Categorification of Smith Theory

This thesis has been submitted to the PhD School of the Faculty of Science, University of Copenhagen.

Vignesh Subramanian

November 2, 2023

Vignesh Subramanian
Department of Mathematical Sciences
University of Copenhagen
svignesh09091997@gmail.com

This thesis has been submitted to the PhD School of the Faculty of Science, University of Copenhagen.

Date of submission	31th August 2023
Date of defence	08th November 2023
Advisor	**Jesper Kragh Grodal** University of Copenhagen
Assessment committee	**Jesper Michael Møller** (chair) University of Copenhagen
	Tobias Barthel Max Planck Institute for Mathematics, Bonn
	Natalia Castellana Vila Universitat Autònoma de Barcelona

This work was supported by the Danish National Research Foundation through the Copenhagen Centre for Geometry and Topology under the grant number CPH-GEOTOP-DNRF151.

The work was also partially supported by the Swedish Research Council under the grant no. 2016-06596, during a research program held at Institut Mittag–Leffler, Sweden, in the spring of 2022.

ISBN 978-87-7125-223-1

Abstract

In this thesis, we study the homotopy theory of fixed points using methods from equivariant homotopy theory. Given a compact topological space with the action of a p-group, the fixed points and their cohomological properties are studied via the so-called Smith theory. We indicate several different categorifications of this theory.

The thesis consists of four main parts:

In the first part (Chapter 3), we investigate the relation between genuine fixed points of a finite G-space and power operations. We analyse the theory of perfect $\mathbb{E}_\infty$ k-algebras for k a characteristic p field and the perfection functor called tilting. Using this theory, we recover the homotopy type of genuine fixed points from the Borel equivariant cohomology.

In the second part (Chapter 4), we study the Segal conjecture for $\mathbb{Z}/p$; more precisely, given a spectrum X, when is the Tate construction with respect to trivial $\mathbb{Z}/p$ is the p-completion. For X finite spectra, this is the celebrated theorem of Lin and Gunawardena. In this chapter, we give examples of several non-finite spectra that satisfy the Segal conjecture and extend this result to a larger class of spectra.

In the third part (Chapter 5), we compute homotopy fixed points of certain actions of based loops on a compact Lie group and certain p-compact groups coming from geometric representation theory. In certain cases, we also compare the homotopy fixed points to the genuine fixed points.

In the fourth and final part (Chapters 6 and 7), we study a categorification of Smith theory for sheaf cohomology building on the works of D. Treumann.

Resume

I denne afhandling bruger vi ækvivariant homotopiteori til at studere homotopiteori for fikspunkter. Givet en virkning af en p-gruppe på et kompakt topologisk rum, kan fikspunkterne og deres kohomologiske egenskaber studeres via såkaldt Smithteori. Vi indikerer en række kategorificeringer af denne teori.

Afhandlingen består af fire hoveddele:

I den første del (Kapitel 3) undersøger vi forholdet mellem ægte fikspunkter af et endeligt G-rum og potensoperationer. Vi analyserer teorien om perfekte $\mathbb{E}_\infty$-k-algebraer for k et legeme af karakteristik p og perfektionsfunktoren 'tilting'. Ved at bruge denne teori genskaber vi homotopitypen af de ægte fikspunkter fra den Borel-ækvivariante kohomologi.

I den anden del (Kapitel 4) studerer vi Segalformodningen for $\mathbb{Z}/p$; mere præcist, givet et spektrum X, hvornår er det så tilfældet at Tatekonstruktionen med hensyn til den trivielle $\mathbb{Z}/p$-virkning er lig med p-fuldstændiggørelsen. Når X er et endeligt spektrum, er dette en kendt sætning af Lin og Gunawardena. I dette kapitel giver vi eksempler på ikke-endelige spektra der opfylder Segalformodningen og udvider dette resultat til en større klasse af spektra.

I den tredje del (Kapitel 5) beregner vi homotopifikspunkter af visse virkninger af løkker på en kompakt Liegruppe og visse p-kompakte grupper fra geometrisk repræsentationsteori. I nogle tilfælde sammenligner vi også homotopifikspunkter til de ægte fikspunkter.

I den fjerde og sidste del (Kapitel 6 og 7) studerer vi en kategorificering af Smithteori for knippekohomologi, der bygger på D. Treummans arbejde.

Thesis Statement

Chapter 2 is material covered in [MNN17; MNN19], we don't claim any originality of contents appearing in this chapter.

Chapters 3 and 4 are joint work with Robert Burklund. The version appearing in the thesis is written by the author.

Chapter 6 is joint work with Oscar Bendix Harr.

Acknowledgements

First of all, I would like to thank my advisor Jesper Grodal for his guidance and constant support during the past three years of my study. For generously sharing insights about various topics and giving me the freedom to pursue my own mathematical curiosity, at the same time making sure that I am not lost and stay motivated.

A great deal of discussions and help from the GeoTop community of Copenhagen has gone into the writing of the thesis, and I would like to take this opportunity to thank these people.

I am highly indebted to Robert Burklund and Shachar Carmeli for making me comfortable with various topics of math, patiently answering my questions, and in many ways shaping my mathematical thinking.

I would like to thank my academic siblings: Kaif Hilman, Jeroen van der Meer, Maxime Ramzi, Oscar Harr and Branko Juran, for making me comfortable with the subject and constantly keeping me motivated with exciting discussions.

A special thanks to Adriano Cordova and Alexis Aumonier, who patiently answered many of my mathematical as well as non-mathematical questions for these three years.

My three years of stay in Copenhagen have shaped my way of thinking about math, and it wouldn't have been possible if not for all the wonderful mathematical discussions that I had with Lars Hesselholt, Søren Galatius, Nathalie Wahl, Dustin Clausen, Markus Land, Fabian Hebestreit, Andrea Bianchi, Ryomei Iwasa, Peter Patzt, Achim Krause, Mikala Jansen, Florian Naef, Ronno Das, Joachim Kock, Jan Steinebrunner and Benjamin Briggs.

I am grateful for the vibrant PhD environment the mathematics department has offered, and the journey wouldn't have been possible without my fellow PhD students. For this, I have to thank Kaif Hilman, Jeroen van der Meer, Maxime Ramzi, Adriano Cordova, Alexis Aumonier, Calista Bernard, Severin Mejak, Ali Muhammad, Alex Frei, Nanna Aamand, Pierre Elis, Dagur Asgeirsson, Nuno Hultberg, Marie–Camille Delarue, Jingxuan Zhang, Branko Juran, Oscar Harr, Qingyuan Bai, Fadi Mezher, Marius Kjærsgaard and Desirée Gijón Gómez and as well as my other friends.

I would like to thank Branko Juran, Qingyuan Bai, Marius Kjærsgaard, Bhavana Joshi, Sophus Williumsgaard and Florian Riedel for giving feedback on various parts of the thesis.

I would like to thank the Institut Mittag–Leffler for hosting me during the program "Higher algebraic structures in algebra, topology and geometry" and MPIM, Bonn for the Felix Klein lectures 2022.

Finally, for the unconditional support my family has offered me. There is not enough words to thank you, Amma, Appa and Pooja.

Contents

1 Introduction

The subject of homotopy theory aims to classify topological spaces, more specifically CW-complexes, up to homotopy by associating for every topological space a natural algebraic object that is invariant under homotopy. That is a functor

$$F : \mathcal{S} \to \mathcal{C}$$

from the category of spaces to a suitable algebraic category.

In an ideal world, we would like to come up with functor F that distinguishes between spaces X and Y, which are not homotopic, but there is the question of how computable F is. There is constant trade between computability and how powerful the functor F is.

To a real smooth manifold M, we can associate the de-Rham cohomology $H^*_{dR}(M)$ and moreover, if M is a compact manifold, the alternating sum of dimensions of the cohomology

$$\chi(M) = \sum_i (-1)^i \dim(H^i_{dR}(M))$$

provides a useful invariant called the Euler characteristic which is an integer.

A striking result of 19^{th} century mathematics is the classification of surfaces using the Euler characteristic. This says, given an orientable smooth surface, that is, a real 2-dimensional compact, smooth manifold that is orientable, the Euler characteristic classifies the surface up to diffeomorphism.

So is the Euler characteristic good enough to distinguish between spaces? If we take our two spaces to be spheres of dimension 2 and 4, then $\chi(S^2) \simeq \chi(S^4) = 2$. But we can do something better by remembering the $\mathbb{F}_2$-based singular homology as a graded abelian group; we have $H_*(S^2) \neq H_*(S^4)$.

Even homology has limitations, if we take $X = S^2 \vee S^4$ and $Y = \mathbb{C}P^2$. Both X and Y have 1 cell in degree $0, 2$ and 4, giving us $H_*(X; \mathbb{F}_2) = H_*(Y; \mathbb{F}_2)$. This leads us to look at the $\mathbb{F}_2$-cohomology groups $H^*(X; \mathbb{F}_2)$ and $H^*(Y; \mathbb{F}_2)$, which comes with a graded commutative ring structure given by cup product of cohomology classes, which helps us distinguish between X and Y.

We can go further and look at $X = \Sigma S^2 \vee S^4 \simeq S^3 \vee S^5$ and $Y = \Sigma \mathbb{C}P^2$. Now, given a space of the form ΣZ, the cup product on $H^*(\Sigma Z)$ is trivial. How do we fix this?

Note that for a space X, we have $H^*(X; \mathbb{F}_2) \simeq \pi_* C^*(X; \mathbb{F}_2)$, by passing to the homotopy groups we have lost a lot of structure. The chain complex $C^*(X; \mathbb{F}_2)$ is not commutative on the nose but only commutative upon passing to the homotopy groups.

This is really a feature rather than a bug. The failure of commutativity on the nose provides a family of operations called the power operations. In particular, we can distinguish between $X = \Sigma S^2 \vee S^4 \simeq S^3 \vee S^5$ and $Y = \Sigma \mathbb{C}P^2$, since the power operation Sq^1 acts trivially on cohomology of X and nontrivially on cohomology of Y.

In this thesis, we explore how power operations help us understand the fixed points of group action on spaces.

Let X be a G-CW complex, where G is a p-group, then X is built by attaching cells of the form $S^n \times G/H$ where H is the isotropy of the cell. Then we can rewrite the expression for Euler characteristic: $\chi(X) = \sum_i (-1)^i \dim(H^i(X))$ by splitting into two parts, one made of cells of the form $S^n \times G/G$, that is, those cells that are fixed by G and the rest:

$$\chi(X) = \chi(X^G) + (\chi(X) - \chi(X^G))$$

Since G is a p-group, we have $\chi(X) - \chi(X^G)$ is multiple of p. Then we have

$$\chi(X) \equiv \chi(X^G) \bmod p$$

This leads us to wonder if there is a cohomological relationship between X and X^G. The earliest instance of investigation of this phenomenon is due to P. A. Smith [Smi34].

We say a CW-complex X, is a mod-p homology n-sphere if $H^i(X; \mathbb{F}_p) = \mathbb{F}_p$ for $i = 0, n$ and trivial in all other degree.

Theorem 1.0.0.1 (P. A. Smith)**.** [Smi34] *Let G be a p-group and X a finite G-CW complex and is a mod-p homology n-sphere, then the genuine fixed points X^G is either empty or a mod-p homology i-sphere for $i \leqslant n$*

The theory of studying the relationship between cohomological properties of X and its fixed points X^G is called Smith theory. The case of S^1-action was studied by A. Borel [BBF+60], M. Atiyah and R. Bott [AB84], D. Quillen [Qui71a; Qui71b] and by W. Dwyer and C. Wilkerson [DW88]. We state the most refined version of the result proved in [DW88]

Theorem 1.0.0.2. [DW88] *Let G be an elementary abelian group $(\mathbb{Z}/p)^n$. Let X be a finite G-CW complex then we have:*

$$H^*(X^G) \simeq \mathbb{F}_p \otimes_{H^*(BG)} \mathrm{Un}(H^*(X_{hG})[e^{-1}])$$

where $e \in H^*(X_{hG})$ is the Euler class described 2.1.3, coming from group cohomology $H^*(BG)$ and $\mathrm{Un}(H^*(X_{hG})[e^{-1}])$ is the largest unstable Steenrod algebra inside $H^*(X_{hG})[e^{-1}]$.

The above statement is extremely surprising because we are able to get access to the cohomology of genuine fixed points using the Borel equivariant cohomology.

Question 1.0.0.3. Could we recover the homotopy type of the genuine fixed points from the Borel equivariant cohomology?

We address this question in Chapter 2. In joint work with R. Burklund, we investigate the theory of perfect algebras in the land of higher algebra.

Given any $\mathbb{E}_\infty$-algebra over $\bar{\mathbb{F}}_p$, we construct two different perfection functor $(-)^\#$ and $(-)^\flat$, which are adjoints to the inclusion $\mathrm{CAlg}^{\mathrm{Fr}}_{\bar{\mathbb{F}}_p} \hookrightarrow \mathrm{CAlg}_{\bar{\mathbb{F}}_p}$. We refer to the right adjoint $(-)^\flat$ as tilting.

More than just constructing the adjoint $(-)^\flat$, we also prove a recipe to compute the homotopy groups of $A^\flat$ for A an $\mathbb{E}_\infty$-algebra over $\bar{\mathbb{F}}_p$.

Proposition 1.0.0.4. (Homotopy groups of $A^\flat$) [Proposition 3.3.2.5] For a finite type[1] $\mathbb{E}_\infty$-algebra A the homotopy groups of $A^\flat$ is given by $\pi_* A^\flat \simeq \varprojlim_n \pi_* A$ where the sequential inverse limit is taken along the operation Q^0.

As an application, we give a spectrum-level statement of the result of Dwyer and Wilkerson:

Theorem 1.0.0.5 (Theorem 3.4.1.4)**.** For G, an elementary abelian group and X, a finite G-CW complex. The tilt of the algebra $C^*(X;k)^{\tau G} \simeq C^*(X;k)^{hG}[e^{-1}]$ is equivalent to $C^*(X^G \times BG; k)$. Where $(-)^{\tau G}$ is the proper Tate construction.

In Chapter 2, we recall the required background about proper Tate construction.

In section 3.4.2, we recall results from p-adic homotopy theory in the sense of M.Mandell [Man01] and recover the p-local homotopy type of X^G.

The ideas involved in chapter 3 are heavily inspired by the proofs of Sullivan conjecture which, as if for a given fintie G-CW complex X and G a p-group, when is the comparison map $X^G \to X^{hG}$, from the genuine fixed points to the homotopy fixed points $\mathbb{F}_p$-equivalence.

In chapter 4, we revisit a stable analogue of the Sullivan conjecture, the Segal conjecture. For $G = \mathbb{Z}/p$, the Segal conjecture ask if the map $(\mathbb{S}_G)^G \to (\mathbb{S}_G)^{hG}$ an equivalence after the p-completion. Where $\mathbb{S}_G$ is the genuine G-sphere spectrum, which is the unit of the category Sp_G of genuine G-spectra.

Equivalently the above theorem can be stated as when is the canonical map $\mathbb{S} \to \mathbb{S}^{t\mathbb{Z}/p}$ from the sphere spectrum to its Tate construction with respect to trivial action of $\mathbb{Z}/p$ a p-adic equivalence. This is answered to be true by W. H. Lin [Lin80] and J. H. Gunawardena [Gun80]. In fact, as an immediate consequence, we can replace the sphere spectrum with any finite spectra X, and we still have a p-complete equivalence $X \to X^{t\mathbb{Z}/p}$.

We introduce the notion of an I-nilpotent spectra (Definition 4.3.1.9) and prove the following theorem:

Theorem 1.0.0.6 (Theorem 4.3.1.10)**.** Let X be a bounded below spectrum, p-complete and I-nilpotent, then X satisfies the Segal conjecture for $\mathbb{Z}/p$. That is, $X \to X^{t\mathbb{Z}/p}$ is an equivalence, where the Tate construction is with respect to trivial action.

The class of I-nilpotent spectra include many non-finite spectra. A rich source of example comes from the class of spectra with cohomology, which is finite type and locally finite (see Definition 4.3.1.11).

In chapter 5, the main object of study is based loops on a compact lie group G. Turns out there is an algebro-geometric object called the affine Grassmannian Gr_G whose underlying analytic space is homotopic to ΩK, where K is the maximal compact subgroup of G. The affine Grassmannian Gr_G comes with a natural $\mathbb{G}_m$-action. In

[1]We say A is of finite type if $\pi_n A$ finite dimensional all n

[RW22a], the authors compute the genuine fixed points $\mathrm{Gr}_G^{\mathbb{Z}/p}$. In this article, we give a recipe to compute the homotopy fixed points for any compact Lie group.

In the case of $G = GL_n$, we compute the homotopy fixed points:

Theorem 1.0.0.7. Let $G = GL_n(\mathbb{C})$, then the homotopy fixed points $(\Omega U(n))^{h\mathbb{Z}/p^n}$ is equivalent to $(\Omega U(n))^{h\mathbb{Z}/p^n} \simeq \coprod_{\rho \in \mathrm{Rep}(\mathbb{Z}/p^n, U(n))} \Omega U(n) \times U(n)/C(\rho)$.

Finally, the last part of the thesis is chapters 6 and 7. We set up foundations for equivariant sheaf theory and categorify the localisation theorem for sheaves.

In chapter 6, joint work with Oscar Bendix Harr, we write down an ∞-categorical version of six functor formalism using the formalism developed in [Man22].

In the final chapter, we introduce two constructions of localisation of sheaves, the first based on the construction of D. Treumann [Tre19], and we give a construction that is valid for equivariant sheaves on locally compact space valued in the ∞-category of R-modules, for R a commutative ring spectrum and a finite groups G in the defect base of R.

The second is based on inverting the Euler class on the category of equivariant sheaves. But in order to get a non-trivial category, we perform the localisation in the ∞-category of stable presentable dualisable categories $\mathrm{Pr}_{st}^{\mathrm{dual}}$.

Conventions

In this thesis, by category, we mean ∞-category as developed by [Lur09], and even 1-categories are viewed as discrete ∞-categories. By commutative algebra, we mean an algebra over the $\mathbb{E}_\infty$-operad and by algebra, we mean an algebra over the $\mathbb{E}_1$-operad.

For a ring or a ring spectrum R, by Mod_R, we mean the stable ∞-category of R-modules. The categories, Cat_∞ denote the ∞-category of all ∞-category, Pr^L denote the ∞-category of all presentable ∞-category, Pr_{st}^L denote ∞-category of all stable presentable ∞-category and $\mathrm{Pr}_{st}^{\mathrm{dual}}$ denote ∞-category of all stable presentable ∞-category which are dualisable.

All functors are derived by convention. For example, we denote the tensor product in the derived category of R-modules simply by $- \otimes_R -$. we will specify when performing an underived construction.

For an object X in the ∞-category of spaces. By a functor $F : X \to \mathcal{C}$, we mean a functor from X, thinking of X as an ∞-groupoid in the sense of [Lur09].

2 Preliminaries

2.1 G-Spectra and proper Tate construction

2.1.1 Localisations in Higher Algebra

Let $\mathcal{C}^{\otimes}$ be stable, presentably symmetric monoidal ∞-category[1] let $A \in \mathrm{Alg}(\mathcal{C}^{\otimes})$. We recall some general theories of completion, acyclisation and localisation with respect to the algebra A. Recall how this theory applies to Sp_G, the category of genuine G-spectra; much of the material presented here has appeared in [MNN17; MNN19]; we recall it here for exposition and claim no originality.

Remark 2.1.1.1. By $\mathrm{Alg}(\mathcal{C})$ we always mean the ∞-category of algebras over the associative operad $\mathbb{E}_1$ and similarly by $\mathrm{CAlg}(\mathcal{C})$ we mean the ∞-category of algebras over the commutaive operad $\mathbb{E}_\infty$ as developed in [Lur17, Section 3, 4].

Definition 2.1.1.2 (A-complete)**.** [MNN17, Definition 2.15] Let $X \in \mathcal{C}$, we say X is A-complete/local, if we have $\mathrm{map}_{\mathcal{C}}(Y, X) = 0$ whenever $A \otimes Y \simeq 0$.

Let $\mathcal{C}_{A-cpl}$ be the full subcategory of $\mathcal{C}$ spanned by A-complete objects. Since A-complete objects are defined via mapping in property; we have by definition the subcategory of A-complete objects are closed under small limits.

The category of $\mathcal{C}_{A-cpl}$ is the localisation of $\mathcal{C}$ at morphisms

$$S = \{f : X \to Y \mid f \otimes A \text{ is an equivalence}\}.$$

We can construct a left adjoint [MNN17, Definition 2.19] $L_A : \mathcal{C} \to \mathcal{C}_{A-cpl}$ to the inclusion functor $\mathcal{C}_{A-cpl} \subset \mathcal{C}$ called the **completion functor**. Following [MNN17], we get this adjoint since by [Lur09, Proposition 5.5.4.16], we have S is a strongly saturated class of morphisms of small generation and by invoking the theory of Bousfield localisation in [Lur09, Proposition 5.5.4.15] we have that $\mathcal{C}_{A-cpl}$ is a presentable category and that the inclusion of the complete objects has a left adjoint.

Example 2.1.1.3. For $\mathcal{C} = \mathcal{D}(\mathbb{Z})$, the unbounded derived category of $\mathbb{Z}$-modules, which is also the category of $\mathbb{Z}$-modules in spectra. Let $A = \mathbb{Z}/p$, then the completion functor $L_{\mathbb{Z}/p}$ recovers the classical derived p-completion functor.

We work under further assumptions of **Hypothesis 2.26** of [MNN17, Hypothesis 2.26]:

Let $(\mathcal{C}^{\otimes}, 1)$ be a presentably symmetric monoidal ∞-category satisfying

- The unit is compact. (which implies all dualisable objects are compact)
- The algebra A is dualisable

[1] means $\mathcal{C}$ is commutative algebra object in Pr^L_{st}

- $\mathcal{C}$ is generated by dualisable objects under small colimits.

Definition 2.1.1.4 (A-torsion)**.** [MNN17, Definition 3.1] Let $\mathcal{C}^{\otimes}$ be a presentably symmetric monoidal ∞-category and $A \in Alg(\mathcal{C})$. The subcategory of A-torsion objects $\mathcal{C}_{A-tors}$ is the localising subcategory generated by $A \otimes X$, where $X \in \mathcal{C}$ ranges over all dualisable objects.

By definition, the subcategory of $\mathcal{C}_{A-tors}$ is closed under colimits; hence the inclusion functor $\mathcal{C}_{A-tors} \subset \mathcal{C}$ is a left adjoint by the adjoint functor theorem and admits a right adjoint $\mathrm{ACyc}_A : \mathcal{C} \to \mathcal{C}_{A-tors}$ called the **acyclisation functor** [MNN17, Construction 3.2]. By **Proposition 3.3** of [MNN17] we have that $\mathcal{C}_{A-tors}$ is a localising $\otimes$-ideal generated by A.

Example 2.1.1.5. Revisting the examples of $\mathcal{C} = \mathcal{D}(\mathbb{Z})$ and $A = \mathbb{Z}/p$, we have $\mathrm{ACyc}_A(\mathbb{Z}) = \Sigma^{-1}\mathbb{Z}/p^{\infty}$. Where $\mathbb{Z}/p^{\infty}$ is the Püfer group $\mathbb{Z}[p^{-1}]/\mathbb{Z}$.

Definition 2.1.1.6 (A^{-1}-local)**.** [MNN17, Definition 3.10] We call an object X in $\mathcal{C}$ to be A^{-1}-local if for every A-torsion object T we have $\mathrm{Hom}_{\mathcal{C}}(T, X) \simeq 0$. Let us denote the class of A^{-1}-local objects by $\mathcal{C}[A^{-1}]$ following [MNN17].

We also have another way to test if an object is A^{-1}-local by **Proposition 3.11** of [MNN17]:

Proposition 2.1.1.7. [MNN17, Proposition 3.11] Let $\mathcal{C}$ be an ∞ category satisfying hypothesis 2.26 stated above, and $X \in \mathcal{C}$, $X \in \mathcal{C}[A^{-1}] \iff X \otimes A$ is contractible.

By definition, the subcategory A^{-1}-local objects is closed under all small limits. Moreover, it is also closed under colimits because of 2.1.1.7 and the tensor product $\otimes$ preserves colimits in each variable. Which in turn gives us the A^{-1}**-localisation** functor $(-)[A^{-1}] : \mathcal{C} \to \mathcal{C}[A^{-1}]$ [MNN17, Construction 3.12], the left adjoint to the inclusion of A^{-1}-local objects.

Example 2.1.1.8. Let R be a discrete commutative ring and $\mathcal{C} = \mathcal{D}(R)$ and $A = R/f$ for some $f \in R$. Then we have $(R/f)^{-1}$-localisation is given by smashing with $R[f^{-1}]$.

Morally speaking, A^{-1}-localisation picks out information away from the algebra A, which is made precise by **Proposition** 2.1.1.7 and the **Example** 2.1.1.8.

We have abstractly set up three different localisations of a symmetric monoidal category $\mathcal{C}^{\otimes}$:

1. A**-completion** $L_A(-)$
2. A**-acyclisation** $\mathrm{Acy}_A(-)$
3. A^{-1}**-localisation** $(-)[A^{-1}]$

with respect to an algebra $A \in Alg(\mathcal{C})$. We now follow [MNN17, Section 2, 3] to give formulas to compute these localisations functors in these cases.

Let us set up a few notations to help us state the formulas for these localisations. As usual let $\mathcal{C}$ be a stable, presentably symmetric monoidal category satisfying **Hypothesis 2.26** of [MNN17] and $A \in Alg(\mathcal{C})$ then we make the following construction:

Construction 2.1.1.9. [MNN17, Construction 2.2] Let I be the fiber of the unit $1 \to A$, tensoring the unit map with $I^{\otimes n}$ we have:

$$I^{\otimes n} \to A \otimes I^{\otimes n}$$

take the fiber to get $I^{\otimes n+1}$ which gives maps $I^{\otimes n+1} \to I^{\otimes n}$ for all $n \geqslant 0$.

This assembles into a tower called the **A-Adams Tower**:

$$I^{\otimes(n+1)} \to I^{\otimes n} \to \ldots \to I^{\otimes 2} \to I \to 1 \tag{2.1}$$

which is the universal A-Adams tower and by tensoring (2.1) with $M \in \mathcal{C}$ we obtain A-Adams Tower of M. We will denote the A-Adams tower of an object $M \in \mathcal{C}$ by $\mathrm{Adams}(A, M)_\bullet$.

In the case of $\mathcal{C} = \mathcal{D}(\mathbb{Z})$ and $A = \mathbb{Z}/p$ we have the unit map is the mod p reduction $\mathbb{Z} \to \mathbb{Z}/p$ with fiber $\mathbb{Z}$ and the map $I \to 1$ given by multiplication by p, $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$. Hence the $\mathbb{Z}/p$-Adams Tower in $\mathcal{D}(\mathbb{Z})$ is given by:

$$\ldots \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \tag{2.2}$$

A closely related construction to the A-Adams tower is the **cobar construction**. Given an algebra $A \in Alg(\mathcal{C})$ we get maps $A^{\otimes n} \to A^{\otimes n+1}$ and $A^{\otimes n} \to A^{\otimes(n-1)}$ using the units and multiplication. In the case $n = 2$, we have $A \to A \otimes A$ given by $a \mapsto 1 \otimes a$ and $a \mapsto a \otimes 1$ and the multiplication gives map $A \otimes A \xrightarrow{m} A$. These coherently assemble into a cosimplicial diagram $\Delta \to \mathcal{C}$ called the **cobar construction**:

$$\mathrm{CB}_\bullet(A) \qquad = \qquad A \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; A \otimes A \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \ldots$$

The coherence of the above diagram crucially depends on the highly coherent multiplication structure of A, coming from the fact that A is an algebra over the $\mathbb{E}_1$-operad. To set up the above diagram infinity categorically with all the coherence, we refer the reader to **Construction 2.7** of [MNN17].

To see how the cobar construction is related to the Adams tower, we use the ∞-categorical Dold-Kan Correspondence, which gives an ∞-categorical equivalence by J.Lurie [Lur17]:

$$\mathrm{Fun}(\Delta, \mathcal{C}) \simeq \mathcal{C}^{\mathbb{Z}^{op}_{\geqslant 0}}$$

where the equivalance is given by given a functor $X^\bullet : \Delta \to \mathcal{C}$ sent to tower $\mathrm{Tow}(X^\bullet)$:

$$n \mapsto \mathrm{Tot}_n(X^\bullet)$$

where $\mathrm{Tot}_n(X^\bullet)$ is the limit of the diagram $X^\bullet$ restricted to $\Delta^{\leqslant n}$.

Now we can state the relation between the Adams tower and the tower associated to the cobar construction. We have a map from the Adams tower to the constant tower $\mathrm{Adams}(A, 1)_\bullet \to \underline{1}$.

Proposition 2.1.1.10. [MNN17, Proposition 2.14] Let $\mathrm{Tow}(\mathrm{CB}_\bullet(A))$ be the tower associated to the cobar construction; we have

$$\mathrm{Tow}_n(\mathrm{CB}_\bullet(A)) = \mathrm{cofib}(\mathrm{Adams}(A,1)_n \to 1) \simeq \mathrm{cofib}(I^{\otimes n} \to 1)$$

infact we have cofiber sequence of towers $\mathrm{Adams}(A,1)_\bullet \to \underline{1} \to \mathrm{Tow}(CB_\bullet(A))$

We continue to work under **Hypothesis 2.26** of [MNN17]. We have the following formula for the completion of X with respect to an algebra A.

Proposition 2.1.1.11. For $X \in \mathcal{C}$, we have the map $X \to \mathrm{Tot}(X \otimes \mathrm{CB}_\bullet(A))$ which exhibits $\mathrm{CB}_\bullet(A))$ as the A-completion of X.

which means using **Proposition** 2.1.1.10 we have

$$\mathrm{Tot}(X \otimes \mathrm{CB}_\bullet(A)) \simeq \varprojlim_n \mathrm{cofib}(I^{\otimes n} \otimes X \to X).$$

Recall the Adams tower from 2.2 where $\mathcal{C} = \mathcal{D}(\mathbb{Z})$ and $A = \mathbb{Z}/p$, we have $\mathrm{cofib}(I^{\otimes n} \to 1) = \mathrm{cofib}(\mathbb{Z} \xrightarrow{p^n} \mathbb{Z}) = \mathbb{Z}/p^n$. Hence $\mathbb{Z}/p$-completion of any object $X \in \mathcal{D}(\mathbb{Z})$ is given by

$$L_{\mathbb{Z}/p}(X) = \varprojlim_n X/p^n$$

Remark 2.1.1.12. Note that $\mathbb{Z}/p$ in the category of spectra Sp does not satisfy the **Hypothesis 2.26** of [MNN17], in particular, $\mathbb{Z}/p$ is not a dualisable object. So the above formula does not hold for $\mathbb{Z}/p$-completion but ends up being completion with respect to homotopy ring $\mathbb{S}/p$ where the two completions coincide in the case of bounded below spectra X.

Finally, we can categorify the above construction with the following proposition.

Proposition 2.1.1.13. [MNN17, Theorem 2.30] Let $A \in \mathrm{CAlg}(\mathcal{C})$, where $\mathcal{C}^{\otimes}$ is a stable, presentably symmetric monoidal ∞-category. Then we have,

$$\mathcal{C}_{A-cpl} \qquad \simeq \qquad \varprojlim_n \left\{ \mathrm{Mod}_{\mathcal{C}}(A) \rightleftarrows \mathrm{Mod}_{\mathcal{C}}(A \otimes A) \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} \dots \right\}$$

where the limit is taken in $\mathrm{CAlg}(\mathrm{Pr}^L_{st})$ the ∞-category of presentably symmetric monoidal ∞-categories.

Now we give formula for acyclisation and A^{-1}-localisation, we have the Adams tower $\mathrm{Adams}(A,1)_\bullet$

$$\dots \to I^{\otimes n} \to I^{\otimes(n-1)} \to \dots \to I \to 1$$

Since we are working under the condition that the algebra A is dualisable. We have the $I = \mathrm{fib}(1 \to A)$ is dualisable; in fact, each term appearing in the Adams tower $\mathrm{Adams}(A,1)_\bullet$ is dualisable, following [MNN17] we write the dualised tower as

$$1 \to U_1 \to \ldots \to U_i \to$$

where $U_i := \mathbb{D}(I^{\otimes n})$ and define $U_A = \mathrm{colim}_i U_i$ [MNN17, Construction 3.4] which comes with a map $1 \to U_A$. Taking fiber, we have

$$V_A \to 1 \to U_A$$

Proposition 2.1.1.14. [MNN17, Proposition 3.5] The acyclisation functor $\mathrm{ACyc}_A : \mathcal{C} \to \mathcal{C}_{A-tors}$ is given by $X \otimes V_A$, that is $X \otimes V_A \to X$ as a co-localisation with respect A-torsion. Similarly, the A^{-1}-localisation is given by smashing with U_A, i.e, $X \to X \otimes U_A$ exhibits the target as the A^{-1}-localisation.

Returning back to the examples of $\mathcal{C} = \mathcal{D}(\mathbb{Z})$ and $A = \mathbb{Z}/p$, then we have the dualised Adams tower:

$$\mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p}$$

hence we have $U_{\mathbb{Z}/p} = \mathbb{Z}[p^{-1}]$ and since we have the fiber sequence $V_A \to 1 \to U_A$ we have that $V_{\mathbb{Z}/p} = \Sigma^{-1}(\mathbb{Z}[p^{-1}]/\mathbb{Z})$.

2.1.2 Localisations in the category of genuine G-spectra

This section covers some basic facts about the ∞-category of G-spectra and the proper Tate construction.

Let G be a finite group; to it we can associate the ∞-category Sp_G of G-spectra which is a presentably symmetric monoidal ∞-category, we refer the reader to [MNN17; GM95; Bar17; AMR22] we will take a rather model-independent approach by assuming the following facts:

1. (Elmendorf Theorem) Let $\mathcal{O}_G$ be the orbit category whose objects are cosets G/H for all subgroups $H \leqslant G$ subgroups and morphisms G-equivariant maps. Then the ∞-cateogry of G-spaces (denoted by $\mathcal{S}_G$) is the presheaf category $\mathrm{Fun}(\mathcal{O}_G^{op}, \mathcal{S})$. This category can be promoted to a symmetric monoidal ∞-category with underlying category $\mathcal{S}_G$, obtained from the pointwise product from the above description.
2. There exists a stable ∞-category Sp_G of genuine G-spectra with a symmetric monoidal structure such that we have a symmetric monoidal functor
$$\Sigma_G^\infty : \mathcal{S}_{G,*} \to \mathrm{Sp}_G$$
from pointed G-spaces to genuine G-spectra which preserve colimits and compact objects. Hence this gives us an adjunction:
$$\mathcal{S}_{G,*} \underset{\Omega_G^\infty}{\overset{\Sigma_G^\infty}{\rightleftarrows}} \mathrm{Sp}_G$$
3. The category of G-spectra is **compactly generated by dualisable objects** $\Sigma^\infty_{G,+}(G/H)$ where H ranges over all subgroups. This means a G-spectra X is contractible if and only if the mapping spectra $\mathrm{map}_{\mathrm{Sp}_G}(\Sigma^\infty_{G,+} G/H, X)$ denoted by X^H is contractible where H runs over all subgroups.

