.52. For a localizing invariant E, it is almost never the case that E^{cyc} is a localizing (Δ-)trace theory. The point is that $\mathrm{End}_\omega(-)$ does not preserve localization sequences of small ∞-categories. ◁

From this definition, the following is essentially tautological:

---

[15] Note that $E \circ \mathrm{End}_\omega$ itself is not a trace theory, only a pre-trace theory.

[16] The maps $iTi^R \to S$ and $S \to p^RQp$ are induced by the structure of i and p as maps in $\Lambda^{\mathrm{st}}/\Delta^{\mathrm{st}}$. The composite is *uniquely* nullhomotopic, as is any map of the form $ix \to p^Ry$, so we do not need to specify the nullhomotopy here.

Proposition 4.1.53. *Let $T : \Delta_{\mathrm{st}} \to \mathcal{E}$ be a localizing Δ-trace theory. The functor $C \mapsto T(\mathrm{Ind}(C), \mathrm{id}_{\mathrm{Ind}(C)})$ is a localizing invariant on* $\mathrm{Cat}^{\mathrm{perf}}$.

However the following is a convenient (if rarely applicable) criterion to detect when a trace theory is localizing:

Proposition 4.1.54. *Any exact trace theory is localizing.*

Here, we used:

Definition 4.1.55. A trace theory is called *exact* if it is exact in the bimodule variable - that is, for every $C \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)}$, the local functor $T(C, -) : \mathrm{Fun}^L(C, C) \to \mathcal{E}$ is exact. $\triangleleft$

This proposition is one of the key examples of interaction between the bimodule functoriality of trace theories and the functoriality in the objects. This interaction will be studied in more depth in my future work comparing shadows and trace theories, but for now, let us simply give the proof - we leave *some* of the details to the reader, as actually spelling out all the necessary checks would take up a lot of unnecessary space:

Proof. Let $(C, T) \xrightarrow{i} (D, S) \xrightarrow{p} (E, Q)$ be a localization sequence in Λ^{st}. Since i is fully faithful and p is a localization, we may replace T with Ti^Ri and Q with Qpp^R freely. We thus have a square $\square_1$:

$$\begin{tikzcd} (C, Ti^Ri) \arrow[r] \arrow[d] & (D, S) \arrow[d] \\ (0,0) \arrow[r] & (E, Qpp^R) \end{tikzcd}$$

which we wish to show is sent to a co/fiber sequence in $\mathcal{E}$. We note, as will be relevant later, that there is a unique homotopy filling this square.

We construct a zigzag of morphisms in Λ_{st} between this square and the square $\square_2$:

$$\begin{tikzcd} (D, iTi^R) \arrow[r] \arrow[d] & (D, S) \arrow[d] \\ (D,0) \arrow[r] & (D, p^RQp) \end{tikzcd}$$

where each of the maps involved in the zigzag consists entirely of coCartesian morphisms. Since the latter square is sent to a co/fiber sequence in $\mathcal{E}$ by assumption, and the maps involved in the zigzag are sent to equivalences, also by assumption, this will conclude the proof. We note, similarly to before, that there is a unique homotopy filling this square which fixes D.

The middle term of the zigzag will be a square of the form $\square$:

$$\begin{tikzcd} (C \xrightarrow{i} D \xrightarrow{Ti^R} C) \arrow[r] \arrow[d] & (D \xrightarrow{\mathrm{id}_D} D \xrightarrow{S} D) \arrow[d] \\ (0 \to D \to 0) \arrow[r] & (E \xrightarrow{p^R} D \xrightarrow{Qp} E) \end{tikzcd}$$

We briefly explain how to properly construct $\square$ as well as the relevant maps $\square \to \square_i$, $i \in \{1, 2\}$, leaving the details to the reader.

Construction of $\square$: All the maps between ∞-categories are the ones we have already introduced, that is, i, id_D and p. The commutation data also comes from commutation data that was given to us: for example, for the bottom horizontal map in the square we need a 2-cell given by a map from iTi^R to S. We chose the map corresponding to the map $(C,T) \to (D,S)$ that was given to us. The other maps are either tautological, 0, the unit $\mathrm{id}_D \to p^R p$, or the map $pS \to Qp$ that is part of the data of the map $(D,S) \to (E,Q)$.

It is sufficient to produce maps like this since $[1]_\Lambda$ is free on the cyclic graph $0 \to 1 \to 0$.

Construction of the maps between squares: The maps from $\square \to \square_i$ will simply be given as coCartesian morphisms lying over specific morphisms $[1]_\Lambda \to [0]_\Lambda$ in Λ^{op}, i.e. specific morphisms $[0]_\Lambda \to [1]_\Lambda$ in Λ. Since $[0]_\Lambda$ is free on the cyclic graph with one vertex $\circlearrowleft$, there are exactly two such morphisms and using coCartesian pushforwards along (say) the one sending $0 \in [0]_\Lambda$ to $0 \in [1]_\Lambda$ amounts to taking the total composite in the corners of $\square$ as indicated - thus the square we obtain looks like $\square_1$, and it is not hard to check that the various edges are correct. Since the edges of $\square_1$ can be filled by at most one homotopy, we indeed get $\square_1$.

The same reasoning works for the other coCartesian pushforward, along the functor sending $0 \in [0]_\Lambda$ to $1 \in [1]_\Lambda$ - only at the end we need to check that the filling homotopy fixes D to guarantee that it is the correct one. This is clear as the homotopy filling $\square$ itself fixes D. $\square$

Remark 4.1.56. A similar proof appears in [HSS17, Theorem 3.4]. I find the proof above slightly clearer, but the idea is essentially the same[17]. $\triangleleft$

Notation 4.1.57. We let $\mathrm{TrThy}^{\mathrm{loc}}(\mathcal{E})$ (resp. $\mathrm{TrThy}^{\mathrm{loc}}_\Delta(\mathcal{E})$)) denote the full subcategory of $\mathrm{TrThy}(\mathcal{E})$ spanned by localizing trace theories (resp. localizing Δ-trace theories).

Similarly, $\mathrm{TrThy}^{\mathrm{ex}}(\mathcal{E})$ (resp. $\mathrm{TrThy}^{\mathrm{ex}}_\Delta(\mathcal{E})$)) denotes the full subcategory spanned by the exact trace theories (resp. exact Δ-trace theories). $\triangleleft$

Corollary 4.1.58. *Let $\mathcal{E}$ be a stable ∞-category and $E : \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{E}$ be a localizing invariant. For any reduced trace theory T, let $T_{\mathrm{id}} : C \mapsto T(C, \mathrm{id}_C)$, we have an equivalence:*

$$\mathrm{Map}_{\mathrm{TrThy}_\Delta(\mathcal{E})}(E^{\mathrm{cyc}}, T) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{Cat}^{\mathrm{perf}}, \mathcal{E})}(E, T_{\mathrm{id}})$$

Proof. This follows from combining:

(i) The universal property of $(-)^{\mathrm{cyc}}$ from Corollary 4.1.47;

(ii) The fact that End_ω is right adjoint to $C \mapsto (C, \mathrm{id}_C)$ and that if $f \dashv g$, then precomposition with f is *right* adjoint to precomposition with g.

$\square$

Remark 4.1.59. We cannot quite state this as an adjunction because T_{id} need not be a localizing invariant. If we restrict to localizing trace theories, then it is, but in this case, E^{cyc} itself need not be a localizing trace theory, cf. Warning 4.1.52.

After linearization, this problem is fixed as we explain below. $\triangleleft$

Notation 4.1.60. Given a reduced trace theory T, we let $P_1^{\mathrm{fbw}} T$ denote the fiberwise first derivative[18] of T. $\triangleleft$

[17] Though we note that the statement of uniqueness of the square (3.3) in *loc. cit.* and its use could be made more precise.

[18] That this is well defined and remains a trace theory is essentially a consequence of general facts about relative adjunctions together with the fact that for T a trace theory, $T(\vec{C}, \vec{T}) = T(C_0, T_n \circ ... \circ T_0)$ so that the multi-derivative agrees with the derivative along any single variable, and is compatible along all the morphisms in Λ^{op}.

Corollary 4.1.61. *Let $\mathcal{E}$ be a cocomplete stable ∞-category. The functor*

$$P_1(-)^{\mathrm{cyc}} : \mathrm{Fun}^{\mathrm{loc}}(\mathrm{Cat}^{\mathrm{perf}}, \mathcal{E}) \to \mathrm{TrThy}^{\mathrm{ex}}_{\Delta}(\mathcal{E})$$

is left adjoint to the functor $T \mapsto T(C, \mathrm{id}_C)$.

Proof. This follows from combining:

(i) The universal property of P_1^{fbw};
(ii) The universal property of $(-)^{\mathrm{cyc}}$ from Corollary 4.1.47;
(iii) The fact that End_ω is right adjoint to $C \mapsto (C, \mathrm{id}_C)$ and that if $f \dashv g$, then precomposition with f is *right* adjoint to precomposition with g.

□

With all of this, we can finally give the universal property of P_1K^{cyc} from the perspective of trace theories:

Corollary 4.1.62. *In the ∞-category of exact trace theories, P_1K^{cyc} corepresents evaluation at $(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})$. That is, there is an equivalence, natural in the exact trace theory T:*

$$\mathrm{map}_{\mathrm{TrThy}^{\mathrm{ex}}_{\Delta}}(P_1K^{\mathrm{cyc}}, T) \simeq T(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})$$

Proof. This follows from Corollary 4.1.61 and the universal property of K-theory from [BGT13][19]. □

Remark 4.1.63. We used localizing invariants in the previous corollary, so that K-theory is to be interpreted as nonconnective K-theory. However, we noted in Remark 4.1.42 that connective K-theory was sufficiently close to a localizing invariant for $(K^{\mathrm{cn}})^{\mathrm{cyc}}$ to be a trace theory, and thus, what we wrote above also holds for connective K-theory.

As a *consequence*, we find that P_1K^{cyc} does not depend on whether we chose connective or nonconnective K-theory (while, of course, K^{cyc} does).

However, we note that in this case there is also an easy way to see this without needing to go through the proofs: Goodwillie derivatives involve a sequential colimit over n of objects of the form $\Omega^n K(C_n)$, so it is clear that any homotopy group of the colimit only depends on the functor $\tau_{\geq 0}K$ which is K^{cn}, up to idempotent completion (and in fact, they only depend on $\tau_{\geq 1}K$ which is simply $\tau_{\geq 1}K^{\mathrm{cn}}$). ◁

Remark 4.1.64. We make a second remark about the meaning of "linearization of K-theory" employed here as opposed to the classical literature, e.g. in [DM94]. Therein, the linearization of K-theory at a ring A and a bimodule M is defined as $\mathrm{colim}_n \Omega^n \tilde{K}(A \oplus \Sigma^{n-1}M)$. The relation is as follows: by the main result of [Bar22] in the split case, or [Ras18, Proposition 3.2.2, Lemma 3.4.1], for connective A, M, there is an equivalence $\mathrm{End}_\omega(\mathbf{Perf}(A), \Sigma M \otimes_A -) \simeq \mathbf{Perf}(A \oplus M)$ which is functorial in M and thus proves that the two definitions are equivalent (and explains the "off by a Σ" difference between our results and the ones in [DM94]).

Miraculously, it turns out that one can also say something in the nonconnective case: while in this generality, it is *not* the case that $\mathrm{End}_\omega(\mathbf{Perf}(A), \Sigma M \otimes_A -)$ and $\mathbf{Perf}(A \oplus M)$ are equivalent, there *is* a natural fully faithful inclusion $\mathbf{Perf}(A \oplus M) \to \mathrm{End}_\omega(\mathbf{Perf}(A), \Sigma M \otimes_A -)$

[19] In [BGT13], they state this universal property only for maps with values in a finitary localizing invariant, that is, one commuting with filtered colimits. However, this universal property holds more generally with values in any localizing invariant. In any case, we will not use it in this generality.

whose image consists of pairs $(P, P \to \Sigma M \otimes_A P)$ such that the map is nilpotent: some composite $P \to \Sigma M \otimes_A P \to \cdots \to \Sigma^n M^{\otimes_A n} \otimes_A P$ is 0. While this is therefore not surjective in general, it follows that in the commutative square:

$$\begin{tikzcd} \mathrm{End}_\omega(\mathbf{Perf}(A), M \otimes_A -) \arrow[r] \arrow[d] & \mathbf{Perf}(A) \arrow[d] \\ \mathbf{Perf}(A) \arrow[r] & \mathrm{End}(\mathbf{Perf}(A), \Sigma M \otimes_A -) \end{tikzcd}$$

obtained from functoriality in M and the canonical square:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma M \end{tikzcd}$$

there is a factorization:

$$\begin{tikzcd} \mathrm{End}_\omega(\mathbf{Perf}(A), M \otimes_A -) \arrow[r] \arrow[d] & \mathbf{Perf}(A) \arrow[d, dashed] \arrow[ddr] & \\ \mathbf{Perf}(A) \arrow[r, dashed] \arrow[drr] & \mathbf{Perf}(A \oplus M) \arrow[dr, hook] & \\ & & \mathrm{End}_\omega(\mathbf{Perf}(A), \Sigma M \otimes_A -) \end{tikzcd}$$

so that, while the natural comparison map is not a degreewise equivalence, it is in fact an ind-equivalence:

$$\text{“colim}_n\text{“}\Omega^n \tilde{K}(A \oplus \Sigma^{n-1} M) \simeq \text{“colim}_n\text{“}\Omega^n K^{\mathrm{cyc}}(\mathbf{Perf}(A), \Sigma^n M)$$

In fact, there is a variant of K^{cyc} (more generally E^{cyc}) where one considers nilpotent endomorphisms instead of endomorphisms, and Theorem 4.2.11 would give another way of proving this result (or at least, a weaker version, namely the statement about actual colimits rather than ind-colimits; though the proof above also simply works in more generality). I do not know whether this variant has other uses/purposes before taking Goodwillie derivatives. ◁

Warning 4.1.65. While the previous remark allows us to reduce the computation of $P_1 K^{\mathrm{cyc}}$ evaluated at a ring spectrum to K-theories of ring spectra (as opposed to arbitrary stable ∞-categories), for a nonconnective ring spectrum, one *still* cannot use the group completion model for K-theory for nonconnective ring spectra, and so proofs of the Dundas–McCarthy theorem based on connectivity estimates alone as in [DM94] will not cut it in general. ◁

Remark 4.1.66. The two previous remarks[20] show that the object “$P_1 K^{\mathrm{cyc}}$” is quite robust: it does not depend on whether one takes connective or nonconnective K-theory, and it does not depend on how exactly one linearizes K-theory. Thus also the Dundas–McCarthy theorem, its equivalence with THH, is quire robust. ◁

[20] With the previous warning as a mild caveat.

4.2 Cocontinuous trace theories and the Dundas–McCarthy theorem

In this section, we finally give a proof of the Dundas–McCarthy theorem [DM94] in the following form:

Theorem 4.2.1. *The Dennis trace map* $K \to \mathrm{THH}$ *induces an equivalence of trace theories*

$$P_1 K^{\mathrm{cyc}} \simeq \mathrm{THH}$$

We note that our proof makes use of the structure of trace theories in an essential way, even for objects of the form (C, id_C).

Here, we implicitly assume that THH can be upgraded to a trace theory. This is done in [HS19, Nikolaus, Definition 10] in terms of the bar construction and from this perspective I have nothing to add to this definition. I will discuss in Section 4.2.2 how one can go about defining a version of the trace functor Tr as in [HSS17] as a trace theory.

Remark 4.2.2. With respect to the previous paragraph, the reader can either assume that there is an a priori-defined THH to which we are comparing P_1K^{cyc} (e.g. the one defined in [HS19]); or they can look ahead in Section 4.2.2 and take *that* definition, or finally they can see this theorem as stating "P_1K^{cyc} is a trace theory, and it has the structure and the properties one would expect of whatever THH is supposed to be".

See also Remark 4.2.10. ◁

We explained in Remark 4.1.64 in what way our "P_1K^{cyc}" really is (at least on ring spectra) what Dundas and McCarthy called the linearization of K-theory; thus this truly is a (slight) generalization of their theorem. See also Remark 4.2.29 and Section 4.2.3 for a discussion of the connectivity estimates in [DM94].

Before giving the proof, we also mention a further extension of it to other localizing invariants.

Theorem 4.2.3. *Let* $E : \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{E}$ *be a finitary localizing invariant with values in a cocomplete stable* ∞*-category* $\mathcal{E}$*. There exists an object of* $\mathcal{E}$ *with* S^1*-action* X_E *such that* $P_1E^{\mathrm{cyc}} \simeq X_E \otimes \mathrm{THH}$.

Here, $X_E \otimes \mathrm{THH}$ is made into a trace theory using the S^1-action on X_E as follows:

$$\Lambda^{\mathrm{st}} \to \Lambda^{\mathrm{op}} \to BS^1 \xrightarrow{X_E} \mathcal{E}$$

where the first functor is the canonical one, the second comes from the definition of Λ as a quotient by an S^1-action.

Remark 4.2.4. I will explain in Section 4.2.1 that X_E has a cyclotomic structure, and that this equivalence can be refined to take into account this extra structure. I will do so in a very "pointwise" way, and leave the higher coherences between varying Frobenii to future work. ◁

Definition 4.2.5. A trace theory $T : \Lambda^{\mathrm{st}} \to \mathcal{E}$ with values in a cocomplete stable ∞-category $\mathcal{E}$ is called cocontinuous if for any $C \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)}$, the induced functor from Example 4.1.22, $T(C, -) : \mathrm{Fun}^L(C, C) \to \mathcal{E}$ preserves colimits. ◁

Remark 4.2.6. Note that this implies that for any tuple $C_0, ..., C_n$ of ∞-categories, the induced functor $T(\vec{C}, -) : \mathrm{Fun}^L(C_0, C_1) \times ... \times \mathrm{Fun}^L(C_n, C_0) \to \mathcal{E}$ preserves colimits in each variable, since the composition functor does so and $T(\vec{C}, -)$ factors through the composition functor by design. ◁

Notation 4.2.7. We let $\mathrm{TrThy}^L(\mathcal{E})$ (resp. $\mathrm{TrThy}^L_\Delta(\mathcal{E})$) the full subcategory of $\mathrm{TrThy}(\mathcal{E})$ (resp. $\mathrm{TrThy}_\Delta(\mathcal{E})$) spanned by cocontinuous trace theories (resp. Δ-trace theories). ◁

The following is essentially the key result concerning cocontinuous trace theories:

Proposition 4.2.8. *Evaluation at* $(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})$ *is a conservative functor*

$$\mathrm{TrThy}^L_\Delta(\mathcal{E}) \to \mathcal{E}$$

and also $\mathrm{TrThy}^L(\mathcal{E}) \to \mathcal{E}$.

Proof. Note that the latter claim follows from the former by Lemma 4.1.23.

For the former, we argue as follows: let $\alpha : T \to T'$ be a morphism of cocontinuous trace theories and suppose that $T(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}}) \to T'(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})$ is an equivalence. We wish to show that α is an equivalence. Because every object in Λ^{op} admits a morphism to $[0]_\Lambda$, and both T, T' invert coCartesian morphisms, it suffices to prove so on the fiber over $[0]$. That is, for a compactly generated ∞-category C equipped with an endomorphism S, we need to show that $T(C, S) \to T'(C, S)$ is an equivalence

Now since C is compactly generated, $\mathrm{Fun}^L(C, C)$ is generated under colimits by functors of the form $\mathrm{map}(x, -) \otimes y, x, y \in C^\omega$, it suffices to prove it for $S = \mathrm{map}(x, -) \otimes y$ *because* T, T' *are cocontinuous.*

But each $\mathrm{map}(x, -) \otimes y : C \to C$ factors as $C \xrightarrow{\mathrm{map}(x,-)} \mathrm{Sp} \xrightarrow{-\otimes y} C$ so that, by the trace property of T, T', it suffices to prove it for

$$(\mathrm{Sp}, \mathrm{map}(x, - \otimes y)) \simeq (\mathrm{Sp}, \mathrm{map}(x, y) \otimes -)$$

Thus it suffices to prove it for $(\mathrm{Sp}, X \otimes -)$, $X \in \mathrm{Sp}$. But now, Sp is generated under colimits (and desuspensions) by $\mathbb{S}$, so we are done. □

In particular, to prove Theorem 4.2.1, it would suffice to prove that the Dennis trace map induces an equivalence $P_1 K^{\mathrm{cyc}}(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}}) \to \mathrm{THH}(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}}) \simeq \mathbb{S}$. One *could* do this by reducing to the classical Dundas–McCarthy theorem, more specifically to the following special case:

Theorem 4.2.9. *There is an equivalence* $\mathrm{colim}_n \Omega^n \tilde{K}(\mathbb{S} \oplus \Sigma^{n-1}\mathbb{S}) \simeq \mathbb{S}$.

This can be proved using connectivity estimates, and in fact was originally proved[21] by Goodwillie in [Goo90, Corollary 3.3].

Remark 4.2.10. This naive argument gives us a local version of the Dundas–McCarthy theorem without needing to upgrade THH itself to a trace theory, namely as follows: $P_1 K^{\mathrm{cyc}}$, restricted to $\mathrm{Fun}^L(C, C) \simeq \mathrm{Fun}^L(C, \mathrm{Sp}) \otimes C$ is equivalent, by cyclic invariance, to the composite $\mathrm{Fun}^L(C, \mathrm{Sp}) \otimes C \xrightarrow{\mathrm{ev}} \mathrm{Sp} \xrightarrow{P_1 K^{\mathrm{cyc}}(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}}) \otimes -} \mathrm{Sp}$, but the first arrow is how THH is defined.

So in some sense, modulo Theorem 4.2.9 (which we recover later as Corollary 4.2.25, also without needing an upgrade of THH to a trace theory), we do not even need to know that THH is itself a trace theory to obtain the result. ◁

[21] This is not exactly what he proves, but further easy connectivity estimates allow us to deduce that result from his work.

Instead, we follow a more indirect route, where we completely classify cocontinuous trace theories.

Specifically, we prove:

Theorem 4.2.11. *Let $\mathcal{E}$ be a stable cocomplete ∞-category. Evaluation at* $(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})$ *induces equivalences:*

$$\mathrm{TrThy}^L_\Delta(\mathcal{E}) \xrightarrow{\simeq} \mathcal{E}$$

and

$$\mathrm{TrThy}^L(\mathcal{E}) \xrightarrow{\simeq} \mathcal{E}^{BS^1}$$

It is easy to convince oneself that this ought to be true, essentially via the same reasoning as in Proposition 4.2.8: the value of $T(\vec{C}, \vec{T})$ is determined by $T(C_0,\ T_n \circ \ \dots\ \circ T_0)$ because T is a trace-theory, and now, to evaluate $T(C_0, F)$ it suffices to note that F is a colimit of functors of the form $\mathrm{Map}(x, -) \otimes y$ which factor as $C_0 \to \mathrm{Sp} \to C_0$ and thus

$$T(C_0, \mathrm{Map}(x, -) \otimes y) \simeq T(\mathrm{Sp}, \mathrm{Map}(x, y) \otimes -) = T(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}}) \otimes \mathrm{Map}(x, y)$$

so that T is completely determined by $T(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})$ which "can be anything".

This argument works in Δ^{st} because there is in some sense a canonical choice of a C_0, but over Λ^{op} there is an ambiguity, and it is this ambiguity that induces the S^1-action.

Let us make the argument a tiny bit more precise before finally moving on to the actual (more technically involved) proof: the idea is to observe that Δ^{st} is a Δ^{op}-indexed colimit of a certain diagram. Since we have asked for fiberwise cocontinuity, this colimit can be understood to be in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, and, more importantly, this diagram can be understood as being built out of things like

$$\mathrm{Fun}^L(C_0, C_1) \otimes \dots \otimes \mathrm{Fun}^L(C_n, C_0)$$

Since each C_i is dualizable in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, this can be rewritten as

$$\mathrm{Fun}^L(\mathrm{Sp}, C_0) \otimes \mathrm{Fun}^L(C_0, C_1) \otimes \dots \otimes \mathrm{Fun}^L(C_n, \mathrm{Sp})$$

- but this new diagram has an extra degeneracy because it is a special case of the same diagram one level up! This extra degeneracy gives us the result immediately for "diagrammatic" reasons.

Remark 4.2.12. This sketch is essentially the same as the one proposed in [HSS17, Proposition 4.24] to prove that bar constructions compute traces, just one category level up, in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. It is a tiny bit more subtle because our Δ^{st} is not *exactly* a version of the cyclic bar construction, it is some kind of lax version thereof. ◁

At this point, the reader comfortable with fibrational technology has all they need to fill in the gaps - the proof from this point onwards is tediously technical, but relatively straightforward.

We now need to enter the details of the argument. To make the above idea precise, we need some auxiliary constructions.

Our goal is to encode the category of labelled cyclic graphs of the form

$$\mathrm{Sp} \to C_0 \to \cdots \to C_n \to \mathrm{Sp}$$

(and morphisms between them that are the identity on Sp), and the procedure of composing the edge $C_n \to \mathrm{Sp} \to C_0$ that returns the labelled cyclic graph

$$C_0 \to \cdots \to C_n \to C_0$$

Since everything involved here is essentially given by coCartesian pushforwards, these auxiliary constructions work in the generality of a coCartesian fibration, and we perform there in this generality to prevent confusion[22].

I will recall the notations every time, but let us fix throughout the following convention: $p : E \to S$ is a coCartesian fibration, and D will be some arbitrary ∞-category. The fibration p should be thought of as $\Delta^{\mathrm{st}} \to \Delta^{\mathrm{op}}$, D as $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$, and I will also indicate what the general fibrational manoeuvers correspond to in this special case.

Recall that our convention is that given an edge $f : t \to s$ in S, $f_! : E_t \to E_s$ denotes the associated coCartesian pushforward.

Construction 4.2.13. Let $p : E \to S$ be a coCartesian fibration, and $s \in S$. There is a functor $S_{/s} \times_S E \to E_s$ corresponding the natural transformation of $S_{/s}$-indexed functors

$$(f : t \to s) \mapsto (E_t \xrightarrow{f_!} E_s)$$

.

More generally, there is a functor $S^{\Delta^1} \times_S E \to E$ where the functor $S^{\Delta^1} \to S$ is given by "source", which acts as above on fibers. To construct this, note that the full subcategory of $(E^{\Delta^1})_{\mathrm{cocart}} \subset E^{\Delta^1}$ spanned by p-coCartesian edges is equivalent, via forgetting, to $S^{\Delta^1} \times_S E$. Using the inverse of that forgetful functor, followed by evaluation at the target in E^{Δ^1} provides the desired functor.

Specializing to fibers over $s \in S$ gives the first statement, but the general functor $S^{\Delta^1} \times_S E \to E$ shows that the functors $S_{/s} \times_S E \to E_s$ are natural in $s \in S$. ◁

Lemma 4.2.14. *Let $p : E \to S$ be a coCartesian fibration and D an ∞-category. Fix $s \in S$, $r : E_s \to D$ a functor, and finally let $d \in D$.*

The projection $(S_{/s} \times_S E) \times_{E_s} E_s \times_D \{d\} \to S_{/s}$ is a coCartesian fibration, where the functor $(S_{/s} \times_S E) \to E_s$ is the one constructed in Construction 4.2.13.

Note that $(S_{/s} \times_S E) \times_{E_s} E_s \times_D \{d\} \simeq (S_{/s} \times_S E) \times_D \{d\}$, we simply spelled it out to clarify the projection functor.

Proof. Let $f : t_0 \to t_1$ be a map in $S_{/s}$, where $p_i : t_i \to s$, and $x \in E_{t_0}$ equipped with an equivalence $\alpha : r \circ (p_0)_! x \simeq d$, so that (t_0, x, α) is a point in $(S_{/s} \times_S E) \times_D \{d\}$ lying over t_0.

We then have an equivalence $\beta : (p_1)_! f_! x \simeq (p_0)_! x$ so that

$$r(\beta) : r(p_1)_! f_! x \simeq r(p_0)_! x \simeq d$$

making $(t_1, f_! x, r(\beta))$ a point in $(S_{/s} \times_S E) \times_D \{d\}$ lying over t_1. Furthermore, f and the identity $f_! x = f_! x$ produce a map $(t_0, x, \alpha) \to (t_1, f_! x, r(\beta))$ lying over f - it is now easy to verify by hand that this is a coCartesian lift of f: we sketch out a more high-brow argument below for ease of formal verification (but the "by hand" argument is easier to check for oneself).

Consider the diagram:

$$\begin{tikzcd}
(S_{/s} \times_S E) \times_D \{d\} \arrow[r] \arrow[d, "q"'] & \{d\} \arrow[d, "i"] \\
S_{/s} \times_S E \arrow[r, "r \circ \pi_!"'] \arrow[d, "p_{/s}"'] & D \\
S_{/s} &
\end{tikzcd}$$

[22] At least the author's confusion.

where the top square is by definition a pullback diagram and the bottom vertical map is a coCartesian fibration because it is pulled back from one.

Now our map $(t_0, x, \alpha) \to (t_1, f_! x, r(\beta))$ is q-coCartesian: indeed, by [Lur09, Proposition 2.4.1.3.(2)], it suffices to check that its image in $\{d\}$ is i-coCartesian: it is an equivalence (as is any map in $\{d\}$) and thus i-coCartesian, by [Lur09, Proposition 2.4.1.5]. Thus, by [Lur09, Proposition 2.4.1.3.(3)], to prove that our edge is $p_{/s} \circ q$-coCartesian, it suffices to show that its image in $S_{/s} \times_S E$ is $p_{/s}$-coCartesian. Using the pullback diagram

$$\begin{tikzcd} S_{/s} \times_S E \arrow[r] \arrow[d, "p_{/s}"'] & E \arrow[d] \\ S_{/s} \arrow[r] & S \end{tikzcd}$$

and again [Lur09, Propostion 2.4.1.3.(2)], it suffices to check that its image in E is p-coCartesian. But now this is by design : the image in E is the p-coCartesian edge $x \to f_! x$ lying over $f : t_0 \to t_1$(!) $\square$

Example 4.2.15. Consider the fibration $\Delta^{\mathrm{st}} \to \Delta^{\mathrm{op}}$, $s = [0] \in \Delta^{\mathrm{op}}$, the functor $r : \Delta^{\mathrm{st}}_{[0]} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$ which forgets the endomorphism, and $d = \mathrm{Sp} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$. In that case, the source of that functor is the ∞-category of cyclic graphs of stable ∞-categories indexed by a pointed finite non-empty linearly ordered set, with a specified equivalence between the value at the distinguished point and Sp. In drawings, this would be something like

$$C_0 \to ... \to C_i \to \mathrm{Sp} \to ... \to C_n \to C_0$$

As will be clear later, we will mostly focus on the case where Sp is "at the start", so pictorially something like

$$\mathrm{Sp} \to C_1 \to ... \to C_n \to \mathrm{Sp}$$

$\triangleleft$

Remark 4.2.16. In this example, it is crucial that we allow D to be different from E_s: otherwise, we would be considering cyclic graphs of the announced form where the total composite $\mathrm{Sp} \to C_1 \to \cdots \to C_n \to \mathrm{Sp}$ is some fixed endofunctor of Sp. $\triangleleft$

By pulling back, we obtain:

Corollary 4.2.17. *Let $p : E \to S$ be a coCartesian fibration, $s \in S$, $r : E_s \to D$ a functor and fix $d \in D$. For any map $T \to S_{/s}$, the map $(T \times_S E) \times_D \{d\} \to T$ is a coCartesian fibration.*

Similarly, if we have a map $T \to S^{\Delta^1}$ we can use the more general construction of Construction 4.2.13 to obtain:

Corollary 4.2.18. *Suppose $p : E \to S$ is a coCartesian fibration, $s \in S$, and a functor over S*

$$f : S \to S^{\Delta^1} \times_S S_{/s}$$

is given[23], corresponding to a functor $F : S \to S$ equipped with transformations $\mathrm{id}_S \xleftarrow{i} F \xrightarrow{q} s$.

For every functor $r : E_s \to D$ and $d \in D$, we have a map $(S \times_S E) \times_D \{d\} \to E$ of coCartesian fibrations over S given informally by

$$(t, x \in E_{F(t)}, r(q_t)_! x \simeq d) \mapsto (t, (i_t)_! x)$$

[23] In the target, the map $S^{\Delta^1} \to S$ used in defining the pullback is the source map; and the map from the pullback to S used in saying that f is "over S" is the target map $S^{\Delta^1} \to S$.

Proof. The map is given by the composite

$$(S \times_S E) \times_D \{d\} \to S \times_S E \to S^{\Delta^1} \times_S E \to E$$

where the first map is the forgetful functor, the second uses the functor $S \to S^{\Delta^1}$ and the third one is the map from Construction 4.2.13.

This is clearly a map over S and combining the proof of Lemma 4.2.14 together with the proof that coCartesian fibratiosn are stable under pullbacks, one finds that it preserves coCartesian edges. □

Example 4.2.19. We keep going with Example 4.2.15. Our functor of interest here is the functor $\Delta^{op} \to \Delta^{op}$ given by $[0] \star -$, i.e. join with $[0]$. This comes with natural transformations[24]

$$[0] \star [n] \to [n], [0] \star [n] \to [0]$$

In that case, the source of the functor from above is something like cyclic graphs of the form $\mathrm{Sp} \to C_0 \to ... \to C_n \to \mathrm{Sp}$, and the map to Δ^{st} simply composes the functors $C_n \to \mathrm{Sp} \to C_0$ to obtain $C_0 \to ... \to C_n \to C_0$. ◁

Notation 4.2.20. We let $\Delta^{st}_{0,\mathrm{Sp}}$ denote the relevant pullback

$$(\Delta^{op} \times_{\Delta^{op}} \Delta^{st}) \times_{\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}} \{\mathrm{Sp}\}$$

◁

With this notation, we have a map of fibrations over Δ^{op}, $\Delta^{st}_{0,\mathrm{Sp}} \to \Delta^{st}$.

The fiber over $[n]$ of that map is the map

$$\mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}([1+n]_\Lambda, \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)}) \times_{\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}} \{\mathrm{Sp}\} \to \mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}([n]_\Lambda, \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)})$$

given by precomposition by the map $[n]_\Lambda \to [1+n]_\Lambda$ that misses the 0th term.

Note that, viewing $[n]_\Lambda$ as an ∞-category, there is a natural functor $[n] \to [n]_\Lambda$.

Proposition 4.2.21. *Restriction along $[n] \to [n]_\Lambda$ induces co/Cartesian fibrations*

$$\mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}([1+n]_\Lambda, \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)}) \times_{\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}} \{\mathrm{Sp}\} \to \mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}([n], \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)})$$

and

$$\mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}([n]_\Lambda, \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)}) \to \mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}([n], \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)})$$

and the map between the two sources preserves co/Cartesian edges.

The idea here is essentially the same as the one in the proof following Example 4.1.22, only more notationally involved. The point is again that, generally speaking, restricting to adjoints in the target/base of some kind of 2-functor often helps provide coCartesian lifts. We recommend the reader have a look again at the simpler proof following Example 4.1.22, as it can be seen as a toy example of this phenomenon, which is essentially all there is to this proof.

[24]Recall that these are in Δ^{op} as opposed to Δ.

Proof. Fix $(\vec{C},\vec{T}) = (C_0 \xrightarrow{T_0} \dots \xrightarrow{T_{n-1}} C_n)$ and similarly for $(\vec{D},\vec{S})$ in $\mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}([n],\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)})$ as well as a map $\vec{f} : (\vec{C},\vec{T}) \to (\vec{D},\vec{S})$ between them.

We first deal with the second claim: for $T_n : C_n \to C_0$, the claim is that fixing $f_0 \circ T_n \circ f_n^R : D_n \to D_0$ and picking the obvious lift of $\vec{f}$ to a morphism in $\mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}([n]_\Lambda,\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)})$ provides a coCartesian lift of $\vec{f}$.

Indeed, let $(\vec{E},\vec{R}) = (E_0 \xrightarrow{R_0} \dots \xrightarrow{R_n} E_0)$ and consider the following lifting problem:

$$\begin{tikzcd}
(\vec{C},\vec{T}) + T_n \arrow[r] \arrow[d] \arrow[dr, mapsto] & (\vec{D},\vec{S}) + f_0 T_n f_n^R \arrow[dl, dashed] \arrow[dr, mapsto] & \\
(\vec{E},\vec{R}) \arrow[dr, mapsto] & (\vec{C},\vec{T}) \arrow[r] \arrow[d] & (\vec{D},\vec{S}) \arrow[dl] \\
& (\vec{E},\vec{R})_{|[n]} &
\end{tikzcd}$$

Since $[n]$ and $[n]_\Lambda$ are free on the chain of n-arrows and the cyclic graph on $n+1$-letters respectively which differ only by the arrow/edge $n \to 0$ in the latter, the space of dotted lifts is equivalent to the space of lifts in the following smaller problem:

$$\begin{tikzcd}
(C_n \xrightarrow{T_n} C_0) \arrow[r] \arrow[d] \arrow[dr, mapsto] & (D_n \xrightarrow{f_0 T_n f_n^R} D_0) \arrow[dl, dashed] \arrow[dr, mapsto] & \\
(E_n \xrightarrow{R_n} E_0) \arrow[dr, mapsto] & (C_n, C_0) \arrow[r] \arrow[d] & (D_n, D_0) \arrow[dl] \\
& (E_n, E_0) &
\end{tikzcd}$$

i.e. we are given $g_0 : D_0 \to E_0, g_n : D_n \to E_n$, as well as a map $g_0 f_0 T_n \to R_n g_n f_n$, and are asking about the fiber of

$$h : \mathrm{Map}(g_0 f_0 T_n f_n^R, R_n g_n) \to \mathrm{Map}(g_0 f_0 T_n, R_n g_n f_n)$$

over that map. But by design of the map

$$(C_n \xrightarrow{T_n} C_0) \to (D_n \xrightarrow{f_0 T_n f_n^R} D_0)$$

h is precisely the adjunction equivalence between those two spaces, so that the fiber is indeed contractible.

For the first claim, we only supply the coCartesian edge: the proof that it is indeed coCartesian is similar (in fact, easier). So let

$$(\vec{C},\vec{T}) = (\mathrm{Sp} \xrightarrow{F} C_0 \xrightarrow{T_0} \dots \xrightarrow{T_{n-1}} C_n \xrightarrow{G} \mathrm{Sp})$$

and

$$(\vec{D},\vec{S}) = (D_0 \xrightarrow{S_0} \dots \xrightarrow{D_{n-1}} D_n)$$

as well as a map

$$\vec{f} : (C_0 \to \dots \to C_n) \to (D_0 \to \dots \to D_n)$$

We claim that the canonical map from

$$(\mathrm{Sp} \to C_0 \to \dots \to C_n \to \mathrm{Sp})$$

to

$$(\mathrm{Sp} \xrightarrow{f_0 F} D_0 \to \dots \to D_n \xrightarrow{G f_n^R} \mathrm{Sp})$$

is a coCartesian lift of $\vec{f}$. □

Corollary 4.2.22. *Let $\mathcal{E}$ be a cocomplete ∞-category. For every n, restriction along*

$$\mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}([1+n]_\Lambda, \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)}) \times_{\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}} \{\mathrm{Sp}\} \to \mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}([n]_\Lambda, \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)})$$

induces an equivalence on functors into $\mathcal{E}$ that are fiberwise cocontinuous in each variable.

Proof. For a coCartesian fibration $E \to S$, E is a lax colimit of the fibers $E_s, s \in S$ and $\mathrm{Fun}(E, \mathcal{E})$ is the lax limit of $\mathrm{Fun}(E_s, \mathcal{E}), s \in S$.

Thus if $E \to E'$ is a map of coCartesian fibrations over S, and for every $s \in S$, $\mathrm{Fun}(E'_s, \mathcal{E}) \to \mathrm{Fun}(E_s, \mathcal{E})$ induces an equivalence on appropriate subcategories, then $\mathrm{Fun}(E', \mathcal{E}) \to \mathrm{Fun}(E, \mathcal{E})$ induces an equivalence on the appropriate fiberwise subcategories.

In our case, the relevant map is a map of coCartesian fibrations over $\mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}([n], \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)})$ and on fibers it induces

$$\mathrm{Fun}^L(C_n, \mathrm{Sp}) \times \mathrm{Fun}^L(\mathrm{Sp}, C_0) \to \mathrm{Fun}^L(C_n, C_0)$$

which induces an equivalence when mapping into a cocomplete $\mathcal{E}$ and restricting to bilinear (resp. cocontinuous) functors because composition induces an equivalence

$$\mathrm{Fun}^L(C_n, \mathrm{Sp}) \otimes \mathrm{Fun}^L(\mathrm{Sp}, C_0) \simeq \mathrm{Fun}^L(C_n, C_0)$$

□

Corollary 4.2.23. *For any stable cocomplete ∞-category $\mathcal{E}$, the functor $\Delta^{\mathrm{st}}_{0,\mathrm{Sp}} \to \Delta^{\mathrm{st}}$ induces an equivalence*

$$\mathrm{TrThy}^L_\Delta(\mathcal{E}) = \mathrm{Fun}^{\mathrm{fbw}-L}_{cocart}(\Delta^{\mathrm{st}}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{fbw}-L}_{cocart}(\Delta^{\mathrm{st}}_{0,\mathrm{Sp}}, \mathcal{E})$$

between ∞-categories of fiberwise multilinear functors that invert coCartesian edges.

Proof. This map is induced by taking a limit over Δ of the maps from Corollary 4.2.22, which are equivalences. □

Corollary 4.2.24. *Evaluation at* $(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})$ *induces an equivalence* $\mathrm{TrThy}^L_\Delta(\mathcal{E}) \to \mathcal{E}$.

Proof. By the previous corollary, it suffices to observe that the ∞-category $\mathrm{Fun}^{\mathrm{fbw}-L}_{cocart}(\Delta^{\mathrm{st}}_{0,\mathrm{Sp}}, \mathcal{E})$ is the totalization over Δ of a diagram admitting extra degeneracies in the sense that it extends to Δ_∞ in the notation of [Lur09, Lemma 6.1.3.16].

Indeed, $\Delta^{\mathrm{st}}_{0,\mathrm{Sp}}$ is the pullback to Δ^{op} of a coCartesian fibration over $(\Delta^{\mathrm{op}})_{/[0]} = (\Delta_{[0]/})^{\mathrm{op}}$, namely the one from Lemma 4.2.14, and there is a functor $\Delta_\infty \to \Delta_{[0]/}$ under Δ (via the functor $[0] \star -$) and over Δ (via the forgetful functors: the obvious one for $\Delta_{[0]/}$, and, still in the (dual) notation from [Lur09, p. 6.3.1.16], $J \mapsto J \cup \{-\infty\}$ for Δ_∞).

The fiber over $[0]$ of that fibration is clearly $\mathcal{E}$. □

Proof of Theorem 4.2.11. Combine Corollary 4.2.24 and Lemma 4.1.23. We simply need to prove that the induced action on $\mathcal{E}$ from Corollary 4.2.24 is trivial - we essentially checked this in Example 4.1.21. Indeed, recall that we constructed there a coCartesian section $\Lambda^{\mathrm{op}} \to \Lambda^{\mathrm{st}}$ with value at $[0]_\Lambda$ given by $(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})$. After inverting coCartesian morphisms, this produces a section $BS^1 \to (\Lambda_\infty)^{\mathrm{st}}[\mathrm{cocart}^{-1}]_{hS^1}$ with value $(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})$, i.e this makes $(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})$ into an S^1-fixed point in $\Delta^{\mathrm{st}}[\mathrm{cocart}^{-1}]$ (where the action on the latter is induced by the equivalence with the Λ_∞ version).

Thus evaluating at $(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})$ produces an S^1-equivariant functor

$$\mathrm{TrThy}^L_\Delta(\mathcal{E}) \to \mathcal{E}^{\mathrm{triv}}$$

whose underlying functor is the functor we produced, and hence is an equivalence, as was needed. □

Corollary 4.2.25. *There is an equivalence There is an equivalence*

$$P_1 K^{\mathrm{cyc}}(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}}) = \mathrm{colim}_n \Omega^n K^{\mathrm{cyc}}(\mathrm{Sp}, \Sigma^n \mathrm{id}_{\mathrm{Sp}}) \simeq \mathbb{S}$$

Remark 4.2.26. This equivalence falls out from the universal property of K-theory and the mere observation that derivatives of localizing invariants naturally form trace theories; and yet it in principle yields information about classifying spaces of $GL_n(\mathbb{S})$, and somewhat precise connectivity estimates, cf. Section 4.2.3. This is surprising, but quite pleasing. ◁

Proof. By Corollary 4.1.62, $P_1 K^{\mathrm{cyc}}$ corepresents the equivalence

$$\mathrm{ev}_{(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})} : \mathrm{TrThy}^L_\Delta(\mathrm{Sp}) \simeq \mathrm{Sp}$$

from Theorem 4.2.11. Thus, its left adjoint (which is also its inverse!) sends $\mathbb{S} \mapsto P_1 K^{\mathrm{cyc}}$.

The fact that this adjunction is an equivalence implies that $\mathrm{ev}_{(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})}(P_1 K^{\mathrm{cyc}}) \simeq \mathbb{S}$, i.e. $P_1 K^{\mathrm{cyc}}(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}}) \simeq \mathbb{S}$, as claimed. □

We can now prove our version of the Dundas–McCarthy theorem:

Proof of Theorem 4.2.1. By [HS19, Nikolaus, Definition 10], THH also upgrades to a trace theory with $\mathrm{THH}(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}}) \simeq \mathrm{THH}(\mathrm{Sp}) = \mathbb{S}$. Thus, by Theorem 4.2.11, $\mathrm{THH} \simeq P_1 K^{\mathrm{cyc}}$. Let us fix such an equivalence, say f.

Now, we have an equivalence $\mathrm{Map}(P_1 K^{\mathrm{cyc}}, \mathrm{THH}) \simeq \mathrm{Map}(K, \mathrm{THH}) \simeq \mathbb{S}$ by Corollary 4.1.61. The equivalence f is sent to some map on the right which corresponds to some $n \in \mathbb{Z} \cong \pi_0 \mathbb{S}$ and which cannot be divisible, since the identity of THH is not divisible (since the identity of $\mathbb{S}$ isn't). Thus $n = \pm 1$. Up to taking $-f$, we may thus assume that under the equivalence

$$\mathrm{Map}(P_1 K^{\mathrm{cyc}}, \mathrm{THH}) \simeq \mathrm{Map}(K, \mathrm{THH}) \simeq \mathbb{S}$$

f maps to 1, and thus to the Dennis trace in $\mathrm{Map}(K, \mathrm{THH})$. In other words, the Dennis trace induces an equivalence, as claimed. □

Remark 4.2.27. Had we set up a symmetric monoidal version of trace theories as mentioned in Remark 4.1.14, the above proof could have been streamlined by making K^{cyc} and THH into symmetric monoidal trace theories. Indeed, taking into account (lax) symmetric monoidal structures, there is a unique map $K \to \mathrm{THH}$ as a well as a unique map $P_1 K^{\mathrm{cyc}} \to \mathrm{THH}$. ◁

Remark 4.2.28. At this point, the context of trace theories makes the Dundas–McCarthy theorem essentially obvious[25], and this raises the question: since $P_1 K^{\mathrm{cyc}}$ is so much simpler than THH to define, and more clearly the appropriate setting for trace methods[26], what would we lose by *defining* THH to be $P_1 K^{\mathrm{cyc}}$? For example, as is clear from the previous discussion we easily recover the S^1-action, furthermore we will explain below how to recover the cyclotomic structure; and one could recover the symmetric monoidality without too much trouble. The key thing that is missing seems to be the simple formula for $\mathrm{THH}(R)$, and specifically the universal property when R is a commutative ring spectrum. Were I able to prove these for $P_1 K^{\mathrm{cyc}}$ in a satisfactory manner without essentially going through the equivalence with THH, I would suggest this change of definitions. ◁

Remark 4.2.29. The astute reader might point out at this point that, while we prove a more general theorem than Dundas and McCarthy by allowing nonconnective rings, in the connective case we get a weaker statement, as they actually get precise connectivity estimates for the map $\Omega^n K^{\mathrm{cyc}}(C, \Sigma^n T) \to \mathrm{THH}(C, T)$ in the connective case, and one can use these connectivity estimates for other purposes than proving this theorem - for example, Waldhausen's calculations from [Wal78] use these more precise estimates. At the end of this chapter, in Section 4.2.3, we digress to explain how the above version of this theorem, together with formal techniques from the work of Land and Tamme [LT19] allow us to actually *recover* these connectivity estimates from the bare statement about the first derivative, and how to recover also Waldhausen's calculations from those, thereby showing that in this situation, nothing is lost from working only with universal properties. ◁

Remark 4.2.30. In fact, in the nonconnective setting, connectivity estimates are essentially impossible at this point, so that a proof in the desired generality is not possible with only connectivity estimates. For the application of the Dundas–McCarthy theorem we present in Chapter 5, the nonconnective setting is needed (and essential). ◁

Let us now move on to the promised generalizations. First, we immediately obtain a proof of Theorem 4.2.3:

Proof of Theorem 4.2.3. It suffices to note that by Proposition 4.1.40, E^{cyc} is a trace theory, and hence $P_1 E^{\mathrm{cyc}}$ is a trace theory which is exact in the bimodule variable. Since E was finitary, $P_1 E^{\mathrm{cyc}}$ is furthermore filtered-colimit preserving in the bimodule variable ($F \mapsto \mathrm{End}_\omega(E, F)$ commutes with filtered colimits).

It is therefore of the given form by Theorem 4.2.11. □

Example 4.2.31. Consider $\mathcal{U}_{\mathrm{loc}} : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Mot}_{\mathrm{loc}}$, the universal finitary localizing invariant from [BGT13]. Theorem 4.2.3 implies that $P_1(\mathcal{U}_{\mathrm{loc}})^{\mathrm{cyc}} \simeq X \otimes \mathrm{THH}$ for some $X \in \mathrm{Mot}_{\mathrm{loc}}$. One can show that

$$K(X \otimes -) \simeq \mathrm{THH}$$

as localizing invariants - that an X with this property exists follows from Efimov's recent proof that $\mathrm{Mot}_{\mathrm{loc}}$ is rigid and in particular self-dual via $M \mapsto K(M \otimes -)$, but here X is relatively explicit.

The proof, however, relies on the methods from Efimov's proof of the rigidity of $\mathrm{Mot}_{\mathrm{loc}}$, so I do not include it[27]. ◁

[25] Modulo the very definition of THH as a trace theory.

[26] Which ultimately would work fine with $P_1 K^{\mathrm{cyc}}$ regardless of its relationship to THH.

[27] For the reader who has an idea of how this proof works: the idea is to use trace-class functors to intertwine $\mathrm{End}_\omega(\mathrm{Sp}, \Sigma^n) \otimes C$ and $\mathrm{End}_\omega(C, \Sigma^n)$ - Efimov proves that there are in some sense enough trace-class functors to make this go through.

In fact, Theorem 4.2.11 also provides information about non-cocontinuous exact trace theories, which will be convenient:

Corollary 4.2.32. *The inclusion* $\mathrm{TrThy}^L_\Delta(\mathrm{Sp}) \hookrightarrow \mathrm{TrThy}^{\mathrm{ex}}_\Delta(\mathrm{Sp})$ *admits a right adjoint given by* $T \mapsto T(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}}) \otimes \mathrm{THH}$.

The same holds for $\mathrm{TrThy}^L \hookrightarrow \mathrm{TrThy}^{\mathrm{ex}}$*, where now one remembers the* S^1*-action on* $T(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})$.

Proof. By Theorem 4.2.11, this inclusion is given by $\mathrm{Sp} \xrightarrow{-\otimes P_1 K^{\mathrm{cyc}}} \mathrm{TrThy}^{\mathrm{ex}}_\Delta(\mathrm{Sp})$ and for general reasons has $\mathrm{map}(P_1 K^{\mathrm{cyc}}, -) \simeq \mathrm{ev}_{(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})}$ as its right adjoint. The claim follows from unwinding the relevant equivalences.

For trace-theories, we note that the inclusion of cocontinuous Δ-trace theories inside exact Δ-trace theories is S^1-equivariant, and hence so is its right adjoint. Passing to fixed points for the S^1-action and using Lemma 4.1.23 gives the desired result. □

This is convenient as it allows us to describe the Frobenius on $P_1 E^{\mathrm{cyc}}$ in terms of X_E. Defining this Frobenius and explaining how to apply this adjunction to describe it is the goal of the following subsection.

4.2.1 Addendum I: Polygonic and cyclotomic Frobenii

Let us, in the first place *define* the Frobenius on $P_1 E^{\mathrm{cyc}}$.

For starters, we need to define its target, which is supposed to be something like "$P_1 E^{\mathrm{cyc}}(C, T^{\circ p})^{tC_p}$" when evaluated on (C, T). To make sense of this properly, as a trace theory, we need to use the Λ_p's of Appendix A.

Definition 4.2.33. For $m \in \mathbb{N}$, we let $(\Lambda_m)^{\mathrm{st}} := (\Lambda_m)^{\mathrm{op}} \times_{\Lambda^{\mathrm{op}}} \Lambda^{\mathrm{st}}$ where the functor $\Lambda_m \to \Lambda$ is q_m (cf. Appendix A). ◁

Construction 4.2.34. The natural transformation $\psi_m : \overline{\Psi}_m \to q_m$ from Construction A.0.30 induces by restriction a natural transformation:

$$\mathrm{Fun}_{\mathrm{colax}}(q_m C, \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)}) \to \mathrm{Fun}_{\mathrm{colax}}(\overline{\Psi}_m C, \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)})$$

which we can restrict appropriately to a natural map

$$\mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}(q_m C, \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)}) \to \mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}(\overline{\Psi}_m C, \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)})$$

and mapping the (appropriately restricted) target into the fiber of Λ^{st} over $\overline{\Psi}_m C$ gives, in total, a map

$$(\overline{\Psi}_m)_! : (\Lambda_m)^{\mathrm{st}} := (q_m)^* \Lambda^{\mathrm{st}} \to \Lambda^{\mathrm{st}}$$

◁

Ultimately, an object in $(\Lambda_m)^{\mathrm{st}}$ is an object $(\vec{C}, \vec{T})$ in Λ^{st}, together with a lift of its underlying $[n]_\Lambda$ to Λ_m[28]. This construction sends this object not to $(\vec{C}, \vec{T})$ itself, but to an "unfolded" version thereof, which looks like

$$(C_0 \xrightarrow{T_0} C_1 \dots \xrightarrow{T_{n-1}} C_n \xrightarrow{T_n} C_0 \xrightarrow{T_0} C_1 \dots \to C_0)$$

with m copies of each C_i.

[28] $[n]_\Lambda$ has a "unique" lift to Λ_m - the point of this lift is therefore rather about encoding the correct functoriality than encoding the correct objects.

Example 4.2.35. This construction sends (C, T) lying over $[1]_{\Lambda_m}$ to

$$(C \xrightarrow{T} C \xrightarrow{T} \dots \xrightarrow{T} C)$$

Thus, any trace theory f evaluated at that object returns an object equivalent to $f(C, T^{\circ m})$. ◁

The example above and the unwinding of the definitions is supposed to show that the construction we are about to give is simply a more fancy version of Example 4.1.20.

To see this clearly, recall that $\Lambda_m \to \Lambda$ admits a canonical BC_m-equivariant structure, and therefore so does $(\Lambda_m)^{\mathrm{st}}$ as an ∞-category. We can therefore do the following construction:

Construction 4.2.36. Let $\mathcal{E}$ be any ∞-category, and consider the following composite:

$$\mathrm{Fun}(\Lambda^{\mathrm{st}}, \mathcal{E}) \xrightarrow{-\circ(\overline{\Psi}_m)_!} \mathrm{Fun}((\Lambda_m)^{\mathrm{st}}, \mathcal{E}) \simeq \mathrm{Fun}_{BC_m}((\Lambda_m)^{\mathrm{st}}, \mathcal{E}^{BC_m})$$

where the last equivalence comes from the forgetful-coinduction adjunction: $\mathcal{E}^{BC_m}$ has a coinduced BC_m-action. ◁

If $\mathcal{E}$ has sufficient co/limits, we have, as in [NS18]:

Proposition 4.2.37. *Let G be an $\mathbb{E}_1$-group, and suppose $\mathcal{E}$ is an ∞-category with BG-indexed limits. In this case $\mathcal{E}^{BG} \xrightarrow{\lim_{BG}} \mathcal{E}$ is $\mathrm{Aut}(BG)$-equivariant.*

In particular, if G is equipped with $\mathbb{E}_2$-structure so that BG is an $\mathbb{E}_1$-group and acts on itself, the limit functor is BG-equivariant.

If $\mathcal{E}$ is stable and furthermore admits G and BG-indexed colimits, then the same holds for the G-Tate construction as defined in [NS18, Theorem I.4.1], for $S = BG$.

Finally, if G is a finite group and $\mathcal{E}$ admits all colimits[29], the same holds for the G-proper Tate construction.

Proof. For the claim about limits, this follows from the fact that the left adjoint, namely the diagonal functor $\mathcal{E} \to \mathcal{E}^{BG}$, is canonically $\mathrm{Aut}(BG)$-equivariant. Thus the theory of relative adjunctions (cf. [Lur12, Section 7.3.2, Proposition 7.3.2.6]) provides a canonical $\mathrm{Aut}(BG)$-equivariant structure on the right adjoint.

The claim about Tate constructions follows from the limit claim together with the universal property of Tate constructions from [NS18, Theorem I.4.1]. □

Remark 4.2.38. The same holds for G orbits with the same proof. We will not need it. ◁

Corollary 4.2.39. *Let $f : \Lambda^{\mathrm{st}} \to \mathcal{E}$ be a trace theory and $m \in \mathbb{N}$; suppose $\mathcal{E}$ admits C_m-indexed limits.*

There is a canonical trace theory $(f_m)^{hC_m}$ which, on the fiber over $[0]_\Lambda$, acts as $(C, T) \mapsto f(C, T^{\circ m})^{hC_m}$.