4. Mapping out of $\Sigma^\infty_{G,+}(G/H)$ gives a functor $(-)^H : \mathrm{Sp}_G \to \mathrm{Sp}$ called **categorical fixed points** or **genuine fixed points**. Which are colimit and limit preserving, lax-monoidal for all subgroups $H \leqslant G$.
5. The functor $(-)^G : \mathrm{Sp}_G \to \mathrm{Sp}$ is a right adjoint with left adjoint $i_* : \mathrm{Sp} \to \mathrm{Sp}_G$ called the **inflation functor**, which can also be seen as the unit map
$$\mathrm{Sp}^\otimes \to \mathrm{Sp}_G^\otimes,$$
since $\mathrm{Sp}_G^\otimes$ is a presentably symmetric monoidal category, we can consider of $\mathrm{Sp}_G^\otimes$ with the symmetric monoidal structure as an object in $\mathrm{CAlg}(\mathrm{Pr}^L_{st})$.
The category $\mathrm{CAlg}(\mathrm{Pr}^L_{st})$ is symmetric monoidal where the monoidal structure comes from the Lurie tensor product on Pr^L_{st} and $\mathrm{Sp}^\otimes$ as the unit. Hence the inflation functor is a symmetric monoidal functor which is a left adjoint.
6. (H-homotopy groups) For every subgroup $H \leqslant G$ we define
$$\pi_*^H(X) = \pi_* \mathrm{map}_{\mathrm{Sp}_G}(\Sigma^\infty_{G,+} G/H, X) = \pi_*(X^H)$$
a morphism $f : X \to Y$ is an equivalence if and only if $\pi_*^H(f)$ is an equivalence for all subgroups H.
7. for all subgroups $H \leqslant G$, we have the restriction functor
$$\mathrm{Res}^G_H : \mathrm{Sp}_G \to \mathrm{Sp}_H$$
which is a symmetric monoidal, colimit-preserving functor which has both a left adjoint induction Ind^G_H and a right adjoint Coind^G_H.
$$\begin{tikzcd} Sp_G \arrow[r, "\mathrm{Res}^G_H"] & Sp_H \arrow[l, bend right=60, "\mathrm{Ind}^G_H"'] \arrow[l, bend left=60, "\mathrm{Coind}^G_H"] \end{tikzcd}$$
such that for $K \leqslant H \leqslant G$ $\mathrm{Res}^H_K \circ \mathrm{Res}^G_H \simeq \mathrm{Res}^G_K$. We have $\mathrm{Ind}^G_H \circ \mathrm{Ind}^H_K \simeq \mathrm{Ind}^G_K$ and $\mathrm{Coind}^G_H \circ \mathrm{Coind}^H_K \simeq \mathrm{Coind}^G_K$.
8. (Projection formula) Furthermore we have
$$(\mathrm{Ind}^G_H X) \otimes Y \simeq \mathrm{Ind}^G_H(X \otimes (\mathrm{Res}^G_H Y))$$
$$(\mathrm{Coind}^G_H X) \otimes Y \simeq \mathrm{Coind}^G_H(X \otimes (\mathrm{Res}^G_H Y))$$
9. For a finite group we have the **Wirthmüller isomorphism** which is a natural equivalence
$$\mathrm{Ind}^G_H \simeq \mathrm{Coind}^G_H$$
.

Given a subgroup $H \leqslant G$, we can indentify the $\mathrm{Coind}^G_H : \mathrm{Sp}_H \to \mathrm{Sp}_\mathrm{G}$, the right adjoint to the restriction functor applied to the H-spectrum $\mathbb{S}_H$ with internal mapping object $F(G/H_+, \mathbb{S}_G)$.

Since the restriction functor is symmetric monoidal, the right adjoint can be lifted to a lax monoidal functor, hence $\mathrm{Coind}_H^G(\mathbb{S}_H)$ receives a natural commutative algebra structure.

The commutative algebra structure can be identified as follows, G/H as a G-space is a commutative coalgebra via the diagonal map, hence mapping out of the commutative algebra $\mathbb{S}_G$ makes $F(G/H_+, \mathbb{S}_G)$ into a commutative algebra.

The commutative algebra $F(G/H_+, \mathbb{S}_G)$ lets us identify Sp_H as a modules category in Sp_G by the result of [BDS15]:

Proposition 2.1.2.1 ([BDS15])**.** The functor $\mathrm{Sp}_H \to \mathrm{Mod}_{\mathrm{Sp}_G}(F(G/H_+, \mathbb{S}_G))$ is an equivalance of symmetric monoidal ∞-categories.

The functor described in the above proposition is described as follows because of the lax-monoidality of the functor CoInd_H^G. We have that $\mathrm{CoInd}_H^G(X)$ is naturally $\mathrm{CoInd}_H^G(\mathbb{S}_H)$-module, hence the coninduction functor lands in $F(G/H_+, \mathbb{S}_G)$-modules.

We now go on to apply the theory of completion, acyclisation and A^{-1}-localisation in the category of genuine G-spectra with respect to a certain algebra $A_{\mathcal{F}}$. Before that, let us set up some notations,

Definition 2.1.2.2. [MNN17, Definition 6.1] We say a collection of subgroups $\mathcal{F}$ of G, a **family of subgroup** if $H^{'}$ is a subgroup subconjugate $H \in \mathcal{F}$ then $H^{'} \in \mathcal{F}$.

Given a family of subgroups, then we can define a commutative algebra $A_{\mathcal{F}} = \prod_{H \in \mathcal{F}} F(G/H_+, \mathbb{S}_G)$. This leads to the following set of definitions:

Definition 2.1.2.3. [MNN17, Definition 6.2] Given a family of subgroups $\mathcal{F}$. Then:

1. Given an object $X \in \mathrm{Sp}_G$ we say it is $\mathcal{F}$-torsion if X belongs to the smallest localising subcategory generated by $A_{\mathcal{F}}$. Equivalently the smallest localising subcategory generated by $F(G/H_+, \mathbb{S}_G)$ where $H \in \mathcal{F}$.
2. Given an object $X \in \mathrm{Sp}_G$ we say it is $\mathcal{F}$-complete if it is complete with respect to $A_{\mathcal{F}}$.
3. Given an object $X \in \mathrm{Sp}_G$ we say it is $\mathcal{F}^{-1}$-local if it is $A_{\mathcal{F}}^{-1}$-local.

As discussed in the section 2.1.1, we have completion, acyclisation and $(-)[A^{-1}]$ with respect to algebra $A_{\mathcal{F}}$. Which we refer to as $\mathcal{F}$-complete, $\mathcal{F}$-acyclisation and $\mathcal{F}^{-1}$-localisation.

Given a family of subgroup we can form a universal G-space $E\mathcal{F}$, with the following properties

$$E\mathcal{F}^H = \begin{cases} * & H \in \mathcal{F} \\ \phi & H \notin \mathcal{F} \end{cases}$$

Let $\mathcal{O}_{\mathcal{F}}(G)$ be the full subcategory of G-spaces spanned by G/H where $H \in \mathcal{P}$ referred to as $\mathcal{F}$-orbit category, let $i : \mathcal{O}_{\mathcal{F}}(G) \hookrightarrow \mathcal{S}_G$ be the inclusion of subcategory. By [MNN19, A.1] we have

$$E\mathcal{F} \simeq \mathrm{Colim}_{\mathcal{O}_{\mathcal{F}}(G)} i \tag{2.3}$$

We then have a cofiber sequence of pointed G-space referred to as **isotropy seperation** sequence [MNN17, Proposition 6.5]:

$$E\mathcal{F}_+ \to S^0_G \to \tilde{E\mathcal{F}}$$

where the $\tilde{E\mathcal{F}}$ as a G-space is unique upto equivalance

$$\tilde{E\mathcal{F}}^H = \begin{cases} * & H \in \mathcal{F} \\ S^0 & H \notin \mathcal{F} \end{cases}$$

Recall from **Proposition** 2.1.1.14 A-acyclisation and A^{-1}-localisation are given by smashing with a certain object V_A and U_A respectively, obtained via the A-Adams tower of the unit. The following proposition [MNN17, Prop 6.5] identifies these object in the category of G-spectra for $\mathcal{F}$-acyclisation and $\mathcal{F}^{-1}$-localisation.

Proposition 2.1.2.4. The $\mathcal{F}^{-1}$-localisation of $\mathbb{S}_G$ is given by $\Sigma^\infty_G \tilde{E\mathcal{F}}$ and $\mathcal{F}$-acyclisation is given by $\Sigma^\infty_{G,+} E\mathcal{F}$.

We also have the $\mathcal{F}$-completion (ref. [MNN17, Prop 6.6]) as:

Proposition 2.1.2.5. The $\mathcal{F}$-completion of a G-spectra X is given by $F(E\mathcal{F}_+, X)$.

Let $\mathcal{P}$ be the family of proper subgroups of a G. We can look at the smallest localising subcategory $\mathrm{Sp}^{\mathcal{P}}_G$ generated by G/H_+ in Sp_G (where H runs over all proper subgroups). Note that $\mathrm{Sp}^{\mathcal{P}}_G$ is precisely the $A_{\mathcal{P}}$-torsion objects. Localising away from this subcategory we have $\mathrm{Sp}_G[A^{-1}_{\mathcal{P}}]$ the full subcategory of $\mathcal{F}^{-1}$-local objects.

We can describe the **geometric fixed points** functor as the total composite:

$$\mathrm{Sp}_G \xrightarrow{(-)[A^{-1}_{\mathcal{P}}]} \mathrm{Sp}_G[A^{-1}_{\mathcal{P}}] \xrightarrow{(-)^G} \mathrm{Sp}$$

In particular by **Proposition** 2.1.2.4 we have $\Phi^G(X) = (\tilde{E}\mathcal{P} \otimes X)^G$. So we get a lax-monoidal transformation of functors $(-)^G \to \Phi^G(-)$ coming from the map of pointed G-spaces $S^0_G \to \tilde{E}\mathcal{P}$.

Since $\mathcal{F}^{-1}$-localisation kills all the $\mathcal{F}$-torsion objects we have $\Phi^G(G/H_+) \simeq 0$. It turns out by this; the geometric fixed points functor is universal with respect to these properties, that is,

Proposition 2.1.2.6. $(-)^G \to \Phi^G(-)$ is a lax-monoidal transformation, and it is initial among functors which kills the $\otimes$-ideal of $\mathcal{P}$-torsion objects (denoted by $< \mathcal{P} >$). That is, for any lax-monoidal functor $F : \mathrm{Sp}_G \to \mathrm{Sp}$ which vanishes on the ideal $< \mathcal{P} >$ and comes with a natural transformation $\eta : (-)^G \to F$ there is a essentially unique factorisation of lax-monoidal transformations

$$\begin{tikzcd} (-)^G \arrow[r] \arrow[d] & \Phi^G(-) \arrow[dl, dashed] \\ F(-) & \end{tikzcd}$$

Given a G-spectrum X we can tensor X with the isotropy separation sequence $E\mathcal{P} \to S^0_G \to \tilde{E}\mathcal{P}$ and apply $(-)^G$ and use the fact that $(-)^G$ commutes with all limits and colimits to get the cofiber sequence:

$$\operatorname*{colim}_{G/H \in \mathcal{O}_{\mathcal{P}}(G)} X^H \to X^G \to \Phi^G(X) \tag{2.4}$$

One can ask how is the category of spectra with a G-action, that is, $\mathrm{Fun}(BG, \mathrm{Sp})$ related to the category of genuine G-spectra. This can be answered as follows:

Let $\mathcal{F} = \{e\}$ be the family of trivial group, by the expression 2.3 we see that $E\mathcal{F} = EG$ and $A_{\mathcal{F}} = F(G_+, \mathbb{S}_G)$, we refer to the category of $A_{\mathcal{F}}$-complete object as **Borel complete spectra** denoted by $\mathrm{Sp}_G^{\mathrm{Borel}}$ and the completion functor is called **Borelification**.

The completion functor $\beta : \mathrm{Sp}_G \to \mathrm{Sp}_G^{\mathrm{Borel}}$ following **Proposition** 2.1.2.5 is given by:

$$X \mapsto F(EG_+, X).$$

In particular, a spectrum X is Borel complete if $X \to F(EG_+, X)$ is an equivalence this gives as a consequence the map:

$$X^H \to X^{hH}$$

is an equivalence for all subgroups $H \leqslant G$. Where the target is the homotopy fixed points with respect to the subgroup H considering X as a spectrum with a G-action.

This suggests that a Borel spectrum has no "new information" other than homotopy fixed point of the underlying spectrum with G-action. This can be made more precise with the help of **Proposition** 2.1.1.13 we obtain the category of $\mathrm{Sp}_G^{\mathrm{Borel}}$ is equivalent to $\mathrm{Fun}(BG, \mathrm{Sp})$ the category of spectra with G-action.

Definition 2.1.2.7 (Proper Tate construction)**.** Let X be a genuine G-spectra, we define the **proper Tate construction** as $X^{\tau G} := \Phi^G(\beta X) = (\tilde{E}\mathcal{P} \otimes F(EG_+, X))^G$. Let $\mathcal{F}$ be a family of subgroups then we can define **$\mathcal{F}$-Tate construction** as $X^{\tau_{\mathcal{F}} G} := (\tilde{E\mathcal{F}} \otimes F(EG_+, X))^G$

So the proper Tate construction is Borelification followed by geometric fixed points. A priory proper Tate construction does not require the notion of genuine spectra; it turns out from the work of [NS18] we can completely perform the construction in the world of Borel spectra.

Let $\mathcal{P}$ be the localising $\otimes$-ideal in $\mathrm{Fun}(BG, Sp)$ generated by $\mathrm{Ind}_H^G(\mathbb{S}) = \Sigma^\infty_+ G/H$ where H runs over all proper subgroups. Similar to **Proposition** 2.1.2.6, the proper Tate construction $(-)^{\tau G}$ is the initial lax-monoidal approximation of $(-)^{hG}$ such that it vanishes on the ideal $\mathcal{P}$.

From the definition 2.1.2.7 and 2.4 we have the cofiber sequence:

$$\operatorname*{colim}_{G/H \in \mathcal{O}_{\mathcal{F}}(G)} X^{hH} \to X^{hG} \to X^{\tau_{\mathcal{F}} G}$$

In particular, if we have $\mathcal{F} = \{e\}$, we get the cofiber sequence for the classical Tate construction:

$$X_{hG} \to X^{hG} \to X^{\tau_{\{e\}} G}$$

2.1.3 Euler class and localisation

In this section, we introduce the notion of **Euler class** using which we can define the **proper Tate construction** as a certain localisation with respect to an Euler class; these ideas are very classical and due to Quillen and Carlsson. We refer the reader to [MNN19; AMR23] for a modern account.

Let G be a finite group and V be a finite dimensional real representation such that it is orthogonal (i.e., the representation is given by $\rho : G \to O(V)$), let S^V be the one-point compactification, we regard (S^V, ∞) as a pointed space. Hence we define the **Euler class** e_V as the pointed G-map $e_V : S^0 \to S^V$.

Let $S(V)$ be the unit sphere of the representation V; since the representation is orthogonal, we have that $S(V) \subset V$ is G-invariant. We have a finite G-CW complex structure on $S(V)$, where H is an isotropy subgroup if and only if $V^H \neq 0$. Finally, we cofiber sequence:

$$S(V)_+ \to S^0 \to S^V$$

where the map $S(V)_+ \to S^0$ is given by sending $S(V)$ to the non-base point.

Let G be a finite group and $\tilde{\rho}_G$ be the reduced regular representation and $\mathcal{P}$ be the family of proper subgroups, then using [MNN19, Prop 2.7] we identify $E\mathcal{P}$ and $\tilde{E}\mathcal{P}$ as:

$$E\mathcal{P} \simeq \operatorname*{colim}_n S(n\tilde{\rho}_G) \quad \tilde{E}\mathcal{P} \simeq \operatorname*{colim}_n S^{n\tilde{\rho}_G} \tag{2.5}$$

Recall from **Definition** 2.1.2.7 we have the proper Tate construction given by

$$\Phi^G(\beta X) \simeq (\tilde{E}\mathcal{P} \otimes F(EG_+, X))^G$$

now using from equitation 2.5 and the fact that $S^{n\tilde{\rho}}$ is dualisable we can deduce the following proposition [AMR23, Prop 5.10]:

Proposition 2.1.3.1. Let G be a finite group and $R \in \mathrm{Sp}_G$, then the proper Tate construction is given by:

$$R^{\tau G} \simeq \operatorname*{colim}_n (S^{n\tilde{\rho}_G} \otimes R)^{hG}$$

Moreover, if R is complex oriented commutative ring spectrum, then

$$R^{\tau G} \simeq R^{hG}[e^{-1}]$$

where $e \in \pi_{-2(|G|-1)}$ is a class obtained from the complex reduced representation $\tilde{\rho}_{\mathbb{C},G}$ via the complex orientation.

Example 2.1.3.2. Some examples of complex oriented ring spectrum include $KU_p^\wedge$ the complex K-theory, MU complex cobordism ring, $\mathbb{Z}$, $\mathbb{F}_p$, $E(n)$ the height n Lubin Tate theory.

Let X be a finite G-CW complex and R be any spectrum; then R^X the R-valued chains on X, can be considered as G-spectrum. We can then compute the proper Tate construction of R^X as follows:

The R-valued cochains, R^X as the limit of the constant diagram R indexed by X, where the limit is taken in Sp. Since X is a finite complex, the limit is a finite limit, and since the proper Tate construction is exact, we can then commute the limits. That is, we have

$$(R^X)^{\tau G} \simeq (R^{\tau G})^X$$

Since X is a G-CW complex obtained from attaching cells of the form $G/H \times S^n$. The proper Tate construction kills $G/H \times S^n$ if H is a proper subgroup. So the limit indexed over the subcomplex spanned by cells with isotropy groups G, i.e., the fixed points X^G. Hence we have:

$$(R^X)^{\tau G} \simeq (R^{\tau G})^{X^G}$$

Now let R be a complex oriented ring spectrum, then we obtain:

$$(R^X)^{\tau G} \simeq (R^X)^{hG}[e^{-1}] \simeq (R^{X_{hG}})[e^{-1}]$$

This leads us to the **localisation theorem** investigated in the literature by Borel, Atiyah-Bott, Atiyah-Segal, Quillen, and Dwyer-Wilkerson.

Theorem 2.1.3.3 (Localisation theorem)**.** Let G be a finite group, X a finite G-CW complexes and R a complex oriented ring spectra, then we have

$$(R^X)^{\tau G} \simeq (R^{X_{hG}})[e^{-1}] \simeq ((R^{hG})[e^{-1}])^{X^G} \simeq (R^{\tau G})^{X^G}$$

3 Fixed points via Tilting

The material presented in this chapter is a joint work with Robert Burklund. The version appearing in the thesis is written by the author.

3.1 Introduction

Given a space X, inspired by arithmetic, we can study the homotopy type of X one prime at a time and try to reconstruct the space using certain glueing data.

This idea was made precise by D.Sullivan [Sul70]. Given a prime $p > 0$ and a space X, we can associate the space $X_p^\wedge$, the p-completion of X and the rationalisation $X_{\mathbb{Q}}$. The subject of rational homotopy theory has seen many advances over the years due to the work of D.Sullivan [Sul77] and D.Quillen [Qui69].

The above works show that rational homotopy theory is completely algebraic. More precisely, the ∞-category of 1-connected rational spaces is equivalent to the ∞-cateogry of differential graded 1-connected Lie algebras or the ∞-category of differential graded coconnective commutative algebras with $\pi_{-1} = 0$.

Given a rational space, the associated CDGA is equivalent to the rational cochains $C^*(X;\mathbb{Q})$. Sullivan in [Sul77] introduces the theory of minimal models, which is extremely useful in computations. In general, rational homotopy theory helps transfer a topological problem into the algebraic world.

One can ask for similar models in the world of p-complete spaces. This was answered by the following theorem of Mandell:

Theorem 3.1.0.1. [Man01] Let $\bar{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$. The functor

$$C^*(-;\bar{\mathbb{F}}_p) : (\mathcal{S}_p)^{op} \to \mathrm{CAlg}_{\bar{\mathbb{F}}_p}$$

$$X \mapsto C^*(X;\bar{\mathbb{F}}_p)$$

from the category of p-complete space to the category of $\mathbb{E}_\infty$-$\bar{\mathbb{F}}_p$-algebras, sending a space X to its $\bar{\mathbb{F}}_p$-valued cochains is fully faithful when restricted to nilpotent spaces of finite type. □

The image of the above functor lands inside $\mathrm{CAlg}^{\mathrm{Fr}}_{\bar{\mathbb{F}}_p}$ the category of perfect algebras (see 3.2.3.8); a more refined version of the above theorem is proved in [Lur11]. Which states:

Theorem 3.1.0.2. [Lur11, Theorem 3.5.8] Let $\bar{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$. The $\bar{\mathbb{F}}_p$-valued cochains functor $C^*(-;\bar{\mathbb{F}}_p) : (\mathcal{S}^{\mathrm{Pro}(p)})^{op} \to \mathrm{CAlg}_{\bar{\mathbb{F}}_p}$ from the opposite category of p-profinite spaces to the category of commutative algebras over $\bar{\mathbb{F}}_p$ is fully faithful and its essential image is identified with the category of perfect algebras.

In this chapter, we study the theory of perfect algebras and associated perfection functors. The inclusion $\mathrm{CAlg}^{\mathrm{Fr}}_{\bar{\mathbb{F}}_p} \hookrightarrow \mathrm{CAlg}_{\bar{\mathbb{F}}_p}$ of the category of perfect algebras into the category of commutative algebras admits both adjoints. We denoted the left adjoint by $(-)^{\#}$ and the right adjoint by $(-)^{\flat}$; the right adjoint is referred to as tilting. We think of these as perfection functors in the land of higher algebra.

More than just constructing the right adjoint, we provide a formula for the homotopy groups of $A^{\flat}$ assuming certain finiteness conditions on A.

Proposition 3.1.0.3. (Homotopy groups of $A^{\flat}$, Proposition 3.3.2.5) For a finite type[1] $\mathbb{E}_\infty$-algebra A, the homotopy groups of $A^{\flat}$ is given by $\pi_* A^{\flat} \simeq \varprojlim_n \pi_* A$ where the sequential inverse limit is taken along the operation Q^0, where Q^0 is the 0-th power operation.

We then apply the above result to the study of the homotopy type of fixed points of group action.

Let X be a finite G-CW complex; then, there are two different notions of fixed points that one can talk about. Namely, the genuine fixed points X^G which depend on the G-CW complex structure and homotopy fixed points X^{hG}, which only depend on the underlying space with group action.

There is a natural comparison map $X^G \to X^{hG}$. One can ask when this is an equivalence. Here is a conjecture of Sullivan:

Conjecture 3.1.0.4 (Sullivan)**.** Let G be a finite p-group and X be a finite G-CW complex. Is the natural map $X^G \to X^{hG}$ is an equivalence after p-completion?

Even the case of the trivial action is complicated. For trivial action, the Sullivan conjecture asks for a finite space X. Is there an isomorphism between $X^G \simeq X$ and $X^{hG} \simeq \mathrm{map}(BG, X)$?

The conjecture is one of the landmarks of equivariant homotopy theory and was answered to be true by the celebrated works of G. Carlsson [Car91], H. Miller [Mil84] and J. Lannes [Lan92].

The proof of the Sullivan conjecture investigated deep relations between the cohomology of fixed points and $\mathbb{F}_p$-cohomology operations. Here is a theorem of Dwyer and Wilkerson [DW88], which computes the cohomology of fixed points from Borel equivariant cohomology.

Theorem 3.1.0.5. [DW88] Let G be an elementary abelian group $(\mathbb{Z}/p)^n$. Let X be a finite G-CW complex then we have:

$$H^*(X^G) \simeq \mathbb{F}_p \otimes_{H^*(BG)} \mathrm{Un}(H^*(X_{hG})[e^{-1}])$$

where $e \in H^*(X_{hG})$ is the Euler class described 2.1.3, coming from group cohomology $H^*(BG)$ and $\mathrm{Un}(H^*(X_{hG})[e^{-1}])$ is the largest unstable Steenrod algebra inside $H^*(X_{hG})[e^{-1}]$.

We prove an analogous theorem which computes the fixed points from Borel cohomology using the notion of perfection.

[1]We say A is of finite type if $\pi_n A$ finite dimensional all n

Theorem 3.1.0.6 (Theorem 3.4.1.4)**.** Let G be an elementary abelian group and X, a finite G-CW complex. The tilt of the algebra $C^*(X;k)^{\tau G} \simeq C^*(X;k)^{hG}[e^{-1}]$ is equivalent to $C^*(X^G \times BG; k)$. Where $(-)^{\tau G}$ is the proper Tate construction.

So combining with Mandell's theorem, we can recover the homotopy type of genuine fixed points from the Borel equivariant cohomology.

3.2 Power Operations on commutative $\mathbb{F}_p$-algebras

3.2.1 Recollection of power operations

This section recalls some preliminaries of power operations and their relations to cohomology operations on spaces. Much of the material presented here is well known; we recall them for clarity.

Given $X \in \mathcal{S}$, we can associate the $\mathbb{F}_p$-valued cochains $C^*(X;\mathbb{F}_p)$ and using the diagonal map $\Delta : X \to X \times X$ we get an operation $m_2 : C^*(X;\mathbb{F}_p) \otimes C^*(X;\mathbb{F}_p) \to C^*(X;\mathbb{F}_p)$, which is not commutative on the nose as a map of chain complexes but is graded commutative after taking homology. The cohomology $\mathrm{H}^*(X;\mathbb{F}_p)$ carries the structure of a graded commutative ring.

The failure of m_2 to be commutative **is a feature rather than a bug**, this gives rise to a family of operations known as the $\mathbb{F}_p$-power operations or the $\mathbb{F}_p$-Steenrod operations.

$$P^i : \mathrm{H}^n(X;\mathbb{F}_p) \to \mathrm{H}^{n+2i(p-1)}(X;\mathbb{F}_p).$$

Before discussing more about the properties of the Steenrod operations, we recall how they arise.

Let $\mathcal{C}^\otimes = \mathcal{D}(\mathbb{F}_p)^\otimes$ be the presentably symmetric monoidal category whose underlying category is the derived category of modules over $\mathbb{F}_p$. Let $A \in \mathrm{CAlg}(\mathcal{C})$, which means we have for all $n \in \mathbb{N}$, operations $m_n : A^\otimes \to A$ along with factorisations:

$$\begin{tikzcd} A^{\otimes n} \arrow[r, "m_n"] \arrow[d] & A \\ A^{\otimes n}_{h\Sigma_n} \arrow[ur, "\overline{m}_n"'] & \end{tikzcd}$$

where the vertical map is the quotient map of the Σ_n-action on $A^{\otimes n}$ by permuting the factors.

Example 3.2.1.1. Let $X \in \mathcal{S}$, then $\Sigma^\infty_+ X$ is naturally an $\mathbb{E}_\infty$-co-algebra in $\mathrm{Sp}^\otimes$. Which in-turn gives an $\mathbb{E}_\infty$-algebra structure on $C^*(X;\mathbb{F}_p) \simeq \mathrm{map}_{\mathrm{Sp}}(\Sigma^\infty_+ X, \mathbb{F}_p)$. ☐

Let us sketch the construction of the Steenrod operation in the case $p = 2$. Let $A \in \mathrm{CAlg}(\mathcal{C})$ and $x_n \in \pi_n(A)$ be class in degree n, which can be represented by a $\mathbb{F}_2$-linear map

$$x_n : \mathbb{F}_2[n] \to A$$

We then form

$$(x_n)^{\otimes 2} : (\mathbb{F}_2[n])^{\otimes 2} \to A^{\otimes 2}$$

taking the equivariant quotient of $x_n^{\otimes 2}$ by Σ_2 and composing with $\bar{m}_2 : A^{\otimes 2}_{h\Sigma 2} \to A$ gives us

$$\bar{m}_2 \circ (x_n)^{\otimes 2}_{h\Sigma_2} : (\mathbb{F}_2[n])^{\otimes 2}_{h\Sigma_2} \to A \tag{3.1}$$

Identifying $(\mathbb{F}_2[n])^{\otimes 2}$ with $\mathbb{F}_2[2n]$, we see that Σ_2-action on $\mathbb{F}_2[-n]$ is given by $\mathbb{E}_1$-map

$$\Sigma_2 \to \mathrm{End}(\mathbb{F}_2[2n]) \simeq \mathbb{F}_2.$$

Since there are no nontrivial units in $\mathbb{F}_2$, we have no nontrivial Σ_2-action on $\mathbb{F}_2[2n]$. Hence, $(\mathbb{F}_2[n])^{\otimes 2}_{h\Sigma_2} \simeq B\Sigma_2 \otimes \mathbb{F}_2[2n]$, we rewrite **equation (3.1)** as

$$B\Sigma_2 \otimes \mathbb{F}_2[2n] \simeq C_*(B\Sigma_2; \mathbb{F}_2)[2n] \to A$$

we have

$$\pi_{i+2n} C_*(B\Sigma_2; \mathbb{F}_2)[2n] = \begin{cases} \mathbb{F}_2\{t_i \otimes e_{2n}\}, & i \geqslant 0 \\ 0, & \text{otherwise} \end{cases}$$

the image of $t_i \otimes e_{2n}$ gives us a class $Q^{n+i}(x_n) \in \pi_{2n+i}(A)$. In the literature, these are often referred to as **Dyer-Lashof operations** or **generalised Steenrod operations**. A more detailed exposition can be found in [BMM+86], and a modern account can be found in [Law20; GL20].

Given an $\mathbb{E}_\infty$-algebra $A \in \mathcal{D}(\mathbb{F}_p)$, there is a family of operations on the homotopy groups $\pi_*(A)$,

For $\mathbf{p} = \mathbf{2}$:

$$Q^i : \pi_n(A) \to \pi_{n+i}(A) \ \forall \ i \in \mathbb{Z}$$

satisfying the following properties:

1. **Additivity**: $Q^i(x+y) = Q^i(x) + Q^i(y)$.
2. **Cartan Formula**: $Q^i(x) = \Sigma_{j+k=i} Q^j(x) Q^k(x)$
3. **Squaring**: $Q^{|x|}(x) = x^2$.
4. **Instability**: $Q^i(x) = 0 \ \forall \ i < |x|$
5. **Adem relations**: $Q^i Q^j(x) = \Sigma_k \binom{k-j-1}{2k-i} Q^{i+j-k} Q^k(x)$

The case of $p > 2$:

$$P^i : \pi_n(X) \to \pi_{n+2i(p-1)}(A)$$

and

$$\beta P^i : \pi_n(A) \to \pi_{n+2i(p-1)-1}(A)$$

where $i \in \mathbb{Z}$. We use uniform notation $\beta^\epsilon P^i$ where $\epsilon \in \{0,1\}$ to state the Dyer-Lashof relations:

1. **Additivity**: $\beta^\epsilon P^i(x+y) = \beta^\epsilon P^i(x) + \beta^\epsilon P^i(y)$.
2. **Cartan Formula**:
$$P^i(xy) = \Sigma_{j+k=i} P^j(x)P^k(x)$$
$$\beta P^i(xy) = \Sigma_{j+k=i} \beta P^j(x)P^k(y) + \Sigma_{j+k=i} P^j(x)\beta P^k(y)$$
3. **Squaring**: $P^{|x|/2}(x) = x^2$.
4. **Instability**: $\beta^\epsilon P^i(x) = 0 \ \forall\ 2i+\epsilon < |x|$
5. **Adem relations**:
$$P^iP^j = \Sigma_k \binom{(p-1)(k-j)-1}{pk-i} P^{i+j-k}P^k$$
$$P^i\beta P^j(x) = \Sigma_k \binom{(p-1)(k-j)-1}{pk-i} \beta P^{i+j-k}P^k$$
$$-\Sigma_k \binom{(p-1)(k-j)-1}{pk-i-1} P^{i+j-k}\beta P^k$$

Remark 3.2.1.2. We can replace the above construction with a field k of characteristic p. The approach above could be uniformly applied to commutative algebra object E in Sp; we see that understanding power operations is essentially understanding $E^*(B\Sigma_n)$. In particular, working with a field k of characteristic zero, we do not have non-trivial power operations since $C^*(B\Sigma_n; k) \simeq k$. □

Proposition 3.2.1.3. Let $X \in \mathcal{S}$ and k a field of characteristic $p > 0$, then the action of Q^i and $\beta^\epsilon P^i$ on the homotopy groups of $C^*(X;k)$ is trivial for all $i > 0$. □

Example 3.2.1.4. Let $X = BC_p$, then we have

$$\pi_* C^*(BC_p; \mathbb{F}_p) = \begin{cases} \mathbb{F}_p[x], & |\, x \,| = -1 \textit{ for } p = 2 \\ \mathbb{F}_p[x] \otimes \Lambda(y), & |\, y \,| = -1 \,|\, x \,| = -2 \textit{ for } p \textit{ odd} \end{cases}$$

By using the above identities, we have:

In the case $p = 2$,

$$Q^i(x^n) = \begin{cases} 0, & \forall i > 0 \\ \binom{n}{-i} x^{n-i}, & \forall i \leqslant 0 \end{cases}$$

□

All our argument goes through for both even and odd primes. We choose to demonstrate the calculations for only the prime $p = 2$.

It is useful to organise these operations into an algebra as considered in [Man01],

Definition 3.2.1.5 (Mandell)**.** Let $\mathfrak{B}_p$ be the free associative algebra generated by Q^i for $p = 2$ and P^i and βP^i for $p > 2$ where $i \in \mathbb{Z}$ and quotient with respect to the two-sided ideal generated by the Adem relations. The algebra $\mathfrak{B}_p$ is called the **Dyer-Lashof algebra** or the **generalised Steenrod algbera** □

For $p = 2$, let $I = (i_1, i_2, ..., i_n)$ be a tuple of length n, where $i_k \in \mathbb{Z}$. We define $Q^I = Q^{i_1}Q^{i_2}...Q^{i_n}$ and the **excess** $e(I) = i_n + \sum_{k=1}^{n-1}(i_k - 2i_{k+1})$. We say that I is **admissible** if $i_k \geqslant 2i_{k+1}$ $\forall$ $1 \leqslant k < n$.

For $p > 2$, let $I = (\epsilon_1, i_1, \epsilon_2, i_2, ..., \epsilon_n, i_n)$ we say its of length n, where $i_k \in \mathbb{Z}$ and $\epsilon_k \in \{0, 1\}$. We define $P^I = \beta^{\epsilon_1}P^{i_1}\beta^{\epsilon_2}P^{i_2}...\beta^{\epsilon_n}P^{i_n}$ and the **excess**

$$e(I) = 2i_n + \sum_{k=1}^{n-1} 2(i_k - pi_{k+1}) + \epsilon_0 - \epsilon_1 + ...(-1)^n\epsilon_n$$

.

We can state the following proposition of [Man01, Proposition 11.2]:

Proposition 3.2.1.6 (Mandell)**.** The algebra $\mathfrak{B}_p$ considered as an $\mathbb{F}_p$ vector space for p odd has a basis $\{P^I \mid I \text{ is admissible}\}$ (resp. for $p = 2$ $\{Q^I \mid I \text{ is admissible}\}$). $\square$

On the other hand, given a spectrum X, the $\mathbb{F}_p$ cohomology $\mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty_+ X; \mathbb{F}_p)$ has an action of the $\mathbb{E}_1$-algebra $\mathrm{End}_{\mathbb{S}}(\mathbb{F}_p)$. The elements of $\mathfrak{A}_p = \pi_*(\mathrm{End}_{\mathbb{S}}(\mathbb{F}_p))$ are called $\mathbb{F}_p$-cohomology operations and the algebra $\mathfrak{A}_p$ is the p**-Steenrod algebra**.

We now state a result of Mandell [Man01, Theorem 1.4], which relates Dyer-Lashof algebra and the p-Steenrod algebra.

Theorem 3.2.1.7 (Mandell)**.** The algebra obtained $\mathfrak{B}_p$ quotient by the two-sided ideal $(1 - P^0)$ (respectively $(1 - Q^0)$ for $p = 2$) is isomoprhic to $\mathfrak{A}_p$. $\square$

3.2.2 Tate-valued Frobenius

In this section, we recall the notion of Tate valued Frobenius, which records the total power operations. a detailed exposition of the material presented here can be found in [Wil19], and these ideas are due to [Lur11; NS18].

We start with the following result of J.Lurie [Lur11] and Nikolaus-Scholze [NS18]:

Theorem 3.2.2.1. [Lur11, Propostion 2.2.3] [NS18, Proposition III.1.1] The functor $T_p : \mathrm{Sp} \to \mathrm{Sp}$ taking a spectrum X to $(X^{\otimes p})^{tC_p}$ is exact. $\square$

Theorem 3.2.2.2. [NS18, Proposition III.1.2] For any exact functor $F : \mathrm{Sp} \to \mathrm{Sp}$, there is an equivalence between the space of natural transformations in exact functors $\mathrm{map}_{\mathrm{Fun}^{ex}(\mathrm{Sp},\mathrm{Sp})}(\mathrm{id}, F)$ and $\mathrm{map}_{Sp}(\mathbb{S}, F(\mathbb{S})) \simeq \Omega^\infty F(\mathbb{S})$ $\square$

Definition 3.2.2.3. [NS18, Definition III.1.4] We define the **Tate diagonal** to be the natural transformation $\Delta_p : \mathrm{id}_{Sp} \to T_p$ corresponding to the map $\mathbb{S} \to \mathbb{S}^{hC_p} \to \mathbb{S}^{tC_p} \simeq \mathbb{S}^{\otimes_l tC_p}$. $\square$

Where $\mathbb{S}$ is endowed with the trivial action, hence we get a map $\mathbb{S} \to \mathbb{S}^{hC_p}$ and compose with the map from the homotopy fixed point to the Tate construction.

We restrict to the case of $p = 2$.

The power operations for $\mathbb{E}_\infty$-algebras over a field k of characteristic 2 are constructed in two different ways. The first one is due to Lurie [Lur11, Construction 2.2.6] and the second using Tate diagonal due to Nikolaus-Scholze [NS18].

Let $T_2 : \mathrm{Mod}_k \to \mathrm{Mod}_k$ be the functor $V \mapsto (V^{\otimes 2})^{tC_2}$, then tthere is a colimit preserving approximation $\tilde{T}_2 \to T_2$. More concretely, we have $\tilde{T}_2$ as the left Kan extension:

$$\begin{tikzcd} \mathrm{Perf}_k \arrow[r, hook] \arrow[d] & \mathrm{Mod}_k \arrow[r, "T_2"] & \mathrm{Mod}_k \\ \mathrm{Mod}_k \arrow[urr, dashed, "\tilde{T}_2"'] & & \end{tikzcd}$$

Since the functor $\tilde{T}_2 : \mathrm{Mod}_k \to \mathrm{Mod}_k$ is colimit preserving, by Morita theory [Lur17, Proposition 7.1.2.4] we have

$$\tilde{T}_2(M) \simeq M \otimes B$$

where B is a $\mathbb{F}_2$-bimodule, so we can identify B with $T_2(k) \simeq k^{tC_2}$ as left modules, this implies that $\tilde{T}_2(V) \simeq V \otimes B$ and the natural transformation gives us a map $V \otimes B \to T_2(V)$.

Construction 3.2.2.4 (Lurie [Lur11])**.** Let A be an $\mathbb{E}_\infty$-algebra; we can define

$$Q^i : \Sigma^i A \to A.$$

We have $\pi_*(k^{tC_2}) \simeq k[t, t^{-1}]$ where $\mid t \mid = -1$. From $t^{-(i+1)}$ we get a map of the right modules $\Sigma^i k \to \Sigma^{-1} B$, tensoring with A using the right module structure gives a map of spectra

$$\Sigma^i k \otimes A \to \Sigma^{-1} B \otimes A.$$

We have the Tate cofiber sequence of $A^{\otimes 2}$ with respect to the swap action

$$(A^{\otimes 2})_{hC_2} \to (A^{\otimes 2})^{hC_2} \to (A^{\otimes 2})^{tC_2} \simeq T_2(A).$$

Rotating the sequence gives you

$$\Sigma^{-1} T_2(A) \to A^{\otimes 2}_{hC_2},$$

and since A is an $\mathbb{E}_\infty$-algebra we have map

$$\bar{m}_2 : (A^{\otimes 2})_{hC_2} \to A.$$

This assembles into the map:

$$Q^i : \Sigma^i k \otimes A \to \Sigma^{-1} B \otimes A \to \Sigma^{-1} T_2(A) \to (A^{\otimes 2})_{hC_2} \xrightarrow{\bar{m}_2} A.$$

Note that since the first map is just a map of spectra, this map Q^i is not k-linear. $\square$

The second approach by [NS18] is to use the Tate diagonal to define the **Tate valued Frobenius**, which encodes the total power operation.

Construction 3.2.2.5 (Tate-valued Frobenius)**.** Let A $\in \mathrm{CAlg}(\mathrm{Mod}_{\mathbb{F}_2})$. Then we define the Tate-valued Frobenius as

$$Q(t) : A \to (A^{\otimes 2})^{tC_2} \xrightarrow{\bar{m}_2} A^{tC_2}$$

where the target is equipped with trivial action. Therefore, $\pi_* A^{tC_2} \simeq \pi_* A((t))$ where $|\, t \,| = -1$. At the level of homotopy groups, we have,

$$Q(t)(x) = \Sigma a_i t^i$$

where $x \in \pi_n(A)$ and we define $Q^i(x) := a_i \in \pi_{n+i}(A)$. As a map of spectra we can extract $Q^i : \Sigma^i A \to A$ as the composite

$$A \to A^{tC_2} \xrightarrow{t^{-(i+1)}} \Sigma^{-(i+1)} A^{tC_2} \to \Sigma^{-i} A_{hC_2} \to \Sigma^{-i} A.$$

□

Remark 3.2.2.6 (Naturality)**.** Given an $\mathbb{E}_\infty$-ring map $A \to B$ (not necessarily map of k-modules), we have the commuting square

$$\begin{tikzcd} A \arrow[r, "Q_A"] \arrow[d] & A^{tC_2} \arrow[d] \\ B \arrow[r, "Q_B"] & B^{tC_2} \end{tikzcd}$$

□

The three definitions of Q^i agree; the proof can be found in [Wil19, Section 3.5].

Proposition 3.2.2.7 (Cartan formula)**.** Let A be an $\mathbb{E}_\infty$-algebra, $x, y \in \pi_* A$, then $Q(t)(x \otimes y) = Q(t)(x).Q(t)(y)$ □

Remark 3.2.2.8. The case of $p > 2$ is treated in the upcoming article of S. Bharkan in a very general context. □

3.2.3 Perfect algebras

In this section, we define the category of perfect $\mathbb{E}_\infty$-algebras over k, a field of characteristic p, as a presentable ∞-category. In order to do that, we recall the algebra of stable power operations for an $\mathbb{E}_\infty$-algebra as introduced by [GL20] and [Lur07].

Construction 3.2.3.1 (Lurie,Glasman-Lawson)**.** Let E be a commutative ring spectrum. We have the forgetful functor $F : \mathrm{CAlg}(\mathrm{Mod}_E) \to \mathrm{Sp}$ from the ∞-category of commutative E-algebra to the category of spectra.

Let us define $\mathcal{C} := \mathrm{Fun}(\mathrm{CAlg}(\mathrm{Mod}_E), \mathrm{Sp})$, which is a stable ∞-category with $F \in \mathcal{C}$, which gives us an $\mathbb{E}_1$-algebra $\mathrm{End}_{\mathcal{C}}(F)$ and we call this the **algebra of stable power operations** following [GL20] and denote it by $\mathrm{Pow}(E)$. □

Let A be a commutative E-algebra; then there is an action of $\mathrm{Pow}(E)$ on A. Now let X be a space, then the E-valued cochains $C^*(X; E) \simeq \mathrm{map}_{Sp}(\Sigma^\infty_+ X, E)$ is a commutative E-algebra so it has a natural action of $\mathrm{Pow}(E)$ on $C^*(X; E)$, and also an action of $\mathrm{End}(E)$ and these actions are compatible [GL20, Theorem 1.1].

We can summarise this following the result of [GL20, Theorem 1.1]:

Theorem 3.2.3.2. (Glasma-Lawson) Given a commutative ring spectrum E, we have a map of $\mathbb{E}_1$-algebras:

$$E \to \mathrm{Pow}(E) \to \mathrm{End}(E)$$

such that the action of $\mathrm{Pow}(E)$ is compatible with the action of $\mathrm{End}(E)$ on $C^*(X; E)$. In particular, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{S}^{op} \arrow[r, "C^*(-;E)"] \arrow[dd, "C^*(-;E)"'] & \mathrm{Mod}_{\mathrm{End}(E)} \arrow[d] \\
& \mathrm{Mod}_{\mathrm{Pow}(E)} \arrow[d] \\
\mathrm{CAlg}_E \arrow[ur] \arrow[r, "\mathrm{forget}"'] & \mathrm{Mod}_E
\end{tikzcd}$$

where the forgetful functor $\mathrm{CAlg}_E \to \mathrm{Mod}_E$ factors through the category $\mathrm{Mod}_{\mathrm{Pow}(E)}$. $\square$

We can justify calling $\mathrm{Pow}(E)$, the algebra of power operation with the following theorem [GL20, Theorem 10.3]:

Theorem 3.2.3.3. Let $E = \mathbb{F}_p$, then $\pi_*\mathrm{Pow}(E)$ is the completion of the algebra $\mathfrak{B}_p$ (refer 3.2.1.5) with respect to the excess filtration. The map $\mathrm{Pow}(E) \to \mathrm{End}(E)$ on homotopy groups is the quotient with respect to the two-sided ideal $(1 - P^0)$ (and $(1 - Q^0)$ for $p = 2$). $\square$

In particular the algebra $\pi_*\mathrm{Pow}(\mathbb{F}_p)$ is generated by $\beta^\epsilon P^i$ (Q^i for $p = 2$). Given an $\mathbb{E}_\infty$-algebra A over $\mathbb{F}_p$, it is naturally a module over $\mathrm{Pow}(\mathbb{F}_p)$, and this also gives a spectrum-level action of the power operations on any commutative $\mathbb{F}_p$-algebra.

Let $\mathrm{Mod}^{\mathrm{Fr}}_{\mathrm{Pow}(k)}$ denote the full subcategory of $\mathrm{Pow}(k)$-modules on which $Q^0 \in \pi_*(\mathrm{Pow}(k))$ acts via isomorphism. We refer to these objects as **Frobenius perfect** modules or Q^0-local modules.

In order to do this, we check the right Ore-condition for the multiplicative closed subset spanned by Q^0 in $\pi_*\mathrm{Pow}(k)$. We will demonstrate the Ore-condition at the prime 2, and a similar proof works for odd primes as well.

Definition 3.2.3.4 (Right Ore condition)**.** In a graded associative ring called R, a set S of homogeneous elements is said to satisfy the right Ore condition [Lur17, Section 7.2.3] if the following three conditions hold:

1. S is multiplicatively closed.
2. Given $r \in R, s \in S$ there exist $r' \in R$ and $s' \in S$ such that $rs' = sr'$
3. $x \in R$ and $s \in S$ such that $sx = 0$, there exist s' such that $xs' = 0$

$\square$

We begin with the following proposition, which we use repeatedly.

Lemma 3.2.3.5. For $i > 0$, $Q^i(Q^0)^i = 0$.

Proof. We recall the Adem relations for prime $p = 2$,

$$Q^iQ^j = \Sigma_k \binom{k-j-1}{2k-i} Q^{i+j-k}Q^k$$

where the sum is non-zero if $\frac{i}{2} \leqslant k < i - j$.

For the base case, $i = 1$ and $j = 0$, the sum is taken over empty set, hence $Q^1Q^0 = 0$. Now we work by induction. Let us look at

$$Q^iQ^0 = \Sigma_k \binom{k-1}{2k-i} Q^{i-k}Q^k$$

where $\frac{i}{2} \leqslant k < i$. Hence the terms appearing in the sum consist of terms when composed with $(Q^0)^{i-1}$ on the right is zero by induction on i. □

Proposition 3.2.3.6. The collection $S = \{(Q^0)^i\}$ for $i \geqslant 0$ in $\pi_*(\mathrm{Pow}(k))$, (where by convention we take $(Q^0)^0 = 1$) satisfy the right Ore-condition.

Proof. We have to check the conditions of 3.2.3.4, and we see that the condition **1** is satisfied by definition.

Now let us check condition **2**, that is given an $r \in \pi_*(\mathrm{Pow}(k))$ and $s \in S$, we need to provide $r' \in \pi_*(\mathrm{Pow}(k))$ and $s' \in S$ such that $rs' = sr'$.

Let us verify the Ore-condition on $r = Q^I = Q^{i_1}Q^{i_2}...Q^{i_n}$ for $I = (i_1, i_2, ..., i_n)$ admissible and $s = (Q^0)^k$. If one of the i_k is positive, then by admissibility $i_n > 0$. By 3.2.3.5 we have that $Q^I(Q^0)^{i_n} = 0$, this implies we can take $s' = (Q^0)^{i_n}$ and $r' = 0$.

So without loss of generality, we can assume that $i_k < 0$ for all $k < 0$ in $r = Q^I = Q^{i_1}Q^{i_2}...Q^{i_n}$ where $I = (i_1, i_2, ..., i_n)$. We proceed by induction on the length of I. Let us start with the following observation of what Q^0Q^{-j} is for $j > 0$. By Adem relations, we have

$$Q^0Q^{-j} = \Sigma_k \binom{k+j-1}{2k} Q^{-j-k}Q^k$$

where $0 \leqslant k \leqslant j - 1$, which in turn gives us

$$Q^0Q^{-j} - Q^{-j}Q^0 = \Sigma_k \binom{k+j-1}{2k} Q^{-j-k}Q^k(x)$$

where $1 \leqslant k \leqslant j-1$. Observe the right-hand side vanishes upon multiplying by $(Q^0)^{j-1}$ by 3.2.3.5, so this gives us the following identity

$$Q^0Q^{-j}(Q^0)^{j-1} = Q^{-j}(Q^0)^j.$$

In particular, we have

$$(Q^0)^{i+1}Q^{-j}(Q^0)^{j-1} = Q^{-j}(Q^0)^{j+i},$$

so given $r = Q^{-j}$ and $s = (Q^0)^i$ we can choose $r' = Q^{-j}(Q^0)^{j-1}$ and $s' = (Q^0)^{j+i-1}$ to get

$$Q^{-j}(Q^0)^{j+i-1} = (Q^0)^i Q^{-j}(Q^0)^{j-1}.$$

Now let Q^I be an admissible monomial with $I = (i_1, i_2, ..., i_n)$ and $i_k < 0\ \forall k$. The induction hypothesis is, for I of length n and Q^I admissible and any $(Q^0)^j$, we can find an N such that

$$Q^I(Q^0)^N = (Q^0)^j Q^I(Q^0)^{N-j}.$$

Note that the above calculation verifies the case of length 1 by taking $N = i_1 + j - 1$. Now assume by induction we have proved for I of length $n-1$, then we for I of length n, we have for $N \geqslant i_n - 1$

$$Q^I(Q^0)^N = Q^{i_1}Q^{i_2}...Q^{i_n}(Q^0)^N = Q^{i_1}Q^{i_2}...Q^{i_{n-1}}(Q^0)^{N-i_n-1}Q^{i_n}(Q^0)^{i_n-1}$$

So we reduce to the case of $Q^{I'}$ where $I' = (i_1, i_2, ...i_{n-1})$ which follows from induction.

Finally, for condition **(3)** of 3.2.3.4, we proceed similar to the previous case, given $r = Q^I$ and if I has an $i_k > 0$ then we have $i_n > 0$ hence for $s' = (Q^0)^{i_n}$ we have $rs' = Q^I(Q^0)^{i_n} = 0$.

So without loss of generality, we can assume that $i_k \leqslant 0$ for all $1 \leqslant k \leqslant n$. But we have seen that there exists an N such that:

$$Q^I(Q^0)^N = (Q^0)^s Q^I(Q^0)^{N-s},$$

so if we have $(Q^0)^s Q^I = 0$ by multiplying suitable $(Q^0)^N$ on the right we get

$$(Q^0)^s Q^I(Q^0)^N = 0.$$

However $(Q^0)^s Q^I(Q^0)^N = Q^I(Q^0)^{N+s} = 0$, hence we have proved criteria **3** of 3.2.3.4. This concludes the proof of the proposition. □

Remark 3.2.3.7. The category of Frobenius perfect modules $\mathrm{Mod}^{\mathrm{Fr}}_{\mathrm{Pow}(k)}$ is Bousfield localisation of $\mathrm{Mod}_{\mathrm{Pow(k)}}$ and it is presentable [Lur17, Section 7.2.3]. □

Definition 3.2.3.8 (Perfect algebras)**.** We define the category of **perfect algebras** $\mathrm{CAlg}^{\mathrm{Fr}}_{\mathbb{F}_p}$ as the pullback:

$$\begin{tikzcd}
\mathrm{CAlg}^{\mathrm{Fr}}_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{CAlg}_k \arrow[d] \\
\mathrm{Mod}^{\mathrm{Fr}}_{\mathrm{Pow}(k)} \arrow[r] & \mathrm{Mod}_{\mathrm{Pow}(k)}
\end{tikzcd}$$

That is the full subcategory of CAlg_k, such that the power operation Q^0 acts by isomorphism. □

Proposition 3.2.3.9. The category of perfect algebras $\mathrm{CAlg}^{\mathrm{Fr}}_k$ is a full subcategory of $\mathrm{CAlg}(k)$, and it is a presentable ∞-category.

Proof. The categories $\mathrm{Mod}_{\mathrm{Pow(k)}}$, $\mathrm{Mod}^{\mathrm{Fr}}_{\mathrm{Pow}(k)}$, CAlg_k are presentable. The forgetful functor $F : \mathrm{CAlg}_k \to \mathrm{Mod}_{\mathrm{Pow(k)}}$ and the inclusion of $\mathrm{Mod}^{\mathrm{Fr}}_{\mathrm{Pow}(k)} \hookrightarrow \mathrm{Mod}_{\mathrm{Pow(k)}}$ are right adjoints. Hence the diagram that we want to take pullback lives in Pr^R and inclusion $\mathrm{Pr}^R \subset \mathrm{Cat}_\infty$ preserves limits. This proves that $\mathrm{CAlg}^{\mathrm{Fr}}_{\mathbb{F}_p}$ is presentable.

Since the inclusion $\mathrm{Mod}^{\mathrm{Fr}}_{\mathrm{Pow}(k)} \hookrightarrow \mathrm{Mod}_{\mathrm{Pow(k)}}$ is fully faithful, we obtain a fully faithful inclusion $\mathrm{CAlg}^{\mathrm{Fr}}_k \hookrightarrow \mathrm{CAlg}_k$. This is because the mapping space in $\mathrm{CAlg}^{\mathrm{Fr}}_k$ is computed as a pullback of mapping spaces; hence fully faithful inclusion of categories is stable under pullback. □

3.3 Tilting

In this section, we construct two adjoints to the inclusion $\mathrm{CAlg}^{\mathrm{Fr}}_k \to \mathrm{CAlg}_k$, the right adjoint to the inclusion is called **tilting**, and we give an explicit formula for the computation of the homotopy groups of the tilt $A^\flat$ of an $\mathbb{E}_\infty$-algebra A.

3.3.1 Construction of tilting

We first start with proving the existence of the left adjoint $(-)^{\#} : \mathrm{CAlg}_k \to \mathrm{CAlg}^{\mathrm{Fr}}_k$.

Proposition 3.3.1.1. There exists a left adjoint to the inclusion functor $\mathrm{CAlg}^{\mathrm{Fr}}_k \to \mathrm{CAlg}_k$.

Proof. Recall the category $\mathrm{CAlg}^{\mathrm{Fr}}_k$ is defined as a pullback:

$$\begin{tikzcd}
\mathrm{CAlg}^{\mathrm{Fr}}_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{CAlg}_k \arrow[d] \\
\mathrm{Mod}^{\mathrm{Fr}}_{\mathrm{Pow}(k)} \arrow[r] & \mathrm{Mod}_{\mathrm{Pow}(k)}
\end{tikzcd}$$

Since the diagram is a diagram in Pr^R and the inclusion $\mathrm{Pr}^R \subset \mathrm{Cat}_\infty$ preserves limits, so the pullback is taken internally to Pr^R; in particular, we have $\mathrm{CAlg}^{\mathrm{Fr}}_k \hookrightarrow \mathrm{CAlg}_k$ is a right adjoint. Hence we have a left adjoint. □

Definition 3.3.1.2. We denote the left adjoint to $\mathrm{Mod}^{\mathrm{Fr}}_{\mathrm{Pow}(k)} \hookrightarrow \mathrm{CAlg}_k$ by

$$(-)^{\#} : \mathrm{CAlg}_k \to \mathrm{CAlg}^{\mathrm{Fr}}_k$$

we refer to this functor as **colimit perfection**. □

Remark 3.3.1.3. We can also observe that the subcategory $\mathrm{CAlg}^{\mathrm{Fr}}_k$ is closed under limits by checking the action of Q^0 on the homotopy groups of the limits. □

Now in order to construct the left adjoint, we prove that perfect algebras are closed under colimits. We begin with the following observation:

Lemma 3.3.1.4. Let A be a perfect algebra over k, i.e. $A \in \mathrm{CAlg}^{\mathrm{Fr}}_k$, then A is coconnective and the action $Q^i : \pi_* A \to \pi_{*+i} A$ is 0, for all $i > 0$.

Proof. For any $A \in \mathrm{CAlg}_k$, we have by $Q^i(Q^0)^i = 0$ for all $i > 0$ by Lemma 3.2.3.5. Now if $A \in \mathrm{CAlg}_k^{\mathrm{Fr}}$, we have Q^0 acting by isomorphism on homotopy groups. Hence $Q^i = 0$.

Let $x \in \pi_i(A)$ where $i > 0$, by instability condition we have $Q^0(x) = 0$, but if A is perfect Q^0 must be isomorphism, hence $x = 0$. Since x was an arbitrary element in a positive degree, we have A is coconnective. □

Proposition 3.3.1.5. The subcategory of perfect algebras $\mathrm{CAlg}_k^{\mathrm{Fr}}$ is closed under colimits.

Proof. In order to prove the closure under all small colimits, it's enough to check perfect algebras are closed under taking sifted colimits and finite coproducts. The forgetful functor $U : \mathrm{CAlg}_k \to \mathrm{Mod}_k$ preserves sifted colimit and factors through the category of $\mathrm{Mod}_{\mathrm{Pow}(k)}$ followed by the forgetful functor $\mathrm{Mod}_{\mathrm{Pow}(k)} \to \mathrm{Mod}_k$, which admits both left and a right adjoint. This implies the factorisation $U : \mathrm{CAlg}_k \to \mathrm{Mod}_{\mathrm{Pow}(k)}$ preserves sifted colimits. Hence, for sifted, it is enough to check the underlying module over power operations is Frobenius perfect.

Let

$$A = \operatorname*{colim}_{k \in K} A_k$$

with A_k perfect for all $k \in K$ and K sifted and since the category of Frobenius perfect modules $\mathrm{Mod}_{\mathrm{Pow}(k)}^{\mathrm{Fr}}$ inclusion into $\mathrm{Mod}_{\mathrm{Pow}(k)}$ is closed under colimits we have A is Frobenius perfect.

The only case remaining is that of finite coproducts, that is, given $A, B \in \mathrm{CAlg}_k^{\mathrm{Fr}}$ we need to prove $A \otimes B \in \mathrm{CAlg}_k^{\mathrm{Fr}}$. Let $x \otimes y \in \pi_n(A \otimes B)$ where $x \in \pi_i(A), y \in \pi_j(B)$ and $i + j = n$. We can compute $Q^0(x \otimes y)$ using the Cartan formula:

$$Q^0(x \otimes y) = \Sigma_{j+k=0} Q^j(x) Q^k(y) = \Sigma_i Q^i(x) Q^{-i}(y)$$

Since $A, B \in \mathrm{CAlg}_k^{\mathrm{Fr}}$ and by lemma 3.3.1.4 we have $Q^i = 0\ \forall i > 0$, so the above expression reduces to $Q^0(x \otimes y) = Q^0(x) \otimes Q^0(y)$. Again due to the perfectness of A, B, we have Q^0 acts via isomorphism on each factor; this proves $A \otimes B$ is perfect. □

Since the inclusion $\mathrm{CAlg}_k^{\mathrm{Fr}} \hookrightarrow \mathrm{CAlg}_k$ preserves colimits, and since both $\mathrm{CAlg}_k^{\mathrm{Fr}}$ and CAlg_k are presentable ∞-category, we can apply the adjoint functor theorem to get the right adjoint to the inclusion.

Definition 3.3.1.6 (Tilting)**.** The right adjoint to the inclusion functor $\mathrm{CAlg}_k^{\mathrm{Fr}} \hookrightarrow \mathrm{CAlg}_k$ is called **tilting** and denoted by

$$(-)^\flat : \mathrm{CAlg}_k \to \mathrm{CAlg}_k^{\mathrm{Fr}}.$$

□

Remark 3.3.1.7. Since the functor $\mathrm{CAlg}_k^{\mathrm{Fr}} \to \mathrm{CAlg}_k$ is fully faithful, the left and right adjoints are Bousfield localisations and co-localisation, respectively.

This means that given $A \in \mathrm{CAlg}_k$, one can universally approximate A by perfect algebras from the left $A^\flat \to A$ and from the right $A \to A^\#$ with the universal property given $B \in \mathrm{CAlg}_k^{\mathrm{Fr}}$ and $A \to B$ a map in CAlg_k, we have up to contractible choice a unique factorisation:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ A^{\#} \arrow[ur, dashed] & \end{tikzcd}$$

Similarly given a map $B \to A$ with $B \in \mathrm{CAlg}_k^{\mathrm{Fr}}$, upto contractible choice we have a unique factorisation:

$$\begin{tikzcd} B \arrow[d, dashed] \arrow[dr] & \\ A^{\flat} \arrow[r] & A \end{tikzcd}$$

□

This will become useful for us to analyse fixed points later in the **section** 3.3.

3.3.2 Homotopy groups of tilt

In this section, we give a more explicit formula for the tilting construction under certain finiteness assumptions on the homotopy groups of A.

The forgetful functor $\mathrm{CAlg}_k \to \mathrm{Mod}_k$ admits a left adjoint [Lur17, Section 3.1];

$$\mathrm{Free} : \mathrm{Mod}_k \to \mathrm{CAlg}_k.$$

More explicitly, given $V \in \mathrm{Mod}_k$ we have the free algebra on V given by:

$$\mathrm{Free}(V) \simeq \bigoplus_{j=0}^{\infty} V^{\otimes j}_{h\Sigma_j}$$

And we call it the **Free** functor. By free algebra on a generator of degree n, we mean the free algebra functor applied to $k[n]$, and we denote it by $k\{x_n\}$.

We start with the following input about the homotopy groups of free algebras[BMM+86]:

Proposition 3.3.2.1. Let $k\{x_n\}$ be the free algebra of degree n, we have $\pi_* k\{x_n\}$ is the free polynomial algebra on the vector space M, where M is generated by basis $Q^I(x_n)$, where I is admissible and excess $e(I) \geqslant n$. □

Remark 3.3.2.2. We can describe the power operation on $\pi_* k\{x_n\}$ as follows,

$$Q^i : \pi_m k\{x_n\} \to \pi_{m+i} k\{x_n\}$$

given by

$$Q^I(x_n) \mapsto Q^i Q^I(x_n)$$

and we apply the Adem relations to $Q^i Q^I(x_n)$ to rewrite in terms of the basis described in the above proposition. □

Since $k\{x_n\}$ is a free algebra, to give a map of commutative algebra from $k\{x_n\}$ to a commutative algebra A is equivalent to giving a map of modules $k[n] \to A$ because of the adjunction we get

$$\mathrm{map}_{\mathrm{CAlg}_k}(k\{x_n\}, A) \to \mathrm{map}_{\mathrm{CAlg}_k}(\mathrm{Free}(k[n]), A) \simeq \mathrm{map}_{\mathrm{Mod}_k}(k[n], A).$$

In particular, a map of commutative algebras $k\{x_n\} \to A$ is equivalent to picking an element $\alpha \in \pi_n(A)$ and we donote it by

$$k\{x_n\} \xrightarrow{x_n \mapsto \alpha} A$$

and refer to it as a map induced by sending x_n to α.

This gives us a map for $n < 0$:

$$k\{x_n\} \xrightarrow{x_n \mapsto Q^0(x_n)} k\{x_n\}$$

which extends to a filtered diagram:

$$k\{x_n\} \xrightarrow{x_n \mapsto Q^0(x_n)} k\{x_n\} \xrightarrow{x_n \mapsto Q^0(x_n)} k\{x_n\} \xrightarrow{x_n \mapsto Q^0(x_n)} k\{x_n\}...$$

we denote

$$C = \mathrm{colim}(k\{x_n\} \xrightarrow{x_n \mapsto Q^0(x_n)} k\{x_n\} \xrightarrow{x_n \mapsto Q^0(x_n)} k\{x_n\} \xrightarrow{x_n \mapsto Q^0(x_n)} k\{x_n\}...)$$

where the colimit is taken in CAlg_k. Now we prove the main theorem of this section which is crucial in computing the homotopy groups of the tilt.

Theorem 3.3.2.3. The colimit C of the filtered diagram above is a perfect algebra.

Proof. We then have forgetful functor $\mathrm{CAlg}_k \to \mathrm{Sp}$, which preserves sifted colimit. In particular, the underlying spectrum can be calculated as a colimit in the category of spectra.

Also, taking homotopy groups commutes with taking filtered colimits. Since perfectness can be tested on homotopy groups,so we proceed to compute the filtered colimit of homotopy groups.

Recall the homotopy groups $\pi_* k\{x_n\}$ from 3.3.2.1 is a polynomial algebra on a vector space M, with basis $Q^I(x_n)$ with I admissible and $e(I) \geqslant n$.

The map $k\{x_n\} \xrightarrow{x_n \mapsto Q^0(x_n)} k\{x_n\}$ on homotopy groups is the unique map induced by sending

$$Q^I(x_n) \mapsto Q^I Q^0(x_n)$$

Which takes M to M. Hence we can compute the filtered colimit on M, and then take polynomial algebra in the resulting vector space:

$$\tilde{M} = \mathrm{colim}(M \xrightarrow{x_n \mapsto Q^0(x_n)} M \xrightarrow{x_n \mapsto Q^0(x_n)} M \xrightarrow{x_n \mapsto Q^0(x_n)} M...)$$

Note that M is a bimodule over the Dyer-Lashof algebra (cf 3.2.1.6) in a natural way. Hence the vector space $\tilde{M}$ is a module over

$$\tilde{\mathfrak{B}}_p = \mathrm{colim}(\mathfrak{B}_p \xrightarrow{Q^0} \mathfrak{B}_p \xrightarrow{Q^0} \mathfrak{B}_p \to ...)$$

where the colimit is taken over multiplication on the right. This colimit can be computed since we checked in 3.2.3.6, the collection S satisfy the right Ore condition.

We can identify $\tilde{\mathfrak{B}}_p \simeq \mathfrak{B}_p[(\mathrm{Sq}^0)^{-1}]$ with $\mathcal{A}_p[(\mathrm{Sq}^0)^{\pm}]$ the steenrod algebra adjoint a formal variable Sq^0 and its inversion, via the map

$$\mathcal{A}_p[(\mathrm{Sq})^{\pm}] \to \mathfrak{B}_p[(\mathrm{Sq}^0)^{-1}]$$

$$\mathrm{Sq}^i \mapsto \mathrm{Sq}^i/\mathrm{Sq}^0$$

for $i > 0$ and

$$\mathrm{Sq}^0 \mapsto \mathrm{Sq}^0$$

So Sq^0 acts invertible on both the left and right side of the bimodule $\tilde{M}$. □

Corollary 3.3.2.4. The algebra C in the above proposition is equivalent to $k\{x_n\}^{\#}$, where $(-)^{\#}$ is left adjoint to the the inclusion $\mathrm{CAlg}_k^{\mathrm{Fr}} \hookrightarrow \mathrm{CAlg}_k$.

Proof. Since the colimits C is perfect, we have

$$C \simeq C^{\#} \simeq [\mathrm{colim}(k\{x_n\} \xrightarrow{x_n \mapsto Q^0(x_n)} k\{x_n\} \xrightarrow{x_n \mapsto Q^0(x_n)} k\{x_n\} \xrightarrow{x_n \mapsto Q^0(x_n)} k\{x_n\}...)]^{\#}$$

using the fact that $(-)^{\#}$ is a left adjoint we have

$$C \simeq \mathrm{colim}(k\{x_n\}^{\#} \xrightarrow{x_n \mapsto Q^0(x_n)} k\{x_n\}^{\#} \xrightarrow{x_n \mapsto Q^0(x_n)} k\{x_n\}^{\#} \xrightarrow{x_n \mapsto Q^0(x_n)} k\{x_n\}^{\#}...)$$

where the transition maps are isomorphisms by definition on perfect algebras. Hence we have $C \simeq k\{x_n\}^{\#}$. □

Proposition 3.3.2.5. (Homotopy groups of $A^\flat$) For a finite type[2] $\mathbb{E}_\infty$-algebra A the homotopy groups of $A^\flat$ is given by $\pi_* A^\flat \simeq \varprojlim_n \pi_* A$ where the sequential inverse limit is taken along the operation Q^0.