If $\mathcal{E}$ is stable and furthermore admits all colimits, there is also a canonical trace theory $(f_m)^{\tau C_m}$ which, on the fiber over $[0]_\Lambda$, acts as $(C, T) \mapsto f(C, T^{\circ m})^{\tau C_m}$ where we use $(-)^{\tau G}$ for the G-proper Tate construction (and $((-)^{tG}$ for the usual Tate construction).

If f is exact in the bimodule variable, then so is $(f_m)^{\tau C_m}$.

Remark 4.2.40. The notation $(f_m)^{hC_m}$ resp. $(f_m)^{\tau C_m}$ for these canonical trace theories is *slightly* abusive in the sense that f_m itself is not a trace theory. f_m is a BC_m-equivariant functor $(\Lambda_m)^{\mathrm{st}} \to \mathcal{E}^{BC_m}$, and as will be clear from the proof, it is only after taking fixed points/Tate constructions that it induces a trace theory. ◁

[29] This could be refined analogously to above.

Remark 4.2.41. For the Tate case, while there is also something like $(f_m)^{tC_m}$, if m is not a prime number, it is not true that if f was exact, $(f_m)^{tC_m}$ is too. ◁

Proof. Construction 4.2.36 gives a functor $\mathrm{Fun}(\Lambda^{\mathrm{st}}, \mathcal{E}) \to \mathrm{Fun}_{BC_m}((\Lambda_m)^{\mathrm{st}}, \mathcal{E}^{BC_m})$. By Proposition 4.2.37, in either the fixed points or Tate case we get a BC_m-equivariant functor $\mathcal{E}^{BC_m} \to \mathcal{E}$ and thus a map to

$$\mathrm{Fun}_{BC_m}((\Lambda_m)^{\mathrm{st}}, \mathcal{E}) \simeq \mathrm{Fun}(((\Lambda_m)^{\mathrm{st}})_{hBC_m}, \mathcal{E}) \simeq \mathrm{Fun}(\Lambda^{\mathrm{st}}, \mathcal{E})$$

For the first part of the statement, we thus only need to prove that if f was a trace theory, then so is this new functor $(f_m)^{?C_m}, ? \in \{h, \tau\}$.

Now $\Lambda_\infty \to \Lambda$ and hence $\Lambda_m \to \Lambda$ is surjective on morphisms. Thus, to prove that $(f_m)^{?C_m}$ sends coCartesian edges to equivalences, it suffices to prove the same for the functor $(\Lambda_m)^{\mathrm{st}} \to \mathcal{E}$, and hence for the functor $(\Lambda_m)^{\mathrm{st}} \to \mathcal{E}^{BC_m}$. Equivalences are underlying, so it suffices to prove it for the non-BC_m-equivariant functor $f_m : (\Lambda_m)^{\mathrm{st}} \xrightarrow{(\Psi_m)_!} \Lambda^{\mathrm{st}} \xrightarrow{f} \mathcal{E}$.

But $(\Psi_m)_!$ is induced by a natural transformation and hence sends coCartesian edges to coCartesian edges, which in turn are sent to equivalences by f (by assumption), so we are done.

For the second part of the statement concerning exact trace theories, this is a fiberwise statement so it suffices to unwind the construction to observe that the restriction of $(f_m)^{\tau C_m}$ to $\mathrm{Fun}^L(C, C)$ is indeed $T \mapsto f((C, ..., C), (T, ..., T))^{\tau C_m}$ so that the lemma below applies (for it to apply, note that

$$f((C, ..., C), (-)) : \mathrm{Fun}^L(C, C)^{\times m} \to \mathcal{E}$$

factors as the exact-in-each-variable composition functor

$$\circ_m : \mathrm{Fun}^L(C, C)^{\times m} \to \mathrm{Fun}^L(C, C)$$

followed by the exact functor $f(C, -) : \mathrm{Fun}^L(C, C) \to \mathcal{E}$ and is thus each in each variable as well). □

Lemma 4.2.42. *Let $D, \mathcal{E}$ be stable ∞-categories, with $\mathcal{E}$ complete and cocomplete, and $m \in \mathbb{N}$. Let $f : D^{\times m} \to \mathcal{E}$ be a C_m-equivariant functor which is exact in each variable.*

Then $(f \circ \delta_m)^{\tau C_m}$ is exact, where $\delta_m : D \to D^{\times m}$ is the diagonal functor.

Proof. This is standard, see e.g. [NS18, Proposition III.1.1] for the case of m being a prime number and a specific C_m-equivariant functor.

The only difference is that the non-diagonal terms need not be induced from $\{e\} \subset C_m$, but from some proper subgroup $C_d \subset C_m$, which is why for general m we need the proper Tate construction which kills these terms too. □

We introduce a slightly better notation for "f_m":

Notation 4.2.43. Given $\Lambda^{\mathrm{st}} \to \mathcal{E}$, we let $\varphi_m^* f : (\Lambda_m)^{\mathrm{st}} \to \mathcal{E}^{BC_m}$ denote the BC_m-equivariant functor from Construction 4.2.36.

We may abuse notation and denote similarly the underlying functor, as well as its post-composition with $\mathcal{E}^{BC_m} \xrightarrow{\text{forget}} \mathcal{E}$ if there is no possibility for confusion. ◁

Before going further, we note a corollary, the proof of which amounts to unwinding the previous constructions:

Corollary 4.2.44. *Let $f, g : \Lambda^{\mathrm{st}} \to \mathcal{E}$ be functors where $\mathcal{E}$ admits BC_m-indexed limits. There is a natural equivalence:*

$$\mathrm{Map}_{\mathrm{Fun}(\Lambda^{\mathrm{st}},\mathcal{E})}(f, (\varphi_m^* g)^{hC_m}) \simeq \mathrm{Map}_{\mathrm{Fun}((\Lambda_m)^{\mathrm{st}},\mathcal{E})}(f \circ (q_m)^{\mathrm{st}}, g \circ (\Psi_m)_!)$$

where $(q_m)^{\mathrm{st}} : (\Lambda_m)^{\mathrm{st}} \to \Lambda^{\mathrm{st}}$ is the canonical quotient map.

We can now construct our Frobenii. First, we construct a categorical Frobenius, informally described as the map $\mathrm{End}_\omega(C, T) \to \mathrm{End}_\omega((C, ..., C), (T, ..., T))$ of pre-trace theories which acts as $(x \to Tx) \mapsto (x \to Tx, x \to Tx, ..., x \to Tx)$. Here, $(C, T) \mapsto \mathrm{End}_\omega((C, ..., C), (T, ..., T))$ will be what we called $\varphi_m^* \mathrm{End}_\omega$

Remark 4.2.45. Note that by composition, $\mathrm{End}_\omega((C, ..., C), (T, ..., T))$ maps further to $\mathrm{End}_\omega(C, T^{\circ m})$ and the composite $\mathrm{End}_\omega(C, T) \to \mathrm{End}_\omega(C, T^{\circ m})$ is

$$(x \to Tx) \mapsto (x \to Tx \to T^2x \to ... \to T^m x)$$

◁

Construction 4.2.46. By Corollary 4.2.44, constructing a map

$$\mathrm{End}_\omega \to (\varphi_m^* \mathrm{End}_\omega)^{hC_m}$$

in $\mathrm{Fun}(\Lambda^{\mathrm{st}}, \mathrm{Cat}^{\mathrm{perf}})$ is equivalent to constructing a map

$$\mathrm{End}_\omega \circ (q_m)^{\mathrm{st}} \to \mathrm{End}_\omega \circ (\Psi_m)_!$$

in $\mathrm{Fun}((\Lambda_m)^{\mathrm{st}}, \mathrm{Cat}^{\mathrm{perf}})$.

For this, we use Γ^{st} as defined in Definition 4.1.9 for Γ specifically the essential image of Λ in $\mathrm{Cat} : \psi_m : \Psi_m \to q_m$ lives in this subcategory and so it induces, via coCartesian lifts in Γ^{st}, a morphism $(q_m)^{\mathrm{st}} \to (\Psi_m)_!$.

In turn, using Corollary 4.1.35, we obtain a morphism of ∞-categories

$$\mathrm{End}_\omega \circ (q_m)^{\mathrm{st}} \to \mathrm{End}_\omega \circ (\Psi_m)_!$$

which is what we needed. ◁

Unwinding the construction shows that it does indeed admit the previous informal description.

Finally, we have:

Construction 4.2.47. Let $E : \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{E}$ be a functor with values in a stable ∞-category admitting BC_m-indexed limits (resp. BC_m-indexed limits and all colimits).

Construction 4.2.46 provides a map $\mathrm{End}_\omega \circ (q_m)^{\mathrm{st}} \to \mathrm{End}_\omega \circ (\Psi_m)_!$ of $(\Lambda_m)^{\mathrm{st}}$-indexed functors, and hence a map $E \circ \mathrm{End}_\omega \circ (q_m)^{\mathrm{st}} \to E \circ \mathrm{End}_\omega \circ (\Psi_m)_!$ of $\mathcal{E}$-valued functors, and therefore a map $E \circ \mathrm{End}_\omega \to (\varphi_m^*(E \circ \mathrm{End}_\omega))^{hC_m}$ of Λ^{st}-indexed functors.

If E is a localizing invariant, then Corollary 4.1.47[30] gives us a map

$$E^{\mathrm{cyc}} \to (\varphi_m^* E^{\mathrm{cyc}})^{hC_m}$$

of trace theories.

Furthermore, the "exact" part of Corollary 4.2.39 together with the universal property of P_1^{fbw} induces a map

$$P_1 E^{\mathrm{cyc}} \to (\varphi_m^*(P_1 E^{\mathrm{cyc}}))^{\tau C_m}$$

of trace theories. We call this the polygonic Frobenius. ◁

[30] And Proposition 4.1.40, Corollary 4.2.39.

Remark 4.2.48. Using the very same argument, and considering higher Goodwillie derivatives, we obtain, for every pair $(n, m) \geq 1$, a "Frobenius" morphism of trace theories of the form

$$P_{nm}E^{\mathrm{cyc}} \to (\varphi_m^*(P_n E^{\mathrm{cyc}}))^{hC_m}$$

coming from the fact that $T \mapsto T^{\circ m}$ is m-excisive, and so $T \mapsto P_n E^{\mathrm{cyc}}(C, T^{\circ m})$ is nm-excisive. ◁

Since trace theories give objects with S^1-action when evaluated at objects of the form (C, id_C), we obtain:

Corollary 4.2.49. *Let E be a localizing invariant. There is a lift of the functor*[31] $P_1 E^{\mathrm{cyc}} : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Sp}$ *to a functor*

$$P_1 E^{\mathrm{cyc}} : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{CycSp}$$

Remark 4.2.50. In principle we should verify that the S^1-action on $((\varphi_m^*)f)^{hC_m}(C, \mathrm{id}_C) = f(C, \mathrm{id}_C)^{hC_m}$ is indeed the residual $S^1/C_m \cong S^1$-action. This is true, and not very difficult, though possibly tedious and notationally involved, so we do not do so here. ◁

Remark 4.2.51. The ∞-category of cyclotomic spectra as defined in [NS18] only involves cyclotomic Frobenii one prime at a time. There is a more refined notion of cyclotomic spectrum introduced in [AMR17] which involves proper Tate constructions, and coherences relating Tate fixed points with respect to cyclic groups of comparable order. The constructions introduced before can be refined to give such a cyclotomic structure on $P_1 E^{\mathrm{cyc}}$, but again the technicalities involved in doing so would lead us too far astray for the purposes of this thesis. ◁

The following corollary is obvious from the functoriality properties arising from

$$P_1 E^{\mathrm{cyc}} \to (\varphi_m^*(P_1 E^{\mathrm{cyc}}))^{\tau C_m}$$

being a map of trace theories, and from the definition of polygonic spectra [KMN23]:

Corollary 4.2.52. *Let E be a localizing invariant. There is a lift of the functor* $P_1 E^{\mathrm{cyc}} : \Lambda^{\mathrm{st}}_{[0]_\Lambda} \to \mathrm{Sp}$ *to a functor*

$$\underline{P_1 E^{\mathrm{cyc}}} : \Lambda^{\mathrm{st}}_{[0]_\Lambda} \to \mathrm{PgSp}$$

where PgSp *is the ∞-category of polygonic spectra as defined in [KMN23], such that, for each $m \in \mathbb{N}$, the value at m of the polygonic spectrum $\underline{P_1 E^{\mathrm{cyc}}}(C, T)$ is $P_1 E^{\mathrm{cyc}}(C, T^{\circ m})$.*

Remark 4.2.53. As in the case of cyclotomic spectra, one could offer a refined notion of polygonic spectra involving coherences between different primes and, again as in that case, the constructions presented here could be refined to produce such polygonic spectra.

In the polygonic case, there should also be further structures that would allow us to define something on the whole of Λ^{st}, as opposed to only the fiber at $[0]_\Lambda$. ◁

Having these constructions in hand, we can now leverage Corollary 4.2.32 to describe the polygonic Frobenius of $P_1 E^{\mathrm{cyc}}$ in terms of $X_E = P_1 E^{\mathrm{cyc}}(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})$. Namely, we have:

[31] We implicitly precompose everything with Ind to go from $\mathrm{Cat}^{\mathrm{perf}}$ to $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$, and with $C \mapsto (C, \mathrm{id}_C)$ to apply trace theories to bare ∞-categories.

Theorem 4.2.54. *Let E be a finitary localizing invariant. Upon restriction to $\Lambda^{\mathrm{st}}_{[0]_\Lambda}$, we have an equivalence of* PgSp*-valued functors*

$$\underline{P_1 E^{\mathrm{cyc}}} \simeq X_E \otimes \mathrm{THH}$$

where $X_E = P_1 E^{\mathrm{cyc}}(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})$ is the cyclotomic spectrum from Corollary 4.2.49 and $\otimes$ is the canonical action of cyclotomic spectra on polygonic spectra[32].

Thus $P_1 E^{\mathrm{cyc}}$, together with its natural extra structure, is entirely controlled by the single cyclotomic spectrum X_E.

Example 4.2.55. Suppose $E = E_C := K(C \otimes -)$ for some fixed stable ∞-category C. One can prove using ideas similar to Remark 4.1.64 that $X_E \simeq \mathrm{THH}(C)$ as cyclotomic spectra.

By Efimov's rigidity theorem and the main result from [RSW], any finitary localizing invariant is of this form. ◁

Question 4.2.56. Can any cyclotomic spectrum be realized as $\mathrm{THH}(C)$ for some $C \in \mathrm{Cat}^{\mathrm{perf}}$? (or equivalently, by [RSW], for some $C \in \mathrm{Mot}_{\mathrm{loc}}$?) ◁

Proof of Theorem 4.2.54. This follows essentially from Corollary 4.2.32: the map of trace theories

$$P_1 E^{\mathrm{cyc}} \to (\varphi_p^* P_1 E^{\mathrm{cyc}})^{tC_p}$$

is entirely determined by the corresponding map at $(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})$, which is a map $X_E \to X_E^{tC_p}$. Unwinding the definitions, one sees that this map is the cyclotomic Frobenius on X_E, and thus the above map simply agrees on $(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})$ with the composite

$$P_1 E^{\mathrm{cyc}} = X_E \otimes \mathrm{THH} \to X_E^{tC_p} \otimes (\varphi_p^* \mathrm{THH})^{tC_p} \to (X_E \otimes \varphi_p^* \mathrm{THH})^{tC_p} \to (\varphi_p^* P_1 E^{\mathrm{cyc}})^{tC_p}$$

as was to be shown. □

Remark 4.2.57. In forthcoming work [HNSa], Harpaz, Nikolaus and Saunier prove a version of the Lindenstrauss–McCarthy theorem, namely that $P_n E^{\mathrm{cyc}} \simeq \mathrm{TR}^n(P_1 E^{\mathrm{cyc}})$ where TR^n is taken with respect to the polygonic structure we defined above. From our perspective, this can be rewritten as $P_n E^{\mathrm{cyc}} \simeq \mathrm{TR}^n(X_E \otimes \mathrm{THH})$ and thus indicates that the whole infinitesimal theory of E is controlled by the single cyclotomic spectrum X_E. ◁

4.2.2 Addendum II: Sketching an a priori construction of Tr as a trace theory

In [HSS17], the authors produce a functor Tr from an ∞-category equivalent to our $\Lambda^{\mathrm{st}}_{[0]_\Lambda}$ to Sp which takes the symmetric monoidal *trace* of an endomorphism in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)}$. Our Remark 4.2.10 shows that at least objectwise, this Tr agrees with $P_1 K^{\mathrm{cyc}}$. In particular, since it is not difficult to check that Tr is cocontinuous in the bimodule variable, this shows that if Tr could be upgraded to a trace theory, it would be equivalent to $P_1 K^{\mathrm{cyc}}$ as such (and in particular as a full coherent functor on $\Lambda^{\mathrm{st}}_{[0]_\Lambda}$, rather than just objectwise). In this subsection, we sketch such a construction.

To actually give all details, we would need some preliminaries on enriched ∞-categories. Thus, we only sketch the construction of Tr. As in [HSS17], we proceed in the generality of a symmetric monoidal $(\infty, 2)$-category.

We will need:

[32] Obtained using the symmetric monoidal functor $i : \mathrm{CycSp} \to \mathrm{PgSp}$ from [KMN23].

Notation 4.2.58. For C an ∞-category, let $\mathrm{Fr}^{\mathrm{rig}}(C)$ denote the free symmetric monoidal ∞-category with duals on C, as in [HSS17]. ◁

Notation 4.2.59. Let B be a symmetric monoidal $(\infty,2)$-category. Following [HSS17], we let $\Omega B := \mathrm{End}(\mathbf{1}_B)$ be the symmetric monoidal ∞-category of endomorphisms of the unit of B. ◁

The key construction appears in [HSS17, Definition 2.9]: let B be a symmetric monoidal $(\infty,2)$-category, and we assume for simplicity that every object in B is dualizable. Fix an ∞-category C and an element $\alpha_C \in \Omega\mathrm{Fr}^{\mathrm{rig}}(C)$.

We then have the following composite:

$$\mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}(C,B) \simeq \mathrm{Fun}^{\otimes}_{\mathrm{colax}}(\mathrm{Fr}^{\mathrm{rig}}(C),B) \to \mathrm{Fun}^{\otimes}_{\mathrm{colax}}(\Omega\mathrm{Fr}^{\mathrm{rig}}(C),\Omega B) \xrightarrow{\mathrm{ev}_{\alpha_C}} \Omega B$$

The first two arrows are natural in C[33], and the third one depends on a choice $\alpha_C \in \Omega\mathrm{Fr}^{\mathrm{rig}}(C)$.

In the case of $C = B\mathbb{N}$, the walking endomorphism, one picks α_C to be the symmetric monoidal trace of that endomorphism in $\mathrm{Fr}^{\mathrm{rig}}(C)$.

In the case of a general $C \in \Lambda$, one would like to simply say : fix any basepoint in C, compose all the generating/indecomposable morphisms starting from that basepoint, and take the symmetric monoidal trace of that endomorphism. Since "traces are cyclically invariant", this is in fact independent of the choice of the basepoint.

If that choice of α_C can be made naturally in $C \in \Lambda$, then we get, by definition, an induced morphism $\Lambda_B[\mathrm{cocart}^{-1}] \to \Omega B$, i.e. a trace theory on B, with values in $\mathrm{End}(\mathbf{1}_B)$ which, on the fiber over $[0]_\Lambda$, definitionally agrees with that of [HSS17].

This choice *can* in fact be made natural in C, though the above sketch does not lend itself to that easily: we said "pick a basepoint and observe that the end result does not depend on it", which typically does not allow for proper, natural constructions. We explain below how to do so.

The key idea is that while the composite $0 \to 1 \to ... \to n \to 0$ in C, for C abstractly equivalent to $[n]_\Lambda$ is not canonical, its image in some kind of unstable HH of C *is*. We can use this together with a "realization map" going from HH to endomorphisms of the unit, realizing abstract traces as symmetric monoidal traces, to realize the desired class in $\Omega\mathrm{Fr}^{\mathrm{rig}}(C)$. The difficulty comes down to constructing this realization map. A version of this was done in limited generality in [Ram21], but I will adress details of the following construction in greater generality in future work.

For this generality, we need a bit of enriched ∞-category theory, and because in its current state it is quite technical, we only sketch the relevant constructions below, though they can be made precise (see, e.g. [Ber22]). The key thing we do not prove (though see *loc. cit.*) is the following:

Fact 4.2.60. Let $\mathcal{V}$ be a cocompletely symmetric monoidal ∞-category, S a space and C an S-flagged $\mathcal{V}$-category[34]. There is a Λ^{op}-shaped, $\mathcal{V}$-valued diagram of the form

$$B^{\mathrm{cyc}}_{\mathcal{V}}(C) : [n]_\Lambda \mapsto \mathrm{colim}_{x_0,\dots,x_n \in S} \hom_C(x_0,x_1) \otimes ... \otimes \hom_C(x_n,x_0)$$

called the cyclic bar construction for C.

It is natural in $(\mathcal{V},S,C)$. Furthermore, when $\mathcal{V} = \mathcal{S}$ and C is a ∞-category with the canonical flagging[35] $S = C^{\simeq}$, this diagram is equivalent to $[n]_\Lambda \mapsto \mathrm{map}([n]_\Lambda, C)$.

[33] See [HSS17, Lemma 2.4] for a discussion of the first equivalence.

[34] This is our name for what is called a "categorical algebra in $\mathcal{V}$ with space of objects S" in [GH15] - this is supposed to be like a $\mathcal{V}$-enriched ∞-category with a presentation of the space of objects given by S.

[35] In the case of $\mathcal{V} = \mathcal{S}$, an S-flagged $\mathcal{S}$-category is exactly an ∞-category C equipped with an essentially surjective functor $S \to C$. The canonical choice for S is thus $C^{\simeq}$.

Notation 4.2.61. We let $\mathrm{HH}^{\mathrm{bar}}_{\mathcal{V}}(C)$ denote the geometric realization of $B^{\mathrm{cyc}}_{\mathcal{V}}(C)_{|\Delta^{\mathrm{op}}}$. ◁

Note that by Construction A.0.13, $\mathrm{HH}^{\mathrm{bar}}_{\mathcal{V}}(C)$ admits an S^1-action, and

$$\mathrm{HH}^{\mathrm{bar}}_{\mathcal{V}}(C)_{hS^1} \simeq \mathrm{colim}_{\Lambda^{\mathrm{op}}} B^{\mathrm{cyc}}_{\mathcal{V}}(C)$$

The following is then a simple calculation:

Proposition 4.2.62. *Let $C \in \Lambda$ be viewed as a category. We have a functorial decomposition*

$$\mathrm{HH}^{\mathrm{bar}}_{\mathcal{S}}(C)_{hS^1} \simeq \mathrm{pt} \coprod F_{\neq 1}(C)$$

Proof. The cyclic bar construction is $\Lambda^{\mathrm{op}} \to \mathcal{S}, D \mapsto \mathrm{map}_{\mathrm{Cat}}(D, C)$. Recall that $\mathrm{HH}^{\mathrm{bar}}_{\mathcal{S}}(C)_{hS^1} \simeq \mathrm{colim}_{\Lambda^{\mathrm{op}}} B^{\mathrm{cyc}}_{\mathcal{S}}(C)$.

Now, given $D \in \Lambda$, $\mathrm{map}_{\mathrm{Cat}}(D, C) \simeq \mathrm{map}_{\Lambda}(D, C) \coprod \mathrm{map}^{\neq 1}(D, C)$ where $\mathrm{map}^{\neq 1}(D, C)$ is the full subspace of maps that induce a degree $\neq 1$ map from $|D|$ to $|C|$. Both these subspaces are subfunctors as they are closed under maps between varying C's *in* Λ.

Now colimits of representable functors are contractible, so we are done by letting $F_{\neq 1}(C) := \mathrm{colim}_{\Lambda^{\mathrm{op}}} \mathrm{map}^{\neq 1}(-, C)$. □

Remark 4.2.63. This is the statement that, up to the S^1-action, the class of long composites in C is canonical in "$\mathrm{HH}^{\mathrm{unstable}}(C)$". ◁

The final thing we need to know about $\mathrm{HH}^{\mathrm{bar}}_{\mathcal{V}}$ is the following:

Proposition 4.2.64. *Let $\mathcal{V}_0 \subset \mathcal{V}$ be a subcategory consisting of dualizable objects in $\mathcal{V}$, and containing $\mathbf{1}_{\mathcal{V}}$. In this case $\mathcal{V}$ admits internal homs for objects of $\mathcal{V}_0$, so $\mathcal{V}_0$ is canonically $\mathcal{V}$-enriched. Furthermore, in this case $\mathrm{HH}^{\mathrm{bar}}_{\mathcal{V}}(\mathcal{V}_0) \simeq \mathbf{1}_{\mathcal{V}}$ with a trivial S^1-action.*

Sketch of proof. The first part is classical, see e.g. [GH15, Corollary 7.4.10]. The second part is also classical, and ultimately comes from a similar proof as in Theorem 4.2.11, see in particular the discussion right after the statement.

The idea is that since every object in $\mathcal{V}_0$ is dualizable,

$$\hom(x_n, x_0) \simeq \hom(x_n, \mathbf{1}) \otimes \hom(\mathbf{1}, x_0)$$

so that $B^{\mathrm{cyc}}_{\mathcal{V}}(\mathcal{V}_0)_n$ can be viewed as a "chunk" of $B^{\mathrm{cyc}}_{\mathcal{V}}(\mathcal{V}_0)_{n+1}$. More precisely, this equivalence provides $B^{\mathrm{cyc}}_{\mathcal{V}}(\mathcal{V}_0)_{|\Delta^{\mathrm{op}}}$ with an extra degeneracy as in [Lur09, Proposition 6.1.3.16], making it an absolute colimit with value $\mathbf{1}_{\mathcal{V}}$.

The triviality of the S^1-action comes from the fact that there is a natural map of Λ^{op}-diagrams $\mathbf{1}_{\mathcal{V}} \to B^{\mathrm{cyc}}_{\mathcal{V}}(\mathcal{V}_0)$ induced by the $\mathcal{V}$-full inclusion of flagged $\mathcal{V}$-categories $(\mathrm{pt}, \{\mathbf{1}_{\mathcal{V}}\}) \to (\mathcal{V}_0^{\simeq}, \mathcal{V}_0)$ which induces the given equivalence upon realization. □

Construction 4.2.65. Let $\mathcal{V}$ be a symmetric monoidal ∞-category in which every object is dualizable. We construct a map, natural in $\mathcal{V}$,

$$\mathrm{HH}^{\mathrm{bar}}_{\mathcal{S}}(\mathcal{V})_{hS^1} \to \mathrm{End}(\mathbf{1}_{\mathcal{V}})$$

such that the restriction to the 0-simplices $\mathrm{Map}(x_0, x_0)$ is given by the symmetric monoidal trace.

We do it as follows: first, we may consider $\mathcal{V}$ canonically as a $\mathcal{V}$-enriched ∞-category (it has all duals) and hence, via the Yoneda embedding, as a $\mathrm{Psh}(\mathcal{V})$-enriched ∞-category. By adjunction, it suffices to produce an S^1-equivariant map $\mathbf{1}_{\mathcal{V}} \otimes \mathrm{HH}^{\mathrm{bar}}_{\mathcal{S}}(\mathcal{V}) \to \mathbf{1}_{\mathcal{V}}$ in[36] $\mathrm{Psh}(\mathcal{V})$. For this we simply note that the naturality of $\mathrm{HH}^{\mathrm{bar}}_{\mathcal{E}}$ in the $\mathcal{E}$ variable implies that if D is an ordinary category, then there is an equivalence in $\mathrm{Psh}(\mathcal{V})$:

$$\mathrm{HH}^{\mathrm{bar}}_{\mathrm{Psh}(\mathcal{V})}(\mathbf{1}_{\mathcal{V}} \otimes D) \simeq \mathbf{1}_{\mathcal{V}} \otimes \mathrm{HH}^{\mathrm{bar}}_{\mathcal{S}}(D)$$

where $\mathbf{1}_{\mathcal{V}} \otimes D$ is the flagged $\mathrm{Psh}(\mathcal{V})$-category obtained by taking the same objects as D and for morphism objects $\hom_{\mathbf{1}_{\mathcal{V}} \otimes D}(d_0, d_1) := \mathbf{1}_{\mathcal{V}} \otimes \mathrm{map}(d_0, d_1)$. This construction participates in an adjunction between $\mathrm{Psh}(\mathcal{V})$-enriched ∞-categories and ordinary ∞-categories, and in particular we have a counit morphism $\mathbf{1}_{\mathcal{V}} \otimes \mathcal{V}_0 \to \mathcal{V}$ where $\mathcal{V}_0 = \mathcal{V}$ is the underlying (= non-enriched) ∞-category of $\mathcal{V}$.