Proof. We can use the above results and compute the homotopy groups of the tilt $A^\flat$. A natural thing to do is to map in from the free algebra $k\{x_n\}$ into $A^\flat$. Since the homotopy groups of $\mathrm{map}_{\mathrm{CAlg}}(k\{x_n\}, A^\flat)$ can be computed via the **free-forgetful** adjunction:

$$\mathrm{map}_{\mathrm{CAlg}_k}(k\{x_n\}, A^\flat) \simeq \mathrm{map}_{\mathrm{Mod}_k}(k[n], A^\flat)$$

for $i > 0$, this implies

$$\pi_i(\mathrm{map}_{\mathrm{CAlg}_k}(k\{x_n\}, A^\flat)) \simeq \pi_i \mathrm{map}_{\mathrm{Mod}_k}(k[n], A^\flat) \simeq \pi_0 \mathrm{map}_{\mathrm{Sp}}(k[n+i], A^\flat) \simeq \pi_{n+i} A^\flat.$$

Now since $A^\flat$ is perfect, any map $k\{x_n\} \to A^\flat$ by universal property factors as follows:

[2]We say A is of finite type if $\pi_n A$ finite dimensional all n

$$\begin{tikzcd} k\{x_n\} \arrow[r] \arrow[dr] & k\{x_n\}^{\#} \arrow[d, dashed] \\ & A^{\flat} \end{tikzcd}$$

Hence we have

$$\mathrm{map}_{\mathrm{CAlg}_k}(k\{x_n\}, A^{\flat}) \simeq \mathrm{map}_{\mathrm{CAlg}_k}(k\{x_n\}^{\#}, A^{\flat}).$$

But we also have

$$\mathrm{map}_{\mathrm{CAlg}_k}(k\{x_n\}^{\#}, A^{\flat}) \simeq \mathrm{map}_{\mathrm{CAlg}_k}(k\{x_n\}^{\#}, A),$$

since $(-)^{\flat}$ is a colocalisation.

Now by 3.3.2.4 we can identify $k\{x_n\}^{\#}$ with

$$\mathrm{colim}(k\{x_n\} \xrightarrow{x_n \mapsto Q^0(x_n)} k\{x_n\} \xrightarrow{x_n \mapsto Q^0(x_n)} k\{x_n\} \xrightarrow{x_n \mapsto Q^0(x_n)} k\{x_n\} ...)$$

we have

$$\mathrm{map}_{\mathrm{CAlg}_k}(k\{x_n\}^{\#}, A) \simeq \lim(... \xrightarrow{x_n \mapsto Q^0(x_n)} \mathrm{map}_{\mathrm{CAlg}_k}(k\{x_n\}, A) \xrightarrow{x_n \mapsto Q^0(x_n)} \mathrm{map}_{\mathrm{CAlg}_k}(k\{x_n\}, A))$$

This gives us a formula for homotopy groups of tilt via,

$$\pi_*(\mathrm{map}_{\mathrm{CAlg}_k}(k\{x_n\}, A^{\flat})) \simeq \pi_*(\mathrm{map}_{\mathrm{CAlg}_k}(k\{x_n\}^{\#}, A)) \simeq \pi_*(\varprojlim_i \mathrm{map}_{\mathrm{CAlg}_k}(k\{x_n\}, A))$$

$$\pi_{*+n} A^{\flat} \simeq \pi_*(\varprojlim_i \mathrm{map}_{\mathrm{CAlg}_k}(k\{x_n\}, A))$$

We can compute the homotopy groups of sequential inverse limits $\varprojlim_n X_i$ using Milnor *lim*^{1} exact sequence. But there is no contribution from the *lim*^{1} term due to the finite type condition on A; that is, all the groups are finite; hence, the image in the limit diagram is eventually constant, which is the Mittag-Leffler condition for the vanishing of *lim*^{1}. Hence we obtain,

$$\pi_{i+n} A^{\flat} \simeq \varprojlim_{Q^0} \pi_i(\mathrm{map}_{\mathrm{CAlg}_k}(k\{x_n\}, A))) \simeq \varprojlim_{Q^0} \pi_{i+n} A$$

This proves the claim. □

Remark 3.3.2.6. Note that for $n < 0$, $\mathrm{map}_{\mathrm{CAlg}_k}(k\{x_n\}, A^{\flat}))$ sees only the homotopy groups $\pi_i A^{\flat}$ for $i > n$ since the mapping space has no negative homotopy groups. This is taken care of by varying n. □

3.4 Fixed points and tilting

Let G be an elementary abelian group, X a finite G-CW complex, we use the formalism of tilting to recover the p-local homotopy type of X^G from the Borel equivariant cohomology of X.

3.4.1 Proper Tate construction

Let k be a field of $\mathrm{char}(k) = p$, $G = (\mathbb{Z}/p)^n$, we have

$$\pi_* C^*(BG; k) = \pi_* k^{hG} = \begin{cases} k[[x_1, x_2, ..., x_n]], & \text{for p } =2 \\ k[[x_1, x_2, ..., x_n]] \otimes \Lambda(\epsilon_1, \epsilon_2, ..., \epsilon_n), & \text{otherwise} \end{cases}$$

where $| x_i |= -1$ for $p = 2$ and $| x_i |= -2$, $| \epsilon_i |= -1$ for $p > 2$.

Let $I \subset \{1, 2, ..., n\}$ and denote $x_I = \sum_{i \in I} x_i$, we define the **Euler class** $e \in \pi_* k^{hG}$ as discussed in 2.1.3 as a product:

$$e = \prod_{I \subset \{1,2,...,n\}} x_I$$

where the product is over I non-empty subsets of $\{1, 2, ..., n\}$.

Theorem 3.4.1.1 (Localisation theorem)**.** Let X be a finite G-CW complex, we have the fibration which records the G-action:

$$\begin{tikzcd} X \arrow[r] & X_{hG} \arrow[d, "a"] \\ & BG \end{tikzcd}$$

to get an induced map $a^* : C^*(BG; k) \to C^*(X_{hG}; k)$, we then have

$$C^*(X_{hG}; k)[e^{-1}] \simeq C^*(X^G \times BG; k)[e^{-1}]$$

we X^G is the genuine fixed point. □

Remark 3.4.1.2. The above theorem, as pointed out in 2.1.3.3, can be stated as

$$C^*(X; k)^{\tau G} \simeq C^*(X^G; k)^{\tau G}$$

where $(-)^{\tau G}$ is the proper Tate construction (cf. 2.1.2.7), where the target is equipped with trivial action. It is also important to note that the above isomorphism is an isomorphism of $\mathbb{E}_\infty$-algebras. □

Question 3.4.1.3. Let A be the $\mathbb{E}_\infty$-algebra $C^*(X; k)^{\tau G}$, what is $A^\flat$? □

Theorem 3.4.1.4. For G an elementary abelian group and X, a finite G-CW complex. The tilt of the algebra $C^*(X; k)^{\tau G}$ is equivalent to $C^*(X^G \times BG; k)$.

Proof. We have $(C^*(X; k)^{\tau G})^\flat \simeq (C^*(X^G; k)^{\tau G})^\flat$ by the localisation theorem. The proper Tate construction $(-)^{\tau G}$ is an exact functor hence

$$(C^*(X^G; k)^{\tau G})^\flat \simeq ((k^{\tau G})^{X^G})^\flat.$$

By construction, $(-)^\flat$ is a right adjoint to the inclusion $\mathrm{CAlg}_k^{\mathrm{Fr}} \hookrightarrow \mathrm{CAlg}_k$ so tilting commutes with taking limits. This gives:

$$(C^*(X^G; k)^{\tau G})^\flat \simeq ((k^{\tau G})^\flat)^{X^G}$$

So it is enough to understand what $(k^{\tau G})^\flat$ is with respect to the trivial G-action on k. The homotopy groups of $k^{\tau G}$ are finite dimensional in each degree. Hence to calculate the homotopy groups of the tilt $(k^{\tau G})^\flat$, we can use **Proposition** 3.3.2.5.

We have $\pi_*(k^{\tau G})^\flat \simeq \varprojlim_i \pi_*(k^{\tau G})$, where the inverse limit is taken with respect to the operation Q^0, so we try to understand how Q^0 acts on homotopy groups of $k^{\tau G}$.

We have the map of $\mathbb{E}_\infty$-algebras

$$k^{hG} \to k^{\tau G}$$

given by localisation at the Euler class described above. Since the Tate valued Frobenius is natural with respect to the map of $\mathbb{E}_\infty$-algebra (cf 3.2.2.6), we get the total power operation:

$$Q(t)(x_i) = x_i + x_i^2 t^{-1}$$

Similarly, for all classes that come from k^{hG}. So the only classes remaining are the ones that arise from localisation, so we can ask what is $Q(t)(\frac{1}{x_i})$?. We can compute it using the Cartan formula (3.2.2.7) as follows:

$$Q(t)(x_i)Q(t)(\frac{1}{x_i}) = Q(t)(1) = 1$$

$$\begin{aligned}
Q(t)(\frac{1}{x_i}) &= \frac{1}{Q(t)(x_i)} \\
&= \frac{1}{x_i + x_i^2 t^{-1}} \\
&= \frac{t}{x_i^2} . \frac{1}{(1 + \frac{t}{x_i})} \\
&= \frac{t}{x_i^2} .(1 + \frac{t}{x_i} + \frac{t^2}{x_i^2} + ...) \\
&= \frac{t}{x_i^2} + \frac{t^2}{x_i^3} + \frac{t^3}{x_i^4} + ...)
\end{aligned}$$

this implies $Q^0(\frac{1}{x_i}) = 0$. Now by using Cartan formula we get $Q^0(\frac{1}{x_i^n}) = 0$ for all n and the same calculation goes through for $Q^0(\frac{1}{x_I}) = 0$.

Since k^{hG} is a perfect algebra, we have a factorisation:

$$\begin{tikzcd}
k^{hG} \arrow[r] \arrow[d, dashed] & k^{\tau G} \\
(k^{\tau G})^\flat \arrow[ur] &
\end{tikzcd}$$

The above computation shows that $k^{hG} \to (k^{\tau G})^\flat$ is an equivalence. This proves the claim $C^*(X;k)^{\tau G}$ is equivalent to $C^*(X^G \times BG; k)$.

□

Corollary 3.4.1.5. Given G an elementary abelian group and X a finite G-CW complex then we have isomorphism of $\mathbb{E}_\infty$-algebras:

$$(C^*(X;k)^{\tau G})^\flat \otimes_{k^{hG}} k \simeq C^*(X^G;k).$$

□

3.4.2 p-adic homotopy theory

In this section, we recall some results from p-adic homotopy theory from [Lur11]. To recover the homotopy type of the genuine fixed points from Borel cohomology using results from the previous section.

Definition 3.4.2.1. [Lur11, Definition 2.4.1] Let p be a prime number.Let $X \in \mathcal{S}$ is a **p-finite space** if the following condition holds:

1. X is n-truncated for some n.
2. $\pi_0(X)$ is a finite set.
3. For all $x \in X$, the homotopy groups $\pi_k(X, x)$ is finite p-group for all $k \geqslant 1$.

We denote by $\mathcal{S}^{\mathrm{p-fin}} \subset \mathcal{S}$ the full spanned by p-finite spaces.

Definition 3.4.2.2. [Lur11, Definition 3.1.1] For $\mathcal{C}$ an accessible ∞-category with all finite limits, we can definite the ∞-category of **pro-objects** $\mathrm{Pro}(\mathcal{C})$ as the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S})^{op}$ spanned by functors which are accessible and preserve finite limits.

For $\mathcal{C}$ as above, there is a fully faithful embedding $j : \mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ [Lur11, Remark 3.1.4], which sends an object $X \in \mathcal{C}$ to be the presheaf co-represented by X.

The category of pro-objects $\mathrm{Pro}(\mathcal{C})$ is obtained by formally adding co-filtered limits to $\mathcal{C}$, and satisfies the following universal property:

Proposition 3.4.2.3. [Lur11, Proposition 3.1.6] For $\mathcal{C}$, an accessible category with all finite limits and $\mathcal{D}$ category with all co-filtered limits. Then there is an equivalence

$$\mathrm{Fun}^{\mathrm{fil}}(\mathrm{Pro}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

where the category on the right is the full subcategory of $\mathrm{Fun}(\mathrm{Pro}(\mathcal{C}), \mathcal{D})$, which preserves cofiltered limits.

Definition 3.4.2.4. [Lur11, Definiiotn 3.1.12] We define the category of p-profinite spaces $\mathrm{Pro}(\mathcal{S}^{\mathrm{p-fin}})$ as the pro-category of p-finite spaces. We denote it by $\mathcal{S}^{\mathrm{Pro}(p)}$.

Let k be a field of characteristic p. The functor $C^*(-;k) : \mathcal{S}^{\mathrm{p-fin}} \to \mathrm{CAlg}_k^{op}$ preserves finite limits. To see this, it is enough to check the functor $C^*(-;k)$ preserves finite products and pullbacks.

By Kunneth theorem $C^*(-;k)$ takes product $X \times Y$ to $C^*(X;k) \otimes_k C^*(Y;k)$ which is the product in CAlg_k^{op}. Now let us consider the pullback of p-finite spaces:

$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Since the spaces in the pullback are p-finite, we have $\pi_0 Y$ is finite, and the $\pi_1(Y, y)$ is a finite p-group for all $y \in Y$, so the action of $\pi_1(Y, y)$ on cohomology of the fiber of $X \to Y$ is nilpotent. Hence we have the Eilenberg-Moore theorem [Lur11, Corollary 1.1.10], which says the induced map

$$\theta : C^*(Y^{'}; k) \otimes_{C^*(Y;k)} C^*(X; k) \to C^*(X^{'}; k)$$

is an equivalence of $\mathbb{E}_\infty$-algebras.

The functor $C^*(-; k) : \mathcal{S}^{\mathrm{p-fin}} \to \mathrm{CAlg}_k^{op}$ extend to a functor $C^*(-; k) : \mathcal{S}^{\mathrm{Pro}(p)} \to \mathrm{CAlg}_k^{op}$ which preserves all small limits and this extension is essentially unique by [Lur09, Proposition 5.5.1.9].

Theorem 3.4.2.5. [Lur11, Proposition 3.1.16][Mandell, Lurie] Let k be a separably closed field of characteristic p. Then the functor $C^*(-; k) : \mathcal{S}^{\mathrm{Pro}(p)} \to \mathrm{CAlg}_k^{op}$ is a fully faithful embedding.

Theorem 3.4.2.6. [Lur11, Theorem 3.5.8] Let k be an algebraically closed field, then the functor $C^*(-; k) : \mathcal{S}^{\mathrm{Pro}(p)} \to \mathrm{CAlg}_k^{op}$ has the essential image identified with the full subcategory $\mathrm{CAlg}_k^{\mathrm{Fr}}$ of perfect algebras.

So the perfection functors $(-)^{\#}$ and $(-)^{\flat}$ constructed in section 3.3, universally approximates an $\mathbb{E}_\infty$-algebra A by cochains of profintie spaces $A^{\#}$ and $A^{\flat}$.

Let $\mathcal{S}^{\mathrm{nil,fin-type}}$ be the category p-complete nilpotent spaces of finite type which is a full subcategory of $\mathcal{S}^{\mathrm{Pro}(p)}$. So we have Mandell's theorem:

Theorem 3.4.2.7. [Man01] Let k be an algebraically closed field of characteristic p. $C^*(-; k) : \mathcal{S}^{\mathrm{nil,fin-type}} \hookrightarrow \mathrm{CAlg}_k^{op}$ is fully faithful and given $X \in \mathcal{S}^{\mathrm{nil,fin-type}}$ we can recover the homotopy type of X by mapping into k. That is,

$$\mathrm{map}_{\mathrm{CAlg}_k}(C^*(X; k), k) \simeq X.$$

Recall from Corollary 3.4.1.5, given G an elementary abelian group, k an algebraic closed field of characteristic p and X a finite G-CW complex we have isomorphism of $\mathbb{E}_\infty$-algebras:

$$(C^*(X; k)^{\tau G})^{\flat} \otimes_{k^{hG}} k \simeq C^*(X^G; k)$$

Combining this with Mandell's theorem, we have:

Corollary 3.4.2.8. Let G be an elementary abelian group and X a finite G-CW complex such that X^G is nilpotent, then the p-local homotopy type of X^G is equivalent to $\mathrm{map}_{\mathrm{CAlg}_{\bar{\mathbb{F}}_p}}(A, k)$, where $A \simeq (C^*(X; k)^{\tau G})^{\flat} \otimes_{k^{hG}} k$

Remark 3.4.2.9. In principle, we can drop the assumption on X^G being nilpotent and always recover the homotopy type of X^G as a profinite space using Mandell's theorem.

4 A remark on Segal conjecture

The material presented in this chapter is a joint work with Robert Burklund. The version appearing in the thesis is written by the author.

4.1 Introduction

The Sullivan conjecture asserts that for G a p-group and X a finite G-CW complex, then the natural comparison map between the genuine fixed points and the homotopy fixed points

$$X^G \to X^{hG}$$

is an equivalence after p-completion.

The stable analogue of this question is the Segal conjecture, which asks if, for a p-group G, the natural comparison

$$(\mathbb{S}_G)^G \to (\mathbb{S}_G)^{hG}$$

an equivalence after p-completion. Where $\mathbb{S}_G$ is the genuine G-sphere, which is the unit in the category of genuine G-spectra, $(\mathbb{S}_G)^G$ denote the genuine G-fixed points and $(\mathbb{S}_G)^{hG}$ denote the homotopy fixed points.

The conjecture was answered to be true by a sequence of progress, for $G = \mathbb{Z}/2$ by W. H. Lin [Lin80], for $G = \mathbb{Z}/p$ for odd primes by J. H. Gunawardena [Gun80], for elementary abelian groups by J. F. Adams, J. H. Gunawardena, H. Miller [Ada74] and G. Carlsson [Car83] and finally for all groups by G.Carlsson [Car84].

In the case of $G = \mathbb{Z}/p$, the Segal conjecture is equivalent to asking if the canonical map $\mathbb{S} \to \mathbb{S}^{t\mathbb{Z}/p}$ from the sphere spectrum to the Tate construction with respect to trivial $\mathbb{Z}/p$ action exhibits $\mathbb{S}^{t\mathbb{Z}/p}$ as the p-completion of $\mathbb{S}$. This is answered to be true by W. H. Lin [Lin80] and J. H. Gunawardena [Gun80].

Since both p-completion and Tate construction are exact functors, the result extends to all finite spectra. That is, for any finite spectrum X with trivial $\mathbb{Z}/p$-action, the map $X \to X^{t\mathbb{Z}/p}$ is an equivalence after p-completion.

In this chapter, we investigate for which non-finite spectra we can extend the Segal conjecture for $\mathbb{Z}/p$. We introduce the notion of I-nilpotent spectra:

Definition 4.1.0.1 (Definition 4.3.1.9)**.** Let I be the augmentation ideal of the Steenrod algebra $\epsilon : \mathcal{A}_p \to \mathbb{F}_p$. We say a spectrum X is I-nilpotent if we have $H_*(X) \simeq \varprojlim_n H_*(X)/I^n$.

this helps us state the Segal conjecture for non-finite spectra. In particular, we have:

Theorem 4.1.0.2 (Theorem 4.3.1.10)**.** Let X be a bounded below spectrum, p-complete and I-nilpotent, then X satisfies the Segal conjecture for $\mathbb{Z}/p$. That is, $X \to X^{t\mathbb{Z}/p}$ is an equivalence, where the Tate construction is with respect to trivial action.

We also discuss examples of non-finite spectra in this chapter. One particular class of example, if X is a space with locally finite cohomology (see Definition 4.3.1.11) and is of finite type, then the suspension spectrum $\Sigma^\infty_+ X$ satisfy the Segal conjecture.

This result is very similar to the result of H. Miller [Mil84] in the unstable world, which asserts that for a locally finite space with trivial action of a p-group G, the map $X \to \mathrm{map}(BG, X) \simeq X^{hG}$ is an equivalence after p-completion.

4.2 Preliminaries

In this section, we recall some important objects and constructions that feature in the proof of the Segal conjecture and state the Segal conjecture for $G = \mathbb{Z}/p$ in terms of the Adams spectral sequence.

4.2.1 Segal conjecture for $\mathbb{Z}/p$

Before starting the discussion on Segal conjecture, we recall the completion theorem of Athiyah and Segal. Let KU_G be the genuine equivariant complex K-theory as defined by [Seg68].

Given a G-space X we can consider two types of cohomology, the equivariant K-theory KU_G^X and the Borel equivariant K-theory $KU^{X_{hG}}$ and a comparison map between them

$$KU_G^*(X) \to KU^*(X_{hG}).$$

In the case of $X = *$ we have $\pi_0 KU_G(*)$ is the representation ring $R(G)$ of G. the Atiyah-Segal completion theorem [Ati61a; AS69] says

$$R(G)_I^\wedge \cong \pi_0 KU^{BG}$$

is an equivalence, where $R(G)_I^\wedge$ is the completion with respect to the augmentation ideal I.

Motivated by this, G. Segal replaced the genuine K-theory by the genuine equivariant sphere $\mathbb{S}_G$ and conjectured is the natural comparison map:

$$A(G)_I^\wedge \cong \pi_0 \mathbb{S}^{BG_+}$$

is an equivalence for any finite group G, where $A(G)$ is the Burnside ring which can be identified with $\pi_0^G(\mathbb{S}_G)$ and I the augmentation ideal.

The conjecture was answered to be true by a sequence of progress, for $G = \mathbb{Z}/2$ by W. H. Lin [Lin80], for $G = \mathbb{Z}/p$ for odd primes by J. H. Gunawardena [Gun80], for elementary abelian groups by J. F. Adams, J. H. Gunawardena, H. Miller [Ada74] and G. Carlsson [Car83] and finally for all groups by G.Carlsson [Car84].

There is a spectrum-level statement of the above statement. Let G be a finite p-group. Sp_G be the category of genuine equivariant spectra, with unit object $\mathbb{S}_G$ the G-equivariant sphere. Then the Segal conjecture can be stated as the comparison map:

$$(\mathbb{S}_G)^G \to (\mathbb{S}_G)^{hG}$$

is an equivalence after p-completion.

Let $G = \mathbb{Z}/p$, for any $X \in \mathrm{Sp}_G$ we have the Tate pullback square:

$$\begin{tikzcd} X^{\mathbb{Z}/p} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Phi^{\mathbb{Z}/p}(X) \arrow[d] \\ X^{h\mathbb{Z}/p} \arrow[r] & X^{t\mathbb{Z}/p} \end{tikzcd}$$

This implies to get the equivalence

$$(\mathbb{S}_G)^G \to (\mathbb{S}_G)^{hG}$$

if and only if

$$\Phi^{\mathbb{Z}/p}(\mathbb{S}_G) \simeq \mathbb{S} \to \mathbb{S}^{t\mathbb{Z}/p}$$

is an equivalence after p-completion.

Theorem 4.2.1.1. [Lin80; Gun80] Let $G = \mathbb{Z}/p$, then $\mathbb{S}_p^\wedge \simeq \mathbb{S}^{t\mathbb{Z}/p}$. □

We have the inflation functor $i : \mathrm{Sp} \to \mathrm{Sp}_G$, which is a symmetric monoidal functor and a left adjoint. Using this, any spectrum X can be seen as genuine G-spectrum whose underlying Borel spectrum X with a trivial action of G.

Since i is symmetric monoidal we have $i(\mathbb{S}) = \mathbb{S}_G$ and since i is also a left adjoint for any $X \in \mathrm{Sp}$, $i(X) = i(\mathbb{S}) \otimes X$[1].

We can then ask the following question:

Question 4.2.1.2. For a spectrum X and $G = \mathbb{Z}/p$, is the map $i(X)^G \to i(X)^{hG}$ is an equivalence after p-completion ? or equivalently, is the map $X \to X^{t\mathbb{Z}/p}$ is an equivalence after p-completion, where the underlying action on the spectrum is trivial?. □

Remark 4.2.1.3. By Theorem 4.2.1.1 and observing both p-completion and Tate construction are exact, we can say the answer to 4.2.1.2 is true if X is a finite spectrum. □

In this note, we extend this result beyond finite spectra and give some examples of non-finite spectra that satisfy the version of the Segal conjecture mentioned above.

Remark 4.2.1.4. There are also many non-examples to the question 4.2.1.2, for example, set $X = KU_p^\wedge$, then the Tate construction $X^{t\mathbb{Z}/p}$ with respect to trivial action is a rational algebra but $KU_p^\wedge$ is not rational. □

We recall the following formula of Tate construction with respect to trivial $\mathbb{Z}/p$-action (cf. [Lin80; Sad92]).

Construction 4.2.1.5 (Stunted projective space)**.** [Ati61b] Let $G = \mathbb{Z}/2$ and σ be the sign representation on $\mathbb{R}$, we can then form the representation sphere $\mathbb{S}^{k\sigma}$ for all $k \in \mathbb{Z}$. We define

$$\mathbb{R}P_k^\infty = (\mathbb{S}^{k\sigma})_{hC_2}$$

for $k > 0$, $\mathbb{R}P_k^\infty = \Sigma^\infty \mathrm{cofib}(\mathbb{R}P_{k-1} \to \mathbb{R}P^\infty)$. In general, for all $k \in \mathbb{Z}$, $\mathbb{R}P_k^\infty$ has one cell on each degree greater than equal to k.

[1] Which makes sense, since Sp_G is a stable, presentably symmetric monoidal ∞-category, so we can tensor with respect any object $X \in \mathrm{Sp}$

For $p > 2$, we replace σ with λ, where λ is the representation whose underlying vector space is the complex plane and $\mathbb{Z}/p$ acts via p-th roots of unity. Then we define

$$P_k = (\mathbb{S}^{k\lambda})_{hC_p}$$

which is a version of stunted lens space. □

Remark 4.2.1.6. I. M. James [Jam59] first introduced the definition of stunted projective space via James periodicity; the above description was revisited in [Ati61b]. □

Proposition 4.2.1.7. [Lin80] In this form, we can express the Tate construction for the trivial action:

$$X^{t\mathbb{Z}/2} \simeq \varprojlim_k (X \otimes \Sigma \mathbb{R}P_k^\infty)$$

for odd primes,

$$X^{t\mathbb{Z}/p} \simeq \varprojlim_k (X \otimes \Sigma P_k)$$

□

4.2.2 Adams spectral Sequence

Let us recall some definitions ([Pst22, Definition 3.13, 3.14]), fix E a homotopy associative ring,

Definition 4.2.2.1 (finite E-projective)**.** [Pst22, Definition 3.13] A spectrum X is said to be finite E-projective if X is a finite, E_*X is finitely generated and projective over the graded ring E_*. □

Definition 4.2.2.2 (Adams-type spectrum)**.** [Pst22, Definition 3.14] A homotopy associative ring E is said to be of Adams-type, if E is written as a filtered colimit $E \simeq \varinjlim E_\alpha$, where each E_α is a finite E-projective and we have the E-cohomology of E_α is dual to its homology, i.e,

$$E^*E_\alpha \simeq \mathrm{Hom}_{E_*}(E_*E_\alpha, E_*)$$

□

Example 4.2.2.3. Examples of Adams-type spectrum include MU the complex cobordism spectrum, KU_p the p-adic complex K-theory, any Landweber exact cohomology theory (which consists of the previous example), $\mathbb{F}_p$ the Eilenberg Maclane spectrum. □

Let E be an Adams-type spectrum. We can associate the ∞-category of synthetic spectra in the sense of [Pst22].

Before spelling out what is the category of synthetic spectra, we start with some motivations,

For E an Adams-type spectrum, Y finite E-projective and X any spectrum J.F. Adams [Ada74] constructed a spectral sequence

$$E^2_{s,t} \simeq \mathrm{Ext}^{s,t}_{E_*E}(E_*Y, E_*X) \implies \pi_{t-s}(\mathrm{map}(Y, X^\wedge_E))$$

where the Ext groups are calculated in the category of comodules over E_*E and $X^\wedge_E$ denote the E-completion of the spectrum X.

The Adams spectral sequence plays a crucial role in the computation of stable homotopy groups of spheres by setting $E = \mathbb{F}_p, Y = \mathbb{S}, X = \mathbb{S}$ to get

$$E^2_{s,t} \simeq \mathrm{Ext}_{\mathbb{F}_{p*}\mathbb{F}_p}(\mathbb{F}_{p*}X, \mathbb{F}_{p*}Y) \implies \pi_*(\mathbb{S}^\wedge_p)$$

where $\mathbb{F}_{p*}\mathbb{F}_p$ is identified with $\mathcal{A}^\vee_p$ is the mod-p dual Steenrod algebra.

Remark 4.2.2.4. Given R an $\mathbb{E}_1$-ring, we can form the Amitsur complex or the cobar construction:

$$\mathbb{S} \longrightarrow R \rightleftarrows R \otimes R \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; R \otimes R \otimes R \cdots .$$

The Adams spectral sequence is the associated descent spectral sequence which converges to the R-completion of the sphere spectrum. To identify the E_2 page above, we must impose certain flatness conditions on R. For example, R is an Adams-type spectrum. □

Remark 4.2.2.5. Let $f : X \to Y$ be a map of spectra if we want to prove that f is a p-adic equivalence (i.e., completion with respect to $\mathbb{S}/p$), a useful strategy is to prove the E_2 page of the Adams spectral sequence is an isomorphism to get an isomorphism of p-complete homotopy groups of X and Y. □

Remark 4.2.2.6. So a reformulation of the question 4.2.1.2 is: **is the limit in 4.2.1.7 for a spectrum X is equivalent to its p-completion?** So in the spirit of 4.2.2.5, Lin [Lin80] establishes an isomorphism between Adams spectral sequence for X and $X^{t\mathbb{Z}/p}$ for a finite spectrum X. We wish to follow the same calculation and extend this result for a larger class of spectra. □

4.2.3 Synthetic Spectra

To carry out our computations with the Adams spectral sequence, we use synthetic spectra introduced by P. Pstragowski [Pst22], where a synthetic spectrum is a categorification of the Adams spectral sequence.

Let E be an Adams-type spectrum[2], to it we can associate the category Syn_E of E-based synthetic spectra. We refer the reader to [Pst22] for its construction, and we recall some important properties to prove our results.

Proposition 4.2.3.1. [Pst22, Proposition 4.2] The category of E-based synthetic spectra Syn_E is a stable, presentably symmetric monoidal ∞-category. □

Lemma 4.2.3.2 (Synthetic analogue)**.** [Pst22, Lemma 4.4] There is a functor

$$\nu_E : Sp \to \mathrm{Syn}_E$$

is canonically lax-symmetric monoidal and preserves filtered colimits. □

[2] refer to Definition 4.2.2.2

Lemma 4.2.3.3 (Symmetric monoidality)**.** [Pst22, Lemma 4.24] Let X be a spectrum such that its filtered colimit of finite E-projective spectra, then the comparison map

$$\nu(X) \otimes \nu(Y) \to \nu(X \otimes Y)$$

is an equivalence □

Corollary 4.2.3.4. In particular for $E = \mathbb{F}_p$ any finite spectrum is finite $\mathbb{F}_p$-projective. Therefore any spectrum can be written as a filtered colimit of finite $\mathbb{F}_p$-projective spectrum; hence $\nu_{\mathbb{F}_p}$ is symmetric monoidal. □

In this note, we will only work with $E = \mathbb{F}_p$; hence we can assume ν is symmetric monoidal.

Definition 4.2.3.5 (Bigraded Sphere)**.** [Pst22, Definition 4.6] We define the bigraded sphere:

$$\mathbb{S}^{t,t} := \nu(\mathbb{S}^t)$$
$$\mathbb{S}^{s,t} := \Sigma^{s-t}\nu(\mathbb{S}^t) \simeq \Sigma^{s-t}\mathbb{S}^{t,t}$$

□

In general, the synthetic analogue functor $\nu : \mathrm{Sp} \to \mathrm{Syn}_E$ does not preserve all colimits. By lemma 4.2.3.2, we have that ν preserves all filtered colimits. Here is a criterion for when ν preserves fiber sequences.

Lemma 4.2.3.6. [Pst22, Lemma 4.23] Given a fiber sequence $X \to Y \to Z$ in Sp, then $\nu X \to \nu Y \to \nu Z$ is a fiber sequence if and only if we have a short exact sequence of E_*E comodules $0 \to E_*(X) \to E_*(Y) \to E_*(Z) \to 0$.

Definition 4.2.3.7 (The map τ)**.** [Pst22] In particular, ν does not preserve suspensions. There is a natural comparison map:

$$\Sigma\nu(X) \to \nu(\Sigma X)$$

setting X $= \mathbb{S}^{-1}$ we get a map $\tau : \mathbb{S}^{0,-1} \to \mathbb{S}^{0,0}$ and for any synthetic spectrum X we get a map $\tau \otimes X : \Sigma^{0,-1}X \to \Sigma^{0,0}X = X$.

Definition 4.2.3.8 (τ-complete and τ-invertible)**.** [Pst22; BHS22] Let $C\tau$ denote the cofiber of $(\tau : \mathbb{S}^{0,-1} \to \mathbb{S}^{0,0})$ and we refer to $C\tau$ as the cofiber of τ. We say a synthetic spectrum X is τ-invertible if the map $\tau \otimes X : \Sigma^{0.-1}X \to X$ is an equivalence and τ-complete if the map $X \to \varprojlim_n X \otimes C\tau^n$ is an equivalence. This is same as being complete with respect to the dualisable algebra $C\tau$ in the sense of 2.1.1.

We denote $\mathrm{Syn}_E[\tau^{-1}]$ and $\mathrm{Syn}_E^{\tau-\mathrm{cpl}}$ for the full subcategories of τ-invertible and τ-complete synthetic spectra respectively. □

Proposition 4.2.3.9. [Pst22, Proposition 4.33] The inclusion of τ-invertible synthetic spectra $\tau : \mathrm{Syn}_E[\tau^{-1}] \hookrightarrow \mathrm{Syn}_E$ admits a left adjoint $(-)[\tau^{-1}] : \mathrm{Syn}_E \to \mathrm{Syn}_E[\tau^{-1}]$. Given a synthetic spectrum X then the τ-inversion is given by

$$X[\tau^{-1}] = \varinjlim(X \xrightarrow{\tau} \Sigma^{0,1}X \xrightarrow{\tau} \Sigma^{0,2}X \xrightarrow{\tau} ...)$$

the colimit over τ. In particular, the localisation is smashing with $\mathbb{S}^{0,0}[\tau^{-1}]$. □

Theorem 4.2.3.10. [Pst22, Theorem 4.7 and Proposition 4.40] The composition of functors $\mathrm{Sp} \xrightarrow{\nu} \mathrm{Syn}_E \xrightarrow{(-)[\tau^{-1}]} \mathrm{Syn}_E[\tau^{-1}]$ is an equivalence and identifies $\mathrm{Syn}_E[\tau^{-1}]$ with the category of spectra Sp. The synthetic analogue functor $\nu : \mathrm{Sp} \to \mathrm{Syn}_E$ is fully faithful. For any $X \in \mathrm{Sp}$, then

$$\nu(X)[\tau^{-1}] \simeq X.$$

Proposition 4.2.3.11. [Pst22, Corollary 4.30] The cofiber of τ, $C\tau$ has a unique $\mathbb{E}_\infty$-algebra structure such that $\mathbb{S}^{0,0} \to C\tau$ is map of $\mathbb{E}_\infty$-algebras.