This counit morphism induces the desired S^1-equivariant map

$$\mathbf{1}_{\mathcal{V}} \otimes \mathrm{HH}^{\mathrm{bar}}_{\mathcal{S}}(\mathcal{V}) \to \mathbf{1}_{\mathcal{V}}$$

upon taking $\mathrm{HH}^{\mathrm{bar}}_{\mathrm{Psh}(\mathcal{V})}$, since $\mathrm{HH}^{\mathrm{bar}}_{\mathrm{Psh}(\mathcal{V})}(\mathcal{V}) \simeq \mathbf{1}_{\mathcal{V}}$ by assumption that $\mathcal{V}$ consists of dualizable objects. $\triangleleft$

With these facts in hand, we can give a construction of $\alpha_C : \mathrm{pt} \to \Omega\mathrm{Fr}^{\mathrm{rig}}(C)$ natural in $C \in \Lambda$, as follows:

Definition 4.2.66. We let α_C denote the following composite:

$$\mathrm{pt} \to \mathrm{HH}^{\mathrm{bar}}_{\mathcal{S}}(C)_{hS^1} \to \mathrm{HH}^{\mathrm{bar}}_{\mathcal{S}}(\mathrm{Fr}^{\mathrm{rig}}(C))_{hS^1} \to \Omega\mathrm{Fr}^{\mathrm{rig}}(C)$$

where the first map is the one arising from the decomposition in Proposition 4.2.62.

$\triangleleft$

We now need to check that α_C hits exactly the symmetric monoidal trace of any choice of long composite $0 \to 1 \to \ldots \to n \to 0$ in C. By naturality it suffices to do so for $C = [0]_\Lambda$, and there it suffices to note that the proof that $\mathrm{HH}^{\mathrm{bar}}_{\mathcal{V}}(\mathcal{V}) \simeq \mathbf{1}_{\mathcal{V}}$ exactly uses the equivalence $\hom(x_0, x_0) \simeq \hom(x_0, \mathbf{1}) \otimes \hom(\mathbf{1}, x_0)$ to get an extra degeneracy, and it is this equivalence that computes traces at the end of the day.

Thus in total we have constructed α_C, and thus $\mathrm{Tr} : \Lambda_B[\mathrm{cocart}^{-1}] \to \mathrm{End}(\mathbf{1}_B)$, as was to be done. The previous paragraph shows that the local functoriality $\mathrm{Map}(b, b) \to \mathrm{End}(\mathbf{1}_B)$ really is given by the usual trace, and so in the case of $B = \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},(\omega)}$, Theorem 4.2.11 directly implies:

Corollary 4.2.67. *There is an equivalence of trace theories*

$$P_1 K^{\mathrm{cyc}} \simeq \mathrm{Tr}_{\mathrm{Sp}}$$

In particular, if we accept that the missing details can be filled, we obtain a canonical (unique) comparison between THH as defined in [HS19] and $\mathrm{Tr}_{\mathrm{Sp}}$ as defined in [HSS17].

Remark 4.2.68. We note that there is a map

$$\mathrm{map}(S^1, B) \simeq \mathrm{map}(|C|, B) \to \mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}(C, B)$$

natural in $C \in \Lambda^{\mathrm{op}}$, which therefore induces an S^1-equivariant map $\mathrm{map}(S^1, B) \to \Delta_B[\mathrm{cocart}^{-1}]$. Composing this with the S^1-equivariant map $\mathrm{Tr} : \Delta_B[\mathrm{cocart}^{-1}] \to \Omega B$ which we have just produced yields, in total, an S^1-equivariant map

$$\mathrm{Tr} : \mathrm{map}(S^1, B) \to \Omega B$$

[36] We suppressed the Yoneda embedding from the notation.

The uniqueness that is proved in [HSS17, Theorem 2.14] (see the last paragraph of the proof) shows that this S^1-equivariant structure agrees with the one constructed in *loc. cit.*. Alternatively, one can also simply unwind the constructions and see that they yield the same S^1-equivariant structure[37]. ◁

We do not go much further in this direction as it would again lead us astray, but these constructions and ideas will be explored further (and expanded upon) in future work.

4.2.3 Addendum III: Connectivity estimates

We conclude this section with the promised digression on connectivity estimates, namely, we explain how to recover, from the abstract statement that $P_1 K^{\mathrm{cyc}} \simeq \mathrm{THH}$, precise connectivity estimates on the maps $\Omega^n \tilde{K}(A \oplus \Sigma^{n-1} M) \to \mathrm{THH}(A, M)$ for A a connective ring spectrum and M a connective A-bimodule.

First, our main tool will be the following classical result:

Lemma 4.2.69 ([LT19, Lemma 2.4])**.** *Suppose $f : R \to S$ is a map of connective ring spectra, which is k-connective for some $k \geq 1$ (in particular $\pi_0(f)$ is an isomorphism).*

Then the induced morphism $K(R) \to K(S)$ is $(k+1)$-connective.

Second, we recall the main theorem of [LT19]:

Theorem 4.2.70 ([LT19, Main theorem])**.** *Consider a pullback square of ring spectra:*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

There exists a ring spectrum (the "circle-dot ring") $B \odot_A^D C$ together with a commutative diagram of ring spectra

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \arrow[dd, bend left] \\ C \arrow[r] \arrow[dr, bend right] & B \odot_A^D C \arrow[d] \\ & D \end{tikzcd}$$

such that the homotopy filling in the outer square is the one from the start, such that, as a map of (B, C)-bimodule spectra, $B \odot_A^D C \to D$ is identified with $B \otimes_A C \to D$ and finally such that the inner square induces a pullback square of K-theory spectra:

$$\begin{array}{ccc} K(A) & \longrightarrow & K(B) \\ \downarrow & & \downarrow \\ K(C) & \longrightarrow & K(B \odot_A^D C) \end{array}$$

An important point is that even when $B \odot_A^D C \to D$ is not an equivalence, if everything is connective, then connectivity estimates can be used to see how far the original square is from

[37]Surprisingly, this produces an S^1-equivariant trace *without* using the cobordism hypothesis at all.

being a K-theory pullback. Indeed, if all the rings involved are connective, and $B \odot_A^D C \to D$ is k-connective for some $k \geq 1$, then

$$K(B \odot_A^D C) \to K(D)$$

is $(k+1)$-connective, by the previous result. This specializes to:

Corollary 4.2.71. *Suppose R is a connective ring spectrum, M a k-connective R-bimodule and let $R \oplus M$ denote the trivial square-zero extension of R by M.*

Then the pullback square

$$\begin{tikzcd} R \oplus M \arrow[r] \arrow[d] & R \arrow[d] \\ R \arrow[r] & R \oplus \Sigma M \end{tikzcd}$$

induces a $2(k+1)$-connective map $K(R \oplus M) \to \Omega_{K(R)} K(R \oplus \Sigma M)$, where

$$\Omega_{K(R)} K(R \oplus \Sigma M) := K(R) \times_{K(R \oplus \Sigma M)} K(R)$$

Proof. Consider a diagram of spectra as follows:

$$\begin{tikzcd} & X \arrow[d] \arrow[dd, bend left] \\ X \arrow[r] \arrow[dr, bend right] & Y \arrow[d] \\ & Z \end{tikzcd}$$

Then if $Y \to Z$ is j-connective, the induced morphism $X \times_Y X \to X \times_Z X$ is $(j-1)$-connective. Indeed, fibers commute with pullbacks, so that the fiber of this morphism is $\Omega \mathrm{fib}(Y \to Z)$, which is $(j-1)$-connective as claimed.

We apply this to

$$\begin{tikzcd} & K(R) \arrow[d] \arrow[dd, bend left] \\ K(R) \arrow[r] \arrow[dr, bend right] & K(R \odot_{R \oplus M}^{R \oplus \Sigma M} R) \arrow[d] \\ & K(R \oplus \Sigma M) \end{tikzcd}$$

So it now suffices to show that the map on bottom right corners is $2(k+1)+1$-connective, and so it suffices to show that $R \otimes_{R \oplus M} R \to R \oplus \Sigma M$ is $2(k+1)$-connective.

We do so by putting a grading everywhere: consider $R \oplus M$ and $R \oplus \Sigma M$ as graded ring spectra, where M is put in grading 1 and R in grading 0.

In that case, everything can be graded, and we claim that the map

$$R \otimes_{R \oplus M(1)} R \to R \oplus \Sigma M(1)$$

is an equivalence in grading ≤ 1, where $X(n)$ denotes the spectrum X in grading n. Indeed, there is a map $T_R(M(1)) \to R \oplus M(1)$ from the free algebra in R-bimodules which is an equivalence in grading ≤ 1, so that $R \otimes_{T_R(M(1))} R \to R \otimes_{R \oplus M(1)} R$ is also an equivalence in grading ≤ 1 (note that everything is nonnegatively graded so the grading ≤ 1 part of these relative tensor products only depends on the grading ≤ 1 part of the terms); and [LT23, Lemma 4.3] shows that the composite

$$R \otimes_{T_R(M(1))} R \to R \otimes_{R \oplus M(1)} R \to R \oplus \Sigma M(1)$$

is an equivalence (in all gradings!).

Thus the right hand map is also an equivalence in grading ≤ 1. But the source of this map, when realized, is a direct sum of its grading ≤ 1 part with its grading ≥ 2 part. The grading w part is the geometric realization of a simplicial object which starts in simplicial degree w with all objects of the form $R^{\otimes ?} \otimes M(1)^{\otimes w+?}$ which are therefore $\geq wk$-connective. Therefore the grading w part is $\geq wk + w = w(k+1) \geq 2(k+1)$-connective, as claimed. □

Remark 4.2.72. There is a version of this claim for nontrivial square-zero extensions, which is convenient for Waldhausen's [Wal78] because it allows one to use for Postnikov square-zero extensions. We will not go this far here, and so we stick to this simpler claim. Note that in the non-split square zero extension case, I only know how to prove that the induced map is $2k+1$-connective, as opposed to $2(k+1)$, because of $\lim^1$-issues. ◁

Corollary 4.2.73. *Let R be a connective ring, M a k-connective R-bimodule. Then the canonical map $\tilde{K}(R, M) \to \Omega\tilde{K}(R, \Sigma M)$ is $2(k+1)$-connective.*

Proof. Simply take the fiber of $K(R \oplus M) \to \Omega_{K(R)} K(R \oplus \Sigma M)$ over $K(R)$. □

We can now finally get the same connectivity estimates as in [DM94]:

Corollary 4.2.74. *For any connective ring spectrum R and k-connective bimodule M, the map $\tilde{K}(R, M) \to \mathrm{THH}(R, \Sigma M)$ coming from the Dundas–McCarthy theorem is $2(k+1)$-connective .*

Proof. The previous corollary implies that all the maps

$$\tilde{K}(R, M) \to \Omega\tilde{K}(R, \Sigma M) \to \Omega^2\tilde{K}(R, \Sigma^2 M) \to \dots$$

are $2(k+1)$-connective, therefore so is the map to colimit, which is, by the Dundas–McCarthy theorem, identified with $\tilde{K}(R, M) \to \mathrm{THH}(R, \Sigma M)$. □

Chapter 5

Endomorphisms of THH

Introduction

In Chapter 4, we saw that THH is very rigid as a trace theory, because it has a very clean universal property. However, restricting it to inputs of the form (C, id_C) turns it into a localizing invariant with a much more complex universal property.

The goal of this chapter is to study the endomorphisms of the localizing invariant we obtain this way. Similar questions have been studied by Wahl–Westerland [WW16], Wahl [Wah16] and Klamt [Kla15]. In these works, the authors study what they call "formal operations", which can be viewed as an approximation to the spectrum of operations, of Hochschild homology relative to a base commutative ring R, evaluated on dg algebras over R. While I am not able to access endomorphisms of THH as a functor on ring spectra, I do give a complete description of the actual endomorphism spectrum of THH as a functor of stable ∞-categories, as well as of the endomorphism monoid of THH as a symmetric monoidal functor.

As a plain functor, the answer looks as follows, which is Theorem H from the introduction:

Theorem. *As a plain functor* $\mathrm{THH} : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Sp}$*, the* S^1*-action induces an equivalence*

$$\mathbb{S}[S^1] \simeq \mathrm{end}(\mathrm{THH})$$

As a symmetric monoidal functor, there is an equivalence

$$\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp})}(\mathbb{S}^{S^1}, \mathbb{S}) \simeq \mathrm{End}^{\otimes}(\mathrm{THH})$$

and the space $\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp})}(\mathbb{S}^{S^1}, \mathbb{S})$ *can be described*[1]*.*

I also prove analogous results in relative contexts, that is, for THH relative to a base commutative ring spectrum k.

I use these results as well as other results from my work with Carmeli–Cnossen–Yanovski [CCR+23] and with Sosnilo–Winges [RSW24] to study analogous questions, related to the more general questions studied in the works of Wahl–Westerland, Wahl and Klamt mentioned above, namely, to study the endomorphism spectrum of THH viewed as a functor on $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathrm{perf}})$ for some (single-colored) ∞-operad $\mathcal{O}$. The integral answer there seems to be more subtle, and I only get partial results, but I also get a full description of these endomorphisms if one suitably localizes THH, e.g. rationally. The following is Theorem J from the introduction, where L denotes either rationalization, or $T(n)$-localization for some height

[1] But it is not exactly S^1.

$n \geq 1$ and implicit prime p - here, γ_k denotes the (unique up to conjugacy) length k cycle in Σ_k, and $\mathcal{O}(k)^{\gamma_k}$ is the space of $\mathbb{Z}$-fixed points of $\mathcal{O}(k)$ with automorphism given by the action of γ_k (it retains a C_k-action, where C_k is the centralizer of γ_k in Σ_k, in this case the subgroup generated by γ_k):

Theorem. *For any single-colored ∞-operad $\mathcal{O}$, there is a canonical map*

$$\bigoplus_{k\geq 1} \mathbb{S}[(\mathcal{O}(k)^{\gamma_k} \times S^1)_{hC_k}] \to \mathrm{end}_{\mathrm{Fun}(\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathrm{perf}}),\mathrm{Sp})}(\mathrm{THH})$$

such that :

(i) For any finite set $S \subset \mathbb{N}_{\geq 1}$, the restriction to $\bigoplus_{k\in S}$ admits a splitting;

(ii) The induced map on L-localization, specifically:

$$L(\bigoplus_{k\geq 1} \mathbb{S}[(\mathcal{O}(k)^{\gamma_k} \times S^1)_{hC_k}]) \to \mathrm{end}_{\mathrm{Fun}(\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathrm{perf}}),\mathrm{Sp})}(L\mathrm{THH})$$

is an equivalence.

I also prove, by studying the "spectrum of cyclotomic Frobenii", that the relevant map is *not* an equivalence in the integral case (though this is not visible in [Kla15] precisely because the cyclotomic Frobenius does not exist over $\mathbb{Z}$).

Outline

In Section 5.1 I set up some preliminaries regarding Day convolution that will be convenient for the later sections. In Section 5.2, I apply these results to compute plain endomorphisms of THH, and I study the relative setting in Section 5.3. Finally, in Section 5.4 I initiate the study of operations for stable ∞-categories equipped with multiplicative structures.

5.1 Day convolution of localizing invariants

Since THH admits a multiplication, one way to compute endomorphisms of THH is to compute THH-linear maps out of "THH $\otimes$ THH" into THH, for some meaning of $\otimes$. The relevant meaning here is the one that encodes the symmetric monoidal structure on the functor THH, as opposed to that on each of its values[2]. The relevant tensor product for this is Day convolution, which is known to encode symmetric monoidal structures. We will not enter into too many details about it, but there is also a version of the Day tensor product for localizing invariants, which is better suited for our purposes.

The goal of this section is therefore to prove the following, which will be the key tool to study endomorphisms of THH:

Proposition 5.1.1. *Let $\mathcal{E}$ be a cocomplete stable ∞-category and $E : \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{E}$ be an accessible localizing invariant. The Day convolution* as localizing invariants $\mathrm{THH} \otimes^{\mathrm{Day}} E$ *of* THH *with E is naturally equivalent to $C \mapsto P_1 E^{\mathrm{cyc}}(C, \mathrm{id}_C)$.*

Remark 5.1.2. We stress that the usual Day convolution of THH and E need not be a localizing invariant. What we mean by "the Day convolution *as localizing invariants*" is what is obtained by first taking ordinary Day convolution and then taking the universal localizing invariant with a map from this Day convolution. Through the ∞-category of localizing motives, this is closely related to "exactified Day convolution", which we briefly discuss below. ◁

[2] Indeed, on THH of an arbitrary stable ∞-category, there is no such structure !

Remark 5.1.3. One can state and prove a refined version of this result - implying in particular an S^1-equivariant version of it, as both sides have a natural S^1-action. However, it requires setting up more technicalities, more than seem worth it for our purposes, so we leave the refined statement for the future. ◁

To prove this, we use the following preparatory lemma:

Lemma 5.1.4. *Let C be a presentably symmetric monoidal ∞-category, and let $x \in C$. For any accessible[3] functor $F : C \to \mathcal{E}$ to a cocomplete ∞-category $\mathcal{E}$, there is a natural equivalence*

$$\mathrm{Map}(x, -) \otimes^{\mathrm{Day}} F \simeq F(\hom(x, -))$$

Proof. Let G be any functor $C \to \mathcal{E}$, and let $\mu : C \times C \to C$ denote the tensor product, $\mathrm{pr}_i : C \times C \to C$ the two projection maps. By definition of the Day tensor product, there is an equivalence

$$\mathrm{Map}(\mathrm{Map}(x, -) \otimes^{\mathrm{Day}} F, G) \simeq \mathrm{Map}(\mathrm{Map}(x, \mathrm{pr}_1) \boxtimes F \circ \mathrm{pr}_2, G \circ \mu)$$

where we temporarily use $\boxtimes$ to distinguish the external tensor product from the Day tensor product.

By currying, this is equivalent to

$$\mathrm{Map}(\mathrm{Map}(x, -), \mathrm{Map}(F, G \circ \mu))$$

where the target is $y \mapsto \mathrm{Map}(F, G(y \otimes -))$.

By the Yoneda lemma, this is equivalent to $\mathrm{Map}(F, G(x \otimes -))$ and by adjunction, this is equivalent to $\mathrm{Map}(F(\hom(x, -)), G)$, as was to be proved. □

If $C, \mathcal{E}$ are stable and F, G are exact functors, $F \otimes^{\mathrm{Day}} G$ need not be exact a priori. However, one can then take its first derivative to make it exact, and this induces a Day convolution monoidal structure on $\mathrm{Fun}^{\mathrm{ex,acc}}(C, \mathcal{E})$ - see e.g. [HR21b, Theorem 4.5] in the case where the source is small, but the generalization to accessible functors is straightforward. We call it the exactified Day convolution.

In that case, there is an analogue of the previous lemma when we consider instead the mapping *spectrum* out of x, and the exactified Day convolution. The proof is the same[4], so we only state it:

Corollary 5.1.5. *Let C be a stable presentably symmetric monoidal ∞-category, and $x \in C$ be an object admitting internal homs $\hom(x, -)$. For any accessible exact functor $F : C \to \mathcal{E}$ to a cocomplete* stable *∞-category $\mathcal{E}$, there is a natural equivalence $\mathrm{map}(x, -) \otimes^{\mathrm{Day}} F \simeq F(\hom(x, -))$, where now* map *denotes the mapping spectrum functor, and $\otimes^{\mathrm{Day}}$ denotes the exactified Day convolution product.*

To apply this to THH, we introduce a bit of notation. Notice that for $C \in \mathrm{Cat}^{\mathrm{perf}}$, $\mathrm{End}(C, \Sigma^n)$ can be described as $\mathrm{Fun}^{\mathrm{ex}}(A_n, C)$ for some $A_n \in \mathrm{Cat}^{\mathrm{perf}}$. Specifically, A_n is easily verified to be $\mathbf{Perf}(\mathbb{S}[\sigma_{-n}])$, where $\mathbb{S}[\sigma_{-n}]$ is the free associative algebra on a class in degree $-n$. Indeed, a map $x \to \Sigma^n x$ is the same as a map $\Omega^n x \to x$, which is the same as a map $\mathbb{S}[\sigma_{-n}] \to \mathrm{map}(x, x)$.

Notation 5.1.6. Let $A_n := \mathbf{Perf}(\mathbb{S}[\sigma_{-n}])$ as discussed above. ◁

[3] Here accessibility is just used to deal with size issues, namely to make sure that the Day tensor product is defined *a priori*.

[4] Or in fact it can be seen to follow from the previous lemma, as $\mathrm{map}(x, -) = P_1 \mathbb{S}[\mathrm{Map}(x, -)]$.

With this notation, we can state Theorem 4.2.1 as follows:

Corollary 5.1.7. *There is a natural equivalence*

$$\mathrm{THH} \simeq \mathrm{colim}_n \Omega^n \tilde{K}(\mathrm{Fun}^{\mathrm{ex}}(A_n, -))$$

If we rephrase it by viewing THH as a functor on $\mathrm{Mot}_{\mathrm{split}}$, the ∞-category of splitting motives[5], we can interpret it as follows with a direct application of Corollary B.0.10:

Corollary 5.1.8. *Considering* THH *as a functor on* $\mathrm{Mot}_{\mathrm{split}}$, *letting* $\tilde{\mathcal{A}}_n$ *denote* $\mathcal{U}_{\mathrm{split}}(A_n)/\mathcal{U}_{\mathrm{split}}(Sp^\omega)$, *we have:*

$$\mathrm{THH} \simeq \mathrm{colim}_n \Omega^n \operatorname{map}(\tilde{\mathcal{A}}_n, -)$$

Thus, directly using the formula from Corollary 5.1.5, we find:

Corollary 5.1.9. *Let* $E : \mathrm{Mot}_{\mathrm{split}} \to \mathcal{E}$ *be an exact accessible functor to a cocomplete stable* ∞*-category. With exactified Day convolution, we have an equivalence:*

$$\mathrm{THH} \otimes^{\mathrm{Day}} E \simeq P_1 E^{\mathrm{cyc}}$$

Proof. Using Corollary 5.1.5 and Corollary 5.1.8, we find

$$\mathrm{THH} \otimes^{\mathrm{Day}} E \simeq \mathrm{colim}_n \Omega^n \tilde{E}(\mathrm{Fun}^{\mathrm{ex}}(A_n, -)) \simeq \mathrm{colim}_n \Omega^n \tilde{E}(\mathrm{End}(-; \Sigma^n)) =: P_1 E^{\mathrm{cyc}}$$

□

Proof of Proposition 5.1.1. We first note that the functor $\mathrm{Mot}_{\mathrm{split}} \to \mathrm{Mot}_{\mathrm{loc}}$ witnessing that every localizing invariant is a splitting invariant is a localization.

Thus, if E, F are localizing invariants such that their Day convolution *as splitting invariants* is already localizing, it is also their Day convolution *as localizing invariants*.

We also note that if E happened to be a localizing invariant, then by Proposition 4.1.40,Proposition 4.1.54, Proposition 4.1.53, $P_1 E^{\mathrm{cyc}}$ is already localizing. Thus in this situation, the calculation from Corollary 5.1.9 works for Day convolution *as splitting invariants* or *as localizing invariants*, which is what we wanted to prove.

□

We point out another, more naive consequence of Corollary 5.1.8, which can sometimes be convenient:

Corollary 5.1.10. *Let E be a splitting invariant. There is an equivalence:*

$$\operatorname{map}(\mathrm{THH}, E) \simeq \lim \Sigma^n \tilde{E}(A_n)$$

Proof. By adjunction, $\operatorname{map}(K(\mathrm{Fun}^{\mathrm{ex}}(A_n, -)), E) \simeq \operatorname{map}(K, E(A_n \otimes -))$ and so, by the universal property of K-theory from [BGT13], the latter is $E(A_n)$.

Putting things together as n varies gives the result. □

Remark 5.1.11. By Remark 4.1.63, we can use either connective or nonconnective K-theory here, which is why E is allowed to be a splitting invariant as opposed to a localizing invariant.

◁

[5]These are typically called "additive motives" in the literature. This, especially the corresponding notion of "additive invariant" sounds very confusing and less descriptive than "splitting", so I have opted for this change of name. I have gathered a few helpful facts in Appendix B.

So while tensoring with THH acts as a Goodwillie derivative, mapping out of THH acts as some kind of *co*-derivative.

Corollary 5.1.12. *Evaluation at* Sp^ω *induces a symmetric monoidal equivalence*

$$\mathrm{Mod}_{\mathrm{THH}}(\mathrm{Fun}^{\mathrm{loc},\omega}(\mathrm{Cat}^{\mathrm{perf}},\mathrm{Sp})) \simeq \mathrm{Sp}$$

Proof. Evaluation at Sp^ω is represented by THH on this ∞-category, so this functor admits a left adjoint given by $X \mapsto X \otimes \mathrm{THH}$.

Furthermore, evaluation at Sp^ω preserves filtered colimits and hence THH is a compact object in that module ∞-category. Thus, since

$$\mathrm{end}_{\mathrm{THH}}(\mathrm{THH}) \simeq \mathrm{Map}(K,\mathrm{THH}) \simeq \mathbb{S}$$

this left adjoint is fully faithful.

Now for any finitary localizing invariant E, $E \otimes^{\mathrm{Day}} \mathrm{THH} \simeq P_1 E^{\mathrm{cyc}}$ by Proposition 5.1.1 and by Theorem 4.2.3 the latter is $X_E \otimes \mathrm{THH}$ for some fixed spectrum X_E (note that we are not considering the trace theory structure here, so X_E is simply a spectrum, no S^1-action in sight).

It follows that the image of the left adjoint $\mathrm{Sp} \to \mathrm{Mod}_{\mathrm{THH}}(\mathrm{Fun}^{\mathrm{loc},\omega}(\mathrm{Cat}^{\mathrm{perf}},\mathrm{Sp}))$ contains all the free modules, i.e. contains a collection of generators under colimits. Since it is fully faithful and colimit-preserving, it follows that it is an equivalence.

Both sides are symmetric monoidal ∞-categories, and Sp has a unique symmetric monoidal structure with $\mathbb{S}$ as the unit, so the symmetric monoidal claim follows as well. □

5.2 Plain endomorphisms of THH

In this section, we prove Theorem H, that is, we compute endomorphisms of THH as a plain functor, as well as its endomorphisms as a lax symmetric monoidal functor. Using a extensions-restriction of scalars adjunction, this will ultimately boil down to a computation of $\mathrm{THH} \otimes^{\mathrm{Day}} \mathrm{THH}$, which the previous section together with Chapter 4 have given us the tools to study.

Thus we begin with the following well-known computation about THH - to limit confusion, in the statement and its proof, we let T denote the *trace theory* "topological Hochschild homology", and THH denote the corresponding localizing invariant. :

Proposition 5.2.1. *There is an equivalence of trace-theories with* S^1*-action:*

$$P_1\mathrm{THH}^{\mathrm{cyc}} \simeq \mathrm{T}^{S^1}$$

where the S^1*-action on the right is coinduced.*

Proof. It suffices to evaluate at $(\mathrm{Sp}, \mathrm{id}_{\mathrm{Sp}})$ by Theorem 4.2.11, where this is a classical computation (more generally in the case of ring spectra), cf. e.g. [Hes94, Proposition 3.2] (there it says $\mathbb{S}[S^1] \otimes \mathrm{T}$, but the shift here is the same as in Remark 4.1.64).

Ultimately, the point as explained in *loc. cit.* is that the (homogeneous) degree 1 part of the cyclic bar construction computing $\mathrm{THH}(\mathbb{S} \oplus \Sigma^{n-1}\mathbb{S})$ is a free cyclic object on the (non-cyclic) bar construction computing $\mathrm{THH}(\mathbb{S}, \Sigma^{n-1}\mathbb{S})$, and so its realization has an induced S^1-action on $\mathbb{S}$. There is a shift involved, and the relation $\Sigma\mathbb{S}^{S^1} \simeq \mathbb{S}[S^1]$ gets us to the announced result.

Alternatively, one can use the general formula for THH of square zero extensions which we recall later in Proposition 5.4.12. □

Corollary 5.2.2. *There is an S^1-equivariant,* THH*-linear equivalence*

$$\mathrm{THH} \otimes^{\mathrm{Day}} \mathrm{THH} \simeq \mathrm{THH}^{S^1}$$

where:

- $\mathrm{THH} \otimes^{\mathrm{Day}} \mathrm{THH}$ *is the extension of scalars to* THH *of* THH *with its S^1-action;*
- THH^{S^1} *is the coinduced S^1-action on the unit in* THH*-modules.*

Remark 5.2.3. The same refinement as in Remark 5.1.3 would allow us to make this equivalence $S^1 \times S^1$-equivariant, but we will not need this, and as mentioned there, the technicalities needed for this statement seem to outweigh the benefits. ◁

Proof. By Proposition 5.1.1 and Proposition 5.2.1, we know that there is an equivalence in $\mathrm{Fun}^{\mathrm{loc},\omega}(\mathrm{Cat}^{\mathrm{perf}}, \mathrm{Sp})^{BS^1}$ between the two.

By Corollary 5.1.12, the equivalence $\mathrm{Mod}_{\mathrm{THH}}(\mathrm{Fun}^{\mathrm{loc},\omega}(\mathrm{Cat}^{\mathrm{perf}}, \mathrm{Sp}))^{BS^1}$ is given by evaluation at Sp^ω, and therefore factors through the forgetful functor $\mathrm{Mod}_{\mathrm{THH}}(\mathrm{Fun}^{\mathrm{loc},\omega}(\mathrm{Cat}^{\mathrm{perf}}, \mathrm{Sp}))^{BS^1} \to \mathrm{Fun}^{\mathrm{loc},\omega}(\mathrm{Cat}^{\mathrm{perf}}, \mathrm{Sp})^{BS^1}$. Thus there is an equivalence between the two, as claimed. □

In fact, this can be upgraded to a multiplicative equivalence:

Corollary 5.2.4. *There is an equivalence of commutative* THH*-algebras with S^1-action*

$$\mathrm{THH} \otimes^{\mathrm{Day}} \mathrm{THH} \simeq \mathrm{THH}^{S^1}$$

Proof. By passing to Sp under the equivalence from Corollary 5.1.12, with $A = \mathrm{THH} \otimes^{\mathrm{Day}} \mathrm{THH}$ (or more precisely its image in Sp), we have the following situation:

(i) There is a commutative algebra $A \in \mathrm{CAlg}(\mathrm{Sp}^{BS^1})$;
(ii) Its underlying object $UA \in \mathrm{Sp}^{BS^1}$ is coinduced, specifically of the form $\mathbb{S}^{S^1}$;
(iii) There is a map $A \to \mathbb{S}$ in $\mathrm{CAlg}(\mathrm{Sp})$.

The algebra map $A \to \mathbb{S}$ comes from the multiplication map

$$\mathrm{THH} \otimes^{\mathrm{Day}} \mathrm{THH} \to \mathrm{THH}$$

We claim that this is enough to conclude $A \simeq \mathrm{coInd}^{S^1}\mathbb{S}$ as commutative algebras.

Since $A \to \mathbb{S}$ is an algebra map, it is split, and hence, on underlying objects $\mathbb{S} \oplus \Omega\mathbb{S} \to \mathbb{S}$ must be the projection onto the first factor (there are no nonzero maps $\Omega\mathbb{S} \to \mathbb{S}$). Thus, as a map $\mathbb{S}^{S^1} \to \mathbb{S}$ it is the co-unit map of the co-induction adjunction.

It follows that the coinduction map $UA \to \mathbb{S}^{S^1}$ is an equivalence. But this coinduction map can be made into a commutative algebra map, since the forgetful functor preserves limits and hence coinductions. Since it is also conservative, we find that $A \simeq \mathbb{S}^{S^1}$. □

Remark 5.2.5. This proof actually gives a bit more than what we claim: it shows that specifically, the map of commutative THH-algebras with S^1-action

$$\mathrm{THH} \otimes^{\mathrm{Day}} \mathrm{THH} \to \mathrm{THH}^{S^1}$$

induced by adjunction from the multiplication $\mathrm{THH} \otimes^{\mathrm{Day}} \mathrm{THH} \to \mathrm{THH}$ is itself an equivalence. ◁

Remark 5.2.6. It is overwhelmingly likely that by setting up everything (trace theories, derivatives etc.) symmetric monoidally from the start, one could obtain a cleaner proof of the previous two results by applying only improved versions of Proposition 5.1.1 and Proposition 5.2.1. ◁

We can now prove Theorem H. In fact, we prove more. Let us first deal with the non-symmetric monoidal part.

Theorem 5.2.7. *The S^1-action on* THH *induces an equivalence*

$$\mathbb{S}[S^1] \simeq \mathrm{end}(\mathrm{THH})$$

More generally, as functors $\mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Sp}$, for any $p, q \geq 1$, we have $\mathrm{map}(\mathrm{THH}^{\otimes p}, \mathrm{THH}^{\otimes q}) \simeq 0$ if $p \neq q$, and $\mathbb{S}[((S^1)^p \times \Sigma_p)]$ if $p = q$. Here, the tensor powers are pointwise.

Remark 5.2.8. We could also explain how to compute endomorphisms of Day-tensor powers of THH, but they follow immediately from the case of THH itself together with Corollary 5.2.2, so we do not delve into it in more detail. ◁

Proof. For the first part of the statement, note that

$$\mathrm{map}(\mathrm{THH}, \mathrm{THH}) \simeq \mathrm{map}_{\mathrm{THH}}(\mathrm{THH} \otimes^{\mathrm{Day}} \mathrm{THH}, \mathrm{THH}) \simeq \mathrm{map}_{\mathrm{THH}}(\mathrm{THH}^{S^1}, \mathrm{THH})$$
$$\simeq \mathrm{map}_{\mathrm{THH}}(\mathrm{THH}, \mathrm{THH}) \otimes S^1 \simeq \mathrm{THH}(\mathbb{S}) \otimes S^1 \simeq \mathbb{S}[S^1]$$

More generally, this shows that for any spectrum X, the canonical assembly map $X \otimes \mathrm{map}(\mathrm{THH}, \mathrm{THH}) \to \mathrm{map}(\mathrm{THH}, X \otimes \mathrm{THH})$ is an equivalence.

Now we move on to pointwise tensor products. The point here is that by the main result of [RSW], recalled in Theorem B.0.15, we can compute this mapping spectrum as a mapping spectrum between functors defined on $\mathrm{Mot}_{\mathrm{loc}}$, where THH is exact. Indeed, this main result states that $\mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Mot}_{\mathrm{loc}}$ is a Dwyer-Kan localization, and $\mathrm{THH}^{\otimes p}$ clearly factors through it, hence the mapping spectra agree[6].

Note that since THH is exact on $\mathrm{Mot}_{\mathrm{loc}}$, $\mathrm{THH}^{\otimes p}$ is p-homogeneous thereon, and so this deals automatically with the case $q < p$.

We now deal with the case $q = p$: in that case, we can use the classification of p-homogoneous functors:

$$\mathrm{map}(\mathrm{THH}^{\otimes p}, \mathrm{THH}^{\otimes p}) \simeq \mathrm{map}_{\Sigma_p}(\bigoplus_{\Sigma_p} \mathrm{THH}^{\boxtimes p}, \bigoplus_{\Sigma_p} \mathrm{THH}^{\boxtimes p}) \simeq \mathrm{map}(\mathrm{THH}^{\boxtimes p}, \bigoplus_{\Sigma_p} \mathrm{THH}^{\boxtimes p})$$

where $\boxtimes$ indicates an external products, so we are considering (symmetric) functors with p inputs.

Now using currying and the fact that the assembly map

$$X \otimes \mathrm{map}(\mathrm{THH}, \mathrm{THH}) \to \mathrm{map}(\mathrm{THH}, X \otimes \mathrm{THH})$$

is an equivalence, we find the desired result.

In the case $p < q$, we appeal to Lemma 5.2.9 below.

Note that in all cases, we find that for any X, the canonical map

$$X \otimes \mathrm{Map}(\mathrm{THH}^{\otimes p}, \mathrm{THH}^{\otimes q}) \to \mathrm{Map}(\mathrm{THH}^{\otimes p}, X \otimes \mathrm{THH}^{\otimes q})$$

is an equivalence. □

[6] THH itself is a localizing invariant, so by the very definition of $\mathrm{Mot}_{\mathrm{loc}}$, this is obvious for THH. On the other hand, $\mathrm{THH}^{\otimes p}$ is not localizing: it's not even an additive functor, so we do need something else.

Lemma 5.2.9. *Let C, D be stable ∞-categories and $F : C^n \to D$ be a functor which is exact in each variable. For $m < n$ and for any m-excisive functor $G : C \to D$, $\mathrm{map}(G, F \circ \delta_n) = 0$.*

In other words, if D admits sequential limits, the mth co-Goodwillie derivative $P^m(F \circ \delta_n)$ vanishes.

Proof. By adjunction, we have $\mathrm{map}(G, F \circ \delta_n) \simeq \mathrm{map}(G \circ \prod_n, F)$.

The reduction of $G \circ \prod_n$ in the sense of [Lur12, Construction 6.1.3.15] is, by definition (cf. [Lur12, Construction 6.1.3.20]) the nth cross effect of G. It follows from [Lur12, Remark 6.1.3.23] that $\mathrm{Red}(G \circ \prod_n) =: \mathrm{cr}_n(G) = \mathrm{cr}_n(P_n G) = P_{1,...,1}\mathrm{cr}_n G = 0$, where the second equality comes from G being m- and hence n-excisive. It is 0 because G is also $P_{n-1}G$, as $m \leq n-1$.

Now as a functor of n-variables, F is $(1, ..., 1)$-homogeneous in the sense of [Lur12, Definition 6.1.3.7], and so, to prove the claim, it suffices to prove that there is an equivalence

$$\mathrm{map}(G \circ \prod_n, F) \simeq \mathrm{map}(\mathrm{Red}(G \circ \prod_n), F)$$

because the latter is also equivalent to $\mathrm{map}(P_{1,...,1}\mathrm{Red}(G \circ \prod_n), F) = \mathrm{map}(0, F) = 0$.

This now follows from [Heu21a, Lemma B.1] : *loc. cit.* implies the third equality in

$$\mathrm{Red}(G \circ \prod_n) \simeq \mathrm{Red}(G \circ \coprod_n) =: \mathrm{cr}_n(G) \simeq \mathrm{cr}^n(G) := \mathrm{coRed}(G \circ \prod_n)$$

and coRed is left adjoint to the inclusion of reduced functors, cf. [Lur12, Proposition 6.2.3.8]. □

Remark 5.2.10. The failure of this result when $F \circ \delta_n$ is replaced with an arbitrary n-homogeneous functor comes from the fact that such a functor is of the form $(F \circ \delta_n)_{h\Sigma_n}$ for some symmetric F, and we cannot "pull out" the $_{h\Sigma_n}$.

Let us give an interpretation in terms of co-Goodwillie derivatives which will come up again later in Section 5.4. For simplicity we focus on $m = 1$. In this case, $P^1(F \circ \delta_n)_{h\Sigma_n}$ is always very explicit: it is given by $X \mapsto \lim_k \Sigma^k F \circ \delta_n(\Omega^k X)$.

It turns out that each of the maps $\Sigma^{k+1}(F \circ \delta_n)(\Omega^{k+1}X) \to \Sigma^k(F \circ \delta_n)(\Omega^k X)$ is nullhomotopic, as they are essentially given by the diagonal $S^1 \to S^n$. But they are not Σ_n-*equivariantly* nullhomotopic (they are essentially Euler maps), and so taking orbits *can* break the 0-ness of the limit, and this is indeed what often happens (e.g. in Theorem 5.4.11). ◁

We also have all the tools at hand to prove the multiplicative part of Theorem H:

Proof of the multiplicative part of Theorem H. Using the equivalence between commutative algebras for the Day tensor product and lax symmetric monoidal functors, we can rewrite $\mathrm{End}^{\otimes}(\mathrm{THH})$ as $\mathrm{Map}_{\mathrm{CAlg}}(\mathrm{THH}, \mathrm{THH})$.

By adjunction, and by Corollary 5.2.4, this is the same as

$$\mathrm{Map}_{\mathrm{CAlg}(\mathrm{THH})}(\mathrm{THH} \otimes^{\mathrm{Day}} \mathrm{THH}, \mathrm{THH}) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{THH})}(\mathrm{THH}^{S^1}, \mathrm{THH})$$

Finally, using Corollary 5.1.12, the latter is equivalent to $\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp})}(\mathbb{S}^{S^1}, \mathbb{S})$, as claimed. In the proposition below, we describe the latter mapping space as claimed in Theorem H - let us point out that this proof no longer has anything to do with THH. □

Proposition 5.2.11. *The space $\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp})}(\mathbb{S}^{S^1}, \mathbb{S})$ is a disjoint union of circles. Its π_0 is $\widehat{\mathbb{Z}}$, the profinite integers.*

The proof of this fact relies on two facts, an easy one in rational homotopy theory, and Mandell's theorem in p-adic homotopy theory:

Lemma 5.2.12. *Let R be a commutative $\mathbb{Q}$-algebra in* Sp*. The commutative R-algebra R^{S^1} is free on a generator in degree -1.*

Proof. It suffices to prove it over $\mathbb{Q}$. As a $\mathbb{Q}$-module, $\mathbb{Q}^{S^1} \simeq \mathbb{Q} \oplus \Omega\mathbb{Q}$, so it receives a map from the free commutative $\mathbb{Q}$-algebra on $\Omega\mathbb{Q}$. The latter is, as a $\mathbb{Q}$-module, $\bigoplus_n((\Omega\mathbb{Q})^{\otimes n})_{h\Sigma_n}$. Now as a Σ_n-module, $(\Omega\mathbb{Q})^{\otimes n}$ is a copy of the sign representation in degree $-n$, which has no coinvariants and no higher homology (as we are over $\mathbb{Q}$ and Σ_n is finite), and hence simply vanishes. □

And now the difficult result:

Theorem 5.2.13 (Mandell)**.** *Let X be a finite nilpotent space. The canonical map induces an equivalence*

$$X_p \simeq \mathrm{Map}_{\mathrm{CAlg}(\overline{\mathbb{F}_p})}(\overline{\mathbb{F}_p}^X, \overline{\mathbb{F}_p})$$

where X_p denotes the p-completion of X.

Corollary 5.2.14. *Let X be a finite nilpotent space. There is an equivalence*

$$L(X_p) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathbb{F}_p)}(\mathbb{F}_p^X, \mathbb{F}_p)$$

where $LY := \mathrm{Map}(S^1, Y)$ is the free loop space on Y.

Proof. This follows from Theorem 5.2.13 by Galois descent, as the Galois action on $\mathrm{Map}_{\mathrm{CAlg}(\overline{\mathbb{F}_p})}(\overline{\mathbb{F}_p}^X, \overline{\mathbb{F}_p})$ is trivial on X_p[7] and thus the fixed points for it are just $L(X_p)$. □

Remark 5.2.15. For $X = S^1$ (which is the universal case up to p-adic completion), this implies that $\pi_0 \mathrm{Map}_{\mathrm{CAlg}(\mathbb{F}_p)}(\mathbb{F}_p^{S^1}, \mathbb{F}_p) \cong \mathbb{Z}_p$. I do not really know how to describe these morphisms in a concrete way, how to "name" them. This lack of concreteness permeates through the later calculations and so I am essentially unable to "name" the elements of $\pi_0 \mathrm{Map}(\mathbb{S}^{S^1}, \mathbb{S}) \cong \widehat{\mathbb{Z}}$ except for 0. ◁

Proof of Proposition 5.2.11. We use the arithmetic fracture square for $\mathbb{S}$. Writing $\mathbb{A}_f := (\prod_p \mathbb{Z}_p)_{\mathbb{Q}}$ for the finite adèles, we have a pullback square of commutative algebras:

$$\begin{array}{ccc} \mathbb{S} & \longrightarrow & \prod_p \mathbb{S}_p \\ \downarrow & & \downarrow \\ \mathbb{Q} & \longrightarrow & \mathbb{A}_f \end{array}$$

so that, using adjunction in each of the mapping spaces, we have a pullback square

$$\begin{array}{ccc} \mathrm{Map}(\mathbb{S}^{S^1}, \mathbb{S}) & \longrightarrow & \prod_p \mathrm{Map}(\mathbb{S}_p^{S^1}, \mathbb{S}_p) \\ \downarrow & & \downarrow \\ \mathrm{Map}(\mathbb{Q}^{S^1}, \mathbb{Q}) & \longrightarrow & \mathrm{Map}(\mathbb{A}_f^{S^1}, \mathbb{A}_f) \end{array}$$

By Lemma 5.2.12, the bottom map is simply $B\mathbb{Q} \to B\mathbb{A}_f$.

[7]It is trivial because the map from X_p to the mapping space in Theorem 5.2.13 is an assembly map which is clearly Galois equivariant.

By [Lur11, Theorem 2.4.9], basechange further induces an equivalence[8] $\mathrm{Map}(\mathbb{S}_p^{S^1}, \mathbb{S}_p) \simeq \mathrm{Map}(\mathbb{F}_p^{S^1}, \mathbb{F}_p)$ which, by Corollary 5.2.14 is $L(S^1)_p \simeq \coprod_{\mathbb{Z}_p}(S^1)_p$.

Using basechange along $\mathbb{Z}_p \to W(\overline{\mathbb{F}}_p)$ one finds that for a given p, all the maps $(S^1)_p \to B\mathbb{A}_f$ indexed over $\mathbb{Z}_p$ are the same map. Thus we find that the pullback can be computed as:

$$\begin{tikzcd}
\mathrm{Map}(\mathbb{S}^{S^1}, \mathbb{S}) \arrow[r] \arrow[d] & \prod_p L(B\mathbb{Z}_p) \arrow[d] \\
B\mathbb{Z} \arrow[r] \arrow[d] & \prod_p B(\mathbb{Z}_p) \arrow[d] \\
B\mathbb{Q} \arrow[r] & B\mathbb{A}_f
\end{tikzcd}$$

The top right vertical map is a fold map of the form $\coprod_{\prod_p \mathbb{Z}_p} \prod_p B\mathbb{Z}_p \to \prod_p B\mathbb{Z}_p$ by distributivity of products over coproducts, and hence the top left vertical map, as a pullback thereof, is also a fold map and hence $\mathrm{map}(\mathbb{S}^{S^1}, \mathbb{S}) \simeq \coprod_{\prod_p \mathbb{Z}_p} B\mathbb{Z}$. The claim then follows as $\widehat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$. □

Remark 5.2.16. I am not sure whether this is the right perspective to have, but the "weird result" coming out of this calculation seems to stem from the following strikingly different behaviour of certain cochain algebras over $\mathbb{Q}$ and over $\mathbb{F}_p$: in characteristic 0, the presentation of $C^*(X; R)$ as an $\mathbb{E}_\infty$-ring is essentially independent of R's arithmetic, while in characteristic p, properties such as algebraic closure seem to affect the presentation quite a bit. ◁

5.3 Endomorphisms over a base

In this section, we extend the calculation from Theorem 5.2.7 to $\mathrm{THH}(-/k)$, that is, THH relative to a base commutative ring spectrum k. A surprising feature of this calculation is that while for classical rings such as $k = \mathbb{F}_p$, one can phrase the question entirely algebraically, the answer turns out to involve $\mathrm{THH}(k)$. We only state and prove the following result for plain endomorphisms, but the method is simple enough that the other variants from the previous section can also easily be calculated:

Corollary 5.3.1. *Let k be a commutative ring spectrum. When regarded as a functor $\mathrm{Cat}_k^{\mathrm{perf}} \to \mathrm{Sp}$, the endomorphism spectrum of $\mathrm{THH}(-/k)$ is*

$$\mathrm{map}_{\mathrm{THH}(k)}(k, k)[S^1]$$

When regarded as a functor $\mathrm{Cat}_k^{\mathrm{perf}} \to \mathrm{Mod}_k$, it has the following endomorphism k-algebra:

$$\mathrm{map}_{k \otimes \mathrm{THH}(k)}(k, k)[S^1]$$

Remark 5.3.2. We provide both calculations here, for the following reason: on the one hand, when k is a classical ring, the second endomorphism spectrum is clearly a purely algebraic gadget, as both $\mathrm{Cat}_k^{\mathrm{perf}}$ and Mod_k are; and this showcases the appearance of $\mathrm{THH}(k)$ in this purely algebraic question.

[8]See [Yua23, Corollary 7.6.1] to see how to deduce this from the given reference.

On the other hand, one could argue that the k-linear structure is "overdetermined" by putting it both in the source and the target which could explain the appearance of THH - e.g., as it does in $\mathrm{Fun}^L(\mathrm{Mod}_{\mathbb{Z}}, \mathrm{Mod}_{\mathbb{Z}})$ as opposed to $\mathrm{Fun}^L(\mathrm{Mod}_{\mathbb{Z}}, \mathrm{Sp})$. The first calculation shows that this is not the case: even removing this over-determinacy, the answer involves $\mathrm{THH}(k)$. ◁

Proof. We prove the second variant, the first variant works the same by replacing the word "$(k, \mathrm{THH}(k))$-bimodule" with "right $\mathrm{THH}(k)$-module".

The idea for this calculation is that $\mathrm{THH}(-/k) \simeq k \otimes_{THH(k)} \widetilde{\mathrm{THH}}$, where

$$\widetilde{\mathrm{THH}} : \mathrm{Cat}_k^{\mathrm{perf}} \to \mathrm{Mod}_{\mathrm{THH}(k)}$$

is THH of the underlying stable ∞-category equipped with its $\mathrm{THH}(k)$-action; and that this expression is well-defined for any $(k, \mathrm{THH}(k))$-bimodule in place of k.

So let M be such a bimodule, and let us instead try to prove more generally that there is an equivalence

$$\mathrm{Map}_k(M \otimes_{\mathrm{THH}(k)} \widetilde{\mathrm{THH}}, \mathrm{THH}(-/k)) \simeq \mathrm{Map}_{k\otimes\mathrm{THH}(k)}(M, k)[S^1]$$

For this, we note that $M \mapsto M \otimes_{\mathrm{THH}(k)} \widetilde{\mathrm{THH}}$ is a functor, and thus (using the S^1-action on $\mathrm{THH}(-/k)$) we have a canonical map

$$\mathrm{Map}_{k\otimes\mathrm{THH}(k)}(M, k)[S^1] \to \mathrm{Map}_k(M \otimes_{\mathrm{THH}(k)} \widetilde{\mathrm{THH}}, \mathrm{THH}(-/k))$$

and the claim is that this map is an equivalence. But now both sides clearly send colimits in the M variable to limits, and so we are reduced to the case $M = k \otimes \mathrm{THH}(k)$[9], where this map is the canonical map

$$k[S^1] \to \mathrm{Map}_{\mathrm{Sp}}(\mathrm{THH} \circ U_k, \mathrm{THH}(-/k))$$

where $U_k : \mathrm{Cat}_k^{\mathrm{perf}} \to \mathrm{Cat}^{\mathrm{perf}}$ is the forgetful functor.

By adjunction nonsense, this last mapping spectrum is simply

$$\mathrm{Map}_{\mathrm{Sp}}(\mathrm{THH}, \mathrm{THH}(-/k) \circ (k \otimes -)) \simeq \mathrm{Map}_{\mathrm{Sp}}(\mathrm{THH}, k \otimes \mathrm{THH}) \simeq k[S^1]$$

by Theorem 5.2.7 (and the claim, in the proof, that $\mathrm{Map}(\mathrm{THH}, X \otimes \mathrm{THH}) \simeq X \otimes \mathrm{Map}(\mathrm{THH}, \mathrm{THH})$).

One checks, e.g. using k-linearity and compatibility with the S^1-actions, that the map $k[S^1] \to k[S^1]$ thus obtained is the identity, and thus we are done. □

Example 5.3.3. Let k be a (classical) perfect field of characteristic p. In this case, by Bökstedt periodicity (see e.g. [KN22]), we have a presentation of k as a $\mathrm{THH}(k)$-module, or as a $k \otimes \mathrm{THH}(k)$-module, so we can actually compute these mapping spectra quite explicitly.

Indeed, letting $\sigma \in \pi_2\mathrm{THH}(k)$ denote the Bökstedt element, we have $\pi_*\mathrm{THH}(k) \cong k[\sigma]$, a polynomial algebra. Thus $k \simeq \mathrm{THH}(k)/\sigma$ as a $\mathrm{THH}(k)$-module, and so

$$\mathrm{map}_{\mathrm{THH}(k)}(k, k)[S^1] \simeq \mathrm{fib}(k \xrightarrow{0} \Omega^2 k)[S^1] \simeq (k \oplus \Omega^3 k)[S^1]$$

and we find, on top of the expected $k[S^1]$, a degree -3-generator (plus the S^1-action).

[9] For the first variant, we are reduced to the right $\mathrm{THH}(k)$-module $\mathrm{THH}(k)$ itself.

The calculation as bimodule is only a bit more involved:

$$\mathrm{map}_{k\otimes \mathrm{THH}(k)}(k,k) \simeq \mathrm{map}_k(k \otimes_{k\otimes \mathrm{THH}(k)} k, k)$$

Using the universal property of $\mathrm{THH}(k)$ as a commutative algebra, we find that $k \otimes_{k\otimes \mathrm{THH}(k)} k \simeq \mathrm{THH}(k) \otimes (k \otimes_{\mathrm{THH}(k)} k)$, and using now Bökstedt's equivalence $\mathrm{THH}(k) \simeq k[\Omega S^3]$, we find $k \otimes_{\mathrm{THH}(k)} k \simeq k[S^3]$, so that, in total,

$$\mathrm{map}_{k\otimes \mathrm{THH}(k)}(k,k) \simeq \prod_n (\Omega^{2n} k \oplus \Omega^{2n+3} k)$$

To get k-linear endomorphisms of HH_k, one simply adjoins $[S^1]$ to this. ◁

Remark 5.3.4. In the previous examples, it is not clear to me what the extra operations "do". Similarly to the situation in Remark 5.2.15, I do not know how to "name" them, and it would probably be worthwhile to spend time figuring out what these operations do, or are. ◁

5.4 Operations on THH of $\mathcal{O}$-algebras

The goal of this section is to initiate the study of endomorphisms of THH viewed as a functor $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathrm{perf}}) \to \mathrm{Sp}$, that is, to study the extra operations that arise on $\mathrm{THH}(C)$ when C has a particular kind of multiplicative structure encoded by a one-colored ∞-operad $\mathcal{O}$. Our main result in this section is Theorem J.

As is clear from the statement of Theorem J, the case of endomorphisms of THH viewed as a functor on $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathrm{perf}})$ is more subtle.

We first describe the general approach to this question, and then specialize to get the precise results that we mentioned.

We let $U : \mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathrm{perf}}) \to \mathrm{Cat}^{\mathrm{perf}}$ denote the forgetful functor, with a left adjoint F such that $UF \simeq \bigoplus_n (\mathcal{O}(n) \otimes (-)^{\otimes n})_{h\Sigma_n}$ [Lur12, Proposition 3.1.3.13].

The idea now is that by general adjunction nonsense,

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathrm{perf}}),\mathrm{Sp})}(\mathrm{THH}\circ U, \mathrm{THH}\circ U) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{Cat}^{\mathrm{perf}},\mathrm{Sp})}(\mathrm{THH}, \mathrm{THH}\circ UF)$$

and so we are "left with" understanding $\mathrm{THH}\circ UF$.

For this, we compute each of the individual terms of

$$\mathrm{THH}\circ UF = \bigoplus_n \mathrm{THH}((\mathcal{O}(n) \otimes (-)^{\otimes n})_{h\Sigma_n})$$

Proposition 5.4.1. *Let O be a space with a Σ_n-action. There is an equivalence, natural in $C \in \mathrm{Cat}^{\mathrm{perf}}$:*

$$\mathrm{THH}((O \otimes C^{\otimes n})_{h\Sigma_n}) \simeq \bigoplus_{\sigma\in\Sigma_n/\mathrm{conj}} (O^\sigma \otimes \mathrm{THH}(C)^{\otimes n(\sigma)})_{hC(\sigma)}$$

where for $\sigma \in \Sigma_n$, $n(\sigma)$ is the number of cycles appearing in σ, $C(\sigma)$ is the centralizer of σ in Σ_n and $O^\sigma = L(O_{h\Sigma_n}) \times_{LB\Sigma_n} \{\sigma\}$ with its residual $C(\sigma)$-action[10].

Remark 5.4.2. The action of $C(\sigma)$ on $\mathrm{THH}(C)^{\otimes n(\sigma)}$ is a bit subtle to describe. Part of the point here is that there is a functor $\mathrm{Sp}^{BS^1} \to \mathrm{Sp}^{BC(\sigma)}$ of the form $X \mapsto X^{\otimes n(\sigma)}$ such that the above is obtained by applying it to $\mathrm{THH}(C)$ with its S^1-action.

[10] Here, L denotes the free loop space, i.e. $\mathrm{Map}(S^1, -)$.

In extreme cases, the action is easier to describe: if $\sigma = 1$ is the trivial permutation, $n(\sigma) = n, C(\sigma) = \Sigma_n$, and the action is simply given by the permutation action. If σ is the[11] length n cycle, $n(\sigma) = 1$ and $C(\sigma)$ is generated by σ, i.e. a cyclic group of order n, and the action in this case is given by restricting the S^1-action. ◁

Given the above remark, we cannot be too precise about the $C(\sigma)$-action for general $\sigma \in \Sigma_n$, and the specifics we will need about this formula are the following:

Proposition 5.4.3. *Let O be a space with a Σ_n-action. There is an equivalence, natural in $C \in \mathrm{Cat}^{\mathrm{perf}}$:*

$$\mathrm{THH}((O \otimes C^{\otimes n})_{h\Sigma_n}) \simeq (O^{C_n} \otimes \mathrm{THH}(C))_{hC_n} \oplus F(C)$$

where the C_n-action on $O^{C_n} \otimes \mathrm{THH}(C)$ is diagonal, the one on $\mathrm{THH}(C)$ being restricted from S^1, and such that $C \mapsto F(C)$ factors through $\mathrm{Mot}_{\mathrm{loc}}$ and is a (finite) direct sum of k-homogeneous functors, $1 < k \leq n$.