Theorem 4.2.3.12. [Pst22, Theorem 4.46] For E a homotopy commutative ring spectrum, then there is a fully faithful inclusion $\mathrm{Mod}_{C\tau} \hookrightarrow \mathrm{Stable}_{E_*E}$ from the category of $C\tau$-modules to the category is stable ∞-category of E_*E-comodules (in the sense of Hovey[3]). The functor $\nu(-) \otimes C\tau : \mathrm{Sp} \to \mathrm{Mod}_{C\tau}$ is identified with the functor $E_*(-)$ such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Mod}_{C\tau} \arrow[r, hook] & \mathit{Stable}_{E_*E} \\
\mathrm{Sp} \arrow[u, "\nu(-)\otimes C\tau"] \arrow[ur, "E_*(-)"'] &
\end{tikzcd}$$

Following [Pst22], we think of Syn_E as a "one-parameter deformation of the category of spectra" with τ, the deformation parameter. Setting $\tau = 0$, that is tensoring with $C\tau$ lands in $\mathrm{Mod}_{C\tau}$ and inverting τ lands in Sp, we have the following diagram summarising the above discussion:

$$\begin{tikzcd}
 & \mathrm{Sp} \arrow[d, "\nu"] \arrow[ddl, bend right, "id"'] \arrow[ddr, bend left, "E_*(-)"] & \\
 & \mathit{Syn}_E \arrow[dl, "\tau^{-1}"'] \arrow[dr, "C\tau\otimes -"] & \\
\mathrm{Sp} & & \mathrm{Mod}_{C\tau}
\end{tikzcd}$$

From now on, set $E = \mathbb{F}_p$, then the bigraded sphere $\mathbb{S}^{a,b}$ form a system of compact generators of $\mathrm{Syn}_{\mathbb{F}_p}$, and define the synthetic homotopy groups of $X \in \mathrm{Syn}_{\mathbb{F}_p}$ to be the homotopy class of maps $[\mathbb{S}^{a,b}, X] = \pi_0(\mathrm{map}_{\mathrm{Syn}_{\mathbb{F}_p}}(\mathbb{S}^{a,b}, X))$ [Pst22, Definition 4.9]. Then a map of synthetic spectra $f : X \to Y$ is an equivalence if and only if it induces an isomorphism of bigraded homotopy groups.

For $X \in \mathrm{Sp}$, νX is a categorification of the $\mathbb{F}_p$ based Adam spectral sequence, by the following result [Pst22, Lemma 4.56]. For $X, Y \in \mathrm{Sp}$ then

$$\pi_{t-s,t}(\mathrm{Hom}_{\mathrm{Syn}_{\mathbb{F}_p}}(\nu X, \nu Y \otimes C\tau)) \simeq \mathrm{Ext}^{s,t}_{\mathbb{F}_{p*}\mathbb{F}_p}(H_*(X), H_*(Y))$$

which is the E_2-page of the $\mathbb{F}_p$-based Adams spectral sequence converging to the homotopy groups of $\mathrm{map}_{Sp}(X, Y_p^\wedge)$ and by definition there is an action of τ on the bi-graded homotopy groups, which encodes information about differentials in Adams spectral sequence for more details refer to [BHS22, Section 9].

[3]refer to [Pst22, Section 3.2]

4.3 Segal Conjecture for I-nilpotent spectra

4.3.1 Proof of the main theorem

Definition 4.3.1.1. [BHS22, Definition 9.16] A spectrum X is said to be E-nilpotent complete if E-Adams resolution of X converges to X.

Example 4.3.1.2. Let X be p-complete, bounded below spectrum, then X is $\mathbb{F}_p$-nilpotent [Bou79].

Proposition 4.3.1.3. [BHS22, Proposition A.11] Given $X \in \mathrm{Sp}$, then the following are equivalent:

1. X is E-nilpotent
2. νX is τ-complete

In particular, νX is τ-complete if X is bounded below p-complete spectrum. □

Remark 4.3.1.4. Given any two τ-complete objects $X, Y \in \mathrm{Syn}_{\mathbb{F}_p}$ and a map $f : X \to Y$ such that $f \otimes C\tau$ is an equivalence then f is an equivalence. In particular,

$$f[\tau^{-1}] : X[\tau^{-1}] \to Y[\tau^{-1}]$$

is an equivalence.

Let us come back to the Segal conjecture for $\mathbb{Z}/p$, which asks, given a spectrum X, when is the canonical map $X \to X^{t\mathbb{Z}/p}$ an equivalence, where the Tate construction is with respect to the trivial action. We can transport the problem into the category of $\mathbb{F}_p$-based synthetic spectra and try to answer it in $\mathrm{Syn}_{\mathbb{F}_p}$.

Remark 4.3.1.5. We will carry out the discussion from now on for prime $p = 2$. The results for odd primes follow by replacing $\mathbb{R}P^{\infty}_k$ with P_k defined in Construction 4.2.1.5.

Let us assume X is bounded below spectra, p-complete. Then $\nu(X \otimes \Sigma\mathbb{R}P^{\infty}_k)$ is a τ-complete spectrum for all $k \in \mathbb{Z}$. Where $\mathbb{R}P^{\infty}_k$ is the stunted projective spectrum as constructed in 4.2.1.5.

Since the category τ-complete synthetic spectra spectrum is closed under limits, we have $Y = \varprojlim_k \nu(X \otimes \Sigma\mathbb{R}P^{\infty}_k)$ is also τ-complete. After inverting τ we get $Y[\tau^{-1}] \simeq \varprojlim_k X \otimes \Sigma\mathbb{R}P^{\infty}_k \simeq X^{t\mathbb{Z}/p}$.

So, in particular, we have a map $X \to \varprojlim_k(X \otimes \Sigma\mathbb{R}P^{\infty}_k) \simeq X^{t\mathbb{Z}/p}$ after applying ν we get map

$$\nu X \to \nu(\varprojlim_k X \otimes \Sigma\mathbb{R}P^{\infty}_k) \to \varprojlim_k \nu(X \otimes \Sigma\mathbb{R}P^{\infty}_k)$$

Hence if we prove that the map $\nu X \to \varprojlim_k \nu(X \otimes \Sigma\mathbb{R}P^{\infty}_k)$ is an equivalence of synthetic spectra, we obtain the Segal conjecture after inverting τ. In general, τ-inversion does not preserve limits, but the limit in question is preserved. This is because τ-inversion is given by taking colimit along multiplication by τ, which acts eventually by isomorphism. Hence we can commute the limit across an eventually constant limit.

Both νX and $\varprojlim_k \nu(X \otimes \Sigma\mathbb{R}P^{\infty}_k)$ are τ-complete; therefore, it is enough to prove the equivalence after tensoring with $C\tau$. Hence we have the following reformulation:

Proposition 4.3.1.6. Let X be a bounded below p-complete spectrum then $X \to X^{t\mathbb{Z}/p}$ is an equivalence if and only if $\nu X \to \varprojlim_k \nu(X \otimes \Sigma\mathbb{R}P_k^\infty)$ is an equivalence after tensoring with $C\tau$.

Recall that $C\tau \simeq \mathrm{Cofib}(\tau : \mathbb{S}^{0,-1} \to \mathbb{S}^{0,0})$, where both $\mathbb{S}^{0,-1}$ and $\mathbb{S}^{0,0}$ are dualisable, hence $C\tau$ is also dualisable. So we have

$$C\tau \otimes \varprojlim_k \nu(X \otimes \Sigma\mathbb{R}P_k^\infty) \simeq \varprojlim_k C\tau \otimes \nu(X \otimes \Sigma\mathbb{R}P_k^\infty)$$

since $C\tau \otimes \nu(-) \simeq H_*(-)$ we have,

$$\simeq \varprojlim_k H_*(X \otimes \Sigma\mathbb{R}P_k^\infty) \simeq \varprojlim_k H_*(X) \otimes H_*(\Sigma\mathbb{R}P_k^\infty).$$

So we have to prove that the natural map:

$$H_*(X) \to \varprojlim_k H_*(X) \otimes H_*(\Sigma\mathbb{R}P_k^\infty)$$

is an equivalence.

We now prove a useful lemma which we will use repeatedly,

Lemma 4.3.1.7. Let Y be bounded below spectrum, which is of finite type, that is, the k-th skeleton of Y is finite for all k. Let $\mathbb{N}$ be the poset of natural numbers and $X_{(-)} : \mathbb{N} \to \mathrm{Sp}$ be a functor such that X_i is uniformly bounded below spectrum for all $i \in \mathbb{N}$. Then

$$Y \otimes \lim_i X_i \simeq \lim_i Y \otimes X_i.$$

Similarly, for product $\prod_{i \in I} X_i$ we obtain,

$$Y \otimes \prod_{i \in I} X_i \simeq \prod_{i \in I} Y \otimes X_i.$$

Proof. We will prove the case of $\mathbb{N}$-indexed limit and the case of product is similar.

Since Y is of finite type, we have $Y \simeq \operatorname*{colim}_k Y^{(k)}$, where $Y^{(k)}$ is a finite spectrum for all k. Hence we have:

$$Y \otimes \lim_i X_i \simeq \operatorname*{colim}_k Y^{(k)} \otimes \lim_i X_i$$

since $Y^{(k)}$ is finite, hence dualisable, we can commute the limit to obtain:

$$\operatorname*{colim}_k Y^{(k)} \otimes \lim_i X_i \simeq \operatorname*{colim}_k \lim_i Y^{(k)} \otimes X_i$$

we have the map:

$$\operatorname*{colim}_k \lim_i Y^{(k)} \otimes X_i \to \lim_i Y \otimes X_i$$

to prove it is equivalence enough to prove isomorphism on homotopy groups. So let us fix an integer b and try to prove that the comparison map is an isomorphism on π_b.

Since $Y^{(k)} \to Y$ be is k-connective and since X_i is uniformly bounded below for all i. Let us say $\pi_n(X_i) = 0$ for all $n < a$. Then, the cofiber of

$$\operatorname*{colim}_k \lim_i Y^{(k)} \otimes X_i \to \lim_i Y \otimes X_i$$

is $a+k$-connected. So for k large enough, we have $a+k >> b$, hence it is π_b-isomorphism. $\square$

The category of synthetic spectra carries a natural t-structure [Pst22, Section 4.2] with heart E_*E-comodules with respect to which we can run the same argument above and get the following lemma which looks exactly as the above.

Lemma 4.3.1.8. Let Y be a bounded below synthetic spectrum of finite type. Let $\mathbb{N}$ be the poset of natural numbers and $X_{(-)} : \mathbb{N} \to \mathrm{Syn}_E$ be a functor such that X_i is uniformly bounded below for all $i \in \mathbb{N}$. Then

$$Y \otimes \lim_i X_i \simeq \lim_i Y \otimes X_i.$$

Similarly, for product $\prod_{i \in I} X_i$ we obtain,

$$Y \otimes \prod_{i \in I} X_i \simeq \prod_{i \in I} Y \otimes X_i.$$

□

Let us introduce a definition before stating the main theorem; given a spectrum X, the $\mathbb{F}_p$-homology has an action of the Steenrod algebra $\mathcal{A}_p$ such that for $Sq^i \in \mathcal{A}_p$ and $x \in H_n(X)$ then $Sq^i(x) \in H_{n-i}(X)$.

Definition 4.3.1.9. Let I be the augmentation ideal of the Steenrod algebra $\epsilon : \mathcal{A}_p \to \mathbb{F}_p$. We say a spectrum X is I-nilpotent if we have have $H_*(X) \simeq \varprojlim_n H_*(X)/I^n$.

Theorem 4.3.1.10. Let X be a bounded below spectrum, p-complete and I-nilpotent, then X satisfies the Segal conjecture for $\mathbb{Z}/p$. That is, $X \to X^{t\mathbb{Z}/p}$ is an equivalence, where the Tate construction is with respect to trivial action.

Proof. So we need to prove

$$H_*(X) \to \lim_k H_*(\Sigma \mathbb{R}P_k^\infty) \otimes H_*(X)$$

is an equivalence. Without loss of generality, we can assume X is connective after suspending enough times.

Since X is an I-nilpotent spectrum, we get

$$\lim_k H_*(\Sigma \mathbb{R}P_k^\infty) \otimes H_*(X) \simeq \lim_k H_*(\Sigma \mathbb{R}P_k^\infty) \otimes \lim_n H_*(X)/I^n.$$

Since $\Sigma \mathbb{R}P_k^\infty$ is bounded below and of finite type for all k, we have by lemma 4.3.1.8

$$\lim_k H_*(\Sigma \mathbb{R}P_k^\infty) \otimes \lim_n H_*(X)/I^n \simeq \lim_{k,n} H_*(\Sigma \mathbb{R}P_k^\infty) \otimes H_*(X)/I^n.$$

So to prove

$$\lim_n H_*(X)/I^n \to \lim_{k,n} H_*(\Sigma \mathbb{R}P_k^\infty) \otimes H_*(X)/I^n$$

is an equivalence, it is enough to prove equivalence on the associated graded

$$K_n = \mathrm{fib}(H_*(X)/I^{n+1} \to H_*(X)/I^n).$$

That is, prove that $K_n \to \lim_k H^*(\Sigma \mathbb{R}P_k^\infty) \otimes K_n$ is an equivalence. Where K_n has a trivial action of $\mathcal{A}_p$, so $K_n \simeq \bigoplus_{i \geqslant 0} \mathbb{F}_p[i]^{\oplus J_i}$ where $i \in I$.

$$\lim_k H_*(\Sigma\mathbb{R}P_k^\infty)\otimes K_n \simeq \lim_k H_*(\Sigma\mathbb{R}P_k^\infty)\otimes \bigoplus_{i\geqslant 0}\mathbb{F}_p[i]^{\oplus J_i}$$

$$\simeq \lim_k H_*(\Sigma\mathbb{R}P_k^\infty)\otimes \prod_{l\geqslant 0}\mathbb{F}_p[i]^{\oplus J_i}$$

again $\Sigma\mathbb{R}P_k^\infty$ is bounded below of finite type, so by lemma 4.3.1.8, we can commute the product all the way outside to get

$$\lim_k H_*(\Sigma\mathbb{R}P_k^\infty)\otimes \prod_{l\geqslant 0}\mathbb{F}_p[i]^{\oplus J_i} \simeq \prod_{l\geqslant 0}\lim_k H_*(\Sigma\mathbb{R}P_k^\infty)\otimes \mathbb{F}_p[i]^{\oplus J_i}$$

So we have to prove

$$K_n \simeq \prod_{l\geqslant 0}\mathbb{F}_p[i]^{\oplus J_i} \to \prod_{l\geqslant 0}\lim_k H_*(\Sigma\mathbb{R}P_k^\infty)\otimes \mathbb{F}_p[i]^{\oplus J_i} \simeq \lim_k H_*(\Sigma\mathbb{R}P_k^\infty)\otimes K_n$$

is an equivalence. In order to prove the product is an equivalence, it is enough to prove equivalence termwise. This reduces to proving

$$\mathbb{F}_p[i]^{\oplus J_i} \to \lim_k H_*(\Sigma\mathbb{R}P_k^\infty)\otimes \mathbb{F}_p[i]^{\oplus J_i}$$

is an equivalence. So if we write $\mathbb{F}_p[i]^{\oplus J_i}$ as a retract of $\mathbb{F}_p[i]^{\times I_i}$ and apply 4.3.1.8 to reduce to proving,

$$\prod_{i\in I_i}\mathbb{F}_p[i] \to \prod_{i\in I_i}\lim_k H_*(\Sigma\mathbb{R}P_k^\infty)\otimes \mathbb{F}_p[i]$$

So again, we can prove equivalence termwise to reduce to proving:

$$\mathbb{F}_p[i] \to \lim_k H_*(\Sigma\mathbb{R}P_k^\infty)\otimes \mathbb{F}_p[i]$$

is an equivalence. This is the case of the classical Segal conjecture, that is, in $\mathrm{Syn}_{\mathbb{F}_p}$ the bigraded homotopy groups on both sides are Ext-groups shown to be equivalence by [LDM+80]. □

Definition 4.3.1.11. Given a spectrum X, we say it has locally finite $\mathbb{F}_p$-cohomology if for any element $x \in H^*(X)$, the $\mathcal{A}_p$ submodule generated by x is finite dimensional $\mathbb{F}_p$-vector space.

Proposition 4.3.1.12. Any bounded below spectrum with $\mathbb{F}_p$-cohomology locally finite and of finite type, then the $\mathbb{F}_p$-homology is I-nilpotent.

Proof. We need to verify that $H_*(X) \simeq \varprojlim_n H^*(X)/I^n$ is in the category of comodules over the dual Steenrod algebra.

We can calculate the homotopy groups of the derived limit using Milnor's $\lim^1$-sequence for the underlying graded abelian group since the objects in the sequential inverse limit are uniformly bounded below (all of them are connective) the forgetful functor from the category of comodules to graded abelian group preserve this particular limits and it is also conservative.

The transition maps are all surjective; hence, by the Mittag-Leffler condition, the derived limit is the ordinary sequential inverse limit. So we want the desired equivalence on the underlying graded abelian groups, and now by conservativity we have isomorphism $H_*(X) \simeq \varprojlim_n H^*(X)/I^n$ in the category of comoudles.

Let us look at a fixed degree $H_i(X)$ because the finite type assumption $H_i(X)$ is finite dimensional for all i and hence $H^i(X) \simeq H_i(X)^\vee$. The locally finite assumption implies for all $x \in H^i(X)$, the $\mathcal{A}_p$-submodule generated is finite dimensional. Hence for all $t \in H_i(X)$ we have $I^n.t = 0$ for n large enough.

Therefore the inverse limit $\varprojlim_n H_*(X)/I^n$ is eventually constant in a fixed degree d and isomorphic to $H_d(X)$, so we have $H_*(X) \simeq \varprojlim_n H_*(X)/I^n$. □

Example 4.3.1.13. For X, a compact space and simply connected, then we have ΩX has a finite type, locally finite $\mathbb{F}_p$-cohomology. Therefore $H_*(\Omega X)$ is I-nilpotent. Hence $\Sigma_+^\infty \Omega X$ after p-completion is equivalent to $(\Sigma_+^\infty \Omega X)^{t\mathbb{Z}/p}$. □

Example 4.3.1.14. Let X be a finite dimensional spectrum; then the $\mathbb{F}_p$-homology is finite dimensional. Hence $H_*(X)$ is I-nilpotent complete. □

Example 4.3.1.15. Recall for $\mathcal{O}$ an operad the free $\mathcal{O}$-algebra on a spectrum X is given by:

$$\mathrm{Free}_{\mathcal{O}}(X) \simeq \bigoplus_n X^{\otimes n} \otimes_{\Sigma_n} \mathcal{O}(n).$$

Let $\mathcal{O}$ be an operad, then the n-th space of arity $\mathcal{O}(n)$ has an Σ_n-action. Let us assume $\mathcal{O}(n)$ is a finite Σ_n space such that only the isotropy group is identity. Examples of such operad include $\mathbb{E}_n$-operad for finite n[4].

For an operad with above properties, $X^{\otimes n} \otimes_{\Sigma_n} \mathcal{O}(n)$ has finite $\mathbb{F}_p$-homology for each n. Moreover, if we assume that X has minimal cell structure such that the least dimensional cell is in degree 1, then the connectivity of $X^{\otimes n} \otimes_{\Sigma_n} \mathcal{O}(n)$ increases as n increases this implies for $\mathrm{Free}_{\mathcal{O}}(X)$ is I-nilpotent, hence satisfies the Segal conjecture for $\mathbb{Z}/p$. □

Remark 4.3.1.16. The above argument also applies to an operad valued in spectra as well, with each arity spectrum bounded below and having a finite Σ_n-cell structure and isotropy groups just the identity. □

[4]that is except for $\mathbb{E}_\infty$

5 Homotopy fixed points of the loop rotation

5.1 Introduction

For G, a complex reductive group, we can associate the Langlands dual group $G^\vee$, which is a central character in the Langlands program. When first introduced, the construction of $G^\vee$ appeals to the classification of reductive algebraic groups due to Chevalley [Che55]. One could try to attempt to canonically recover the dual group starting with G.

This was successfully achieved by the work of I. Mirkovic and K. Vilonen [MV07] called the Geometric Satake correspondence. Starting from G, a reductive algebraic group over $\mathbb{C}$, we can associate an infinite dimensional algebro-geometric object Gr_G called the affine Grassmannian and the abelian category of algebraic representations of $G^\vee$ as certain stratified local systems on Gr_G.

Theorem 5.1.0.1 (Geometric Satake Correspondence)**.** [MV07] Let $\mathrm{Perv}_{L^+G}(\mathrm{Gr}_{Gr})$ be the abelian category L^+G-equivariant perverse sheaf on the affine Grassmannian, which has a monoidal structure called the convolution product, then there is a symmetric monoidal equivalence between $\mathrm{Perv}_{L^+G}(\mathrm{Gr}_G)$ and the representation 1-category of the dual group $\mathrm{Rep}(G^\vee)$.

In particular, the more we understand the geometry of Gr_G, the more we understand the category $\mathrm{Rep}(G^\vee)$. The underlying analytic space of the affine Grassmannian Gr_G is homotopy equivalent to ΩK, the based loops space on the maximal compact subgroup of $G(\mathbb{C})$.

The affine Grassmannian Gr_G has a S^1-action coming from loop rotation. In [RW22a], the authors study the relationship between the loop rotation action and the Frobenius twisting of representations in positive characteristics. They also compute the $\mathbb{Z}/p$-fixed points $\mathrm{Gr}_G^{\mathbb{Z}/p}$.

In this chapter, we give a recipe to compute the homotopy fixed points of this action, and in the case of $G = GL_n(\mathbb{C})$, we have the following theorem:

Theorem 5.1.0.2. Let $G = GL_n(\mathbb{C})$, then the homotopy fixed points $(\Omega U(n))^{h\mathbb{Z}/p^n}$ is equivalent to $\bigsqcup_{\rho \in \mathrm{Rep}(\mathbb{Z}/p^n, U(n))} \Omega U(n) \times U(n)/C(\rho)$.

The proof of Sullivan conjecture by H. Miller showed if a space X has locally finite cohomology, then with respect to trivial action of a p-group

$$X \simeq X^G \to X^{hG} \simeq \mathrm{map}(BG, X)$$

is an equivalence after p-completion.

We can also ask if the Sullivan conjecture is true for spaces with locally finite cohomology and non-trivial action. In particular, the cohomology of $\Omega U(n)$ is locally finite (see Definition 4.3.1.11) and is of finite type. We compare the genuine fixed points computed in [RW22a] and homotopy fixed points with respect to the loop rotation action.

5.2 Homotopy Fixed points

5.2.1 Affine Grassmannian and Geometric Satake

This section briefly recalls results about the affine Grassmannian Gr_G, a certain infinite-dimensional algebraic variety that plays a central role in geometric representation theory.

The theory is extremely rich, and the interested reader can find detailed expositions in [Zhu16; BR18]. For our purpose, we are interested in the homotopical side of the story. Hence we will quickly recall the necessary theory to set up the context.

Let G be a linear algebraic group over R; then we can define the loop stack LG as the presheaf

$$LG(S) = G(S((t)))$$

where S is a R-algebra. Similarly define L^+G as the presheaf

$$L^+G(S) = G(S[[t]])$$

where S is a R-algebra. Where $S[[t]]$ is the power series ring in one variable t and $S((t))$ is the Laurent power series ring.

Then we define the affine Grassmannian Gr_G associated to the group G, as the fpqc sheafification of

$$\mathrm{Gr}_G = LG/L^+G.$$

From this point, we set G to be a reductive algebraic group over the algebraically closed field k.

The following is an important theorem about the geometry of affine Grassmannian.

Theorem 5.2.1.1. Let G be a reductive algebraic group, then the affine Grassmannian Gr_G is represented by an ind-scheme of finite type and is also in-projective. That is, Gr_G is the colimit $\varinjlim_n \mathrm{Gr}^n$ where each Gr^n is of projective and finite type.

There exist more geometric interpretations of Gr_G using the Beauville-Laszlo theorem. For more details, see [Zhu16, Section 1.4].

Definition 5.2.1.2. Let X be a curve over a field k (for us $k = \mathbb{C}$) and $x \in X(k)$ a smooth k-point and G be a reductive group over k. Then we define $\mathrm{Gr}_{G,x}$ to be:

$$\mathrm{Gr}_{G,x}(R) = \{(E, \beta) \mid E \textit{ a principle } G\textit{-bundle on } X_R \textit{ and } \beta \textit{ a trivialization on } (X \backslash x)_R\}$$

We have $\mathrm{Gr}_{G,x}$ is the sheaf of sets on the site of commutative algebra with respect to the $fpqc$-topology. We then have the following theorem:

Theorem 5.2.1.3. The $fpqc$-sheaf $\mathrm{Gr}_{G,x}$ is equivalent to the affine Grassmannian Gr_G.

Let us introduce some notation:

Let $T \subset G$ be the maximal torus of G. Given a co-character $\lambda : \mathbb{G}_m \to T$ we can get a map

$$\tilde{\lambda} : \mathbb{C}((x))[t, t^{-1}] = \mathbb{G}_{m\mathbb{C}((x))} \to LG = G(\mathbb{C}((x)))$$

we denote t_λ to be the image of t under the map $\tilde{\lambda}$.

Proposition 5.2.1.4. (Cartan Decomposition) Let G be a reductive group over k and choose a maximal torus $T \subset G$. Then there is a double closet decomposition

$$LG = \bigsqcup_{\lambda \in \mathbb{X}_\bullet(T)_+} L^+ G t_\lambda L^+ G$$

where $\mathbb{X}_\bullet(T)_+ \subset \mathbb{X}_\bullet(T)$ is the set of dominant coweights.

Remark 5.2.1.5. There is Cartan decomposition for all reductive groups over local fields, as investigated by [BT72; BT84]. The Cartan decomposition is also often referred to as Bruhat decomposition.

For any reductive group, G and fix $T \subset B \subset G$, B a Borel subgroup and T a maximal torus, then we have the Bruhat decomposition:

$$G = \bigsqcup_{w \in W} BwB,$$

where W is the Weyl group of G defined as the quotient $N_G(T)/T$.

Then we can form the flag variety G/B, and the Bruhat decomposition gives a cell structure on the underlying analytic variety of G/B with one cell for each $w \in W$ given by the image of BwB in the quotient G/B denoted by C_w.

The Weyl group, W, has an ordering called the Bruhat ordering, and this gives a stratification on G/B. That is, let C_w be the cell corresponding to w, then $\bar{C}_w = \bigsqcup_{w' \leqslant w} C_{w'}$.

Studying stratified local systems with respect to this stratification has deep connections with representations of Lie algebra of G via the Beilinson-Bernstein localisation theorem [Bei81] and Riemann-Hilbert Correspondence [Kas79; Kas84].

In a similar spirit, the affine Grassmannian $\mathrm{Gr}_G = LG/L^+G$ is a certain affine flag variety, which is stratified over $\mathbb{X}_\bullet(T)_+$ by $L^+G.t_\lambda$ (refer to [Zhu16, Lecture II]).

A reductive group G over an algebraically closed field is classified by the root datum $(\mathbb{X}^\bullet(T), R, \mathbb{X}_\bullet(T), R^\vee)$ [Che55] where T is a fixed maximal torus of G. Given a root datum $(\mathbb{X}^\bullet(T), R, \mathbb{X}_\bullet(T), R^\vee)$ we can form the dual root datum $(\mathbb{X}_\bullet(T), R^\vee, \mathbb{X}^\bullet(T), R)$ by swapping roots and co-roots, then appealing to the existence results of Chevalley we get the Langlands dual group $G^\vee$ associated to the dual root datum.

One could ask if there is a more canonical way to obtain the Langlands dual group $G^\vee$. This is done using the Geometric Stake Correspondence using certain stratified local systems on the affine Grassmannian Gr_G.

Theorem 5.2.1.6 (Geometric Satake Correspondence)**.** [MV07] Let $\mathrm{Perv}_{L^+G}(\mathrm{Gr}_{Gr})$ be the abelian category L^+G-equivariant perverse sheaf on the affine Grassmannian, which has a monoidal structure called the convolution product, then there is a symmetric monoidal equivalence between $\mathrm{Perv}_{L^+G}(\mathrm{Gr}_G)$ and the representation 1-category of the dual group $\mathrm{Rep}(G^\vee)$.

So there is a more geometrical way of approaching the category of representation over the dual group $G^\vee$, in particular, the affine Grassmannian has a $\mathbb{G}_m$-action called the loop rotation action.

Let $\mu_p \subset \mathbb{G}_m$ be the cyclic group. We can ask what are the fixed points with respect to this action. Turns out the μ_p-fixed points have a deep connection with the Frobenius operator on representations of $G^\vee$. This is investigated in the recent work of S. Riche and G. Williamson in [RW22a].

The ideas in [RW22a] involve developing a categorified Smith theory for etale sheaves following D. Treumann [Tre19]. Riche-Williamson studies the genuine fixed points of the μ_p-action, and in this article, we study the homotopy fixed points of this action for all cyclic group $\mathbb{Z}/p^n$.

We state the following result of G. Segal and A. Pressley for $G = GL_n$ [Pre86, Section 8.3] and for more general reductive groups by D. Nadler [Nad04, Section 4].

Proposition 5.2.1.7. Let G be a reductive group over $\mathbb{C}$ and $K \subset G$ be the maximal compact subgroup; then we have a homotopy equivalence

$$\Omega K \simeq \mathrm{Gr}_G(\mathbb{C}),$$

between the based loops on K and the complex points of the affine Grassmannian.

5.2.2 Loop Rotation and fixed points

In this section, we recall some facts about mapping spaces between classifying spaces, in particular, the work of [DZ06], which will be useful in the computation of certain fixed points.

Let G be a compact lie group and P a finite p group. Given a representation $\rho : P \to G$, i.e. a homomorphism, we obtain a group homomorphism $C(\rho) \times P \to G$ where $C(\rho)$ is the centraliser of the representation. This in turn gives a map of classifying spaces

$$B(P \times C(\rho)) \simeq BP \times BC(\rho) \to BG$$

by adjunction, we obtain a map:

$$BC(\rho) \to \mathrm{map}(BP, BG)$$

varying ρ we get:

$$\bigsqcup_{\rho \in \mathrm{Rep}(P,G)} BC(\rho) \to \mathrm{map}(BP, BG)$$

Rep(P, G) is the collection of all possible representations $\rho : P \to G$ upto conjugation, (i.e, we say two representations ρ_1 amd ρ_2 are equivalent if there exist a $g \in G$ such that $\rho_2 = g\rho_1 g^{-1}$).

The main theorem of [DZ06] state that:

Proposition 5.2.2.1. The map

$$\bigsqcup_{\rho \in \mathrm{Rep}(P,G)} BC(\rho) \to \mathrm{map}(BP, BG)$$

is a mod p-equivalence.

Let G be a reductive group over $\mathbb{C}$. We have seen that the homotopy type of the affine Grassmannian Gr_G is equivalent to the based loop space ΩK where K is the maximal compact subgroup of G. By the recognition principle we have $\Omega K \simeq \Omega^2 BK \simeq \mathrm{map}_*(S^2; BK)$ where the base point of S^2 by convention is the "south pole".

There is a natural S^1-action on S^2 as a pointed space by rotation along the equator. Hence we get a $\mathbb{Z}/p^n$-action for $n \in \mathbb{N}$. By identifying ΩK with $\mathrm{map}_*(S^2, BG)$, we get an S^1-action on ΩK, which is classically called the "loop rotation" action.

One can always get a loop rotation action on the free loop space LX on any space X given by:

$$S^1 \times LX \to LX$$

$$(z, \gamma) \mapsto \gamma^z$$

where $\gamma^z(z') = \gamma(z.z')$. In general, there is no such action on the based loop space ΩX for any space X since the base points are not preserved, but one can fix this if X happens to be a topological group we have an S^1-action by sending $(z, \gamma) \mapsto \gamma^z.\gamma(z)^{-1}$.

Remark 5.2.2.2. Note that we do not need X to be a topological group. We only need it to be a **grouplike** $\mathbb{E}_1$-space which by the recognition theorem is a based loop on some space Y.

Question 5.2.2.3. With the $\mathbb{Z}/p^n$ loop rotation action mentioned above on ΩK, what is the homotopy fixed points $(\Omega K)^{h\mathbb{Z}/p^n}$?.

There is a loop rotation action of S^1 and $\mathbb{Z}/p^n$ on the affine Grassmannian Gr_G for some reductive group G. We know the affine Grassmannian is homotopy equivalent to ΩK, but apriori, there is no reason for the fixed points to be homotopy equivalent since the notion of fixed points is not a homotopy invariant notion.

This leads us to the above question of homotopy fixed points. We do have $\mathrm{Gr}_G^{h\mathbb{Z}/p^n} \simeq (\Omega K)^{h\mathbb{Z}/p^n}$ for all n. We devote the rest of the section to answering **Question** 5.2.2.3 by analysing maps between classifying spaces and a closed form expression in the case of $G = GL_n(\mathbb{C})$ with maximal compact $K = U(n)$.

Identify ΩK with $\mathrm{map}_*(S^2, BK)$, we have

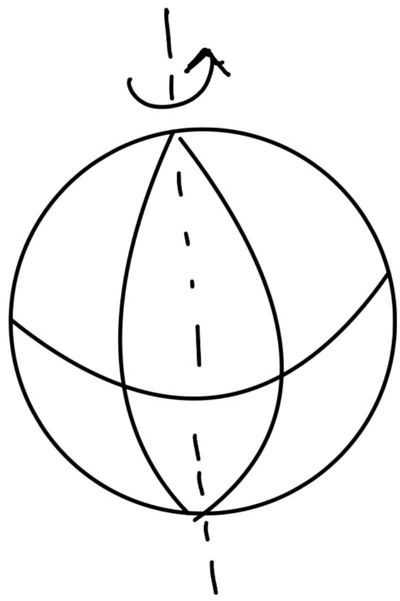
$$\mathrm{map}_*(S^2, BG)^{h\mathbb{Z}/p^n} \simeq \lim_{B\mathbb{Z}/p^n} \mathrm{map}_*(S^2, BG) \simeq \mathrm{map}_*(\operatorname*{colim}_{B\mathbb{Z}/p^n}(S^2), BK) \simeq \mathrm{map}_*({S^2}_{h\mathbb{Z}/p^n,*}, BG)$$

where ${S^2}_{h\mathbb{Z}/p^n,*}$ is the homotopy quotient with respect to the rotation action in the ∞-category $\mathcal{S}_*$ of pointed spaces.

As an unpointed space S^2 is written as the pushout:

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{pt} \arrow[d] \\ \mathrm{pt} \arrow[r] & S^2 \end{tikzcd}$$

It turns out this is a $\mathbb{Z}/p^n$-equivariant pushout where the cell structure can be represented by the following diagram:



This lets you compute $S^2_{h\mathbb{Z}/p^n}$ as:

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & B\mathbb{Z}/p^n \arrow[d] \\ B\mathbb{Z}/p^n \arrow[r] & S^2_{h\mathbb{Z}/p^n} \end{tikzcd}$$

where the map $S^1 \simeq B\mathbb{Z} \to B\mathbb{Z}/p^n$ is given by the quotient map $\mathbb{Z} \to \mathbb{Z}/p^n$, by identifying S^1 with $B\mathbb{Z}$.

This lets us compute the pushout $S^2_{h\mathbb{Z}/p^n,*}$ in pointed space:

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & B\mathbb{Z}/p^n \arrow[d] \\ * \arrow[r] & S^2_{h\mathbb{Z}/p^n}, * \end{tikzcd}$$

by collapsing the $B\mathbb{Z}/p$ factor on the base point.