The main tool for this proposition is [CCR+23, Corollary 7.15], which we recall for convenience of the reader. We specialize it to the relevant context. In the notation of that paper, this means setting $\mathscr{C} = \mathrm{Sp}$. See also Section 4.2.2 for the comparison between Tr, as used in [CCR+23] and THH, as used here.

Definition 5.4.4. Let A be a space and $\zeta : A \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a functor with values in dualizable objects[12], and let $\chi_\zeta : LA \to \mathrm{Sp}$ denote the functor

$$\gamma \mapsto \mathrm{THH}(\zeta(\gamma(0)); \zeta \circ \gamma)$$

◁

Theorem 5.4.5 ([CCR+23, Corollary 7.15])**.** *Let A be a space and $\zeta : A \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ be a functor with values in dualizable objects. The colimit $\mathrm{colim}_A \zeta$ is still dualizable, and there is a natural equivalence:*

$$\mathrm{THH}(\mathrm{colim}_A \zeta) \simeq \mathrm{colim}_{LA}(\chi_\zeta)$$

We need two more trace-calculations. The first is the trace of a permutation:

Lemma 5.4.6. *Fix $\sigma \in \Sigma_n$ for some $n \geq 0$, and let $\mathscr{C}$ be a symmetric monoidal ∞-category and $x \in \mathscr{C}$ a dualizable object. There is an equivalence*

$$\mathrm{tr}(x^{\otimes n}; \sigma) \simeq \dim(x)^{\otimes n(\sigma)}$$

where $n(\sigma)$ is the number of cycles in σ, in $\mathrm{End}(\mathbf{1}_{\mathscr{C}})$.

Furthermore, if σ is a length n cycle, the $C(\sigma) = \langle \sigma \rangle$-action on the right, induced by functoriality in $LB\Sigma_n$, is mapped to the $C_n \subset S^1$-action on $\dim(x)$ (which is $\dim(x)^{\otimes n(\sigma)}$ since $n(\sigma) = 1$).

Proof. First, we note that the pair $(x^{\otimes n}, \sigma)$ decomposes as a tensor product over the cycles in σ of terms of the form $(x^{\otimes \ell}, C_\ell)$ for each cycle C_ℓ in σ[13]. Therefore it suffices to identifty $\mathrm{tr}(x^{\otimes n}; C_n)$ where C_n is the length n cycle.

[11] There is only one up to conjugacy.

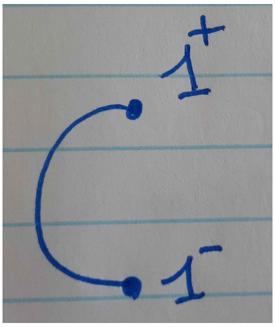
[12] For our purposes, the reader can replace this with compactly generated ∞-categories, or equivalently think of a functor $A \to \mathrm{Cat}^{\mathrm{perf}}$.

[13] Though this decomposition is not natural in $\sigma \in LB\Sigma_n$, there we would have to worry about cycle types too.

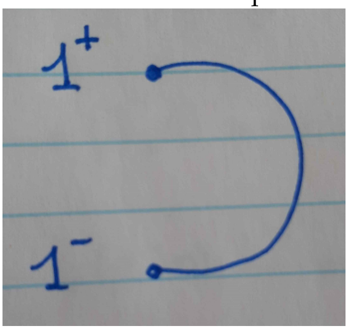
By [Har12][14], the one-dimensional cobordism ∞-category Cob_1 is the universal symmetric monoidal ∞-category on a dualizable object, thus the calculation of the trace and the $C(\sigma)$ -action can be performed entirely there.

Let 1^+ denote the universal dualizable object in Cob_1, 1^- its dual, and $n^+ := (1^+)^{\otimes n}$ (similarly for n^-). Recall that $\mathrm{End}(\mathbf{1}_{\mathrm{Cob}_1}) = \coprod_n (BS^1)^n_{h\Sigma_n}$ is given by the space of 1-dimensional, closed oriented manifolds, i.e. disjoint unions of oriented circles. So, to identify the object $\mathrm{tr}(n^+, C_n)$ as a manifold it suffices to identify its number of connected components.

Recall that the cobordism representing the coevaluation looks as follows (we read our cobordisms from left to right):



and the cobordism representing the evaluation looks like:



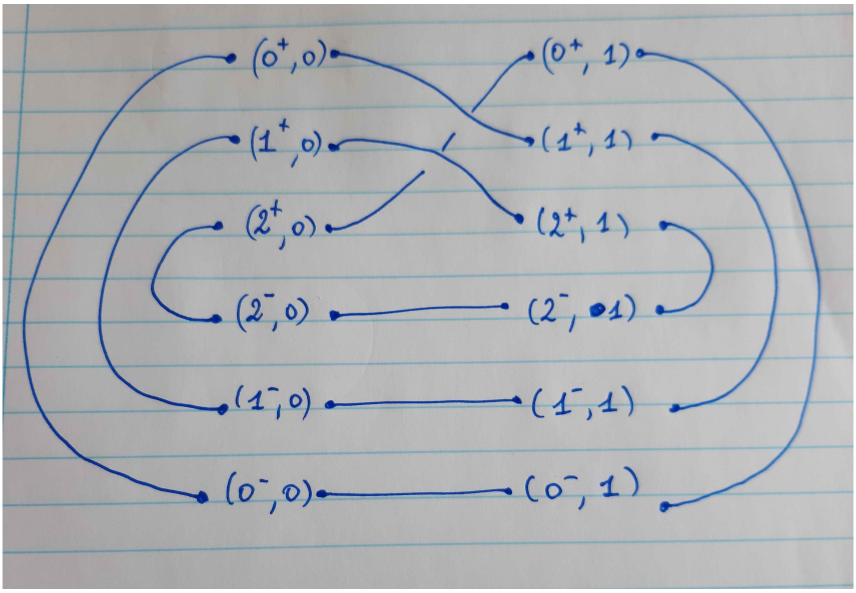
The manifold representing $\mathrm{tr}(n^+, C_n)$ is obtained by taking the cobordism representing C_n from n^+ to itself, the one representing id from n^- to itself, and adding evaluations and coevaluations.

We let $(i^\pm, 0), i \in \{0, n-1\}$ denote the endpoints in the source, $(i^\pm, 1)$ denote the endpoints in the target. By definition, $(i^+, 0)$ is related by the cobordism to $(\sigma(i)^+, 1)$; this is in turn related by the evaluation to $(\sigma(i)^-, 1)$, which is related by the id-cobordism to $(\sigma(i)^-, 0)$ which, finally, is related by the coevaluation to $(\sigma(i)^+, 0)$.

Since $\sigma = C_n$ is transitive (by definition of cycle!) it follows that all the points $(i^+, 0)$ are connected. Continuing this argument shows that all points are connected, and thus $\mathrm{tr}(n^+, C_n) \in \mathrm{End}(\mathbf{1}_{\mathrm{Cob}_1})$ is a connected manifold, i.e. the circle, i.e. $\dim(1^+)$.

Here is a pictorial example:

[14] Harpaz gave a full proof of the one dimension cobordism hypothesis, previously sketched in all dimensions by Lurie in [Lur08].



We now need to identify the C_n-action on this circle. The proof that it is as we claim is not so difficult, but it takes up a bit of space. As it is mostly unrelated to the rest of the section, we defer it to Appendix C, specifically see Proposition C.0.2. $\square$

Corollary 5.4.7. *Let B be a symmetric monoidal $(\infty, 2)$-category. The functor $B^{\mathrm{dbl}} \to \mathrm{End}(\mathbf{1}_B)$ given by $b \mapsto \mathrm{tr}(b^{\otimes n}; \sigma)$ is naturally equivalent to*

$$b \mapsto \dim(b)^{\otimes n(\sigma)}$$

For $\sigma = C_n$ being the length n cycle, the induced C_n-action is restricted from the S^1-action on $\dim(x)$.

Proof. Using naturality of the Ω-construction in [HSS17, (2.7)], we find that this functor is given by

$$\mathrm{Fun}^{\otimes}_{\mathrm{colax}}(\mathrm{Fr}^{\mathrm{rig}}(\mathrm{pt}), B) \to \mathrm{Fun}^{\otimes}_{\mathrm{colax}}(\Omega\mathrm{Fr}^{\mathrm{rig}}(\mathrm{pt}), \Omega B) \xrightarrow{\mathrm{ev}} \Omega B$$

where ev is evaluation at $\mathrm{tr}(x^{\otimes n}; \sigma) \in \Omega\mathrm{Fr}^{\mathrm{rig}}(\mathrm{pt})$, where x is the universal dualizable object. But now this trace has been computed in Lemma 5.4.6, and is indeed $\dim(x)^{\otimes n(\sigma)}$. The claim follows. $\square$

And second, we need a calculation of traces on ∞-categories of local systems in Pr^{L}:

Lemma 5.4.8. *Let $X : A \to \mathcal{S}$ be a local system of spaces, and consider the induced local system $\mathcal{S}^X : A \to \mathrm{Pr}^{\mathrm{L}}$. This has dualizable values, and the corresponding χ_X is given by the local system $LA \to \mathcal{S}$ whose unstraightening is $L(\mathrm{colim}_A X) \to LA$, that is,*

$$\gamma \mapsto X^{\gamma}$$

where X^{γ} is the fixed points for the $\mathbb{Z}$-action on X induced by γ, or equivalently the global sections $\Gamma(S^1, \mathrm{colim}_A X \times_A S^1)$.

Proof. Let $\int X := \mathrm{colim}_A X$ for simplicity of notation, with its map $p : \int X \to A$.

Note that by un/straightening, $X = p_!\mathrm{pt}$, where $\mathrm{pt} : \int X \to \mathcal{S}$ is the constant local system. Thus, in terms of functors $\mathrm{Pr}^{\mathrm{L}}[A] \to \mathrm{Pr}^{\mathrm{L}}$, letting p^* denote the right adjoint to

$p_!$, we find that the functor $\mathrm{Pr}^{\mathrm{L}}[A] \to \mathrm{Pr}^{\mathrm{L}}$ induced by $\mathcal{S}^X$ is equivalent to the composite $\mathrm{Pr}^{\mathrm{L}}[A] \xrightarrow{p^*} \mathrm{Pr}^{\mathrm{L}}[\int X] \xrightarrow{r_!} \mathrm{Pr}^{\mathrm{L}}$,where $r : \int X \to \mathrm{pt}$ is the unique map.

Thus, passing to Pr^{L}-linear traces[15] and using (an obvious generalization of) [CCR+23, Proposition 7.14][16] we obtain that χ_X is the composite:

$$\mathcal{S}[LA] \xrightarrow{(Lp)^*} \mathcal{S}[L(\int X)] \xrightarrow{Lr} \mathcal{S}$$

This implies the desired claim under un/straightening. □

With these two facts in mind, we can give:

Proof of Proposition 5.4.3, Proposition 5.4.1. By Theorem 5.4.5, we have

$$\begin{aligned} \mathrm{THH}((O \otimes C^{\otimes n})_{h\Sigma_n} &\simeq \mathrm{colim}_{\sigma \in LB\Sigma_n} \mathrm{THH}(O \otimes C^{\otimes n}; \sigma) \\ \simeq \mathrm{colim}_{\sigma \in BC_n} \mathrm{THH}(O \otimes C^{\otimes n}; \sigma) &\oplus \mathrm{colim}_{\sigma \in LB\Sigma_n | n(\sigma) > 1} \mathrm{THH}(O \otimes C^{\otimes n}, \sigma) \end{aligned}$$

Now σ acts diagonally on O and $C^{\otimes n}$, so each of these terms splits as a colimit of $\mathrm{THH}(O;\sigma) \otimes \mathrm{THH}(C^{\otimes n};\sigma)$.

The second summand is therefore a colimit of terms of the form $X \otimes \mathrm{THH}(C)^{\otimes k}, 1 < k \leq n$ by Corollary 5.4.7 and is therefore indeed a sum of homogeneous functors of (varying) degrees $1 < k \leq n$.

The first summand is, again by Corollary 5.4.7[17] and by Lemma 5.4.8, of the form $(O^\sigma \otimes \mathrm{THH}(C))_{hC_n}$, as was to be proved. □

Putting things together, we obtain:

Corollary 5.4.9. *Let $\mathcal{O}$ be a single-colored operad and $F : \mathrm{Cat}^{\mathrm{perf}} \rightleftharpoons \mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathrm{perf}}) : U$ the free-forgetful adjunction. There is a natural equivalence:*

$$\mathrm{THH} \circ UF \simeq \bigoplus_{n \geq 0} \bigoplus_{\sigma \in \Sigma_n/\mathrm{conj}} (\mathcal{O}(n)^\sigma \otimes \mathrm{THH}(C)^{\otimes n(\sigma)})_{hC(\sigma)}$$

From this we can see how to approach the question of endomorphisms of $\mathrm{THH} \circ U$, and we can also see the relevant difficulties: we have already understood endomorphisms of THH, and we see that we "simply" need to understand maps from THH to (say in the case of $\mathcal{O} = \mathrm{Comm}$) THH_{hC_n}, $C_n \subset S^1$ a cyclic group, and maybe variants involving tensor powers.

These variants are of higher polynomiality degree than THH itself, and so one might expect there to not be maps from THH to them, thus leading to a (potentially) simple expression.

Three problems arise:

(i) First, this is not quite right in general: there can a priori be maps from a homogeneous functor of degree n to a homogeneous functor of degree $m > n$ - if the symmetric m-linear functor corresponding to the latter has an *induced* symmetric structure, then this in fact does *not* happen, but here the homotopy orbits by $C(\sigma)$ prevent that from being the case, except for very special operads $\mathcal{O}$. Another place where this does not happen is in chromatically localized contexts, as was first observed by Kuhn in [Kuh04];

[15]The reader uncomfortable with set theory may pass to $\mathrm{Pr}^{\mathrm{L}}_\kappa$ for κ sufficiently large, this does not change anything to the argument.

[16]The proof of this uses nothing specific to maps $A \to \mathrm{pt}$: it only uses [CCR+23, Theorem 4.34] which is stated in full generality.

[17]Note that by Remark 4.2.68, the S^1-action on $\mathrm{Tr}(C)$, which is used in Corollary 5.4.7, agrees with that on $\mathrm{THH}(C)$.

(ii) The second problem is that we have an infinite direct sum in the target of our mapping spectrum, and this can cause issues a priori both with the homogeneity phenomenon observed above, but also with computing the actual mapping spectrum once we've removed the "high degree" terms. This will turn out to be solved also in the chromatic context, because of a "quick convergence" phenomenon for Goodwillie derivatives in that context, a phenomenon recorded by Heuts in [Heu21b] (though he attributes it to Mathew);

(iii) Finally, it turns out to be difficult to compute maps from THH to THH_{hC_n}. Among other things, the reason for this is that THH_{hC_n} is simply not a module over THH, and so the tools we have developped so far are not great for this purpose. The same "quick convergence" phenomenon mentioned above will turn out to save us here again in the chromatic context, though we end up also being able to compute this integrally.

These three problems, and their chromatically localized fixes are the reason our results in this section look the way they do, and are only partial. We point out that there *are* in fact subtleties in the integral context, and that the answer we get in the chromatic world does not generalize verbatim to the integral context (thus, it is not only that our proof is not optimal, but that the answer is more complicated):

Observation 5.4.10. Let p be a prime number. There is a nonzero natural transformation $\mathrm{THH} \to \Sigma\mathrm{THH}_{hC_p}$, given by the composite

$$\mathrm{THH} \xrightarrow{\varphi_p} \mathrm{THH}^{tC_p} \to \Sigma\mathrm{THH}_{hC_p}$$

where the first map is the cyclotomic Frobenius, and the second is the attaching map for the fiber sequence defining the Tate construction.

This is nonzero e.g. because it is also the attaching map for the fiber sequence defining (p-typical) TR^2, which in general does not split as $\mathrm{THH} \oplus \mathrm{THH}_{hC_p}$. ◁

In fact, the partial splitting result from Theorem J will show that this nonzero transformation participates nonzero-ly to the endomorphism spectrum

$$\mathrm{end}_{\mathrm{Fun}(\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{perf}}),\mathrm{Sp})}(\mathrm{THH} \circ U, \mathrm{THH} \circ U)$$

in the case $\mathcal{O} = \mathrm{Comm}$, thus leading to genuinely new operations, not predicted e.g. by the chromatic case, or by the $\mathbb{Z}$-linear case explored in [Kla15]. In fact, we can fully compute each mapping spectrum $\mathrm{map}(\mathrm{THH}, \mathrm{THH}_{hC_n})$ and see that the above operations essentially "generate" it:

Theorem 5.4.11. *Let $n \geq 2$ be a prime number. The mapping spectrum $\mathrm{map}(\mathrm{THH}, \mathrm{THH}_{hC_n})$ is equivalent to*

$$\mathbb{S}[S^1/C_n] \oplus \bigoplus_{p|n,\ \text{prime}} \Omega\mathbb{S}_p[S^1/C_p]$$

where the first factor is mapped into isomorphically by $\mathrm{map}(\mathrm{THH}, \mathrm{THH})_{hC_n}$ along the assembly map, and where the degree -1 generators correspond to the Frobenius morphisms described above.

I wish to thank Achim Krause for an enlightening discussion that allowed me to prove this result.

Before we get into the proof, we need a computation of $\mathrm{THH}(A_k)$ which is suitably natural. We let $T : \mathrm{Sp} \to \mathrm{Alg}(\mathrm{Sp})$ denote the left adjoint to the forgetful functor, i.e. the free (associative) algebra functor[18]. This is a canonically augmented algebra, since $T(0) = \mathbb{S}$, and for a functor $F : \mathrm{Alg}(\mathrm{Sp}) \to \mathcal{E}$ with values in some stable ∞-category, we let $\tilde{F}(T(M))$ denote the complementary summand to $F(\mathbb{S}) = F(T(0))$.

Proposition 5.4.12. *There is a natural equivalence*

$$\widetilde{\mathrm{THH}}(T(M)) \simeq \bigoplus_{d\geq 1} \mathrm{Ind}_{C_d}^{S^1} M^{\otimes d}$$

where: $M^{\otimes d}$ has a C_d-action by permutation, and $\mathrm{Ind}_{C_d}^{S^1}$ is left adjoint to the restriction functor $\mathrm{Sp}^{BS^1} \to \mathrm{Sp}^{BC_d}$.

Proof. This follows from combining [LT23, Theorem D] and [Ras18, Proposition 3.2.2, Proposition 4.5.1][19] and noting that $\mathrm{THH}(\mathbb{S}, N) \simeq N$ naturally in N. □

Proof of Theorem 5.4.11. We first note that, as a spectrum with S^1-action coming from the target, $\mathrm{map}(\mathrm{THH}, \mathrm{THH}) \simeq \mathbb{S}[S^1]$ has an induced S^1-action, so the norm map $\mathrm{map}(\mathrm{THH}, \mathrm{THH})_{hC_n} \to \mathrm{map}(\mathrm{THH}, \mathrm{THH})^{hC_n}$ is an equivalence.

Since it factors through (via assembly and coassembly) the norm map $\mathrm{map}(\mathrm{THH}, \mathrm{THH}_{hC_n}) \to \mathrm{map}(\mathrm{THH}, \mathrm{THH}^{hC_n}) \simeq \mathrm{map}(\mathrm{THH}, \mathrm{THH})^{hC_n}$, it follows that $\mathrm{map}(\mathrm{THH}, \mathrm{THH}_{hC_n})$ admits $\mathrm{map}(\mathrm{THH}, \mathrm{THH})_{hC_n}$ as a summand.

We now move on to the explicit calculation of the extra summand. By Corollary 5.1.10, there is an equivalence

$$\mathrm{map}(\mathrm{THH}, \mathrm{THH}_{hC_n}) \simeq \lim_k \Sigma^k(\mathrm{THH}(\tilde{A}_k)_{hC_n})$$

and recall from Proposition 5.4.12 that

$$\mathrm{THH}(\tilde{A}_k) \simeq \bigoplus_{d\geq 1} \mathrm{Ind}_{C_d}^{S^1}((\Omega^k\mathbb{S})^{\otimes d})$$

Remark 5.4.13. We make a digression to mention that without $_{hC_n}$, this inverse limit is easy to deal with: the map $\Sigma^{k+1}(\Omega^{k+1}\mathbb{S})^{\otimes d} \to \Sigma^k(\Omega^k\mathbb{S})^{\otimes d}$ is null whenever $d \geq 2$, and thus the whole inverse system is $\mathbb{Z}$-equivariantly nilpotent of order 2, where $\mathbb{Z}$ acts via picking a generator $\mathbb{Z} \to C_d$. Since the underlying object of $\mathrm{Ind}_{C_d}^{S^1} X$ is $X_{h\mathbb{Z}}$ for $X \in \mathrm{Sp}^{BC_d}$, it follows from Lemma 5.4.21 that this inverse limit is equivalent to the inverse limit of the $d = 1$ summand, which is a constant diagram. This is another way of proving that $\mathrm{map}(\mathrm{THH}, \mathrm{THH}) \simeq \mathbb{S}[S^1]$, but will also be relevant in a second.

The difficulty now is that for THH_{hC_n}, we have to take into account the C_n-equivariance, and the map $\Sigma^{k+1}(\Omega^{k+1}\mathbb{S})^{\otimes d} \to \Sigma^k(\Omega^k\mathbb{S})^{\otimes d}$ is not C_d-equivariantly null in general. ◁

Now, using a basechange formula, we obtain that for $d, k \geq 1$,

$$(\mathrm{Ind}_{C_d}^{S^1} X)_{hC_n} \simeq X_{h(C_{d\wedge n}\times\mathbb{Z})}$$

where the $\mathbb{Z} \times C_{d\wedge n}$-action on X is induced by the map $\mathbb{Z} \times C_{d\wedge n} \to C_d$ which is the inclusion on the right factor, and picks out a generator on the left factor - we use $a \wedge b$ to denote the gcd of a and b.

[18] T stands for "tensor algebra".

[19] Note that Raskin uses Ind to denote the *right* adjoint to the forgetful functor. However, they differ by a shift Σ, and this is accounted for by the motivic pullback square from [LT23]

Let us point out that $\mathbb{Z}$-orbits are given by a colimit over S^1, i.e. a finite colimit, and therefore commute with the limit term.

So all in all we are considering the limit of $(\bigoplus_{d\geq 1}\Sigma^k(\mathbb{S}^{\otimes dk})_{hC_{d\wedge n}})_{h\mathbb{Z}}$. It is not hard to check that the maps $\Sigma^{k+1}((\mathbb{S}^{\otimes d(k+1)})_{hC_{d\wedge n}}) \to \Sigma^k((\mathbb{S}^{\otimes dk})_{hC_{d\wedge n}})$ are $C_{d\wedge n}$-orbits of Euler maps for the standard representation $\mathbb{R}^d$ of $C_{d\wedge n}$.

If $d\wedge n < d$, then $\mathbb{R}^d$ admits a more than 1-dimensional fixed point space for $C_{d\wedge n}$, and thus this Euler map is $C_{d\wedge n}$-equivariantly nullhomotopic. It follows that the limit is the same as the limit taken over the d's for which $d\wedge n = d$, i.e. $d \mid n$.

Thus we are dealing with finite sums and can stop worrying about commuting limits and finite sums and focus on each

$$\lim_k \Sigma^k((\Omega^k\mathbb{S})^{\otimes d})_{hC_d}, d \mid n$$

(again, the $\mathbb{Z}$-orbits commute with the inverse limit, so we will deal with them later).

We claim that these are 0 whenever d is not 1 or prime, and $\Omega\mathbb{S}_p$ whenever $d = p$ is prime. The term corresponding to $d = 1$ comes from $\mathrm{map}(\mathrm{THH},\mathrm{THH})_{hC_n}$, by Remark 5.4.13, and therefore we can also focus on the case where d is not 1. Once this is proved, we will be done: the only nonzero terms correspond to the primes dividing n, they are given by $(\Omega\mathbb{S}_p)_{h\mathbb{Z}} \simeq \Omega\mathbb{S}_p[S^1/C_p]$[20], as claimed.

Let us prove the second fact, i.e. the case of $d = p$ being prime: in this case, the map

$$\Sigma^k((\Omega^k\mathbb{S})^{\otimes p})_{hC_p} \to \Sigma^k((\Omega^k\mathbb{S})^{\otimes p})^{hC_p}$$

has $\Sigma^k((\Omega^k\mathbb{S})^{\otimes p})^{tC_p}$ as a cofiber. Since $X \mapsto (X^{\otimes p})^{tC_p}$ is an exact functor by [NS18, Proposition III.1.1], we find that the cofiber, as a $\mathbb{N}^{\mathrm{op}}$-indexed diagram, is equivalent to the constant diagram at $\mathbb{S}^{tC_p}$, which is equivalent to $\mathbb{S}_p$ by the Segal conjecture, a theorem of Gunawardena [Gun80] (at odd primes) and Lin [Lin80] (at $p = 2$).

Since $\lim_k \Sigma^k((\Omega^k\mathbb{S})^{\otimes p})^{hC_p} \simeq (\lim_k(\Omega^k\mathbb{S})^{\otimes p})^{hC_p} = 0$, the result follows. The latter equality comes from the fact that the maps in the system are non-equivariantly trivial, and hence the limit is trivial before taking homotopy fixed points, therefore it is also trivial after taking homotopy fixed points.

Our proof of the first fact, namely the vanishing of the inverse limit for d composite proceeds differently depending on whether d is a prime power or has different prime factors.

The key input will be the following easy observation: for any proper subgroup $C_a \subset C_d$, $\mathbb{R}^d$, viewed as a C_a-representation, has a fixed point space of dimension > 1 and thus the associated Euler map is C_a-equivariantly nullhomotopic.

The case of several prime factors: We consider more generally the Euler map for a spectrum M:

$$\Sigma(\Omega M)^{\otimes d} \to M^{\otimes d}$$

So let a, b be coprime integers > 1 such that $ab = d$. Using the associated fracture square, we may assume that either a or b is invertible. If b is invertible (say, the argument is symmetric), then we use the above fact to obtain that the induced map

$$(\Sigma(\Omega M)^{\otimes d})_{hC_a} \to (M^{\otimes d})_{hC_a}$$

is null, and b acts invertibly on both terms. Thus, it is also C_b-equivariantly null, and so in total, the map

$$(\Sigma(\Omega M)^{\otimes d})_{hC_d} \to (M^{\otimes d})_{hC_d}$$

[20] The $/C_p$ does not affect the homotopy type, but is here to remember the specific S^1-action. The fact that it is given exactly as that one can be recovered by remembering where this $(-)_{h\mathbb{Z}}$ came from. Since we will not specifically need this S^1-action, we do not give more details.

is null, which implies that the inverse limit is 0.

The case of a prime power: Say $d = p^s, s > 1$. In this case, we use the Tate orbit lemma [NS18, Lemma I.2.1]. An immediate corollary of this is that for X bounded below with a C_{p^s}-action, we have $(X_{hC_{p^{s-1}}})^{tC_{p^s}/C_{p^{s-1}}} = 0$. A way to rephrase this is that the norm map

$$X_{hC_{p^s}} = (X_{hC_{p^{s-1}}})_{hC_{p^s}/C_{p^{s-1}}} \to (X_{hC_{p^{s-1}}})^{hC_{p^s}/C_{p^{s-1}}}$$

is an equivalence.

In particular, we obtain

$$(\lim_k \Sigma^k (\Omega^k \mathbb{S})^{\otimes p^s})_{hC_{p^s}} = \lim_k \Sigma^k (((\Omega^k \mathbb{S})^{\otimes p^s})_{hC_{p^{s-1}}})^{hC_{p^s}/C_{p^{s-1}}} = (\lim_k \Sigma^k ((\Omega^k \mathbb{S})^{\otimes p^s})_{hC_{p^{s-1}}})^{hC_{p^s}/C_{p^{s-1}}}$$

and it suffices to prove that the inner term is 0. But now $C_{p^{s-1}}$ is a proper subgroup of C_{p^s} so the Euler maps are again $C_{p^{s-1}}$-equivariantly null, and thus the inner term is indeed 0. □

Remark 5.4.14. Let us point out that in the last part of this proof, the composite case and the prime power case have qualitatively different proofs: in the composite case, we essentially obtain that the relevant inverse system is nilpotent, because nilpotent inverse systems are closed under pullback; whereas in the prime power case we only get that the inverse limit is 0. It seems likely that the inverse system is actually not pro-zero. ◁

We can also use this result to study the independently interesting question of the uniqueness of cyclotomic Frobenii on THH. Indeed, maps from THH to THH^{hC_p} are easily computable. We obtain:

Corollary 5.4.15. *The spectrum* $\mathrm{map}(\mathrm{THH}, \mathrm{THH}^{tC_p})$ *is equivalent to* $\mathbb{S}_p[S^1/C_p]$*, and in particular the spectrum of* S^1*-equivariant maps* $\mathrm{THH} \to \mathrm{THH}^{tC_p}$ *is equivalent to* $\mathbb{S}_p$*.*

Proof. This follows from Theorem 5.4.11 and the fact that

$$\mathrm{map}(\mathrm{THH}, \mathrm{THH}^{hC_p}) \simeq \mathrm{map}(\mathrm{THH}, \mathrm{THH})^{hC_p}$$

□

Similarly, and using the same methods as for the proof of the multiplicative part of Theorem H, we obtain:

Corollary 5.4.16. *The space of lax symmetric monoidal transformations* $\mathrm{Map}^{\otimes}(\mathrm{THH}, \mathrm{THH}^{tC_p})$ *is equivalent to* $\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp})}(\mathbb{S}^{S^1}, \mathbb{S}_p)$*, which is in turn equivalent to* $\mathrm{Map}(S^1, S^1_p)$*, where* S^1_p *is the* p*-complete circle.*

Before moving on to proofs of the general results about $\mathcal{O}$-algebras, we point out one final thing: our methods, both for the partial result in the integral case, and the full result in the chromatically localized context, extend to "computing" (with the same caveats) mapping spectra between (pointwise) tensor powers of THH. However, the expression of our answer there is more complicated, and still only partial, thus we refrain from stating and proving it, as the methods are exactly the same.