Since $(\Omega K)^{h\mathbb{Z}/p^n} \simeq \mathrm{map}_*(S^2_{h\mathbb{Z}/p^n,*}, BG)$, we can compute the homotopy fixed points via the pullback:

$$\begin{tikzcd}
(\Omega K)^{h\mathbb{Z}/p^n} \arrow[r] \arrow[d] & \mathrm{map}_*(B\mathbb{Z}/p^n, BG) \arrow[d] \\
* \arrow[r] & \mathrm{map}_*(B\mathbb{Z}, BG)
\end{tikzcd}$$

We also have a further fiber sequence to compute pointed mapping space:

$$\begin{tikzcd}
\mathrm{map}_*(X,Y) \arrow[r] \arrow[d] & \mathrm{map}(X,Y) \arrow[d] \\
* \arrow[r] & Y
\end{tikzcd}$$

putting this together, we get a fiber sequence:

$$\begin{tikzcd}
(\Omega K)^{h\mathbb{Z}/p^n} \arrow[r] & \coprod_{\rho \in \mathrm{Rep}(\mathbb{Z}/p^n, K)} K/C(\rho) \arrow[d] \\
 & \Omega BK \simeq K
\end{tikzcd}$$

where we identify: $\mathrm{map}_*(B\mathbb{Z}/p^n, BK) \simeq \coprod_{\rho \in \mathrm{Rep}(\mathbb{Z}/p^n, G)} K/C(\rho)$ from Proposition 5.2.2.1.

The map,

$$K/C(\rho) \to K$$

is given as follows:

We have map $K \to K$ given by $g \mapsto g\rho(1)g^{-1}$. Since all element of $C(\rho)$ commutes with $\rho(1)$ by definition, we have a factorisation:

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & K \\
K/C(\rho) \arrow[ur] &
\end{tikzcd}$$

note that the map $K/C(\rho) \to K$ is not a group homomorphism, in fact $K/C(\rho)$ is not even a group. We have the following lemma to begin with:

Lemma 5.2.2.4. For a connected group K, the map $f : K \to K$ given by $g \mapsto g\rho(1)g^{-1}$ is null-homotopic.

Proof. Let γ be a path from $\rho(1)$ to e, the identity element of the group G. We have a continuous family of maps $f_t : G \to G$, given by $g \mapsto g\gamma(t)g^{-1}$, where $f_0 = f$ and f_1 is the constant map at e, showing that f is nullhomotopic. $\square$

So if we can say the homotopy factors through the centraliser, the pullback computing $(\Omega K)^{h\mathbb{Z}/p}$ reduces to the following square:

$$\begin{tikzcd}
(\Omega K)^{h\mathbb{Z}/p} \arrow[r] \arrow[dd] & \bigsqcup_{\rho \in \mathrm{Rep}(P,K)} K/C(\rho) \arrow[d] \\
& * \arrow[d] \\
* \arrow[r] & K
\end{tikzcd}$$

If that is true, we have:

Theorem 5.2.2.5. If the map $K/C(\rho) \to K$ is null-homotopic for all $\rho \in \mathrm{Rep}(\mathbb{Z}/p^n, K)$, then $(\Omega K)^{h\mathbb{Z}/p^n} \simeq \bigsqcup_{\rho \in \mathrm{Rep}(P,K)} \Omega K \times K/C(\rho)$ after p-completion.

So the question reduces to, when does the null-homotopy of $g \mapsto g\rho(1)g^{-1}$ factor through $K/C(\rho)$ or more broadly, if there exists any null-homotopy of the map $K/C(\rho) \to K$, to obtain the above theorem.

Example 5.2.2.6. Let us start with $G = \mathrm{SL}_2(\mathbb{C})$, then the maximal compact $K = SU(2)$ and we have $SU(2) \simeq S^3$. Given a representation $\rho : \mathbb{Z}/p \to S^3$, the image of a generator of $\mathbb{Z}/p$ is an order p matrix with determinant 1, hence any representation up to conjugation is given by:

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$$

where λ is a p^{th}-root of unity.

If $\lambda = \{1, -1\}$ then the centralizer $C(\rho) = S^3$, so $K/C(\rho) = *$ and for non-trivial values, i.e., $\lambda \notin \{1, -1\}$ then $C(\rho) = S^1$ hence $K/C(\rho) = S^2$. In particular, $K/C(\rho) \to K$ is null homotopic because there is no nontrivial map from $S^2 \to S^3$.

Hence,
$$(\Omega S^3)^{h\mathbb{Z}/p} \simeq \Omega S^3 \sqcup \bigsqcup_{p-1} \Omega S^3 \times S^2.$$
after p-completion. □

We can also uniformly analyse the case of $G = GL_n(\mathbb{C})$ and $K = U(n)$ given in the following proposition:

Proposition 5.2.2.7. Let $G = GL_n(\mathbb{C})$ and $\rho : \mathbb{Z}/p^n \to K$ be a representation. Then $U(n)/C(\rho) \to U(n)$ is nullhomotopic. In particular

$$(\Omega U(n))^{h\mathbb{Z}/p^n} \simeq \bigsqcup_{\rho \in \mathrm{Rep}(\mathbb{Z}/p^n, U(n))} \Omega U(n) \times U(n)/C(\rho).$$

after p-completion on both sides

Proof. Let σ be the generator of $\mathbb{Z}/p^n$, we will also refer to the image of σ under ρ as σ. We can get a path γ in $U(n)$ from σ to e, the identity element of G, giving a null homotopy by Lemma 5.2.2.4. The null hoomtpy factors through $U(n)/C(\rho)$ if $C(\gamma(t))$ constains $C(\rho)$ for all t.

$$\begin{tikzcd} U(n) \times I \arrow[r] \arrow[d] & U(n) \\ U(n)/C(\rho) \times I \arrow[ur] & \end{tikzcd}$$

Let us diagonalise the σ and express it in terms of the block matrices of the form

$$\lambda_i \mathrm{Id}$$

where $\lambda_i\mathrm{Id}$ is the scalar matrix of scalar λ_i.

Let $(\lambda_1, \lambda_2, ..., \lambda_k)$ be the distinct eigenvalues of σ.

Let $\mid \lambda_i \mid$ be the dimension of the eigenspace corresponding to λ_i, then the centraliser of σ is $U(\mid \lambda \mid) \times ... \times U(\mid \lambda_k \mid)$ identified as a subgroup of $U(n)$ as block diagonal matrices.

Choosing paths γ_i from λ_i to 1 in $\mathbb{C}^*$, we can construct a path γ from σ to the identity $\mathrm{id} \in U(n)$ and the centraliser of $\gamma(t)$ contains $U(\mid \lambda \mid) \times ... \times U(\mid \lambda_k \mid)$ for all t. Hence we have the null-homotopy that factors through $U(n)/C(\rho)$.

In conclusion, $U(n)/C(\rho) \to U(n)$ is nullhomotopic, and particularly

$$(\Omega U(n))^{h\mathbb{Z}/p^n} \simeq \coprod_{\rho \in \mathrm{Rep}(\mathbb{Z}/p^n, K)} \Omega U(n) \times U(n)/C(\rho).$$

□

Remark 5.2.2.8. The above approach does not work in general. For example let $G = SL_3(\mathbb{C})$ and $K = \mathrm{SU}(3)$ and $p = 3$, then for representation $\rho : \mathbb{Z}/p\{\sigma\} \to SU(3)$, then σ can be diagonalised with diagonal entries given by p-th roots of unity. Let us consider the representation $\mathrm{diag}(\omega, \omega, \omega)$ where $\omega = e^{\frac{2\pi i}{3}}$; then, the path consider will not be in $SU(3)$. Since the representation lands in the centre and the centre of $SU(3)$ is disconnected, there is no path γ from σ to identity such that $C(\gamma(t))$ contains $C(\rho) = SU(3)$.

However, the above argument does work for all prime p other than 3, which enforces not all eigenvalues are equal.

□

5.2.3 Genuine fixed points and comparison

In this section, we comment on the genuine fixed points and compare them to homotopy fixed points.

For X a G-CW complex, such that G a p-group, then there is a natural comparison map

$$X^G \to X^{hG}.$$

The Sullivan conjecture asks, when is this map an equivalence after p-completion?

Theorem 5.2.3.1. (G. Carlsson, H. Miller, J. Lannes) If X is finite G-CW complex then $X^G \simeq X^{hG}$ after p-completion.

Here is a way to calculate the cohomology of the homotopy fixed points following [DW88]. Let G be an elementary abelian group and X a finite G-CW complex, then there exist fiber sequence:

$$\begin{array}{ccc} X & \longrightarrow & X_{hG} \\ & & \downarrow \\ & & BG \end{array}$$

There exists an Euler class $e \in H^*(BG)$ (see 2.1.3), there is the induced map of graded rings $H^*(BG) \to H^*(X_{hG})$ localising $H^*(X_{hG})$ at the Euler class gives an equivalence

$$H^*(X_{hG})[e^{-1}] \simeq H^*(X^G) \otimes H^*(BG)[e^{-1}]$$

From this, we can recover the cohomology of the fixed points X^G following W. Dwyer and C. Wilkerson [DW88]. The localised algebra $H^*(X_{hG})[e^{-1}]$ is a module over the Steenrod algebra and let $Un(H^*(X_{hG})[e^{-1}])$ be the largest unstable algebra inside $H^*(X_{hG})[e^{-1}]$.

Proposition 5.2.3.2. (Dwyer-Wilkerson) For an elementary abelian group G and X a finite G-CW complex, The largest unstable algebra $Un(H^*(X_{hG})[e^{-1}])$ is isomorphic to $H^*(X^G \times BG)$, in particular

$$Un(H^*(X_{hG})[e^{-1}]) \otimes_{H^*(BG)} \mathbb{F}_p \simeq H^*(X^G).$$

□

Dwyer and Wilkerson show in [DW91], that for a $H^*(BC_p) \otimes A_p$-module M such that M is finite dimensional, the above construction $Un(M[e^{-1}]) \otimes_{H^*(BG)} \mathbb{F}_p$ is equivalent to the Fix functor constructed by J.Lannes [Lan92] which computes the cohomology homotopy fixed points. For an A_p-module N, the Fix functor applied to $H^*(BG) \otimes N$ is the T-functor constructed by Lannes.

Proving the Sullivan conjecture for spaces with trivial action[1] is equivalent to proving that the Lannes T-functor applied to $H^*(X)$ is $H^*(X)$, that is $T_V(H^*(X)) \simeq H^*(\mathrm{map}(BV, X)) \simeq H^*(X)$ for V an elementary abelian group.

In this form, H. Miller proved the T-functor acts as identity on all unstable modules, which are locally finite. Let us recall the definition of locally finite modules:

Definition 5.2.3.3. (Locally finite module) A module M over the Steenrod algebra. We say it is locally finite if, for any $x \in M$, the A_p-submodule spanned by x is a finite dimensional $\mathbb{F}_p$-vector space. If X is a space which has locally finite cohomology, we say it is a locally finite space.

Let X be a nilpotent compact space, then the cohomology of the based loop space ΩX is a locally finite module over the Steenrod algebra.

[1] i.e., $X_p^\wedge \simeq \mathrm{map}(BV, X)_p^\wedge$

Motivated by this, we can ask if the Sullivan conjecture is valid for locally finite spaces for non-trivial actions.

The initial approach that we took to compute the $\mathbb{Z}/p$-fixed points of Gr_G using the theorem 5.2.3.2, but for the locally finite space, there are several difficulties in carrying out the computation:

- We might try to compute $H^*((\Omega K)_{h\mathbb{Z}/p})$ using the Serre spectral sequence applied to the fiber sequence

$$\begin{tikzcd} \Omega K \arrow[r] & \Omega K_{h\mathbb{Z}/p} \arrow[d] \\ & B\mathbb{Z}/p \end{tikzcd}$$

  but the cohomology $H^*(\Omega K)$ is a divided power algebra, and the Serre spectral sequence has infinitely many generators to analyse.
- Even if we succeed in computing the cohomology $H^*(\Omega K_{h\mathbb{Z}/p})$, the Serre spectral sequence is not a module over the Steenrod algebra, i.e. not all differential commutes with Steenrod operations. Hence figuring out the Steenrod module structure becomes impossible.
- Finally, this is under the assumption that the approach of theorem 5.2.3.2 computes the cohomology of genuine fixed points.

The $\mathbb{Z}/p$ genuine fixed points of Gr_G is computed by Williamson-Riche [RW22a] as

$$\mathrm{Gr}_G^{\mathbb{Z}/p} \simeq \bigsqcup_{\lambda \in X_\bullet / W_p} \mathcal{F}l_\lambda$$

where the indexing set is the quotient of co-root lattice by the p-dilated dot action (refer to [RW22a] for more details on the dot action).

In the example of $G = SL_3(\mathbb{C})$ and $K = SU(3)$, following remark 5.2.2.8 we have that

$$(\Omega SU(3))^{h\mathbb{Z}/2} \simeq \bigsqcup_{\rho \in \mathrm{Rep}(\mathbb{Z}/2, SU(3))} \Omega SU(3) \times SU(3)/C(\rho)$$

after 2-completion. In particular, the indexing set has cardinality two, given by id, the trivial representation and $diag(1, -1, -1)$. However, there are more than two W_p orbits in $X_\bullet$, which means the genuine fixed points has more path components than the homotopy fixed points, this provided a counter-example:

Proposition 5.2.3.4. For $SL_3(\mathbb{C})$, the genuine fixed points and the homotopy fixed points with respect to the rotation action restricted to $\mathbb{Z}/2$ on $\mathrm{Gr}_{SL_3(\mathbb{C})}$ are not equivalent.

Remark 5.2.3.5. Finally, we would like to point out that J. Hahn and A. Yuan in [HY19] construct a filtered $\mathbb{E}_2$-structure on Gr_G, where each of the filtered pieces is compact. One could hope to compute the cohomology of genuine fixed point using 5.2.3.2.

5.2.4 p-compact groups

In this section, we comment on what is the homotopy fixed points for certain p-compact groups. We will refer the reader to [Gro10] for a quick review.

Definition 5.2.4.1 (p-compact group)**.** [DW94] [Gro10, Definition 2.1] A p-compact groups is a triple (X, BX, e), such that $e : X \simeq \Omega BX$, where BX is a pointed, connected, p-completed space such that $\mathbb{F}_p$ cohomology $H^*(X; \mathbb{F}_p)$ is finite.

Example 5.2.4.2. Given a compact Lie group G, then $G_p^\wedge$ as an example of a p-compact group.

There are exotic examples which do not come from compact Lie groups. Here is an example constructed by Sullivan:

Example 5.2.4.3. Let p be a prime and d a divisor of $p-1$, then $\mathbb{Z}/d \leqslant \mathbb{Z}_p^*$. So there is a natural action of $\mathbb{Z}/d$ on $K(\mathbb{Z}_p, 2)$ giving a fiber sequence:

$$\begin{tikzcd} K(\mathbb{Z}_p, 2) \arrow[r] & K(\mathbb{Z}_p, 2)_{h\mathbb{Z}/d} \arrow[d] \\ & B\mathbb{Z}/d \end{tikzcd}$$

Using the Serre spectral sequence, we have the $\mathbb{F}_p$-cohomology of $K(\mathbb{Z}_p, 2)_{h\mathbb{Z}/d}$ is a polynomial algebra with a generator in degree $2d$.

We set

$$BY = K(\mathbb{Z}_p, 2)_{h\mathbb{Z}/d},$$

so $\Omega BY_p^\wedge$ has $\mathbb{F}_p$-cohomology an exterior algebra with generator in degree $2d-1$. Hence $\Omega BY_p^\wedge \simeq {S^{2d-1}}_p^\wedge$. So the triple $(\Omega BY_p^\wedge, BY_p^\wedge, e)$ is a p-compact group.

Remark 5.2.4.4. At first glance, this should be surprising because for an odd sphere of dimension greater than 7, by the Hopf invariant one problem, there is no H-space structure. But after p-completing at certain primes, you get an $\mathbb{E}_1$-structure. We refer to these as Sullivan spheres

The theory of p-compact groups is very rich. They behave very much like a compact Lie group. Given a p-compact group X, it admits a p-adic maximal torus [DW94], and to every p-compact group, we can associate a p-adic root datum. Similar to the classification theorem of Chevalley, p-compact groups are classified by their root datum [Gro10].

However, there is no well-developed representation theory of p-compact groups. We hope methods from geometric representation theory could help us investigate more in this direction. In this spirit, we have the following proposition:

Proposition 5.2.4.5. Let p be a prime, d divides $p-1$, then we have

$$(\Omega S^{2d-1})^{h\mathbb{Z}/p} \simeq \Omega S^{2d-1} \sqcup \coprod_{\frac{p-1}{d}} \Omega S^{2d-1} \times \mathbb{C}P^{d-1}$$

after p-completion on both sides.

Proof. There is $\frac{p-1}{d}+1$ representations of $\mathbb{Z}/p$ into $(S^{2d-1})^\wedge_p$ up to conjugation, i.e $\mathrm{Rep}(\mathbb{Z}/p,(S^{2d-1})^\wedge_p) = \frac{p-1}{d}+1$ where the centraliser for $\frac{p-1}{d}$ many of them is $(S^1)^\wedge_p$ and for the trivial representation the centraliser is $(S^{2d-1})^\wedge_p$. This implies we have an equivalence $K/C(\rho) \simeq \mathbb{C}P^{d-1} \simeq S^{2d-1}/S^1$ after p-completion in the case $C(\rho) = S^1_p$ and $K/C(\rho) \simeq *$ for the trivial representation. Hence, we have the pullback:

$$\begin{tikzcd}
(\Omega(S^{2d-1})^\wedge_p)^{h\mathbb{Z}/p} \arrow[r] \arrow[dd] & * \sqcup \coprod_{\frac{p-1}{d}} (\mathbb{C}P^{d-1})^\wedge_p \arrow[d] \\
 & * \arrow[d] \\
* \arrow[r] & (S^{2d-1})^\wedge_p
\end{tikzcd}$$

Since the dimension of $\mathbb{C}P^{d-1}$ is $2d-2$, the map $(\mathbb{C}P^{d-1})^\wedge_p \to S^{2d-1}_p$ is nullhomotopic because of cellular approximation. Hence we can compute the pullback as $(\Omega S^{2d-1})^\wedge_p \sqcup \coprod_{\frac{p-1}{d}} (\Omega S^{2d-1})^\wedge_p \times (\mathbb{C}P^{d-1})^\wedge_p$. □

6 Six functor formalism for equivariant sheaves

The material presented in this chapter is joint work with Oscar Bendix Harr.

6.1 Equivariant Sheaves and Six functor formalism

In this section, we will recall some definitions and properties of the sheaf category and introduce the six functor formalism for equivariant sheaves.

Much of the material about the non-equivariant setup here is known to experts; an ∞-cateogorical version can be found [Lur17; Lur09] in we present it here to have coherent documentation for the reader.

Let G be a topological group acting continuously on a topological space X. For a fixed discrete ring R, Bernstein and Lunts [BL94] define a triangulated category $D_G(X; R)$ of derived G-equivariant R-valued sheaves on X. When the group G is locally compact Hausdorff, they also show that the assignment $X \mapsto D_G(X; R)$ can be fleshed out into a six-functor formalism for equivariant sheaves on locally compact Hausdorff G-spaces, generalising the non-equivariant six-functor formalism of Verdier [Ver65], and thus providing a natural context for understanding, among other things, certain equivariant duality phenomena. More specifically, the Bernstein–Lunts category $D_G(X; R)$ sees Borel-type properties of the G-space X, in the sense that if one takes the derived global sections of a constant equivariant sheaf on a nice space X, one gets the Borel cohomology of X.

In this paper, we briefly show how to generalise the construction of Bernstein–Lunts to allow coefficients in an ∞-category. In particular, when the coefficient ∞-category is stable, one gets a full[1] six-functor formalism for equivariant sheaves by invoking the general results in an appendix of [Man22].

6.1.1 The ∞-topos of Borel Equivariant Sheaves

Let G be a topological group. In this section, we introduce the 'non-abelian Borel equivariant derived category' $Shv_{hG}(X)$ associated with a G-space X, which is a Borel equivariant version of the non-abelian derived category $\mathrm{Shv}(X)$ of sheaves of anima on X. The objects of $\mathrm{Shv}_{hG}(X)$ are, informally, G-equivariant sheaves of anima on X. Although, in general, $\mathrm{Shv}_{hG}(X)$ cannot be realised as $\mathrm{Shv}(Y)$ for a topological space Y, it is an ∞-topos, meaning that it enjoys many of the same properties as the ∞-category of sheaves of anima on a topological space, e.g.

[1] having all the coherence

- To an equivaraint sheaf $\mathcal{F} \in \mathrm{Shv}_{hG}(X)$ we can associate an anima $\Gamma(X, \mathcal{F})$ of 'derived global sections'.
- For an arbitrary ∞-category $\mathcal{C}$, we can define an ∞-category of $\mathcal{C}$-valued sheaves on $\mathrm{Shv}_{hG}(X)$, which we will denote by $\mathrm{Shv}_{hG}(X;\mathcal{C}) := \mathrm{Shv}(\mathrm{Shv}_{hG}(X);\mathcal{C})$. If $\mathcal{C}$ happened to be an ordinary category, then $\mathrm{Shv}_{hG}(X;\mathcal{C})$ precisely recovers the classical notion of equivariant $\mathcal{C}$-valued sheaves on X, and if $\mathcal{C} = \mathrm{D}(R)$ is the unbounded derived category of a ring R, then the triangulated category $h\mathrm{Shv}_{hG}(X;\mathrm{D}(R))$ agrees for reasonable spaces with the unbounded derived category of equivariant sheaves studied by Bernstein and Lunts [BL94].
- The ∞-topos $\mathrm{Shv}_{hG}(X)$ has a homotopy type, namely its shape $\Pi_\infty(\mathrm{Shv}_{hG}(X)) \in \mathrm{Pro}(\mathcal{S})$, which, when X and G are nice, parametrizes locally constant equivariant sheaves on X.

By a purely formal calculation, it will turn out that in good cases, we have

$$\Pi_\infty(\mathrm{Shv}_{hG}(X)) \simeq \Pi_\infty(X)_{hG},$$

where $\Pi_\infty(X)_{hG}$ is the Borel construction on $\Pi_\infty(X)$. By invoking Lurie's monodromy equivalence for ∞-topoi [Lur17, Theorem A.4.2], this will allow us to interpret the Borel cohomology of a well-behaved G-space X with respect to some Borel equivariant spectrum E as the homotopy groups of the spectrum of derived global sections $\Gamma(X; E_X)$ of a certain equivariant spectral sheaf E_X on X, which we think of as a 'constant sheaf at E'.

Definition and basic properties

Let G be a topological group acting continuously on X. The *action groupoid* associated to this action is a topological groupoid with object space X, and such that morphisms from x to x' correspond to elements $g \in G$ with $gx = x'$. We denote the bar construction of the action groupoid, which is a simplicial space, by

$$[X /\!\!/ G]_\bullet \quad = \quad \cdots \; \vdots \; G \times G \times X \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; G \times X \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X$$

So the face maps are given by

$$d_i\colon (g_0, \ldots, g_{p-1}, x) \mapsto \begin{cases} (g_0, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{p-1}, x) & \text{if } 0 \leqslant i < p, \\ (g_0, \ldots, g_{p-2}, g_{p-1}x) & \end{cases}$$

and the degeneracy maps are the obvious ones.

Let $\mathrm{Shv}_* : \mathrm{Top} \to \mathrm{RTop}_\infty$ denote the covariant functor given informally by sending a space X to the ∞-category of $\mathcal{S}$-valued sheaves $\mathrm{Shv}(X)$, and by sending a morphism $f : X \to Y$ to the pushforward $f_* : \mathrm{Shv}(X) \to \mathrm{Shv}(Y)$. We then define:

Definition 6.1.1.1. Let X be a G-space, where G is a topological group. We put

$$\mathrm{Shv}_{hG}(X) := \varinjlim_{\Delta^{op}} \mathrm{Shv}_*([X/G]_\bullet),$$

where the colimit is taken in the ∞-category of ∞-topoi RTop_∞ and morphism geometric morphisms.

For an arbitrary ∞-category $\mathcal{C}$, we define

$$\mathrm{Shv}_{hG}(X;\mathcal{C}) := \mathrm{Fun}^{\mathrm{lex}}(\mathrm{Shv}_{hG}(X),\mathcal{C}),$$

where $\mathrm{Fun}^{\mathrm{lex}}(\mathrm{Shv}_{hG}(X),\mathcal{C}) \subseteq \mathrm{Fun}(\mathrm{Shv}_{hG}(X),\mathcal{C})$ is the full subcategory spanned by functors that preserve small limits.

That is, $\mathrm{Shv}_{hG}(X;\mathcal{C})$ is the ∞-category $\mathrm{Shv}(\mathrm{Shv}_{hG}(X);\mathcal{C})$ of $\mathcal{C}$-valued sheaves on the ∞- topos $\mathrm{Shv}_{hG}(X)$ in the sense of [Lur09]. We think of $\mathrm{Shv}_{hG}(X;\mathcal{C})$ as the ∞-category of Borel equivariant $\mathcal{C}$-valued sheaves on X. The Yoneda embedding gives an equivalence

$$\mathrm{Shv}_{hG}(X) \xrightarrow{\simeq} \mathrm{Shv}_{hG}(X;\mathcal{S}),$$

so $\mathrm{Shv}_{hG}(X)$ is the ∞-topos of Borel equivariant sheaves of anima on X.

One should think of $\mathrm{Shv}_{hG}(X)$ as the non-abelian Borel equivariant derived category of X, in the same way that for a non-equivariant space X, the ∞-topos $\mathrm{Shv}(X)$ of sheaves of anima on X is the non-abelian derived category of X [Lur09], cf. [Gir71].

Comparison with classical notions

When $\mathcal{C}$ is an ordinary category, there is a classical notion of an equivariant $\mathcal{C}$-valued sheaf on X. These form the objects of an 1-category, which we will temporarily denote by $\mathfrak{S}\mathrm{hv}_G(X;\mathcal{C})$. We first briefly recall the definition of this category. Recall the simplicial space $[X \mathbin{/\!\!/} G]_\bullet : \Delta^{op} \to \mathrm{Top}$ from above. Let $\Delta_{\leqslant 2} \subset \Delta$ denote the full subcategory spanned by the ordinals $\mathbf{0}$,$\mathbf{1}$ and $\mathbf{2}$. We let $[X \mathbin{/\!\!/} G]_{\bullet \leqslant 2} : \Delta^{op}_{\leqslant 2} \to \mathrm{Top}$ denote the precomposition of $[X \mathbin{/\!\!/} G]_\bullet$ with the inclusion $\Delta_{\leqslant 2} \subset \Delta$. Then

$$\mathfrak{S}hv_G(X;C) := \varprojlim_{\Delta_{\leqslant 2}} \mathrm{Shv}^*([X \mathbin{/\!\!/} G]_{\bullet \leqslant 2})$$

where the limit is the $(2,1)$-limit in the $(2,1)$-category of ordinary categories. Thus, objects of $\mathfrak{S}hv_G(X;C)$ are pairs $(\mathcal{F},\theta)$ where $\mathcal{F} \in \mathrm{Shv}(X;C)$ and $\theta : d_1^*\mathcal{F} \xrightarrow{\simeq} d_0^*\mathcal{F}$ is an equivalence in $\mathrm{Shv}(X \times G;C)$ such that the following diagrams commute

$$\begin{tikzcd}
s_0^*d_1^*\mathcal{F} \arrow[r] \arrow[d, "\simeq"'] & s_0^*d_0^*\mathcal{F} \arrow[d, "\simeq"] \\
\mathcal{F} \arrow[r] & \mathcal{F}
\end{tikzcd}$$

$$\begin{tikzcd}
d_2^*d_1^*\mathcal{F} \arrow[r, "d_2^*(\theta)"] \arrow[d, "\simeq"'] & d_2^*d_0^*\mathcal{F} \arrow[r, "\simeq"] & d_0^*d_1^*\mathcal{F} \arrow[r, "d_0^*(\theta)"] & d_0^*d_0^*\mathcal{F} \arrow[d, "\simeq"] \\
d_1^*d_1^*\mathcal{F} \arrow[rrr, "d_1^*(\theta)"] & & & d_1^*d_0^*\mathcal{F}
\end{tikzcd}$$

Proposition 6.1.1.2. Let $\mathcal{C}$ be an ordinary category. Then there is an equivalence

$$\mathrm{Shv}_{hG}(X;C) \simeq \mathfrak{S}_G(X;\mathcal{C})$$

In particular the ∞-category $\mathrm{Shv}_{hG}(X;C)$ is 1-truncated.

More generally for $\mathcal{C}$ an $(n,1)$-category where $n < \infty$, then $\mathrm{Shv}_{hG}(X;C)$ is equivalent to the $(n+1,1)$-limit

$$\varprojlim_{\Delta_{\leqslant n+1}} \mathrm{Shv}^*(X;\mathcal{C})$$

computed in the $(n+1,1)$-category of $(n,1)$-categories. This follows from the following description of the $\mathcal{C}$-valued equivariant sheaves on a topological space X:

Proposition 6.1.1.3. Let $\mathcal{C}$ be an ∞-category. There is an equivalence

$$\mathrm{Shv}_{hG}(X;\mathcal{C}) \simeq \varprojlim_{\Delta} \mathrm{Shv}^*([X \,/\!\!/\, G]_\bullet;\mathcal{C})$$

where the limit is computed in the ∞-category Cat_∞ of ∞-categories, and $\mathrm{Shv}^*(-;\mathcal{C}) : \mathrm{Top} \to \mathrm{Cat}_\infty$ is the contravariant functor given by $X \mapsto \mathrm{Shv}(X;\mathcal{C})$ on objects, and by sending $f : X \to Y$ to the pullback $f^* : \mathrm{Shv}(Y;\mathcal{C}) \to \mathrm{Shv}(X;\mathcal{C})$.

Proof. We have

$$\begin{aligned}\mathrm{Shv}_{hG}(X;\mathcal{C}) &\simeq \mathrm{Fun}^{\mathrm{lex}}(\mathrm{Shv}_{hG}(X),\mathcal{C})\\ &\simeq \mathrm{Fun}^{\mathrm{lex}}(\varprojlim_{\Delta} \mathrm{Shv}^*([X \,/\!\!/\, G]_\bullet),\mathcal{C})\\ &\simeq \varprojlim_{\Delta} \mathrm{Fun}^{\mathrm{lex}}(\mathrm{Shv}^*([X \,/\!\!/\, G]_\bullet),\mathcal{C}),\end{aligned}$$

where $\mathrm{Shv}^*\colon \mathrm{Top}^{\mathrm{op}} \to \mathrm{LTop}_\infty$ denotes the functor given informally by $X \mapsto \mathrm{Shv}(X)$ on objects and $f \mapsto f^*$ on morphisms. Equivalently, Shv^* is the composition of the functor Shv_* described earlier with the antiequivalence $\mathrm{RTop}_\infty^{\mathrm{op}} \simeq \mathrm{LTop}_\infty$ of [Lur09, Cor 6.3.1.8]. □

Funtoriality

The construction of the ∞-topos $\mathrm{Shv}_{hG}(X;\mathcal{C})$ is functorial over the G-space X; indeed it is the composition of functors

$$G\text{-}\mathrm{Top} \xrightarrow{[-/\!\!/G]_\bullet} \mathrm{Top}^{\Delta^{op}} \xrightarrow{\mathrm{Shv}_*} \mathrm{RTop}_\infty^{\Delta^{op}} \xrightarrow{\varprojlim} \mathrm{RTop}_\infty \xrightarrow{\mathrm{Fun}^{\mathrm{lex}}} \mathrm{Cat}_\infty$$

which to a G-equivariant map $f : X \to Y$ induces a geometric morphism that we will denote by

$$f_* : \mathrm{Shv}_{hG}(X) \to \mathrm{Shv}_{hG}(Y) \tag{pushforward}$$

similarly, the left adjoint of f_* will denoted by

$$f^* : \mathrm{Shv}_{hG}(Y) \to \mathrm{Shv}_{hG}(X) \tag{6.1}$$

The ∞-topos $\mathrm{Shv}_{hG}(X)$ is also functorial in the topological group G. Given a morphism $\varphi : H \to G$ of groups and a G-space X, we can view X as a H-space by restricting along φ. There is then a geometric morphism

$$\mathrm{Res}_\varphi : \mathrm{Shv}_{hH}(X) \to \mathrm{Shv}_{hG}(X) \tag{restriction}$$

which is induced by the map of simplicial spaces $[X /\!\!/ H]_\bullet \to [X /\!\!/ G]_\bullet$ given by $\varphi^n \times X : H^{\times n} \times X \to G^{\times n} \times X$ on n-simplicies. If φ is an inclusion, we will also write Res^G_H instead of Res_φ.

We can view the functionality in the space and group direction simultaneously by considering $\mathrm{Shv}_{h(-)}(-)$ as a functor on the global equivariant category $\mathrm{Top}_{\mathrm{glob}}$. Here we define $\mathrm{Top}_{\mathrm{glob}}$ to be the ordinary category which has, as its objects, pairs (G, X), where G is a topological group and X is a G-space, and such that a map from (H, Y) to (G, X) is a pair (f, φ) where $f : Y \to X$ is a continuous map and $\varphi : H \to G$ is a continuous homomorphism, satisfying that $f(hy) = \varphi(h)f(y)$.

Then there is a functor $\mathrm{Shv}_{h(-)}(-) : \mathrm{Top}_{\mathrm{glob}} \to \mathrm{RTop}_\infty$ given informally on objects by sending (G, X) to $\mathrm{Shv}_{hG}(X)$ and on morphisms by sending $(f, \varphi) : (H, Y) \to (G, X)$ to the geometric morphism

$$(f, \varphi)_* : \mathrm{Shv}_{hH}(Y) \to \mathrm{Shv}_{hG}(H)$$

induced by the map of simplicial spaces $[Y /\!\!/ H]_\bullet \to [X /\!\!/ G]_\bullet$ given by $\varphi^{\times n} \times f : H^{\times n} \times Y \to G^{\times n} \times X$ on n-simplices. Just as when we had a fixed group, functoriality follows from the fact that $\mathrm{Shv}_{h(-)}(-)$ is by construction equal to the composition

$$\mathrm{Top}_{\mathrm{glob}} \xrightarrow{[-/-]_\bullet} \mathrm{Top}^{\Delta^{op}} \xrightarrow{\mathrm{Shv}_*(-)} \mathrm{RTop}_\infty^{\Delta^{op}} \xrightarrow{\lim} \mathrm{RTop}_\infty,$$

Here it is easy to check that the first arrow is a functor as both the source and target are ordinary categories.