We now move on to the proofs. We start with the integral version, which we can prove right away with no further preparation:

Proof of (i) in Theorem J. The map we construct is the following : for each n, we pick the inclusion of the length n-cycle γ_n in Σ_n and produce the following composite:

$$\begin{aligned} \mathbb{S}[\mathcal{O}(n)^{\gamma_n} \times S^1]_{hC_n} &\simeq \mathrm{map}(\mathrm{THH}, \mathcal{O}(n)^{\gamma_n} \otimes \mathrm{THH})_{hC_n} \\ \to \mathrm{map}(\mathrm{THH}, (\mathcal{O}(n)^{\gamma_n} \otimes \mathrm{THH})_{hC_n}) &\to \mathrm{map}(\mathrm{THH}, \mathrm{THH} \circ UF) \\ &\simeq \mathrm{map}(\mathrm{THH} \circ U, \mathrm{THH} \circ U) \end{aligned}$$

where the map $(\mathcal{O}(n)^{\gamma_n} \otimes \mathrm{THH})_{hC_n} \to \mathrm{THH} \circ UF$ is the summand inclusion from Proposition 5.4.1.

To prove that each finite direct sum of these inclusion splits, we give the argument for each summand for simplicity of writing, but it clearly extends to the general case, e.g. by using norm maps for non-connected groupoids.

The point is that the last maps involved in this composite split because of the computation of $\mathrm{THH} \circ UF$ from Proposition 5.4.1, while the assembly map from the C_n-orbits outside to the C_n-orbits inside is split by the norm map (it is the same argument as in the beginning of the proof of Theorem 5.4.11: if F is a functor that preserves limits, and $F(x)_{hC_n} \to F(x)^{hC_n}$ is an equivalence, then the assembly map $F(x)_{hC_n} \to F(x_{hC_n})$ splits). □

We now move on to the chromatically localized picture. The key piece of intuition here is the following result, which is a corollary of (a suitably generalized version of) Kuhn's [Kuh04, Theorem 1.1]:

Corollary 5.4.17. *Let $\mathcal{E}$ be a cocomplete $T(n)$-local stable ∞-category, where $n \geq 0$, and C be stable ∞-category. For any $m_0 < m_1$, any m_i-homogeneous functors $F_i : C \to \mathcal{E}$,* $\mathrm{Map}(F_0, F_1) = 0$.

Remark 5.4.18. In the other direction, note that the definition of m_1-homogeneous includes $(m_1 - 1)$- and thus m_0-reduced, so that $\mathrm{Map}(F_1, F_0) = 0$ is clear. We note that this direction is simply not true without $T(n)$-localization, as the existence of k-excisive functors that do not split as sums of homogeneous ones shows (e.g. norm functors in equivariant homotopy theory, or more classically derived symmetric/exterior powers functors). ◁

Kuhn's proof is more direct[21], but we give a proof in terms of Proposition 5.4.19 which we will need later anyways.

Proof. By Proposition 5.4.19, for any finite type n spectrum V, $V \otimes P^{m_0} F_1 = 0$, where P^{m_0} denote the m_0th co-Goodwillie derivative.

Since $\mathcal{E}$ is $T(n)$-local, it follows that $P^{m_0} F_1 = 0$, and thus

$$\mathrm{Map}(F_0, F_1) = \mathrm{Map}(F_0, P^{m_0} F_1) = 0$$

□

As we mentioned before, this corollary will in fact not quite cut it because of the presence of infinite direct sums in our considerations.

The point is that (as "our" proof shows) we can reinterpret this statement as a vanishing statement for the m_0-th *co*-Goodwillie derivative[22] of F_1, something which involves infinite inverse limits (as Goodwillie derivatives involve infinite colimits), which thus do not commute *a priori* with the relevant colimits. The idea, and the point where $T(n)$-localization

[21] The way to deduce this from Kuhn's result is to take a map $F_0 \to F_1$ and consider its cofiber. Kuhn's splitting result implies that it splits off ΣF_0 and thus the map $F_0 \to F_1$ must have been 0.

[22] Or Goodwillie coderivative ?

will again be relevant, is that in fact the vanishing of this co-Goodwillie derivative happens *quickly*: the relevant inverse limits, rather than simply being 0, are *pro*-zero, which allows us to get the corresponding result for infinite colimits.

Let us briefly explain the idea that will allow us to deal with these infinite colimits, in the (much) simpler special case of $m_0 = 1$, so that F_0 is an exact functor, and $n = 0$ so that $T(n)$-local means rational.

In that case, the first co-Goodwillie derivative of F_1 is given by the inverse limit $\lim_n \Sigma^n F_1(\Omega^n -)$. Now, F_1 being m_1-homogeneous is of the form $(f_1 \circ \delta_n)_{h\Sigma_n}$ for some symmetric n-linear functor f_1 [Lur12, Proposition 6.1.4.14], and we note that each of the transition maps $\Sigma f_1(\Omega X_1, ..., \Omega X_{m_1}) \to f_1(X_1, ..., X_{m_1})$ is nullhomotopic, because they are (dual to) the diagonal map $S^1 \to (S^1)^{\wedge n} = S^n$. However, they are *not* Σ_n-equivariantly null in general. Rationally, though, they are because null implies equivariantly null for a finite group over a rational base.

Thus the relevant inverse system is even more than pro-zero: all of the transition maps are 0.

Moving to a higher m_0 and possibly a higher height, this is no longer true, but the pro-zero-ness persists, although the argument is more involved. This is explicitly recorded in [Heu21b, Appendix B], where Heuts attributes it to Akhil Mathew. For this statement to make sense, recall from [Lur12, Construction 6.1.1.27] that n-excisive approximations are given as certain explicit sequential colimits of finite limits - thus, dually, the n-excisive coapproximation $P^n F$ of a functor F is given as a sequential limit of a certain inverse system. In the following statement, it is this inverse system we call "the inverse system computing $P^n F$":

Proposition 5.4.19 ([Heu21b, Appendix B]). *Let $n \geq 0$, V a finite type n spectrum and let $k \geq 0$ be an integer. There is a constant C such that for any $m > k$ and any m-homogeneous functor $F : D \to \mathcal{E}$ from a stable ∞-category D to a cocomplete stable ∞-category $\mathcal{E}$, the inverse system computing $V \otimes P^k F$, the k-th co-Goodwillie derivative is nilpotent of exponent C.*

Here, we used:

Definition 5.4.20. Let I be a cofiltered ∞-category, and $A_\bullet : I \to \mathcal{E}$ be a diagram with values in a pointed ∞-category. $A_\bullet$ is called nilpotent of exponent N if for every sequence of N composable non-identity morphisms $i_0 \to ... \to i_N$, $A_{i_0} \to A_{i_N}$ is nullhomotopic. ◁

If I has an initial copy of $\mathbb{N}^{\mathrm{op}}$, then any nilpotent I-shaped diagram has a trivial limit (in fact is pro-zero, but being nilpotent is stronger than being pro-zero).

But nilpotency with a uniform exponent allows for more, as the following easy observation shows:

Lemma 5.4.21. *Let J be any set, I a cofiltered ∞-category and $X_j, j \in J$ a family of I-shaped diagrams that are nilpotent of a uniform exponent C. The diagram $\bigoplus_J X_j$ is also nilpotent of degree C. In particular, if I has an initial copy of $\mathbb{N}^{\mathrm{op}}$, then $\lim_I \bigoplus_J X_j = 0$.*

Proof. This is clear, as direct sums of zero morphisms are 0. □

We are now ready to prove the second part of Theorem J:

Proof of (ii) in Theorem J. For the duration of this proof, everything is implicitly rationalized, or $T(n)$-localized for some implicit prime and some height $n \geq 1$.

We recall that

$$\mathrm{map}_{\mathrm{Fun}(\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathrm{perf}}),\mathrm{Sp})}(\mathrm{THH} \circ U, \mathrm{THH} \circ U) \simeq \mathrm{map}_{\mathrm{Fun}(\mathrm{Cat}^{\mathrm{perf}},\mathrm{Sp})}(\mathrm{THH}, \mathrm{THH} \circ UF)$$

We combine Proposition 5.4.19, Lemma 5.4.21,Corollary 5.4.9 and Theorem B.0.15 to find that this is equivalent to

$$\mathrm{map}(\mathrm{THH}, \bigoplus_{n\geq 1} \bigoplus_{\sigma\in\Sigma_n/\mathrm{conj}|n(\sigma)=1} (\mathcal{O}(n)^{\sigma} \otimes \mathrm{THH}^{\otimes n(\sigma)})_{hC(\sigma)})$$

Now, for every n, there is only one permutation $C_n \in \Sigma_n$ with $n(\sigma) = 1$, up to conjugacy: the length n cycle. Its centralizer is itself the length n cycle, and by Corollary 5.4.7[23], it acts on THH via the canonical map $C_n \to S^1$. Thus, we get an equivalence with

$$\mathrm{map}(\mathrm{THH}, \bigoplus_{n\geq 1}(\mathcal{O}(n)^{\gamma_n} \otimes \mathrm{THH})_{hC_n})$$

Now, we use Corollary 5.1.10 to rewrite this as

$$\lim_k(\bigoplus_{n\geq 1}\Sigma^k(\mathcal{O}(n)^{\gamma_n} \otimes \widetilde{\mathrm{THH}}(A_k)_{hC_n})$$

Now we use again Proposition 5.4.19, Lemma 5.4.21 and Proposition 5.4.12 to obtain

$$\lim_k \bigoplus_{n\geq 1}\Sigma^k(\mathcal{O}(n)^{\gamma_n} \otimes \mathrm{Ind}_e^{S^1}\Omega^k\mathbb{S})_{hC_n} \simeq \bigoplus_{n\geq 1}(\mathcal{O}(n)^{\gamma_n} \otimes \mathbb{S}[S^1])_{hC_n} \simeq \bigoplus_{n\geq 1}\mathbb{S}[\mathcal{O}(n)^{\gamma_n} \times S^1]_{hC_n}$$

which was to be proved. □

As explained in the beginning of this section, this proof does not work integrally: it should be clear that we used Proposition 5.4.19 repeatedly, and this is crucially a truly chromatic or rational phenomenon. We have, in any case, indicated in Theorem 5.4.11 that the result fails in the integral case. I do not have a guess for what the correct answer is: it is not clear to me whether the operations from Theorem 5.4.11 account for all the difference in the integral case, or if the commutation between inverse limits and infinite direct sums fails badly enough that there are other “exotic” operations.

[23]Though the proof really is in Proposition C.0.2.

Chapter 6

Questions and perspectives

The goal of this chapter is to record a number of questions we have not answered in the body of this text, as well as perspectives for future research on the subjects at hand. I will probably be thinking at the very least on-and-off about these questions, but anyone reading this document is more than welcome to think about them on their own, and I'd be more than happy to discuss them with them.

6.1 Trace theories

Question 6.1.1. In Theorem 4.2.11, we have classified cocontinuous trace theories. Many variants of this classification remain open, in the polynomial direction.

(i) For any spectrum with S^1-action X, $(C, T) \mapsto (X \otimes \mathrm{THH}(C, T^n))_{hC_n}$ upgrades to a fiberwise n-homogeneous, finitary localizing trace-theory. Are these the only examples ?

(ii) Removing the "localizing" condition in the previous item, we also find tensor products of trace theories of the above form with $n_1, ..., n_k$ such that $\sum_i n_i = n$ gives other examples of fiberwise n-homogeneous finitary trace theories. Are they the only ones ?

(iii) Removing the "finitary" condition instead, can we relate localizing, fiberwise n-homogeneous trace theories and exact trace theories through the T^n ?

(iv) Finally, removing *all* conditions but fiberwise n-homogeneity, what can we say ? In that case, I no longer have a guess/conjecture.

◁

Question 6.1.2. Does the ∞-category PgSp of polygonic spectra from [KMN23] have a universal property from the perspective of trace theories ? ◁

6.2 Operads

In this section, $\mathcal{O}$ denotes an operad, such as Comm, and $F \dashv U$ denotes the corresponding free-forgetful adjunction for $\mathcal{O}$-algebras in $\mathrm{Cat}^{\mathrm{perf}}$.

Obviously, the partialness of Theorem J raises:

Question 6.2.1. What is

$$\mathrm{end}_{\mathrm{Fun}(\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathrm{perf}}),\mathrm{Sp})}(\mathrm{THH} \circ U, \mathrm{THH} \circ U)$$

? What about tensor powers of THH ? ◁

In principle, if one is able to compute, for all $p, q \geq 1$,

$$\mathrm{Map}(\mathrm{THH}^{\otimes p} \circ U, \mathrm{THH}^{\otimes q} \circ U)$$

one should be able to compute $\mathrm{Map}_{\mathrm{Fun}(\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathrm{perf}}),\mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp})}(\mathrm{THH}, \mathrm{THH})$, which leads to:

Question 6.2.2. What is $\mathrm{End}_{\mathrm{Fun}(\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}^{\mathrm{perf}}),\mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp}))}(\mathrm{THH})$?

How does it relate to $\mathrm{End}^{\otimes-\mathcal{O}}(\mathrm{THH})$, the space of $\mathcal{O}$-monoidal endomorphisms of THH? ◁

Finally, we point out that we got a formula for $\mathrm{THH} \circ UF(C)$ only in terms of $\mathrm{THH}(C)$ with its S^1-action. This leads to the following natural question:

Question 6.2.3. Is there a monad T on Sp^{BS^1} such that THH intertwines the monad UF on $\mathrm{Cat}^{\mathrm{perf}}$ and T ?

Can we use this monad to get obstructions to the following realization problem: "Is any commutative algebra in $\mathrm{Mot}^{\mathrm{loc}}$ of the form $\mathcal{U}^{\mathrm{loc}}(C)$ for some commutative algebra C ?" ? ◁

The following is unrelated to our work in the main body of the text but is inspired by this last question:

Question 6.2.4. Let C be a stable ∞-category. What kind of structure on $K(C)$ is needed to recover $K(UF(C))$, where K is algebraic K-theory ? ◁

6.3 Extra structure on THH

Besides its cyclotomic structure, THH admits extra structure in certain restrictive cases. For example, when restricted to $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{perf}})$, THH comes with the "even filtration"; and when restricted to animated commutative rings $\mathrm{HH}_{\mathbb{Z}}$ comes with the HKR filtration.

Question 6.3.1. How do the endomorphisms we computed interact with these filtrations? What are endomorphisms of THH, resp. $\mathrm{HH}_{\mathbb{Z}}$ when viewed as a functor to filtered spectra ? ◁

Finally, we have indicated that, while we obtain extra operations on HH_k or on THH *as a symmetric monoidal functor*, we do not really know what these operations are or what they do.

Question 6.3.2. How can we understand these operations on HH_k, or on $\mathrm{THH}^{\otimes}$? Are they good for something ? ◁

Appendix A

The cyclic category

The goal of this appendix is to gather basic facts about Connes' cyclic category Λ and related functors and categories, which will be used in the main body of Part II. Most (if not all) of the material here can be extracted from [NS18] or [HS19].

First, we define the paracyclic category.

Definition A.0.1. The paracyclic category Λ_∞ is the full subcategory of the category of linearly ordered posets with a $\mathbb{Z}$-action spanned by the posets isomorphic to $\frac{1}{n}\mathbb{Z}$ with the $\mathbb{Z}$-action given by $+1$. ◁

Remark A.0.2. More abstractly, these are characterized as linearly ordered $\mathbb{Z}$-posets such that every element has a successor and a predecessor, such that the transitive-symmetric-reflexive closure of the "successor" relation is the full relation; and such that $x \leq \sigma x$ where σ is the generator of $\mathbb{Z}$. ◁

Notation A.0.3. For $n \in \mathbb{N}_{\geq 0}$, we let $[n]_\infty \in \Lambda_\infty$ denote $\frac{1}{n+1}\mathbb{Z}$ with the $+1$-action. ◁

Warning A.0.4. Our convention here (and therefore, all the later ones) differs from that of [NS18] by a $+1$. While this $+1$ makes sense given the above formula, it is also confusing with respect to the functor $(-)_\infty : \Delta \to \Lambda_\infty$: the formula $[n] \mapsto [n]_\infty$ is much more convenient, and since it is in this form that our convention will mostly be used, we stick to this one. ◁

Construction A.0.5. There is a functor $(-)_\infty : \Delta \to \Lambda_\infty$ sending a finite linearly ordered poset S to $\mathbb{Z} \times S$ with the lexicographic ordering and $\mathbb{Z}$-action on the left factor. If $S = [n]$, this is isomorphic to $\frac{1}{n+1}\mathbb{Z} =: [n]_\infty$. ◁

Proposition A.0.6. *The functor above is initial.*

Remark A.0.7. This is proved in [NS18, Theorem B.3], but it seems like their proof contains a(n eventually fixable) mistake: in their notation, $\mathcal{C}[t, t+1) \cap \mathcal{C}[s, s+1)$ is often empty, and thus does not allow for the induction to run as they do. We discuss below the necessary corrections to their proof - this is based on a discussion with Tim Campion, available at [Cam], though I believe the proof there also contains a mistake. ◁

Proof. We need to prove that for every $T \in \Lambda_\infty$, the slice $\mathcal{C} := \Delta \times_{\Lambda_\infty} (\Lambda_\infty)_{/T}$ is weakly contractible.

Note that for $S \in \Delta$, a map $S_\infty \to T$ is the same as a map $S \to T$ such that $f(\max S) \leq f(\min S) + 1$[1].

[1] This can be proved by noting that, as a *set* with $\mathbb{Z}$-action, $\mathbb{Z} \times S$ is free on S.

Fix $T = \frac{1}{n}\mathbb{Z}$, and for $a, b \in T$ we let $\mathcal{C}[a,b]$ denote the full subcategory of $\mathcal{C}$ spanned by those maps $f : S \to T$ such that $f(S) \subset [a,b]$.

In particular, if $b > a+1$, we have a pushout square:

$$\begin{tikzcd} \mathcal{C}[b-1, b-\frac{1}{n}] \arrow[r] \arrow[d] & \mathcal{C}[b-1,b] \arrow[d] \\ \mathcal{C}[a, b-\frac{1}{n}] \arrow[r] & \mathcal{C}[a,b] \end{tikzcd}$$

since each map $f : S \to T$ in $\mathcal{C}[a,b]$ either hits b and therefore must be in the top right corner, or does not, in which case it must be in the bottom left corner.

By induction, this reduces us to the situation where $b \leq a+1$, in which case, an object in $\mathcal{C}[a,b]$ is just a map $S \to [a,b] \subset T$, with no extra condition: in this case, $\mathcal{C}[a,b] \simeq \Delta_{/[a,b]}$ has a terminal object and hence is contractible. $\square$

Construction A.0.8. Λ_∞ has a $S^1 = B\mathbb{Z}$-action, that is[2], an endomorphism of the identity functor simply given by the generator of the $\mathbb{Z}$-action. $\triangleleft$

Definition A.0.9. The cyclic category Λ is defined to be the quotient of Λ_∞ by the S^1-action, namely $\Lambda := (\Lambda_\infty)_{hS^1}$. We let $\pi : \Lambda_\infty \to \Lambda$ denote the canonical functor. $\triangleleft$

Notation A.0.10. We let $[n]_\Lambda := \pi([n]_\infty)$ denote the image in Λ of $[n]_\infty$.

More generally, we also let $(-)_\Lambda$ denote the composite functor

$$\Delta \xrightarrow{(-)_\infty} \Lambda_\infty \xrightarrow{\pi} \Lambda$$

$\triangleleft$

In particular, we have a fiber sequence of ∞-categories:

$$\begin{tikzcd} \Lambda_\infty \arrow[r] \arrow[d] & \Lambda \arrow[d] \\ \mathrm{pt} \arrow[r] & BS^1 \end{tikzcd}$$

This allows us to compute mapping spaces in Λ:

Lemma A.0.11. *Let $n, m \in \mathbb{N}_{\geq 0}$. The mapping space $\mathrm{Map}_\Lambda([n]_\Lambda, [m]_\Lambda)$ is discrete, and is the (ordinary) quotient of $\hom_{\Lambda_\infty}([n]_\infty, [m]_\infty)$ by the $\mathbb{Z}$-action given by that on $[m]_\infty$.*

In particular, Λ is a 1-category.

Proof. We use the fiber sequence

$$\begin{tikzcd} \Lambda_\infty \arrow[r] \arrow[d] & \Lambda \arrow[d] \\ \mathrm{pt} \arrow[r] & BS^1 \end{tikzcd}$$

to obtain fiber sequences of mapping spaces of the form:

$$\begin{tikzcd} \hom_{\Lambda_\infty}([n]_\infty, [m]_\infty) \arrow[r] \arrow[d] & \mathrm{Map}_\Lambda([n]_\Lambda, [m]_\Lambda) \arrow[d] \\ \mathrm{pt} \arrow[r] & S^1 \end{tikzcd}$$

[2]Because Λ_∞ is a 1-category - for general ∞-categories, a $B\mathbb{Z}$-action is more data.

which witnesses $\mathrm{Map}_\Lambda([n]_\Lambda,[m]_\Lambda)$ as $\hom_{\Lambda_\infty}([n]_\infty,[m]_\infty)_{h\mathbb{Z}}$, where one checks that the action is the action on $[m]_\infty$ (or $[n]_\infty$).

One checks by hand that this action is *free* on the hom-set, and thus its quotient is the ordinary quotient. □

We also have:

Corollary A.0.12. *The functor $\Lambda \to BS^1$ induces an equivalence $|\Lambda| \simeq BS^1$.*

Proof. $|\Lambda| \simeq |\Lambda_\infty|_{hS^1}$ since $|-|$ commutes with colimits. Since $\Delta \to \Lambda_\infty$ is initial, it induces an equivalence upon realization, and Δ is sifted, so that $|\Delta| \simeq \mathrm{pt}$. The claim now follows. □

Given a cyclic object $X : \Lambda^{\mathrm{op}} \to \mathcal{E}$, where $\mathcal{E}$ admits geometric realizations, one can restrict X to Δ^{op} along $\Delta^{\mathrm{op}} \to (\Lambda_\infty)^{\mathrm{op}} \to \Lambda^{\mathrm{op}}$ and take its geometric realization. It is a classical fact (and one of the main motivations for the cyclic category) that this geometric realization admits a canonical S^1-action. For completeness, we record the construction below:

Construction A.0.13. Let $p : \Lambda \to BS^1$ be the canonical functor. We let $p_! : \mathrm{Fun}(\Lambda^{\mathrm{op}},\mathcal{E}) \to \mathrm{Fun}(BS^1,\mathcal{E})$ - we claim that this exists as soon as $\mathcal{E}$ admits geometric realizations and that in this case, letting $e : \mathrm{pt} \to BS^1$ be the inclusion of the basepoint, $e^*p_! : \mathrm{Fun}(\Lambda^{\mathrm{op}},\mathcal{E}) \to \mathrm{Fun}(BS^1,\mathcal{E}) \to \mathcal{E}$ is canonically isomorphic to $\mathrm{colim}_{\Delta^{\mathrm{op}}}$.

Indeed, $\Lambda^{\mathrm{op}} \to BS^1$ is a coCartesian fibration so the pointwise formula for left Kan extension simplifies to colimits over the fiber, i.e. over $(\Lambda_\infty)^{\mathrm{op}}$. Since $\Delta^{\mathrm{op}} \to (\Lambda_\infty)^{\mathrm{op}}$ is cofinal by Proposition A.0.6, this exists if and only if the restriction to Δ^{op} admits a colimit, and if it does, the canonical map between them is an equivalence. This proves all claims. ◁

Remark A.0.14. As for the "pointwise formula for left Kan extensions", this is implicitly an adjointability claim. ◁

Let us now give an alternative description of Λ.

Construction A.0.15. By definition, viewing posets as categories, we have an inclusion $\Lambda_\infty \subset \mathrm{Cat}^{B\mathbb{Z}}$. The colimit functor can be upgraded canonically to a $B\mathbb{Z}$-equivariant functor $\mathrm{Cat}^{B\mathbb{Z}} \to \mathrm{Cat}$, so that we obtain a $B\mathbb{Z}$-equivariant map $\Lambda_\infty \to \mathrm{Cat}$ of the form $P \mapsto P_{h\mathbb{Z}}$, and thus, a functor $i : \Lambda \to \mathrm{Cat}$ of this form. ◁

Observation A.0.16. The poset $\mathbb{Z}$ is weakly contractible, and hence each $|P_{h\mathbb{Z}}|, P \in \Lambda_\infty$ is equivalent to $B\mathbb{Z}$. Since P is a poset, any choice of a successor relation $x \leq x^+$ provides an orientation of $|P_{h\mathbb{Z}}|$, i.e. a generator of $H_1(|P_{h\mathbb{Z}}|)$, and it is elementary to check that any two x's yield the same generator.

Thus $|P_{h\mathbb{Z}}|$ is canonically an oriented circle. ◁

Lemma A.0.17. *Let $n \in \mathbb{N}_{\geq 0}$. The category $([n]_\infty)_{h\mathbb{Z}}$ is free on an oriented graph of the form $0 \to 1 \to ... \to n \to 0$.*

In particular it is a 1-category, and none of its objects have nontrivial automorphisms[3].

Proof. The category $\mathbb{Z}$ and hence $\frac{1}{n}\mathbb{Z}$ is free on the graph $\cdots \to 0 \to 1 \to \dots$, i.e. $\mathrm{map}(\frac{1}{n}\mathbb{Z}, C) \simeq \mathrm{eq}(\prod_{\frac{1}{n}\mathbb{Z}} C^{\Delta^1} \rightrightarrows \prod_{\frac{1}{n}\mathbb{Z}} C)$.

The $\mathbb{Z}$-action on each factor of the equalizer is cofree, which allows us to compute $\mathrm{Map}_{\mathrm{Cat}}(([n]_\infty)_{h\mathbb{Z}}, C)$ as an equalizer of the form $\mathrm{eq}(\prod_{\frac{1}{n}\mathbb{Z}/\mathbb{Z}} C^{\Delta^1} \rightrightarrows \prod_{\frac{1}{n}\mathbb{Z}/\mathbb{Z}} C)$, thus proving the claim. □

[3] As an object of Cat, it does have automorphisms which is why we are careful in stating this.

Lemma A.0.18. *Let $n, m \in \mathbb{N}_{\geq 0}$. The map*

$$\mathrm{Map}_\Lambda([n]_\Lambda, [m]_\Lambda) \to \mathrm{Map}_{\mathrm{Cat}}(([n]_\infty)_{h\mathbb{Z}}, ([m]_\infty)_{h\mathbb{Z}})$$

obtained via the previous construction is fully faithful.

A functor is in the image if and only if the induced map $|([n]_\infty)_{h\mathbb{Z}}| \to |([m]_\infty)_{h\mathbb{Z}}|$ has degree 1.

Proof. By Lemma A.0.11 and Lemma A.0.17, both sides are ordinary sets, so that this is really an injectivity statement, and this can be checked by explicitly describing both. □

As an immediate corollary, we obtain:

Corollary A.0.19. *Λ is equivalent to the non-full subcategory of* Cat *spanned by categories free on a finite cyclic directed graph, and functors between them that induce degree* 1 *maps on their realizations.*

We will freely use this corollary in describing Λ and related constructions.

Let us describe typical morphisms in Λ. This will be particularly helpful when checking whether a functor out of a fibration $E \to \Lambda$ sends coCartesian morphisms to equivalences.

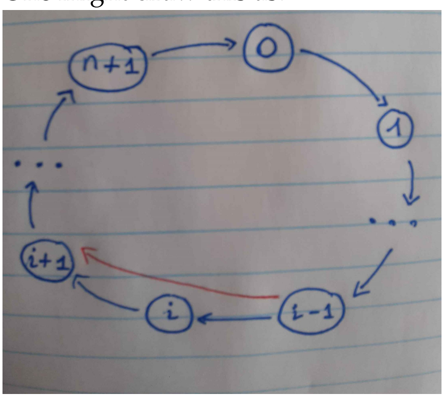
Construction A.0.20. Let $n \in \mathbb{N}$ and $i \in \{0, ..., n+1\}$. We construct a morphism $[n]_\Lambda \to [n+1]_\Lambda$ which *skips i*: on the free generating graph

$$0 \to ... \to i-1 \to i \to ... \to n \to 0$$

it sends $j < i$ to j, $j \geq i$ to $j+1$ and the generating edge $i-1 \to i$ is sent to the composite $i-1 \to i \to i+1$ in $[n+1]_\Lambda$.

We will denote this morphism typically by "$(i-1 \to i) \mapsto (i-1 \to i+1)$" if there is no ambiguity.

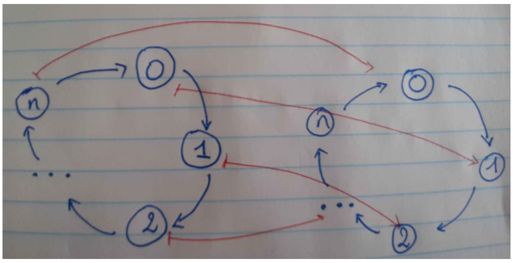
One might draw this as:



◁

Construction A.0.21. Let $n \in \mathbb{N}$. The automorphism group of $[n]_\Lambda$ is C_n: any automorphism induces an automorphism on $([n]_\Lambda)^\simeq \simeq \{0, ..., n\}$ and so a permutation. Since it is an equivalence it must send irreducible arrows to irreducible arrows and hence must be given by a cyclic permutation. Conversely every cyclic permutation is clearly realized by an automorphism of $[n]_\Lambda$.

Pictorially, this looks like:

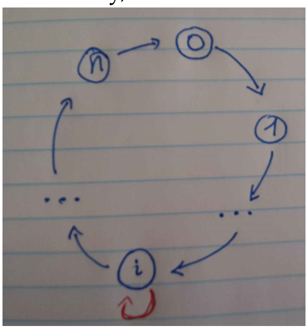


◁

Construction A.0.22. Let $n \in \mathbb{N}$ and $i \in \{0, ..., n+1\}$. We construct a morphism $[n+1]_\Lambda \to [n]_\Lambda$ that sends objects $j \le i$ to j, objects $j \ge i+1$ to $j-1$ and finally the edge $i \to i+1$ to the identity endomorphism of i.

We denote this morphism by "$(i \to i+1) \mapsto i$" if there is no ambiguity.

Pictorially, this looks like:



◁

Lemma A.0.23. *Fix $n \in \mathbb{N}$ and $i \in \{0, ..., n+1\}$. The composite*

$$[n]_\Lambda \to [n+1]_\Lambda \to [n]_\Lambda$$

given by $(i \to i+1) \mapsto (i \to i+2)$ followed by $(i \to i+1) \mapsto i$ is the identity.

Proof. Check on objects and generating morphisms. □

The following is a simple exercise in combinatorics:

Proposition A.0.24. *Any morphism in Λ is a composite of morphisms constructed in Constructions A.0.20 to A.0.22.*

Corollary A.0.25. *Let $p : E \to \Lambda$ be a cartesian (resp. coCartesian) fibration, and let $f : E \to C$ be a functor. If f sends all p-cartesian (resp. p-coCartesian) morphisms lying over edges of the form $(i \to i+1) \mapsto (i \to i+2)$ to equivalences in C, then f sends all p-cartesian (resp. p-coCartesian) morphisms to equivalences.*

Proof. The morphisms from Construction A.0.21 are equivalences, so p-cartesian morphisms lying over them are equivalences, and hence are sent to equivalences by any functor.

By Lemma A.0.23, the maps from Construction A.0.22 are one-sided inverses of the ones from Construction A.0.20, thus the same holds for the p-cartesian morphisms lying over them: if $x \to y, y \to z$ are p-cartesian morphisms lying over α, β, and $\beta \circ \alpha \simeq \mathrm{id}$, then $x \to z$ is an equivalence (since it is p-cartesian and lying over an equivalence).

Thus if f inverts the p-cartesian morphisms lying over the latter to equivalences, the same holds for the ones lying over the former.

By Proposition A.0.24, this concludes the proof. □

The final object we will need from this appendix is a variant of Λ called $\Lambda_m, m \in \mathbb{N}$. Recall that $\Lambda := (\Lambda_\infty)_{hS^1}$.

Definition A.0.26. For $m \in \mathbb{N}$, let Λ_m denote $(\Lambda_\infty)_{hS^1}$ where the action is restricted along $S^1 \xrightarrow{m} S^1$. ◁

In other words, Λ_m fits in a pullback square of the form:

$$\begin{tikzcd} \Lambda_m \arrow[r] \arrow[d] & \Lambda \arrow[d] \\ BS^1 \arrow[r, "m"'] & BS^1 \end{tikzcd}$$

In particular, there is also a pullback square of the form:

$$\begin{tikzcd} \Lambda_m \arrow[r] \arrow[d] & \Lambda \arrow[d] \\ \mathrm{pt} \arrow[r] & B^2C_m \end{tikzcd}$$

which witnesses Λ as a quotient of Λ_m by a BC_m-action.

Notation A.0.27. Let $\pi_m : \Lambda_\infty \to \Lambda_m$ denote the canonical projection, and let $q_m : \Lambda_m \to \Lambda$ the canonical projection as well; so that $q_m \circ \pi_m = \pi$. We also let $[n]_{\Lambda_m} := \pi_m([n]_\infty)$. ◁

There is another relevant functor here, and it is important not to confuse it with q_m.

Notation A.0.28. Let $\Psi_m : \Lambda_\infty \to \Lambda_\infty$ denote the functor sending a poset P with automorphism σ to (P, σ^m).

It is not hard to verify that it is well-defined and $B\mathbb{Z}$-equivariant where the source has the $B\mathbb{Z} \simeq Bm\mathbb{Z} \to B\mathbb{Z}$-action, and the target has its usual $B\mathbb{Z}$-action. ◁

In particular, Ψ_m induces, upon taking S^1-orbits, a functor $\overline{\Psi}_m : \Lambda_m \to \Lambda$.

Example A.0.29. For $P = [n]_\infty = \frac{1}{n+1}\mathbb{Z}$, $\Psi_m(P)$ is easily checked to be

$$[(n+1)m-1]_\infty = \frac{1}{(n+1)m}\mathbb{Z}$$

◁

We thus have a span:

$$\begin{tikzcd} & \Lambda_m \arrow[dl, "\overline{\Psi}_m"'] \arrow[dr, "q_m"] & \\ \Lambda & & \Lambda \end{tikzcd}$$

where q_m is a quotient map and $\overline{\Psi}_m$ is some map.

The following construction will be important when we consider Frobenii of trace theories:

Construction A.0.30. For $P \in \Lambda_\infty$, consider the canonical map

$$P_{hm\mathbb{Z}} \to P_{h\mathbb{Z}}$$

We can reinterpret its source as $(\Psi_m P)_{h\mathbb{Z}}$, and so this is a morphism of categories, though it does not lie in Λ (it has degree m upon realization).

Furthermore, by design it fits into a pullback square as follows:

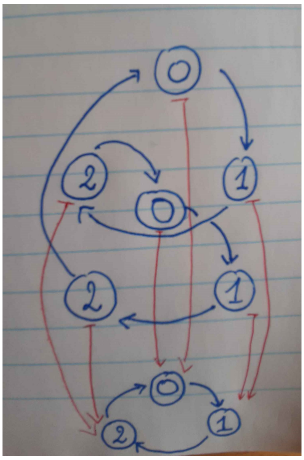
$$\begin{array}{ccc} P_{hm\mathbb{Z}} & \longrightarrow & P_{h\mathbb{Z}} \\ \downarrow & & \downarrow \\ B(m\mathbb{Z}) & \longrightarrow & B\mathbb{Z} \end{array}$$

Unwinding the definitions, this implies that it is $B(m\mathbb{Z})$-equivariant, and thus in total it descends to Λ_m. In other words, we obtain a natural transformation of functors $\Lambda_m \to \mathrm{Cat}_\infty$ of the form

$$\psi_m : \overline{\Psi}_m \to q_m$$

and while its source and target are in the image of Λ, *it* is not. ◁

Here is a drawing of ψ_2 evaluated at $P = \frac{1}{3}\mathbb{Z} = [2]_\infty$ (the two "2"'s are unrelated):



Appendix B

Splitting motives

In this appendix, we gather a few facts about splitting motives[1] which will be convenient for Section 5.1.

First, let us fix some notation and terminology.

Notation B.0.1. Let C be a stable ∞-category. We let $\mathrm{Cof}(C) \subset C^{\square}$ denote the full subcategory of cofiber sequences, that is, squares whose bottom left corner is 0, and that are coCartesian. $\triangleleft$

This is the analogue of what Waldhausen called $S_2(C)$ in the context of his $S_\bullet$-construction.

Definition B.0.2. A split exact sequence is a sequence $C \xrightarrow{i} D \xrightarrow{p} Q$ of stable ∞-categories where:

(i) $p \circ i = 0$;
(ii) p and i both have right adjoints p^R and i^R respectively;
(iii) i and p^R are fully faithful[2];
(iv) the canonical nullsequence[3] $ii^R \to \mathrm{id}_D \to p^R p$ is a cofiber sequence.

$\triangleleft$

Remark B.0.3. Split exact sequences are essentially equivalent to semi-orthogonal decompositions. $\triangleleft$

The following is the key example of a split exact sequence, it is essentially universal:

Example B.0.4. Let C be a stable ∞-category, and let $i : C \to \mathrm{Cof}(C)$ be given by $x \mapsto (x \to x \to 0)$, $p : \mathrm{Cof}(C) \to C$ be given by evaluation the the bottom right corner, so $(x \to y \to z) \mapsto z$.

i^R is given by evaluation at the top left corner, while $p^R : z \mapsto (0 \to z \to z)$.

We let $f, t, c : \mathrm{Cof}(C) \to C$ denote the evaluation functors at the three nonzero corners[4]. They fit into a canonical cofiber sequence $f \to t \to c$ of functors $\mathrm{Cof}(C) \to C$. $\triangleleft$

[1]These are typically called "additive motives" in the literature. This, especially the corresponding notion of "additive invariant" sounds very confusing and less descriptive than "splitting", so I have opted for this change of name.
[2]Equivalently, the unit $\mathrm{id}_C \to i^R i$ and the counit $pp^R \to \mathrm{id}_Q$ are equivalences.
[3]For any functors f, g, $\mathrm{map}(if, p^R g) = 0$, so this nullsequence is canonical and unique.
[4]f stands for "fiber", t for "total" and c for "cofiber".

We further recall that $\mathrm{Cat}^{\mathrm{ex}}$ is a semi-additive ∞-category, so that for any ∞-category $\mathcal{E}$ with finite products, every finite-product preserving functor $f : \mathrm{Cat}^{\mathrm{ex}} \to \mathcal{E}$ admits a canonical lift to $\mathrm{CMon}(\mathcal{E})$, so we can add maps between values of f. In this context, we have:

Theorem B.0.5 (Waldhausen)**.** *Let $E : \mathrm{Cat}^{\mathrm{ex}} \to \mathcal{E}$ be a finite-product-preserving functor. The following are equivalent:*

(i) For any cofiber sequence of exact functors $F, G, H : C \to D$, $F \to G \to H$, there exists some homotopy $E(G) \simeq E(F) + E(H)$;

(ii) For any stable ∞-category C, and for the canonical cofiber sequence of functors $f \to t \to c : \mathrm{Cof}(C) \to C$, $E(t) \simeq E(f) + E(c)$;

(iii) For any stable C, the functor E applied to $\mathrm{Cof}(C) \xrightarrow{(f,c)} C \times C$ yields an equivalence;

(iv) For any split exact sequence $C \xrightarrow{i} D \xrightarrow{p} Q$, letting r denote the right adjoint to i, E applied to $D \xrightarrow{(r,p)} C \times Q$ yields an equivalence.

Definition B.0.6. A functor $E : \mathrm{Cat}^{\mathrm{ex}} \to \mathcal{E}$ is called a splitting invariant if it preserves finite products and satisfies any (and hence all) of the equivalent conditions of the previous theorem.

It is called *finitary* if it preserves filtered colimits. ◁

Definition B.0.7. Let $\mathcal{U}_{\mathrm{split}} : \mathrm{Cat}^{\mathrm{ex}} \to \mathrm{Mot}_{\mathrm{split}}$ denote the universal finitary splitting invariant with values in a stable ∞-category[5]. It exists by [BGT13], and $\mathrm{Mot}_{\mathrm{split}}$ admits a unique presentably symmetric monoidal structure for which $\mathcal{U}_{\mathrm{split}}$ is symmetric monoidal. ◁

We take for granted[6] the following main result from [BGT13], and deduce two keys fact about mapping spaces in $\mathrm{Mot}_{\mathrm{split}}$ from it which we will use in Section 5.1.

Theorem B.0.8 ([BGT13, Theorem 1.3])**.** *There is an equivalence*

$$K^{\mathrm{cn}} \simeq \mathrm{map}(\mathcal{U}_{\mathrm{split}}(\mathrm{Sp}^{\omega}), \mathcal{U}_{\mathrm{split}}(-))$$

Here, K^{cn} denotes connective K-theory.

Remark B.0.9. We depart slightly from [BGT13] by not requiring splitting invariants to be invariant under idempotent-completion. This is a mild difference, and only affects K_0. ◁

We can essentially take this theorem as our definition of K^{cn}. Thus, the following is only really a corollary if one has an *a priori* definition of K-theory.

Before we state it, note that we have a canonical internal hom comparison map of the form:

$$\mathcal{U}_{\mathrm{split}}(\mathrm{Fun}^{\mathrm{ex}}(A, B)) \to \hom(\mathcal{U}_{\mathrm{split}}(A), \mathcal{U}_{\mathrm{split}}(B))$$

whenever $A, B \in \mathrm{Cat}^{\mathrm{ex}}$. By taking maps from the unit, this induces an map

$$K^{\mathrm{cn}}(\mathrm{Fun}^{\mathrm{ex}}(A, B)) \to \mathrm{map}_{\mathrm{Mot}_{\mathrm{split}}}(\mathcal{U}_{\mathrm{split}}(A), \mathcal{U}_{\mathrm{split}}(B))$$

Corollary B.0.10. *Let $A, B \in \mathrm{Cat}^{\mathrm{ex}}$, with A being a compact object therein. The canonical map*

$$K^{\mathrm{cn}}(\mathrm{Fun}^{\mathrm{ex}}(A, B)) \to \mathrm{map}_{\mathrm{Mot}_{\mathrm{split}}}(\mathcal{U}_{\mathrm{split}}(A), \mathcal{U}_{\mathrm{split}}(B))$$

described above is an equivalence.

[5] Strictly speaking, we do not need stability here. It turns out to be more convenient for our use of $\mathrm{Mot}_{\mathrm{split}}$ in Section 5.1 - though not necessary - so we choose this convention, but for other purposes it is maybe more canonical not to require this, though the not-necessarily-stable version automatically embeds in the stable version.

[6] Or for definition.

Proof. Fix A, and consider B a variable.

Since A is compact, $K^{\mathrm{cn}}(\mathrm{Fun}^{\mathrm{ex}}(A,-))$ is finitary, and since $\mathrm{Fun}^{\mathrm{ex}}(A,-)$ preserves split exact sequences[7], $K^{\mathrm{cn}}(\mathrm{Fun}^{\mathrm{ex}}(A,-))$ is a splitting invariant, hence in total a finitary splitting invariant.

One also proves ahead of time (essentially using item (iii) in Theorem B.0.5) that $\mathcal{U}_{\mathrm{split}}(A)$ is compact in $\mathrm{Mot}_{\mathrm{split}}$, so that $\mathrm{map}_{\mathrm{Mot}_{\mathrm{split}}}(\mathcal{U}_{\mathrm{split}}(A), \mathcal{U}_{\mathrm{split}}(-))$ is also a finitary splitting invariant.

We now use the Yoneda lemma: let $F : \mathrm{Cat}^{\mathrm{ex}} \to \mathrm{Sp}$ be a finitary splitting invariant. We then have equivalences:

$$\mathrm{map}(K^{\mathrm{cn}}(\mathrm{Fun}^{\mathrm{ex}}(A,-)), F) \simeq \mathrm{map}(K^{\mathrm{cn}}, F(A \otimes -)) \simeq F(A \otimes \mathrm{Sp}^{\omega}) \simeq F(A)$$

where the first equivalence is by adjunction, the second uses the Yoneda lemma with $K^{\mathrm{cn}} \simeq \mathrm{map}_{\mathrm{Mot}_{\mathrm{split}}}(\mathcal{U}_{\mathrm{split}}(\mathrm{Sp}^{\omega}), \mathcal{U}_{\mathrm{split}}(-))$ and the fact that $F(A \otimes -)$ is still a finitary splitting invariant.

By the Yoneda lemma, we also have

$$\mathrm{map}(\mathrm{map}_{\mathrm{Mot}_{\mathrm{split}}}(\mathcal{U}_{\mathrm{split}}(A), \mathcal{U}_{\mathrm{split}}(-)), F) \simeq F(A)$$

It follows that there is an equivalence as desired. It is not difficult to verify that the equivalence in question is indeed given by the internal hom-comparison map, for example by rewriting this second equivalence as the string:

$$\mathrm{map}(\mathrm{map}_{\mathrm{Mot}_{\mathrm{split}}}(\mathcal{U}_{\mathrm{split}}(A), \mathcal{U}_{\mathrm{split}}(-)), F) \simeq \mathrm{map}(\mathrm{map}(\mathcal{U}_{\mathrm{split}}(\mathrm{Sp}^{\omega}), \mathrm{hom}(\mathcal{U}_{\mathrm{split}}(A), \mathcal{U}_{\mathrm{split}}(-))), F)$$

$$\simeq \mathrm{map}(K^{\mathrm{cn}}, F(\mathcal{U}_{\mathrm{split}}(A) \otimes -)) \simeq F(A)$$

□

Corollary B.0.11. *Let $A, B \in \mathrm{Cat}^{\mathrm{ex}}$, with A being a compact object therein. The internal hom-comparison map*

$$\mathcal{U}_{\mathrm{split}}(\mathrm{Fun}^{\mathrm{ex}}(A,B)) \to \mathrm{hom}(\mathcal{U}_{\mathrm{split}}(A), \mathcal{U}_{\mathrm{split}}(B))$$

is an equivalence.

Proof. It suffices to check that mapping in from $\mathcal{U}_{\mathrm{split}}(C), C \in (\mathrm{Cat}^{\mathrm{ex}})^{\omega}$ produces an equivalence.

Since C is compact, Corollary B.0.10 shows that mapping in from C gives the following map:

$$K^{\mathrm{cn}}(\mathrm{Fun}^{\mathrm{ex}}(C, \mathrm{Fun}^{\mathrm{ex}}(A,B))) \to \mathrm{map}_{\mathrm{Mot}_{\mathrm{split}}}(\mathcal{U}_{\mathrm{split}}(C), \mathrm{hom}(\mathcal{U}_{\mathrm{split}}(A), \mathcal{U}_{\mathrm{split}}(B)))$$

which is, in turn:

$$K^{\mathrm{cn}}(\mathrm{Fun}^{\mathrm{ex}}(C \otimes A, B)) \to \mathrm{map}_{\mathrm{Mot}_{\mathrm{split}}}(\mathcal{U}_{\mathrm{split}}(C) \otimes \mathcal{U}_{\mathrm{split}}(A), \mathcal{U}_{\mathrm{split}}(B))$$

Since $\mathcal{U}_{\mathrm{split}}$ is strong monoidal, using that $A \otimes C$ is compact and using again Corollary B.0.10, we conclude that this map is an equivalence. □

[7] This is the crucial difference with the story in the localizing case.

Beyond splitting invariants/motives, there is the crucial notion of a *localizing* invariant/motive[8].

Definition B.0.12. A null-sequence $C \to D \to Q$ of stable ∞-categories is called a Karoubi sequence, or localization sequence, if $\mathrm{Ind}(C) \to \mathrm{Ind}(D) \to \mathrm{Ind}(Q)$ is a split exact sequence.
One can show that this is equivalent to:

(i) $C \to D$ is fully faithful;
(ii) the sequence is a cofiber sequence in $\mathrm{Cat}^{\mathrm{perf}}$, the ∞-category of idempotent-complete stable ∞-categories.

$\triangleleft$

With this in hand, we can define:

Definition B.0.13. A localizing invariant with values in a stable ∞-category $\mathcal{E}$ is a functor $E : \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{E}$ sending 0 to 0 and Karoubi sequences to co/fiber sequences.
It is called finitary if it preserves filtered colimits. $\triangleleft$

And finally, we have:

Definition B.0.14. We let $\mathcal{U}_{\mathrm{loc}} : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Mot}_{\mathrm{loc}}$ denote the universal finitary localizing invariant. It exists by [BGT13], and $\mathrm{Mot}_{\mathrm{loc}}$ is presentable and stable. $\triangleleft$

A result which we do not prove here, but will be used once in the body of the thesis is the following result, obtained in joint work with Sosnilo and Winges [RSW24]:

Theorem B.0.15. *The functor $\mathcal{U}_{\mathrm{loc}} : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Mot}_{\mathrm{loc}}$ is a Dwyer-Kan localization (at the class of morphisms it inverts, called motivic equivalences).*

The point of this theorem is that it allows us to compare mapping spaces/spectra of the form $\mathrm{Map}_{\mathrm{Fun}(\mathrm{Mot}_{\mathrm{loc}},\mathcal{E})}(F, G)$ and $\mathrm{Map}_{\mathrm{Fun}(\mathrm{Cat}^{\mathrm{perf}},\mathcal{E})}(F \circ \mathcal{U}_{\mathrm{loc}}, G \circ \mathcal{U}_{\mathrm{loc}})$ even when F, G need not be cocontinuous functors out of $\mathrm{Mot}_{\mathrm{loc}}$. We will use this for F, G being functors such as $\mathrm{THH}^{\otimes n}$, which, while clearly not cocontinuous in general (it is n-excisive), still clearly factors through $\mathrm{Mot}_{\mathrm{loc}}$.

[8]Our localizing invariants will be Morita invariant, also known as Karoubi localizing invariants. In this setting, this is the standard convention, though we note that K^{cn} is an interesting example of a "Verdier localizing invariant" which is not Morita invariant.

Appendix C

Actions on traces

Let $\mathbf{C}$ be a symmetric monoidal ∞-category. In [HSS17], the authors construct a trace map

$$\mathrm{tr} : \mathrm{Map}(S^1, \mathbf{C}^{\mathrm{dbl}}) \to \mathrm{End}(\mathbf{1}_{\mathbf{C}})$$

It is S^1-equivariant, and in particular induces a map $\mathbf{C}^{\mathrm{dbl}} \to \mathrm{End}(\mathbf{1}_{\mathbf{C}})^{BS^1}$ upon taking S^1-fixed points.

In particular, whenever $A \to \mathbf{C}^{\mathrm{dbl}}$ is a map from a space A, we obtain a map $LA \to \mathrm{End}(\mathbf{1}_{\mathbf{C}})$.

Example C.0.1. Let $x \in \mathbf{C}^{\mathrm{dbl}}$, and consider the Σ_n-action on $x^{\otimes n}$. It corresponds formally to a map $B\Sigma_n \to \mathbf{C}^{\mathrm{dbl}}$ which in turn induces a map $LB\Sigma_n \to \mathrm{map}(S^1, \mathbf{C}^{\mathrm{dbl}}) \to \mathrm{End}(\mathbf{1}_{\mathbf{C}})$.

Restricting, e.g. to the conjugacy class of the length n-cycle σ in Σ_n, we obtain a map $BC_n \to \mathrm{End}(\mathbf{1}_{\mathbf{C}})$ sending the point to $\mathrm{tr}(x^{\otimes n}; \sigma)$, i.e. a C_n-action on this trace. ◁

We have observed in Lemma 5.4.6 that for this length n cycle, $\mathrm{tr}(x^{\otimes n}; \sigma) = \dim(x)$, and thus the above construction produces a C_n-action on $\dim(x)$.

The goal of this appendix is to prove the following claim:

Proposition C.0.2. *The equivalence*

$$\mathrm{tr}(x^{\otimes n}; \sigma) \simeq \dim(x)$$

is C_n-equivariant, where the right hand side has the C_n-action described in Example C.0.1, and the left hand side has the C_n-action restricted from the usual S^1-action.

As usual, it suffices to prove this in the universal case, i.e. where $\mathbf{C} = \mathrm{Cob}_1$ and x is the universal dualizable object.

To prove this case, we observe the following:

Observation C.0.3. The symmetric monoidal functor $\mathrm{Cob}_1 \to \mathrm{coSpan}(\mathcal{S}^{\mathrm{fin}})$ induces, on endomorphisms of the unit, a map which sends the circle (as a framed manifold) to the circle (as a homotopy type), and on automorphisms of those, it is given by $(\mathrm{id}, 0) : S^1 \to S^1 \times \mathbb{Z}/2$. ◁

(this can be proved by explicitly describing the functor $\mathrm{Cob}_1 \to \mathrm{coSpan}(\mathcal{S}^{\mathrm{fin}})$, using that Cob_1 is essentially a category of cospans)

In particular, to prove that the C_n-actions agree, it suffices to do so after passing to $\mathrm{coSpan}(\mathcal{S}^{\mathrm{fin}})$, where we have more freedom. For example, we can try to identify the C_n-action *on the trace of the universal C_n-action* (for some sense of universal).

Consider the natural map $A \to \mathrm{coSpan}(\mathcal{S}^{\mathrm{fin}}/A)$. By the construction above, it induces a natural map $LA \to (\mathcal{S}^{\mathrm{fin}}/A)^{\simeq}$, as the unit in $\mathrm{coSpan}(\mathcal{S}^{\mathrm{fin}}/A)$ is , and so its endomorphism space is precisely $(\mathcal{S}^{\mathrm{fin}}/A)^{\simeq}$.

By naturality in A, the Yoneda lemma implies that this is completely determined by the value at S^1 of the identity in $LS^1 = \mathrm{Map}(S^1, S^1)$, which is, in turn, a certain map $S^1 \to S^1$.

Comparing $\mathcal{S}^{\mathrm{fin}}/S^1 \simeq (\mathcal{S}^{B\mathbb{Z}})^{\mathrm{fin}}$,we find that this trace, in $\mathrm{coSpan}((\mathcal{S}^{B\mathbb{Z}})^{\mathrm{fin}})$ is the pushout of the following span:

$$\begin{tikzcd} \mathbb{Z} & & \mathbb{Z} \\ & \mathbb{Z} \coprod \mathbb{Z} \arrow[ul, "(\mathrm{id},\mathrm{id})"] \arrow[ur, "(\mathrm{id},\sigma)"'] & \end{tikzcd}$$

which one easily computes to be pt (here, σ denotes the successor function). Now, the object pt corresponds to the identity map $S^1 \to S^1$ under the equivalence $\mathcal{S}^{B\mathbb{Z}} \simeq \mathcal{S}_{/S^1}$, so this proves (by the Yoneda lemma) that the map $LA \to (\mathcal{S}^{\mathrm{fin}}/A)^{\simeq}$ we constructed with traces is simply the composite

$$LA = (\mathcal{S}^{\mathrm{fin}}/A \times_{\mathcal{S}^{\mathrm{fin}}} \{S^1\})^{\simeq} \to (\mathcal{S}^{\mathrm{fin}}/A)^{\simeq}$$

Proof of Proposition C.0.2. By the discussion in Observation C.0.3, it suffices to prove this equivalence in the special case of $\mathbf{C} = \mathrm{coSpan}(\mathcal{S}^{\mathrm{fin}}), x = \mathrm{pt}$.

The composite functor $BC_n \to \mathrm{coSpan}((\mathcal{S}^{BC_n})^{\mathrm{fin}}) \to \mathrm{coSpan}(\mathcal{S}^{\mathrm{fin}})$, where the second functor is the forgetful functor, classifies S^1 with the action induced from $C_n \subset S^1$.

Thus it suffices to prove that the BC_n-action on $\mathrm{tr}(\mathrm{pt};\sigma) \simeq S^1$ in endomorphisms of the unit in $\mathrm{coSpan}((\mathcal{S}^{BC_n})^{\mathrm{fin}})$ is the canonical one, or equivalently, the same statement for $\mathrm{tr}(S^1;\sigma)$ in $\mathrm{coSpan}(\mathcal{S}^{\mathrm{fin}}/BC_n)$.

But we have identified the corresponding trace map $LBC_n \to (\mathcal{S}^{\mathrm{fin}}/BC_n)^{\simeq}$ with the "canonical map", and this canonical map is now a map of ordinary 2-groupoids, namely:

$$\mathrm{map}(B\mathbb{Z}, BC_n) \to (1\mathrm{Gpd}/BC_n)^{\simeq}$$

which one can simply fully identify by hand, and check the statement there. $\square$

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