Relation to $\mathrm{Shv}(X)$ and $\mathrm{Shv}(X/G)$

Given a G-space X, the simplicial space $[X /\!\!/ G]_\bullet$ admits an augmentation $[X /\!\!/ G]_\bullet \to X/G$. Hence there is an induced geometric morphism

$$\mathrm{Shv}_{hG}(X) \to \mathrm{Shv}\, X/G, \tag{6.2}$$

natural in X, which compares equivariant sheaves on X to non-equivariant sheaves on the orbit space X/G. On the other hand, the universal cone for $\mathrm{Shv}_{hG}(X)$ come with a geometric morphism

$$\mathrm{Shv}(X) \to \mathrm{Shv}_{hG}(X); \tag{6.3}$$

we think of the left adjoint of this morphism as the functor which *forgets* that a sheaf $\mathcal{F} \in \mathrm{Shv}_{hG}(X)$ is equivariant. Indeed, the right adjoint can alternatively be described as the restriction map $\mathrm{Res}^G_e : \mathrm{Shv}_{hG}(X) \to \mathrm{Shv}_{he}(X) \simeq \mathrm{Shv}(X)$, where $e \in G$ is the identity element. Both (6.2) and (6.3) are natural in X.

Principal G-spaces

In general, the comparison maps given above are both very far from being equivalences. Nevertheless, there is a condition which ensures that (6.2) is an equivalence, and this observation will furnish us with our first calculations of the ∞-topos of equivariant sheaves. For us, a *principal G-space* is a G-space X such that X can be covered by G-invariant open sets U, where U is G-equivariantly homeomorphic to a G-space of the form $G \times V$, where G acts by left multiplication on the G-factor and trivially on the

V-factor. In particular, any principal G-space is free, but a free G-space need not be principal.

We then have:

Proposition 6.1.1.4 (Torsor descent)**.** Let G be a topological group and X be principal G-space. Then (6.2) gives an equivalence

$$\mathrm{Shv}_{hG}(X) \simeq \mathrm{Shv}(X/G)$$

with the ∞-topos of non-equivariant sheaves on X/G.

Proof. Let $p : X \to X/G =: \bar{X}$ denote the projcetion, and let $N_\bullet$ denotes its Čech nerve

$$N_\bullet \quad = \quad \cdots \; \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \vdots \\ \longleftarrow\\ \longrightarrow\end{smallmatrix} \; X \times_{\overline{X}} X \times_{\overline{X}} X \; \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} \; X \times_{\overline{X}} X \; \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} \; X$$

Using that G acts freely on X, we find that there is an isomorphism of simplicial spaces $[X /\!\!/ G]_\bullet \to N_\bullet$ over $\overline{X}$, given on n-simplices by

$$(x, g_1, g_2, ..., g_n) \to (x, g_1 x, ..., g_n x),$$

so the statement of the proposition is equivalent to the claim that the projection p is of descent type. But this follows from the fact that p has local sections. $\square$

Note that torsor descent is the categorified version of the statement that the equivariant cohomology of a principal G-space X is the same as the ordinary cohomology of X/G.

6.1.2 The six operations for equivariant sheaves

Above we showed how to construct, for any ∞-category $\mathcal{C}$, a theory of G-equivariant $\mathcal{C}$-valued sheaves on G-spaces, which to any G-space X associates the ∞-category $\mathrm{Shv}_{hG}(X;\mathcal{C})$ and to a continuous function $f\colon X \to Y$ associates the adjunction

$$f_*\colon \mathrm{Shv}_{hG}(X;\mathcal{C}) \rightleftarrows \mathrm{Shv}_{hG}(Y;\mathcal{C}) : f^*. \qquad \text{(pushforward/pullback)}$$

As in the non-equivariant case, we can say a great deal more if the coefficient category $\mathcal{C}$ is *stable* (and we restrict ourselves to considering a locally compact Hausdorff group G acting continuously on locally compact Hausdorff spaces). In that case, one also has, for every an adjunction

$$f^!\colon \mathrm{Shv}_{hG}(Y;\mathcal{C}) \rightleftarrows \mathrm{Shv}_{hG}(X;\mathcal{C}) : f_!, \qquad \text{(exceptional pushforward/pullback)}$$

encoding equivariant Verdier duality. If $\mathcal{C}$ is furthermore presentably symmetric monoidal, in the sense that $\mathcal{C} \in \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Pr}^{L,\otimes}_{\mathrm{stab}})$, then there are two more operations: the functor $\mathrm{Shv}_{hG}(-;\mathcal{C})\colon \mathrm{Top}^{\mathrm{op}} \to \mathrm{Pr}^{L}_{\mathrm{stab}}$ factors through the forgetful functor $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Pr}^{L,\otimes}_{\mathrm{stab}}) \to \mathrm{Pr}^{L}_{\mathrm{stab}}$, i.e. for each G-space X the ∞-category $\mathrm{Shv}_{hG}(X;\mathcal{C})$ is canonically a presentably symmetric monoidal ∞-category, and hence in particular *closed* symmetric monoidal, giving two more operations: the tensor product of sheaves $\otimes_X$ and the internal Hom-functor Hom_X. Together, these six operations form a 'six functor formalism' after the philosophy of Grothendieck–Verdier. For the classical choice of coefficient category

$\mathcal{C} = D(R)$, the six functor formalism for equivariant sheaves was described by Bernstein and Lunts [BL94] in the language of triangulated categories. Recent work of [LZ17; Man22] has clarified what is needed to construct a six functor formalism, with all the desired compatibilities and coherence, and in this subsection we will describe the six operations for equivariant sheaves from this perspective.

We start by briefly recalling the framework developed in [Man22]. A *geometric setup* is a pair $(\mathcal{T}, E)$ where $\mathcal{T}$ is an ∞-category admitting all pullbacks and E is a class of morphisms in $\mathcal{T}$, subject to three conditions: (i) E contains all equivalences; (ii) E is stable under composition; and (iii) E is stable under pullback along any morphism in $\mathcal{T}$.[2] We think of $\mathcal{T}$ as being a category of 'geometric objects' (e.g. spaces or stacks with extra structure), and E as being a class of 'admissible' morphisms between these geometric objects.[3] Associated to the geometric setup $(\mathcal{T}, E)$ is a *category of correspondences*, which [Man22] denotes by $\mathrm{Corr}(\mathcal{T})_{E,\mathrm{all}}$. The objects of $\mathrm{Corr}(\mathcal{T})_{E,\mathrm{all}}$ are the same as those of $\mathcal{T}$, and a morphism from X to Y is given by a diagram

$$X \leftarrow X' \rightarrow Y,$$

where $X' \rightarrow Y$ lies in E.[4] A 2-simplex in $\mathrm{Corr}(\mathcal{T})_{E,\mathrm{all}}$ is a commutative diagram

$$\begin{tikzcd}
 & & X' \times_Y Y' \arrow[dl] \arrow[dr] & & \\
 & X' \arrow[dl] \arrow[dr] & & Y' \arrow[dl] \arrow[dr] & \\
X & & Y & & Z
\end{tikzcd}$$

in $\mathcal{C}$, where the top square is a pullback; in $\mathrm{Corr}(\mathcal{T})_{E,\mathrm{all}}$, the 2-simplex corresponding to this diagram has vertices X, Y, and Z, and edges $X \leftarrow X' \rightarrow Y$, $Y \leftarrow Y' \rightarrow Z$, and $X \leftarrow X' \times_Y Y' \rightarrow Z$. In particular, although the geometric setups that we have in mind—namely, $\mathcal{T} = G$-Top for some topological group G—are ordinary categories, the category of correspondences $\mathrm{Corr}(\mathcal{T})_{E,\mathrm{all}}$ is not, essentially because pullbacks are only unique up to equivalence, so the composition of two spans is only defined up to equivalence.

We will identify $\mathcal{T}^{\mathrm{op}}$ with the wide subcategory of $\mathrm{Corr}(\mathcal{T})_{E,\mathrm{all}}$ whose morphisms are exactly those spans of the form

$$X \leftarrow Y \xrightarrow{\mathrm{id}} Y.$$

Similarly, letting $\mathcal{T}_E$ denote the wide subcategory of $\mathcal{T}$ whose morphisms are exactly E, then $\mathcal{T}_E$ will be identified with the wide subcategory of $\mathrm{Corr}(\mathcal{T})_{E,\mathrm{all}}$ whose morphisms are spans of the form

$$X \xleftarrow{\mathrm{id}} X \rightarrow Y.$$

The category $\mathrm{Corr}(\mathcal{T})_{E,\mathrm{all}}$ has the canonical structure of an ∞-operad

$$\mathrm{Corr}(\mathcal{T})^{\otimes}_{E,\mathrm{all}} \rightarrow \mathrm{Fin}_*. \tag{6.4}$$

[2] Note that (i) and (iii) imply in particular that if $f \in E$ and $g \simeq f$, then $g \in E$.

[3] In topological contexts such as ours, one typically wants E to be the class of all morphisms in $\mathcal{C}$. The flexibility of allowing E to be a smaller class of morphism is needed in algebraic geometry, where it may be the case for instance that the exceptional functors $f_!$ and $f^!$ are only defined for *separated* morphisms.

[4] The reason for the subscript 'all' is that no condition is imposed on the morphism $X \leftarrow X'$. A more general construction specifies a class of morphisms F to which $X \leftarrow X'$ must belong as well, but this is not relevant here.

If finite products exist in $\mathcal{T}$, then (6.4) is a cocartesian fibration, so the category of correspondences $\mathrm{Corr}(\mathcal{T})_{E,\mathrm{all}}$ receives in fact the structure of a symmetric monoidal ∞-category. In that case, the monoidal structure is given informally on objects by $X \otimes Y = X \times Y$, i.e. it is simply the product in $\mathcal{T}$. More generally, the inclusion $\mathcal{T}^{\mathrm{op}} \subseteq \mathrm{Corr}(\mathcal{T})_{E,\mathrm{all}}$ admits canonically the structure of a map of ∞-operads

$$(\mathcal{T}^{\mathrm{op}})^{\sqcup} \to \mathrm{Corr}(\mathcal{T})^{\otimes}_{E,\mathrm{all}}, \tag{6.5}$$

where $(\mathcal{T}^{\mathrm{op}})^{\sqcup}$ is the coproduct monoidal structure on $\mathcal{T}^{\mathrm{op}}$, which in case products exist in $\mathcal{T}$ is given by those.

A *pre-six-functor formalism* [Man22, Def A.5.7] is a map of ∞-operads

$$\mathrm{SH}\colon \mathrm{Corr}(\mathcal{T})^{\otimes}_{E,\mathrm{all}} \to \mathrm{Cat}^{\times}_{\infty},$$

where $\mathrm{Cat}^{\times}_{\infty}$ denotes the symmetric monoidal ∞-category of large ∞-categories, with the product monoidal structure. We think of $\mathrm{SH}(X)$ as being an ∞-category of 'sheaves' on the geometric object $X \in \mathcal{T}$, compatible with the geometric structure. Note that the datum of a map of ∞-operads as above gives rise to the following:

- By restricting along $(\mathcal{T}^{\mathrm{op}})^{\sqcup} \to \mathrm{Corr}(\mathcal{T})^{\otimes}_{E,\mathrm{all}}$, we get in a particular a functor of ∞-operads
$$\mathrm{SH}^*\colon (\mathcal{T}^{\mathrm{op}})^{\sqcup} \to \mathrm{Cat}^{\times}_{\infty},$$
meaning in particular that for each $X \in \mathcal{T}$, the category $\mathrm{SH}(X)$ admits canonically the structure of a symmetric monoidal category. Furthermore, given a morphism $f\colon X \to Y$, we write
$$f^* = \mathrm{SH}^*(f)\colon \mathrm{SH}^*(Y)^{\otimes} \to \mathrm{SH}^*(X)^{\otimes}$$
for the corresponding symmetric monoidal functor, which we refer to as the *pull-back* along f.
- By restricting along $\mathcal{T}_E \subseteq \mathrm{Corr}(\mathcal{T})_{E,\mathbf{all}}$, we get a functor
$$\mathrm{SH}_!\colon \mathcal{T}_E \to \mathrm{Cat}^{\mathrm{op}}_{\infty}.$$
Given $f\colon X \to Y$ in E, we will write
$$f_! = \mathrm{SH}_!(f)\colon \mathrm{SH}^*(X) \to \mathrm{SH}^*(Y)$$
for the corresponding functor, which we refer to as the *exceptional pushforward* along f.

A pre-six-functor formalism SH as above is said to be a *six-functor formalism* if it satisfies the following two conditions:

(i) For each $X \in \mathcal{T}$, the symmetric monoidal ∞-category $\mathrm{SH}(X)$ is closed, with corresponding internal Hom-functor denoted Hom; and

(ii) The functors f^* and $f_!$ admit right adjoints, which we denote f_* and $f^!$ respectively.[5]

[5] Although right adjoints are not uniquely defined, it is possible to make a coherent choice; i.e. there is an antiequivalence $\mathrm{Cat}^R_\infty \simeq (\mathrm{Cat}^L_\infty)^{\mathrm{op}}$, where Cat^R_∞ resp. Cat^L_∞ denotes the ∞-category of ∞-categories with functors that admit left adjoints resp. right adjoints between them.

It is in general hopeless to try to build a morphism of ∞-operads $\mathrm{SH}\colon \mathrm{Corr}(\mathcal{T})^{\otimes}_{E,\mathbf{all}} \to \mathrm{Cat}_{\infty}^{\times}$ cell-wise by hand. The key contribution of [LZ17; Man22] is to give various usable machines for producing such a functor from the in practice often readily available datum of a functor $\mathrm{SH}^*\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Cat}_{\infty}$. As is sometimes the case in higher category theory, it then turns out that for all the necessary coherence structure to exist, it is in good cases only necessary to make certain pointwise checks.

Although several machines for constructing six-functor formalisms are described in [Man22], we only have need of one. A *suitable decomposition* of a geometric setup $(\mathcal{T}, E)$ is a pair of subsets $I, P \subseteq E$, such that (i) each $f \in E$ admits a factorization $f \simeq p \circ i$ for $p \in P$ and $i \in I$; (ii) each $f \in I \cap P$ is n-truncated for some $n \geqslant -2$; (iii) I and P contain all equivalences and are stable under pullback along all morphisms in $\mathcal{T}$; and (iv) if $g \in I$ (resp. $g \in P$) and

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "g\circ f"] & & Z \end{tikzcd}$$

is a 2-cell, then $g \circ f \in I$ (resp. $g \circ f \in P$) if and only if $f \in I$ (resp. $f \in P$). In applications, I will typically be a suitable class of ‘immersions’ and P will be a suitable class of ‘proper maps’, hence the notation. Mann then shows:

Theorem 6.1.2.1 ([Man22])**.** Suppose $\mathrm{SH}^*\colon (\mathcal{T}^{\mathrm{op}})^{\sqcup} \to \mathrm{Cat}_{\infty}^{\times}$ is a morphism of ∞-operads, and write $f^* = \mathrm{SH}^*(f)$ for each morphism f in $\mathcal{T}$. Suppose that:

(i) For each $j\colon U \to X$ in I, the pullback j^* admits a left adjoint $j_!$ such that

- for each $f\colon Y \to X$, the diagram

$$\begin{tikzcd} \mathrm{SH}(X) \arrow[r, "f^*"] \arrow[d, "i^*"] & \mathrm{SH}(Y) \arrow[d] \\ \mathrm{SH}(U) \arrow[r] & \mathrm{SH}(U \times_X Y) \end{tikzcd},$$

formed by applying SH^* to a pullback square of $U \to X \leftarrow Y$, is left adjointable; and

- for all $F \in \mathrm{SH}(U)$ and $G \in \mathrm{SH}(X)$, the canonical map

$$j_!(F \otimes j^*G) \to j_!F \otimes G$$

is an equivalence.

(ii) For each $p\colon Y \to X$ in P, the pullback p^* admits a right adjoint p_* such that

- for each $g\colon X' \to X$, the diagram

$$\begin{tikzcd} \mathrm{SH}(X) \arrow[r, "f^*"] \arrow[d, "i^*"] & \mathrm{SH}(X') \arrow[d] \\ \mathrm{SH}(Y) \arrow[r] & \mathrm{SH}(Y \times_X X') \end{tikzcd},$$

formed by applying SH^* to a pullback square of $Y \to X \leftarrow X'$, is right adjointable; and

- for all $F \in \mathrm{SH}(Y)$ and $G \in \mathrm{SH}(X)$, the canonical map

$$p_*(F \otimes j^*G) \to p_*F \otimes G$$

  is an equivalence.

(iii) For each pullback square

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "q"] & Y \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

in $\mathcal{T}$ in which $i, j \in E$ and $p, q \in P$, the canonical map $i_! q_* \to p_* j_!$ is an equivalence.

Then SH^* extends canonically to a pre-six functor formalism, in the sense that there is a commutative diagram

$$\begin{tikzcd} \mathrm{Corr}(\mathcal{T})^{\otimes}_{E,\mathbf{all}} \arrow[dr, dashed, "\mathrm{SH}"] & \\ (T^{\mathrm{op}})^{\sqcup} \arrow[u] \arrow[r, "\mathrm{SH}^*"] & \mathrm{Cat}_\infty^\times \end{tikzcd}$$

in the ∞-category of ∞-operads.

As the difference between a six-functor formalism and a pre-six-functor formalism is also just a short list of pointwise checks, one gets a usable machine for constructing six-functor formalisms.

The conditions of the theorem are easily verified for the functor $\mathrm{Shv}^*_{hG}(-;\mathcal{C})$ constructed above, where now $\mathcal{C}$ is some presentably symmetric monoidal stable ∞-category such as the unbounded derived category $D(R)$ for an ordinary ring R or more generally the ∞-category of modules Mod_R over an $\mathbb{E}_\infty$-ring R. For our geometric setup, we take $\mathcal{T} = G$-LCHaus to be the category of locally compact Hausdorff G-spaces for a locally compact Hausdorff group G and $E = \text{all}$ to be the set of all morphisms. We let I resp. P be the class of G-equivariant maps that are immersions resp. proper on underlying spaces.

Proposition 6.1.2.2. The functor $\mathrm{Shv}_{hG}\colon (G\text{-LCHaus}^{\mathrm{op}})^{\sqcup} \to \mathrm{Cat}_\infty^\times$ satisfies the conditions of the preceding theorem.

Proof. We first check that maps from I and P give rise to the desired adjoints. We have already seen that for any G-equivariant map $p\colon X \to Y$, the pullback p^* has a right adjoint p_*. If p is proper, the required base change property follows from the non-equivariant case and [Lur17, Cor 4.7.4.18]. The projection formula holds non-equivariantly, and hence after each projection from the limit $\mathrm{Shv}_{hG}(X;\mathcal{C}) = \varprojlim_\Delta \mathrm{Shv}([X /\!\!/ G])_\bullet$; since the projections are jointly conservative, this shows that it holds in the category of equivariant sheaves. Similarly arguments reduce the pointwise verifications for morphisms in I to the non-equivariant case, where they are known to hold. The only part of the proof which is not an argument of this type is the verification that each morphism f in G-Top admits a factorization of the prescribed form, namely $f = p \circ i$ where p is proper and i is an immersion. Here we write f as the composition

$$X \hookrightarrow \beta_G X \times_{\beta_G Y} Y \to Y,$$

where β_G- is the G-equivariant Stone-Čech compactification (see the next section).

The map $\beta_G X \times_{\beta_G Y} Y \to Y$ is pullback of the proper map $\beta_G X \to \beta_G(Y)$ hence it is proper. By Corollary 6.1.3.8 we have the map $Y \hookrightarrow \beta_G Y$ is an open immersion hence we have $\beta_G X \times_{\beta_G Y} Y \to \beta_G X$ is an immersion and again the map $X \hookrightarrow \beta_G X$ is an immersion this gives us the map $X \hookrightarrow \beta_G X \times_{\beta_G Y} Y$ is an immersion by the 2 out of 3 property of immersion. $\square$

6.1.3 Equivariant Stone-Cech compactification

Equivariant compactifications have been constructed in high generality by Jan de Vries [Vri77]. For our purposes, we may assume that every space in sight is locally compact Hausdorff, which makes several proofs easier. We include these proofs for completeness.

Let G be a topological group acting continuously on a locally compact Hausdorff space X. There is a canonical action of G on the non-equivariant Stone–Cech compactification βX. Recall that βX is the maximal ideal space of the C*-algebra $\mathrm{C}^b(X)$ of bounded continuous complex valued functions on X. Thus G acts on $C^b(X)$ from the right by sending $(f,t)C^b(X) \times G$ to the function $f^t(x) = f(tx)$. However, this action and the resulting action on βX fail to be continuous even for good transformation groups.

The equivariant Stone–Cech compactification will instead be constructed as the maximal ideal space of a certain C^*-subalgebra of $C^b(X)$, on which the canonical discontinuous G-action restricts to a continuous action:

Definition 6.1.3.1. A function $f \in C^b(X)$ is right-uniform continuous if the map $G \to C^b(X)$ given by $t \to f^t$ is continuous. We denote by $C^b_{ru}(X) \subset C^b(X)$ the subset consisting of right-uniform continuous functions.

Lemma 6.1.3.2. $\mathrm{C}^b_{ru}(X)$ ia a unital C*-subalgebra of $\mathrm{C}^b(X)$

Proof. It is clear that $C^b_{ru}(X)$ is an unital $*$-subalgebra, so it suffices to show that it is closed. For this, suppose $f_\lambda \to f$ is a convergent net in $C^b(X)$ with each f_λ a right-uniform continuous function. Let $t_\mu \to t$ be a net in G. We must show that $f^{t_\mu} \to f^t$ in $C^b(X)$. Given $\epsilon > 0$, pick λ so that $\|f - f_\lambda\|_u < \epsilon/3$. Since f_λ is right-uniform continuous, we have for sufficiently large μ that $\|f^t_\lambda - f^t_\mu\|_u < \epsilon/3$, and hence

$$
\begin{aligned}
\|f^t - f^{t_\mu}\|_u &\leqslant \|f^t - f^t_\lambda\|_u + \|f^t_\lambda - f^{t_\mu}_\lambda\|_u + \|f^{t_\mu}_\lambda - f^{t_\mu}\|_u \\
&= \|f - f_\lambda\|_\mu + \|f^t_\lambda - f^{t_\mu}_\lambda\| + \|f_\lambda - f\|_u \\
&< \epsilon
\end{aligned}
$$

$\square$

Lemma 6.1.3.3. G acts continuously on $C^b_{ru}(X)$

Proof. This is essentially by construction. Let $(f_\lambda, t_\lambda) \to (f,t)$ be a convergent net in $C^b_{ru}(X) \times G$. Given $\epsilon > 0$, we then have that for sufficiently large λ, both $\|f^t - f^{t_\mu}\|_u$ and $\|f - f_\lambda\|_u$ are strictly less than $\epsilon/2$, so

$$
\|f^t - f^{t_\mu}_\lambda\|_u \leqslant \|f^t - f^{t_\lambda}\|_u + \|f^{t_\lambda} - f^{t_\lambda}_\lambda\| = \|f^t - f^{t_\lambda}\|_u + \|f - f_\lambda\| < \epsilon
$$

$\square$

Lemma 6.1.3.4. $C^b_{ru}(X)$ contains $C^0(X)$ as a subalgebra.

Proof. To see this, suppose $f \in C^0(X)$ and let $t_\lambda \to t$ be a net in G. Suppose for contradiction that f^{t_λ} does not converge to f^t in $C^b(X)$. Then there is $\epsilon > 0$ so that

$$\Lambda' = \{\lambda \mid \|f^t - f^{t_\lambda}\| \geqslant \epsilon\} \subset \Lambda$$

is cofinal. Thus for each $\lambda' \in \Lambda'$ there is $x_{\lambda'} \in X$ with

$$| f(tx_{\lambda'}) - f(t_{\lambda'}x_{\lambda'}) | \geqslant \epsilon. \tag{6.6}$$

As $f \in C^0(X)$ there is a compact set $K \subset X$ with $| f(x) | < \epsilon/2$ for $x \notin K$. Hence by (6.6), we get that either $(tx_{\lambda'})$ or $(t_{\lambda'}x_{\lambda'})$ is completely contained in K. If $(t_{\lambda'}x_{\lambda'})$ is all contained in K, then we can pass to a convergent subsequence $t_{\lambda''}x_{\lambda''} \to y$. By continuity of the G-action, we get that $x_{\lambda''} \to t^{-1}y =: x$. Hence

$$| f((tx_{\lambda'})) - f(t_{\lambda'}x_{\lambda'}) | \to | f(tx) - f(tx) | = 0$$

which is a contradiction. A similar argument works in the case where $(tx_{\lambda'})$ is contained in K. □

As a corollary, we obtain that:

Corollary 6.1.3.5. If X is a compact Hausdorff G-space, then $C^b_{ru}(X) = C(X)$

Proof. The statement of the corollary follows from the previous lemma and the following inclusions

$$C(X) = C_0(X) \subset C^b_{ru}(X) \subset C^b(X) = C(X)$$

□

Lemma 6.1.3.6. If $\phi : X \to Y$ is a G-equivariant map with Y compact Hausdorff, then the precomposition map $\varphi^* : C(Y) \to C^b(X)$ lands in $C^b_{ru}(X)$

Proof. Let $f \in C(Y)$. Note that

$$(\varphi^* f)^t(x) = f(\varphi(tx)) = f(t\varphi(x)) = f^t(\varphi(x))$$

for all $t \in G$, where the second equality uses that φ is equivariant. Given a convergent net $t_\lambda \to t$ in G, we thus have

$$\|(\varphi^* f)^t - (\varphi^* f)^{t_\lambda}\|_u \leqslant \|f^t - f^{t_\lambda} |_u \to 0,$$

as desired. □

Now we can state the main theorem of this section,

Theorem 6.1.3.7. Let X be a locally compact Hausdorff G-space. Then there is a compact Hausdorff G-space $\beta_G X$ and an equivariant embedding $i : X \hookrightarrow \beta_G X$, so that $i(X)$ is dense in $\beta_G X$. Furthermore, if $f : X \to Y$ is a G-equivariant map with Y a compact Hausdorff G-space, then there is a unique dashed equivariant map making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & \beta_G(X) \arrow[d, dotted] \\ & Y \end{tikzcd}$$

Proof. As in [Vri77, Theorem 1.1], using Gelfand duality, there is a correspondence between compactifications and closed subalgebras of $C^b(X)$ that contains constants and closed under complex conjugation. The universal property follows from Lemma 6.1.3.6, which gives a factorisation of the desired form, and by Gelfand duality, any such factorisation must arise in this way. □

We would also like to record the following result, which shows the desired factorisation in the previous section.

Corollary 6.1.3.8. Let X be a locally compact Hausdorff G-space then we have the map $X \hookrightarrow \beta_G X$ is an open immersion

Proof. By Theorem 6.1.3.7 essentially due to [Vri77], we have that the map $X \to \beta_G X$ is injective and has a dense image, which is a compactification in the sense of [Loe69] and since X is locally compact by [Loe69, Theorem 1.1 (vi)] we have that $X \hookrightarrow \beta_G X$ is an open embedding. □

7 Equivariant sheaves and localisation

7.1 Localisation of Equivariant Sheaves

In this section, we introduce two different equivariant localisation functors in the context of equivariant sheaves on a topological space.

Let X be a space and G be a finite group, the classical localisation theorem relates the equivariant cohomology of a space X to its fixed point, this can be thought of as a statement relating the equivariant sheaf cohomology on the space X and its fixed points X^G with respect to a constant local system.

We can go further and try to generalise these methods on more general sheaves, in the first part we carry out these ideas on the subcategory of constructible sheaves, which are stratified versions of locally constant sheaves.

Localisations in sheaf theory are certainly not a new idea, and goes back to the work of Tom Braden [Bra03] commonly known as "**hyperbolic localisation**".

Where they consider the case of a $\mathbb{G}_m$-action on a normal variety X over a field of k characteristic p and constructs a certain localisation functor

$$D_{et}(X) \to D_{et}(X^T)$$

where $D_{et}(X)$ denote the derived category of l-adic étale sheaves, where $l \neq p$. These methods prove to be extremely useful in the proof of the Geometric Stake isomorphism [MV07].

In algebraic topology, the localisation theorem is true for other groups as well, such as the cyclic group $\mathbb{Z}/p$ in modular characteristics. This raises the question of whether we can perform localisation of $\mathbb{Z}/p$-equivariant sheaves. This was answered by the work of David Treumann [Tre19], where one can consider a complex analytic variety or more generally, a real sub-analytic set with an action of $\mathbb{Z}/p$ and get a localisation functor:

$$\mathrm{Shv}^b_{c,\mathbb{Z}/p}(X; \mathbb{F}_p) \to \mathrm{Smith}(X^{\mathbb{Z}/p})$$

the category on the left is bounded equivariant constructive sheaves and the target is a certain Verdier localisation, the localisation functor is referred to as **Smith localisation**. There have also been analogues of these constructions on étale sheaves for $\mathbb{Z}/p$-action due to Riche and Williamson [RW22b] for finite type schemes and for locally of finite type by Feng [Fen23].

Similar to Branden's hyperbolic localisation, the $\mathbb{Z}/p$ - Smith localisation has found applications in the Geometric Langlands program, where Riche and Williamson [RW22b] studies the loop rotation action coming from $\mathbb{Z}/p$ on the affine Grassmannian Gr_G and obtain a geometric proof of the linkage principle.

In the first part of this section, we revisit the work of Treumann and give an ∞-categorial account, and indicate how to extend to arbitrary groups with respect to suitable coefficients.

In the second part of the section, we restrict ourselves to compact manifolds with an action of a finite group and introduce a different localisation functor, which is more in the spirit of the localisations theorem considered in equivariant homotopy theory.

7.1.1 Localisation after Treumann

Let X be a real sub-analytic space with an action of a finite group G. We consider the category $\mathrm{Shv}_{c,G}(X;\mathcal{C})$ of constructible equivariant sheaves valued in compactly generated presentable ∞-category $\mathcal{C}$.

This is a full subcategory of equivariant sheaves $\mathrm{Shv}_{hG}(X;\mathcal{C})$, spanned by objects $\mathcal{F}$ such that there exist a stratification $X \to P$ with strata $\{X_p\}_{p \in P}$, where each strata X_p is G-invariant and the underlying sheaf $\mathcal{F}\mid_{X_p}$ is locally constant.

Let Z be the closed subspace of X, which is G-invariant, and U be the complement of Z, and let $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X$ be the corresponding inclusions. Since $i : Z \hookrightarrow X$ is a closed embedding $i_{G,*} = i_{G,!}$ and $j : U \hookrightarrow X$ is a open embedding, hence we have $j^!_G = j^*_G$.

We can organise this information as a stable recollement in the sense of [Lur17, A.8]:

$$\begin{tikzcd} \mathrm{Sh}_{G,c}(Z) \arrow[r, "i_{*,G}=i_{!,G}"'] & \mathrm{Sh}_{G,c}(X) \arrow[l, bend right, "i^*_G"'] \arrow[l, bend left, "i^!_G"] \arrow[r, "j^*_G=j^!_G"'] & \mathrm{Sh}_{G,c}(U) \arrow[l, bend right, "j_{!,G}"'] \arrow[l, bend left, "j_{*,G}"] \end{tikzcd}$$

where the middle sequence is a localisation sequence, i.e. a fiber-cofiber sequence in Pr^L [Lur17; AMR22].

(In the rest of the section, we will drop the group G as it makes the notations cleaner)

Given an object $\mathcal{F} \in \mathrm{Shv}_{c,G}(X)$ by [BG16] we have a two cofiber sequences $i_!i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F}$ and $j_!j^!\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F}$.

Given an object $\mathcal{F} \in \mathrm{Shv}_{c,G}(X)$, we can recover $\mathcal{F}$ from its restriction to the closed and open part via a fracture square, i.e. there is pullback square.

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & i_*i^*\mathcal{F} \arrow[d] \\ j_*j^*\mathcal{F} \arrow[r] & i_*i^*j_*j^*\mathcal{F} \end{tikzcd}$$

In fact we get $\mathrm{Shv}_{c,G}(X)$ is a lax-limit of $\mathrm{Shv}_{c,G}(U)$ and $\mathrm{Shv}_{c,G}(Z)$, or in the language of [AMR22]; $\mathrm{Shv}_{c,G}(X)$ is stratified over the poset $0 < 1$.

Remark 7.1.1.1. One first observes there is a recollement at the level of the category of sheaves, and by [BL94, Section 1.10], the six functors preserve constructability since we have assumed X is real sub-analytic.

Let the closed subspace $Z = X^G$, since the action on X^G is trivial, we have $\mathrm{Shv}_{hG}(X;\mathcal{C}) \simeq \mathrm{Shv}(X;\mathcal{C}^{hG})$, where $\mathcal{C}^{hG} \simeq \mathrm{Fun}(BG,\mathcal{C})$. We will primarily be interested in the case where $\mathcal{C} = \mathrm{Mod}_R$ the stable ∞-category of modules over an $\mathbb{E}_\infty$-ring spectrum R[1]. By Schwede-Shipley theorem [Lur17, Theorem 7.1.2.1] we can identify $\mathrm{Fun}(BG,\mathrm{Mod}_R) \simeq \mathrm{Mod}_{R[G]}$.

Definition 7.1.1.2 (Proper ideal)**.** Let $\mathcal{P}$ be the subcategory of $\mathrm{Shv}^b_{c,G}(X^G)$, spanned by sheaves whose stalks lie in the thick $\otimes$-ideal of $\mathrm{Mod}_{R[G]}$ generated by $R[G/H]$ where H runs over all proper subgroups. We call this the **proper ideal**.

Definition 7.1.1.3. We define $\mathrm{Smith}(X^G;R) \simeq \mathrm{Shv}^b_{c,G}(X;R)/\mathcal{P}$ to be the Verdier quotient.

Let $X = *$ and $R = \mathbb{F}_p$ and G a p-group, then we have:

$$\mathrm{Smith}(*;\mathbb{F}_p) \simeq \mathrm{Mod}^b_{\mathbb{F}_p[G]}/\mathcal{P}$$

In particular, $\mathrm{Ind}(\mathrm{Smith}(*;F_p)) \simeq \mathrm{Mod}(\mathbb{F}_p^{\tau G})$, where $\mathbb{F}_p^{\tau G}$ is the **proper Tate construction** [Mat15].

Note that for $\mathbb{F}_p$, the proper Tate construction with respect to the trivial action is non-zero if and only if G is an elementary abelian group. This has to do with the fact that the derived defect base[2] of the ring $\mathbb{F}_p$ is exactly the elementary abelian groups (in the sense of [MNN19]).

This suggests that given an $\mathbb{E}_\infty$-ring, it is crucial to work with something within the defect base of the ring R. For example, in the case $R = KU_p^\wedge$ we have defect base, all cyclic group. More generally, $E(n)$ the height n Morava E-theory has defect base as all abelian groups G, such that the maximal elementary subgroup has rank $\leqslant n$.

The following is the main theorem in [Tre19] for cyclic and coefficient $\mathbb{F}_p$. We state a modified version to accommodate all coefficients R, an $\mathbb{E}_\infty$-ring and all finite groups in the defect base of R.

Theorem 7.1.1.4. Let X be a G-space and i the inclusion of fixed points $i : X^G \hookrightarrow X$. Then the cofiber of the natural transformation $\eta : i^! \to i^*$ lies in the proper ideal $\mathcal{P}$.

Proof. Let $\mathcal{F}$ be an object in $\mathrm{Shv}^b_{c,G}(X;\mathbb{F}_p)$ and $x \in X^G$. To show the cofiber of $\eta : i^!\mathcal{F} \to i^*\mathcal{F}$ belongs to the proper ideal $\mathcal{P}$, we need to compute the stalk of cofiber of η at the point x. For which we can use the cofiber sequence coming from recollement:

$$i_!i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F}$$

where $i : X^G \hookrightarrow X$ and $j : U \hookrightarrow X$. Using this, the stalk can be computed as follows: we can choose a regular neighbourhood V of x and $V' = V\backslash X^G$, such that V is invariant under the action of G and V' is a subset of $U = X\backslash X^G$, i.e.,V' has no fixed points. Using the localisation sequence $i_!i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F}$ the stalk $\mathrm{cofib}(\eta)(\mathcal{F})_x$ is the global section of $\mathcal{F}\mid_{V'}$.

[1] readers who are unfamiliar with the formalism of $\mathbb{E}_\infty$ rings can think of ordinary commutative rings and Mod_R to be the derived category of R

[2] derived defect base of ring R is those groups G such that $R^{\tau G}$ for the trivial action is non zero

We claim that, $\Gamma(\mathcal{F}\mid_{V'}) \in \mathrm{Fun}(BG, Mod_R)$ is in the thick $\otimes$-ideal I generated by $R[G/H]$ where H runs over proper subgroups.

To prove this, we can establish a general result that for any space Y with a free G-action, the global section of an equivariant constructible sheaf $\mathcal{F}$ belongs to the ideal I. Note that the category of constructible sheaves are generated by sheaves which are constant on a closed invariant space and zero outside.

So, let us assume $\mathcal{F}$ is a sheaf of which is constant on a closed invariant subspace zero on Z and $Y \backslash Z$. The global section of $\mathcal{F}$ is cochains on Z with respect to the constant coefficient. For constant sheaves on finite G-CW complex, the result follows as the cohomology of the finite colimit of $\Sigma^n R[G/H]$ where $H \leqslant G$ belongs to isotropy subgroups of the action. Hence, we have $\Gamma(\mathcal{F})$ lies in the ideal $I \subset \mathrm{Fun}(BG, Mod_R)$. □

The above calculations can be suitably modified in the context of étale sheaves over algebraic varieties in the case of field coefficients to a more general setup as indicated by [RW22b].

In particular, in order to understand the loop rotation action for $\mathbb{Z}/p^n \subset \mathbb{G}_m$ on the affine Grassmannian Gr_G and carry out the calculation in [RW22b] we can work with coefficients such as $KU_p^\wedge$, the p-complete complex K-theory, whose defect base is the cyclic groups and proper Tate construction $(KU_p^\wedge)^{\tau\mathbb{Z}/p^n}$ is rational.

7.1.2 Localisation in Dualisable categories

In this section, we will indicate a different categorification of the Smith localisation and several obstructions for such a construction.

Let X be a finite G-CW complex, where G is an elementary abelian group. Recall that we can study the $\mathbb{F}_p$-valued singular cohomology of fixed points X^G up to a localisation. That is,

$$H^*(X_{hG}; \mathbb{F}_2)[e^{-1}] \simeq H^*(X^G; \mathbb{F}_2)[[t_1, ..., t_n]][e^{-1}],\ p = 2$$

$$H^*(X_{hG}; \mathbb{F}_p)[e^{-1}] \simeq H^*(X^G; \mathbb{F}_p)[[\beta_1, ..., \beta_n, t_1, ..., t_n]][e^{-1}]\ p \text{ odd}$$

where e is the Euler class discussed in section 2.1.3. In this section, we will indicate a version of the categorification of this statement, that is,

$$\mathrm{Shv}_{hG}(X; \mathbb{F}_p)[e^{-1}] \simeq \mathrm{Shv}_{hG}(X^G; \mathbb{F}_p)[e^{-1}].$$

Here we recall a few facts about presentable dualisable categories, which would be essential in stating the main construction.

Let Pr^L_{st} be the ∞-category of presentable stable ∞-categories, with morphism left adjoint functors, which can be equipped with a symmetric monoidal structure called the **Lurie tensor product**. A commutative algebra object in Pr^L_{st} is a presentably[3] symmetric monoidal ∞-category.

[3]i.e., the colimits commute with tensor products in each variable

Definition 7.1.2.1 (Dualisable Categories)**.** A presentable stable ∞-category $\mathcal{C} \in \mathrm{Pr}^L_{st}$ is said to be dualisable if it is dualisable with respect to the Lurie tensor product.

Here are useful and equivalent characterisations of dualisable categories found in [Lur18, Appendix D.7.3.1] :

Theorem 7.1.2.2. [Lur18, Appendix D.7.3.1] Let $\mathcal{C} \in \mathrm{Pr}^L_{st}$, The following conditions are equivalent:

1. $\mathcal{C}$ is a dualisable category.
2. The colimit evaluation functor colim : $\mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $\hat{y} : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$.
3. There is a localisation sequence such that $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$, where $\mathcal{D}$ and $\mathcal{E}$ are compactly generated.
4. $\mathcal{C}$ is a retract of a compactly generated category, where the retract is taken in Pr^L_{st}.

To start with, all compactly generated categories are dualisable. There are also more examples coming from geometry, i.e., for a locally compact Hausdorff space X, we have the category of sheaves $\mathrm{Sh}(X; \mathcal{C})$, which almost never compactly generated [Nee01]. However, the Verdier duality for the category of sheaves [Lur17; Vol17], along with the six functors formalism, express this category as a dualisable object in Pr^L_{st}.

Definition 7.1.2.3. Let $\mathrm{Pr}^{\mathrm{dual}}_{st}$ be the ∞-category of presentable stable categories, which are dualisable and functors between them left adjoints, whose right adjoint preserves colimits.

Note that the category $\mathrm{Pr}^{\mathrm{dual}}_{st}$ is only a subcategory of Pr^L_{st} but not a full subcategory. The category $\mathrm{Pr}^{\mathrm{dual}}_{st}$ admits all limits and colimits. In fact, we learned the following from Maxime Ramzi:

Theorem 7.1.2.4 (M. Ramzi)**.** The category $\mathrm{Pr}^{\mathrm{dual}}_{st}$ of presentable stable dualisable ∞-categories is presentable.

Theorem 7.1.2.5 (A. Efimov)**.** The inclusion $\mathrm{Pr}^{\mathrm{dual}}_{st} \hookrightarrow \mathrm{Pr}^L_{st}$ preserves colimits.

Remark 7.1.2.6. This gives us an idea of how to think of colimits in $\mathrm{Pr}^{\mathrm{dual}}_{st}$, as they are the same as colimits in Pr^L_{st}, but practically speaking, colimits in Pr^L_{st} are not very transparent. One can compute colimits in Pr^L_{st}, as follows: given $F : I \to \mathrm{Pr}^L_{st}$, we use the fact that

$$\mathrm{Pr}^L_{st} \simeq (\mathrm{Pr}^R_{st})^{op}$$

where the equivalence is given by passing to the adjoints.

We then get a diagram $F^{op} : I^{op} \to \mathrm{Pr}^R_{st}$, with $\operatorname*{colim}_I F$ identified with the limit $\lim_{I^{op}} F^{op}$.Since $\mathrm{Pr}^R_{st} \hookrightarrow \mathrm{Cat}_\infty$ preserve limits, we compute the limit in Cat_∞.

The notion of a dualisable ∞-category seemed to have the right "size". For example, let $\mathcal{C}$ be a large category, say Mod_R, then the K-theory of Mod_R vanishes because of the Eilenberg swindle. But due to the recent work of A. Efimov, we can extend the definition of K-theory to large categories.

Theorem 7.1.2.7 (A. Efimov)**.** The localising invariant E on $\mathrm{Cat}_\infty^{\mathrm{perf}}$ uniquely extend to a localising invariant E^{cont} on $\mathrm{Pr}_{st}^{\mathrm{dual}}$ and any localising invariant on $\mathrm{Pr}_{st}^{\mathrm{dual}}$ comes from this extension referred to as the continuous extension.

This leads us to ask what the algebraic K-theory of $\mathrm{Shv}(X;\mathcal{C})$ for X locally compact Hausdorff. In the classical sense, you get zero as the category is large, but here is an interesting result of Efimov

Theorem 7.1.2.8 (A. Efimov)**.** Let X be a locally compact Hausdorff space, and let E be a localising invariant on $\mathrm{Cat}_\infty^{\mathrm{perf}}$ valued in spectra, and it preserves filtered colimits. Then $E^{cont}(\mathrm{Shv}(X;\mathcal{C})) \simeq \Gamma_c(X;E(\mathcal{C}))$. □

Where the target is compactly supported cohomology with respect to the constant sheaf on X, valued in Sp. The proof of this above theorem is available in notes by Marc Hoyois[4]

Let $\mathrm{Pr}_{st}^{L,\omega}$ be the ∞-category of compactly generated stable presentable categories, with morphisms left adjoint functors that preserve compact objects.

Remark 7.1.2.9. [Mat16, Corollary 2.9] The category $\mathrm{Pr}_{st}^{L,\omega}$ is a presentable ∞-cateogry. We have an equivalence $\mathrm{Cat}_\infty^{\mathrm{perf}} \simeq \mathrm{Pr}_{st}^{L,\omega}$. Furthermore there is a fully faithful inclusion

$$\mathrm{Pr}_{st}^{L,\omega} \hookrightarrow \mathrm{Pr}_{st}^{\mathrm{dual}}.$$

□

Proposition 7.1.2.10. The inclusion $\mathrm{Pr}_{st}^{L,\omega} \hookrightarrow \mathrm{Pr}_{st}^{\mathrm{dual}}$ admits a right adjoint $\mathrm{Pr}_{st}^{\mathrm{dual}} \to \mathrm{Pr}_{st}^{L,\omega}$ given by

$$\mathcal{C} \mapsto \mathrm{Ind}(\mathcal{C}^\omega)$$

□

Let M be a compact manifold with an action of a finite group G. This gives a G-CW complex structure on M. Let M^G be the subspace of fixed points, and for the rest of the section we set $U := M \backslash M^G$. We denote the inclusions of M^G and U by $i : M^G \hookrightarrow M$ and $j : U \to M$. This gives us a stable recollement of the category of sheaves (refer [Lur17, Appendix A.8]):

$$\begin{tikzcd}
\mathrm{Sh}(M^G;\mathcal{C}) \arrow[r, "i_! = i_*"] & \mathrm{Sh}(X;\mathcal{C}) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^! = j^*"] & \mathrm{Sh}(U;\mathcal{C}) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}$$

Where the sheaves are valued in a presentable stable ∞-category $\mathcal{C}$.

Construction 7.1.2.11. Let M be a locally compact Hausdorff space with a G-action, where G is a finite group.

Then $\mathrm{Shv}(X;\mathcal{C})$ is an object of $\mathrm{Pr}_{st}^{\mathrm{dual}}$, and moreover if we assume M to be compact then $\mathrm{Shv}(X;\mathcal{C})$ is a G-object in $\mathrm{Pr}_{st}^{\mathrm{dual}}$. That is a functor

$$BG \to \mathrm{Pr}_{st}^{\mathrm{dual}},$$

[4]https://hoyois.app.uni-regensburg.de/papers/efimov.pdf

We define the category of equivariant dualisable sheaves $\mathrm{Shv}_{hG}(X;\mathcal{C})^{\mathrm{dual}}$ as the limit in $\mathrm{Pr}^{\mathrm{dual}}_{st}$ over BG. □

Remark 7.1.2.12. Note the construction above looks similar to the definition of the equivariant sheaf category as in definition 6.1.1.1, but since the inclusion $\mathrm{Pr}^{\mathrm{dual}}_{st} \hookrightarrow \mathrm{Pr}^{L}_{st}$ does not preserve limits, the two definitions are genuinely different. □

Here is an important result of A. Efimov, which is crucial for our construction of Smith localisation:

Theorem 7.1.2.13 (A. Efimov)**.** Let $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Fun}(BG, \mathrm{Pr}^{\mathrm{dual}}_{st})$ and $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{g} \mathcal{E}$ be a localisation sequence such that f, g are morphism in $\mathrm{Fun}(BG, \mathrm{Pr}^{\mathrm{dual}}_{st})$, i.e, they are equivariant. Then the induced sequence

$$\mathcal{C}^{hG} \xrightarrow{f^{hG}} \mathcal{D}^{hG} \xrightarrow{g^{hG}} \mathcal{D}$$

is a localisation sequence in $\mathrm{Pr}^{\mathrm{dual}}_{st}$. □

The above theorem can be applied to the category of equivariant sheaves to obtain a localisation sequence:

$$\mathrm{Shv}_{hG}(U;\mathcal{C})^{\mathrm{dual}} \to \mathrm{Shv}_{hG}(X;\mathcal{C})^{\mathrm{dual}} \to \mathrm{Shv}_{hG}(X^G;\mathcal{C})^{\mathrm{dual}}$$

Where X is a compact Hausdorff space with an action of a finite group G.

We can identify

$$\mathrm{Shv}_{hG}(X^G;\mathcal{C})^{\mathrm{dual}} \simeq \mathrm{Shv}(X^G;Sp) \otimes \mathcal{C}^{hG,dual},$$

since the action on X^G is trivial and $\mathrm{Shv}(X^G;Sp)$ is dualisable hence we have:

$$\lim_{BG} (\mathrm{Shv}(X^G;\mathcal{C})) \simeq \lim_{BG} (\mathrm{Shv}(X^G;Sp) \otimes \mathcal{C}) \simeq \mathrm{Shv}(X^G;Sp) \otimes \lim_{BG} \mathcal{C}$$

We are particularly concerned about the case of $\mathcal{C} = \mathrm{Mod}_R$.

Note that any stable symmetric monoidal category $\mathcal{C}^{\otimes}$ and $1 \in \mathcal{C}^{\otimes}$ then $\mathrm{End}(1)$ is an $\mathbb{E}_\infty$-algebra in spectra. For any objects X, Y, the mapping spectra $\mathrm{map}_{\mathcal{C}}(X, Y)$ has a natural action of $\mathrm{End}(1)$, informally the action is as follows, given $f : X \to Y$ and an endomorphism $\eta : 1 \to 1$, we get $(\eta\dot{f}) = f \otimes \eta : X \to Y$.

Construction 7.1.2.14 (Inverting Euler class)**.** The category $\mathrm{Shv}_{hG}(X;\mathbb{F}_p)^{\mathrm{dual}}$ is a module over the category $\mathrm{Mod}_{\mathbb{F}_p^{hG}}$ so we can base change it to the category $\mathrm{Mod}_{\mathbb{F}_p^{\tau G}}$, modules over the proper Tate construction:

$$\mathrm{Shv}_{hG}(X;\mathbb{F}_p)^{\mathrm{dual}} \otimes_{\mathrm{Mod}_{\mathbb{F}_p^{hG}}} \mathrm{Mod}_{\mathbb{F}_p^{\tau G}}.$$

We denote it by $\mathrm{Shv}_{hG}(X;\mathbb{F}_p)^{\mathrm{dual}}[e^{-1}]$. □

Now let us work with a compact manifold M with a G-action. Since M is compact, it admits a finite G-CW complex structure. Recall that we have a localisation sequence

$$\mathrm{Shv}_{hG}(U;\mathbb{F}_p)^{\mathrm{dual}} \to \mathrm{Shv}_{hG}(M;\mathbb{F}_p)^{\mathrm{dual}} \to \mathrm{Shv}_{hG}(M^G;\mathbb{F}_p)^{\mathrm{dual}}$$

where U is the complement of the fixed points M^G. The enodmorphism of the unit, End(1) in $\mathrm{Shv}_{hG}(U;\mathbb{F}_p)^{\mathrm{dual}}$ is an algebra over the equivariant cohomology $C^*(U_{hG};\mathbb{F}_p)$ and the base changed category is a module over $C^*(U_{hG};\mathbb{F}_p)[e^{-1}]$, where e is an Euler class described in section 2.1.3.

Since U does not have any fixed point, we have $C^*(U_{hG};\mathbb{F}_p)[e^{-1}] \simeq 0$ by the localisation theorem 2.1.3.3. Hence $\mathrm{Shv}_{hG}(U;\mathbb{F}_p)^{\mathrm{dual}}[e^{-1}] \simeq 0$. Combining this with the above, we have the following proposition:

Proposition 7.1.2.15. Let M be a compact manifold with an action of a group G, then $\mathrm{Shv}_{hG}(X;\mathbb{F}_p)^{\mathrm{dual}}[e^{-1}] \simeq \mathrm{Shv}_{hG}(X^G;\mathbb{F}_p)^{\mathrm{dual}}[e^{-1}]$.

Let us investigate the category $\mathrm{Shv}_{hG}(X^G;\mathbb{F}_p)^{\mathrm{dual}}[e^{-1}] \simeq \mathrm{Shv}(X;\mathrm{Sp})\otimes\mathrm{Mod}_{\mathbb{F}_p}^{hG,dual}[e^{-1}]$. It is enough to analyse what $\mathrm{Mod}_{\mathbb{F}_p}^{hG,dual}[e^{-1}]$ is. First let us understand compact objects of $\mathrm{Mod}_{\mathbb{F}_p}^{hG,dual}$, which we can do using the right adjoint:

$$\mathrm{Pr}_{st}^{\mathrm{dual}} \to \mathrm{Cat}_\infty^{\mathrm{perf}}$$
$$\mathcal{C} \mapsto \mathcal{C}^\omega$$

$$(\mathrm{Mod}_{\mathbb{F}_p}^{hG,dual})^\omega \simeq \lim_{BG}(\mathrm{Mod}_{\mathbb{F}_p})^\omega \simeq \mathrm{Fun}(BG, \mathrm{Perf}_{\mathbb{F}_p}).$$

We have the Tate cofiber sequence:

$$\mathcal{P} = \mathrm{Perf}\{G/H \mid H \textit{ proper subgroup}\} \to \mathrm{Perf}(\mathbb{F}_p)^{hG} \to \mathrm{Perf}(\mathbb{F}_p)^{\tau G}$$

We identify $\mathrm{Perf}(\mathbb{F}_p)^{\tau G} \simeq \mathrm{Perf}(\mathbb{F}_p^{\tau G})$ due to unipotence of representations, i.e., every representation of a p-group V in characteristic p can be filtered:

$$0 \subset V_1 \subset V_2 \subset V_3 \subset \ldots \subset V_n = V$$

such that $V_i/V_{i-1} \simeq \mathbb{F}_p$, for a more detailed discussion, we refer the reader to [MNN17, Section 7].

we can apply $- \otimes_{\mathrm{Perf}(\mathbb{F}_p^{hG})} \mathrm{Perf}(\mathbb{F}_p^{\tau G})$ to the Tate cofiber sequence above.

On perfect modules, the base change inverts the Euler class on mapping space which kills all the mapping spaces in $\mathcal{P}$, hence the category $\mathcal{P} \otimes_{\mathrm{Perf}(\mathbb{F}_p^{hG})} \mathrm{Perf}(\mathbb{F}_p^{\tau G}) \simeq 0$. So we have:

$$\mathrm{Perf}(\mathbb{F}_p)^{hG} \otimes_{\mathrm{Perf}(\mathbb{F}_p^{hG})} \mathrm{Perf}(\mathbb{F}_p^{\tau G}) \simeq \mathrm{Perf}(\mathbb{F}_p^{\tau G}) \otimes_{\mathrm{Perf}(\mathbb{F}_p^{hG})} \mathrm{Perf}(\mathbb{F}_p^{\tau G})$$

We have $\mathbb{F}_p^{\tau G}$ is an idempotent algebra over $\mathbb{F}_p^{hG}$ because it is given by localisation of Euler class; hence we have:

$$\mathrm{Perf}(\mathbb{F}_p^{\tau G}) \otimes_{\mathrm{Perf}(\mathbb{F}_p^{hG})} \mathrm{Perf}(\mathbb{F}_p^{\tau G}) \simeq \mathrm{Perf}(\mathbb{F}_p^{\tau G})$$

Now we can take G to be an elementary abelian group, so the proper Tate construction $\mathbb{F}_p^{\tau G}$ is non-zero.

From this we get, the compact objects of $\mathrm{Mod}_{\mathbb{F}_p}^{hG,dual}[e^{-1}]$ are exactly given by $\mathrm{Mod}_{\mathbb{F}_p^{\tau G}}$, but we dont know if the category $\mathrm{Mod}_{\mathbb{F}_p}^{hG,dual}[e^{-1}]$ is compactly generated, since $\mathrm{Mod}_{\mathbb{F}_p}^{hG,dual}$ need not be compactly generated.

Remark 7.1.2.16. Note that the limit Mod_R^{hG} in Pr_{st}^L is identified with the category $\mathrm{Fun}(BG, \mathrm{Mod}_R)$ whose compact objects are $\mathrm{Perf}(R[G])$. This implies the limits in $\mathrm{Pr}_{st}^{\mathrm{dual}}$ and Pr_{st}^L are genuinely different.

Remark 7.1.2.17. Since the compact objects of $\mathrm{Mod}_{\mathbb{F}_p}^{hG}$ are $\mathrm{Perf}(\mathbb{F}_p[G])$, inverting the Euler class on $\mathrm{Mod}_{\mathbb{F}_p}^{hG}$ would kill all the compact objects, hence we get $\mathrm{Mod}_{\mathbb{F}_p}^{hG}[e^{-1}] \simeq 0$. This is the crucial reason to work in the category of presentable dualisable categories.

Finally, we would like to end the section with the following questions.

Question 7.1.2.18. Is the category $\mathrm{Mod}_{\mathbb{F}_p}^{hG,dual}[e^{-1}]$, compactly generated?

We have established in Proposition 7.1.2.15 for M be a compact manifold with an action of a group G, then $\mathrm{Shv}_{hG}(X; \mathbb{F}_p)^{\mathrm{dual}}[e^{-1}] \simeq \mathrm{Shv}_{hG}(X^G; \mathbb{F}_p)^{\mathrm{dual}}[e^{-1}]$.

The localised categories are still in $\mathrm{Pr}_{st}^{\mathrm{dual}}$. Hence can we compute some localising invariant on both sides?

Question 7.1.2.19. What is the topological Hochschild homology applied to the expression $\mathrm{Shv}_{hG}(X; \mathbb{F}_p)^{\mathrm{dual}}[e^{-1}] \simeq \mathrm{Shv}_{hG}(X^G; \mathbb{F}_p)^{\mathrm{dual}}[e^{-1}]$ give us?

Bibliography

[AB84] Michael F Atiyah and Raoul Bott. "The moment map and equivariant cohomology". In: *Topology* 23.1 (1984), pp. 1–28.

[Ada74] John Frank Adams. *Stable homotopy and generalised homology*. University of Chicago press, 1974.

[AMR22] David Ayala, Aaron Mazel-Gee, and Nick Rozenblyum. *Stratified noncommutative geometry*. 2022. arXiv: 1910.14602 [math.AG].

[AMR23] David Ayala, Aaron Mazel-Gee, and Nick Rozenblyum. *Derived Mackey functors and C_{p^n}-equivariant cohomology*. 2023. arXiv: 2105.02456 [math.AT].

[AS69] Michael F Atiyah and Graeme B Segal. "Equivariant K-theory and completion". In: *Journal of Differential Geometry* 3.1-2 (1969), pp. 1–18.

[Ati61a] Michael F Atiyah. "Characters and cohomology of finite groups". In: *Publications Mathématiques de l'IHÉS* 9 (1961), pp. 23–64.

[Ati61b] Michael Francis Atiyah. "Thom complexes". In: *Proceedings of the London Mathematical Society* 3.1 (1961), pp. 291–310.

[Bar17] C. Barwick. "Spectral Mackey Functors and Equivariant Algebraic K-theory (I)". In: *Adv. in Math.* 304 (2017), pp. 646–727.

[BBF+60] Armand Borel, G. Bredon, E. E. Floyd, D. Montgomery, and R. Palais. *Seminar on Transformation Groups. (AM-46)*. Princeton University Press, 1960. ISBN: 9780691090948. URL: http://www.jstor.org/stable/j.ctt1bd6jxd (visited on 08/30/2023).

[BDS15] Paul Balmer, Ivo Dell'Ambrogio, and Beren Sanders. *Restriction to finite-index subgroups as étale extensions in topology, KK-theory and geometry*. 2015. arXiv: 1408.2524 [math.CT].

[Bei81] Alexander Beilinson. "Localisation de g-modules". In: *CR Acad. Sci. Paris* 292 (1981), pp. 15–18.

[BG16] Clark Barwick and Saul Glasman. *A note on stable recollements*. 2016. arXiv: 1607.02064 [math.CT].

[BHS22] Robert Burklund, Jeremy Hahn, and Andrew Senger. *On the boundaries of highly connected, almost closed manifolds*. 2022. arXiv: 1910.14116 [math.AT].

[BL94] Joseph Bernstein and Valery Lunts. "Equivariant sheaves and functors". In: *Springer Book Archive-Mathematics* (1994).

[BMM+86] Robert R Bruner, J Peter May, James E McClure, and Mark Steinberger. *H_∞-ring spectra and their applications*. Vol. 1176. Springer, 1986.

[Bou79] Aldridge K Bousfield. "The localization of spectra with respect to homology". In: *Topology* 18.4 (1979), pp. 257–281.

[BR18] Pierre Baumann and Simon Riche. *Notes on the geometric Satake equivalence*. 2018. arXiv: 1703.07288 [math.RT].

[Bra03] Tom Braden. "Hyperbolic localization of intersection cohomology". In: *Transformation groups* 8 (2003), pp. 209–216.

[BT72] François Bruhat and Jacques Tits. "Groupes réductifs sur un corps local: I. Données radicielles valuées". In: *Publications Mathématiques de l'IHÉS* 41 (1972), pp. 5–251.

[BT84] François Bruhat and Jacques Tits. "Groupes réductifs sur un corps local: II. Schémas en groupes. Existence d'une donnée radicielle valuée". In: *Publications Mathématiques de l'IHÉS* 60 (1984), pp. 5–184.

[Car83] Gunnar Carlsson. "GB Segal's Burnside ring conjecture for (Z2) k". In: *Topology* 22.1 (1983), pp. 83–103.

[Car84] Gunnar Carlsson. "Equivariant stable homotopy and Segal's Burnside ring conjecture". In: *Annals of Mathematics* (1984), pp. 189–224.

[Car91] Gunnar Carlsson. "Equivariant stable homotopy and Sullivan's conjecture". In: *Inventiones mathematicae* 103.1 (1991), pp. 497–525.

[Che55] Claude Chevalley. "Sur certains groupes simples". In: *Tohoku Mathematical Journal, Second Series* 7.1-2 (1955), pp. 14–66.

[DW88] William G Dwyer and Clarence W Wilkerson. "Smith theory revisited". In: *Annals of Mathematics* 127.1 (1988), pp. 191–198.

[DW91] William G Dwyer and Clarence W Wilkerson. "Smith theory and the functor T". In: *Commentarii Mathematici Helvetici* 66 (1991), pp. 1–17.

[DW94] William G Dwyer and Clarence W Wilkerson. "Homotopy fixed-point methods for Lie groups and finite loop spaces". In: *Annals of Mathematics* 139.2 (1994), pp. 395–442.

[DZ06] W Dwyer and Alexander Zabrodsky. "Maps between classifying spaces". In: *Algebraic Topology Barcelona 1986: Proceedings of a Symposium held in Barcelona, April 2–8, 1986*. Springer. 2006, pp. 106–119.

[Fen23] Tony Feng. *Smith theory and cyclic base change functoriality*. 2023. arXiv: 2009.14236 [math.NT].

[Gir71] Jean Giraud. *Cohomologie non abélienne*. Vol. 179. Die Grundlehren der mathematischen Wissenschaften. Springer Verlag, 1971.

[GL20] Saul Glasman and Tyler Lawson. *Stable power operations*. 2020. arXiv: 2002.02035 [math.AT].

[GM95] J.P.C. Greenlees and J.P. May. "Generalised Tate cohomology". In: *Mem. A.M.S.* 543 (1995).

[Gro10] Jesper Grodal. "The classification of p-compact groups and homotopical group theory". In: *Proceedings of the International Congress of Mathematicians 2010 (ICM 2010) (In 4 Volumes) Vol. I: Plenary Lectures and Ceremonies Vols. II–IV: Invited Lectures*. World Scientific. 2010, pp. 973–1001.

[Gun80] JHC Gunawardena. "Segal's conjecture for cyclic groups of (odd) prime order". In: *JT Knight prize essay, Cambridge* 224 (1980).

[HY19] Jeremy Hahn and Allen Yuan. "Multiplicative structure in the stable splitting of ΩSLn (C)". In: *Advances in Mathematics* 348 (2019), pp. 412–455.

[Jam59] IM James. "Spaces associated with Stiefel manifolds". In: *Proceedings of the London Mathematical Society* 3.1 (1959), pp. 115–140.

[Kas79] Masaki Kashiwara. "Faisceaux constructibles et systèmes holonomes d'équations aux dérivées partielles linéaires à points singuliers réguliers". In: *Séminaire Équations aux dérivées partielles (Polytechnique) dit aussi" Séminaire Goulaouic-Schwartz"* (1979), pp. 1–6.

[Kas84] Masaki Kashiwara. "The Riemann-Hilbert problem for holonomic systems". In: *Publications of the Research Institute for Mathematical Sciences* 20.2 (1984), pp. 319–365.

[Lan92] Jean Lannes. "Sur les espaces fonctionnels dont la source est le classifiant d'un p-groupe abélien élémentaire". In: *Publications Mathématiques de l'IHÉS* 75 (1992), pp. 135–244.

[Law20] Tyler Lawson. *E_n ring spectra and Dyer-Lashof operations*. 2020. arXiv: 2002.03889 [math.AT].

[LDM+80] Wên Hsiung Lin, Donald M Davis, Mark E Mahowald, and John Frank Adams. "Calculation of Lin's Ext groups". In: 87.3 (1980), pp. 459–469.

[Lin80] Wen-Hsiung Lin. "On conjectures of Mahowald, Segal and Sullivan". In: *Mathematical Proceedings of the Cambridge Philosophical Society*. Vol. 87. 3. Cambridge University Press. 1980, pp. 449–458.

[Loe69] Peter A Loeb. "Compactifications of Hausdorff spaces". In: *Proceedings of the American Mathematical Society* 22.3 (1969), pp. 627–634.

[Lur07] J. Lurie. "Topics in algebraic Topology: The Sullivan Conjecture". In: *MIT opencourseware webpage* (2007).

[Lur09] J. Lurie. "Higher Topos Theory". In: Ann. of Math. Studies 170 (2009), pp. xviii+925.

[Lur11] J. Lurie. "Derived Algebraic Geometry XIII: Rational and p-adic Homotopy Theory". In: *author's webpage* (2011).

[Lur17] J. Lurie. "Higher Algebra". In: *author's webpage* (2017).

[Lur18] J. Lurie. "Spectral Algebraic Geometry". In: *author's webpage* (2018).

[LZ17] Yifeng Liu and Weizhe Zheng. *Enhanced six operations and base change theorem for higher Artin stacks*. 2017. arXiv: 1211.5948 [math.AG].

[Man01] Michael A Mandell. "E_∞-algebras and p-adic homotopy theory". In: *Topology* 40.1 (2001), pp. 43–94.

[Man22] Lucas Mann. *A p-Adic 6-Functor Formalism in Rigid-Analytic Geometry*. 2022. arXiv: 2206.02022 [math.AG].

[Mat15] Akhil Mathew. *Torus actions on stable module categories, Picard groups, and localizing subcategories*. 2015. arXiv: 1512.01716 [math.RT].

[Mat16] A. Mathew. "The Galois Group of a Stable Homotopy Theory". In: *Adv. Math.* 291 (2016), pp. 403–541.

[Mil84] Haynes Miller. "The Sullivan conjecture on maps from classifying spaces". In: *Annals of Mathematics* (1984), pp. 39–87.

[MNN17] A. Mathew, N. Naumann, and J. Noel. "Nilpotence and Descent in Equivariant Stable Homotopy Theory". In: Adv. in Math. 305 (2017), pp. 994–1084.

[MNN19] Akhil Mathew, Niko Naumann, and Justin Noel. "Derived induction and restriction theory". In: *Geometry & Topology* 23.2 (Apr. 2019), pp. 541–636.

[MV07] Ivan Mirković and Kari Vilonen. "Geometric Langlands duality and representations of algebraic groups over commutative rings". In: *Annals of mathematics* (2007), pp. 95–143.

[Nad04] David Nadler. *Matsuki correspondence for the affine Grassmannian*. 2004. arXiv: math/0301091 [math.AG].

[Nee01] Amnon Neeman. "On the derived category of sheaves on a manifold". In: *Documenta Mathematica* 6 (2001), pp. 483–488.

[NS18] T. Nikolaus and P. Scholze. "On Topological Cyclic Homology". In: *Acta Math.* 221.2 (2018), pp. 203–409.

[Pre86] Andrew Pressley. "Loop groups". In: *Oxford Sc. Publ.* (1986).

[Pst22] Piotr Pstrągowski. *Synthetic spectra and the cellular motivic category*. 2022. arXiv: 1803.01804 [math.AT].

[Qui69] Daniel Quillen. "Rational homotopy theory". In: *Annals of Mathematics* (1969), pp. 205–295.

[Qui71a] Daniel Quillen. "The spectrum of an equivariant cohomology ring: I". In: *Annals of Mathematics* 94.3 (1971), pp. 549–572.

[Qui71b] Daniel Quillen. "The spectrum of an equivariant cohomology ring: II". In: *Annals of Mathematics* 94.3 (1971), pp. 573–602.

[RW22a] Simon Riche and Geordie Williamson. "Smith–Treumann theory and the linkage principle". In: *Publications mathématiques de l'IHÉS* 136.1 (2022), pp. 225–292.

[RW22b] Simon Riche and Geordie Williamson. *Smith-Treumann theory and the linkage principle*. 2022. arXiv: 2003.08522 [math.RT].

[Sad92] Hal Sadofsky. "The root invariant and v1-periodic families". In: *Topology* 31.1 (1992), pp. 65–111.

[Seg68] Graeme Segal. "Equivariant K-theory". In: *Publications Mathématiques de l'IHÉS* 34 (1968), pp. 129–151.

[Smi34] Paul A Smith. "A theorem on fixed points for periodic transformations". In: *Annals of Mathematics* (1934), pp. 572–578.

[Sul70] Dennis Sullivan. "Geometric topology, part I: Localization, periodicity and Galois symmetry". In: *Notes, MIT* (1970).

[Sul77] Dennis Sullivan. "Infinitesimal computations in topology". In: *Publications Mathématiques de l'IHÉS* 47 (1977), pp. 269–331.

[Tre19] David Treumann. "Smith theory and geometric Hecke algebras". In: *Mathematische Annalen* 375.1-2 (2019), pp. 595–628.

[Ver65] Jean-Louis Verdier. "Dualité dans la cohomologie des espaces localement compacts". In: *Séminaire Bourbaki* 9 (1965), pp. 337–349.

[Vol17] M. Volpe. "The six operations in topology". In: *arXiv:2110.10212* (2017).

[Vri77] Jan de Vries. "Equivariant embeddings of G-spaces". In: *General topology and its relations to modern analysis and algebra IV* (1977), pp. 485–493.

[Wil19] Dylan Wilson. *Mod 2 power operations revisited*. 2019. arXiv: `1905.00054 [math.AT]`.

[Zhu16] Xinwen Zhu. *An introduction to affine Grassmannians and the geometric Satake equivalence*. 2016. arXiv: `1603.05593 [math.AG]`.